PERIODIC STRUCTURE OF TRANSLATIONAL MULTI-TILINGS IN THE PLANE

BOCHEN LIU

ABSTRACT. Suppose $f \in L^1(\mathbb{R}^d)$, $\Lambda \subset \mathbb{R}^d$ is a finite union of translated lattices such that $f + \Lambda$ tiles with a weight. We prove that there exists a lattice $L \subset \mathbb{R}^d$ such that $f + L$ also tiles, with a possibly different weight. As a corollary, together with a result of Kolountzakis, it implies that any convex polygon that multi-tiles the plane by translations admits a lattice multi-tiling, of a possibly different multiplicity.

Our second result is a new characterization of convex polygons that multi-tile the plane by translations. It also provides a very efficient criteria to determine whether a convex polygon admits translational multi-tilings. As an application, one can easily construct symmetric $(2m)$-gons, for any $m \geqslant 4$, that do not multi-tile by translations.

Finally, we prove a convex polygon which is not a parallelogram only admits periodic multiple tilings, if any.

1. Introduction

1.1. **Tiling and multiple tiling.** Let $P \subset \mathbb{R}^d$ be a convex body and $\Lambda \subset \mathbb{R}^d$ be a discrete multi-set, which means Λ is discrete and each point has finite multiplicity in $\mathbb{Z}_+$. Denote χ_P as the indicator function of P and

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda,$$

where δ_λ is the Dirac measure at λ. We say that $P + \Lambda$ tiles if for almost all $x \in \mathbb{R}^d$,

$$\chi_P * \delta_\Lambda(x) = \sum_{\lambda \in \Lambda} \chi_P(x - \lambda) = \sum_{\lambda \in \Lambda} \chi_{P+\lambda}(x) = 1. \tag{1.1}$$

We say $P + \Lambda$ multi-tiles, or is a multiple tiling, of multiplicity $k \in \mathbb{Z}_+$, if for almost all $x \in \mathbb{R}^d$,

$$\chi_P * \delta_\Lambda(x) = k. \tag{1.2}$$

More generally we say $f + \Lambda$ tiles with a weight $w \in \mathbb{R}$, where $f \in L^1(\mathbb{R}^d)$, if for almost all $x \in \mathbb{R}^d$,

$$f * \delta_\Lambda(x) = w.$$

One can see under these definitions $P + \Lambda$ is equivalent to $\chi_P + \Lambda$.

Throughout this paper a lattice of $\mathbb{R}^d$ is a discrete subgroup of the additive group $\mathbb{R}^d$ which is isomorphic to the additive group $\mathbb{Z}^d$.

The study of translational tilings by convex bodies has a long history. It has been known for a long time that the only convex domains that tile the plane by translations are parallelograms and hexagons. In 1885, Fedorov classified three-dimensional convex polytopes which can tile by translations into 5 different combinatorial types. In 1897,

2010 *Mathematics Subject Classification.* 52C20, 05B45.

Key words and phrases. multiple tiling, periodic tiling.

Research supported by ERC Starting Grant No. 713927.

Minkowski ([22]) showed that if a convex body P tiles R^d by lattice translations, then P must be a centrally symmetric polytope, with centrally symmetric facets. Finally Venkov ([24]) gave a characterization, which was later rediscovered by McMullen ([21]), of convex bodies that tile $\mathbb{R}^d$ by translations.

Theorem 1.1 (Venkov, 1954 & McMullen, 1980)**.** *Let P be a convex body in $\mathbb{R}^d$. Then P tiles by translations if and only if the following four conditions are satisfied:*

1. *P is a polytope.*
2. *P is centrally symmetric.*
3. *All facets of P are centrally symmetric.*
4. *Each "belt" of P consists of 4 or 6 facets.*

Here by a facet one means a $(d-1)$-dimensional face, and by a belt one means the collection of its facets which contain a translate of a given subfacet, that is, a $(d-2)$-dimensional face, of P.

As a consequence of Venkov-McMullen theorem, it follows that if a convex polytope P tiles, it admits a face-to-face tiling by translates along a certain lattice.

The study of multiple translational tilings dates back to 1936, when the famous Minkowski conjecture for tilings was extended to multiple tilings by Furtwängler ([6]). For more information about this problem, one can see, for example, [28], Chapter 6, 7, 8. It was showed by Bolle ([2]) that in the plane, every convex domain that admits lattice multi-tilings has to be a centrally symmetric polygon. More generally, it is well known that a convex body in $\mathbb{R}^d$ that multi-tiles by translations must be a convex polytope (see Appendix). In [9], Gravin, Robins and Shiryaev showed that these polytopes must be centrally symmetric with centrally symmetric facets. This implies in dimension 2, 3, a convex body P multi-tiles only if it is a zonotope. Therefore in the rest of this paper we assume $P \subset \mathbb{R}^2$ is the zonotope generated by pairwise non-colinear vectors $e_1, \dots, e_m$, of increasing arguments (see Figure 1.1), that is,

$$P = \left\{ \alpha_1 e_1 + \cdots + \alpha_m e_m : \alpha_j \in [-\frac{1}{2}, \frac{1}{2}] \right\}.$$

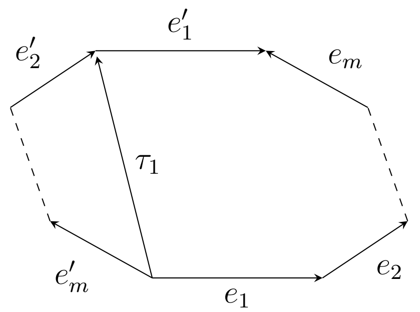


FIGURE 1.1. The zonotope generated by $e_1, \dots, e_m$, of increasing arguments

Now edges of P consist of translated $e_1, \dots, e_m$ and their parallel edges $e_1', \dots, e_m'$. We denote τ_j as the vector that translates e_j to its parallel edge.

Besides the structure of P, one can also study the structure of the discrete multi-set Λ such that $P+\Lambda$ multi-tiles. In 2000, Kolountzakis ([13]) proved the following result in the plane.

Theorem 1.2 (Kolountzakis, 2000)**.** *Suppose $P \subset \mathbb{R}^2$ is a convex polygon which is not a parallelogram, Λ is a multi-set such that $P+\Lambda$ multi-tiles. Then Λ must be a finite union of translated lattices.*

A three dimensional version of this result was later obtained by Gravin, Kolountzakis, Robins and Shiryaev ([8]). They proved that the same conclusion on Λ holds if a convex polytope $P \subset \mathbb{R}^3$ multi-tiles with Λ and P is not a two-flat zonotope. They also constructed multiple tilings by two-flat zonotopes where the discrete sets are not finite unions of translated lattices. Here by a two-flat zonotope one means the Minkowski sum of finitely many line segments which lie in the union of two different two-dimensional subspaces.

If, in particular, Λ is given as a lattice, Bolle ([2]) used combinatorial methods to give a characterization of convex polygons that multi-tiles the plane with Λ. An equivalent formulation is the following.

Theorem 1.3 (Bolle, 1994)**.** *Let P be a convex polygon in $\mathbb{R}^2$, and L be a lattice in $\mathbb{R}^2$. Then $P+L$ multi-tiles if and only if P is centrally symmetric, and for each pair of parallel edges e and e' of P one of the following conditions is satisfied:*

1. *The translation vector, τ, which carries e onto e', is in L, or*
2. *$e \in L$ and there exists $t \in \mathbb{R}$ such that $te+\tau \in L$.*

A general dimensional version of Bolle's theorem was recently obtained by Lev and the author ([19]), via a Fourier-analytic approach.

There is also some work on planar translational multi-tilings with given multiplicities. See, for example, Section 5, 6, 7 in Zong's survey [29].

1.2. **Main results.** Throughout this paper we say a (multiple) tiling $P+\Lambda$ is periodic if there exists a lattice L such that $\Lambda+L=\Lambda$, which is equivalent to that Λ is a finite union of translations of L.

1.2.1. *Periodic tiling conjecture.* One famous open problem on tiling is the periodic tiling conjecture ([11], [17]), which states any region that tiles $\mathbb{R}^d$ by translations has a periodic tiling. Here a region is a closed subset of $\mathbb{R}^d$ whose boundary has measure 0.

