ON THE SIGN PATTERNS OF ENTRYWISE POSITIVITY PRESERVERS IN FIXED DIMENSION

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Abstract. Given a domain $I \subset \mathbb{C}$ and an integer $N > 0$, a function $f : I \to \mathbb{C}$ is said to be *entrywise positivity preserving* on positive semidefinite $N \times N$ matrices $A = (a_{jk}) \in I^{N\times N}$, if the entrywise application $f[A] = (f(a_{jk}))$ of f to A is positive semidefinite for all such A. Such preservers in all dimensions have been classified by Schoenberg as being absolutely monotonic [*Duke Math. J.* 1942]; see also Rudin [*Duke Math. J.* 1959]. In fixed dimension N, results akin to work of Horn and Loewner [*Trans. Amer. Math. Soc.* 1969] show that the first N non-zero Maclaurin coefficients of any positivity preserver f are positive; and the last N coefficients are also positive if I is unbounded. However, very little was known about the higher-order coefficients: the only examples to date for unbounded domains I were absolutely monotonic, hence work in all dimensions; and for bounded I examples of non-absolutely monotonic preservers were very few (and recent).

In this paper, we provide a complete characterization of the sign patterns of the higher-order Maclaurin coefficients of positivity preservers in fixed dimension N, over bounded and unbounded domains $I = (0, \rho)$. In particular, this shows that the above Horn–Loewner-type conditions cannot be improved upon. As a further special case, this provides the first examples of polynomials which preserve positivity on positive semidefinite matrices in $I^{N\times N}$ but not in $I^{(N+1)\times(N+1)}$. Our main tools in this regard are the Cauchy–Binet formula and lower and upper bounds on Schur polynomials. We also obtain analogous results for real exponents, using the Harish-Chandra–Itzykson–Zuber formula in place of bounds on Schur polynomials.

We then go from qualitative existence bounds – which suffice to understand all possible sign patterns – to exact quantitative bounds. This is achieved using a Schur positivity result due to Lam, Postnikov, and Pylyavskyy [*Amer. J. Math.* 2007], and in particular provides a second proof of the existence of threshold bounds for tuples of integer and real powers. As an application, we extend our previous qualitative and quantitative results to understand preservers of total non-negativity in fixed dimension – including their sign patterns. We deduce several further applications, including extending a Schur polynomial conjecture of Cuttler, Greene, and Skandera [*Eur. J. Comb.* 2011] to obtain a novel characterization of weak majorization for real tuples.

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Contents

1. Introduction and main results

1.1. **Notation and prior results.** For any natural number N, let $\mathbb{P}_N(\mathbb{C})$ denote the convex cone of positive semidefinite Hermitian $N \times N$ matrices; this defines a partial ordering $\preceq$ on $N \times N$ matrices, with $A \preceq B$ if and only if $B - A$ is positive semidefinite.

Given a domain $I \subset \mathbb{C}$, let $\mathbb{P}_N(I) \subset \mathbb{P}_N(\mathbb{C})$ denote the set of matrices $A = (a_{jk})_{j,k=1,\dots,N} \in \mathbb{P}_N(\mathbb{C})$ with all entries a_{jk} in I, thus for instance $\mathbb{P}_N(\mathbb{R})$ is the cone of positive definite real symmetric matrices. A key role in this paper will also be played by the subset $\mathbb{P}^1_N(I) \subset \mathbb{P}_N(I)$ of *rank one* matrices $\mathbf{u}\mathbf{u}^*$ in $\mathbb{P}_N(I)$. We will focus our attention in this paper almost entirely on the cases $I = (0, \rho)$ for $0 < \rho \leqslant +\infty$, although we will also briefly consider the case $I = (-\rho, \rho)$, as well as the complex disk $I = D(0, \rho)$.

Remark 1.1. In this paper, all our vectors $\mathbf{u} = (u_1, \dots, u_N)^T$ will be column vectors, with the space of such vectors denoted as $(\mathbb{C}^N)^T$; row vectors $(u_1, \dots, u_N)$ will be referred to instead as *tuples*, and the space of such tuples denoted as $\mathbb{C}^N$. A more complete list of notations used in this paper is provided in the final section.

Given a matrix $A = (a_{jk})_{j,k=1,\dots,N}$ in $\mathbb{P}_N(I)$, a function $f : I \to \mathbb{C}$ acts *entrywise* on A via the formula

$$f[A] := (f(a_{jk}))_{j,k=1,\dots,N}.$$

Remark 1.2. Note that the entrywise application $f[A]$ of f to A should not be confused with the more common functional calculus $f(A)$ of f applied to A; we will not use the latter in this paper.

For instance, if f is a monomial $f(x) = x^m$, then $f[A] = A^{\circ m}$ is the Hadamard product of m copies of A. We say that the function $f : I \to \mathbb{C}$ is *entrywise positivity preserving* on $\mathbb{P}_N(I)$ if $f[A] \in \mathbb{P}_N(\mathbb{C})$ for all $A \in \mathbb{P}_N(I)$; similarly if $\mathbb{P}_N(I)$ is replaced with any subset of $\mathbb{P}_N(I)$, such as $\mathbb{P}_N^1(I)$.

The Schur product theorem [36] asserts that if two matrices A, B lie in $\mathbb{P}_N(\mathbb{C})$, then so does their Hadamard product $A \circ B$. As observed in 1925 by Pólya and Szegö [32, Problem 37], this immediately implies that any function $f : I \to \mathbb{C}$ which is *absolutely monotonic*, in the sense that one has a convergent power series representation

$$f(x) = \sum_{k \geqslant 0} c_k x^k$$

on I for some non-negative coefficients $c_k \geqslant 0$, will be entrywise positivity preserving on $\mathbb{P}_N(I)$ for any N.

It is then natural to ask which of the positivity conditions $c_k \geqslant 0$ are in fact necessary. More precisely, in this paper we address the following question:

Question 1.3. *Fix a positive integer N and a set $I \subset \mathbb{C}$, and consider a convergent power series $f : I \to \mathbb{C}$ which is entrywise positivity preserving on $\mathbb{P}_N(I)$. Which coefficients of f can be negative?*

In fact we completely resolve this question in the case $I = (0, \rho)$ for any $0 < \rho \leqslant +\infty$. Additionally, we completely answer a variant of Question 1.3 for real powers; and give some partial results in the cases $I = (-\rho, \rho)$ and $I = D(0, \rho)$, where (as we explain) there is no 'uniform' answer to the question.

Question 1.3 arises out of a longstanding program in analysis over the past century. In *loc. cit.*, Pólya and Szegö asked if there are functions $f : \mathbb{R} \to \mathbb{R}$ besides the convergent absolutely monotonic functions which were entrywise positivity preserving on $\mathbb{P}_N(I)$ for all N and $I \subset \mathbb{R}$. In his celebrated work [35], Schoenberg in 1942 proved this was not possible for continuous f (even if one restricted to the case $I = (-1, 1)$), using positive definite functions on spheres (Gegenbauer polynomials). Schoenberg was interested in embedding positive definite metrics into Hilbert space; see also [9, 42]. The continuity hypothesis in Schoenberg's theorem was later removed by Rudin (1959) in [34], using analysis of measures on the torus, and working in the broader context of studying functions acting on Fourier–Stieltjes transforms, as explored with Kahane and others in [20, 25]. In fact, Rudin's result only required positivity preservation of Toeplitz matrices in $\mathbb{P}_N((-1, 1))$ of rank at most three, which correspond to measures on the torus by Herglotz's theorem; a parallel result for Hankel matrices (which correspond to measures on the real line) was shown in [8]. In a sense, Schoenberg's result is the (far harder) converse to that of his advisor, Schur. For variants of Schoenberg's theorem for other choices of I, see [22], [41], [18].

Since these results of Schoenberg and Rudin, the question of classifying the entrywise positivity preservers for a *fixed* dimension N has been actively studied. Necessary conditions for entrywise positivity preservation were first established in the 1969 thesis of Horn [22], who attributes the result to Loewner. We summarize these conditions, as well as some further necessary conditions of Horn–Loewner type established by subsequent authors, as follows:

Lemma 1.4 (Horn–Loewner-type necessary conditions)**.** *Let* $N \geqslant 2$ *and* $0 < \rho \leqslant +\infty$.

(i) *(Horn and Loewner* [22]*; Guillot–Khare–Rajaratnam* [18]*) Suppose that* $f : (0, \rho) \to \mathbb{R}$ *is entrywise positivity preserving on all matrices in* $\mathbb{P}_N((0, \rho))$ *of the form* $A = a\mathbf{1}_{N\times N} + \mathbf{u}\mathbf{u}^T$*, with* $a \in [0, \rho), \mathbf{u} \in [0, \sqrt{\rho - a})^N$*. Then* $f \in C^{N-3}((0, \rho))$,

$$f^{(k)}(x) \geqslant 0, \qquad \forall x \in (0, \rho),\ 0 \leqslant k \leqslant N-3,$$

and $f^{(N-3)}$ *is a convex non-decreasing function on* $(0, \rho)$*. In particular, if* $f \in C^{N-1}((0, \rho))$*, then* $f^{(k)}(x) \geqslant 0$ *for all* $x \in (0, \rho), 0 \leqslant k \leqslant N-1$.

(ii) *(See* [4, Lemma 2.4]*) If* $f(x) = \sum_{n=0}^{\infty} c_n x^n$ *is a convergent power series on* $(0, \rho)$ *that is entrywise positivity preserving on* $\mathbb{P}_N^1((0, \rho))$*, and* $c_{n_0} < 0$ *for some* n_0*, then we have* $c_n > 0$ *for at least* N *values of* $n < n_0$*. (In particular, the first* N *non-zero Taylor coefficients of* f*, if they exist, must be positive.)*

(iii) *(See Section 4) If* $f(x) = \sum_{n=0}^{\infty} c_n x^n$ *is a convergent power series on* $(0, +\infty)$ *which is entrywise positivity preserving on* $\mathbb{P}_N^1((0, +\infty))$*, and* $c_{n_0} < 0$ *for some* n_0*, then we have* $c_n > 0$ *for at least* N *values of* $n < n_0$*, and at least* N *values of* $n > n_0$*. (In particular, if* f *is a polynomial, then the first* N *non-zero coefficients and the last* N *non-zero coefficients of* f*, if they exist, are all positive.)*

We make two remarks here. First, the original result of Horn and Loewner required f to be continuous; this assumption was removed in [18], in the spirit of Rudin's strengthening [34] of Schoenberg's theorem [35] alluded to above. Second, the proof of Lemma 1.4(ii) uses the positivity property

$$\det(u_i^{\alpha_j})_{i,j=1,\dots,N} > 0 \tag{1.1}$$

of generalized Vandermonde determinants for any $0 < u_1 < \cdots < u_N$ and $\alpha_1 < \cdots < \alpha_N$; see e.g. [17, Chapter XIII, §8, Example 1], or the bounds in (5.2) below. Variations of this positivity property will recur throughout this paper.

In a slightly different direction, it was shown by FitzGerald–Horn in [15] (solving a conjecture of Horn [22]) that the fractional monomials $x \mapsto x^\alpha$ are entrywise positivity preservers on $\mathbb{P}_N((0, +\infty))$ if and only if α is a non-negative integer, or a real number greater than $N-2$. (Note this shows that Lemma 1.4(i) is sharp.) See the recent survey [6] for further results and references on entrywise positivity preservers, as well as for more on Schoenberg and Rudin's motivations in proving their results mentioned above. However, in spite of significant subsequent interest and activity, a complete characterization of the functions – even for polynomials – that entrywise preserve positivity on $\mathbb{P}_N((0,+\infty))$ remains unknown even for $N=3$. (For $N=2$ the problem was resolved by Vasudeva [41].)

In light of the above discussion, it is natural to ask if for real analytic preservers f, the positive coefficient requirements in Lemma 1.4(ii) and Lemma 1.4(iii) are sharp. In [4], Schur polynomials were used to establish a necessary and sufficient condition for entrywise positivity preservation on $\mathbb{P}_N((0,\rho))$, $0<\rho<\infty$ for polynomials of the form

$$x \mapsto c_0 + c_1 x + \cdots + c_{N-1} x^{N-1} + c_M x^M$$

with $M \geqslant N$; in particular, it was shown that for any choice of M, one could construct entrywise positivity preserving polynomials with c_M negative (of course, Lemma 1.4(ii) forces the remaining coefficients $c_0,\dots,c_{N-1}$ to then be positive). Via the Schur product theorem, this implies a similar result for polynomials of the form

$$x \mapsto c_0 x^h + c_1 x^{h+1} + \cdots + c_{N-1} x^{h+N-1} + c_M x^{h+M}, \qquad h \in \mathbb{Z}^{\geqslant 0}.$$

In the $N=2$ case, a similar analysis was also carried out in [4, §3.4] for polynomials of the form

$$x \mapsto c_m x^m + c_n x^n + c_p x^p$$

with $m<n<p$, where again it was shown that for any choice of m,n,p, one could construct such a polynomial with c_p negative but which was still entrywise positivity preserving on $\mathbb{P}_2((0,\rho))$.

However, aside from these few results (and linear combinations of them), there were no examples previously known of entrywise positivity preserving convergent power series with at least one negative coefficient. In particular, with the exceptions discussed above, all previously known entrywise positivity preservers on $\mathbb{P}_N((0,\rho))$ were absolutely monotonic, hence in fact work for all dimensions. For the unbounded domain $\rho=+\infty$, there was even less progress, with no examples of preservers with negative coefficients known to date (nor if such functions could even exist).

1.2. **New results 1: Qualitative bounds.** We begin with the simple observation that Question 1.3 can have a 'structured' solution (in the flavor of Lemma 1.4) only for $I \subset [0,+\infty)$, but not other domains $I=(-\rho,\rho)$ or $D(0,\rho)$ in the complex plane. For example, the family of polynomials

$$p_{k,t}(x) := t(1+x^2+\cdots+x^{2k}) - x^{2k+1}, \qquad k \geqslant 0,\ t>0,$$

can never preserve positivity on $\mathbb{P}_2((-\rho,\rho))$, since setting e.g. $\mathbf{u} := (1,-1)^T$ and $A := (\rho/2)\mathbf{u}\mathbf{u}^T \in \mathbb{P}_2((-\rho,\rho))$, one computes:

$$\mathbf{u}^T p_{k,t}[A]\mathbf{u} = -4(\rho/2)^{2k+1} < 0, \tag{1.2}$$

whence $p_{k,t}[A]$ is not positive semidefinite for any $k \geqslant 0$. Similar examples with higher-order roots of unity (fail to) work in the case of complex domains.

Thus the present work is primarily concerned with bounded and unbounded domains $I \subset (0,+\infty)$. In the case of bounded intervals $I=(0,\rho)$, we completely resolve Question 1.3 by showing that the non-zero coefficients beyond the first N of an entrywise positivity preserver on $\mathbb{P}_N((0,\rho))$ are allowed to be of arbitrary sign:

Theorem 1.5. *Let $N > 0$ and $0 \leqslant n_0 < n_1 < \cdots < n_{N-1}$ be integers, and for each $M > n_{N-1}$, let $\epsilon_M \in \{-1, 0, +1\}$ be a sign. Let $0 < \rho < \infty$, and let $c_{n_0}, \dots, c_{n_{N-1}}$ be positive reals. Then there exists a convergent power series*

$$f(x) = c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{N-1}} x^{n_{N-1}} + \sum_{M > n_{N-1}} c_M x^M$$

on $(0,\rho)$ that is an entrywise positivity preserver on $\mathbb{P}_N((0,\rho))$, such that for each $M > n_{N-1}$, c_M has the sign of ϵ_M.

In particular, Theorem 1.5 shows that the Horn–Loewner-type necessary criterion in Lemma 1.4(ii) cannot be improved upon. Note from a limiting argument that we may replace $(0,\rho)$ here by $[0,\rho]$, and hence by any subset of $[0,\rho]$, if desired.

Theorem 1.5 follows readily from the following special case:

Theorem 1.6. *Let $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1} < M$ be integers. Let $0 < \rho < \infty$, and let $c_{n_0}, \dots, c_{n_{N-1}}$ be positive reals. Then there exists a negative number c_M such that*

$$x \mapsto c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{N-1}} x^{n_{N-1}} + c_M x^M \tag{1.3}$$

entrywise preserves positivity on $\mathbb{P}_N((0,\rho))$.

Indeed, to derive Theorem 1.5 from Theorem 1.6, we see (since the space of entrywise positivity preserving functions forms a cone, and because any monomial is entrywise positivity preserving thanks to the Schur product theorem) that for any $M > n_{N-1}$, there exists $\delta_M > 0$ such that the polynomial (1.3) is entrywise positivity preserving whenever $|c_M| \leqslant \delta_M$; by shrinking δ_M if necessary, we may assume that $\delta_M \leqslant \frac{1}{M!}$ (say) for all M. Multiplying (1.3) (with c_M replaced by $\epsilon_M \delta_M$) by $2^{n_{N-1}-M}$ and summing over all $M > n_{N-1}$, we obtain Theorem 1.5 with $c_M := 2^{n_{N-1}-M} \epsilon_M \delta_M$.

Theorem 1.6 can be reformulated as a matrix inequality: for any $0 \leqslant n_0 < \cdots < n_{N-1} < M$, $0 < \rho < \infty$, and $c_{n_0}, \dots, c_{n_{N-1}} > 0$, there exists a finite threshold $\mathcal{K}$ (depending on $n_0, \dots, n_{N-1}, \rho, c_{n_0}, \dots, c_{n_{N-1}}, M$) such that

$$A^{\circ M} \preceq \mathcal{K} \sum_{j=0}^{N-1} c_{n_j} A^{\circ n_j} \tag{1.4}$$

for any $A \in \mathbb{P}_N((0,\rho))$. The quantity $\mathcal{K}$ provided by the argument will be explicit (see (3.5)) but not completely optimal; the optimal threshold is given in Theorems 1.11, 1.12 below, and established in Section 8.

The bounds in Theorem 1.6 will be sufficiently strong that we can replace the monomials x^M in (1.4) with arbitrary convergent power series:

Corollary 1.7 (Analytic functions)**.** *Fix integers $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1}$, and a polynomial $c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}}$, with $c_{n_j} > 0 \ \forall j$. Let $0 < \rho < \infty$. Given a power series $g(x) = \sum_{M > n_{N-1}} g_M x^M$ which is convergent* *at ρ, there exists a finite threshold $\mathcal{K} = \mathcal{K}(n_0, \dots, n_{N-1}, \rho, c_{n_0}, \dots, c_{n_{N-1}}, g)$ such that the function*

$$x \mapsto \mathcal{K} \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - g(x)$$

is entrywise positivity preserving on $\mathbb{P}_N((0,\rho))$. Equivalently, one has

$$g[A] \preceq \mathcal{K} \sum_{j=0}^{N-1} c_{n_j} A^{\circ n_j} \tag{1.5}$$

for all $A \in \mathbb{P}_N((0,\rho))$.

We establish this result in Section 3.3. It should be possible to relax the requirement that g be a convergent power series to the hypothesis that g is in the regularity class $C^M([0,\rho])$ for some sufficiently large M, but we will not attempt to do so here.

Remark 1.8. If one specializes (1.4) to the rank one matrix $A = \mathbf{u}\mathbf{u}^T$ with $\mathbf{u} = (u_1, \dots, u_N)^T$ and $0 < u_1 < \cdots < u_N$, we conclude in particular that the vectors $(u_1^{n_j}, \dots, u_N^{n_j})^T$ for $j = 1, \dots, N$ are linearly independent, which is essentially (1.1) (in the case of non-negative integer exponents). One may thus view Theorem 1.6 as a "robust" variant of (1.1).

Coming to the unbounded domain case $I = (0, +\infty)$, we once again completely resolve Question 1.3. Just as Theorem 1.5 demonstrates the sharpness of Lemma 1.4(ii), our second main result demonstrates the sharpness of Lemma 1.4(iii):

Theorem 1.9. *Let $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1}$ be integers, and for each $M > n_{N-1}$, let $\epsilon_M \in \{-1, 0, +1\}$ be a sign. Suppose that whenever $\epsilon_{M_0} = -1$ for some $M_0 > n_{N-1}$, one has $\epsilon_M = +1$ for at least N choices of $M > M_0$. Let $c_{n_0}, \dots, c_{n_{N-1}}$ be positive reals. Then there exists a convergent power series*

$$f(x) = c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{N-1}} x^{n_{N-1}} + \sum_{M > n_{N-1}} c_M x^M$$

on $(0, +\infty)$ that is an entrywise positivity preserver on $\mathbb{P}_N((0, +\infty))$, such that for each $M > n_{N-1}$, c_M has the sign of ϵ_M.

Unlike the setting of bounded I, this is also the first existence result for power series preservers of $\mathbb{P}_N(I)$ with negative coefficients.

Like the setting of bounded I, Theorem 1.9 is a consequence of the following special case:

Theorem 1.10. *Let $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1} < M < n_N < \cdots < n_{2N-1}$ be integers, and let $c_{n_0}, \dots, c_{n_{2N-1}}$ be positive reals. Then there exists a negative number c_M such that*

$$x \mapsto c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{N-1}} x^{n_{N-1}} + c_M x^M + c_{n_N} x^{n_N} + \cdots + c_{n_{2N-1}} x^{n_{2N-1}} \tag{1.6}$$

entrywise preserves positivity on $\mathbb{P}_N((0, +\infty))$.

Indeed, if $N, n_0, \dots, n_{N-1}, (\epsilon_M)_{M>n_{N-1}}$ are as in Theorem 1.9, then from Theorem 1.10, one may find for each $M > n_{n-1}$ with $\epsilon_M = -1$, a real number $0 < \delta_M \leqslant \frac{1}{M!}$ such that

$$f_M : x \mapsto c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{N-1}} x^{n_{N-1}} - \delta_M x^M + \sum_{n>M:\epsilon_n=+1} \frac{1}{n!} x^n$$

entrywise preserves positivity on $\mathbb{P}_N((0,+\infty))$. For all other powers $M > n_{N-1}$ with $\epsilon_M \neq -1$, define $f_M(x) := \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + \frac{\epsilon_M}{M!} x^M$. Now Theorem 1.9 follows by considering $f(x) := \sum_{M>n_{N-1}} 2^{n_{N-1}-M} f_M(x)$; note here that $|f(x)| \leqslant \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + (x+1)e^x$ for $x > 0$.

We prove Theorem 1.10 in Section 4. As one corollary of this theorem (and Lemma 1.4(iii)), we see that for any N, there exist analytic functions that entrywise preserve positivity on $\mathbb{P}_N((0,+\infty))$ but not on $\mathbb{P}_{N+1}((0,+\infty))$.

We are also able to establish analogues of the above theorems in which the exponents n_j, M are real numbers rather than natural numbers; see Section 5. This allows us to answer Question 1.3 for real powers, thus replacing power series by countable sums of powers, including but not restricted to Hahn and Puiseux series. Similarly, we obtain an analogue of Corollary 1.7 in which the analytic function g is replaced by a Laplace transform of more general real measures with support in (n_{N-1}, ∞).

On the other hand, if one replaces the domain $(0, \rho)$ with a two-sided domain $(-\rho, \rho)$ or with a complex disk $D(0, \rho)$, then the results largely break down for all tuples $\mathbf{n} := (n_0, \dots, n_{N-1})$ that do not equal shifts by $h \in \mathbb{Z}^{\geqslant 0}$ of the 'minimal' tuple $(0, \dots, N-1)$; see Sections 6, 7. As the results for tuples of the form $\mathbf{n} = (h, h+1, \dots, h+N-1)$ were uniformly valid over $I = D(0, \rho)$ (see [4]), it follows that the problem for every other $\mathbf{n}$ is more challenging, and new techniques are required to resolve Question 1.3.

Our proof strategy is as follows. We first focus on establishing entrywise positivity preservation for rank one matrices $\mathbf{u}\mathbf{u}^T$. In this case, one can use the Cauchy–Binet formula to obtain an explicit criterion for positive definiteness, in terms of generalized Vandermonde determinants. In the case of natural number exponents, these determinants can be factored as the product of the ordinary Vandermonde determinant and a Schur polynomial. One can then use the totally positive nature of Schur polynomials to obtain satisfactory upper and lower bounds on these polynomials (relying crucially on the fact that we are restricting the entries of the rank one matrix to be non-negative). The main novelty in our arguments, compared to previous work, is the use of *lower* bounds on Schur polynomials, which are needed due to the presence of such polynomials in the denominators of the formulae for various thresholds whenever $\mathbf{n} \neq (h, h+1, \dots, h+N-1)$ for $h \in \mathbb{Z}^{\geqslant 0}$.

Once entrywise positivity preservation is shown for rank one matrices, we induct on N using an argument of FitzGerald and Horn [15], relying on the observation that any positive definite matrix can be viewed as the sum of a rank one matrix and a matrix with vanishing final row and column, allowing one to derive entrywise positivity preservation for general positive definite matrices from the rank one case and the induction hypothesis using the fundamental theorem of calculus.

In the case of real exponents, the same argument as above is used to extend the threshold from rank-one matrices to all matrices. To produce a threshold in the rank-one case, Schur polynomials are no longer available to control generalized Vandermonde determinants, but we can use the famous Harish-Chandra–Itzykson–Zuber formula [19, 24] as a substitute for obtaining the corresponding upper bound. For the lower bound, we refine this analysis using Gelfand–Tsetlin polytopes. These effective lower and upper bounds allow us to answer Question 1.3 for real powers, and also to extend Corollary 1.7 to Laplace transforms. The bounds are also applied later in the paper, to prove a new characterization of weak majorization (see Theorem 1.14). It is remarkable that not only Schur polynomials, but also the Harish-Chandra–Itzykson–Zuber unitary integral, Gelfand–Tsetlin patterns, and Schur positivity (below) – all of which play a central role in our proofs – arise naturally in type A representation theory.

1.3. **New results 2: Exact quantitative bounds and applications.**

We now produce sharper bounds. As our chief purpose in the previously stated results was to solve Question 1.3, it sufficed to use lower and upper bounds on Schur polynomials to obtain threshold bounds. We will show the following exact result for rank-one matrices.

Theorem 1.11. *Fix an integer $N > 0$ and real powers $n_0 < \cdots < n_{N-1} < M$. Also fix real scalars $\rho > 0$ and $c_{n_0}, \ldots, c_{n_{N-1}}, c'$, and define*

$$f(x) \coloneqq \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c' x^M. \tag{1.7}$$

Then the following are equivalent:

1. *The entrywise map $f[-]$ preserves positivity on rank-one matrices in $\mathbb{P}_N((0,\rho))$.*
2. *Either all $c_{n_j}, c' \geqslant 0$; or $c_{n_j} > 0 \ \forall j$ and $c' \geqslant -\mathcal{C}^{-1}$, where*

$$\mathcal{C} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}. \tag{1.8}$$

Here $\mathbf{n} \coloneqq (n_0, \ldots, n_{N-1})$, $\mathbf{n}_j \coloneqq (n_0, \ldots, n_{j-1}, n_{j+1}, \ldots, n_{N-1}, M)$, and given a tuple $(t_0, \ldots, t_{k-1})$ or a vector $\mathbf{t} = (t_0, \ldots, t_{k-1})^T$, we define its 'Vandermonde determinant' to be $V(t_0, \ldots, t_{k-1}) = V(\mathbf{t}) \coloneqq \prod_{0 \leqslant i < j \leqslant k-1} (t_j - t_i)$.

