ON THE FAILURE OF GORENSTEINNESS AT WEIGHT 1 EISENSTEIN POINTS OF THE EIGENCURVE

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Abstract. We prove that the cuspidal eigencurve $\mathscr{C}_{\mathrm{cusp}}$ is étale over the weight space at any classical weight 1 Eisenstein point f and meets two Eisenstein components of the eigencurve $\mathscr{C}$ transversally at f. Further, we prove that the local ring of $\mathscr{C}$ at f is Cohen–Macaulay but not Gorenstein and compute the Fourier coefficients of a basis of overconvergent weight 1 modular forms lying in the same generalised eigenspace as f. In addition, we prove an $R = T$ theorem for the local ring at f of the closed subspace of $\mathscr{C}$ given by the union of $\mathscr{C}_{\mathrm{cusp}}$ and one Eisenstein component and prove unconditionally, via a geometric construction of the residue map, that the corresponding congruence ideal is generated by the Kubota–Leopoldt p-adic L-function. Finally we obtain a new proof of the Ferrero–Greenberg Theorem and Gross' formula for the derivative of the p-adic L-function at the trivial zero.

Contents

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Introduction

Let p be any prime number and let N be a positive integer relatively prime to p. Let $\mathscr{C}$ be the reduced eigencurve of tame level N endowed with a flat and locally finite morphism κ to the weight space $\mathcal{W}$ (see §3.1). The irreducible components of $\mathscr{C}$ correspond either to Eisenstein or to cuspidal p-adic families of modular forms, and the study of how these meet plays a prominent role in the work [36] of Mazur and Wiles on the Iwasawa Main Conjecture for GL_1 over $\mathbb{Q}$. Any p-stabilisation of a newform of level N has finite slope, thus defines a point in $\mathscr{C}$ (see [1, §1.3]). The point is called *irregular* if the p-stabilisation is unique, and *regular* otherwise. The main theorem in [4] describes the geometry of $\mathscr{C}$ at all regular cuspidal weight 1 points ($\mathscr{C}$ is smooth at such points and there is a precise condition for κ to be étale) and one expects the geometry at irregular cuspidal points to be more involved, because κ is never étale at such points and there are even examples where $\mathscr{C}$ is not smooth (see [21] and [6]).

In this paper we study the geometry of $\mathscr{C}$ at classical weight 1 Eisenstein points. Fix a primitive odd Dirichlet character ϕ of conductor N and consider the Eisenstein series

$$E_1(\mathbf{1},\phi)(z) = \frac{L(\phi,0)}{2} + \sum_{n\geqslant 1} q^n \sum_{d|n} \phi(d), \text{ where } q = e^{2i\pi z}. \tag{1}$$

It is a newform of level N admitting the p-stabilisations

$$E_1(\mathbf{1},\phi)(z) - \phi(p)E_1(\mathbf{1},\phi)(pz) \qquad \text{and} \qquad E_1(\mathbf{1},\phi)(z) - E_1(\mathbf{1},\phi)(pz),$$

of U_p-eigenvalues 1 and $\phi(p)$, respectively, belonging to the Eisenstein components $\mathcal{E}_{\mathbf{1},\phi}$ and $\mathcal{E}_{\phi,\mathbf{1}}$. In particular, these two components intersect at weight 1 if and only if $\phi(p) = 1$, *i.e.* when $E_1(\mathbf{1},\phi)$ is irregular. If $\phi(p) \neq 1$, the constant term of each one of these p-stabilisations is non-zero at some cusps in the multiplicative part of the ordinary locus of the modular curve $X_{\mathrm{Iw}}^{\mathrm{rig}}$ of tame level N and Iwahori level at p (see §3.1), hence such forms are not cuspidal-overconvergent and belong to a unique Eisenstein component. We thus restrict our attention to an Eisenstein series which is irregular at p and denote by f its unique p-stabilisation. In addition to belonging to the Eisenstein components $\mathcal{E}_{\mathbf{1},\phi}$ and $\mathcal{E}_{\phi,\mathbf{1}}$, the corresponding point also belongs to the cuspidal locus $\mathscr{C}_{\mathrm{cusp}}$ of $\mathscr{C}$, because f vanishes at all cusps of the multiplicative part of the ordinary locus of $X_{\mathrm{Iw}}^{\mathrm{rig}}$ (see Proposition 4.7).

Denote by $\Lambda = \bar{\mathbb{Q}}_p[\![X]\!]$ (resp. $\mathcal{T}$) the completed strict local ring of $\mathcal{W}$ at $\kappa(f)$ (resp. of $\mathscr{C}$ at f). The weight map κ induces a finite, flat map $\Lambda \to \mathcal{T}$ and a surjection $\Lambda[T_\ell, U_p]_{\ell\nmid Np} \twoheadrightarrow \mathcal{T}$.

Theorem A. *(i) The cuspidal eigencurve $\mathscr{C}_{\mathrm{cusp}}$ is étale over $\mathcal{W}$ at f. In particular there exists a unique cuspidal irreducible component $\mathcal{F}$ of $\mathscr{C}$ containing f.*

(ii) The Λ-algebra $\mathcal{T}$ is Cohen–Macaulay but not Gorenstein, and is in fact isomorphic to

$$\Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda = \left\{(a,b,c) \in \Lambda^3 \mid a(0) = b(0) = c(0)\right\}$$

endowed with Λ-algebra structure via the diagonal embedding.

(iii) The image $\mathcal{T}'$ of $\Lambda[T_\ell]_{\ell \nmid Np}$ in $\mathcal{T}$ is a complete intersection, and is isomorphic to

$$\left\{(a,b,c) \in \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \,\middle|\, \left(\mathscr{L}(\phi^{-1}) + \mathscr{L}(\phi)\right) a'(0) = \mathscr{L}(\phi^{-1}) b'(0) + \mathscr{L}(\phi) c'(0)\right\},$$

where $\mathscr{L}(\phi)$ is the $\mathscr{L}$-invariant of ϕ defined in (15).