In the real line this conjecture was confirmed by Lagarias and Wang ([17]) for bounded regions. In the plane it is proved, first by Girault-Beauquier and Nivat ([7]) with boundary conditions and finally by Kenyon ([12]), for closed topological discs.

For convex bodies (equivalently, convex polytopes) in general $\mathbb{R}^d$, it follows from Venkov-McMullen's result that if P tiles by translations then it admits a lattice tiling.

The periodic tiling conjecture in $\mathbb{Z}^2$ has been solved by Bhattacharya ([1]).

It is natural to extend this problem to multiple tilings. The following question was raised by Gravin, Robins and Shiryaev (see Problem 7.3 in [9]).

Problem 1.4. *Prove or disprove that if a convex polytope multi-tiles $\mathbb{R}^d$ by translations, then it also multi-tiles $\mathbb{R}^d$ by a lattice, for a possibly different multiplicity.*

Problem 1.4 is open in all dimensions. It is mentioned in [3] that Shiryaev has a proof in the plane, but it is never posted. Theorem 1.2 and the discussion afterwards show that, in low dimensions it is quite often that Λ is a finite union of translated lattices. Inspired by this, Chan asked the following weaker version in [3], where he also considered two special cases.

Problem 1.5. *Let P be a convex polytope that multi-tiles $\mathbb{R}^d$ with a discrete multiset Λ, which is a finite union of translated lattices. Prove or disprove that P could multi-tile $\mathbb{R}^d$ with a lattice.*

In this paper we solve Problem 1.5 in all dimensions. In fact we prove a stronger statement that works for any $f \in L^1(\mathbb{R}^d)$. Fourier analysis and number theory play important roles in the proof.

Theorem 1.6. *Suppose $f \in L^1(\mathbb{R}^d)$ and Λ is a discrete multi-set such that*

$$\delta_\Lambda = \sum_{j=1}^{n} \delta_{L_j + z_j},$$

where L_j are lattices and $z_j \in \mathbb{R}^d$. If $f + \Lambda$ tiles with a weight, then for each j there exists a lattice $\widetilde{L_j}$ containing L_j such that $f + \widetilde{L_j}$ also tiles, with a possibly different weight.

The author does not know if the assumption that Λ is a finite union of translated lattices is necessary. An interesting question is, does there exist a function $f \in L^1(\mathbb{R}^d)$ that tiles by translations but does not tile by any lattice with any weight? This question is also asked by Kolountzakis and Lev in [16], where they construct non-periodic tilings by some $f \in L^1(\mathbb{R})$, which also admit periodic tilings.

Together with Theorem 1.2 we solve Problem 1.4 in the plane.

Corollary 1.7. *If a convex polygon P multi-tiles the plane by translations, it also multi-tiles the plane by a lattice.*

As a remark, we remind the reader that Theorem 1.2 remains valid for non-convex polygons with the pairing property (see [13]), which means for each edge e there is precisely one other edge parallel to e. Therefore Corollary 1.7 holds for non-convex polygons with the pairing property as well.

Independently, Corollary 1.7 is obtained by Yang ([25]). Her argument is also based on Theorem 1.2 of Kolountzakis, while elementary and purely combinatorial, thus completely different from ours. Her method does not seem to yield Theorem 1.6.

1.2.2. *A new characterization of convex polygons that multi-tile the plane.* Since Corollary 1.7 holds, Bolle's theorem (Theorem 1.3) has automatically become a characterization of convex polygons that multi-tiles. However, it is not a very efficient criteria to determine whether a convex polygon multi-tiles. To apply Bolle's theorem, one needs to find a subset J of $\{1, \dots, m\}$ such that

$$span_{\mathbb{Z}}\{e_j, \tau_{j'} : j \in J, j' \notin J\}$$

is a lattice. If we check this condition in the brute-force way, the computation complexity is exponential in terms of m. Then it is natural to look for a more efficient criteria, where the complexity has polynomial growth as m increases. Our second main result in this paper is a new characterization, as well as an efficient criteria, on convex polygons that multi-tile by translations. The proof is based on Theorem 1.6 and Bolle's theorem (Theorem 1.3).

Theorem 1.8. *Suppose P is a convex symmetric polygon as in Figure 1.1 which is not a parallelogram. Then P admits multiple translational tilings if and only if*

1. *m is odd and $\Lambda_\tau := span_{\mathbb{Z}}\{\tau_1, \dots, \tau_m\}$ is a lattice, or*
2. *m is even and there exists $1 \leqslant j_0 \leqslant m$ such that*
   (a) *$\Lambda_{j_0} := span_{\mathbb{Z}}\{\tau_1, \dots, \tau_{j_0-1}, \tau_{j_0+1}, \dots, \tau_m\}$ is a lattice, and*
   (b) *$\det(e_{j_0}, \tau_{j_0})$ is an rational multiple of $\det(\Lambda_{j_0})$.*

Moreover, if P multi-tiles by translations, then

$$L_P := \begin{cases} \Lambda_\tau, & m \text{ is odd} \\ \bigcap_{j:\, \Lambda_j \text{ is a lattice}} \Lambda_j, & m \text{ is even} \end{cases} \tag{1.3}$$

is a lattice and $L_P \cap L$ is a lattice for any lattice multi-tiling $P + L$.

As an application, one can easily construct convex symmetric $(2m)$-gons, for any $m \geqslant 4$, that do not multi-tile by translations (see Example 6.1 in Section 6). As far as the author knows, these are first known symmetric polygons that do not multi-tile by translations. On the other hand, there are many symmetric $(2m)$-gons that do multi-tile by translations. For example $P + L$ multi-tiles if P is symmetric whose vertices lie in a lattice L (see [9], [19]). This means, unlike tiling (Venkov-McMullen), one can not determine whether a polygon multi-tiles only by its combinatorial type.

1.2.3. *Periodic multiple tilings.* The last problem we consider is whether a multiple tiling must be periodic.

In $\mathbb{R}$, it was proved by Lagarias and Wang ([17]) that a bounded region only admits periodic tilings. This result was extended by Kolountzakis and Lagarias ([15]) (and proved earlier by Leptin and Müller in [18]) to tilings by a function $f \in L^1(\mathbb{R})$ with compact support. More precisely they showed if $f + \Lambda$ tiles and Λ has bounded density, then $f + \Lambda$ is a finite union of periodic tilings, with weights. Later Kolountzakis and Lev ([16]) showed the assumption f has compact support is necessary. They also proved that if the translation set has finite local complexity, then it must be periodic, even if the support of f is unbounded.

In this paper we answer this question for multiple tilings in the plane. The proof starts from Theorem 1.2 and eventually we improve it from "a finite union of translated (possibly different) lattices" to "a finite union of translations of a single lattice".

Theorem 1.9. *Suppose $P \subset \mathbb{R}^2$ is a convex polygon which is not a parallelogram. Then every multiple tiling of P is periodic.*

It can be seen that in Theorem 1.9 both non-parallelogram and convexity are necessary. In fact in either case there are multiple tilings $P + \Lambda$ where $\Lambda + \alpha \neq \Lambda$ for any $\alpha \in \mathbb{R}^2 \backslash \{0\}$. One can see Example 6.2 in Section 6.

In [17], Lagarias and Wang not only proved that $\Omega + \Lambda$ tiles the real line implies

$$\Lambda = \alpha\mathbb{Z} + \{\beta_1, \dots, \beta_n\},$$

but also showed $\beta_i - \beta_j, \forall 1 \leqslant i, j \leqslant n$, must be a rational multiple of α. This rationality result does not hold for tilings of compactly supported functions ([15]). Also it is easily seen to fail for tilings in higher dimensions (parallelepipeds), or decomposable multi-tilings ($\Lambda \cup (\Lambda + z)$ where $\Omega + \Lambda$ tiles). In this paper, we give examples to show, even for indecomposable multi-tilings by convex symmetric polygons that are not parallelograms, rationality still fails. See Example 6.3 in Section 6.