Notice in this case that the powers n_j, M are allowed to be negative as well.

Our next result proves that the sharp threshold (1.8) works for matrices of all ranks. There is a small subtlety about which powers n_j are allowed if the rank of the matrices is greater than one; see the remarks after Theorem 5.5 below.

Theorem 1.12. *With notation as in Theorem 1.11, if we further assume that* $n_j \in \mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$ *for all* j*, then the two conditions (1), (2) are further equivalent to:*

(3) *The entrywise map* $f[-]$ *preserves positivity on* $\mathbb{P}_N([0,\rho])$*.*

The proof of these theorems involves refining the approach to prove the aforementioned results. The key additional tool is a Schur positivity result by Lam–Postnikov–Pylyavskyy [28], which implies the following monotonicity property for ratios of Schur polynomials $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$:

Proposition 1.13. *Fix tuples of non-negative integers* $0 \leqslant n_0 < \cdots < n_{N-1}$ *and* $0 \leqslant m_0 < \cdots < m_{N-1}$*, such that* $n_j \leqslant m_j \ \forall j$*. Then the function*

$$f : ((0,\infty)^N)^T \to \mathbb{R}, \qquad f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})}$$

is non-decreasing in each coordinate.

While $\mathbf{m}, \mathbf{n}$ are integer tuples in this result, it helps prove the above theorems for all real powers. We provide below in the paper several application of this analysis; we mention here two of them. First, we extend all of the previous results on positivity preservers to preservers of *total non-negativity* on Hankel matrices of a fixed dimension. (Recall that a possibly non-square real matrix is totally non-negative – sometimes termed totally positive – if it has all non-negative real minors [26].) Note that the constraint of having non-negative entries is natural also in total non-negativity, as in the above results.

Second, we show that a conjecture by Cuttler–Greene–Skandera [11], recently proved by Sra [40] and Ait-Haddou and Mazure [1], can be extended to obtain a characterization of weak majorization for all non-negative real tuples, which involves Schur polynomials/generalized Vandermonde determinants, and which we believe is new:

Theorem 1.14. *Fix* N*-tuples* $\mathbf{m}, \mathbf{n}$ *of pairwise distinct real powers. Then the following are equivalent.*

(1) *For all vectors* $\mathbf{u} \in ([1,\infty)^N)^T$*, we have:*
$$\frac{|\det(\mathbf{u}^{\circ m_0} | \dots | \mathbf{u}^{\circ m_{N-1}})|}{|V(\mathbf{m})|} \geqslant \frac{|\det(\mathbf{u}^{\circ n_0} | \dots | \mathbf{u}^{\circ n_{N-1}})|}{|V(\mathbf{n})|}. \tag{1.9}$$

(2) *The tuple* $\mathbf{m}$ *weakly majorizes* $\mathbf{n}$*.*

As we will show, the assertion (2) is implied by (1) holding for vectors $\mathbf{u}$ in the smaller set $((I_\infty)^N)^T$ – in fact a certain countable subset of this suffices – where I_∞ is any non-empty deleted neighborhood in $[1,\infty)$ of ∞.

In this vein, we further strengthen in two ways the aforementioned recent works by Cuttler–Greene–Skandera, Sra, and Ait-Haddou–Mazure, which characterizes majorization. Namely, these authors showed – for integer powers $\mathbf{m}, \mathbf{n}$ – that (1.9) holds now for all $\mathbf{u} \in ([0,\infty)^N)^T$, if and only if the integer tuple $\mathbf{m}$ majorizes $\mathbf{n}$. (In fact, these authors related this inequality at all $\mathbf{u} \in ([0,\infty))^N$ to $\mathbf{m} - \mathbf{n}_{\min}$ majorizing $\mathbf{n} - \mathbf{n}_{\min}$; but after arranging both $\mathbf{m}, \mathbf{n}$ in increasing order, note that such a (weak) majorization is indeed equivalent to $\mathbf{m}$ (weakly) majorizing $\mathbf{n}$.)

In the Cuttler–Greene–Skandera result, by continuity it suffices to assume (1.9) holds on the positive open orthant $((0,\infty)^N)^T$ instead of its closure. Our first strengthening is that the aforementioned characterization of majorization also holds for real powers, not just integer ones.

Theorem 1.15. *Fix N-tuples $\mathbf{m}, \mathbf{n}$ of pairwise distinct real powers. Then the following are equivalent.*

1. *The inequality (1.9) holds, now for all vectors $\mathbf{u} \in ((0,\infty)^N)^T$.*
2. *The tuple $\mathbf{m}$ majorizes $\mathbf{n}$.*

Second, even in the original case of non-negative integer tuples $\mathbf{m}, \mathbf{n}$ (but also in full generality – for arbitrary tuples $\mathbf{m}, \mathbf{n}$ of pairwise distinct real powers), we show that one does not require the above inequality at all points in the orthant, but only on the open unit cube and on the subset $((1,\infty)^N)^T$ in our 'weak majorization' result above:

Theorem 1.16. *Notation as in Theorem 1.15. Then the two assertions are further equivalent to:*

3. *The inequality (1.9) holds for the 'restricted' set of vectors $\mathbf{u} \in ((0,1)^N)^T \cup ((1,\infty)^N)^T$.*

In fact, here one only needs to work with a certain countable set of vectors $\mathbf{u}$ in $((I_0)^N)^T \cup ((I_\infty)^N)^T$, where I_∞ was defined immediately after Theorem 1.14, and $I_0 \subset (0,1]$ similarly denotes an arbitrary non-empty deleted neighborhood in $(0,1]$ of 0.

In the final section, we explain how to further extend (a part of) Theorem 1.14, as well as the 'positivity' part of the result of Lam–Postnikov–Pylyavskyy, to 'continuous' versions of Schur polynomials – i.e., generalized Vandermonde determinants. This follows from a more general log-supermodularity phenomenon for strictly totally positive matrices, which follows from the work of Skandera [39].

2. Preliminaries on Schur polynomials

As the proofs of the main results crucially involve Schur polynomials, in this section we present some preliminaries on them.

Fix an integer $N > 0$, and define $\mathbf{n}_{\min}$ to be the tuple $(0, 1, \dots, N-1)$. Given a tuple of strictly increasing non-negative integers $\mathbf{n} = (n_0, \dots, n_{N-1})$, we will define the corresponding Schur polynomial $s_{\mathbf{n}} : \mathbb{R}^N \to \mathbb{R}$ in variables $(u_1, \dots, u_N)$ or in the vector $\mathbf{u} = (u_1, \dots, u_N)^T$ by the formula

$$s_{\mathbf{n}}(u_1, \dots, u_N) = s_{\mathbf{n}}(\mathbf{u}) := \sum_T \mathbf{u}^{|T|}. \tag{2.1}$$

Here T ranges over the column-strict Young tableaux of shape given by the reversal $\overline{\mathbf{n} - \mathbf{n}_{\min}} = (n_{N-1} - N + 1, \dots, n_0)$ of $\mathbf{n} - \mathbf{n}_{\min} = (n_0, \dots, n_{N-1} - N + 1)$ and cell entries $1, \dots, N$, $|T|$ is the tuple $|T| := (a_1, \dots, a_N)$ where a_i is the number of occurrences of i in the tableau T, and we use the multinomial notation

$$\mathbf{u}^{|T|} = (u_1, \dots, u_N)^{(a_1, \dots, a_N)} := \prod_{j=1}^N u_j^{a_j}.$$

In particular, $s_{\mathbf{n}}$ is a homogeneous polynomial, with total degree $\sum_{j=0}^{N-1}(n_j - j)$ and positive integer coefficients. Each Schur polynomial $s_{\mathbf{n}}$ may be interpreted as the character of an irreducible polynomial representation of the Lie group $GL_N(\mathbb{C})$, although we will not need this interpretation here.

Example 2.1. Suppose $N = 3$ and $\mathbf{n} = (0, 2, 4)$, then we consider Young tableaux of shape $(2, 1, 0)$ where the entries in each row (resp. column) weakly decrease (resp. strictly decrease); and the entries can only be $1, 2, 3$. Thus, all possible tableaux are:

$$\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

which correspond to the individual monomials in the polynomial

$$\begin{aligned} & s_{\mathbf{n}}(u_1, u_2, u_3) \\ = \ & u_3^2 u_2 + u_3^2 u_1 + u_3 u_2^2 + u_3 u_2 u_1 + u_3 u_1 u_2 + u_3 u_1^2 + u_2^2 u_1 + u_2 u_1^2 \\ = \ & (u_1 + u_2)(u_2 + u_3)(u_3 + u_1). \end{aligned}$$

One may interpret $s_{\mathbf{n}}$ as the character of the adjoint representation of $GL_3(\mathbb{C})$ on $\mathfrak{sl}_3$.

We will need two basic facts about Schur polynomials: see for instance [29, Chapter I] for proofs and more details.

Proposition 2.2. *Fix $N \in \mathbb{N}$ and an integer tuple $\mathbf{n} = (n_0, \dots, n_{N-1})$ with $0 \leqslant n_0 < \cdots < n_{N-1}$. Then we have the formula*

$$\det(\mathbf{u}^{\circ n_0} | \dots | \mathbf{u}^{\circ n_{N-1}}) = \det(u_j^{n_{k-1}})_{j,k=1,\dots,N} = V(\mathbf{u}) s_{\mathbf{n}}(\mathbf{u}) \tag{2.2}$$

relating generalized Vandermonde determinants to Schur polynomials for all $\mathbf{u} \in \mathbb{C}^n$, where $V(\mathbf{u})$ is the Vandermonde determinant

$$\begin{aligned} V(\mathbf{u}) \coloneqq \ & \det(\mathbf{u}^{\circ 0} | \dots | \mathbf{u}^{\circ (N-1)}) \\ = \ & \det(u_j^{k-1})_{j,k=1,\dots,N} \\ = \ & \prod_{1 \leqslant j < k \leqslant N} (u_k - u_j). \end{aligned}$$

In particular, the polynomial $s_{\mathbf{n}}$ is symmetric. Furthermore, we have the Weyl dimension formula

$$s_{\mathbf{n}}(1, \dots, 1) = \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})}. \tag{2.3}$$

One can interpret the quantity in (2.3) as the dimension of the representation associated to the Schur polynomial $s_{\mathbf{n}}$, although we will not use this interpretation here. (For instance, the adjoint representation of $GL_3(\mathbb{C})$ on $\mathfrak{sl}_3$ has dimension 8.) Note that (2.2) immediately establishes (1.1) in the case when the exponents α_j are natural numbers.

The relevance of Schur polynomials to our problem comes from the following application of Proposition 2.2 and the Cauchy–Binet formula.

Lemma 2.3 (Determinant formula)**.** *Let S be a set of non-negative integers of cardinality at least N, and let h be a polynomial of the form*

$$h(x) = \sum_{n \in S} c_n x^n$$

for some real coefficients c_n. Then for any vector $\mathbf{u} \in (\mathbb{C}^n)^T$, we have

$$\det h[\mathbf{u}\mathbf{u}^*] = \sum_{\mathbf{n} \in S^N_<} |s_{\mathbf{n}}(\mathbf{u})|^2 |V(\mathbf{u})|^2 \prod_{n \in \mathbf{n}} c_n$$

where $S^N_<$ denotes the set of all N-tuples $(n_0, \dots, n_{N-1})$ of elements of S, sorted in increasing order $n_0 < \cdots < n_{N-1}$.

Proof. Write $S = \{n_1, \dots, n_M\}$ with $n_1 < \cdots < n_M$. We may factor

$$\begin{aligned} h[\mathbf{u}\mathbf{u}^*] &= \sum_{j=1}^M c_{n_j} \mathbf{u}^{\circ n_j} (\mathbf{u}^{\circ n_j})^* \\ &= (\mathbf{u}^{\circ n_1} | \dots | \mathbf{u}^{\circ n_M}) \operatorname{diag}(c_{n_1}, \dots, c_{n_M}) (\mathbf{u}^{\circ n_1} | \dots | \mathbf{u}^{\circ n_M})^*. \end{aligned}$$

Applying the Cauchy–Binet formula, we may thus expand $\det h[\mathbf{u}\mathbf{u}^*]$ as

$$\sum_{1 \leqslant j_1 < \cdots < j_N \leqslant M} |\det(\mathbf{u}^{\circ n_{j_1}} | \dots | \mathbf{u}^{\circ n_{j_N}})|^2 c_{n_{j_1}} \dots c_{n_{j_M}}$$

and the claim then follows from Proposition 2.2. ☐

Remark 2.4. The argument used to prove Lemma 2.3 in fact gives the more general algebraic identity

$$\det h[\mathbf{u}\mathbf{v}^T] = \sum_{\mathbf{n} \in S^N_<} s_{\mathbf{n}}(\mathbf{u}) s_{\mathbf{n}}(\mathbf{v}) V(\mathbf{u}) V(\mathbf{v}) \prod_{n \in \mathbf{n}} c_n$$

for arbitrary fields $\mathbb{F}$ and vectors $\mathbf{u}, \mathbf{v} \in (\mathbb{F}^N)^T$, where $h(x) = \sum_{n \in S} c_n x^n \in \mathbb{F}[x]$, $s_{\mathbf{n}}(\cdot)$ is the specialization to $\mathbb{F}[\cdot]$ of the polynomial $s_{\mathbf{n}}(\mathbf{u}) \in \mathbb{Z}[\mathbf{u}]$, and $V(\mathbf{u}), s_{\mathbf{n}}(\mathbf{u})V(\mathbf{u})$ are (generalized) Vandermonde determinants.

3. Bounded domains: the leading term of a Schur polynomial

In this section we prove Theorem 1.6. Our strategy is to first establish the result for rank one matrices $A = \mathbf{u}\mathbf{u}^T$, in which one can exploit Lemma 2.3, and then apply the fundamental theorem of calculus to extend the entrywise positivity preservation property to more general positive semidefinite matrices.

3.1. **The case of rank-one matrices.**

We begin with the simple but crucial observation that a Schur polynomial $s_{\mathbf{n}}$ is comparable in size to its leading monomial, when applied to non-negative arguments.

Proposition 3.1 (Comparing Schur polynomials with a monomial)**.** *Fix integers $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1}$, and scalars $0 \leqslant u_1 \leqslant \cdots \leqslant u_N$. Set $\mathbf{n} := (n_0, \dots, n_{N-1})$ and $\mathbf{u} := (u_1, \dots, u_N)$. Then we have*

$$1 \times \mathbf{u}^{\mathbf{n} - \mathbf{n}_{\min}} \leqslant s_{\mathbf{n}}(\mathbf{u}) \leqslant \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})} \times \mathbf{u}^{\mathbf{n} - \mathbf{n}_{\min}}, \tag{3.1}$$

where $\mathbf{n}_{\min} = (0, \dots, N-1)$*. Furthermore, the constants* 1 *and* $\frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})}$ *on both sides of* (3.1) *cannot be improved.*

Proof. By Proposition 2.2, $s_{\mathbf{n}}(\mathbf{u})$ is the sum of exactly $\frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})}$ monomials (not necessarily distinct), one of which is equal to $\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}$ (arising from the Young tableau in which the i^{th} row is entirely occupied by the number $N+1-i$). All other monomials are of the form $\mathbf{u}^{\mathbf{a}}$ for some tuple $\mathbf{a} = (a_1, \dots, a_N) \neq \mathbf{n} - \mathbf{n}_{\min}$ of non-negative integers summing to $\sum_{j=0}^{N-1} n_j - j$, and obeying the majorization condition

$$\sum_{j=0}^{J} a_{j+1} \geqslant \sum_{j=0}^{J} n_j - j$$

for all $J = 0, \dots, N-1$. From this and the hypothesis $0 \leqslant u_1 \leqslant \cdots \leqslant u_N$ we have

$$0 \leqslant \mathbf{u}^{\mathbf{a}} \leqslant \mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}} \tag{3.2}$$

and the claim (3.1) follows.

Setting $u_j = 1$ for all j and using (2.3), we see that the second inequality in (3.1) is sharp. For the first inequality, we set $u_i = A^i$ for some large $A > 1$ and observe that we can now improve (3.2) to

$$0 \leqslant \mathbf{u}^{\mathbf{a}} \leqslant \frac{1}{A} \mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}$$

for any monomial appearing in $s_{\mathbf{n}}$ other than the single dominant monomial $\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}$. Sending $A \to \infty$, we obtain the claim. □

Next, we give the precise threshold for positive semidefiniteness of a polynomial with $N+1$ terms applied to a generic rank one matrix.

Proposition 3.2. *Let* $0 \leqslant n_0 < \cdots < n_{N-1} < M$ *be non-negative integers, let* $c_{n_0}, \dots, c_{n_{N-1}}$ *be positive reals, let* $\mathbf{u} = (u_1, \dots, u_N)^T$ *have distinct positive coordinates, and let* $t > 0$ *be real. Let* p_t *denote the polynomial*

$$p_t(x) \coloneqq t \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M.$$

Then $p_t[\mathbf{u}\mathbf{u}^T]$ *is positive semidefinite if and only if*

$$t \geqslant \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_{n_j} s_{\mathbf{n}}(\mathbf{u})^2}, \tag{3.3}$$

where $\mathbf{n} = (n_0, \dots, n_{N-1})$ *as above, and the tuples* $\mathbf{n}_j$ *are defined as*

$$\mathbf{n}_j \coloneqq (n_0, \dots, n_{j-1}, n_{j+1}, \dots, n_{N-1}, M), \qquad \forall j = 0, \dots, N-1.$$

Proof. From Lemma 2.3 we have

$$\begin{aligned} &\det p_t[\mathbf{u}\mathbf{u}^T] \qquad (3.4)\\ &= t^N \left(\prod_{j=0}^{N-1} c_{n_j} \right) V(\mathbf{u})^2 s_{\mathbf{n}}(\mathbf{u})^2 - t^{N-1} \sum_{j=0}^{N-1} \left(\prod_{0 \leqslant k \leqslant N-1 : k \neq j} c_{n_k} \right) V(\mathbf{u})^2 s_{\mathbf{n}_j}(\mathbf{u})^2 \end{aligned}$$

from which we conclude that $\det p_t[\mathbf{u}\mathbf{u}^T]$ is non-negative precisely when (3.3) holds. We also see that the matrix $\sum_{j=0}^{N-1} c_{n_j} \mathbf{u}^{\circ n_j} (\mathbf{u}^{\circ n_j})^T$ has determinant

$$\left(\prod_{j=0}^{N-1} c_{n_j} \right) V(\mathbf{u})^2 s_{\mathbf{n}}(\mathbf{u})^2,$$

which is positive, and hence this matrix is not just positive semidefinite but is in fact positive definite. In particular, $p_t[\mathbf{u}\mathbf{u}^T]$ is positive definite for sufficiently large t. Since the determinant function is non-negative on $\mathbb{P}_N$ and vanishes on the boundary of $\mathbb{P}_N$, the claim now follows from the continuity of the eigenvalues of $p_t[\mathbf{u}\mathbf{u}^T]$. □

Remark 3.3. In the special case $\mathbf{n} = \mathbf{n}_{\min}$ studied in [4], one implication in Proposition 3.2 was shown using a Rayleigh quotient argument. That argument can be extended to work for general $\mathbf{n}$; see Section 11. It is also possible to obtain this implication using the matrix determinant lemma (see e.g. [12]), but we will not do so here.

Now we can obtain Theorem 1.6 (with an explicit threshold $\mathcal{K}$) in the special case of rank one matrices.

Proposition 3.4. *Fix integers $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1} < M$, and scalars $c_{n_0}, \dots, c_{n_{N-1}} > 0$. Let $I \subset [0, +\infty)$ be a bounded domain, and write $\rho \coloneqq \sup I$. If we define*

$$\mathcal{K} \coloneqq \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{M-n_j}}{c_{n_j}} \tag{3.5}$$

then the polynomial

$$x \mapsto \mathcal{K}(c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}}) - x^M$$

is entrywise positivity preserving on $\mathbb{P}_N^1(I)$.

Proof. A matrix in $\mathbb{P}_N^1(I)$ can be written in the form $\mathbf{u}\mathbf{u}^T$, where the coordinates $u_1, \dots, u_N$ of the vector $\mathbf{u}$ lie in $[0, \sqrt{\rho}]$. By a limiting argument, and permutation symmetry, we may assume without loss of generality that the u_i are distinct, positive, and strictly increasing. By Proposition 3.2, it will suffice to show that

$$\sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_{n_j} s_{\mathbf{n}}(\mathbf{u})^2} \leqslant \mathcal{K}.$$

But by the upper and lower bounds in (3.1), we have

$$\begin{aligned} \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_{n_j} s_{\mathbf{n}}(\mathbf{u})^2} &\leqslant \sum_{j=0}^{N-1} \frac{\left(\frac{V(\mathbf{n}_j)}{V(\mathbf{n}_{\min})} \mathbf{u}^{\mathbf{n}_j - \mathbf{n}_{\min}} \right)^2}{c_{n_j} (\mathbf{u}^{\mathbf{n} - \mathbf{n}_{\min}})^2} \\ &= \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2 c_{n_j}} (u_N^{M-n_{N-1}} u_{N-1}^{n_{N-1}-n_{N-2}} \cdots u_{j+1}^{n_{j+1}-n_j})^2 \end{aligned}$$

$$\leqslant \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{M-n_j}}{c_{n_j}}.$$

The claim now follows from (3.5). □

3.2. **From rank-one matrices to all matrices.** Given the threshold $\mathcal{K}$ from Proposition 3.4, we now prove the existence of a threshold for all matrices in $\mathbb{P}_N([0,\rho])$, i.e. Theorem 1.6. This will follow from the following more general 'extension principle':

Theorem 3.5 (Extension principle)**.** *Let $0 < \rho \leqslant +\infty$. Fix an integer $N > 1$, and let $h : (0,\rho) \to \mathbb{R}$ be continuously differentiable. If h is entrywise positivity preserving on $\mathbb{P}_N^1((0,\rho))$, and the derivative h' is entrywise positivity preserving on $\mathbb{P}_{N-1}((0,\rho))$, then h is entrywise positivity preserving on $\mathbb{P}_N((0,\rho))$. Similarly with $(0,\rho)$ replaced by $(-\rho,\rho)$ throughout.*

Proof. We use the approach in [4, Section 3]. Suppose $A = (a_{jk})_{j,k=1,\dots,N}$ is a matrix in $\mathbb{P}_N((0,\rho))$. Define ζ to be the last column of A divided by $\sqrt{a_{NN}}$; then $A - \zeta\zeta^T$ has last row and column zero and is positive semidefinite, and $\zeta\zeta^T \in \mathbb{P}_N^1((0,\rho))$. We now use an integration trick of FitzGerald and Horn [15, Equation (2.1)]. For any $x, y \in I$, we see from the fundamental theorem of calculus (and a change of variables $t = \lambda x + (1-\lambda)y$) that

$$h(x) - h(y) = \int_x^y h'(t)\ dt = \int_0^1 (x-y)h'(\lambda x + (1-\lambda)y)\ d\lambda.$$

Applying this entrywise with x, y replaced by the entries of A and $\zeta\zeta^T$ respectively, we obtain the identity

$$h[A] = h[\zeta\zeta^T] + \int_0^1 (A - \zeta\zeta^T) \circ h'[\lambda A + (1-\lambda)\zeta\zeta^T]\ d\lambda. \tag{3.6}$$

As h is entrywise positivity preserving on $\mathbb{P}_N^1((0,\rho))$, $h[\zeta\zeta^T]$ is positive semidefinite. Now since $A - \zeta\zeta^T$ is positive semidefinite and has last row and column zero, we see from the Schur product theorem that the integrand is positive semidefinite if the leading principal $(N-1)\times(N-1)$ minor of $h'[\lambda A + (1-\lambda)\zeta\zeta^T]$ is. Since the principal minors of A and $\zeta\zeta^T$ both lie in the convex set $\mathbb{P}_{N-1}((0,\rho))$, by assumption on h' we conclude that the integrand is everywhere positive semidefinite, whence so is $h[A]$ by (3.6). This gives the claim for $(0,\rho)$. A similar argument works if one replaces $(0,\rho)$ with $(-\rho,\rho)$, noting that one can easily reduce by a limiting argument to the case where a_{NN} is strictly positive. □

Using Theorem 3.5, we now show our first main result.

Proof of Theorem 1.6. Let $\mathcal{K}$ be the quantity defined in (3.5). It will suffice to show that for every $N \geqslant 1$, the polynomial

$$h(x) := \mathcal{K}(c_{n_0}x^{n_0} + \dots + c_{n_{N-1}}x^{n_{N-1}}) - x^M$$

is entrywise positivity preserving on $\mathbb{P}_N((0,\rho))$.

We induct on N. For $N = 1$ the claim follows from Proposition 3.4, so suppose that $N > 1$ and that the claim has already been proven for $N - 1$.

First observe that h is equal to x^{n_0} times another polynomial $\tilde{h}$, formed by reducing all the exponents $n_0, \dots, n_{N-1}, M$ by n_0; note from (3.5) that such a shift would not affect the quantity $\mathcal{K}$. Also, from the Schur product theorem we know that if $\tilde{h}$ is entrywise positivity preserving on $\mathbb{P}_N((0,\rho))$, then h will be also. As a consequence, we may assume without loss of generality that $n_0 = 0$.

By Proposition 3.4 and Theorem 3.5, it suffices to show that h' is entrywise positivity preserving on $\mathbb{P}_{N-1}((0,\rho))$. Since

$$h'(x) = \mathcal{K}(c_{n_1} n_1 x^{n_1 - 1} + \cdots + c_{n_{N-1}} n_{N-1} x^{n_{N-1}-1}) - M x^{M-1},$$

we will be done using the induction hypothesis provided that we can establish the inequality

$$\mathcal{K} \geqslant M \tilde{\mathcal{K}},$$

where $\tilde{\mathcal{K}}$ is defined like $\mathcal{K}$ but with N replaced by $N-1$, M replaced by $M-1$, $n_0, \dots, n_{N-1}$ replaced by $n_1 - 1, \dots, n_{N-1} - 1$, and $c_{n_0}, \dots, c_{n_{N-1}}$ replaced by $n_1 c_{n_1}, \dots, n_{N-1} c_{n_{N-1}}$ respectively.