The non-smoothness of $\mathscr{C}$ at f is related to the vanishing at $s = 0$ of the Kubota–Leopoldt p-adic L-function $L_p(\phi\omega_p, s)$, ω_p being the Teichmüller character. Using the relation between the element $\zeta_\phi(X) \in \Lambda$ defined in (39) and the constant term of $\mathcal{E}_{\mathbf{1},\phi}$, Darmon, Dasgupta and Pollack constructed a first order cuspidal deformation of f which played a pivotal role in their work [18] on the Gross–Stark conjecture over totally real number fields. Their strategy consists in taking a suitable combination of Eisenstein series and, via the action of certain Hecke operators, producing a cuspidal Hida family that, while not being an eigenform itself, yields a first order eigenform. Theorem A, and its expected generalisation to totally real number fields where p is inert (see [7]), gives a more precise result as it implies the uniqueness of the cuspidal deformation at first and in fact at any order.

Remark 0.1. When ϕ is quadratic then $\mathcal{F}$ admits a familiar description as a Hida family interpolating theta series for the imaginary quadratic field fixed by ϕ. Theorem A(i) then implies that the characteristic power series of the congruence ideal attached to $\mathcal{F}$ (see [29, (6.9)]) does not vanish at $X = 0$, as conjectured by Hida and Tilouine [29, p. 192]. This is used in a forthcoming work of Burungale, Skinner and Tian [9] on a conjecture of Perrin–Riou regarding the local non-triviality at p of p-adic Beilinson-Kato elements attached to elliptic curves over $\mathbb{Q}$. Theorem A(i) is also used in [44] to show that the source of the map ϖ in Sharifi's Conjecture [43, Proposition 5.7] (see also [44, (2.3)]) is one-dimensional after localising at a prime ideal of the Iwasawa algebra corresponding to a trivial zero of the Kubota–Leopoldt p-adic L-function.

Our approach to the geometry of the eigencurve is Galois theoretic, and consists in proving a modularity theorem for the ordinary deformations of the Galois representation attached to f. However, the reducibility of the latter causes several issues. First, since the Artin representation $\mathbf{1} \oplus \phi$ attached to f is decomposable, its deformation functor is not representable in the sense of Mazur [35]. Moreover, the irregularity assumption implies that $\mathbf{1} \oplus \phi$ is trivial on the decomposition group $\mathrm{G}_{\mathbb{Q}_p}$, creating an obstacle to imposing an ordinary deformation condition at p. In order to circumvent these difficulties we introduce two reducible indecomposable representations

$$\rho = \begin{pmatrix} \phi & \eta \\ 0 & \mathbf{1} \end{pmatrix} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p) \text{ and } \rho' = \begin{pmatrix} \mathbf{1} & \phi\eta' \\ 0 & \phi \end{pmatrix} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p),$$

where $[\eta]$ and $[\eta']$ are bases of the lines $\mathrm{H}^1(\mathbb{Q},\phi) \simeq \mathrm{Ext}^1_{\mathbb{Q}}(\mathbf{1},\phi)$ and $\mathrm{H}^1(\mathbb{Q},\phi^{-1}) \simeq \mathrm{Ext}^1_{\mathbb{Q}}(\phi,\mathbf{1})$. The representations ρ and ρ' are, up to isomorphism, the unique non-decomposable extensions and admit universal ordinary deformation rings $\mathcal{R}^{\mathrm{ord}}_{\rho}$ and $\mathcal{R}^{\mathrm{ord}}_{\rho'}$, respectively. Finally we introduce a universal ring $\mathcal{R}_{\mathrm{cusp}}$ classifying pairs of ordinary deformations of (ρ,ρ') sharing the same traces and Frobenius action on the unramified $\mathrm{G}_{\mathbb{Q}_p}$-quotient. A substantial part of this paper is devoted to the computation of their tangent spaces relying on Baker–Brumer's Theorem [8] in Transcendence Theory. The vanishing of the relative tangent space of $\mathcal{R}_{\mathrm{cusp}}$ allows us to deduce as in [4], that $\mathcal{R}_{\mathrm{cusp}}$ is unramified over Λ and isomorphic to the completed strict local ring $\mathcal{T}_{\mathrm{cusp}}$ of $\mathscr{C}_{\mathrm{cusp}}$ at f. Whereas the $\mathcal{T}$-valued two dimensional pseudo-character of $\mathrm{G}_{\mathbb{Q}}$ does not arise from an ordinary representation, because of the non-Gorensteinness of $\mathcal{T}$, we manage to prove a modularity result for $\mathcal{R}^{\mathrm{ord}}_{\rho}$, and for $\mathcal{R}^{\mathrm{ord}}_{\rho'}$ (see §4.4). Denote by $\mathcal{T}^{\mathrm{ord}}_{\rho}$ the completed strict local ring at f of the reduced equidimensional closed analytic subspace of $\mathscr{C}$ given by the union of $\mathscr{C}_{\mathrm{cusp}}$ and the component of $\mathscr{C}$ corresponding to $\mathcal{E}_{\mathbf{1},\phi}$ (see §3.1).

Theorem B. *The Λ-algebra $\mathcal{R}^{\mathrm{ord}}_{\rho}$ is a complete intersection isomorphic to $\mathcal{T}^{\mathrm{ord}}_{\rho}$.*

The Gorenstein property plays a prominent role in the theory of Hecke algebras, as it guarantees the freeness of the module of modular forms over those algebras. Theorem A thus provides a testing ground for challenging questions in Iwasawa Theory, such as the construction of p-adic L-functions in a neighbourhood of a non-Gorenstein point of $\mathscr{C}$, and a formulation of a Main Conjecture at such a point. It further suggests that this remarkable phenomenon is related to the action of the U_p operator, as the p-deprived Hecke algebra $\mathcal{T}'$ is Gorenstein and even a complete intersection (see Corollary 5.4).

Let $J_{\mathrm{eis}} \subset \mathcal{T}_{\mathrm{cusp}}$ be the Eisenstein ideal associated to $\mathcal{E}_{\mathbf{1},\phi}$ (see §.4.1).