1.3. **Other applications.**

1.3.1. *Dimension* 1. With results of Lagarias-Wang, Kolountzakis-Lagarias, Leptin-Müller, Kolountzakis-Lev introduced above, Theorem 1.6 implies the following. The case of f of compat support has been proved in [15], Theorem 1.2.

Corollary 1.10. *Suppose $f \in L^1(\mathbb{R})$ with compact support and $\Lambda \subset \mathbb{R}$ has bounded density, or $f \in L^1(\mathbb{R})$ and $\Lambda \subset \mathbb{R}$ has finite local complexity. If $f + \Lambda$ tiles with a weight, then there exists a lattice $L \subset \mathbb{R}$ such that $f + L$ also tiles, with a possibly different weight.*

1.3.2. *Higher dimensions.* As we introduced right after Theorem 1.2, Gravin, Kolountzakis, Robins and Shiryaev ([8]) proved that if $P \subset \mathbb{R}^3$ is a convex polytope but not a two-flat zonotope and $P + \Lambda$ multi-tiles, then Λ must be a finite union of translated lattices. By Theorem 1.6 we obtain the following partial result on Problem 1.4 in $\mathbb{R}^3$.

Corollary 1.11. *Suppose $P \subset \mathbb{R}^3$ is a convex polytope, which is not a two-flat zonotope, and P multi-tiles by translations. Then there exists a lattice $L \subset \mathbb{R}^3$ such that $P + L$ multi-tiles.*

Although Gravin, Kolountzakis, Robins and Shiryaev ([8]) gave examples of two-flat zonotopes which admit weird multiple tilings, their examples admit periodic multi-tilings as well. So whether Corollary 1.11 holds for general convex polytopes in $\mathbb{R}^3$ is still unknown.

There is very little known in dimension 4 and higher. In fact there exists centrally symmetric polytopes, with centrally symmetric facets, that multi-tile by translations but are not zonotopes (e.g. the 24-cell in $\mathbb{R}^4$), which makes the study of multiple tilings in higher dimensions more difficult than in lower dimensions.

1.3.3. *Riesz basis.* We say $\Omega \subset \mathbb{R}^d$ admits an exponential Riesz basis if there exists a discrete set $\Lambda \subset \mathbb{R}^d$, $A, B > 0$ such that

$$A||f||^2_{L^2(\Omega)} \leqslant \sum_{\lambda \in \Lambda} \left| \widehat{f\chi_\Omega}(\lambda) \right|^2 \leqslant B||f||^2_{L^2(\Omega)}$$

and

$$A \sum_{\lambda \in \Lambda} |c_\lambda|^2 \leqslant \int_\Omega \left| \sum_{\lambda \in \Lambda} c_\lambda e^{-2\pi i x \cdot \lambda} \right|^2 dx \leqslant B \sum_{\lambda \in \Lambda} |c_\lambda|^2.$$

The connection between multiple tiling and exponential Riesz basis was first discovered by Grepstad and Lev ([10]) in 2014, and later reproved by Kolountzakis ([14]) in 2015 with an elementary argument. They proved that if a bounded region $\Omega \subset \mathbb{R}^d$ multi-tiles by a lattice, then it admits an exponential Riesz basis.

Since we have proved that every convex polygon P that multi-tiles the plane admits a lattice multi-tiling, it follows that a convex polygon admits an exponential Riesz basis if it multi-tiles (no need to assume lattice multi-tiling). Also a sufficient condition on the existence of exponential Riesz bases follows from Theorem 1.8.

Corollary 1.12. *Let P be a convex polygon in the plane. Then P admits an exponential Riesz basis if it admits translational multi-tilings.*

Similar to the remark right after Corollary 1.7, this corollary also holds for non-convex polygons with the pairing property.

During the referee process, Debernardi and Lev [4] proved a much stronger result. They show that any zonotope in $\mathbb{R}^d$, $d \geqslant 1$, admits an exponential Riesz basis. Notice in the plane any convex multiple tile is necessarily a zonotope, but not vice versa.

Organization. This paper is organized as follows. In Section 2 we review useful tools from Fourier analysis and number theory. Then we prove Theorem 1.6, 1.8, 1.9 in Section 3, 4, 5, respectively. In Section 6 we discuss some examples. In the Appendix, we give a proof of that any convex body in $\mathbb{R}^d$ that multi-tiles by translations must be a convex polytope.

Notation. Throughout this paper a lattice of $\mathbb{R}^d$ is a discrete subgroup of the additive group $\mathbb{R}^d$ which is isomorphic to the additive group $\mathbb{Z}^d$.

We say Λ is a finite union of translated lattices if it is a multi-set and

$$\delta_\Lambda = \sum_{j=1}^{n} \delta_{L_j + z_j},$$

where L_j are (possibly different) lattices and $z_j \in \mathbb{R}^d$.

We say a (multiple) tiling $P + \Lambda$ is periodic if there exists a lattice L such that $\Lambda + L = \Lambda$, which is equivalent to that Λ is a finite union of translations of L.

Acknowledgment. The author would like to thank Nir Lev for bringing this subject to attention, useful discussions, and comments on manuscript.

2. Preliminaries

2.1. Fourier analysis.

Let L be a lattice in $\mathbb{R}^d$ and denote its dual lattice as

$$L^* = \{\lambda^* \in \mathbb{R}^d : \lambda^* \cdot \lambda \in \mathbb{Z}, \forall \lambda \in L\}.$$

For $f \in L^1(\mathbb{R}^d)$, define its Fourier transform as

$$\widehat{f}(\xi) = \int_{R^d} e^{-2\pi i x\cdot\xi} f(x)\, dx.$$

Denote $\det(L)$ as the volume of a fundamental domain of a lattice $L \subset \mathbb{R}^d$. The well-known Poisson summation formula can be stated as

$$\sum_{\lambda\in L} \phi(\lambda + z) = \frac{1}{\det(L)} \sum_{\lambda^*\in L^*} e^{2\pi i\lambda^*\cdot z}\, \widehat{\phi}(\lambda^*)$$

for any Schwartz function ϕ. In the sense of distributions, it is equivalent to

$$\widehat{\delta_{L+z}}(\xi) = \frac{1}{\det(L)} \sum_{\lambda^*\in L^*} e^{-2\pi i\xi\cdot z}\, \delta_{\lambda^*}(\xi) = \frac{e^{-2\pi i\xi\cdot z}}{\det(L)}\, \delta_{L^*}(\xi). \tag{2.1}$$

We also need the following well-known lemma that connects Fourier analysis and multiple tilings.

Lemma 2.1. *Let $f \in L^1(\mathbb{R}^d)$ and L be a lattice in $\mathbb{R}^d$. Then $f + L$ tiles with a weight if and only if $\widehat{f}$ vanishes on $L^*\backslash\{0\}$.*

We give a proof below for completeness.

Proof. We may assume $L = \mathbb{Z}^d$. Let

$$F(x) := \sum_{\lambda \in \mathbb{Z}^d} f(x - \lambda),$$

then F is a $\mathbb{Z}^d$-periodic function whose Fourier series is given by

$$\sum_{\lambda \in \mathbb{Z}^d} \widehat{f}(\lambda) e^{2\pi i \langle \lambda, x \rangle}$$

(see e.g. [23], Chapter VII, Theorem 2.4). Hence f coincides a.e. with a constant function, if and only if $\widehat{f}$ vanishes on $\mathbb{Z}^d \setminus \{0\}$. □

2.2. **Solutions of linear equations.** Let K be an algebraically closed field of characteristic 0 and denote $K\backslash\{0\}$ as its multiplicative group of nonzero elements. Let $(a_1, \ldots, a_n) \in (K\backslash\{0\})^n$ and Γ be a subgroup of $(K\backslash\{0\})^n$. One may ask how many solutions does the linear equation

$$a_1 x_1 + \cdots + a_n x_n = 1 \tag{2.2}$$

have with $(x_1, \ldots, x_n) \in \Gamma$. This problem has been studied for a long time in number theory and literature dates back to early 1930s (e.g., [20]). Finally, in 2002, Evertse, Schlickewei and Schmidt ([5]) proved the following celebrated result.