Writing $\mathbf{n}_j = (m_{j,0}, \dots, m_{j,N-1})$ for $j = 0, \dots, N-1$, and recalling that $n_0 = 0$, we have $m_{j,0} = 0$ for $j = 1, \dots, N-1$. We may therefore verify using (3.5), (2.3) that

$$\begin{aligned}
\mathcal{K} &= \sum_{j=0}^{N-1} \frac{\rho^{M-n_j}}{c_{n_j} V(\mathbf{n}_{\min})^2} \prod_{0 \leqslant a < b \leqslant N-1} (m_{j,b} - m_{j,a})^2 \\
&\geqslant \sum_{j=1}^{N-1} \frac{\rho^{M-n_j}}{c_{n_j} V(\mathbf{n}_{\min})^2} \prod_{0 \leqslant a < b \leqslant N-1} (m_{j,b} - m_{j,a})^2 \\
&= \sum_{j=1}^{N-1} \frac{\rho^{M-n_j}}{c_{n_j} V(\mathbf{n}'_{\min})^2} \prod_{0 < a < b \leqslant N-1} (m_{j,b} - m_{j,a})^2 \cdot \frac{M^2}{(N-1)!^2} \prod_{b \neq 0, j} n_b^2 \\
&= M \sum_{j=1}^{N-1} \frac{\rho^{M-n_j}}{n_j c_{n_j} V(\mathbf{n}'_{\min})^2} \prod_{0 < a < b \leqslant N-1} (m_{j,b} - m_{j,a})^2 \cdot \frac{n_j M}{(N-1)!^2} \prod_{b \neq 0, j} n_b^2 \\
&\geqslant M \sum_{j=1}^{N-1} \frac{\rho^{M-n_j}}{n_j c_{n_j} V(\mathbf{n}'_{\min})^2} \prod_{0 < a < b \leqslant N-1} (m_{j,b} - m_{j,a})^2 \cdot \left(\frac{\prod_{b=1}^{N-1} n_j}{(N-1)!} \right)^2 \\
&\geqslant M \sum_{j=1}^{N-1} \frac{\rho^{M-n_j}}{n_j c_{n_j} V(\mathbf{n}'_{\min})^2} \prod_{0 < a < b \leqslant N-1} (m_{j,b} - m_{j,a})^2 \\
&= M \tilde{\mathcal{K}}
\end{aligned}$$

as required, where $\mathbf{n}'_{\min} \coloneqq (0, 1, \dots, N-2)$. □

Remark 3.6. In the case $\mathbf{n} = \mathbf{n}_{\min}$, this result (with the same value of the threshold $\mathcal{K}$) was established in [4]. This special case is simpler due to the fact that the denominator $s_{\mathbf{n}}(\mathbf{u})$ is now equal to 1.

3.3. **Threshold bounds for arbitrary analytic functions.** We next prove Corollary 1.7. By replacing ρ with $\rho - \epsilon$ and taking limits, we may assume without loss of generality that the power series of interest $g(x) =$

$\sum_{M>n_{N-1}} g_M x^M$ in fact converges in some neighborhood of ρ, and hence we have

$$|g_M| \leqslant C\rho^{-M}(1+\epsilon)^{-M}$$

for some $C, \epsilon > 0$ and all $M > n_{N-1}$.

By Theorem 1.6 (with the explicit bound (3.5)) and the triangle inequality, it will now suffice to show that

$$\sum_{M>n_{N-1}} \rho^{-M}(1+\epsilon)^{-M}\mathcal{K}_M < \infty,$$

where $\mathcal{K}_M$ is the quantity (3.5) for the specified value of M. If we write

$$\mathbf{n}'_j := (n_0, \dots, n_{j-1}, n_{j+1}, \dots n_{N-1})$$

for $j = 1, \dots, N-1$ (so that $\mathbf{n}_j = (\mathbf{n}'_j, M)$), then we can use Tonelli's theorem to compute

$$\begin{aligned}
&\sum_{M>n_{N-1}} \rho^{-M}(1+\epsilon)^{-M}\mathcal{K}_M \\
&= \sum_{j=0}^{N-1} \sum_{M>n_{N-1}} \rho^{-M}(1+\epsilon)^{-M} \frac{V(\mathbf{n}'_j)^2 \prod_{k\neq j}(M-n_k)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{M-n_j}}{c_{n_j}} \\
&= \sum_{j=0}^{N-1} \frac{V(\mathbf{n}'_j)^2}{V(\mathbf{n}_{\min})^2 c_{n_j}\rho^{n_j}} \sum_{M>n_{N-1}} (1+\epsilon)^{-M} \prod_{k\neq j}(M-n_k)^2.
\end{aligned}$$

But the inner summand is exponentially decaying in M, and so this sum is finite as required.

4. The case of unbounded domain

We now explore the unbounded case: namely, when $\rho = +\infty$. We begin by proving Lemma 1.4(iii). Suppose for contradiction that this claim failed. Applying Lemma 1.4(ii), it follows that there are fewer than N values of $n > n_0$ with $c_n > 0$. By adding the absolutely monotone function $-\sum_{n>n_0: c_n<0} c_n x^n$ to f, we may assume without loss of generality that there are no values of $n > n_0$ with $c_n < 0$. In particular, f is now a polynomial of some degree $d \geqslant n_0$, with fewer than N terms of higher degree than n_0. Now introduce the polynomial

$$\tilde{f}(x) := x^d f(1/x).$$

Observe that if $\mathbf{u} = (u_1, \dots, u_N)^T$ is a vector with entries in $(0, +\infty)$, then one has the identity

$$\tilde{f}[\mathbf{u}\mathbf{u}^T] = \mathbf{u}^{\circ d} \circ f[\mathbf{u}^{\circ -1}(\mathbf{u}^{\circ -1})^T]$$

where $\mathbf{u}^{\circ\alpha} := (u_1^\alpha, \dots, u_N^\alpha)^T$ denotes the entrywise power of $\mathbf{u}$ by α, and $\circ$ denotes the Hadamard product. Since f is entrywise positivity preserving on $\mathbb{P}_N^1((0,+\infty))$, it follows from the Schur product theorem that $\tilde{f}$ does also. On the other hand, from construction, $\tilde{f}$ has a negative x^{d-n_0} coefficient, but has fewer than N positive coefficients of lower degree. This contradicts Lemma 1.4(ii), as required.

Now we prove our main result in this setting.

Proof of Theorem 1.10. By replacing $c_{n_0}, \dots, c_{n_{2N-1}}$ by their minimal value and then rescaling, we may assume without loss of generality that $c_{n_j} = 1$ for all $0 \leqslant j \leqslant 2N-1$. Setting h to be the polynomial

$$h(x) := \sum_{j=0}^{2N-1} x^{n_j},$$

it suffices to show that the polynomial $x \mapsto th(x) - x^M$ entrywise preserves positivity on $\mathbb{P}_N((0,+\infty))$ for t large enough.

We first establish the rank one case, that is to say that for sufficiently large t and for all $\mathbf{u} = (u_1, \dots, u_N)^T$ with entries $u_1, \dots, u_N$ in $(0, +\infty)$, we show that the matrix $t\,h[\mathbf{u}\mathbf{u}^T] - \mathbf{u}^{\circ M}(\mathbf{u}^{\circ M})^T$ is positive semidefinite. By a limiting argument and symmetry we may assume that $0 < u_1 < \cdots < u_N$. Lemma 2.3 assures us that $h[\mathbf{u}\mathbf{u}^T]$ has positive determinant, and is thus positive definite as opposed to merely positive semidefinite. Thus, for each fixed $\mathbf{u}$, $t\,h[\mathbf{u}\mathbf{u}^T] - \mathbf{u}^{\circ M}(\mathbf{u}^{\circ M})^T$ is positive definite for sufficiently large t (where the threshold for t may possibly vary with $\mathbf{u}$). Using a continuity argument as in the proof of Proposition 3.2, it now suffices to show that for all sufficiently large t, one has

$$\det(t\,h[\mathbf{u}\mathbf{u}^T] - \mathbf{u}^{\circ M}(\mathbf{u}^{\circ M})^T) > 0$$

uniformly in $\mathbf{u}$. Applying Lemma 2.3, we may write this determinant as

$$t^N \sum_{B \in S^N_<} s_B(\mathbf{u})^2 - t^{N-1} \sum_{C \in S^{N-1}_<} s_{C \sqcup \{M\}}(\mathbf{u})^2$$

where $S := \{n_0, \dots, n_{2N-1}\}$, and $C \sqcup \{M\}$ denotes the union of the $N-1$-tuple C and $\{M\}$, sorted to be in increasing order. It thus suffices to show that for each $C \in S^{N-1}_<$, the ratio

$$\frac{s_{C \sqcup \{M\}}(\mathbf{u})^2}{\sum_{B \in S^N_<} s_B(\mathbf{u})^2}$$

is uniformly bounded in $\mathbf{u}$.

Fix C, and permute $\mathbf{u}$ to have non-decreasing coordinates. Applying Proposition 3.1, it suffices to show that

$$\frac{(\mathbf{u}^{C \sqcup \{M\}})^2}{\sum_{B \in S^N_<} (\mathbf{u}^B)^2}$$

is uniformly bounded in all such $\mathbf{u}$. But as C only has cardinality $N-1$, and there are N elements of S that are less than M and N elements that are greater than M, there exist exponents $n_- < M < n_+$ such that $n_-, n_+ \in S \backslash C$. (Note, this argument also shows the need for the necessary condition in Lemma 1.4(iii).) This implies that

$$(\mathbf{u}^{C \sqcup \{M\}})^2 \leqslant (\mathbf{u}^{C \sqcup \{n_-\}})^2 + (\mathbf{u}^{C \sqcup \{n_+\}})^2$$

(breaking into cases depending on whether the component of $\mathbf{u}$ that will be paired with M is less than 1 or not), and hence the above ratio is uniformly bounded by one, giving the claim.

To remove the restriction to rank one matrices, we induct on N as in the previous section. For $N = 1$ the claim is already proven, so suppose that $N > 1$ and that the claim has already been proven for $N - 1$. By the induction hypothesis (and discarding some manifestly entrywise positivity preserving terms), the derivative of $th(x) - x^M$ will entrywise preserve positivity on $\mathbb{P}_{N-1}((0, +\infty))$ for t large enough. We have already shown that $th(x) - x^M$ also entrywise preserves positivity $\mathbb{P}^1_N((0, +\infty))$ for t large enough. Applying Theorem 3.5, we conclude that $th(x) - x^M$ entrywise preserves positivity on all matrices in $\mathbb{P}_N((0, +\infty))$ for t large enough, as required. □

5. Real exponents: the Harish-Chandra–Itzykson–Zuber formula

We now explore extensions of the above arguments to answer Question 1.3 in the case when the exponents $n_0, \dots, n_{N-1}, M$ are only assumed to be real rather than natural numbers. We begin by observing that the parts (ii), (iii) of Lemma 1.4 hold for real powers as well:

Lemma 5.1 (Horn–Loewner-type necessary conditions for real powers)**.** *Fix an integer $N \geqslant 2$ and a scalar $0 < \rho \leqslant +\infty$. Further fix scalars $c_{n_i} \in \mathbb{R}$ and distinct real powers n_i for $i \geqslant 0$, and suppose $f(x) := \sum_{i=0}^{\infty} c_{n_i} x^{n_i}$ is a convergent sum of powers on $(0, \rho)$.*

- (ii) *If f is entrywise positivity preserving on $\mathbb{P}^1_N((0, \rho))$, and $c_{n_{i_0}} < 0$ for some $i_0 \geqslant 0$, then we have $c_{n_i} > 0$ for at least N values of i for which $n_i < n_{i_0}$.*
- (iii) *If f is entrywise positivity preserving on $\mathbb{P}^1_N((0, +\infty))$, and $c_{n_{i_0}} < 0$ for some $i_0 \geqslant 0$, then in addition to (ii) we also have $c_{n_i} > 0$ for at least N values of i such that $n_i > n_{i_0}$.*

The proofs are minor modifications of those of Lemma 1.4(ii), (iii) respectively.

The objectives of the remainder of this section are to show that (in analogy to the integer exponent case) the necessary conditions in Lemma 5.1 are once again completely sharp, and that one can obtain threshold bounds for all Puiseux or Hahn series, in analogy to Corollary 1.7, with quantitative bounds that are as sharp as possible. For non-negative rational power exponents, one can achieve the first two objectives by the simple change of variables $y_j := u_j^{1/L}$, where $L > 0$ is a common denominator for the rationals $n_0, \dots, n_{N-1}, M$. However, the quantitative bounds obtained by doing so depend on L in an unfavorable manner, and so this approach does not seem to easily extend to the general real exponent case. Hence we shall adopt a different approach in the arguments below.

5.1. **Sign patterns of sums of powers.** In this subsection we resolve Question 1.3 for the more involved case of real powers (which we take to be non-negative because negative powers cannot entrywise preserve positivity on matrices of rank 2 or higher). As the theory of Schur polynomials crucially requires integer powers (or rational powers via the above workaround), in place of it we now rely on the Harish-Chandra–Itzykson–Zuber identity

$$\begin{aligned}&\det(e^{\alpha_i x_j})_{1\leqslant i,j\leqslant N} \\ &= \frac{V(\alpha)V(\mathbf{x})}{V(\mathbf{n}_{\min})}\int_{U(N)} \exp\operatorname{tr}\left(\operatorname{diag}(\alpha_1,\dots,\alpha_N)U\operatorname{diag}(x_1,\dots,x_N)U^*\right)\ dU,\end{aligned} \tag{5.1}$$

which is valid for any tuples $\alpha = (\alpha_1,\dots,\alpha_N)$, $\mathbf{x} = (x_1,\dots,x_N)$ of real numbers, and where dU denotes Haar probability measure on the unitary group $U(N)$; see e.g., [24, (3.4)]. (We thank Ryan O'Donnell for drawing our attention to this identity.) If $\alpha_1 \leqslant \cdots \leqslant \alpha_N$ and $x_1 \leqslant \cdots \leqslant x_N$, then by the Schur–Horn theorem [37, 21], the diagonal entries of $U\operatorname{diag}(x_1,\dots,x_N)U^*$ are majorized by $(x_1,\dots,x_N)$, and hence the trace in the above expression ranges between $\sum_{j=1}^N \alpha_j x_{N+1-j}$ and $\sum_{j=1}^N \alpha_j x_j$. As all Vandermonde determinants appearing here are non-negative, we conclude the (somewhat crude) inequalities

$$\frac{V(\alpha)V(\mathbf{x})}{V(\mathbf{n}_{\min})}\exp\sum_{j=1}^N \alpha_j x_{N+1-j} \leqslant \det(e^{\alpha_i x_j})_{1\leqslant i,j\leqslant N} \leqslant \frac{V(\alpha)V(\mathbf{x})}{V(\mathbf{n}_{\min})}\exp\sum_{j=1}^N \alpha_j x_j.$$

Writing $u_j = \exp(x_j)$, we thus have

$$\frac{V(\alpha)V(\log[\mathbf{u}])}{V(\mathbf{n}_{\min})}\mathbf{u}^{\overline{\alpha}} \leqslant \det(\mathbf{u}^{\circ\alpha_1}|\dots|\mathbf{u}^{\circ\alpha_N}) \leqslant \frac{V(\alpha)V(\log[\mathbf{u}])}{V(\mathbf{n}_{\min})}\mathbf{u}^{\alpha}. \tag{5.2}$$

In particular, (5.2) implies the following upper and lower bounds for this determinant in the case that $\mathbf{u}$ ranges in a compact set:

Lemma 5.2. *Let $I \subset (0,+\infty)$ be a compact interval, and let K be a compact subset of the cone*

$$\{(\alpha_1,\dots,\alpha_N) \in \mathbb{R}^N : \alpha_1 < \alpha_2 < \cdots < \alpha_N\}.$$

Then there exist constants $C, c > 0$ such that

$$c|V(\mathbf{u})| \leqslant |\det(\mathbf{u}^{\circ\alpha_1}|\dots|\mathbf{u}^{\circ\alpha_N})| \leqslant C|V(\mathbf{u})|$$

for all $\mathbf{u} \in I^N$ and all $\alpha = (\alpha_1,\dots,\alpha_N) \in K$.

Proof. For $\alpha \in K$ and $\mathbf{u} \in I^N$, $V(\alpha)$ is bounded above and below by constants depending only on I, K, N, as are $\mathbf{u}^\alpha$ and $\mathbf{u}^{\overline{\alpha}}$. Furthermore, for each $1 \leqslant i < j \leqslant N$, $|\log(u_i) - \log(u_j)|$ is comparable to $|u_i - u_j|$ thanks to the mean value theorem. The claim now follows from (5.2). ☐

Now we extend the above lemma to obtain estimates when the arguments $\mathbf{u}$ are not restricted to a compact set.

Lemma 5.3. *Let K be a compact subset of the cone*

$$\{(n_0,\dots,n_{N-1}) \in \mathbb{R}^N : n_0 < n_1 < \cdots < n_{N-1}\}.$$

Then there exist constants $C, c > 0$ such that

$$cV(\mathbf{u})\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}} \leqslant \det(\mathbf{u}^{\circ n_0}|\dots|\mathbf{u}^{\circ n_{N-1}}) \leqslant CV(\mathbf{u})\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}$$

for all $\mathbf{u} = (u_1, \dots, u_N)^T \in ((0,+\infty)^N)^T$ *with* $u_1 \leqslant \dots \leqslant u_N$ *and all* $\mathbf{n} = (n_0, \dots, n_{N-1}) \in K$.

If $n_0, \dots, n_{N-1}$ were restricted to be integers, then this claim would follow directly from Proposition 2.2 and Proposition 3.1. One can view this lemma as a substitute for these propositions in the non-integer setting.

Proof. The claim is easy when $N = 1$, so we suppose inductively that $N > 1$ and that the claim has already been proven for all smaller values of N. By a limiting argument we may assume that $0 < u_1 < \dots < u_N$.

Let $A > 2$ be a large constant to be chosen later. We first consider the non-separated case in which $u_{i+1}/u_i < A$ for all $i = 1, \dots, N-1$. By dividing all the u_j by (say) u_1, we may normalize $u_1 = 1$ without loss of generality, and now the $u_1, \dots, u_N$ are all confined to a compact subset of $(0,+\infty)$, in which case the claim follows from the previous lemma.

Now suppose that one has $u_{i+1}/u_i \geqslant A$ for some $1 \leqslant i < N$. We can split $\mathbf{u}$ into the two smaller vectors $\mathbf{u}' := (u_1, \dots, u_i)^T$ and $\mathbf{u}'' := (u_{i+1}, \dots, u_N)^T$. By cofactor expansion, we may then express $\det(\mathbf{u}^{\circ n_0}|\dots|\mathbf{u}^{\circ n_{N-1}})$ as the alternating sum of $\binom{N}{i}$ products of the form

$$\det((\mathbf{u}')^{\circ n'_1}|\dots|(\mathbf{u}')^{\circ n'_i}) \cdot \det((\mathbf{u}'')^{\circ n''_1}|\dots|(\mathbf{u}'')^{\circ n''_{N-i}})$$

where the $n'_1, \dots, n'_i, n''_1, \dots, n''_{N-i}$ are a permutation of $n_0, \dots, n_{N-1}$ with $n'_1 < \dots < n'_i$ and $n''_1 < \dots < n''_{N-i}$. By the induction hypothesis, each such product is comparable to

$$V(\mathbf{u}')(\mathbf{u}')^{\mathbf{n}'-(0,\dots,i-1)}V(\mathbf{u}'')(\mathbf{u}'')^{\mathbf{n}''-(0,\dots,N-i-1)}$$

where $\mathbf{n}' := (n'_1, \dots, n'_i)$ and $\mathbf{n}'' := (n''_1, \dots, n''_{N-i})$.

If $1 \leqslant j \leqslant i < k \leqslant N$, then $u_k \geqslant Au_i \geqslant 2u_j$, and hence $u_k - u_j$ is comparable to u_k. From this we conclude that $V(\mathbf{u})$ is comparable to $V(\mathbf{u}') \cdot V(\mathbf{u}'') \cdot (\mathbf{u}'')^{(i,\dots,i)}$, and hence the preceding expression is comparable to

$$V(\mathbf{u})\mathbf{u}^{(\mathbf{n}',\mathbf{n}'')-(0,\dots,N-1)}.$$

As $(\mathbf{n}', \mathbf{n}'')$ is a rearrangement of $\mathbf{n}$, one has

$$\mathbf{u}^{(\mathbf{n}',\mathbf{n}'')-(0,\dots,N-1)} \leqslant \mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}$$

and furthermore (because all the entries of $\mathbf{u}''$ are at least A times larger than those of $\mathbf{u}'$) one has the refinement

$$\mathbf{u}^{(\mathbf{n}',\mathbf{n}'')-(0,\dots,N-1)} \leqslant \frac{1}{A}\mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}$$

unless $\mathbf{n}' = (0, \dots, i-1)$ and $\mathbf{n}'' = (i, \dots, N-1)$. For A large enough, this (together with (1.1)) proves the desired lower bound; and the upper bound also follows, using the triangle inequality. □

Repeating the proof of Proposition 3.4, using Lemma 5.3 as a replacement for Lemma 3.1, we conclude

Proposition 5.4. *Let $n_0 < \cdots < n_{N-1} < M$ and scalars $c_{n_0}, \dots, c_{n_{N-1}} > 0$ be real numbers. Let $I \subset (0, +\infty)$ be a bounded domain. Then for sufficiently large t, the function*

$$x \mapsto t(c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}}) - x^M$$

is entrywise positivity preserving on $\mathbb{P}_N^1(I)$.

Using Theorem 3.5, we may remove the rank one restriction assuming that the $n_0, \dots, n_{N-1}$ are either natural numbers or not too small, giving a version of Theorem 1.6 for real exponents:

Theorem 5.5. *Let $0 \leqslant n_0 < \cdots < n_{N-1} < M$ and scalars $c_{n_0}, \dots, c_{n_{N-1}} > 0$ be real numbers. Assume that each n_i is either a non-negative integer, or is greater than $N-2$ (or both). Let $I \subset [0, +\infty)$ be a bounded domain. Then for sufficiently large t, the function*

$$x \mapsto t(c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}}) - x^M \tag{5.3}$$

is entrywise positivity preserving on $\mathbb{P}_N(I)$.

The condition that each n_i is either a non-negative integer or greater than $N-2$ is natural in view of the results in [15], in which it is shown that these conditions are necessary and sufficient to ensure that $x \mapsto x^{n_i}$ is entrywise positivity preserving on $\mathbb{P}_N((0, +\infty))$.

Proof. The claim is trivial for $N = 1$. Now we consider the $N = 2$ case. In this case it follows from the results in [15] that the map $x \mapsto x^{n_0}$ is entrywise positivity preserving on $\mathbb{P}_N((0, +\infty))$. By the Schur product theorem, we may thus factor out x^{n_0} and assume without loss of generality that $n_0 = 0$. Similarly, the map $x \mapsto x^{n_1}$ is entrywise positivity preserving on $\mathbb{P}_N((0, +\infty))$, and so by composing with this map we may assume that $n_1 = 1$. For t large enough, the derivative of the function $t(c_0 + c_1 x) - x^M$ is then entrywise positivity preserving on $\mathbb{P}_1(I)$, and the claim now follows from Theorem 3.5 and Proposition 5.4.

Now suppose inductively that $N > 2$, and that the claim has already been proven for $N - 1$. Observe that the derivative of the polynomial (5.3) is of the form required for the inductive hypothesis (all the surviving monomials have exponents that are either non-negative integers, or greater than $N - 3$, or both). Thus the derivative will be entrywise positivity preserving on $\mathbb{P}_{N-1}(I)$ for t large enough, and the claim again follows from Theorem 3.5 and Proposition 5.4. $\square$

We can now give the complete solution to Question 1.3 for real powers, which shows Lemma 5.1(i) is sharp.

Theorem 5.6. *Let $N \geqslant 2$, and let $\{n_i : i \geqslant 0\} \subset \mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$ be a set of pairwise distinct real numbers. For each i, let $\epsilon_i \in \{-1, 0, +1\}$ be a sign such that whenever $\epsilon_{i_0} = -1$, one has $\epsilon_i = +1$ for at least N choices of i satisfying: $n_i < n_{i_0}$. Let $0 < \rho < \infty$. Then there exists a convergent series with real coefficients*

$$f(x) = \sum_{i=0}^{\infty} c_{n_i} x^{n_i}$$

on $(0,\rho)$ that is an entrywise positivity preserver on $\mathbb{P}_N((0,\rho))$, such that c_{n_i} has the sign of ϵ_i for all $i \geqslant 0$.

We remark that a difference between power series (from previous sections) and countable sums of real powers is that the latter can include an infinite decreasing set of powers.

Proof. The proof uses the following computation that is also useful below: given any set $\{n_i : i \geqslant 0\}$ of (pairwise distinct) non-negative powers, we claim that

$$\sum_{i \geqslant 0} \frac{x^{n_i}}{i! \lceil n_i \rceil !} < \infty, \qquad \forall x > 0. \tag{5.4}$$

Indeed, partition the non-negative integers as

$$\mathbb{Z}^{\geqslant 0} = \sqcup_{j \geqslant 0} I_j, \quad \text{where} \quad I_j := \{i \geqslant 0 : n_i \in (j-1, j]\}.$$

Using Tonelli's theorem, we crudely estimate:

$$\begin{aligned}\sum_{i \geqslant 0} \frac{x^{n_i}}{i! \lceil n_i \rceil !} = \sum_{j \geqslant 0} \frac{1}{j!} \sum_{i \in I_j} \frac{x^{n_i}}{i!} &\leqslant e + \sum_{j \geqslant 1} \frac{1}{j!} \sum_{i \in I_j} \frac{x^j + x^{j-1}}{i!} \\ &< e + e(e^x + x^{-1} e^x) < \infty.\end{aligned}$$

Now to prove the result, let $J \subset \mathbb{Z}^{\geqslant 0}$ denote the subset $\{i : \epsilon_i = -1\}$. For each $j \in J$ we have $i_1(j), \dots, i_N(j)$ such that $\epsilon_{i_k(j)} = 1$ and $n_{i_k(j)} < n_j$, for $k = 1, \dots, N$. We define

$$f_j(x) := \sum_{k=1}^N \frac{x^{n_{i_k(j)}}}{\lceil n_{i_k(j)} \rceil !} - \delta_j \frac{x^{n_j}}{\lceil n_j \rceil !},$$

where $\delta_j \in (0,1)$ is chosen such that $f_j[-]$ preserves positivity on $\mathbb{P}_N((0,\rho))$ by Theorem 5.5. Let J' denote the set of all $i \geqslant 0$ for which $\epsilon_i = +1$ but $i \neq i_k(j)$ for any $j \in J,\ k \in [1,N]$. Finally, define

$$f(x) := \sum_{j \in J} \frac{f_j(x)}{j!} + \sum_{i \in J'} \frac{x^{n_i}}{i! \lceil n_i \rceil !}, \qquad x > 0.$$

By repeating the above computation (5.4), one verifies f is absolutely convergent on $(0,\infty)$ and hence on $(0,\rho)$. By the Schur product theorem and the above hypotheses, it follows that $f[-]$ preserves positivity on $\mathbb{P}_N((0,\rho))$. □

In a similar vein to the bounded case, for the unbounded domain $(0,+\infty)$ we may adapt the proof of Theorem 1.10 to real exponents, using Lemma 5.3 as a replacement for Lemma 3.1, to obtain

Theorem 5.7. *Let $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1} < M < n_N < \cdots < n_{2N-1}$ be real numbers, such that each of the $n_0, \dots, n_{N-2}$ are either non-negative integers, greater than $N-2$, or both. Let $c_{n_0}, \dots, c_{n_{2N-1}}$ be positive reals. Then there exists a negative number c_M such that*

$$x \mapsto c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{N-1}} x^{n_{N-1}} + c_M x^M + c_{n_N} x^{n_N} + \cdots + c_{n_{2N-1}} x^{n_{2N-1}}$$

entrywise preserves positivity on $\mathbb{P}_N((0,+\infty))$.