Theorem C. *There exists an isomorphism of Λ-algebras $\mathcal{T}_{\mathrm{cusp}}/J_{\mathrm{eis}} \xrightarrow{\sim} \Lambda/(\zeta_\phi(X))$.*

In the absence of a trivial zero (*i.e.* if $\phi(p) \neq 1$), this is a well known result of Mazur-Wiles and Wiles [49, Theorem 4.1] (see also Ohta [40] and Emerton [23] when $p \geqslant 5$). Our proof uses a geometric residue map from the space of Hida families onto the ordinary cuspidal group whose kernel consists of cuspidal Hida families. Its definition taps into Pilloni's geometric constructions [42] of p-adic families of modular forms, hence differs from Ohta's residue map. The surjectivity of the residue map is deduced from the fact that the ordinary locus of the modular curve is an affinoid (see Proposition 3.1), while the Kubota–Leopoldt p-adic L-function appears in the constant terms of the Eisenstein family $\mathcal{E}_{\mathbf{1},\phi}$ (see Proposition 4.7).

Combining Theorems A and C yields a new proof of the famous result of Ferrero–Greenberg [24] and Gross–Koblitz on the non-vanishing of $L_p'(\phi\omega_p, 0)$ (see Proposition 4.8).

The failure of étaleness for the eigencurve at f, combined with the perfectness of the duality between $\mathcal{T}$ and the Λ-module of ordinary families specialising to f

established in §5.2, implies the existence of non-classical forms in the generalised eigenspace associated to f. Their q-expansions admit an explicit description in terms of p-adic logarithms of rational numbers and of p-units of the splitting field of ϕ; this contrasts with the cuspidal regular setting in the work of Darmon, Lauder and Rotger [15] where only ℓ-units for $\ell \neq p$ are involved.

Theorem D. *Let $M^\dagger[\![f]\!]$ (resp. $S^\dagger[\![f]\!]$) be the generalised eigenspace attached to f inside the space of weight 1 overconvergent modular forms (resp. cuspforms) of tame level N and central character ϕ. Then $S^\dagger[\![f]\!] = \bar{\mathbb{Q}}_p f$, while a complement of $S^\dagger[\![f]\!]$ in $M^\dagger[\![f]\!]$ is spanned by*

$$f^\dagger_{\phi,\mathbf{1}} = \sum_{n\geqslant 1} q^n \sum_{d|n,\, p\nmid d} \phi(d)\left(\mathrm{ord}_p(n)\mathscr{L}(\phi) + \log_p\left(\tfrac{d^2}{n}\right)\right) \text{ and}$$

$$f^\dagger_{\mathbf{1},\phi} = (\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\frac{L(\phi,0)}{2} + \sum_{n\geqslant 1} q^n \sum_{d|n,\, p\nmid d} \phi(d)\left(\mathrm{ord}_p(n)\mathscr{L}(\phi^{-1}) - \log_p\left(\tfrac{d^2}{n}\right)\right),$$

where ord_p is the p-adic valuation and $\log_p$ is the p-adic logarithm normalised by $\log_p(p) = 0$.

Determining the coefficients of the classical (non p-stabilised) form $E_1(\mathbf{1},\phi) \in M^\dagger[\![f]\!]$ in the basis $\{f, f^\dagger_{\phi,\mathbf{1}}, f^\dagger_{\mathbf{1},\phi}\}$ yields a new proof (see Corollary 5.9) of Gross' formula [26] for the derivative at a trivial zero:

$$L'_p(\phi\omega_p, 0) = \mathscr{L}(\phi)L(\phi,0).$$

For "tame" analogues of these results the interested reader is referred to Remark 5.10 where we illustrate a rather striking analogy with a phenomenon arising in Mazur's Eisenstein ideal setting [34], as well as to the recent work of P. Wake [45] where such phenomena are related to the notion of "extra reducibility".

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1. Ordinary Galois deformations

For a perfect field L we denote $\mathrm{G}_L = \mathrm{Gal}(\bar{L}/L)$ its absolute Galois group. Given a prime number ℓ, we fix an embedding $\iota_\ell : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_\ell$ which determines an embedding $\mathrm{G}_{\mathbb{Q}_\ell} \hookrightarrow \mathrm{G}_{\mathbb{Q}}$, and denote by $\mathrm{I}_{\mathbb{Q}_\ell}$ the inertia subgroup at ℓ and by $\mathrm{Frob}_\ell \in \mathrm{G}_{\mathbb{Q}_\ell}$ an arithmetic Frobenius. We also fix an embedding $\iota_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ which determines a complex conjugation $\tau \in \mathrm{G}_{\mathbb{Q}}$. All Galois representations are with coefficients in $\bar{\mathbb{Q}}_p$ unless stated otherwise.

1.1. A canonical reducible non-split Galois representation attached to ϕ.

Consider the unique element $\eta_{\mathbf{1}} \in \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$ whose restriction to the image of $\mathrm{H}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$ in $\mathrm{H}^1(\mathrm{I}_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p)$ corresponds via Local Class Field Theory to the p-adic logarithm $\log_p$. By Global Class Field Theory one has $\eta_{\mathbf{1}}(\mathrm{Frob}_\ell) = -\log_p(\ell)$ for all primes $\ell \neq p$. As ϕ is odd, one has $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}, \phi) = 1$ (see for example [4, (8)]). Noting that the restriction map $\mathrm{H}^1(\mathbb{Q},\phi) \to \mathrm{H}^1(\mathrm{I}_{\mathbb{Q}_p},\bar{\mathbb{Q}}_p)$ is injective, we let $[\eta] \in \mathrm{H}^1(\mathbb{Q},\phi)$ be the unique element mapping to $\log_p$.

Given any cocycle $\eta'' : \mathrm{G}_{\mathbb{Q}} \to \phi$ representing a non-trivial element $[\eta''] \in \mathrm{H}^1(\mathbb{Q},\phi)$, the $\mathrm{G}_{\mathbb{Q}}$-representations $\left(\begin{smallmatrix}\phi & \eta \\ 0 & \mathbf{1}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\phi & \eta'' \\ 0 & \mathbf{1}\end{smallmatrix}\right)$ are always conjugated by an upper-triangular matrix which is unipotent if and only if $[\eta] = [\eta'']$. Since $\phi - \mathbf{1}$ is a basis of the coboundaries and $\phi_{|\mathrm{G}_{\mathbb{Q}_p}} = \mathbf{1}$, the restriction $\eta_{|\mathrm{G}_{\mathbb{Q}_p}}$ only depends on $[\eta]$. Moreover, as $\phi(\tau) = -1$, there exists a unique cocycle η representing the cohomology class $[\eta]$ such that $\eta(\tau) = 0$. We let $\rho = \left(\begin{smallmatrix}\phi & \eta \\ 0 & \mathbf{1}\end{smallmatrix}\right) : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$.