We say Γ has finite rank r, if there exists a finitely generated subgroup Γ_0 of Γ, again of rank r, such that the factor group Γ/Γ_0 is a torsion group.

Theorem 2.2 (Evertse, Schlickewei, Schmidt, 2002)**.** *With notation above, suppose Γ has finite rank r. Then $A(a_1, \ldots, a_n, \Gamma)$, the number of non-degenerate solutions $(x_1, \ldots, x_n) \in \Gamma$ of equation (2.2) satisfies the estimate*

$$A(a_1, \ldots, a_n, \Gamma) \leqslant A(n, r) = \exp\left((6n)^{3n}(r+1)\right).$$

Here a solution $(x_1, \ldots, x_n) \in \Gamma$ is called non-degenerate if $\sum_{i \in I} a_i x_i \neq 0$ for every nonempty subset $I \subset \{1, \ldots, n\}$.

In particular, given $z_1, \ldots, z_n \in \mathbb{R}^d$ and a lattice $L \subset \mathbb{R}^d$, take $K = \mathbb{C}$ and

$$\Gamma_{z_1,\ldots,z_n,L} = \{(e^{-2\pi i \lambda \cdot z_1}, \ldots, e^{-2\pi i \lambda \cdot z_n}) : \lambda \in L\}.$$

The following corollary plays an important role in our proof of Theorem 1.6.

Corollary 2.3. *Given $a_1, \ldots, a_n \in \mathbb{C}\backslash\{0\}$, $z_1, \ldots, z_n \in \mathbb{R}^d$ and a lattice $L \subset \mathbb{R}^d$, then for any nonempty subset $I \subset \{1, \ldots, n\}$, the linear equation*

$$\sum_{i \in I} a_i x_i = 1$$

has finitely many non-degenerate solutions in $\Gamma_{z_1,\ldots,z_n,L}$. In particular,

$$\#\{(x_1, \ldots, x_n) \in \Gamma_{z_1,\ldots,z_n,L} : \exists\, \emptyset \neq I \subset \{1, \ldots, n\}, \sum_{i \in I} a_i x_i = 1\} < \infty$$

3. Proof of Theorem 1.6

Now Λ is a finite union of translated lattices,

$$\delta_\Lambda = \sum_{j=1}^{n} \delta_{L_j+z_j}.$$

Without loss of generality, we may assume $L_1 = \mathbb{Z}^d$, $z_1 = 0$ and $n \geqslant 2$. Then, by Lemma 2.1, it suffices to find a lattice $L^* \subset \mathbb{Z}^d$ such that $\widehat{f}$ vanishes on $L^*\backslash\{0\}$.

By Poisson summation formula (2.1),

$$\widehat{\delta_\Lambda}(\xi) = \delta_{\mathbb{Z}^d} + \sum_{j=2}^{n} \frac{e^{-2\pi i\xi\cdot z_j}}{\det(L_j)} \delta_{L_j^*}(\xi).$$

Denote

$$\omega_j(\lambda^*) = \begin{cases} 1, & \lambda^* \in L_j^* \\ 0, & otherwise \end{cases} . \tag{3.1}$$

In a small neighborhood U_{λ^*} of each $\lambda^* \in \mathbb{Z}^d$,

$$\widehat{\delta_\Lambda}\Big|_{U_{\lambda^*}} = \left(1 + \sum_{j\geqslant 2} \omega_j(\lambda^*) \frac{e^{-2\pi i\lambda^*\cdot z_j}}{\det(L_j)}\right) \delta_{\lambda^*}.$$

Since $f * \delta_\Lambda$ is a constant almost everywhere, its Fourier transform is a multiple of δ_0. Therefore on a small neighborhood U_{λ^*} of each $\lambda^* \in \mathbb{Z}^d\backslash\{0\}$,

$$0 = \widehat{f * \delta_\Lambda}\Big|_{U_{\lambda^*}} = \widehat{f} \cdot \widehat{\delta_\Lambda}\Big|_{U_{\lambda^*}} = \left(1 + \sum_{j\geqslant 2} \omega_j(\lambda^*) \frac{e^{-2\pi i\lambda^*\cdot z_j}}{\det(L_j)}\right) \widehat{f}(\lambda^*)\, \delta_{\lambda^*},$$

which implies that for any $\lambda^* \in \mathbb{Z}^d\backslash\{0\}$, either $\widehat{f}(\lambda^*) = 0$ or

$$\sum_{j\geqslant 2} -\omega_j(\lambda^*) \frac{1}{\det(L_j)} e^{-2\pi i\lambda^*\cdot z_j} = 1.$$

Therefore, to find a lattice $L^* \subset \mathbb{Z}^d$ such that $\widehat{f}$ vanishes on $L^*\backslash\{0\}$, it suffices to find a lattice $L^* \subset \mathbb{Z}^d$ such that

$$\sum_{j\in I} -\frac{1}{\det(L_j)} e^{-2\pi i\lambda^*\cdot z_j} \neq 1$$

for any $\lambda^* \in L^*\backslash\{0\}$ and any nonempty subset $I \subset \{2, \ldots, n\}$.

Lemma 3.1.

$$\left\{\lambda^* \in \mathbb{Z}^d : \exists\, \emptyset \neq I \subset \{2, \ldots, n\}, \sum_{j\in I} -\frac{1}{\det(L_j)} e^{-2\pi i\lambda^*\cdot z_j} = 1\right\}$$

is a finite union of cosets of subgroups of $\mathbb{Z}^d$*, where each coset does not contain the origin.*

Proof. By Corollary 2.3,

$$\left\{(e^{-2\pi i\lambda^*\cdot z_1}, \ldots, e^{-2\pi i\lambda^*\cdot z_n}) : \lambda^* \in \mathbb{Z}^d, \exists\, \emptyset \neq I \subset \{2, \ldots, n\}, \sum_{j\in I} -\frac{e^{-2\pi i\lambda^*\cdot z_j}}{\det(L_j)} = 1\right\} \tag{3.2}$$

has finitely many elements. Then it suffices to show for each solution $(e^{-2\pi i\lambda_0^*\cdot z_1},\dots,e^{-2\pi i\lambda_0^*\cdot z_n})$, $\lambda_0^*\in\mathbb{Z}^d$, of

$$\sum_{j\in I}-\frac{1}{\det(L_j)}e^{-2\pi i\lambda^*\cdot z_j}=1,$$

for some $\emptyset\neq I\subset\{2,\dots,n\}$, the set

$$L_{\lambda_0^*}:=\left\{\lambda^*\in\mathbb{Z}^d: e^{-2\pi i\lambda^*\cdot z_j}=e^{-2\pi i\lambda_0^*\cdot z_j}, j\in I\right\}$$

is a coset of a subgroup of $\mathbb{Z}^d$ which does not contain the origin. It is easy to see $L_{\lambda_0^*}$ is a coset of

$$\left\{\lambda^*\in\mathbb{Z}^d: e^{-2\pi i\lambda^*\cdot z_j}=1, j\in I\right\}.$$

For any $\emptyset\neq I\subset\{2,\dots,n\}$, since $(1,\dots,1)$ is not a solution of

$$\sum_{j\geqslant 2,\ j\in I}-\frac{x_j}{\det(L_j)}=1,$$

one concludes that $L_{\lambda_0^*}$ does not contain the origin. □

Then Theorem 1.6 follows by applying the following lemma finitely many times.

Lemma 3.2. *Let $L\subset\mathbb{R}^d$ be a lattice and $V\subset L$ be a subgroup. Then for any $\tau_V\in L\backslash V$, there exists a lattice $L'\subset L$ such that $L'\cap(V+\tau_V)=\emptyset$.*

Proof. If $\dim(span_{\mathbb{Z}}\{V,\tau_V\})=\dim(V)$, find $u_{\dim(V)+1},\dots,u_d\in L$, if necessary, such that

$$L':=span_{\mathbb{Z}}\{V,u_{\dim(V)+1},\dots,u_d\}$$

is a lattice. By our construction, $\tau_V\notin L'$. Hence $V+\tau_V\subset L'+\tau_V$, a coset of L' in L which does not intersect L', as desired.