Using this theorem, we can show that Lemma 5.1(ii) is sharp, vis-a-vis Question 1.3:

Theorem 5.8. *Let $N \geqslant 2$, and let $\{n_i : i \geqslant 0\} \subset \mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$ be a set of pairwise distinct real numbers. For each i, let $\epsilon_i \in \{-1, 0, +1\}$ be a sign such that whenever $\epsilon_{i_0} = -1$, one has $\epsilon_i = +1$ for at least N choices of i satisfying: $n_i < n_{i_0}$, and at least N choices of i satisfying: $n_i > n_{i_0}$. Then there exists a convergent series with real coefficients*

$$f(x) = \sum_{i=0}^{\infty} c_{n_i} x^{n_i}$$

on $(0, +\infty)$ that is an entrywise positivity preserver on $\mathbb{P}_N((0, +\infty))$, such that c_{n_i} has the sign of ϵ_i for all $i \geqslant 0$.

The proof is similar to that of Theorem 5.6 and is left to the interested reader.

5.2. **Bounds for Laplace transforms.** Our final result in this section obtains a similar assertion to Corollary 1.7 for real powers. In this setting we begin with real powers $0 \leqslant n_0 < \cdots < n_{N-1}$, and replace the analytic function $g(x) = \sum_{M > n_{N-1}} g_M x^M$ from Corollary 1.7 by Laplace transforms against more general measures,

$$g_\mu(x) := \int_{n_{N-1}}^{\infty} x^t \, d\mu(t), \tag{5.5}$$

which we assume to be absolutely convergent at ρ. We now prove:

Theorem 5.9. *Fix an integer $N > 0$ and suppose $n_0, \dots, n_{N-1}$ are strictly increasing real numbers in the set $\mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$. Also fix positive real scalars $\rho, c_{n_0}, \dots, c_{n_{N-1}} > 0$. Given $\varepsilon > 0$ and a real measure μ supported on $[n_{N-1} + \varepsilon, \infty)$ whose 'Laplace transform' g_μ (defined in (5.5)) is absolutely convergent at ρ, there exists a finite threshold*

$$\mathcal{K} = \mathcal{K}(n_0, \dots, n_{N-1}, \rho, c_{n_0}, \dots, c_{n_{N-1}}, g_\mu) > 0$$

such that the function

$$x \mapsto \mathcal{K} \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - g_\mu(x)$$

is entrywise positivity preserving on $\mathbb{P}_N((0, \rho))$. Equivalently, one has

$$g_\mu[A] \preceq \mathcal{K} \sum_{j=0}^{N-1} c_{n_j} A^{\circ n_j} \tag{5.6}$$

for all $A \in \mathbb{P}_N((0, \rho))$.

The remainder of this section is devoted to proving Theorem 5.9. The key improvement over the previous subsection that is required in the proof is a sharper version of Lemma 5.3:

Proposition 5.10 (Leading term of generalized Vandermonde determinants)**.** *Let $n_0 < \cdots < n_{N-1}$ be 1-separated, in the sense that $n_{i+1} - n_i \geqslant 1$ for all $0 \leqslant i < N-1$. Then for all $\mathbf{u} = (u_1, \ldots, u_N)^T \in ((0, +\infty)^N)^T$ with $u_1 \leqslant \cdots \leqslant u_N$, we have*

$$1 \times V(\mathbf{u}) \mathbf{u}^{\mathbf{n} - \mathbf{n}_{\min}} \leqslant \det(\mathbf{u}^{\circ n_0} | \ldots | \mathbf{u}^{\circ n_{N-1}}) \leqslant \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})} \times V(\mathbf{u}) \mathbf{u}^{\mathbf{n} - \mathbf{n}_{\min}}, \quad (5.7)$$

where $\mathbf{n}_{\min} := (n_0, \ldots, n_{N-1})$. Moreover, the lower and upper bounds of $1, \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})}$ cannot be improved.

Note that (5.7) matches the bounds in Proposition 3.1, and hence extends that result to all real powers.

To prove Proposition 5.10, we require the following generalization of a symmetric function identity to real powers (which is also required later; and which we show for completeness).

Proposition 5.11 (Principal specialization of generalized Vandermonde determinants)**.** *Fix an integer $N > 0$ and real powers $n_0 < \cdots < n_{N-1}$. Also define*

$$\mathbf{u}(\epsilon) := (1, \epsilon, \ldots, \epsilon^{N-1})^T, \qquad \epsilon > 0. \quad (5.8)$$

Defining the matrix $\mathbf{u}(\epsilon)^{\circ \mathbf{n}} := (\mathbf{u}(\epsilon)^{\circ n_0} | \ldots | \mathbf{u}(\epsilon)^{\circ n_{N-1}})$, we have:

$$\frac{\det \mathbf{u}(\epsilon)^{\circ \mathbf{n}}}{V(\mathbf{u}(\epsilon))} = \prod_{0 \leqslant i < j \leqslant N-1} \frac{\epsilon^{n_j} - \epsilon^{n_i}}{\epsilon^j - \epsilon^i}, \qquad \forall \epsilon > 0, \ \epsilon \neq 1. \quad (5.9)$$

Notice that unlike Proposition 5.10, we do not require the entries of $\mathbf{u}(\epsilon)$ to be non-decreasing, whence $\epsilon > 0$ can be arbitrary.

Proof. The transpose of $\mathbf{u}(\epsilon)^{\circ \mathbf{n}}$ is the Vandermonde matrix $(\mathbf{v}^{\circ 0} | \ldots | \mathbf{v}^{\circ (N-1)})$ with $\mathbf{v} := (\epsilon^{n_0}, \ldots, \epsilon^{n_{N-1}})^T$. In particular, the determinant of this matrix is $V(\mathbf{v}) = \prod_{1 \leqslant i < j \leqslant N-1} (\epsilon^{n_j} - \epsilon^{n_i})$, and the claim follows. □

Remark 5.12. One can view this identity as the real exponent version of the principal specialization of the Weyl Character Formula in type A (see e.g. [29, Chapter I.3]), which says that for integers $0 \leqslant n_0 < \cdots < n_{N-1}$, and a variable q,

$$s_{\mathbf{n}}(1, q, \ldots, q^{N-1}) = \prod_{0 \leqslant i < j \leqslant N-1} \frac{q^{n_j} - q^{n_i}}{q^j - q^i}$$

over any ground field.

We now prove the aforementioned tight bounds on generalized Vandermonde determinants.

Proof of Proposition 5.10. Note that if $n_0 < 0$, then by multiplying all terms in the inequality (5.7) by $(u_1 \cdots u_N)^{-n_0}$, one can reduce to the case of non-negative powers $n_j - n_0$. Thus, we suppose henceforth that $n_0 \geqslant 0$. By a limiting argument, we may assume without loss of generality that $0 < u_1 < \cdots < u_N$. From (5.1) we have

$$\det(\mathbf{u}^{\circ n_0} | \ldots | \mathbf{u}^{\circ n_{N-1}}) \quad (5.10)$$

$$= \frac{V(\mathbf{n})V(\log[\mathbf{u}])}{V(\mathbf{n}_{\min})} \int_{U(N)} \exp\operatorname{tr}\left(\operatorname{diag}(\mathbf{n})U\operatorname{diag}(\log[\mathbf{u}])U^*\right)\ dU;$$

replacing $\mathbf{n}$ by $\mathbf{n}_{\min}$, we also see that

$$V(\mathbf{u}) = V(\log[\mathbf{u}]) \int_{U(N)} \exp\operatorname{tr}\left(\operatorname{diag}(\mathbf{n}_{\min})U\operatorname{diag}(\log[\mathbf{u}])U^*\right)\ dU$$

and hence

$$\begin{aligned} &\det(\mathbf{u}^{\circ n_0}|\dots|\mathbf{u}^{\circ n_{N-1}}) \\ &\leqslant \frac{V(\mathbf{n})V(\mathbf{u})}{V(\mathbf{n}_{\min})} \sup_{U\in U(N)} \exp\operatorname{tr}\left(\operatorname{diag}(\mathbf{n}-\mathbf{n}_{\min})U\operatorname{diag}(\log[\mathbf{u}])U^*\right). \end{aligned}$$

By the Schur–Horn theorem [37, 21], the diagonal entries of the matrix $U\operatorname{diag}(\log[\mathbf{u}])U^*$ are majorized by $\log[\mathbf{u}]$. By hypothesis, the vectors $\mathbf{n}-\mathbf{n}_{\min}$ and $\log[\mathbf{u}]$ have non-decreasing entries, and hence

$$\exp\operatorname{tr}\left(\operatorname{diag}(\mathbf{n}-\mathbf{n}_{\min})U\operatorname{diag}(\log[\mathbf{u}])U^*\right) \leqslant \mathbf{u}^{\mathbf{n}-\mathbf{n}_{\min}}.$$

Putting all this together, we obtain the upper bound in (5.7).

Now we turn to the lower bound. Using the integration formula in [38, (3.2)] (we thank Abdelmalek Abdesselam for this reference), we may write the right-hand side of (5.10) as

$$V(\log[\mathbf{u}]) \int_{GT(\mathbf{n})} \exp\left(\sum_{k=1}^{N}\left(\sum_{i=1}^{N-k+1} m_i^{k-1} - \sum_{i=1}^{N-k} m_i^k\right)\log u_k\right) \tag{5.11}$$

where the *Gelfand–Tsetlin polytope* $GT(\mathbf{n})$ is the collection of all tuples $(m_i^k)_{1\leqslant k\leqslant N-1;1\leqslant i\leqslant N-k} \in \mathbb{R}^{N(N-1)/2}$ of real numbers m_i^k obeying the interlacing relations

$$m_i^k > m_i^{k+1} > m_{i+1}^k, \qquad 0 \leqslant k < N-1,\ 1 \leqslant i < N-k,$$

with the convention that $m_i^0 = n_{i-1}$ for $i = 1, \dots, N$, and the integration is with respect to Lebesgue measure on this polytope.

Remark 5.13. The reader may wish to first run the argument here in the simple case $N = 2$, $n_0 = 0$, $u_1 = 1$, in which case the formula (5.11) simplifies to $u_2^{n_1} - 1 = \log(u_2)\int_0^{u_2}\exp(m_1^1\log u_2)\ dm_1^1$, while the derivation (5.12) below becomes $u_2^{n_1} - 1 = \log(u_2)u_2^{n_2}\int_0^{u_2}\exp(-\beta_1^1\log u_2)\ d\beta_1^1$. The formula (5.11) can also be thought of as a continuous or "classical" version of (2.2), or equivalently (2.2) may be thought of as a discrete or "quantized" version of (5.11).

Returning to the general case, if we now write $\log u_k = \sum_{j=1}^k \alpha_j$ for some reals $\alpha_j = \log u_j - \log u_{j-1}$ (with the convention $\log u_0 = 0$), we can telescope the expression

$$\sum_{k=1}^{N}\left(\sum_{i=1}^{N-k+1} m_i^{k-1} - \sum_{i=1}^{N-k} m_i^k\right)\log u_k$$

appearing in the above formula as

$$\sum_{j=1}^{N} \alpha_j \sum_{i=0}^{N-j+1} m_i^{j-1};$$

making the linear change of variables

$$m_i^j = n_{i+j-1} - \beta^1_{i+j-1} - \beta^2_{i+j-2} - \cdots - \beta^i_j$$

for a tuple $(\beta_i^k)_{1 \leqslant k \leqslant N-1; 1 \leqslant i \leqslant N-k} \in \mathbb{R}^{N(N-1)/2}$ in a sheared version $\widetilde{GT}(\mathbf{n})$ of the Gelfand–Tsetlin polytope, this expression can be telescoped again as

$$\sum_{j=1}^{N} \alpha_j \sum_{i=0}^{N-j+1} n_{i+j} - \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} \beta_i^k (\log u_{i+k-1} - \log u_{i-1}).$$

Since

$$\exp(\sum_{j=1}^{N} \alpha_j \sum_{i=0}^{N-j+1} n_{i+j}) = \mathbf{u}^{\mathbf{n}}$$

we thus have the identity

$$\begin{aligned} &\det(\mathbf{u}^{\circ n_0} | \dots | \mathbf{u}^{\circ n_{N-1}}) \\ &= V(\log[\mathbf{u}]) \mathbf{u}^{\mathbf{n}} \int_{\widetilde{GT}(\mathbf{n})} \exp\left(- \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} \beta_i^k (\log u_{i+k-1} - \log u_{i-1}) \right). \end{aligned} \tag{5.12}$$

Replacing $\mathbf{n}$ by $\mathbf{n}_{\min}$, we also have

$$V(\mathbf{u}) = V(\log[\mathbf{u}]) \mathbf{u}^{\mathbf{n}_{\min}} \int_{\widetilde{GT}(\mathbf{n}_{\min})} \exp(- \sum_{k=1}^{N-1} \sum_{i=1}^{N-k} \beta_i^k (\log u_{i+k-1} - \log u_{i-1})).$$

Observe that the polytope $\widetilde{GT}(\mathbf{n})$ is cut out by the inequalities $\beta_i^k > 0$, as well as

$$\beta^1_{i+k} + \cdots + \beta^{k+1}_i - \beta^1_{i+k-1} - \cdots - \beta^k_i \leqslant n_{i+k} - n_{i+k-1}$$

for $0 \leqslant k < N$ and $1 \leqslant i < N-k$. In particular, as $\mathbf{n}$ is 1-separated, we have the inclusion

$$\widetilde{GT}(\mathbf{n}_{\min}) \subset \widetilde{GT}(\mathbf{n})$$

and the lower bound in (5.7) follows.

Finally, we prove sharpness of the lower and upper bounds in (5.7), using the principal specialization formula (5.9) with $\epsilon > 1$. Indeed, if $\mathbf{u}_\epsilon \coloneqq \rho \mathbf{u}(\epsilon) = \rho(1, \epsilon, \dots, \epsilon^{N-1})^T$, then

$$\frac{\det \mathbf{u}_\epsilon^{\circ \mathbf{n}}}{V(\mathbf{u}_\epsilon) \mathbf{u}_\epsilon^{\mathbf{n} - \mathbf{n}^{\min}}} = \prod_{j=0}^{N-1} \epsilon^{j(j-n_j)} \prod_{0 \leqslant i < j \leqslant N-1} \frac{\epsilon^{n_j} - \epsilon^{n_i}}{\epsilon^j - \epsilon^i} = \prod_{0 \leqslant i < j \leqslant N-1} \frac{1 - \epsilon^{n_i - n_j}}{1 - \epsilon^{i-j}}. \tag{5.13}$$

Now the sharpness of the upper bound in (5.7) follows by taking $\epsilon \to 1^+$, while that of the lower bound follows by taking $\epsilon \to \infty$. □

Using the tight bounds in Proposition 5.10, we now sharpen Proposition 5.4 to obtain an explicit bound for rank-one matrices, with arbitrary tuples of real powers.

Proposition 5.14. *Let $n_0 < \cdots < n_{N-1} < M$ and scalars $\rho, c_{n_0}, \dots, c_{n_{N-1}} > 0$ be real numbers. Define $\delta_{\mathbf{n},M} \coloneqq \min(n_1 - n_0, \dots, n_{N-1} - n_{N-2}, M - n_{N-1})$. Then the function*

$$x \mapsto \mathcal{K}(c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}}) - x^M$$

is entrywise positivity preserving on $\mathbb{P}_N^1((0,\rho))$, where

$$\mathcal{K} \coloneqq \delta_{\mathbf{n},M}^{-N(N-1)} \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{M-n_j}}{c_{n_j}}. \tag{5.14}$$

Proof. Notice that the proof of Proposition 3.2 applies on the nose to real powers $n_0, \dots, n_{N-1}, M$, replacing $V(\mathbf{u}) s_{\mathbf{n}}(\mathbf{u})$ by $\det(\mathbf{u}^{\circ n_0} | \dots | \mathbf{u}^{\circ n_{N-1}})$ and similarly with $\mathbf{n}_j$ instead of $\mathbf{n}$. Now define

$$\mathbf{m} \coloneqq \frac{1}{\delta}\mathbf{n}, \quad \mathbf{m}_j \coloneqq \frac{1}{\delta}\mathbf{n}_j, \quad \mathbf{v} \coloneqq (u_1^\delta, \dots, u_N^\delta)^T \in (0, \rho^{\delta/2})^N, \quad \text{where } \delta = \delta_{\mathbf{n},M};$$

note that $\mathbf{m}$ and $\mathbf{m}_j$ are all 1-separated. Now we repeat the proof of Proposition 3.4 using Proposition 5.10 and assuming by a continuity argument that the coordinates of $\mathbf{u} \in ((0, \sqrt{\rho})^N)^T$ are strictly increasing (and slightly abusing notation for generalized Vandermonde determinants):

$$\begin{aligned}
\sum_{j=0}^{N-1} \frac{(\det \mathbf{u}^{\circ \mathbf{n}_j})^2}{c_{n_j} (\det \mathbf{u}^{\circ \mathbf{n}})^2} = \sum_{j=0}^{N-1} \frac{(\det \mathbf{v}^{\circ \mathbf{m}_j})^2}{c_{n_j} (\det \mathbf{v}^{\circ \mathbf{m}})^2} &\leqslant \sum_{j=0}^{N-1} \frac{\left(\frac{V(\mathbf{m}_j)}{V(\mathbf{n}_{\min})} \mathbf{v}^{\mathbf{m}_j - \mathbf{n}_{\min}} \right)^2}{c_{n_j} (\mathbf{v}^{\mathbf{m} - \mathbf{n}_{\min}})^2} \\
&\leqslant \delta_{\mathbf{n},M}^{-N(N-1)} \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{M-n_j}}{c_{n_j}},
\end{aligned}$$

and this is precisely $\mathcal{K}$ by (5.14). □

As in the case of integer powers, we now (mildly modify the above threshold to) extend Proposition 5.14 to all matrices in $\mathbb{P}_N((0,\rho))$.

Theorem 5.15. *Let the notation be as in Proposition 5.14. Define $\mathbf{c} \coloneqq (c_{n_0}, \dots, c_{n_{N-1}})$ and*

$$\mathcal{K}_{\mathbf{n},\mathbf{c},M} \coloneqq \max(1, \delta_{\mathbf{n},M}^{-N(N-1)}) \sum_{j=0}^{N-1} \prod_{\alpha=0}^{j-1} \max(1, g(\mathbf{n}, \alpha)) \cdot \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{M-n_j}}{c_{n_j}}, \tag{5.15}$$

where the empty product in the $j = 0$ summand is taken to be 1, and

$$g(\mathbf{n}, \alpha) \coloneqq \frac{(N-1-\alpha)!^2}{\prod_{k=\alpha+1}^{N-1} (n_k - n_\alpha)^2}. \tag{5.16}$$

Suppose further that $n_j \in \mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$ for all j. Then the function

$$x \mapsto \mathcal{K}_{\mathbf{n},\mathbf{c},M}(c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}}) - x^M$$

is entrywise positivity preserving on $\mathbb{P}_N((0,\rho))$.

Notice that the constant $\mathcal{K}_{\mathbf{n},\mathbf{c},M}$ specializes to the one in Theorem 1.6 (i.e., (3.5)) when the n_j and M are integers.

Proof. The proof is by induction on N, with the $N=1$ case a consequence of Proposition 5.14. For the induction step, we apply Theorem 3.5 with $h(x) = \mathcal{K}_{\mathbf{n},\mathbf{c},M} \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M$. Akin to the proof of Theorem 1.6, define

$$\mathbf{n}' := (n_1 - 1, \dots, n_{N-1} - 1), \qquad \mathbf{c}' := (n_1 c_{n_1}, \dots, n_{N-1} c_{n_{N-1}}).$$

Now note that since $\mathcal{K}_{\mathbf{n}',\mathbf{c}',M-1} \sum_{j=1}^{N-1} n_j c_{n_j} x^{n_j - 1} - x^{M-1}$ preserves positivity on $\mathbb{P}_{N-1}((0,\rho))$, hence so does $h'(x)$ in view of [15], if we can show (akin to Theorem 1.6) that

$$\mathcal{K}_{\mathbf{n},\mathbf{c},M} \geqslant M\, \mathcal{K}_{\mathbf{n}',\mathbf{c}',M-1}.$$

To verify this, noting that $0 \leqslant \delta_{\mathbf{n},M} \leqslant \delta_{\mathbf{n}',M-1}$, we compute:

$$\begin{aligned}
&\mathcal{K}_{\mathbf{n},\mathbf{c},M} \\
\geqslant\ & \max(1, \delta_{\mathbf{n}',M-1}^{-N(N-1)}) \sum_{j=0}^{N-1} \prod_{\alpha=0}^{j-1} \max(1, g(\mathbf{n},\alpha)) \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{M-n_j}}{c_{n_j}} \\
\geqslant\ & \max(1, \delta_{\mathbf{n}',M-1}^{-(N-1)(N-2)}) \sum_{j=1}^{N-1} g(\mathbf{n},0) \prod_{\alpha=1}^{j-1} \max(1, g(\mathbf{n},\alpha)) \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{M-n_j}}{c_{n_j}} \\
>\ & \max(1, \delta_{\mathbf{n}',M-1}^{-(N-1)(N-2)}) \sum_{j=1}^{N-1} \prod_{\alpha=1}^{j-1} \max(1, g(\mathbf{n},\alpha)) \frac{V(\mathbf{n}'_j)^2}{V(\mathbf{n}'_{\min})^2} \frac{M\rho^{M-n_j}}{n_j c_{n_j}} \\
=\ & M\, \mathcal{K}_{\mathbf{n}',\mathbf{c}',M-1},
\end{aligned}$$

as desired, where the final inequality follows from the fact that

$$\frac{(M-n_0)^2}{(n_j-n_0)^2} > \frac{M}{n_j}, \qquad \forall 0 \leqslant n_0 < n_j < M. \tag{5.17}$$

Finally, that $h[-]$ preserves positivity on $\mathbb{P}_N^1((0,\rho))$ follows from Proposition 5.14, since $\mathcal{K}_{\mathbf{n},\mathbf{c},M}$ dominates the bound in (5.14). Therefore we are done by Theorem 3.5. ☐

As mentioned above, a pleasing consequence of the preceding result is to obtain explicit threshold bounds on Laplace transforms of real measures. We thus conclude the section by showing

Proof of Theorem 5.9. Akin to Corollary 1.7, by the preceding result it suffices by Fubini's theorem (and discarding the negative components of the measure) to show the finiteness of the expression

$$\int_{n_{N-1}+\varepsilon}^{\infty} \mathcal{K}_{\mathbf{n},\mathbf{c},M}\ d\mu_+(M),$$

where μ_+ is the positive part of μ. Also, by a limiting argument and adjusting ρ and ε as necessary, we may assume that

$$\int_{n_{N-1}+\varepsilon}^{\infty} \rho^M (1+\varepsilon)^M\ d\mu_+(M) < \infty.$$

By (5.15), it thus suffices to show the finiteness of

$$\sup_{M \geqslant n_{N-1}+\varepsilon} \frac{\max(1, \delta_{\mathbf{n},M}^{-N(N-1)})}{(1+\varepsilon)^M} \sum_{j=0}^{N-1} \prod_{\alpha=0}^{j-1} \max(1, g(\mathbf{n}, \alpha)) \cdot \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n}_{\min})^2} \frac{\rho^{-n_j}}{c_{n_j}}.$$

But as M varies, $\delta_{\mathbf{n},M}$ is bounded away from zero, $V(\mathbf{n}_j)$ grows polynomially in M, and $V(\mathbf{n}_{\min})$, $g(\mathbf{n}, \alpha)$, ρ^{-n_j}, and c_{n_j} do not depend on M. The claim follows. □

6. Two-sided domains: complete homogeneous symmetric polynomials

We now address the extent to which the above results continue to hold if we work with a two-sided domain, i.e., $\mathbb{P}_N((-\rho, \rho))$ instead of $\mathbb{P}_N((0, \rho))$. For this we must return to the case of natural number exponents, since exponentiation to fractional powers is problematic when the base is negative.

On the one hand, we have the trivial observation (using the Schur product theorem) that if $f : [0, \rho^2) \to \mathbb{R}$ is entrywise positivity preserving on $\mathbb{P}_N([0, \rho^2))$, then the map $x \mapsto f(x^2)$ is entrywise positivity preserving on $\mathbb{P}_N((-\rho, \rho))$. By combining this with the results of the preceding sections, we can obtain a number of polynomials or power series with some negative coefficients that are entrywise positivity preserving on $\mathbb{P}_N((-\rho, \rho))$ or even on all of $\mathbb{P}_N$.

On the other hand, we have a new necessary condition:

Lemma 6.1. *Let $0 \leqslant n_0 < n_1 < \cdots < n_{N-1} < M$ be integers, and let $0 < \rho \leqslant +\infty$. Suppose that there exists a polynomial*

$$x \mapsto c_{n_0} x^{n_0} + \cdots + c_{n_{N-1}} x^{n_{N-1}} + c_M x^M$$

with $c_{n_0}, \ldots, c_{n_{N-1}}$ positive and c_M negative, which is entrywise positivity preserving on $\mathbb{P}_N^1((-\rho, \rho))$. Then whenever $\mathbf{u} \in (\mathbb{R}^N)^T$ is a root of $s_{\mathbf{n}}$, it is also a root of $s_{\mathbf{n}_j}$ for every $j = 0, \ldots, N-1$, where $\mathbf{n} := (n_0, \ldots, n_{N-1})$ and $\mathbf{n}_j := (n_0, \ldots, n_{j-1}, n_{j+1}, \ldots, n_{N-1}, M)$.