1.2. **Ordinary deformations of ρ.** Let $\mathscr{A}$ be the category of Artinian local $\bar{\mathbb{Q}}_p$-algebras A with maximal ideal $\mathfrak{m}_A$ and residue field $\bar{\mathbb{Q}}_p$, where the morphisms are local homomorphisms of $\bar{\mathbb{Q}}_p$-algebras inducing identity on the residue field (note that $A/\mathfrak{m}_A = \bar{\mathbb{Q}}_p$ canonically as $\bar{\mathbb{Q}}_p$-algebras).

Consider the functor $\mathcal{D}_\rho^{\mathrm{univ}}$ assigning to $A \in \mathscr{A}$ the set of lifts $\rho_A : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ of ρ modulo strict equivalence (*i.e.* conjugation by an element of $1 + \mathrm{M}_2(\mathfrak{m}_A)$). Since ϕ and $[\eta]$ are both non-trivial, the centraliser of the image of ρ consists only of scalar matrices, hence $\mathcal{D}_\rho^{\mathrm{univ}}$ is pro-representable by a complete Noetherian local $\bar{\mathbb{Q}}_p$-algebra $\mathcal{R}_\rho^{\mathrm{univ}}$, called the universal deformation ring (see [35]).

Denote by $V_\rho = \bar{\mathbb{Q}}_p^2$ the representation space of ρ endowed with its canonical basis (e_1, e_2). There exists a unique $\mathrm{G}_{\mathbb{Q}_p}$-stable (in fact, it is $\mathrm{G}_{\mathbb{Q}}$-stable) filtration with unramified quotient

$$0 \to F_\rho = \bar{\mathbb{Q}}_p e_1 \to V_\rho \to V_\rho/F_\rho \to 0. \tag{2}$$

Definition 1.1. The functor $\mathcal{D}_\rho^{\mathrm{ord}}$ assigns to $A \in \mathscr{A}$ the set of tuples (ρ_A, F_A), where

- (i) $\rho_A : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ is a continuous representation, such that $\rho_A \mod \mathfrak{m}_A = \rho$, and
- (ii) $F_A \subset A^2$ is a free direct factor over A of rank 1 which is $\mathrm{G}_{\mathbb{Q}_p}$-stable and such that $\mathrm{G}_{\mathbb{Q}_p}$ acts on A^2/F_A by an unramified character, denoted χ_A,

modulo the strict equivalence relation $[(\rho_A, F_A)] = [(P\rho_A P^{-1}, P \cdot F_A)]$ for $P \in 1 + \mathrm{M}_2(\mathfrak{m}_A)$.

As the restriction of $[\eta]$ to $\mathrm{I}_{\mathbb{Q}_p}$ is non-trivial, it follows from Schlessinger's criterion (see [25, Corollary 6.6]) that $\mathcal{D}_\rho^{\mathrm{ord}}$ is pro-representable by a quotient of $\mathcal{R}_\rho^{\mathrm{univ}}$. We will now provide an explicit description of the ideal defining that quotient.

Choose a lift $\rho_{\mathrm{univ}} = \left(\begin{smallmatrix}a & b \\ c & d\end{smallmatrix}\right) : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{R}_\rho^{\mathrm{univ}})$ representing the universal deformation of ρ and define $\mathcal{R}_\rho^{\mathrm{ord}}$ as the quotient of $\mathcal{R}_\rho^{\mathrm{univ}}[\![Y]\!]$ by the ideal

$$\left(d(h) - 1 - b(h)Y, c(g) + (d(g) - a(g))Y - b(g)Y^2; h \in \mathrm{I}_{\mathbb{Q}_p}, g \in \mathrm{G}_{\mathbb{Q}_p}\right).$$

Choosing any $h_0 \in \mathrm{I}_{\mathbb{Q}_p}$ such that $\eta(h_0) \neq 0$ (so that $b(h_0) \in (\mathcal{R}_\rho^{\mathrm{univ}})^\times$), the linear relation $d(h_0) - 1 - b(h_0)Y = 0$ shows that the natural composed map $\mathcal{R}_\rho^{\mathrm{univ}} \to \mathcal{R}_\rho^{\mathrm{ord}}$ is surjective. As

$$\begin{pmatrix}1 & 0 \\ -Y & 1\end{pmatrix}\begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{pmatrix}1 & 0 \\ Y & 1\end{pmatrix} = \begin{pmatrix}a + bY & b \\ c + (d-a)Y - bY^2 & d - bY\end{pmatrix}, \tag{3}$$

the push-forward of ρ_{univ} along that surjection, together with the line having basis $e_1 + Ye_2$, yield a point of $\mathcal{D}_\rho^{\mathrm{ord}}(\mathcal{R}_\rho^{\mathrm{ord}})$. Conversely, any point of $\mathcal{D}_\rho^{\mathrm{ord}}(A)$ is represented by a push-forward ρ_A of ρ_{univ} along a (unique) homomorphism $\varphi_A : \mathcal{R}_\rho^{\mathrm{univ}} \to A$, and an ordinary line $F_A \subset A^2$. The latter has a basis $e_1 + ye_2$ with $y \in \mathfrak{m}_A$, because $F_A \otimes_A \bar{\mathbb{Q}}_p = F_\rho$. Let $\widetilde{\varphi}_A : \mathcal{R}_\rho^{\mathrm{univ}}[\![Y]\!] \to A$ be the homomorphism extending φ_A and sending Y to y. It follows from (3) that $\widetilde{\varphi}_A$ factors through $\mathcal{R}_\rho^{\mathrm{ord}}$, hence φ_A factors through $\mathcal{R}_\rho^{\mathrm{ord}}$ as well. Therefore $\mathcal{R}_\rho^{\mathrm{ord}}$ represents $\mathcal{D}_\rho^{\mathrm{ord}}$, in particular the kernel of the natural surjection $\mathcal{R}_\rho^{\mathrm{univ}} \to \mathcal{R}_\rho^{\mathrm{ord}}$ is independent of the particular choice of ρ_{univ}. It follows that any tuple (ρ_A, F_A) in $\mathcal{D}_\rho^{\mathrm{ord}}(A)$ is characterised by ρ_A alone, *i.e.* when the ordinary filtration of ρ_A exists, then it is unique. For this reason, and as the unramified character χ_A plays an important role, we will sometimes denote a point in $\mathcal{D}_\rho^{\mathrm{ord}}(A)$ by (ρ_A, χ_A).