If $\dim(span_{\mathbb{Z}}\{V,\tau_V\})>\dim(V)$, find $u_{\dim(V)+2},\dots,u_d\in L$, if necessary, such that

$$L':=span_{\mathbb{Z}}\{V,2\tau_V,u_{\dim(V)+2},\dots,u_d\}$$

is a lattice. Notice $L'\cap(V+\tau_V)$ is not empty if and only if there exist $v,v'\in V,\alpha\in\mathbb{Z}$ such that

$$v+2\alpha\tau_V=v'+\tau_V.$$

If it happens, $(2\alpha-1)\tau_V\in V$, which contradicts the assumption that $\dim(span_{\mathbb{Z}}\{V,\tau_V\})>\dim(V)$. □

4. Proof of Theorem 1.8

We first study relations between e and τ in planar zonotopes, equivalently symmetric convex polygons.

Lemma 4.1. *Let P be a zonotope as in Figure 1.1. Then*

$$\tau_j-\tau_{j+1}=e_j+e_{j+1},\ \forall\ j=1,\dots,m-1.$$

Furthermore, if m is even, then for any $1\leqslant j\leqslant m$, e_j is a linear combination of $\tau_{j'},j'\neq j$, with coefficients ±1.

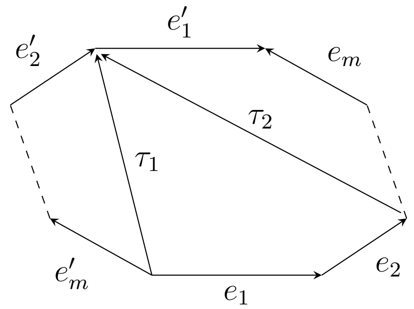


FIGURE 4.1. $\tau_1 - \tau_2 = e_1 + e_2$

Proof. For convenience, denote $e_{j+m} = -e_j, 1 \leqslant j \leqslant m$. It follows that

$$\begin{aligned} \tau_1 =& e_2 + \cdots + e_m \\ \tau_2 =& e_3 + \cdots + e_{m+1} \\ &\vdots \\ \tau_m =& e_{m+1} + \cdots + e_{2m-1} \end{aligned} \quad . \tag{4.1}$$

Then the differences between adjacent equalities imply

$$\tau_j - \tau_{j+1} = e_j + e_{j+1},\ j = 1, \ldots, m-1 \text{ (see Figure 4.1).} \tag{4.2}$$

If m is even, by (4.1), (4.2),

$$\begin{aligned} \tau_1 &= (e_2 + e_3) + \cdots + (e_{m-2} + e_{m-1}) + e_m \\ &= (\tau_2 - \tau_3) + \cdots + (\tau_{m-2} - \tau_{m-1}) + e_m, \end{aligned} \tag{4.3}$$

as desired. Similar argument works for any e_j, $1 \leqslant j \leqslant m$. □

We also need a quantitative version of condition 2 in Bolle's theorem (Theorem 1.3).

Lemma 4.2. *Suppose $L \subset \mathbb{R}^2$ is a lattice and $e \in L$, $\tau \in \mathbb{R}^2$. If there exists $t \in \mathbb{R}$ such that $te + \tau \in L$, then $\det(e, \tau)$ is an integer multiple of $\det(L)$. Conversely, if $\det(\tau, e) \in \det(L)\mathbb{Q}$, there exists $t \in \mathbb{R}$ and a lattice $\widetilde{L}$ containing L such that $te+\tau \in \widetilde{L}$.*

Proof. We may assume $L = \mathbb{Z}^2$. If $te + \tau = u \in \mathbb{Z}^2$, then

$$\det(e, \tau) = \det(e, te + \tau) = \det(e, u) \in \mathbb{Z}.$$

Conversely, if $\det(\tau, e) \in \mathbb{Q}$, choose t_0 such that $(t_0 e + \tau) \perp e$. Denote $e = (x_1, x_2)$ and $e^\perp = (-x_2, x_1)$. Say $(t_0 e + \tau) = Ce^\perp$. Since $e \in \mathbb{Z}^2$ and $\det(\tau, e) \in \mathbb{Q}$, both

$$\det(e, e^\perp),\ \det(e, \tau) = \det(e, t_0 e + \tau) = C \det(e, e^\perp)$$

are rational. Hence $C \in \mathbb{Q}$, $t_0 e + \tau \in \mathbb{Q}^2$ and $\widetilde{L} = span_{\mathbb{Z}}\{t_0 e + \tau, \mathbb{Z}^2\}$ is the desired lattice. □

Now we can prove Theorem 1.8. The "if" part follows from Theorem 1.3 and the second half of Lemma 4.2. The reason we do not need e_{j_0} in the definition of Λ_{j_0} is, when m is even, e_{j_0} is a linear combination of $\tau_j, j \neq j_0$, with coefficients ± 1 (see Lemma 4.1).

Conversely, assume P multi-tiles. By Corollary 1.7, P admits a lattice multi-tiling. Since the statement is invariant under non-degenerate linear transformations, we may assume P is as in Figure 1.1 and $P+\mathbb{Z}^d$ multi-tiles. Denote

$$J=\left\{j\in\{1,\dots,m\}:e_j\notin\mathbb{Z}^d\right\}.$$

Case 1. $\#(J)\geqslant 2$.

Say $J=\{j_1,\dots,j_{\#(J)}\}$. Denote $e_s^J=e_{j_s}, s=1,\dots,\#(J)$ and P_J as the zonotope generated by $e_s^J, s=1,\dots,\#(J)$, namely

$$P_J=\left\{\sum_{s=1}^{\#(J)}\alpha_s e_s^J:\alpha_s\in[-\frac{1}{2},\frac{1}{2}]\right\}.$$

Denote τ_s^J as the vector that translates e_s^J to its parallel edge in P_J.

Since $e_j, 1\leqslant j\leqslant m$ have increasing arguments, so do $e_s^J, 1\leqslant s\leqslant\#(J)$. Therefore Lemma 4.1 applies to P_J. We first show that $P_J+\mathbb{Z}^d$ is also a multiple tiling. To see this, observe that P can be obtained by "adding" integer vectors $e_j, j\notin J$ into P_J. More precisely, apply (4.1) to both P and P_J, one can see that for each $s=1,\dots,\#(J)$, the difference between τ_s^J (in P_J) and τ_{j_s} (in P) is a linear combination of $e_j, j\notin J$ with coefficients ±1, which implies $\tau_s^J-\tau_{j_s}\in\mathbb{Z}^2$. Since each pair $(e_j,\tau_j), j=1,\dots,m$, satisfies conditions in Theorem 1.3 with respect to $\mathbb{Z}^2$, it follows that $(e_s^J,\tau_s^J), s=1,\dots,\#(J)$ also satisfy conditions in Theorem 1.3 with $\mathbb{Z}^2$. Hence $P_J+\mathbb{Z}^2$ is a multiple tiling.

Claim 1. *$\#(J)$ must be odd and there exists $\gamma\in\mathbb{R}^2$ such that*

$$e_s^J\in\mathbb{Z}^d+(-1)^s\gamma,\ s=1,2,\dots,\#(J).$$

Since $(e_s^J,\tau_s^J), s=1,\dots,\#(J)$ satisfy conditions in Theorem 1.3 with respect to $\mathbb{Z}^2$ but $e_s^J\notin\mathbb{Z}^2$, it follows that all $\tau_s^J\in\mathbb{Z}^2$. By Lemma 4.1, $e_s^J+e_{s+1}^J\in\mathbb{Z}^2$ for any $1\leqslant s\leqslant\#(J)-1$, which implies there exists $\gamma\in\mathbb{R}^2$ such that

$$e_s^J\in\mathbb{Z}^d+(-1)^s\gamma,\ s=1,2,\dots,\#(J).$$

It remains to show $\#(J)$ must be odd. If $\#(J)$ is even, the second half of Lemma 4.1 implies all $e_s^J\in\mathbb{Z}^2$, contradiction. This completes the proof of Claim 1.