Proof. Suppose for contradiction that there existed $\mathbf{u}$ such that $s_{\mathbf{n}}(\mathbf{u}) = 0$ but $s_{\mathbf{n}_j}(\mathbf{u}) \neq 0$ for some $0 \leqslant j \leqslant N-1$. By multiplying $\mathbf{u}$ by a small constant, we may assume that $\mathbf{u}$ has coefficients in $(-\sqrt{\rho}, \sqrt{\rho})$, so that $\mathbf{u}\mathbf{u}^T \in \mathbb{P}_N^1((-\rho, \rho))$. But from Lemma 2.3 or equation (3.4), we see that $\det h[\mathbf{u}\mathbf{u}^T]$ is negative, giving the required contradiction. □

Thus, for instance, when $N = 2$, no polynomial of the form

$$x \mapsto c_0 + c_2 x^2 + c_3 x^3$$

with c_0, c_2 positive and c_3 negative can be entrywise positivity preserving on $\mathbb{P}_2^1((-\rho, \rho))$ for any $\rho > 0$, since $s_{(0,2)}(\mathbf{u}) = u_1 + u_2$ vanishes when $u_2 = -u_1 \neq 0$, but $s_{(0,3)}(\mathbf{u}) = u_1^2 + u_1 u_2 + u_2^2$ does not. This is closely related to the failure of (1.1) when the bases u_i are allowed to be negative; the point here is that (u_1^3, u_2^3) will not lie in the span of (u_1^0, u_2^0) and (u_1^2, u_2^2) if $u_2 = -u_1 \neq 0$.

This lemma already rules out analogues of Theorem 1.6 on $\mathbb{P}_N((-\rho,\rho))$ for most choices of exponents $n_0,\dots,n_{N-1}$, since typically $s_{\mathbf{n}}$ will have a non-trivial zero set which will not be covered by the zero sets of other $s_{\mathbf{n}_j}$. However, there are a small number of exponents $n_0,\dots,n_{N-1}$ for which a version of this theorem may be salvaged:

Proposition 6.2. *Let $\mathbf{n}=(n_0,\dots,n_{N-1})$ be a tuple of integers $0\leqslant n_0<n_1<\cdots<n_{N-1}$, with the property that the Schur polynomial $s_{\mathbf{n}}$ does not vanish on $\mathbb{R}^N$ except at the origin. Then for any integers $h\geqslant 0$ and $M>n_{N-1}$, and any positive constants ρ, $c_{n_0},\dots,c_{n_{N-1}}$, the polynomial*

$$x\mapsto t(c_{n_0}x^{h+n_0}+\cdots+c_{n_{N-1}}x^{h+n_{N-1}})-x^{h+M} \tag{6.1}$$

is entrywise positivity preserving on $\mathbb{P}_N((-\rho,\rho))$ for t sufficiently large.

Proof. We may assume that $N\geqslant 2$, as the case $N=1$ is trivial; and by the Schur product theorem we may assume without loss of generality that $h=0$. We first verify entrywise positivity preservation on the rank one matrices $\mathbb{P}^1_N((-\rho,\rho))$; such matrices take the form $\mathbf{u}\mathbf{u}^T$ where $\mathbf{u}$ has coefficients in $(-\sqrt{\rho},\sqrt{\rho})$. From Lemma 2.3 and a continuity argument, it suffices to show that for t sufficiently large, one has

$$t^N|s_{\mathbf{n}}(\mathbf{u})|^2-\sum_{j=0}^{N-1}\frac{t^{N-1}}{c_{n_j}}|s_{\mathbf{n}_j}(\mathbf{u})|^2\geqslant 0$$

for all such $\mathbf{u}$. This will follow if we can establish a bound of the form

$$|s_{\mathbf{n}_j}(\mathbf{u})|\leqslant C|s_{\mathbf{n}}(\mathbf{u})|$$

uniformly for all $\mathbf{u}\in([-\sqrt{\rho},\sqrt{\rho}]^N)^T$, and for some finite C. The left-hand side has a higher order of homogeneity than the right-hand side, so it suffices to verify this on the boundary $\partial[-\sqrt{\rho},\sqrt{\rho}]^N$. This is a compact set on which $s_{\mathbf{n}}(\mathbf{u})$ is non-zero by hypothesis, so the claim now follows from continuity.

To remove the restriction to rank 1 matrices, we would like to use Theorem 3.5. We first observe that n_0 must vanish, since otherwise $s_{\mathbf{n}}(\mathbf{u})$ will contain a factor of $\mathbf{u}^{(1,\dots,1)}$ and thus vanishes outside of the origin. From (2.2) (or from the Young tableaux definition of $s_{\mathbf{n}}$) we then observe the identity

$$s_{\mathbf{n}}(u_1,\dots,u_{N-1},0)=(-1)^{N-1}s_{(n_1-1,\dots,n_{N-1}-1)}(u_1,\dots,u_{N-1}),$$

and hence $s_{(n_1-1,\dots,n_{N-1}-1)}$ is also non-vanishing on $\mathbb{R}^{N-1}$ except at the origin. By the induction hypothesis, we now conclude that the derivative of (6.1) is entrywise positivity preserving on $\mathbb{P}_{N-1}((-\rho,\rho))$ for t sufficiently large, and the claim now follows from Theorem 3.5. □

In fact it is possible to identify the integer tuples $\mathbf{n}$ satisfying the hypothesis in Proposition 6.2, and these yield a well-known family of symmetric functions:

Proposition 6.3. *Given integers $N\geqslant 1$ and $0\leqslant n_0<\cdots<n_{N-1}$, the following are equivalent:*

(1) *The Schur polynomial $s_{\mathbf{n}}$ does not vanish on $\mathbb{R}^N$ except at the origin.*

(2) *We have:* $n_0 = 0, \dots, n_{N-2} = N-2$, *and* $n_{N-1} - (N-1)$ *is even, say* $2r$ *for* $r \in \mathbb{Z}^{\geqslant 0}$. *In other words,* $s_{\mathbf{n}}(\mathbf{u})$ *is precisely the complete homogeneous symmetric polynomial (of even degree)*

$$h_{2r}(\mathbf{u}) \coloneqq \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_{2r} \leqslant N} u_{i_1} \cdots u_{i_{2r}}.$$

Proof. If (1) holds, then the argument in the above proof of Proposition 6.2 shows $n_0 = 0$ and $(n_1 - 1, \dots, n_{N-1} - 1)$ satisfies the same property for real $(N-1)$-tuples. This reduces the problem to $N = 2$, in which case the assertion is easily verified. (Alternatively, one can evaluate $s_{\mathbf{n}}$ at the basis vector $(1, 0, \dots, 0)$ and observe using (2.1) that this vanishes unless $n_j = j$ for $j = 0, \dots, N-2$.) Conversely, that $s_{\mathbf{n}}(\mathbf{u}) = h_{2r}(\mathbf{u})$ follows from definition; now the proof is completed using the inequality

$$h_{2r}(\mathbf{u}) \geqslant \frac{\|\mathbf{u}\|^{2r}}{2^r r!}$$

proven by Hunter [23] for all integers $r \geqslant 0$. □

Remark 6.4. We were made aware of the following alternate proof of (2) $\implies$ (1) using the method of moments; see an anonymous comment on `https://terrytao.wordpress.com/2017/08/06`, or [3, Lemma 3.1]. Namely, given i.i.d. exponential(1) random variables $Z_1, \dots, Z_N$, we have

$$k!\, h_k(u_1, \dots, u_N) = \mathbb{E}\left[(u_1 Z_1 + \cdots + u_N Z_N)^k\right] \qquad \forall u_1, \dots, u_N \in \mathbb{R} \quad (6.2)$$

for any $k \geqslant 0$; whence $h_{2r}(u_1, \dots, u_N) \geqslant 0$ for any integer $r \geqslant 0$ and any $u_1, \dots, u_N$, and equality holds if and only if $u_1 = \cdots = u_N = 0$.

An *a priori* different proof is to obtain a sum-of-squares decomposition of h_{2r}. For low values of r, we have:

$$\begin{aligned} h_0(\mathbf{u}) &= 1, \\ h_2(\mathbf{u}) &= \frac{1}{2}(h_1(\mathbf{u})^2 + p_2(\mathbf{u})), \\ h_4(\mathbf{u}) &= \frac{1}{72}\left(3h_1(\mathbf{u})^4 + 2\sum_i u_i^2(2u_i + 3h_1(\mathbf{u}))^2 + 9p_2(\mathbf{u})^2 + 10p_4(\mathbf{u})\right), \end{aligned}$$

where $p_r(\mathbf{u}) \coloneqq \sum_i u_i^r$ are the power-sum symmetric polynomials in the tuple $\mathbf{u}^T = (u_1, \dots, u_N)$. As pointed out to us by David Speyer, one way to similarly obtain a sum-of-squares decomposition for every even $r \geqslant 0$ is to use (6.2), replacing the exponential random variable Z by a discrete one Y, which matches moments with Z up to order $2r$. Notice that the existence of such a discrete variable Y follows from Caratheodory's theorem.

Remark 6.5. The proofs of Propositions 6.2 and 6.3 lead to an explicit bound on the threshold t in (6.1):

$$t \geqslant \mathcal{K} \coloneqq 2^r r! \sum_{j=0}^{N-1} V(\mathbf{n}_j)^2 \frac{(N\rho)^{M-n_j}}{c_{n_j}},$$

where the $\mathbf{n}_j$ are as in the proof of Proposition 6.2, and r is as in Proposition 6.3.

7. Complex domains

We next briefly study matrices in $\mathbb{P}_N$ with complex entries. In [4] it was shown that for every $M \geqslant N$, positive coefficients $c_0, \dots, c_{N-1}$, and $0 < \rho < \infty$, the polynomial

$$z \mapsto t(c_0 + c_1 z + \cdots + c_{N-1} z^{N-1}) - z^M$$

is entrywise positivity preserving on $\mathbb{P}_N(D(0,\rho))$ for t sufficiently large, where $D(0,\rho)$ denotes the complex disk $\{z \in \mathbb{C} : |z| < \rho\}$. From the Schur product theorem, the same assertion holds for

$$z \mapsto t(c_0 z^h + c_1 z^{h+1} + \cdots + c_{N-1} z^{h+N-1}) - z^{h+M}$$

for any integer $h \geqslant 0$. However, such a result cannot hold for any other set of exponents, at least if one allows M to vary:

Proposition 7.1. *Let $N \geqslant 2$ and $0 \leqslant n_0 < \cdots < n_{N-1}$ be integers with $(n_0, \dots, n_{N-1}) \neq (h, h+1, \dots, h+N-1)$ for any integer $h \geqslant 0$. Then there exists $M > n_{N-1}$, such that any polynomial of the form*

$$z \mapsto c_{n_0} z^{n_0} + \cdots + c_{n_{N-1}} z^{n_{N-1}} + c_M z^M, \tag{7.1}$$

with $c_{n_0}, \dots, c_{n_{N-1}}$ positive and c_M negative, cannot be entrywise positivity preserving on $\mathbb{P}_N^1(D(0,\rho))$ for any $\rho > 0$.

Proof. By Lemma 2.3, it suffices to show that there exists $M > n_{N-1}$ and vectors $\mathbf{u} \in (\mathbb{C}^N)^T$ of arbitrarily small norm such that

$$\frac{1}{c_M} |s_{\mathbf{n}}(\mathbf{u})|^2 + \sum_{j=0}^{N-1} \frac{1}{c_{n_j}} |s_{\mathbf{n}_j}(\mathbf{u})|^2 < 0.$$

We see from the definition (2.1) that the specialization

$$s_{\mathbf{n}}(1, 2, \dots, n-1, z) \in \mathbb{C}[z]$$

is a polynomial that is positive on the positive real axis; as the Young tableaux appearing in (2.1) can have as few as n_0 and as many as $n_{N-1} - N + 1$ entries equal to N, and $\mathbf{n}$ is not of the form $(h, h+1, \dots, h+N-1)$, this polynomial is not a monomial. From the fundamental theorem of algebra, we conclude that there exists $z_0 \in \mathbb{C} \setminus [0, \infty)$ such that

$$s_{\mathbf{n}}(1, 2, \dots, n-1, z_0) = 0.$$

Rescaling, we see that there exist arbitrarily small $\mathbf{u} \in (\mathbb{C}^N)^T$, with all coefficients non-zero and distinct, such that $s_{\mathbf{n}}(\mathbf{u}) = 0$; thus the vectors $\mathbf{u}^{\circ n_0}, \dots, \mathbf{u}^{\circ n_{N-1}}$ are linearly dependent in $(\mathbb{C}^N)^T$. On the other hand, from Vandermonde determinants we see that $\mathbf{u}^{\circ(n_{N-1}+1)}, \dots, \mathbf{u}^{\circ(n_{N-1}+N)}$ are linearly independent in $(\mathbb{C}^N)^T$. Thus there must exist some M between $n_{N-1} + 1$ and $n_{N-1} + N$ for which $\mathbf{u}^{\circ M}$ lies outside the span of $\mathbf{u}^{\circ n_0}, \dots, \mathbf{u}^{\circ n_{N-1}}$, which implies from (2.2) that $s_{\mathbf{n}_j}(\mathbf{u})$ is non-zero for some j. The claim follows. $\square$

Remark 7.2. In fact the above proof shows the infinitude of such 'rigid' powers M; more precisely, for any $\mathbf{n}$ that is not a shift by $h \in \mathbb{Z}^{\geqslant 0}$ of $\mathbf{n}_{\min}$, among any N consecutive integers in $[n_{N-1}, \infty)$ there is some M such that every entrywise positivity preserver on $\mathbb{P}_N(D(0,\rho))$ of the form (7.1) must be absolutely monotonic.

8. Quantitative bounds, via Schur positivity

Having answered Question 1.3 for integer and real powers, we now present stronger versions of the above results, as well as several applications. We begin by proving the quantitative results in Section 1.3. The improvement over estimates in previous sections comes from understanding the behavior of the functions $s_{\mathbf{n}_j}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ for $\mathbf{u} \in ((0,\sqrt{\rho})^N)^T$ and *integer* powers $n_0, \dots, n_{N-1}, M$ (and $j = 0, \dots, N-1$). We now show the following result in a slightly more general setting.

Proposition 8.1. *Fix tuples of non-negative integers* $0 \leqslant n_0 < \cdots < n_{N-1}$ *and* $0 \leqslant m_0 < \cdots < m_{N-1}$*, such that* $n_j \leqslant m_j$ $\forall j$*. Then the function*

$$f : ((0,\infty)^N)^T \to \mathbb{R}, \qquad f(\mathbf{u}) := \frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})}$$

is non-decreasing in each coordinate.

In fact this result is the analytical shadow of a deeper algebraic Schur-positivity phenomenon; see Theorem 8.6. Proposition 8.1 was also independently observed by Rachid Ait-Haddou; see the remarks following Corollary 8.7 for more details.

It turns out that Proposition 8.1 for integer powers suffices to derive exact thresholds for negative coefficients in the case of integer exponents. In fact, we also show below that it easily implies exact thresholds for all non-negative real exponents as well.

8.1. Exact thresholds over bounded domains.

Before proceeding to the proof of Proposition 8.1, in this subsection we use it to prove another main result above – Theorem 1.11, which obtains a sharp value for the threshold for rank-one matrices, and all non-negative real powers.

Proof of Theorem 1.11. First notice that by taking the entrywise product with $(\mathbf{u}\mathbf{u}^T)^{\circ \pm n_0}$ for $\mathbf{u} \in ((0,\sqrt{\rho})^N)^T$, it suffices to show the theorem when the powers $n_j, M \geqslant 0$, by the Schur product theorem.

Now the proof for non-negative powers is in steps, starting with proving the result for integer powers $n_0, \dots, n_{N-1}, M \geqslant 0$. In this case, define $p_t(x) := t \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M$, and

$$(0,\sqrt{\rho})^N_{\neq} := \{(u_1, \dots, u_N) \in (0,\sqrt{\rho})^N : V(\mathbf{u}) \neq 0\}, \tag{8.1}$$

where the non-vanishing of $V(\mathbf{u})$ simply means that the tuple $(u_1, \dots, u_N)$ has pairwise distinct coordinates.

That (1) $\implies$ (2) is shown under the weaker assumption that $\det p_t[\mathbf{u}\mathbf{u}^T] \geqslant 0$ for $\mathbf{u} \in ((0,\sqrt{\rho})^N_{\neq})^T$. Under this assumption, we see from (3.3) that

$$t \geqslant \sup_{\mathbf{u} \in ((0,\sqrt{\rho})^N_{\neq})^T} \sum_{j=0}^{N-1} \frac{s_{\mathbf{n}_j}(\mathbf{u})^2}{c_{n_j} s_{\mathbf{n}}(\mathbf{u})^2}.$$

But now by Proposition 8.1, this supremum is attained as all $u_j \to \sqrt{\rho}^-$, and equals precisely the expression in (1.8). In fact, note that since the highest power in $\mathbf{n}_j$ is strictly larger than the highest power in $\mathbf{n}$ (namely, $M > n_{N-1}$), it follows by Theorem 8.6 below that the supremum is uniquely attained as $u_j \to \sqrt{\rho}^-$. This is because the leading u_1-term of

$$s_{\mathbf{n}} \cdot \partial_{u_1}(s_{\mathbf{n}_j}) - s_{\mathbf{n}_j} \cdot \partial_{u_1}(s_{\mathbf{n}})$$

is nonzero, so that the ratio $\frac{s_{\mathbf{n}_j}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})}$ is strictly increasing along each coordinate.

That (2) $\implies$ (1) is immediate if $c_{n_j}, c' \geqslant 0$, while if $c' < 0 < c_{n_j}$ and $M > n_j$ then we repeat the proof of Proposition 3.2. This concludes the proof for non-negative integer powers.

We next show the result when M, n_j are all rational and non-negative. Choose a large integer $L \in \mathbb{N}$ such that $LM, Ln_j \in \mathbb{Z}$. Repeating the proof of Theorem 1.11 for $p_t[\mathbf{y}\mathbf{y}^T]$, where $p_t(y) := t \sum_{j=0}^{N-1} c_{n_j} y^{Ln_j} - y^{LM}$ and $\mathbf{y} := \mathbf{u}^{\circ 1/L} \in ((0, \sqrt[2L]{\rho})^N)^T$, we obtain the threshold

$$\mathcal{C} = \sum_{j=0}^{N-1} \frac{V(L\mathbf{n}_j)^2}{V(L\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}},$$

as desired.

We now claim the equivalence of the two statements in Theorem 1.11 holds for all non-negative real powers n_j, M. In other words, if we define $p_t(x) := t \sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M$, then

$$p_t[\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_N, \qquad \forall \mathbf{u} \in ((0,\sqrt{\rho})^N)^T, \tag{8.2}$$

if and only if $M > n_j \ \forall j$ and $t \geqslant \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}$.

To prove one implication, suppose

$$M > n_j \ \forall j \qquad \text{and} \qquad t > \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}.$$

By continuity, choose strictly decreasing rational sequences $n_{j,k} \to n_j$ and $M_k \to M$, such that $0 \leqslant n_{0,k} < \cdots < n_{N-1,k} < M_k$ for all k, and

$$\sum_{j=0}^{N-1} \frac{V(\mathbf{n}_{j,k})^2}{V(\mathbf{n}'_k)^2} \frac{\rho^{M_k - n_{j,k}}}{c_{n_j}} < t,$$

where

$$\mathbf{n}'_k := (n_{0,k}, \dots, n_{N-1,k}), \quad \mathbf{n}_{j,k} := (n_{0,k}, \dots, n_{j-1,k}, n_{j+1,k}, \cdots, n_{N-1,k}, M_k).$$

Now define $p_{t,k}(x) := t\sum_{j=0}^{N-1} c_{n_j} x^{n_{j,k}} - x^{M_k}$. By the result for rational powers, it follows that $p_{t,k}[\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_N$ for all $\mathbf{u} \in ((0,\sqrt{\rho})^N)^T$ and all k. Taking the limit as $k \to \infty$ proves one implication for the real powers n_j, M (by continuity in t).

Finally, we prove the converse implication. Suppose (8.2) holds; given $\rho > 0$ and $\epsilon \in (0,1)$, define the vector

$$\mathbf{u}_\epsilon := \sqrt{\rho\epsilon} \cdot \mathbf{u}(\epsilon) := \sqrt{\rho\epsilon}(1, \epsilon, \dots, \epsilon^{N-1})^T. \tag{8.3}$$

Consider the threshold function (for single rank-one matrices, as in Propositions 3.2 and 5.14):

$$t = t(\epsilon, \mathbf{n}, M) := \sum_{j=0}^{N-1} \frac{(\det \mathbf{u}_\epsilon^{\circ \mathbf{n}_j})^2}{c_{n_j} (\det \mathbf{u}_\epsilon^{\circ \mathbf{n}})^2}. \tag{8.4}$$

Suppose $p_t(x) := t\sum_{j=0}^{N-1} c_{n_j} x^{n_j} - x^M$ for real powers $0 \leqslant n_0 < \cdots < n_{N-1} < M$, and $p_t[\mathbf{u}\mathbf{u}^T] \in \mathbb{P}_N$ for $\mathbf{u} \in ((0,\sqrt{\rho})^N)^T$. Following the proof of Proposition 5.14 for the matrices $\mathbf{u}_\epsilon \mathbf{u}_\epsilon^T$, we obtain:

$$t \geqslant \lim_{\epsilon \to 1^-} t(\epsilon, \mathbf{n}, M),$$

and hence it suffices to prove that

$$\lim_{\epsilon \to 1^-} t(\epsilon, \mathbf{n}, M) = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}. \tag{8.5}$$

For this, we compute using Proposition 5.11 that

$$\frac{(\det \mathbf{u}_\epsilon^{\circ \mathbf{n}_j})^2}{c_{n_j} (\det \mathbf{u}_\epsilon^{\circ \mathbf{n}})^2} = \frac{(\rho\epsilon)^{M-n_j}}{c_{n_j}} \prod_{1 \leqslant a < b \leqslant N} \frac{(\epsilon^{\mathbf{n}_j(b)} - \epsilon^{\mathbf{n}_j(a)})^2}{(\epsilon^{\mathbf{n}(b)} - \epsilon^{\mathbf{n}(a)})^2},$$

where $\mathbf{n}(a)$ denotes the ath coordinate of $\mathbf{n}$. Now summing over j and taking $\epsilon \to 1^-$ yields (8.5), as desired. □

Akin to previous sections, Theorem 1.11 for rank-one matrices extends to Theorem 1.12 for all matrices in $\mathbb{P}_N((0,\rho))$, with the same threshold:

Proof of Theorem 1.12. Clearly (3) $\implies$ (1). Conversely, assuming (2), to show (3) we use the extension principle from Theorem 3.5. Repeating the proof of Theorem 1.6, it suffices to show that

$$\mathcal{C} \geqslant M\,\tilde{\mathcal{C}},$$

where $\mathcal{C}$ is as in (1.8), and $\tilde{\mathcal{C}}$ is defined like $\mathcal{C}$ but with N replaced by $N-1$, $n_0, \dots, n_{N-1}$ replaced by $n_1 - 1, \dots, n_{N-1} - 1$, M replaced by $M-1$, and $c_{n_0}, \dots, c_{n_{N-1}}$ replaced by $n_1 c_{n_1}, \dots, n_{N-1} c_{n_{N-1}}$ respectively. Setting $\mathbf{n}' := (n_1 - 1, \dots, n_{N-1} - 1)$ and

$$\mathbf{n}'_j := (n_1 - 1, \dots, n_{j-1} - 1, n_{j+1} - 1, \dots, n_{N-1} - 1, M - 1), \qquad \forall 0 < j < N,$$

a straightforward computation as in (5.17) shows that

$$\frac{V(\mathbf{n}_j)}{V(\mathbf{n})} \geqslant \frac{V(\mathbf{n}'_j)}{V(\mathbf{n}')} \cdot \frac{M - n_0}{n_j - n_0} > \frac{V(\mathbf{n}'_j)}{V(\mathbf{n}')} \cdot \frac{\sqrt{M}}{\sqrt{n_j}}, \qquad \forall 0 < j < N. \tag{8.6}$$

Using this, it follows that

$$\mathcal{C} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}} > \sum_{j=1}^{N-1} \frac{V(\mathbf{n}'_j)^2}{V(\mathbf{n}')^2} \frac{M}{n_j} \cdot \frac{\rho^{M-n_j}}{c_{n_j}} = M\,\tilde{\mathcal{C}},$$

and the proof is complete. □

Remark 8.2. Notice that the sharp quantitative bound in Theorem 1.12 also implies Theorems 1.6 and 5.5 in previous sections (which showed that the Horn–Loewner-type necessary conditions were sharp for integer or real powers, over bounded domains $I = (0, \rho)$). However, the tight lower and upper bounds on Schur polynomials and generalized Vandermonde determinants in Propositions 3.1 and 5.10 are not implied.

Remark 8.3. That the constant $\mathcal{K}_{\mathbf{n},\mathbf{c},M}$ in (5.15) dominates over the sharp bound (1.8) can be directly verified, since

$$\max(1, \delta_{\mathbf{n},M}^{-N(N-1)}) \prod_{\alpha=0}^{j-1} \max(1, g(\mathbf{n}, \alpha)) \geqslant \prod_{\alpha=0}^{j-1} g(\mathbf{n}, \alpha), \quad \forall j.$$

Remark 8.4. Notice that Theorems 1.11 and 1.12 extend the main result in previous work [4] from $(n_0, \dots, n_{N-1}) = (0, \dots, N-1)$ and integers $M > n_{N-1}$ to all real powers $n_0, \dots, n_{N-1}, M$. Once again, as discussed above, the methods in [4] necessarily fail to work for any tuple $\mathbf{n}$ other than an integer translate of $\mathbf{n}_{\min}$ (or even for $\mathbf{n} = \mathbf{n}_{\min}$ and $M \notin \mathbb{N}$), since the analysis in [4] also works for matrices with complex entries.

Remark 8.5. Suppose all the c_{n_j} are strictly positive, and let $\mathbf{u}_n$ be a sequence of vectors in $((0, \sqrt{\rho})^N)^T$ with distinct coefficients that converges to $(\sqrt{\rho}, \dots, \sqrt{\rho})^T$. Then by Proposition 3.2, the condition in Theorem 1.12(2) is equivalent to $\det f[\mathbf{u}_n \mathbf{u}_n^T] \geqslant 0$ for all n. In a similar spirit, if ϵ_n is a sequence of positive numbers going to zero, and $\det f[\mathbf{u}(\epsilon_n)\mathbf{u}(\epsilon_n)^T] \geqslant 0$ for all n, where $\mathbf{u}(\epsilon)$ is defined in (5.8), then by the discussion in [8, Remark 3.13], one can adapt the proof of Lemma 5.1(i) to conclude that the c_{n_j} are all non-negative, and either c is non-negative or the c_{n_j} are all strictly positive. Combining these observations, we conclude that to obtain Theorem 1.12(2), one does not need to test positive semidefiniteness of $f[A]$ for all $A \in \mathbb{P}_N((0,\rho))$; it suffices to do so for the two sequences $A = \mathbf{u}_n \mathbf{u}_n^T$ and $\mathbf{u}(\epsilon_n)\mathbf{u}(\epsilon_n)^T$. We do not know if one can simplify this test to involve only a finite number of matrices, rather than two infinite sequences of matrices.