One can define the nearly-ordinary deformation functor $\mathcal{D}_\rho^{\mathrm{n.ord}}$ by using the same definition as for $\mathcal{D}_\rho^{\mathrm{ord}}$, but without imposing the $\mathrm{G}_{\mathbb{Q}_p}$-quotient to be unramified. An argument similar to the one presented above shows that $\mathcal{D}_\rho^{\mathrm{n.ord}}$ is pro-representable by a quotient $\mathcal{R}_\rho^{\mathrm{n.ord}}$ of $\mathcal{R}_\rho^{\mathrm{univ}}[\![Y]\!]$, and $\mathcal{R}_\rho^{\mathrm{n.ord}}$ is generated over $\mathrm{Im}(\mathcal{R}_\rho^{\mathrm{univ}} \to \mathcal{R}_\rho^{\mathrm{n.ord}})$ by a root of the polynomial $b(h_0)Y^2 + (a(h_0) - d(h_0))Y - c(h_0)$, or equivalently a root of $U^2 - \mathrm{tr}(\rho_{\mathrm{univ}})(h_0)U + \det(\rho_{\mathrm{univ}})(h_0)$. Using Theorem A, one can see that $\mathcal{R}_\rho^{\mathrm{n.ord}}$ is indeed quadratic over $\mathrm{Im}(\mathcal{R}_\rho^{\mathrm{univ}} \to \mathcal{R}_\rho^{\mathrm{n.ord}})$.

Finally let $\mathcal{D}_{\rho,0}^{\mathrm{ord}}$ be the sub-functor of $\mathcal{D}_\rho^{\mathrm{ord}}$ given by the deformations with fixed determinant equal to ϕ. Since Λ is the universal deformation ring of ϕ (see [4, §6]) the natural transformation $\rho_A \mapsto \det(\rho_A)$ endows $\mathcal{R}_\rho^{\mathrm{ord}}$ with a natural Λ-algebra structure and $\mathcal{D}_{\rho,0}^{\mathrm{ord}}$ is pro-representable by $\mathcal{R}_{\rho,0}^{\mathrm{ord}} = \mathcal{R}_\rho^{\mathrm{ord}}/\mathfrak{m}_\Lambda \mathcal{R}_\rho^{\mathrm{ord}}$.

1.3. Reducible deformations of ρ.

Definition 1.2. Let $\mathcal{D}_\rho^{\mathrm{red}}$ be the subfunctor of $\mathcal{D}_\rho^{\mathrm{ord}}$ consisting of $\mathrm{G}_{\mathbb{Q}}$-reducible deformations.

Lemma 1.3. *The functor $\mathcal{D}_\rho^{\mathrm{red}}$ is pro-representable by a quotient $\mathcal{R}_\rho^{\mathrm{red}}$ of $\mathcal{R}_\rho^{\mathrm{ord}}$.*

Proof. Choose a lift $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{R}_\rho^{\mathrm{ord}})$ representing the universal ordinary deformation sending the complex conjugation τ to $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Applying (3) to a $\mathrm{G}_{\mathbb{Q}}$-stable line with basis $e_1 + ye_2$, one finds that $c + (d-a)y - by^2 = 0$. Evaluating at $\tau \in \mathrm{G}_{\mathbb{Q}}$ yields $y = 0$ and shows that

$$\mathcal{R}_\rho^{\mathrm{red}} = \mathcal{R}_\rho^{\mathrm{ord}}/(c(g); g \in \mathrm{G}_{\mathbb{Q}}). \qquad \square$$

While ρ admits a $\mathrm{G}_{\mathbb{Q}}$-quotient which is unramified at p (see (2)), this is not necessarily true for all its ordinary, reducible lifts. To account for this discrepancy, we introduce the following functor.

Definition 1.4. Let $\mathcal{D}_\rho^{\mathrm{eis}}$ be the subfunctor of $\mathcal{D}_\rho^{\mathrm{red}}$ consisting of deformations which are reducible and ordinary for the same filtration, *i.e.* admit a rank 1 $\mathrm{G}_{\mathbb{Q}}$-quotient which is unramified at p.

Lemma 1.5. *The functor $\mathcal{D}_\rho^{\mathrm{eis}}$ is pro-representable by $\mathcal{R}_\rho^{\mathrm{eis}} = \mathcal{R}_\rho^{\mathrm{ord}}/(C(g); g \in \mathrm{G}_{\mathbb{Q}})$, where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{R}_\rho^{\mathrm{ord}})$ is a lift representing the universal ordinary deformation in an ordinary basis.*

Finally, let $\mathcal{D}^{\mathrm{red}}_{\rho,0}$ and $\mathcal{D}^{\mathrm{eis}}_{\rho,0}$ be the sub-functors of $\mathcal{D}^{\mathrm{red}}_{\rho}$ and $\mathcal{D}^{\mathrm{eis}}_{\rho}$, respectively, classifying the deformations having fixed determinant equal to ϕ. By the discussion in §1.2 they are pro-representable by $\mathcal{R}^{\mathrm{red}}_{\rho,0} = \mathcal{R}^{\mathrm{red}}_{\rho}/\mathfrak{m}_\Lambda\mathcal{R}^{\mathrm{red}}_{\rho}$ and $\mathcal{R}^{\mathrm{eis}}_{\rho,0} = \mathcal{R}^{\mathrm{eis}}_{\rho}/\mathfrak{m}_\Lambda\mathcal{R}^{\mathrm{eis}}_{\rho}$, respectively. We will see in §2.1 that $\mathcal{D}^{\mathrm{eis}}_{\rho}$ and $\mathcal{D}^{\mathrm{red}}_{\rho}$ differ.