If $\gamma\in\mathbb{Q}^2$, then $e_j\in\mathbb{Q}^2$ for any $1\leqslant j\leqslant m$ and there is nothing to prove. So we may assume $\gamma\notin\mathbb{Q}^2$. Denote $J^c=\{1,\dots,m\}\backslash J$.

Claim 2. *$\#(J^c)=0,1$ if $\gamma\notin\mathbb{Q}^2$.*

Back to the original polygon P. Since $\#(J)$ is odd and

$$e_s^J\in\mathbb{Z}^d+(-1)^s\gamma,\ s=1,2,\dots,\#(J),$$

it follows that for any $j\notin J$, $\tau_j\in\mathbb{Z}^2\pm\gamma$ which does not lie in $\mathbb{Z}^2$. Thus condition 2 in Theorem 1.3 must hold for $(e_j,\tau_j),\forall j\notin J$, with respect to $\mathbb{Z}^2$. If $\#(J^c)\geqslant 2$, there are $e_{j_0}, e_{j_0'}\in\mathbb{Z}^2\backslash\{0\}$, not parallel to each other, and $t,t'\in\mathbb{R}$ such that both

$$te_{j_0}+\gamma\in\mathbb{Z}^2,\ t'e_{j_0'}+\gamma\in\mathbb{Z}^2.$$

See $e_{j_0}, e_{j_0'}$ as column vectors. Then

$$te_{j_0}-t'e_{j_0'}=(e_{j_0},e_{j_0'})\begin{pmatrix}t\\-t'\end{pmatrix}\in\mathbb{Z}^2.$$

Since $e_{j_0}, e'_{j'_0} \in \mathbb{Z}^2\backslash\{0\}$ are not parallel to each other, the matrix $(e_{j_0}, e_{j'_0}) \in \mathbb{Z}_{2\times 2}$ is non-degenerate. Then both t, t' are rational and $\gamma \in \mathbb{Z}^2 - te_{j_0} \subset \mathbb{Q}^2$, contradiction.

Now let us finish the case $\#(J) \geqslant 2$.

If $\gamma \in \mathbb{Q}^2$, all e_j are rational and conditions in Theorem 1.8 are satisfied.

If $\gamma \notin \mathbb{Q}$ and $J^c = \emptyset$, then $m = \#(J)$ must be odd. Also the definition of J and Theorem 1.3 imply all $\tau_j \in \mathbb{Z}^2$. Hence Λ_τ is a sub-lattice of $\mathbb{Z}^2$, as desired.

If $\gamma \notin \mathbb{Q}$ and $J^c = \{j_0\}$, then $m = \#(J) + 1$ must be even. As we discussed right before Claim 1, $\tau_s^J \in \mathbb{Z}^2$ implies $\tau_{j_s} \in \mathbb{Z}^2$. Therefore $\tau_j \in \mathbb{Z}^2$ for any $j \neq j_0$ and Λ_{j_0} is a sub-lattice of $\mathbb{Z}^2$. It remains to show $\det(e_{j_0}, \tau_{j_0}) \in \mathbb{Q}$. Since $\tau_{j_0} \notin \mathbb{Z}^2$ and $P + \mathbb{Z}^2$ multi-tiles, the pair (e_{j_0}, τ_{j_0}) must satisfy condition 2 in Theorem 1.3, which, by Lemma 4.2, implies $\det(e_{j_0}, \tau_{j_0}) \in \mathbb{Z}$, as desired.

Case 2. $\#(J) = 1$.

Say $J = \{j_0\}$. In this case $\#(J)$ is odd and there exists $\gamma \in \mathbb{R}^2$ such that $e_{j_0} \in \mathbb{Z}^2 - \gamma$. If $\gamma \in \mathbb{Q}$, all e_j are rational and there is nothing to prove. If $\gamma \notin \mathbb{Q}^2$, the proof of Claim 2 in Case 1 still works and P turns out to be a parallelogram.

Case 3. $\#(J) = 0$. Trivial.

Above all, we proved that if P is not a parallelogram and $P + \mathbb{Z}^2$ multi-tiles, then P must satisfy one of the following.

1. $e_j \in \mathbb{Q}^2$, for any $1 \leqslant j \leqslant m$. ($\#(J) = 0, 1$, or $\#(J) \geqslant 2, \gamma \in \mathbb{Q}^2$)
2. m is odd and $\tau_j \in \mathbb{Z}^2$ for any $1 \leqslant j \leqslant m$. ($\#(J) \geqslant 2, \gamma \notin \mathbb{Q}$, $\#(J^c) = 0$)
3. m is even, there is a unique $j_0 \in \{1, \ldots, m\}$ such that $\tau_j \in \mathbb{Z}^2$ for any $j \neq j_0$, $e_{j_0} \in \mathbb{Z}^2$ and $\det(e_{j_0}, \tau_{j_0})$ is a rational multiple of $\det(\Lambda_{j_0})$. ($\#(J) \geqslant 2, \gamma \notin \mathbb{Q}$, $\#(J^c) = 1$)

Any case above satisfies condition 1 or 2 in Theorem 1.8. Also one can see L_P is a discrete subgroup of additive group $\mathbb{Q}^2$. If one can show L_P is full-rank, it is not hard to check that $L_P \cap \mathbb{Z}^2$ is also full-rank, which completes the proof.

Now it remains to show L_P is full-rank. When m is odd, $L_P = \Lambda_\tau$ must be full-rank so there is nothing to prove. When m is even, we shall show that if there exists another j'_0 such that $\Lambda_{j'_0}$ is also a lattice, then $\tau_j \in \mathbb{Q}^2$ for any $1 \leqslant j \leqslant m$. It is already proved above that Λ_{j_0} is a lattice in $\mathbb{Q}^2$, so it remains to show $\tau_{j_0} \in \mathbb{Q}^2$. Since P is not a parallelogram and $m \geqslant 4$, $\{\tau_j,\ j \neq j_0, j'_0\}$ generate a sub-lattice of $\Lambda_{j_0} \subset \mathbb{Q}^2$. Since $\Lambda_{j'_0}$ is also a lattice, τ_{j_0} is rationally dependent with $\{\tau_j,\ j \neq j_0, j'_0\}$, thus must be rational, as desired. In fact in this case $e_j \in \mathbb{Q}^2$ for any $1 \leqslant j \leqslant m$ (see Lemma 4.1).

5. Proof of Theorem 1.9

Theorem 1.9 follows from Theorem 1.2, Theorem 1.6, Theorem 1.8 and the following lemma.

Lemma 5.1. *Let $L \subset \mathbb{R}^d$ be a lattice and $L_1, \ldots, L_n \subset L$ are sub-lattices. Then*

$$\bigcap_{j=1}^{n} L_j$$

is a sub-lattice of L.

Proof. It suffices to prove the case $n = 2$. If $L_1 \cap L_2$ is not full-rank, there exists $u \in L_1$ such that

$$\dim(span_{\mathbb{Z}}\{L_1 \cap L_2, u\}) > \dim(L_1 \cap L_2).$$

Since $u \in L_1$, $\mathbb{Z}u \cap L_2$ must be trivial, which implies

$$\dim(span_{\mathbb{Z}}\{L_2, u\}) > \dim(L_2) = d,$$

contradiction. □

Now we can complete the proof. By Theorem 1.2, if $P \subset \mathbb{R}^2$ is not a parallelogram and $P + \Lambda$ multi-tiles, Λ must be a finite union of translated lattices, that is,

$$\delta_\Lambda = \sum_{j=1}^{n} \delta_{L_j + z_j}.$$

By Theorem 1.6, for each j there exists a lattice $\widetilde{L_j}$ containing L_j such that $P + \widetilde{L_j}$ multi-tiles. By Theorem 1.8, $\widetilde{L_j} \cap L_P$ is a lattice, where L_P is defined in (1.3). Since both L_j and $\widetilde{L_j} \cap L_P$ are sub-lattices of $\widetilde{L_j}$, by Lemma 5.1 $L_j \cap L_P$ is a lattice. Therefore, by Lemma 5.1 again,

$$\bigcap_{j=1}^{n} (L_j \cap L_P) = \left(\bigcap_{j=1}^{n} L_j \right) \cap L_P,$$

which is a finite union of sub-lattices of L_P, is a lattice. Hence $\bigcap L_j$ is full-rank and Λ is a finite union of translations of $\bigcap L_j$.