8.2. **Proof of Proposition 8.1 and extension to real powers.** It remains to prove Proposition 8.1. Our proof once again combines analysis with symmetric function theory, via type A representation theory. Namely, to prove the proposition, it suffices via the quotient rule to show by symmetry that for any fixed $j \in [1, N]$, the polynomial

$$s_{\mathbf{n}} \cdot \partial_{u_j}(s_{\mathbf{m}}) - s_{\mathbf{m}} \cdot \partial_{u_j}(s_{\mathbf{n}}) \tag{8.7}$$

is non-negatively valued on $(0,\infty)^N$. This is guaranteed if the expression (8.7) is a linear combination of monomials with positive (integer) coefficients, i.e. monomial-positive. We will show a stronger statement. Notice that expanding $s_{\mathbf{n}}, s_{\mathbf{m}}$ as polynomials in u_1 (by symmetry), the coefficients of each power u_1^k are skew-Schur functions $s_{\mathbf{n}/(k)}(u_2,\dots,u_N)$ for $0 \leqslant k \leqslant n_{N-1} - N + 1$. (See [29, Chapter I.5] for more details on these symmetric polynomials.) Now we claim:

Theorem 8.6. *When written as a polynomial in u_1, every coefficient in the expression* (8.7) *is* Schur-positive*, i.e., a non-negative integer linear combination of Schur polynomials in $u_2,\dots,u_N$.*

Note that Schur-positivity immediately implies monomial-positivity, and Proposition 8.1 is a trivial consequence of the latter.

Proof. In order to make the notation compatible with that in [28], we now index Schur polynomials $s_{\mathbf{n}}$ by the partition

$$\overline{\mathbf{n} - \mathbf{n}_{\min}} = (n_{N-1} - (N-1), \dots, n_1 - 1, n_0);$$

more precisely we write

$$s_{\mathbf{n}} = \tilde{s}_{\overline{\mathbf{n}-\mathbf{n}_{\min}}}.$$

We also say that a shape ν *contains* a shape λ if $\nu_j \geqslant \lambda_j$ for $0 \leqslant j \leqslant N-1$. Now setting $\lambda := \overline{\mathbf{n} - \mathbf{n}_{\min}}$ and $\nu := \overline{\mathbf{m} - \mathbf{n}_{\min}}$, we see that ν contains λ, and our task is to show that the coefficient of each power of u_j in

$$\tilde{s}_\lambda \cdot \partial_{u_j}(\tilde{s}_\nu) - \tilde{s}_\nu \cdot \partial_{u_j}(\tilde{s}_\lambda) \tag{8.8}$$

is Schur-positive.

As Schur polynomials are symmetric, we may assume without loss of generality that $j = 1$. We have the well-known expansion of Schur polynomials [29, Chapter I, Equation (5.12)] into skew-Schur polynomials

$$\tilde{s}_\lambda(\mathbf{u}) = \sum_{j \geqslant 0} \tilde{s}_{\lambda/(j,0,\dots,0)}(\mathbf{u}') u_1^j$$

where $\mathbf{u}' := (u_2,\dots,u_N)$, where $\tilde{s}_{\lambda/\mu}$ is the skew-Schur polynomial corresponding to the shape λ/μ if λ contains μ, and we adopt the convention that $\tilde{s}_{\lambda/\mu} = 0$ if λ does not contain μ. Note that only finitely many of the terms will be non-zero. Similarly we have

$$\tilde{s}_\nu(\mathbf{u}) = \sum_{k \geqslant 0} \tilde{s}_{\nu/(k,0,\dots,0)}(\mathbf{u}') u_1^k$$

and hence we may write (8.8) as

$$\sum_{j,k \geqslant 0} \tilde{s}_{\lambda/(j,0,\dots,0)}(\mathbf{u}') \tilde{s}_{\nu/(k,0,\dots,0)}(\mathbf{u}')(k-j) u_1^{j+k-1}.$$

Symmetrizing, this can be rewritten as

$$\sum_{k > j \geqslant 0} (\tilde{s}_{\lambda/(j,0,\dots,0)}(\mathbf{u}') \tilde{s}_{\nu/(k,0,\dots,0)}(\mathbf{u}') - \tilde{s}_{\lambda/(k,0,\dots,0)}(\mathbf{u}') \tilde{s}_{\nu/(j,0,\dots,0)}(\mathbf{u}')) \times \\ \times (k-j) u_1^{j+k-1}.$$

Thus it will suffice to show that the polynomial

$$\tilde{s}_{\lambda/(j,0,\dots,0)}\tilde{s}_{\nu/(k,0,\dots,0)} - \tilde{s}_{\lambda/(k,0,\dots,0)}\tilde{s}_{\nu/(j,0,\dots,0)}$$

is Schur-positive whenever $k > j$ and ν contains λ. But this is a special case of a result [28, Theorem 4] of Lam, Postnikov, and Pylyavskyy (we thank Pavlo Pylyavskyy for this reference). These authors establish in *loc. cit.* the more general claim that

$$\tilde{s}_{\lambda\wedge\nu/\mu\wedge\rho}\tilde{s}_{\lambda\vee\nu/\mu\vee\rho} - \tilde{s}_{\lambda/\mu}\tilde{s}_{\nu/\rho} \tag{8.9}$$

is Schur-positive for any partitions λ, ν, μ, ρ, where we write

$$\begin{aligned} \lambda \wedge \nu &:= (\min(\lambda_{N-1}, \nu_{N-1}), \dots, \min(\lambda_0, \nu_0)), \\ \lambda \vee \nu &:= (\max(\lambda_{N-1}, \nu_{N-1}), \dots, \max(\lambda_0, \nu_0)) \end{aligned}$$

if $\lambda = (\lambda_{N-1}, \dots, \lambda_0)$, and similarly for ν, μ, ρ. Note that the Littlewood–Richardson rule [29, Chapter I, Equations (5.2), (5.3)] implies that all skew-Schur polynomials and their products are Schur-positive, so the Schur-positivity of (8.9) is only non-trivial when $\tilde{s}_{\lambda/\mu}$ and $\tilde{s}_{\nu/\rho}$ are non-zero, thus λ contains μ and ν contains ρ. This implies that $\lambda \wedge \nu$ contains $\mu \wedge \rho$ and $\lambda \vee \nu$ contains $\mu \vee \rho$. This is the case that is treated in [28]. □

In addition to the proof of the above Theorems 1.11, 1.12, we now list some other applications of Proposition 8.1. The first is that it can be extended to hold for all real powers:

Corollary 8.7. *Fix tuples of real powers* $\mathbf{n} = (n_0 < \cdots < n_{N-1})$ *and* $\mathbf{m} = (m_0 < \cdots < m_{N-1})$, *such that* $n_j \leqslant m_j \ \forall j$. *Defining* $\mathbf{u}^{\circ\mathbf{n}} := (u_j^{n_{k-1}})_{j,k=1}^N$ *as above, the function*

$$f : ((0,\infty)^N_{\neq})^T \to \mathbb{R}, \qquad f(\mathbf{u}) := \frac{\det \mathbf{u}^{\circ\mathbf{m}}}{\det \mathbf{u}^{\circ\mathbf{n}}}$$

is non-decreasing in each coordinate, where $S^N_{\neq}$ *for a set* S *denotes ordered* N*-tuples of pairwise distinct elements.*

Note that we do not need to assume the u_j to be strictly increasing in $\mathbf{u} \in ((0,\infty)^N_{\neq})^T$, since the ratio of determinants used to define $f(\mathbf{u})$ is invariant with respect to permutations of the components of $\mathbf{u}$.

We also remark that this corollary allows us to bypass Theorem 1.11 (for integer powers) and prove Theorem 1.12 for tuples of real powers via a slightly different approach.

Rachid Ait-Haddou has mentioned to us (see the comments at `https://terrytao.wordpress.com/2017/08/17`) that Corollary 8.7 can be proved directly using the theory of Chebyshev blossoming in Müntz spaces [1, 2]. We give a further proof of Corollary 8.7, relying primarily on determinant identities such as Dodgson condensation, in Section 12.

Proof of Corollary 8.7. As above, we first observe that it suffices to assume $n_j, m_j \geqslant 0$, since multiplying and dividing $f(\mathbf{u})$ by $(u_1 \cdots u_N)^{-n_0}$ shows that

$$f(\mathbf{u}) = \frac{\det \mathbf{u}^{\circ\mathbf{m}'}}{\det \mathbf{u}^{\circ\mathbf{n}'}}, \quad \text{where } \mathbf{m}' := (m_0 - n_0, \dots, m_{N-1} - n_0),$$

$$\mathbf{n}' := (n_0 - n_0, \dots, n_{N-1} - n_0).$$

Thus we may assume n_j, m_j are non-negative. We now show the result for non-negative rational powers $\mathbf{m}, \mathbf{n}$. Choose $L \in \mathbb{N}$ such that $Ln_j, Lm_j \in \mathbb{Z}$ for all j, and set $\mathbf{y} := \mathbf{u}^{\circ 1/L}$ as above. Notice that $\mathbf{u} \in ((0,\infty)^N_{\neq})^T$ if and only if $\mathbf{y} \in ((0,\infty)^N_{\neq})^T$. Now we compute:

$$f(\mathbf{u}) = \frac{s_{L\mathbf{m}}(\mathbf{y})}{s_{L\mathbf{n}}(\mathbf{y})},$$

and by Proposition 8.1, this is non-decreasing in each coordinate y_j, whence in u_j.

Next, given non-negative real powers n_j, m_j, as above we choose rational sequences $n_{j,k} \to n_j$ and $m_{j,k} \to m_k$, such that

$$n_{0,k} < \cdots < n_{N-1,k}, \quad m_{0,k} < \cdots < m_{N-1,k}, \quad n_{j,k} \leqslant m_{j,k} \ \forall j, \quad \forall k \in \mathbb{N}.$$

Define $\mathbf{n}_k := (n_{0,k}, \dots, n_{N-1,k})$ and similarly $\mathbf{m}_k$ for $k \in \mathbb{N}$; and now define

$$f_k(\mathbf{u}) := \frac{\det \mathbf{u}^{\circ \mathbf{m}_k}}{\det \mathbf{u}^{\circ \mathbf{n}_k}}.$$

Then the functions $f_k(\mathbf{u})$ are all non-decreasing in every coordinate, whence so is their pointwise limit on $((0,\infty)^N_{\neq})^T$, namely, f. □

Remark 8.8. Another consequence of Proposition 8.1 and the principal specialization in Proposition 5.11 is that for any (integer or) real tuple $\mathbf{m} \geqslant \mathbf{n}$ coordinatewise, we have:

$$\sup_{\mathbf{u} \in ((0,1)^N_{\neq})^T} \frac{\det \mathbf{u}^{\circ \mathbf{m}}}{\det \mathbf{u}^{\circ \mathbf{n}}} = \frac{V(\mathbf{m})}{V(\mathbf{n})} = \lim_{\epsilon \to 1^-} \frac{\det \mathbf{u}(\epsilon)^{\circ \mathbf{m}}}{\det \mathbf{u}(\epsilon)^{\circ \mathbf{n}}}. \tag{8.10}$$

Notice by homogeneity that this identity has an obvious variant for $\mathbf{u} \in ((0,\sqrt{\rho})^N_{\neq})^T$ for any $0 < \rho < +\infty$. In turn, this variant leads to the following 'depolarization-type' phenomenon: for all $\mathbf{u} \in ((0,\infty)^N_{\neq})^T$, the ratio $s_{\mathbf{m}}(\mathbf{u})/s_{\mathbf{n}}(\mathbf{u})$ equals the value at some diagonal point $(u,\dots,u)^T$, where $0 < \min_j u_j \leqslant u \leqslant \max_j u_j$.

9. Application 1: Entrywise preservers of total non-negativity

The next few sections are devoted to applications of the above results. In this section we study entrywise preservers of total non-negativity, with the ultimate aim of extending the above classification of sign patterns – and computation of exact threshold bounds – to totally non-negative matrices.

Recall that a real matrix is said to be *totally non-negative* (resp. *strictly totally positive*) if every minor is a non-negative (resp. positive) real number. A well-known example of total non-negativity – in fact, strict total positivity – is that of generalized Vandermonde determinants (1.1). These classes of matrices feature in analysis and differential equations, probability and stochastic processes, representation theory, discrete mathematics, and particle systems; and are closely connected to positive semidefinite matrices.

Remark 9.1. As the notion of *total positivity* is taken by various authors in the literature to mean either total non-negativity or strict total positivity (see [14, 26]), we avoid using the former notion in the paper, and use the latter two instead.

In the dimension-free setting, it was recently shown in [8] that, in the spirit of Schoenberg and Rudin's original theorems, an entrywise function $F : [0,\infty) \to \mathbb{R}$ preserves total non-negativity on Hankel matrices of all sizes, if and only if F preserves positivity on the same set, if and only if F agrees with an absolutely monotonic entire function

$$\sum_{k \geqslant 0} c_k x^k, \ c_k \geqslant 0 \ \forall k$$

on $(0,\infty)$, and $0 \leqslant F(0) \leqslant \lim_{\epsilon \to 0^+} F(\epsilon)$.

Definition 9.2. Given an integer $N > 0$ and a domain $I \subset [0,\infty)$, define $HTN_N(I)$ to be the set of Hankel totally non-negative matrices with entries in I.

As the above remarks show, Hankel totally non-negative matrices (which are automatically positive semidefinite) are a 'well-behaved' test set for preserving total non-negativity in arbitrary dimension – in fact, they are closed under taking Schur products. On the other hand, if one works with the slightly larger class of all positive semidefinite (equivalently, symmetric) totally non-negative matrices, then any preserver is necessarily constant or linear. This is true even for 5×5 matrices; for these and related results, we refer the reader to [7].

In light of these remarks, we now study the preservation of total non-negativity in the fixed dimension setting, again for Hankel totally non-negative matrices. It turns out that the above results for positivity preservation apply to preserving total non-negativity as well; note that in all cases, the entries of totally non-negative matrices, whence the domains of their entrywise preservers, are contained in $[0,\infty)$. We begin by presenting some of the main results, which will be followed by proofs.

Theorem 9.3. *Fix an integer $N > 0$ and real powers $n_0 < \cdots < n_{N-1} < M$. Also fix real scalars $\rho > 0$ and $c_{n_0}, \dots, c_{n_{N-1}}, c'$, and define:*

$$f(x) := \sum_{j=0}^{N-1} c_{n_j} x^{n_j} + c' x^M, \qquad c_{n_j}, c' \in \mathbb{R}. \tag{9.1}$$

Then the following are equivalent:

1. *The entrywise map $f[-]$ preserves total non-negativity on rank-one matrices in $HTN_N((0,\rho))$.*
2. *The entrywise map $f[-]$ preserves positivity on rank-one matrices in $HTN_N((0,\rho))$.*
3. *Either all $c_{n_j}, c' \geqslant 0$; or $c_{n_j} > 0 \ \forall j$ and $c' \geqslant -\mathcal{C}^{-1}$, where*

$$\mathcal{C} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}. \tag{9.2}$$

In other words, preserving total non-negativity on the test set in assertion (1) is equivalent to preserving positivity on the same set, and provides additional equivalent conditions to the ones in Theorem 1.11.

As shown in [14], a power $x \mapsto x^\alpha$ entrywise preserves total non-negativity on $HTN_N((0,\rho))$ if and only if it preserves positivity on $\mathbb{P}_N((0,\rho))$, namely, $\alpha \in \mathbb{Z}^{\geqslant 0} \cup [N-2,\infty)$. With this constraint in mind, the preceding result extends to matrices of all ranks:

Theorem 9.4. *With notation as in Theorem 9.3, if we further assume that $n_j \in \mathbb{Z}^{\geqslant 0} \cup [N-2,\infty)$ for all j, then the conditions (1)–(3) are further equivalent to:*

(4) *The entrywise map $f[-]$ preserves total non-negativity on the set $HTN_N([0,\rho])$.*

Moreover, the bound obtained above (and earlier in the paper) is tight enough to enable bounding more general functions:

Corollary 9.5. *Notation as in Theorem 9.3; assume further that $n_j \in \mathbb{Z}^{\geqslant 0} \cup [N-2,\infty)$ for all j. Given $\varepsilon > 0$ and a real measure μ supported on $[n_{N-1}+\varepsilon,\infty)$ whose Laplace transform g_μ (defined in (5.5)) is absolutely convergent at ρ, there exists a finite threshold*

$$\mathcal{K} = \mathcal{K}(n_0,\dots,n_{N-1},\rho,c_{n_0},\dots,c_{n_{N-1}},g_\mu) > 0$$

such that the entrywise function $x \mapsto \mathcal{K}\sum_{j=0}^{N-1} c_{n_j} x^{n_j} - g_\mu(x)$ preserves total non-negativity on $HTN_N((0,\rho))$, i.e.,

$$\mathcal{K}\sum_{j=0}^{N-1} c_{n_j} A^{\circ n_j} - g_\mu[A] \in HTN_N([0,\infty)), \qquad \forall A \in HTN_N((0,\rho)). \tag{9.3}$$

These theorems follow from two results. The first relates total non-negativity and positive semidefiniteness for Hankel matrices:

Lemma 9.6 (see [14, Corollary 3.5])**.** *Let $A_{N\times N}$ be a Hankel matrix. Then A is totally non-negative if and only if A and its truncation $A^{(1)}$ have non-negative principal minors. Here, $A^{(1)}$ denotes the submatrix of A with the first column and last row removed.*

The second result uses Lemma 9.6 to connect entrywise preservers of these two notions. Recall below that $\mathbb{P}_N^k$ comprises all positive semidefinite $N\times N$ matrices of rank at most k.

Proposition 9.7. *Fix integers $1 \leqslant k \leqslant N$ and a scalar $0 < \rho \leqslant +\infty$. Suppose $f : [0,\rho) \to \mathbb{R}$ is such that the entrywise map $f[-]$ preserves positivity on $\mathbb{P}_N^k([0,\rho))$. Then $f[-]$ preserves total non-negativity on the set $HTN_N([0,\rho)) \cap \mathbb{P}_N^k([0,\rho))$.*

Proof. Given a matrix $A \in HTN_N([0,\rho)) \cap \mathbb{P}_N^k([0,\rho))$, by Lemma 9.6 the padded matrix $A^{(1)} \oplus (0)_{1\times 1}$ belongs to $\mathbb{P}_N([0,\rho))$. Moreover, it has vanishing $(k+1)\times(k+1)$ minors, hence is of rank at most k. By assumption, $f[A]$

and $f[A]^{(1)} = f[A^{(1)}]$ are thus positive semidefinite, whence $f[A]$ is totally non-negative by Lemma 9.6. □

We now prove the above theorems.

Proof of Theorem 9.3. Clearly (1) $\implies$ (2). Now observe that the proof of the 'Horn–Loewner-type' Lemma 5.1 goes through if we restrict to rank-one matrices of the form $\mathbf{u}(\epsilon)\mathbf{u}(\epsilon)^T$ for $\epsilon \in (0,1)$, where $\mathbf{u}(\epsilon)$ is defined in (5.8). Moreover, every such matrix $\mathbf{u}(\epsilon)\mathbf{u}(\epsilon)^T$ is in $HTN_N((0,\infty))$, since all $k \times k$ minors vanish for $k \geqslant 2$. Thus if $f[-]$ preserves total non-negativity on rank-one matrices in $HTN_N((0,\rho))$, then either all c_j, c' are non-negative, or $c_j > 0\ \forall j$. In the latter case, the discussion in Remark 8.5 now implies assertion (3). Finally, if (3) holds then $f[-]$ preserves positivity on $\mathbb{P}_N^1([0,\rho))$ by Theorem 1.11. We are now done by Proposition 9.7. □

Proof of Theorem 9.4. Clearly (4) $\implies$ (1). Conversely, if (3) holds then $f[-]$ preserves positivity on $\mathbb{P}_N([0,\rho))$ by Theorem 1.12. We are now done by Proposition 9.7 for $k = N$. □

Finally, Corollary 9.5 follows similarly from Theorem 5.9 via Proposition 9.7.

We conclude this part by answering a variant of Question 1.3, for total non-negativity. By the Schur product theorem and Lemma 9.6, it follows that absolutely monotonic maps $f : (0,\rho) \to \mathbb{R}$ preserve total non-negativity on $\bigcup_{N \geqslant 1} HTN_N((0,\rho))$. Now given a convergent power series $f(x)$ satisfying

$$f[-] : HTN_N((0,\rho)) \to HTN_N([0,\infty))$$

for *fixed* dimension N, which coefficients of f can be negative? The following results show that for both bounded and unbounded domains, the above results on positivity preservers once again extend to give the same characterizations:

Theorem 9.8. *Let $N > 0$ and $0 \leqslant n_0 < n_1 < \cdots < n_{N-1}$ be integers, and for each $M > n_{N-1}$, let $\epsilon_M \in \{-1, 0, +1\}$ be a sign. Let $0 < \rho < \infty$, and let $c_{n_0}, \dots, c_{n_{N-1}}$ be positive reals. Then there exists a convergent power series*

$$f(x) = c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{N-1}} x^{n_{N-1}} + \sum_{M > n_{N-1}} c_M x^M$$

on $(0,\rho)$ that preserves total non-negativity on $HTN_N((0,\rho))$ when applied entrywise, such that for each $M > n_{N-1}$, c_M has the sign of ϵ_M.

Theorem 9.9. *Let $N > 0$ and $0 \leqslant n_0 < \cdots < n_{N-1}$ be integers, and for each $M > n_{N-1}$, let $\epsilon_M \in \{-1, 0, +1\}$ be a sign. Suppose that whenever $\epsilon_{M_0} = -1$ for some $M_0 > n_{N-1}$, one has $\epsilon_M = +1$ for at least N choices of $M > M_0$. Let $c_{n_0}, \dots, c_{n_{N-1}}$ be positive reals. Then there exists a convergent power series*

$$f(x) = c_{n_0} x^{n_0} + c_{n_1} x^{n_1} + \cdots + c_{n_{N-1}} x^{n_{N-1}} + \sum_{M > n_{N-1}} c_M x^M$$

on $(0,+\infty)$ that preserves total non-negativity on $HTN_N((0,+\infty))$ when applied entrywise, such that for each $M > n_{N-1}$, c_M has the sign of ϵ_M.

In fact, the more general results for sums of *real* powers – namely, Theorems 5.6 and 5.8 – naturally extend to preservers of total non-negativity. We emphasize that these four results were *a priori* nontrivial to formulate and prove; but with the above analysis in this paper, they follow directly from their 'positivity' counterparts, using Proposition 9.7.

10. Application 2: Characterization of (weak) majorization via Schur polynomials

Recall the notion of (weak) majorization: given a real N-tuple $\mathbf{u} = (u_1, \dots, u_N)$, let $u_{[1]} \geqslant \cdots \geqslant u_{[N]}$ denote the decreasing rearrangement of its coordinates. Now given $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$, one says $\mathbf{u}$ *weakly majorizes* $\mathbf{v}$ – and writes $\mathbf{u} \succ_w \mathbf{v}$ – if

$$\sum_{j=1}^k u_{[j]} \geqslant \sum_{j=1}^k v_{[j]}, \ \forall 0 < k < N, \qquad \sum_{j=1}^N u_{[j]} \geqslant \sum_{j=1}^N v_{[j]}, \tag{10.1}$$

while $\mathbf{u}$ *majorizes* $\mathbf{v}$ if the final inequality above (for $k = N$) is an equality.

In this part, we apply Proposition 8.1 and Corollary 8.7 to obtain a new characterization of weak majorization that involves Schur polynomials. In particular, the result also extends a conjecture of Cuttler, Greene, and Skandera [11, Conjecture 7.4]. The conjecture says that if $\mathbf{m}$ majorizes $\mathbf{n}$, where $\mathbf{m}, \mathbf{n}$ are N-tuples of (distinct) non-negative integers, then

$$\frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1, \dots, 1)}{s_{\mathbf{n}}(1, \dots, 1)}, \qquad \forall \mathbf{u} \in ((0, \infty)^N)^T. \tag{10.2}$$

(We thank Rachid Ait-Haddou and Suvrit Sra for correcting the formulation of this conjecture in a previous version of this manuscript.) The conjecture has been proved very recently in [40] and alternately in [1] (see Remark 5.1 therein) using the Harish-Chandra–Itzykson–Zuber formula.

In our setting, first observe that if $\mathbf{m}$ dominates $\mathbf{n}$ coordinatewise and $\sum_j m_j > \sum_j n_j$, then by homogeneity, (10.2) necessarily cannot hold at e.g. points $\mathbf{u} = \rho(1, \dots, 1)^T$ with $\rho \in (0, 1)$. However, it does hold at all points in $([1, \infty)^N)^T$:

$$m_j \geqslant n_j \ \forall j \implies \frac{s_{\mathbf{m}}(\mathbf{u})}{s_{\mathbf{n}}(\mathbf{u})} \geqslant \frac{s_{\mathbf{m}}(1, \dots, 1)}{s_{\mathbf{n}}(1, \dots, 1)} = \frac{V(\mathbf{m})}{V(\mathbf{n})}, \qquad \forall \mathbf{u} \in ([1, \infty)^N)^T, \tag{10.3}$$

since this is a direct consequence of Proposition 8.1.

Since (10.3) also holds if $\mathbf{m}$ majorizes $\mathbf{n}$, it is natural to try to characterize the pairs of tuples $\mathbf{m}, \mathbf{n}$ for which (10.3) holds. In other words, we seek a unification of both settings: $\mathbf{m}$ majorizing $\mathbf{n}$, and $\mathbf{m}$ dominating $\mathbf{n}$ coordinatewise (and restricting to $\mathbf{u} \in ([1, \infty)^N)^T$ for homogeneity reasons). Observe that all of the cited works [1, 11, 40] require $\sum_j m_j = \sum_j n_j$. Replacing this equality by an inequality as in (10.1) allows us to achieve such a unification, and thereby provide the sought-for extension of (both implications in) the Cuttler–Greene–Skandera conjecture. In fact, the following result holds more generally for tuples of real powers – including negative powers – and characterizes weak majorization.