1.4. **Cuspidal deformations of ρ.** We will exploit the interchangeability of the characters $\mathbf{1}$ and ϕ used in the definition of $\mathcal{D}^{\mathrm{ord}}_{\rho}$ to define another deformation functor denoted $\mathcal{D}_{\mathrm{cusp}}$ (together with its relative version $\mathcal{D}^0_{\mathrm{cusp}}$), and show that it is pro-representable by a universal deformation ring $\mathcal{R}_{\mathrm{cusp}}$. We will later show that $\mathcal{R}_{\mathrm{cusp}}$ is isomorphic to the strict completed local ring of the cuspidal eigencurve at f, thus justifying the notation.

Recall that in §1.1 we fixed a basis $[\eta] \in \mathrm{H}^1(\mathbb{Q},\phi)$ and constructed a representation $\rho = \left(\begin{smallmatrix}\phi & \eta\\ 0 & \mathbf{1}\end{smallmatrix}\right)$. Since $\dim_{\bar{\mathbb{Q}}_p}\mathrm{H}^1(\mathbb{Q},\phi^{-1}) = 1$ we can perform the following analogous construction. Fix a basis $[\eta'] \in \mathrm{H}^1(\mathbb{Q},\phi^{-1})$ such that $[\eta']_{|\mathrm{I}_{\mathbb{Q}_p}}$ corresponds via Local Class Field Theory to the p-adic logarithm and let η' be the unique representative such that $\eta'(\tau) = 0$. Let $\rho' = \left(\begin{smallmatrix}\mathbf{1} & \phi\eta'\\ 0 & \phi\end{smallmatrix}\right) : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ and consider, as in Definition 1.1, the functor $\mathcal{D}^{\mathrm{ord}}_{\rho'}$ which is analogously pro-representable by a universal deformation Λ-algebra $\mathcal{R}^{\mathrm{ord}}_{\rho'}$.

Definition 1.6. Let $\mathcal{D}_{\mathrm{cusp}}$ be the functor assigning to $A \in \mathscr{A}$ the set of equivalence classes of pairs $((\rho_A,\chi_A),(\rho'_A,\chi'_A))$ in $\mathcal{D}^{\mathrm{ord}}_{\rho}(A)\times\mathcal{D}^{\mathrm{ord}}_{\rho'}(A)$ such that

(i) $\mathrm{tr}(\rho_A) = \mathrm{tr}(\rho'_A)$, $\det(\rho_A) = \det(\rho'_A)$, and
(ii) $\chi_A(\mathrm{Frob}_p) = \chi'_A(\mathrm{Frob}_p)$.

Put $\rho_{\mathcal{R}} = \rho_{\mathcal{R}^{\mathrm{ord}}_{\rho}}$, $\chi_{\mathcal{R}} = \chi_{\mathcal{R}^{\mathrm{ord}}_{\rho}}$, and analogously $\rho'_{\mathcal{R}} = \rho_{\mathcal{R}^{\mathrm{ord}}_{\rho'}}$, $\chi'_{\mathcal{R}} = \chi_{\mathcal{R}^{\mathrm{ord}}_{\rho'}}$. By definition the functor $\mathcal{D}_{\mathrm{cusp}}$ is pro-representable by the quotient $\mathcal{R}_{\mathrm{cusp}}$ of $\mathcal{R}^{\mathrm{ord}}_{\rho}\widehat{\otimes}_\Lambda\mathcal{R}^{\mathrm{ord}}_{\rho'}$ by the ideal

$$\left(\mathrm{tr}(\rho_{\mathcal{R}})(g)\otimes 1 - 1\otimes\mathrm{tr}(\rho'_{\mathcal{R}})(g), \chi_{\mathcal{R}}(\mathrm{Frob}_p)\otimes 1 - 1\otimes\chi'_{\mathcal{R}}(\mathrm{Frob}_p); g\in\mathrm{G}_{\mathbb{Q}}\right).$$

Lemma 1.7. *The natural homomorphisms $\mathcal{R}^{\mathrm{ord}}_{\rho} \to \mathcal{R}_{\mathrm{cusp}}$ and $\mathcal{R}^{\mathrm{ord}}_{\rho'} \to \mathcal{R}_{\mathrm{cusp}}$ are surjective.*

Proof. Let $\mathcal{R}^{\mathrm{ps}}$ be the universal ring pro-representing deformations of the pseudo-character $\phi+\mathbf{1}$ (see [30, Lemma 1.4.2]). Since $\mathrm{tr}(\rho_{\mathrm{univ}})$ is a pseudo-character lifting $\phi+\mathbf{1}$, the universal property gives a homomorphism $\mathcal{R}^{\mathrm{ps}} \to \mathcal{R}^{\mathrm{univ}}_{\rho}$ which is surjective by [30, Corollary 1.4.4(ii)]. Composing with the surjection $\mathcal{R}^{\mathrm{univ}}_{\rho} \twoheadrightarrow \mathcal{R}^{\mathrm{ord}}_{\rho}$ yields a natural surjection $\mathcal{R}^{\mathrm{ps}} \twoheadrightarrow \mathcal{R}^{\mathrm{ord}}_{\rho}$. It follows that $\mathfrak{m}_{\mathcal{R}^{\mathrm{ord}}_{\rho}}$ is generated by the values of $\mathrm{tr}(\rho_{\mathcal{R}})-\phi-\mathbf{1}$ and similarly for $\mathfrak{m}_{\mathcal{R}^{\mathrm{ord}}_{\rho'}}$. Using this and the fact that $(\mathrm{tr}(\rho_{\mathcal{R}})-\phi-\mathbf{1})\otimes 1$ and $1\otimes(\mathrm{tr}(\rho'_{\mathcal{R}})-\phi-\mathbf{1})$ have the same image under $\mathcal{R}^{\mathrm{ord}}_{\rho}\widehat{\otimes}_\Lambda\mathcal{R}^{\mathrm{ord}}_{\rho'} \twoheadrightarrow \mathcal{R}_{\mathrm{cusp}}$, one can show that $\mathfrak{m}_{\mathcal{R}^{\mathrm{ord}}_{\rho}}\mathcal{R}_{\mathrm{cusp}} = \mathfrak{m}_{\mathcal{R}_{\mathrm{cusp}}}$. In other terms, the natural homomorphisms $\mathcal{R}^{\mathrm{ord}}_{\rho} \to \mathcal{R}_{\mathrm{cusp}}$ and $\mathcal{R}^{\mathrm{ord}}_{\rho'} \to \mathcal{R}_{\mathrm{cusp}}$ are unramified morphisms of complete local Noetherian rings having the same residue field, hence they are surjective. □

Let $\mathcal{D}^0_{\mathrm{cusp}}$ be the subfunctor of $\mathcal{D}_{\mathrm{cusp}}$ consisting of deformations with fixed determinant equal to ϕ. It is pro-representable by $\mathcal{R}^0_{\mathrm{cusp}} = \mathcal{R}_{\mathrm{cusp}}/\mathfrak{m}_\Lambda\mathcal{R}_{\mathrm{cusp}}$.