6. Examples

6.1. **Symmetric polygons that do not multi-tile by translations.** We shall show that for any $m \geqslant 4$, there exist symmetric $(2m)$-gons that do not multi-tile by translations. As far as the author knows, these are the first known symmetric polygons that do not multi-tile by translations. Since there are many symmetric $(2m)$-gons that do multi-tile by translations, this means, unlike tiling (Venkov-McMullen), one can not determine whether a polygon multi-tiles only by its combinatorial type.

Take a zonotope P as in Figure 1.1 such that $e_j, 1 \leqslant j \leqslant m$ are rationally independent.

When m is odd, if P multi-tiles, by Theorem 1.8 all τ_j generate a lattice. Since $m - 1 \geqslant 3$, by (4.2), $\tau_j - \tau_{j+1} = e_j + e_{j+1}, 1 \leqslant j \leqslant m-1$, are rationally dependent, namely there exists $q_1, \dots, q_{m-1} \in \mathbb{Q}$, not all 0, such that

$$\begin{aligned} 0 =& q_1(e_1 + e_2) + \cdots + q_{m-1}(e_{m-1} + e_m) \\ =& q_1 e_1 + (q_1 + q_2)e_2 + \cdots + (q_{m-2} + q_{m-1})e_{m-1} + q_{m-1}e_m. \end{aligned}$$

It follows that $q_1 = 0, q_1 + q_2 = 0, \dots, q_{m-2} + q_{m-1} = 0, q_{m-1} = 0$, which implies $q_j = 0$ for any $1 \leqslant j \leqslant m-1$, contradiction.

When m is even, we may assume j_0 in Theorem 1.8 equals 1. Since e_1 is a linear combination of $\tau_j, j \geqslant 2$, with coefficients ± 1 (see Lemma (4.1)), $e_1, \tau_2, \dots, \tau_m$ generate a lattice. Since $m-1 \geqslant 3$, by (4.2), e_1 and $\tau_j - \tau_{j+1} = e_j + e_{j+1}, 2 \leqslant j \leqslant m-1$ are rationally dependent, namely there exists $q_1, \dots, q_{m-1} \in \mathbb{Q}$, not all 0, such that

$$\begin{aligned} 0 =& q_1 e_1 + q_2(e_2+e_3) + \cdots + q_{m-1}(e_{m-1}+e_m) \\ =& q_1 e_1 + q_2 e_2 + (q_2+q_3)e_3 + \cdots + (q_{m-2}+q_{m-1})e_{m-1} + q_{m-1}e_m. \end{aligned}$$

It follows that $q_1 = 0, q_2 = 0, q_2 + q_3 = 0 \dots, q_{m-2} + q_{m-1} = 0, q_{m-1} = 0$, which implies $q_j = 0$ for any $1 \leqslant j \leqslant m-1$, contradiction.

In fact, the summary at the end of Section 4 says, if $P + \mathbb{Z}^2$ multi-tiles, then $e_j \in (\mathbb{Z}^2 \pm \gamma) \cup \mathbb{Q}^2$, for some $\gamma \in \mathbb{R}^2$. Therefore we only need a quadruple of rationally independent e_j to deny multiple tilings of P. We omit the proof.

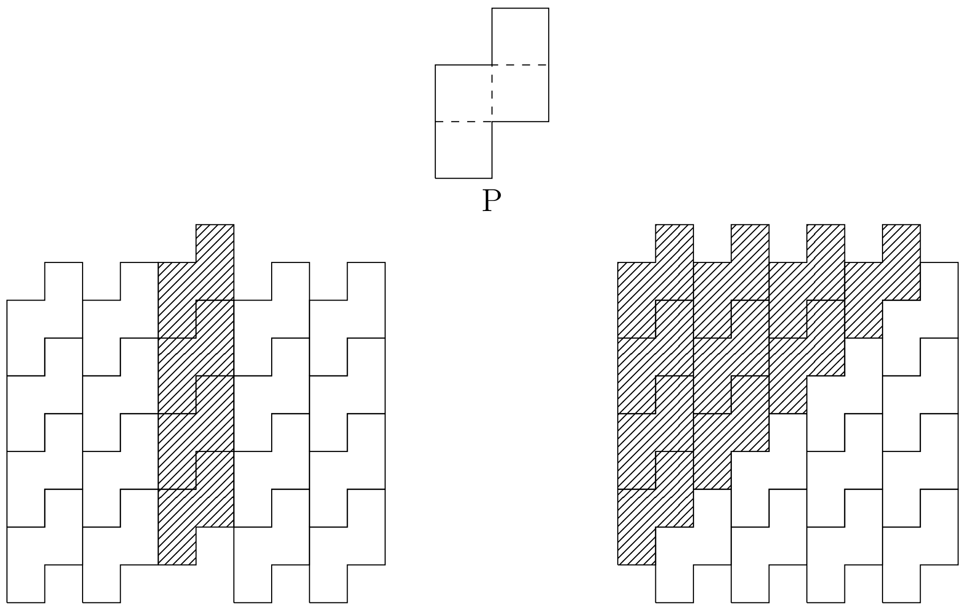
6.2. **Some non-periodic multi-tilings.** We shall show that non-parallelogram and convexity are necessary in Theorem 1.9. In fact we shall construct multiple tilings where $\Lambda + \alpha \neq \Lambda$ for any $\alpha \in \mathbb{R}^2 \backslash \{0\}$.

It is very easy for parallelograms. One can simply take $P = [0,1]^2$,

$$\Lambda_1 = (\mathbb{Z} \times (\mathbb{Z}\backslash\{0\})) \cup (\mathbb{Z}+\beta) \times \{0\}, \quad \Lambda_2 = ((\mathbb{Z}\backslash\{0\}) \times \mathbb{Z}) \cup \{0\} \times (\mathbb{Z}+\beta'),$$

for $\beta, \beta' \notin \mathbb{Z}$ and take $\Lambda = \Lambda_1 \cup \Lambda_2$

For the convexity, one example is the skew tetromino (see Figure 6.1), which is a union of 4 unit squares.



P

$P + \Lambda_1$ $\qquad$ $P + \Lambda_2$

FIGURE 6.1. The skew tetromino and two tiles

As shown in Figure 6.1, both $P + \Lambda_1$, $P + \Lambda_2$ tile, where

$$\Lambda_1 = ((2\mathbb{Z}\backslash\{0\}) \times 2\mathbb{Z}) \cup (\{0\} \times (2\mathbb{Z}+1)),$$

$$\Lambda_2 = \{(m,n) \in (2\mathbb{Z})^2 : m \geqslant n\} \cup \{(m,n) \in (2\mathbb{Z}-1)^2 : m < n\}.$$

Then $P + (\Lambda_1 \cup \Lambda_2)$ is a multiple tiling where $\Lambda + \alpha \neq \Lambda$ for any $\alpha \in \mathbb{R}^2 \backslash \{0\}$.

6.3. **A family of indecomposable multi-tilings by a symmetric non-regular octagon.** We shall construct a family of indecomposable multiple tilings of an octagon, where each discrete set is

$$\Lambda = \mathbb{Z} \times 2\mathbb{Z} + \{\vec{0}, \alpha\}, \text{ for some } \alpha \notin \mathbb{Q}^2.$$

This means, even with trivial counterexamples ruled out (parallelograms, decomposable multi-tilings), the analog of Lagarias-Wang's rationality theorem on 1-dimensional tiling still fails for multiple tilings in the plane. See the discussion after Theorem 1.9.

Let P be the symmetric octagon in Figure 6.2 below whose vertices lie in $\mathbb{Z}^2$.