Theorem 10.1. *Suppose* $\mathbf{m}, \mathbf{n}$ *are* N*-tuples of pairwise distinct real powers. Then we have*

$$\frac{|\det \mathbf{u}^{\circ \mathbf{m}}|}{|V(\mathbf{m})|} \geqslant \frac{|\det \mathbf{u}^{\circ \mathbf{n}}|}{|V(\mathbf{n})|}, \qquad \forall \mathbf{u} \in ([1,\infty)^N)^T \tag{10.4}$$

if and only if $\mathbf{m} \succ_w \mathbf{n}$*. In fact to deduce* $\mathbf{m} \succ_w \mathbf{n}$*, it suffices to work with countably many vectors* $\mathbf{u} \in ((I_\infty)^N)^T$*, where* I_∞ *is an arbitrary non-empty deleted neighborhood of* ∞ *in* $[1,\infty)$*.*

In other words, I_∞ is essentially of the form (K,∞) or $[K,\infty)$, for some real $K \geq 1$. As the proof reveals, there is sufficient freedom in choosing the aforementioned countable set of vectors, e.g. they can be chosen to be rational.

Remark 10.2. As a consequence of Theorem 10.1, we obtain the classification of the set of pairs $(\mathbf{m}, \mathbf{n})$ that satisfy (10.4) for all $\mathbf{u} \in ((K,\infty)^N)^T$; and this classification is independent of $K \geq 1$.

Proof of Theorem 10.1. By multiplying both sides by $(u_1 \cdots u_N)^{-k}$ where $k := \min\{m_j, n_j : 0 \leqslant j < N\}$, the result reduces to the case when all powers m_j, n_j are non-negative. While the proof for non-negative $\mathbf{m}, \mathbf{n}$ now follows that in [11, 40], we include it for completeness since real powers are involved, leading to some differences in argument. We may assume $\mathbf{m}, \mathbf{n}$ are in increasing order, and drop all absolute value signs in (10.4). Now suppose (10.4) holds, and for each $j \in [1, N]$, define the following partial sums of $\mathbf{m}, \mathbf{n}$:

$$\widetilde{n}_j := n_{N-j} + \cdots + n_{N-1}, \qquad \widetilde{m}_j := m_{N-j} + \cdots + m_{N-1}.$$

Fix scalars $0 < u_1 < \cdots < u_N < \infty$ in the neighborhood I_∞, note that $tu_j \in I_\infty$ if $t \geq 1$, and now appeal to our tight bounds in Proposition 5.10, setting

$$\mathbf{u} = \mathbf{u}_j(t) := (u_1, \dots, u_{N-j}, tu_{N-j+1}, \dots, tu_N), \qquad t \in [1,\infty). \tag{10.5}$$

More precisely, we multiply all terms in (5.7) by $\mathbf{u}^{\mathbf{n}_{\min}}/V(\mathbf{u})$, and compute using (10.4):

$$0 \leqslant t^{\widetilde{n}_j} \prod_{k=1}^N u_k^{n_{k-1}} = \mathbf{u}^{\mathbf{n}} \leqslant \frac{\mathbf{u}^{\mathbf{n}_{\min}}}{V(\mathbf{u})} \det \mathbf{u}^{\circ \mathbf{n}} \leqslant \frac{\mathbf{u}^{\mathbf{n}_{\min}}}{V(\mathbf{u})} \frac{V(\mathbf{n})}{V(\mathbf{m})} \det \mathbf{u}^{\circ \mathbf{m}}$$
$$\leqslant \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})} \mathbf{u}^{\mathbf{m}} = t^{\widetilde{m}_j} \frac{V(\mathbf{n})}{V(\mathbf{n}_{\min})} \prod_{k=1}^N u_k^{m_{k-1}}, \qquad \forall t \geqslant 1.$$

Taking a countable sequence $t = t_l \to \infty$ (for each $j \in [1, N]$) shows that the growth rate $\widetilde{n}_j$ of the initial expression must be exceeded by that of the final expression, which is $\widetilde{m}_j$. It follows that $\mathbf{m}$ weakly majorizes $\mathbf{n}$.

Conversely, suppose $\mathbf{m}$ weakly majorizes $\mathbf{n}$ (with all non-negative and possibly non-pairwise-distinct real coordinates). We then claim that $F_{\mathbf{u}}(\mathbf{m}) \geqslant F_{\mathbf{u}}(\mathbf{n})$, where $F_{\mathbf{u}} : ([0,+\infty))^N \to \mathbb{R}$ is the Harish-Chandra–Itzykson–Zuber integral (5.1):

$$F_{\mathbf{u}}(\mathbf{m}) \tag{10.6}$$

$$:= \int_{U(N)} \exp \operatorname{tr} \left(\operatorname{diag}(m_0, \dots, m_{N-1}) U \operatorname{diag}(\log(u_1), \dots, \log(u_N)) U^* \right) \, dU.$$

By continuity and a limiting argument, note we only need to prove (10.4) for all $\mathbf{u} \in (1, +\infty)^N$ with pairwise distinct, strictly increasing coordinates. But by the aforementioned integral formula (5.1), such a statement would immediately imply $G_{\mathbf{u}}(\mathbf{m}) \geqslant G_{\mathbf{u}}(\mathbf{n})$ whenever $\mathbf{m}$ weakly majorizes $\mathbf{n}$ and both tuples have pairwise distinct coordinates, where:

$$G_{\mathbf{u}}(\mathbf{m}) := \frac{\det \mathbf{u}^{\circ \mathbf{m}}}{V(\mathbf{m})} \cdot \frac{V(\mathbf{n}_{\min})}{V(\log[\mathbf{u}])}. \tag{10.7}$$

This would end the proof of the reverse implication, and with it, the theorem.

It thus remains to show the assertion:

$$\mathbf{m} \succ_w \mathbf{n} \quad \implies \quad F_{\mathbf{u}}(\mathbf{m}) \geqslant F_{\mathbf{u}}(\mathbf{n}) \ \forall \mathbf{u} \in (1, \infty)^N_{\neq}.$$

For this we appeal to [30, Chapter 3, C.2.d] (which follows from a result of Schur). This says that if ϕ is symmetric, convex, and coordinatewise non-decreasing, then $\phi(\mathbf{m}) \geqslant \phi(\mathbf{n})$ whenever $\mathbf{m} \succ_w \mathbf{n}$. Defining $\phi(\mathbf{m}) := F_{\mathbf{u}}(\mathbf{m})$ for fixed $\mathbf{u}$ as above, proving the three properties for ϕ would conclude the proof.

From (10.6), ϕ is symmetric (since $U(N)$ contains the permutation matrices S_N). We next claim $\phi(\mathbf{m})$ is coordinatewise non-decreasing in $\mathbf{m}$. Indeed, by a limiting argument it suffices to consider $\mathbf{m}, \mathbf{n}$ with pairwise distinct coordinates, whence $\phi(\mathbf{m}) = G_{\mathbf{u}}(\mathbf{m})$ by the Harish-Chandra–Itzykson–Zuber formula (5.1). But now if $\mathbf{m} \geqslant \mathbf{n}$ coordinatewise, then by Corollary 8.7 (which we remark requires the positivity part of the result of Lam et al [28]),

$$\frac{\det \mathbf{u}^{\circ \mathbf{m}}}{\det \mathbf{u}^{\circ \mathbf{n}}} \geqslant \frac{\det \mathbf{u}(\epsilon)^{\circ \mathbf{m}}}{\det \mathbf{u}(\epsilon)^{\circ \mathbf{n}}},$$

with $\mathbf{u}(\epsilon) := (1, \epsilon, \dots, \epsilon^{N-1})^T$ as in (5.8), and $\epsilon > 1$ chosen to be so small that $\epsilon^{N-1} < u_j \ \forall j$. In other words,

$$\frac{\det \mathbf{u}^{\circ \mathbf{m}}}{\det \mathbf{u}(\epsilon)^{\circ \mathbf{m}}} \geqslant \frac{\det \mathbf{u}^{\circ \mathbf{n}}}{\det \mathbf{u}(\epsilon)^{\circ \mathbf{n}}}.$$

Now taking $\epsilon \to 1^+$ and applying (5.9) proves the claim for $\phi = G_{\mathbf{u}}$, whence for $F_{\mathbf{u}}$.

It remains to show that $F_{\mathbf{u}}$ is convex, or by continuity, mid-convex. But this is as in [40]: denoting

$$D(\mathbf{m}) := \operatorname{diag}(m_0, \dots, m_{N-1}), \qquad D_{\mathbf{u}} := \operatorname{diag}(\log(u_1), \dots, \log(u_N)),$$

we compute:

$$\begin{aligned} & F_{\mathbf{u}} \left(\frac{\mathbf{m} + \mathbf{n}}{2} \right) \\ = & \int_{U(N)} \exp \operatorname{tr} \left(D \left(\frac{\mathbf{m} + \mathbf{n}}{2} \right) U D_{\mathbf{u}} U^* \right) \, dU \\ = & \int_{U(N)} \left(\exp(\operatorname{tr}(D(\mathbf{m}) U D_{\mathbf{u}} U^*)) \cdot \exp(\operatorname{tr}(D(\mathbf{n}) U D_{\mathbf{u}} U^*)) \right)^{1/2} \, dU \end{aligned}$$

$$\leqslant \frac{1}{2}\int_{U(N)} \exp \operatorname{tr}(D(\mathbf{m})UD_{\mathbf{u}}U^*)\ dU + \frac{1}{2}\int_{U(N)} \exp \operatorname{tr}(D(\mathbf{n})UD_{\mathbf{u}}U^*)\ dU$$
$$= \frac{F_{\mathbf{u}}(\mathbf{m}) + F_{\mathbf{u}}(\mathbf{n})}{2},$$

using the AM-GM inequality. From the above discussion, the proof is complete. □

Remark 10.3. The key underlying fact used in deducing $\mathbf{m} \succ_w \mathbf{n}$ from (10.4) is that if $t^\alpha \geq c$ for some $c > 0$, $\alpha \in \mathbb{R}$, and all $t \geq 1$ (or a countable sequence $t_l \to \infty$), then α must be non-negative. (In the proof above, $\alpha = \widetilde{m}_j - \widetilde{n}_j$ and $c = V(\mathbf{n}_{\min}) \prod_{k=1}^N u_k^{n_{k-1}-m_{k-1}} / V(\mathbf{n})$.) If it was possible to choose suitable $u_k \in I_\infty$ – given any powers $\mathbf{m}, \mathbf{n}$ – such that $c \geq 1$, then our deduced condition

$$t^\alpha \geq c,$$

for a *single* value of $t > 1$, would imply $\alpha \geq 0$. In the absence of a 'uniform' recipe to choose u_k such that $c \geq 1$, we require countably many t_l to deduce the key underlying fact mentioned above.

As a consequence of Theorem 10.1, we immediately have:

Corollary 10.4. *Suppose* $\mathbf{m}, \mathbf{n}$ *are* N*-tuples of distinct real powers. The inequality* (10.4) *holds for all* $\mathbf{u} \in ((0,1)^N)^T$ *if and only if* $-\mathbf{m} \succ_w -\mathbf{n}$. *In fact it suffices to work with countably many vectors* $\mathbf{u} \in ((I_0)^N)^T$, *where* I_0 *is an arbitrary non-empty deleted neighborhood of* 0 *in* $(0,1]$.

This immediately follows from Theorem 10.1 because

$$\mathbf{u}^{\circ \mathbf{m}} = ((1/u_1, \dots, 1/u_N)^{\circ(-\mathbf{m})})^T, \qquad \forall \mathbf{u} = (u_1, \dots, u_N)^T \in ((0,1)^N)^T.$$

We conclude this section by proving the twofold strengthening to the original result of Cuttler–Greene–Skandera, stated in Theorems 1.15 and 1.16.

Proof of Theorems 1.15 and 1.16. That (1) $\implies$ (3) is immediate. To show (3) $\implies$ (2), first consider the inequality (10.4) for $\mathbf{u} = \mathbf{u}_j(t_l)$ as in (10.5), and for $\mathbf{u}_j(t_l)^{\circ(-1)}$, with $j \in [1, N]$ and $l \geq 1$. It follows from Theorem 10.1 and Corollary 10.4 that

$$\mathbf{m} \succ_w \mathbf{n}, \qquad -\mathbf{m} \succ_w -\mathbf{n},$$

and these conditions are together equivalent to majorization: $\mathbf{m} \succ \mathbf{n}$.

Finally, to show (2) $\implies$ (1), from the equivalence for non-negative integer powers $\mathbf{m}, \mathbf{n}$ (proved by Cuttler–Greene–Skandera, Sra, and Ait-Haddou–Mazure), one deduces the result for non-negative rational powers $\mathbf{m}, \mathbf{n}$ by "taking a common denominator" $L \in \mathbb{N}$ and changing variables to $y_j := u_j^{1/L}$ (see e.g. the proof of Corollary 8.7). The result for non-negative real powers now follows by continuity. Finally, the result for arbitrary real powers can be reduced to the case of non-negative $\mathbf{m}, \mathbf{n}$ by multiplying both sides of the inequality by $(u_1 \cdots u_N)^{-k}$, where $k := \min\{m_j, n_j : 0 \leqslant j < N\}$. □

11. Further applications: Rayleigh quotients and the cube problem

We now discuss further remarks and applications of the above quantitative results. To begin, our main quantitative result Theorem 1.12 has an optimization-theoretic formulation, as follows. First notice that the inequality (3.3) holds for all real powers $0 \leqslant n_0 < \cdots < n_{N-1} < M$, with the same proof. Now as mentioned in Remark 3.3, using the theory of generalized Rayleigh quotients, Theorem 1.12 implies the following result for a *single* positive semidefinite matrix:

Proposition 11.1. *Fix an integer $N \geqslant 2$ and real powers $n_0 < \cdots < n_{N-1} < M$, where $n_j \in \mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$. Given scalars $c_{n_0}, \dots, c_{n_{N-1}} > 0$, define*

$$h(x) := \sum_{j=0}^{N-1} c_{n_j} x^{n_j}, \qquad x \in (0, +\infty).$$

Then for $0 < \rho < +\infty$ and $A \in \mathbb{P}_N([0, \rho])$,

$$t\, h[A] \succeq A^{\circ M}, \qquad \text{if and only if} \quad t \geqslant \varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2}), \tag{11.1}$$

where $\varrho[B], B^\dagger$ denote the spectral radius and the Moore–Penrose (pseudo-)inverse of a square matrix B, respectively. Moreover, for every nonzero matrix $A \in \mathbb{P}_N([0, \rho])$, we have the variational formula

$$\varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2}) = \sup_{\mathbf{u} \in (\ker h[A])^\perp \setminus \{\mathbf{0}\}} \frac{\mathbf{u}^* A^{\circ M} \mathbf{u}}{\mathbf{u}^* h[\mathbf{u}\mathbf{u}^*] \mathbf{u}} \leqslant \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}.$$

Thus, for each A the threshold bound $\varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2})$ for the semidefinite inequality

$$t\, h[A] \succeq A^{\circ M}$$

is a generalized Rayleigh quotient. When the matrix A is a generic rank-one matrix, this Rayleigh quotient has a closed-form expression using the Schur polynomials $s_{\mathbf{n}_j}, s_{\mathbf{n}}$ – i.e., the generalized Vandermonde determinants $\det \mathbf{u}^{\circ \mathbf{n}_j}, \det \mathbf{u}^{\circ \mathbf{n}}$:

Proposition 11.2. *Notation as in Proposition 11.1; but now with n_j not necessarily in $\mathbb{Z}^{\geqslant 0} \cup [N-2, \infty)$. If $A = \mathbf{u}\mathbf{u}^T$ with $\mathbf{u} \in ((0, \sqrt{\rho})^N_{\neq})^T$ having distinct coordinates, then $h[A]$ is invertible, and the threshold bound equals:*

$$\varrho(h[A]^{\dagger/2} A^{\circ M} h[A]^{\dagger/2}) = (\mathbf{u}^{\circ M})^T h[\mathbf{u}\mathbf{u}^T]^{-1} \mathbf{u}^{\circ M} = \sum_{j=0}^{N-1} \frac{(\det \mathbf{u}^{\circ \mathbf{n}_j})^2}{c_{n_j} (\det \mathbf{u}^{\circ \mathbf{n}})^2}. \tag{11.2}$$

In particular, by going from single rank-one matrices $A = \mathbf{u}\mathbf{u}^T$ for generic vectors $\mathbf{u}$, to all matrices in the linear matrix inequality

$$t\, h[A] \succeq A^{\circ M} \quad \forall A \in \mathbb{P}_N^1([0, \rho]),$$

we recover by a second proof the inequality $t \geqslant \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2} \frac{\rho^{M-n_j}}{c_{n_j}}$, using Corollary 8.7.

Proof. First observe that if $h[\mathbf{u}\mathbf{u}^T]v = 0$, then $v^T(\mathbf{u}\mathbf{u}^T)^{\circ n_j}v = 0$ for all j, whence v is killed by $\mathbf{u}^{\circ n_j}$ for all j. But then $v = 0$ by (1.1); thus $h[\mathbf{u}\mathbf{u}^T]$ is nonsingular. Now we compute:

$$\begin{aligned}\varrho(h[A]^{\dagger/2}A^{\circ M}h[A]^{\dagger/2}) &= \varrho(h[\mathbf{u}\mathbf{u}^T]^{-1/2}\mathbf{u}^{\circ M}\cdot(\mathbf{u}^{\circ M})^T h[\mathbf{u}\mathbf{u}^T]^{-1/2})\\ &= (\mathbf{u}^{\circ M})^T h[\mathbf{u}\mathbf{u}^T]^{-1}\mathbf{u}^{\circ M},\end{aligned}$$

since $\varrho(\mathbf{v}\cdot\mathbf{v}^T) = \mathbf{v}^T\mathbf{v}$ for a real vector $\mathbf{v}$. As in the proof of Lemma 2.3, we have

$$\begin{aligned}&h[\mathbf{u}\mathbf{u}^T]^{-1}\\ &= ((\mathbf{u}^{\circ n_0}|\dots|\mathbf{u}^{\circ n_{N-1}})^{-1})^T \operatorname{diag}(c_{n_0}^{-1},\dots,c_{n_{N-1}}^{-1})(\mathbf{u}^{\circ n_0}|\dots|\mathbf{u}^{\circ n_{N-1}})^{-1}\end{aligned}$$

and using Cramer's rule, $(\mathbf{u}^{\circ n_0}|\dots|\mathbf{u}^{\circ n_{N-1}})^{-1}\mathbf{u}^{\circ M}$ has coordinates $\dfrac{\det \mathbf{u}^{\circ \mathbf{n}_j}}{\det \mathbf{u}^{\circ \mathbf{n}}}$. Hence (11.2) follows. □

Remark 11.3. Akin to Remark 2.4, the above argument shows the following more general algebraic phenomenon. Suppose $\mathbb{F}$ is a field, $\mathbf{u},\mathbf{v} \in (\mathbb{F}^N_{\neq})^T$ each have pairwise distinct coordinates, and $h(x) := \sum_{j=0}^{N-1} c_{n_j}x^{n_j} \in \mathbb{F}[x]$ for integers $n_0 < \dots < n_{N-1}$ and nonzero $c_{n_j} \in \mathbb{F}^\times$. Then $h[\mathbf{u}\mathbf{v}^T] := (h(u_jv_k))_{j,k=1}^N$ is invertible (where we also require $\mathbf{u},\mathbf{v}$ to have nonzero coordinates if $n_0 < 0$); and moreover for all integers $M > n_{N-1}$,

$$(\mathbf{v}^{\circ M})^T h[\mathbf{u}\mathbf{v}^T]^{-1}\mathbf{u}^{\circ M} = \sum_{j=0}^{N-1} \frac{\det \mathbf{u}^{\circ \mathbf{n}_j}\cdot \det \mathbf{v}^{\circ \mathbf{n}_j}}{c_{n_j}\cdot \det \mathbf{u}^{\circ \mathbf{n}}\cdot \det \mathbf{v}^{\circ \mathbf{n}}}.$$

Remark 11.4. By Lemma 1.4, any linear matrix inequality of the form

$$A^{\circ M} \preceq \mathcal{K}\sum_{j=0}^{N-1} c_{n_j}A^{\circ n_j}, \qquad \forall A \in \mathbb{P}_N([0,\rho])$$

does not hold for fewer than N powers. Theorem 1.12 can be reformulated to say such an inequality does not require more than N powers either, and the constant $\mathcal{K} = \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j)^2}{V(\mathbf{n})^2}\frac{\rho^{M-n_j}}{c_{n_j}}$ is sharp.

We end with an application to spectrahedra and the cube problem [31] in optimization theory, where we produce sharp asymptotic bounds when the matrices involved are Hadamard powers. To remind the reader, given $N \times N$ real symmetric matrices $A'_N, A_1, \dots, A_{M+1}$ for $M \geqslant 0$, one defines the matrix cube to be:

$$\mathcal{U}[A'_N, A_1\dots,A_{M+1};\eta] := \left\{A'_N + \sum_{m=1}^{M+1} u_mA_m : u_m \in [-\eta,\eta]\right\} \qquad (\eta > 0). \tag{11.3}$$

The matrix cube problem asks to find the largest $\eta \geqslant 0$ such that $\mathcal{U}[\eta] \subset \mathbb{P}_N(\mathbb{R})$. As another consequence of the main result, Theorem 1.12, we obtain bounds for such η:

Corollary 11.5. *Let the notation as in Proposition 11.1. Fix scalars* $0 < \alpha_1 < \cdots < \alpha_{M+1}$ *for some* $M \geqslant 0$*. Now given a matrix* $A \in \mathbb{P}_N([0,\rho])$*, define*

$$A'_N \coloneqq \sum_{j=0}^{N-1} c_{n_j} A^{\circ n_j}, \qquad A_m \coloneqq A^{\circ(n_{N-1}+\alpha_m)} \text{ for } 1 \leqslant m \leqslant M+1.$$

Also define for $\alpha > 0$*:*

$$\mathcal{K}_\alpha \coloneqq \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j(\alpha))^2}{V(\mathbf{n})^2} \frac{\rho^{\alpha - n_j}}{c_{n_j}}, \tag{11.4}$$

where $\mathbf{n}_j(\alpha) \coloneqq (n_0, \dots, n_{j-1}, n_{j+1}, \dots, n_{N-1}, n_{N-1}+\alpha)$*. Then,*

$$\begin{aligned} & \eta \leqslant (\mathcal{K}_{\alpha_1} + \cdots + \mathcal{K}_{\alpha_{M+1}})^{-1} \\ \Longrightarrow\ & \mathcal{U}[A'_N, A_1, \dots, A_{M+1}; \eta] \subset \mathbb{P}_N(\mathbb{R}),\ \forall A \in \mathbb{P}_N([0,\rho]) \\ \Longrightarrow\ & \eta \leqslant \mathcal{K}_{\alpha_{M+1}}^{-1}. \end{aligned} \tag{11.5}$$

We conclude this part by showing that these bounds are asymptotically equal as $N \to \infty$, when the n_j grow 'linearly' at a bounded rate. Notice that in such a setting, in light of [15] we will require all n_j to be integers.

Proposition 11.6. *Suppose* $0 \leqslant n_0 < n_1 < \cdots$ *are integers. Fix scalars* $0 < \alpha_1 < \cdots < \alpha_{M+1}$ *for some* $M \geqslant 0$*, as well as* $c_{n_j} > 0\ \forall j \geqslant 0$*. Given* $N \in \mathbb{N}$ *and a matrix* $A \in \mathbb{P}_N([0,\rho])$*, define* A'_N, A_m*, and* $\mathcal{K}_\alpha = \mathcal{K}_\alpha(N)$ *as in Corollary 11.5. If* $\alpha_{M+1} - \alpha_M \geqslant n_{j+1} - n_j\ \forall j \geqslant 0$*, then the lower and upper bounds for* $\eta = \eta_N$ *in* (11.5) *are asymptotically equal as* $N \to \infty$*, i.e.,*

$$\lim_{N \to \infty} \mathcal{K}_{\alpha_{M+1}}(N)^{-1} \sum_{m=1}^{M+1} \mathcal{K}_{\alpha_m}(N) = 1. \tag{11.6}$$

Proof. We compute for $1 \leqslant m < M+1$, first in the case $0 < j < N-1$:

$$\begin{aligned} \frac{V(\mathbf{n}_j(\alpha_m))}{V(\mathbf{n}_j(\alpha_{M+1}))} &= \prod_{k<N, k\neq j} \frac{n_{N-1} - n_k + \alpha_m}{n_{N-1} - n_k + \alpha_{M+1}} \\ &= \frac{\alpha_m}{n_{N-1} - n_0 + \alpha_{M+1}} \cdot \frac{n_{N-1} - n_{j-1} + \alpha_m}{n_{N-1} - n_{j+1} + \alpha_{M+1}} \times \\ &\qquad \times \prod_{0 \leqslant k < N-1, k \neq j-1, j} \frac{n_{N-1} - n_k + \alpha_m}{n_{N-1} - n_{k+1} + \alpha_{M+1}}. \end{aligned}$$

By assumption on the α_m, each factor in the 'final' product above is in $(0,1]$. We now claim that the second fraction in the preceding expression is at most 2. Indeed,

$$\begin{aligned} & 2(n_{N-1} - n_{j+1} + \alpha_{M+1}) - (n_{N-1} - n_{j-1} + \alpha_m) \\ = {} & (n_{N-1} - n_{j+1}) + \alpha_m + [2(\alpha_{M+1} - \alpha_m) - (n_{j+1} - n_{j-1})] \geqslant \alpha_m > 0. \end{aligned}$$

It follows that

$$0 \leqslant \frac{V(\mathbf{n}_j(\alpha_m))^2}{V(\mathbf{n}_j(\alpha_{M+1}))^2} \frac{\rho^{\alpha_m - n_j}}{\rho^{\alpha_{M+1} - n_j}} \leqslant \frac{4\alpha_m^2 \rho^{\alpha_m - \alpha_{M+1}}}{(N - 1 + \alpha_{M+1})^2}$$

$$\leqslant \frac{4\alpha_{M+1}^2 \max(1, \rho^{\alpha_1 - \alpha_{M+1}})}{(N-2)^2}.$$

Similarly in the case $j = 0$, we have:

$$\begin{aligned} 0 \leqslant \frac{V(\mathbf{n}_0(\alpha_m))^2}{V(\mathbf{n}_0(\alpha_{M+1}))^2} \frac{\rho^{\alpha_m - n_0}}{\rho^{\alpha_{M+1} - n_0}} &\leqslant \frac{\alpha_m^2 \rho^{\alpha_m - \alpha_{M+1}}}{(n_{N-1} - n_1 + \alpha_{M+1})^2} \\ &\leqslant \frac{4\alpha_{M+1}^2 \max(1, \rho^{\alpha_1 - \alpha_{M+1}})}{(N-2)^2}. \end{aligned}$$

Finally, if $j = N-1$, then by similar computations,

$$0 \leqslant \frac{V(\mathbf{n}_{N-1}(\alpha_m))}{V(\mathbf{n}_{N-1}(\alpha_{M+1}))} \leqslant \frac{\alpha_m + n_{N-1} - n_{N-2}}{n_{N-1} - n_0 + \alpha_{M+1}},$$

and the numerator on the right is at most $\alpha_m + (\alpha_{M+1} - \alpha_m)$ by hypothesis. Therefore,

$$\begin{aligned} 0 \leqslant \frac{V(\mathbf{n}_{N-1}(\alpha_m))^2}{V(\mathbf{n}_{N-1}(\alpha_{M+1}))^2} \frac{\rho^{\alpha_m - n_0}}{\rho^{\alpha_{M+1} - n_0}} &\leqslant \frac{\alpha_{M+1}^2 \rho^{\alpha_m - \alpha_{M+1}}}{(n_{N-1} - n_0 + \alpha_{M+1})^2} \\ &\leqslant \frac{4\alpha_{M+1}^2 \max(1, \rho^{\alpha_1 - \alpha_{M+1}})}{(N-2)^2}. \end{aligned}$$

From these computations it follows that if $1 \leqslant m < M+1$, then for fixed $N \geqslant 3$,

$$\begin{aligned} \mathcal{K}_{\alpha_m}(N) = & \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j(\alpha_m))^2}{V(\mathbf{n})^2} \frac{\rho^{\alpha_m - n_j}}{c_{n_j}} \\ \leqslant & \frac{4\alpha_{M+1}^2 \max(1, \rho^{\alpha_1 - \alpha_{M+1}})}{(N-2)^2} \sum_{j=0}^{N-1} \frac{V(\mathbf{n}_j(\alpha_{M+1}))^2}{V(\mathbf{n})^2} \frac{\rho^{\alpha_{M+1} - n_j}}{c_{n_j}} \\ \leqslant & \frac{4\alpha_{M+1}^2 \max(1, \rho^{\alpha_1 - \alpha_{M+1}})}{(N-2)^2} \mathcal{K}_{\alpha_{M+1}}(N) \end{aligned}$$

Summing this over $m = 1, \dots, M+1$ and taking $N \to \infty$ yields (11.6), as desired. □

12. Log-supermodularity of strictly totally positive matrix minors

Given Proposition 8.1 and its generalization to real powers in Corollary 8.7, it is natural to ask if there is a more 'accessible' proof of the positivity property in (8.9), avoiding the heavy machinery required to prove Schur or monomial positivity. In this final section, we provide a positive answer to this question. In fact we prove a more general log-supermodularity phenomenon which had previously been established by Skandera [39] by a different argument.