2. Tangent spaces

In this section we interpret the tangent spaces of the functors introduced in §1 using Galois cohomology and compute their dimensions. We will discover that all infinitesimal reducible deformations of ρ are necessarily ordinary.

2.1. **Tangent spaces for nearly ordinary deformations.** Let $\bar{\mathbb{Q}}_p[\epsilon]$ denote the $\bar{\mathbb{Q}}_p$-algebra of dual numbers. Recall that there is a natural isomorphism:

(4)
$$\mathrm{H}^1(\mathbb{Q},\mathrm{ad}(\rho)) \xrightarrow{\sim} \mathcal{D}_\rho^{\mathrm{univ}}(\bar{\mathbb{Q}}_p[\epsilon]) = t_\rho^{\mathrm{univ}},\ [\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)] \mapsto [\rho_\epsilon],\ \text{where } \rho_\epsilon = (1+\epsilon\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right))\rho,$$

identifying $\mathrm{H}^1(\mathbb{Q},\mathrm{ad}^0(\rho))$ with $t_{\rho,0}^{\mathrm{univ}}$, where $\mathrm{ad}(\rho)$ (resp. $\mathrm{ad}^0(\rho)$) is the adjoint representation of ρ (resp. the sub-representation on trace 0 elements $\mathrm{End}^0_{\bar{\mathbb{Q}}_p}(V_\rho)$). Hence the tangent spaces

$$t_\rho^{\mathrm{ord}} = \mathcal{D}_\rho^{\mathrm{ord}}(\bar{\mathbb{Q}}_p[\epsilon]),\ t_\rho^{\mathrm{red}} = \mathcal{D}_\rho^{\mathrm{red}}(\bar{\mathbb{Q}}_p[\epsilon]),\ \text{and } t_\rho^{\mathrm{eis}} = \mathcal{D}_\rho^{\mathrm{eis}}(\bar{\mathbb{Q}}_p[\epsilon]). \quad (5)$$

of the functors defined in §1.2 and §1.3 are naturally isomorphic to subspaces of $\mathrm{H}^1(\mathbb{Q},\mathrm{ad}(\rho))$ that we will now determine precisely.

Let W_ρ be the kernel of the homomorphism $\mathrm{End}_{\bar{\mathbb{Q}}_p}(V_\rho) \to \mathrm{Hom}_{\bar{\mathbb{Q}}_p}(F_\rho, V_\rho/F_\rho)$ of $\mathrm{G}_\mathbb{Q}$-representations arising from (2) and let $W_\rho^0 = W_\rho \cap \mathrm{End}^0_{\bar{\mathbb{Q}}_p}(V_\rho)$. Let W'_ρ be the kernel of the natural homomorphism $W_\rho \to \mathrm{Hom}_{\bar{\mathbb{Q}}_p}(V_\rho/F_\rho, V_\rho/F_\rho)$ and let $W'^0_\rho = W'_\rho \cap \mathrm{End}^0_{\bar{\mathbb{Q}}_p}(V_\rho)$.

The basis (e_1,e_2) of V_ρ in which $\rho = \left(\begin{smallmatrix} \phi & \eta\\ 0 & 1\end{smallmatrix}\right)$ yields an identification $\mathrm{End}_{\bar{\mathbb{Q}}_p}(V_\rho) = \mathrm{M}_2(\bar{\mathbb{Q}}_p)$ under which W_ρ (resp. W'_ρ) corresponds to the subspace of the upper triangular matrices (resp. matrices of the form $\left(\begin{smallmatrix} a & b\\ 0 & 0\end{smallmatrix}\right)$).

Proposition 2.1. *One has $t_\rho^{\mathrm{n.ord}} \simeq \mathrm{H}^1(\mathbb{Q},W_\rho)\oplus\bar{\mathbb{Q}}_p$, $t_\rho^{\mathrm{eis}} \simeq \mathrm{H}^1(\mathbb{Q},W'_\rho)$, $t_\rho^{\mathrm{red}} = t_\rho^{\mathrm{ord}} \simeq \mathrm{H}^1(\mathbb{Q},W_\rho)$. Moreover $t_{\rho,0}^{\mathrm{n.ord}} \simeq \mathrm{H}^1(\mathbb{Q},W^0_\rho)\oplus\bar{\mathbb{Q}}_p$ and $t_{\rho,0}^{\mathrm{red}} = t_{\rho,0}^{\mathrm{ord}} \simeq \mathrm{H}^1(\mathbb{Q},W^0_\rho)$. Finally $t_{\rho,0}^{\mathrm{eis}} = t_\rho^{\mathrm{eis}} \cap t_{\rho,0}^{\mathrm{ord}} = \{0\}$.*

Proof. An infinitesimal deformation $[\rho_\epsilon]$ of ρ can always be represented by a lift

$$\rho_\epsilon = \left(1+\epsilon\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)\rho = \begin{pmatrix} \phi(1+\epsilon a) & \epsilon b+\eta(1+\epsilon a)\\ \epsilon\phi c & 1+\epsilon d+\epsilon\eta c\end{pmatrix}, \quad (6)$$

such that $\rho_\epsilon(\tau) = \left(\begin{smallmatrix} -1 & 0\\ 0 & 1\end{smallmatrix}\right)$, where $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) : \mathrm{G}_\mathbb{Q} \to \mathrm{M}_2(\bar{\mathbb{Q}}_p)$ is a cocycle (we recall that $\mathrm{M}_2(\bar{\mathbb{Q}}_p)$ is endowed with the adjoint action of ρ). As changing the lift amounts to changing the cocycle $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ by a coboundary, one sees from