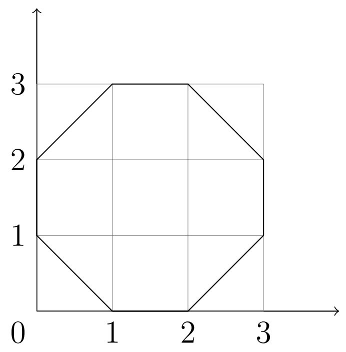


FIGURE 6.2. A symmetric octagon whose vertices lie in $\mathbb{Z}^2$

One can check that on each horizontal strip $\mathbb{R} \times [n, n+1]$,

$$P + \mathbb{Z} \times 2\mathbb{Z} = \begin{cases} 4, & n \text{ is even} \\ 3, & n \text{ is odd} \end{cases}.$$

Therefore for any $\beta \in \mathbb{R}$,

$$P + (\mathbb{Z} \times 2\mathbb{Z} + \{(0,0), (\beta, 1)\})$$

is a multi-tiling of multiplicity 7. We claim this multi-tiling is indecomposable. To see this, clearly it does not tile and it is proved by Yang and Zong ([26], [27]) that except parallelograms and hexagons, no convex polygon admits translational multi-tilings of multiplicities $2, 3, 4$. This means a 7-tiling by an octagon can not be decomposed into two multiple tilings.

APPENDIX: CONVEX BODIES THAT MULTI-TILE BY TRANSLATIONS MUST BE CONVEX POLYTOPES

Theorem. *Suppose $P \subset \mathbb{R}^d$ is a convex body and there exists a discrete multi-set Λ such that $P + \Lambda$ multi-tiles. Then P is a convex polytope.*

Proof. Notice $\partial P + \Lambda$ decomposes $\mathbb{R}^d$ into disjoint (open) cells. We first show each cell is convex. Pick a cell C, for any $\lambda \in \Lambda$,

$$C \subset P + \lambda, \ \textit{or} \ C \cap (P + \lambda) = \emptyset.$$

Say the multiplicity of $P+\Lambda$ is k. Then there exists $\lambda_1,\dots,\lambda_k$ such that

$$C \subset \bigcap_{j=1}^{k}(P+\lambda_j).$$

We claim they are actually equal. If not, there exists another cell C' such that

$$C' \subset \bigcap_{j=1}^{k}(P+\lambda_j).$$

Since C and C' are two different cells, they can be separated by $\partial P+\lambda'$ for some $\lambda' \in \Lambda$, that is

$$C \subset (P+\lambda'),\ C' \cap (P+\lambda') = \emptyset,$$

or

$$C' \subset (P+\lambda'),\ C \cap (P+\lambda') = \emptyset.$$

In either case, λ' is not equal to any of $\lambda_1,\dots,\lambda_k$, which means C or C' is covered at least $k+1$ times. Contradiction.

Next, fix a convex cell C_0, for any other convex cell C, there exists a half-space H_C such that

$$C_0 \subset H_C,\ C \cap H_C = \emptyset,$$

which implies

$$C \subset \bigcap_{C \neq C_0} H_C.$$

Since all cells tile $\mathbb{R}^d$, it follows that up to measure 0

$$C_0 = \bigcap_{C \neq C_0} H_C.$$

Now it suffices to show C_0 is in fact an intersection of finitely many half-spaces. Since $diam(C)$ is bounded above uniformly, we may assume $dist(\partial H_C, C_0)$ is large when $dist(C, C_0)$ is large. Choose finitely may H_C whose intersection is bounded. Then when $dist(C, C_0)$ is large, ∂H_C is far from C_0 and therefore dropping H_C does not change the intersection, as desired. Thus C must be a polytope. Since the original convex body P is a union of finitely many cells, it must be a convex polytope. □

References

[1] S. Bhattacharya. Periodicity and decidability of tilings of $\mathbb{Z}^2$. *arXiv preprint arXiv:1602.05738*, 2016.

[2] U. Bolle. On multiple tiles in E^2. In *Intuitive geometry (Szeged, 1991)*, volume 63 of *Colloq. Math. Soc. János Bolyai*, pages 39–43. North-Holland, Amsterdam, 1994.

[3] S. H. Chan. Quasi-periodic tiling with multiplicity: a lattice enumeration approach. *Discrete Comput. Geom.*, 54(3):647–662, 2015.

[4] A. Debernardi and N. Lev. Riesz bases of exponentials for convex polytopes with symmetric faces. *https://arxiv.org/abs/1907.04561*, 2019.

[5] J.-H. Evertse, H. P. Schlickewei, and W. M. Schmidt. Linear equations in variables which lie in a multiplicative group. *Ann. of Math. (2)*, 155(3):807–836, 2002.

[6] P. Furtwängler. über Gitter konstanter Dichte. *Monatsh. Math. Phys.*, 43(1):281–288, 1936.

[7] D. Girault-Beauquier and M. Nivat. Tiling the plane with one tile. In *Topology and category theory in computer science (Oxford, 1989)*, Oxford Sci. Publ., pages 291–333. Oxford Univ. Press, New York, 1991.
[8] N. Gravin, M. N. Kolountzakis, S. Robins, and D. Shiryaev. Structure results for multiple tilings in 3D. *Discrete Comput. Geom.*, 50(4):1033–1050, 2013.
[9] N. Gravin, S. Robins, and D. Shiryaev. Translational tilings by a polytope, with multiplicity. *Combinatorica*, 32(6):629–649, 2012.
[10] S. Grepstad and N. Lev. Multi-tiling and Riesz bases. *Adv. Math.*, 252:1–6, 2014.
[11] B. Grünbaum and G. C. Shephard. *Tilings and patterns.* W. H. Freeman and Company, New York, 1987.
[12] R. Kenyon. Rigidity of planar tilings. *Invent. Math.*, 107(3):637–651, 1992.
[13] M. N. Kolountzakis. On the structure of multiple translational tilings by polygonal regions. *Discrete Comput. Geom.*, 23(4):537–553, 2000.
[14] M. N. Kolountzakis. Multiple lattice tiles and Riesz bases of exponentials. *Proc. Amer. Math. Soc.*, 143(2):741–747, 2015.
[15] M. N. Kolountzakis and J. C. Lagarias. Structure of tilings of the line by a function. *Duke Math. J.*, 82(3):653–678, 1996.
[16] M. N. Kolountzakis and N. Lev. On non-periodic tilings of the real line by a function. *Int. Math. Res. Not. IMRN*, (15):4588–4601, 2016.
[17] J. C. Lagarias and Y. Wang. Tiling the line with translates of one tile. *Invent. Math.*, 124(1-3):341–365, 1996.
[18] H. Leptin and D. Müller. Uniform partitions of unity on locally compact groups. *Adv. Math.*, 90(1):1–14, 1991.
[19] N. Lev and B. Liu. Multi-tiling and equidecomposability of polytopes by lattice translates. *Bull. London Math. Soc. (2019), doi:10.1112/blms.12297.*
[20] K. Mahler. Zur Approximation algebraischer Zahlen. I. *Math. Ann.*, 107(1):691–730, 1933.
[21] P. McMullen. Convex bodies which tile space by translation. *Mathematika*, 27(1):113–121, 1980.
[22] H. Minkowski. Allgemeine lehrsätze über die konvexen polyeder. *Nachr. Ges. Wiss. Göttingen.*, pages 198–219, 1897.
[23] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
[24] B. A. Venkov. On a class of Euclidean polyhedra. *Vestnik Leningrad. Univ. Ser. Mat. Fiz. Him.*, 9(2):11–31, 1954.
[25] Q. Yang. On multiple translative tiling in the plane. *https://arxiv.org/abs/1809.04272*, 2018.
[26] Q. Yang and C. Zong. Multiple lattice tilings in euclidean spaces. *Canad. Math. Bull. (2018), doi:10.4153/S0008439518000103.*
[27] Q. Yang and C. Zong. Multiple translative tilings in euclidean spaces. *arXiv preprint arXiv:1711.02514*, 2017.
[28] C. Zong. *The cube: a window to convex and discrete geometry*, volume 168 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2006.
[29] C. Zong. Can you pave the plane nicely with identical tiles. *arXiv preprint arXiv:1803.06610*, 2018.

Department of Mathematics, the Chinese University of Hong Kong, Shatin, N.T., Hong Kong

E-mail address: Bochen.Liu1989@gmail.com