We begin by setting notation and recalling preliminaries. Suppose $1 \leqslant k \leqslant n$ are natural numbers. Recall that $[n]^k_{\neq}$ is the set of all tuples $(i_1, \dots, i_k)$

with $i_1, \dots, i_k$ distinct elements of $[n] := \{1, \dots, n\}$. We define $[n]^k_<$ to be the subset of increasing tuples:

$$[n]^k_< := \{(i_1, \dots, i_k) \in [n]^k_{\neq} : 1 \leqslant i_1 < \dots < i_k \leqslant n\}. \tag{12.1}$$

Clearly, for each tuple $I = (i_1, \dots, i_k)$ in $[n]^k_{\neq}$, we may sort the tuple in increasing order to obtain a sorted tuple $\mathrm{sort}(I) = (i_{\sigma(1)}, \dots, i_{\sigma(k)}) \in [n]^k_<$ for a uniquely determined permutation $\sigma : [k] \to [k]$, which we will call the *ordering* of I. The sign $\mathrm{sgn}(\sigma) \in \{-1, +1\}$ of this permutation will be referred to as the *sign of the tuple* and denoted $\mathrm{sgn}(I)$. Observe that $\mathrm{sgn}(I)$ is equal to -1 raised to the number of inversions of I, that is to say those pairs of natural numbers $1 \leqslant m < m' \leqslant k$ with $i_{m'} < i_m$.

If $A = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ is an $n \times n$ matrix with real entries $a_{i,j}$, and $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$ are tuples in $[n]^k_{\neq}$, we define the *minor* $A_{I,J}$ to be the $k \times k$ matrix

$$A_{I,J} := (a_{i_l, j_m})_{1 \leqslant l,m \leqslant k}.$$

Note that we are generalizing slightly the usual notion of a minor by allowing I, J to have non-trivial ordering, so the ordering of the rows and columns of the minor $A_{I,J}$ may differ from that of the original matrix A. We say that A is *strictly totally positive* if one has

$$\det(A_{I,J}) > 0$$

for any increasing $I, J \in [n]^k_<$, i.e. by the alternating nature of the determinant,

$$\mathrm{sgn}(I)\mathrm{sgn}(J) \det(A_{I,J}) > 0, \quad \forall I, J \in [n]^k_{\neq}. \tag{12.2}$$

Given two tuples $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_k)$ of real numbers, we may define their meet

$$I \wedge J := (\min(i_1, j_1), \dots, \min(i_k, j_k))$$

and join

$$I \vee J := (\max(i_1, j_1), \dots, \max(i_k, j_k)).$$

It is easy to see that if $I, J \in [n]^k_{\neq}$ have the same ordering, then $I \wedge J$ and $I \vee J$ are also tuples in $[n]^k_{\neq}$ with the same ordering as I and J (this can be verified by first sorting the elements to reduce to the case when I, J are increasing.) The purpose of this section is to establish the following log-supermodularity property (also known as "the FKG lattice condition" [16] or "multivariate total positivity of order two" MTP_2 [27]):

Theorem 12.1 (Log-supermodularity)**.** *Let A be a strictly totally positive $n \times n$ matrix, let $I_1, I_2 \in [n]^k_{\neq}$ be tuples of the same ordering, and let $J_1, J_2 \in [n]^k_{\neq}$ be tuples of the same ordering. Then*

$$\det(A_{I_1 \wedge I_2, J_1 \wedge J_2}) \det(A_{I_1 \vee I_2, J_1 \vee J_2}) \geqslant \det(A_{I_1, J_1}) \det(A_{I_2, J_2}). \tag{12.3}$$

A special case of this proposition (in which $J_1 = J_2$) appears implicitly in [33], and our arguments here are loosely based on the ones in that paper. As pointed out to us by Steven Karp, this result is also a special case of [39, Theorem 4.2], which was proven by a different method (using weighted directed graphs instead of the determinant identities in the lemma below).

We will need two classical identities concerning determinants:

Lemma 12.2 (Determinant identities). *Let $n \geqslant 2$ be a natural number.*

(i) *(Dodgson condensation, see* [13], [10].*) For any $1 \leqslant i_1 < i_2 \leqslant n$ and $1 \leqslant j_1 < j_2 \leqslant n$, and any $n \times n$ matrix A, one has*

$$\begin{aligned} &\det(A)\det(A_{[n]\backslash(i_1,i_2),[n]\backslash(j_1,j_2)}) \\ = \ &\det(A_{[n]\backslash(i_1),[n]\backslash(j_1)})\det(A_{[n]\backslash(i_2),[n]\backslash(j_2)}) \\ &- \det(A_{[n]\backslash(i_1),[n]\backslash(j_2)})\det(A_{[n]\backslash(i_2),[n]\backslash(j_1)}), \end{aligned}$$

where $[n]\backslash(i_1,\dots,i_k)$ denotes the increasing $n-k$-tuple formed by deleting $i_1,\dots,i_k$ from $(1,\dots,n)$.

(ii) *(Karlin identity, see* [26, p.7].*) If $X_1, X_2, Y_1, Y_2 \in (\mathbb{R}^n)^T$ are n-dimensional vectors, and A is an $n \times n-2$-matrix, then one has*

$$\begin{aligned} &\det(X_1,Y_1,A)\det(X_2,Y_2,A) \ - \det(X_1,Y_2,A)\det(X_2,Y_1,A) \\ = \ &\det(X_1,X_2,A)\det(Y_1,Y_2,A) \end{aligned}$$

where $\det(X,Y,A)$ denotes the determinant of the $n \times n$ matrix formed by concatenating two $n \times 1$ vectors X, Y with an $n \times n-2$ matrix A.

Proof of Theorem 12.1. By sorting, we may assume without loss of generality that I_1, I_2, J_1, J_2 are all increasing, so that all the determinants involved are positive. Writing

$$F(I,J) := \log\det(A_{I,J})$$

for $I, J \in [n]^k_<$, our task is now to show that

$$F(K_1 \wedge K_2) + F(K_1 \vee K_2) \geqslant F(K_1) + F(K_2)$$

for all $K_1, K_2 \in [n]^k_< \times [n]^k_<$, where we write

$$(I_1,J_1) \wedge (I_2,J_2) := (I_1 \wedge I_2, J_1 \wedge J_2)$$

and

$$(I_1,J_1) \vee (I_2,J_2) := (I_1 \vee I_2, J_1 \vee J_2).$$

Write $K_0 := K_1 \wedge K_2$, $\delta K_1 := K_1 - K_0$, and $\delta K_2 := K_2 - K_0$ (where we view the $2k$-tuples K_1, K_2 as lying in the vector space $\mathbb{R}^{2k}$). Then $\delta K_1, \delta K_2$ have non-negative coefficients and disjoint supports (that is to say, the set of indices in which δK_1 has non-zero coefficients is disjoint from that of δK_2), and our task is now to show that

$$F(K_0 + \delta K_1 + \delta K_2) - F(K_0) \geqslant F(K_0 + \delta K_1) + F(K_0 + \delta K_2) \qquad (12.4)$$

whenever $K_0, K_0 + \delta K_1, K_0 + \delta K_2, K_0 + \delta K_1 + \delta K_2$ lie in $[n]^k_< \times [n]^k_<$ with $\delta K_1, \delta K_2$ having non-negative coefficients and disjoint supports.

We can rewrite (12.4) as the assertion the quantity $F(K_0+\delta K_1) - F(K_0)$ does not decrease if we increase K_0 by δK_2. Writing $\delta K_2 = \sum_{m=1}^{2k} c_m e_m$ for some non-negative c_m, where $e_1, \dots, e_{2k}$ is the standard basis of $\mathbb{R}^{2k}$, it suffices to show that for each $1 < M \leqslant 2k$, the quantity

$$F(K_0 + \sum_{m=M+1}^{2k} c_m e_m + \delta K_1) - F(K_0 + \sum_{m=M+1}^{2k} c_m e_m)$$

does not decrease if one increases $K_0+\sum_{m=M+1}^{2k} c_m e_m$ by $c_M e_M$. Note that as K_0 and $K_0+\sum_{m=1}^{2k} c_m e_m$ both lie in $[n]^k_< \times [n]^k_<$, the intermediate tuples $K_0+\sum_{m=M+1}^{2k} c_m e_m$ and $K_0+\sum_{m=M}^{2k} c_m e_m$ do also (here it is important we are summing the $c_m e_m$ from the right end of the range $m=1,\dots,2k$, rather than from the left). Similarly for $K_0+\sum_{m=M+1}^{2k} c_m e_m+\delta K_1$ and $K_0+\sum_{m=M}^{2k} c_m e_m+\delta K_1$. Because of this, we see that to prove (12.4) it suffices to do so in the special case when δK_2 is supported on a single element, thus $\delta K_2 = c_2 e_{m_2}$ for some $m_2=1,\dots,2k$ and some positive c_2. Similarly we may also assume without loss of generality that $\delta K_1 = c_1 e_{m_1}$ for some integer $m_1 \in [1,2k]\setminus\{m_2\}$ and some positive c_1.

Splitting into cases depending on whether m_1 and m_2 lie in $\{1,\dots,k\}$ or in $\{k+1,\dots,2k\}$, we can thus reduce (12.4) to the verification of three special cases:

(a) One has
$$F(I_0+c_1e_{m_1}+c_2e_{m_2},J_0)+F(I_0,J_0) \geqslant F(I_0+c_1e_{m_1},J_0)+F(I_0+c_2e_{m_2},J_0)$$
whenever $1 \leqslant m_1 < m_2 \leqslant k$, c_1,c_2 are positive, and $I_0, I_0+c_1e_{m_1}, I_0+c_2e_{m_2}, I_0+c_1e_{m_1}+c_2e_{m_2}, J_0$ all lie in $[n]^k_<$.

(b) One has
$$F(I_0,J_0+c_1e_{m_1}+c_2e_{m_2})+F(I_0,J_0) \geqslant F(I_0,J_0+c_1e_{m_1})+F(I_0,J_0+c_2e_{m_2})$$
whenever $1 \leqslant m_1 < m_2 \leqslant k$, c_1,c_2 are positive, and $I_0, J_0, J_0+c_1e_{m_1}, J_0+c_2e_{m_2}, J_0+c_1e_{m_1}+c_2e_{m_2}$ all lie in $[n]^k_<$.

(c) One has
$$F(I_0+c_1e_{m_1},J_0+c_2e_{m_2})+F(I_0,J_0) \geqslant F(I_0+c_1e_{m_1},J_0)+F(I_0,J_0+c_2e_{m_2})$$
whenever $1 \leqslant m_1, m_2 \leqslant k$, c_1,c_2 are positive, and $I_0, I_0+c_1e_{m_1}, J_0$, $J_0+c_2e_{m_2}$ all lie in $[n]^k_<$.

We first prove (a). By exponentiating, this claim is equivalent to the assertion that
$$\det(A_{I_0+c_1e_{m_1}+c_2e_{m_2},J_0})\det(A_{I_0,J_0})-\det(A_{I_0+c_1e_{m_1},J_0})\det(A_{I_0+c_2e_{m_2},J_0}) \geqslant 0.$$
By permuting I_0, it will suffice to show that
$$\det(A_{I_0+c_1e_1+c_2e_2,J_0})\det(A_{I_0,J_0})-\det(A_{I_0+c_1e_1,J_0})\det(A_{I_0+c_2e_2,J_0}) \geqslant 0 \tag{12.5}$$
whenever c_1,c_2 are positive, $J_0\in[n]^k_<$, and $I_0, I_0+c_1e_1, I_0+c_2e_2 \in [n]^k_{\neq}$ all have the same ordering.

Write $I_0=(i_1,i_2,i_3,\dots,i_k)$; thus,
$$\begin{aligned} I_0+c_1e_1 &= (i_1+c_1,i_2,i_3,\dots,i_k),\\ I_0+c_2e_2 &= (i_1,i_2+c_2,i_3,\dots,i_k),\\ I_0+c_1e_1+c_2e_2 &= (i_1+c_1,i_2+c_2,i_3,\dots,i_k). \end{aligned}$$
Applying the Karlin identity (Lemma 12.2(ii)), the left-hand side of (12.5) may be factorized as
$$\det(A_{(i_1,i_1+c_1,i_3,\dots,i_k),J_0})\det(A_{(i_2,i_2+c_2,i_3,\dots,i_k),J_0}).$$

Since the tuples $I_0, I_0+c_1e_1, I_0+c_2e_2, I_0+c_1e_1+c_2e_2$ all have the same ordering, we see that the tuples $(i_1, i_1+c_1, i_3, \dots, i_k)$ and $(i_2, i_2+c_2, i_3, \dots, i_k)$ have the same parity of inversion counts, and thus the same sign. The claim now follows from (12.2).

The proof of (b) is identical to (a) (and indeed follows from (a) after replacing A with A^T), so we turn to (c). By exponentiating and permuting as before, it will suffice to show that

$$\det(A_{I_0+ce_1,J_0+de_1})\det(A_{I_0,J_0}) - \det(A_{I_0+ce_1,J_0})\det(A_{I_0,J_0+de_1}) \geqslant 0 \quad (12.6)$$

whenever c,d are positive, $I_0, I_0+ce_1 \in [n]^k_{\neq}$ have the same ordering, and $J_0, J_0+de_1 \in [n]^k_{\neq}$ have the same ordering.

Write $I_0 = (i_1, i_2, \dots, i_k)$ and $J_0 = (j_1, j_2, \dots, j_k)$. Applying Dodgson condensation (Lemma 12.2(i)), the left-hand side of (12.6) may be written as

$$\det(A_{(i_1,i_1+c,i_2,\dots,i_k),(j_1,j_1+d,j_2,\dots,j_k)})\det(A_{(i_2,\dots,i_k),(j_2,\dots,j_k)}).$$

Since I_0, I_0+ce_1 have the same ordering, we see that $(i_1, i_1+c, i_2, \dots, i_k)$ and $(i_2, \dots, i_k)$ have the same parity of inversion counts, and thus the same sign. Similarly for $(j_1, j_1+d, j_2, \dots, j_k)$ and $(j_2, \dots, j_k)$. The claim now follows from (12.2). $\square$

Remark 12.3. The above argument in fact shows that the inequality in (12.3) is strict whenever the tuples (I_1, J_1) and (I_2, J_2) are incomparable (thus $(I_1 \vee I_2, J_1 \vee J_2)$ is not equal to either (I_1, J_1) or (I_2, J_2)). Of course, (12.3) is satisfied with equality if (I_1, J_1) and (I_2, J_2) are comparable.

12.1. **Applications.** From the positivity of generalized Vandermonde determinants, we see that any generalized Vandermonde matrix $(u_j^{n_{k-1}})_{1\leqslant j,k\leqslant N}$ with $0 < u_1 < \dots < u_N$ and $0 < n_0 < \dots < n_{N-1}$ is strictly totally positive. Writing

$$\det(u_j^{n_{k-1}})_{1\leqslant j,k\leqslant N} = V(\mathbf{u})s_{\mathbf{n}}(\mathbf{u})$$

we arrive at the log-supermodularity inequality

$$\begin{aligned}&V(\mathbf{u}_1 \vee \mathbf{u}_2)s_{\mathbf{n}_1\vee\mathbf{n}_2}(\mathbf{u}_1 \vee \mathbf{u}_2)V(\mathbf{u}_1 \wedge \mathbf{u}_2)s_{\mathbf{n}_1\wedge\mathbf{n}_2}(\mathbf{u}_1 \wedge \mathbf{u}_2)\\ &\geqslant V(\mathbf{u}_1)s_{\mathbf{n}_1}(\mathbf{u}_1)V(\mathbf{u}_2)s_{\mathbf{n}_2}(\mathbf{u}_2)\end{aligned}$$

whenever $\mathbf{u}_1^T, \mathbf{u}_2^T, \mathbf{n}_1, \mathbf{n}_2$ are increasing tuples with positive coefficients. In particular, we have

$$s_{\mathbf{n}_1}(\mathbf{u}_1)s_{\mathbf{n}_2}(\mathbf{u}_2) \geqslant s_{\mathbf{n}_1}(\mathbf{u}_2)s_{\mathbf{n}_2}(\mathbf{u}_1)$$

whenever $\mathbf{u}_1^T, \mathbf{u}_2^T, \mathbf{n}_1, \mathbf{n}_2$ are increasing tuples with positive coefficients with $\mathbf{n}_1 \geqslant \mathbf{n}_2$ and $\mathbf{u}_1 \geqslant \mathbf{u}_2 \geqslant \mathbf{0}$ in the product order. Equivalently, the ratio

$$s_{\mathbf{n}_1}(\mathbf{u})/s_{\mathbf{n}_2}(\mathbf{u}) := \det \mathbf{u}^{\circ\mathbf{n}_1} / \det \mathbf{u}^{\circ\mathbf{n}_2}$$

is non-decreasing in the vector $\mathbf{u}$ coordinatewise (where all tuples are restricted to be positive and increasing). Note that this argument does not require $\mathbf{n}_1, \mathbf{n}_2$ to have integer coefficients.

If one uses the Jacobi–Trudi identity, one similarly concludes the pointwise inequality

$$\tilde{s}_{\lambda_1 \vee \lambda_2/\mu_1 \vee \mu_2} \tilde{s}_{\lambda_1 \wedge \lambda_2/\mu_1 \wedge \mu_2} \geqslant \tilde{s}_{\lambda_1/\mu_1} \tilde{s}_{\lambda_2/\mu_2} \tag{12.7}$$

between products of skew-Schur polynomials, whenever $\lambda_1, \lambda_2, \mu_1, \mu_2$ are non-increasing tuples of natural or even real numbers, with non-negative coordinates. (We remark here that these 'continuous' versions of Schur and skew-Schur polynomials can be defined for all non-negative real tuples of powers, in a manner generalizing (usual) Schur and skew-Schur polynomials for integer powers. See `https://terrytao.wordpress.com/2017/09/05/` for more details.). This is a weaker form of the Schur-positivity of (8.9) by Lam, Postnikov, and Pylyavskyy [28] that was previously used to prove Theorem 8.6. We also remark that in the special case of (12.7) when $\mu_1 = \mu_2 = \mathbf{0}$, this inequality was also established by Richards [33], using essentially the same techniques as those given here.

Remark 12.4. The above discussion shows that the positivity in (12.7) implies part of the 'extended' Cuttler–Greene–Skandera conjecture (10.4) as a special case, via Proposition 8.1 (as also the positivity but not monomial- or Schur-positivity in the inequality (8.9) by Lam–Postnikov–Pylyavskyy). It would be interesting to explore if the result (12.7) – and the techniques in this paper – can be used to prove other positivity inequalities involving symmetric functions.

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List of notation and symbols

For the convenience of the reader, a tabulation of the symbols used in this paper is provided. The following symbols involve matrices and operations on them. In what follows, $N > 0$ is a fixed integer.

- $\mathbb{P}_N(I)$: Given a subset $I \subset \mathbb{C}$, the $\mathbb{P}_N(I)$ is the set of positive semidefinite $N \times N$ matrices with entries in a subset $I \subset \mathbb{C}$.
- $\mathbb{P}_N^1(I)$ is the subset of $\mathbb{P}_N(I)$ consisting of all rank-one matrices.
- $HTN_N(I)$ is the subset of $\mathbb{P}_N(I)$ comprising all Hankel totally nonnegative matrices.
- $\mathbf{1}_{N \times N}$ is the $N \times N$ matrix with each entry equal to 1.
- $f[A]$ is the result of applying f to each entry of the matrix A.

- $A^{\circ\alpha}$ is the entrywise αth power of the matrix A (usually with positive entries if α is not an integer).
- $A^\dagger$ is the Moore–Penrose (pseudo) inverse of the matrix A.
- $\varrho(A)$ is the spectral radius of the matrix A.
- $\succeq$: Given matrices A, B, we say $A \preceq B$, or $B \succeq A$, if $B - A$ is positive semidefinite.

The second set of symbols concerns vectors, tuples, and operations on them.

- $\mathbf{u}^{\mathbf{n}}, \mathbf{u}^{\circ n}, \mathbf{u}^{\circ \mathbf{n}}$: Given a vector $\mathbf{u} = (u_1, \dots, u_N)^T$ and a tuple $\mathbf{n} = (n_0, \dots, n_{N-1})$, with either complex $\mathbf{u}$ and integer $\mathbf{n}$, or positive $\mathbf{u}$ and real $\mathbf{n}$, we define

$$\mathbf{u}^{\mathbf{n}} := u_1^{n_0} \cdots u_N^{n_{N-1}}, \quad \mathbf{u}^{\circ n_0} := (u_1^{n_0}, \dots, u_N^{n_0})^T, \quad \mathbf{u}^{\circ \mathbf{n}} := (\mathbf{u}^{\circ n_0} | \dots | \mathbf{u}^{\circ n_{N-1}}).$$

- $V(\mathbf{u})$ for a vector $\mathbf{u} = (u_1, \dots, u_N)^T$ or a tuple $(u_1, \dots, u_N)$ is the Vandermonde determinant $V(\mathbf{u}) = V(u_1, \dots, u_N) = \prod_{1 \leqslant i < j \leqslant N} (u_j - u_i)$.
- $\mathbf{u}(\epsilon)$ for a scalar ϵ denotes the vector $(1, \epsilon, \dots, \epsilon^{N-1})^T$.
- $\mathbf{n}_{\min}$ is the integer tuple $(0, 1, \dots, N-1)$.
- $s_{\mathbf{n}}(\mathbf{u})$: Given fixed integers $N, m > 0$, an N-tuple of integers $0 \leqslant n_0 < n_1 < \dots < n_{N-1}$, and a vector $\mathbf{u} = (u_1, \dots, u_m)^T$ or a tuple $(u_1, \dots, u_m)$, denote by $s_{\mathbf{n}}(\mathbf{u}) = s_{\mathbf{n}}(u_1, \dots, u_m)$ the corresponding Schur polynomial – see equation (2.1). (In this paper, unless otherwise specified we set $m = N$.)
- $\overline{\mathbf{u}}$: Given a tuple $\mathbf{u} = (u_1, \dots, u_N)$, define $\overline{\mathbf{u}} := (u_N, \dots, u_1)$.
- $\tilde{s}_\lambda(\mathbf{u})$: Given a partition, i.e. a non-increasing tuple of integers $\lambda_{N-1} \geqslant \dots \geqslant \lambda_0 \geqslant 0$, the corresponding Schur polynomial is

$$\tilde{s}_\lambda(u_1, \dots, u_m) = s_{\overline{\lambda} + \mathbf{n}_{\min}}(u_1, \dots, u_m), \qquad \text{where } \lambda := (\lambda_{N-1}, \dots, \lambda_0).$$

- $\tilde{s}_{\lambda/\mu}(\mathbf{u})$: Given partitions λ, μ, the corresponding skew-Schur polynomial is denoted by $\tilde{s}_{\lambda/\mu}(\mathbf{u})$ if λ contains μ (i.e., $\lambda_j \geqslant \mu_j \ \forall j$), and zero otherwise.
- $\lambda \wedge \mu, \lambda \vee \mu$: Given N-tuples of real numbers $\lambda = (\lambda_0 \dots, \lambda_{N-1})$ and $\mu = (\mu_0, \dots, \mu_{N-1})$, define their meet and join to be, respectively,

$$\lambda \wedge \mu := (\min(\lambda_0, \mu_0), \dots, \min(\lambda_{N-1}, \mu_{N-1})),$$
$$\lambda \vee \mu := (\max(\lambda_0, \mu_0), \dots, \max(\lambda_{N-1}, \mu_{N-1})).$$

- $S^N_{\neq}, S^N_{<}$: Given a set S, denote by $S^N_{\neq}$ the collection of all ordered N-tuples in S with pairwise distinct entries. If moreover $S \subset \mathbb{R}$, then denote by $S^N_{<}$ the subset of $S^N_{\neq}$ with strictly increasing entries.
- $\succ_w$: Given real N-tuples $\mathbf{u} = (u_1, \dots, u_N)$ and $\mathbf{v} = (v_1, \dots, v_N)$, let $u_{[1]} \geqslant \dots \geqslant u_{[N]}$ and $v_{[1]} \geqslant \dots \geqslant v_{[N]}$ denote their decreasing rearrangements. Now $\mathbf{u}$ weakly majorizes $\mathbf{v}$ – denoted $\mathbf{u} \succ_w \mathbf{v}$ – if for all $k = 1, \dots, N$, one has: $u_{[1]} + \dots + u_{[k]} \geqslant v_{[1]} + \dots + v_{[k]}$. See equation (10.1).

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