$$\rho\begin{pmatrix} a & b\\ c & d\end{pmatrix}\rho^{-1} = \begin{pmatrix} a+c\eta\phi^{-1} & b\phi+(d-a)\eta-c\eta^2\phi^{-1}\\ c\phi^{-1} & d-c\eta\phi^{-1}\end{pmatrix}. \quad (7)$$

that the cocycle $c : \mathrm{G}_\mathbb{Q} \to \bar{\mathbb{Q}}_p(\phi^{-1})$ is changed by a coboundary as well, hence $[c] \in \mathrm{H}^1(\mathbb{Q},\phi^{-1})$ is uniquely determined by $[\rho_\epsilon] \in t_\rho^{\mathrm{univ}}$. As $\mathrm{H}^0(\mathbb{Q},\phi^{-1}) = \{0\}$, the exact sequence of $\mathrm{G}_\mathbb{Q}$-modules

$$0 \to W_\rho \to \mathrm{ad}\,\rho \to \phi^{-1} \to 0,$$

where the map $\mathrm{ad}\,\rho \to \phi^{-1}$ is given by $\left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] \mapsto [c]$, yields an exact sequence in cohomology

$$0 \to \mathrm{H}^1(\mathbb{Q}, W_\rho) \to \mathrm{H}^1(\mathbb{Q}, \mathrm{ad}\,\rho) \to \mathrm{H}^1(\mathbb{Q}, \phi^{-1}) \to \mathrm{H}^2(\mathbb{Q}, W_\rho). \tag{8}$$

As $\rho_{|\mathrm{G}_{\mathbb{Q}_p}}$ is indecomposable, any $\rho_\epsilon(\mathrm{G}_{\mathbb{Q}_p})$-stable $\bar{\mathbb{Q}}_p[\epsilon]$-line in $\bar{\mathbb{Q}}_p[\epsilon]^2$ has basis $e_1 + \epsilon\mu \cdot e_2$ for some $\mu \in \bar{\mathbb{Q}}_p$. By (6), for $L = \mathbb{Q}$ or $\mathbb{Q}_p$ one has

$$\bar{\mathbb{Q}}_p[\epsilon](e_1 + \epsilon\mu \cdot e_2) \text{ is } \rho_\epsilon(\mathrm{G}_L)\text{-stable} \iff c(g) = \mu(1 - \phi^{-1}(g)) \text{ for all } g \in \mathrm{G}_L.$$

Noting the restriction map $\mathrm{H}^1(\mathbb{Q}, \phi^{-1}) \to \mathrm{H}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$ is injective, one deduces that:

(9)

$$\rho_\epsilon \text{ is } \mathrm{G}_{\mathbb{Q}}\text{-reducible} \iff \rho_\epsilon \text{ is nearly-ordinary} \iff [c] = 0 \overset{(8)}{\iff} [\rho_\epsilon] \in \mathrm{H}^1(\mathbb{Q}, W_\rho).$$

However, as $\phi(\tau) \neq 1$, the $\mathrm{G}_{\mathbb{Q}}$-stable line is unique (when it exists), while $\phi_{|\mathrm{G}_{\mathbb{Q}_p}} = \mathbf{1}$ implies that if there exists a $\mathrm{G}_{\mathbb{Q}_p}$-stable line then all lines lifting F_ρ are. In particular $t_\rho^{\mathrm{n.ord}} \simeq \mathrm{H}^1(\mathbb{Q}, W_\rho) \oplus \bar{\mathbb{Q}}_p$.

To prove that $t_\rho^{\mathrm{red}} = t_\rho^{\mathrm{ord}} \simeq \mathrm{H}^1(\mathbb{Q}, W_\rho)$ it suffices to show that any nearly-ordinary ρ_ϵ (in particular any reducible ρ_ϵ) is in fact ordinary, *i.e.* there exists a $\mathrm{G}_{\mathbb{Q}_p}$-stable line $F_{\rho_\epsilon} = \bar{\mathbb{Q}}_p[\epsilon](e_1 + \epsilon\mu \cdot e_2)$ such that $\mathrm{I}_{\mathbb{Q}_p}$ acts trivially on $(\bar{\mathbb{Q}}_p[\epsilon])^2/F_{\rho_\epsilon}$. As $c(\tau) = 0$, the condition $[c] = 0$ implies that $c = 0$. It then follows from (3) and (6) that

$$\rho_{\epsilon|\mathrm{G}_{\mathbb{Q}_p}} = \begin{pmatrix} 1 & 0 \\ \epsilon\mu & 1 \end{pmatrix} \begin{pmatrix} 1 + \epsilon(a + \mu\eta) & \eta + \epsilon(b + \eta a) \\ 0 & 1 + \epsilon(d - \mu\eta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\epsilon\mu & 1 \end{pmatrix}, \tag{10}$$

hence $\mathrm{I}_{\mathbb{Q}_p}$ acts trivially on $(\bar{\mathbb{Q}}_p[\epsilon])^2/F_{\rho_\epsilon}$ if and only if $d = \mu\eta$ on $\mathrm{I}_{\mathbb{Q}_p}$. As $d_{|\mathrm{G}_{\mathbb{Q}_p}} \in \mathrm{H}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$ and $\eta_{|\mathrm{I}_{\mathbb{Q}_p}}$ is a basis of the image of the restriction map $\mathrm{H}^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) \to \mathrm{H}^1(\mathrm{I}_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p)$, there exists a unique $\mu \in \bar{\mathbb{Q}}_p$ such that $d_{|\mathrm{I}_{\mathbb{Q}_p}} = \mu\eta_{|\mathrm{I}_{\mathbb{Q}_p}}$.

Since $W_\rho \simeq W_\rho^0 \oplus \mathbf{1}$, it is clear that $t_{\rho,0}^{\mathrm{n.ord}} \simeq \mathrm{H}^1(\mathbb{Q}, W_\rho^0) \oplus \bar{\mathbb{Q}}_p$ and $t_{\rho,0}^{\mathrm{red}} = t_{\rho,0}^{\mathrm{ord}} \simeq \mathrm{H}^1(\mathbb{Q}, W_\rho^0)$.

By definition $[\rho_\epsilon] \in t_\rho^{\mathrm{eis}}$ if and only if $d \in \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$ is unramified, *i.e.* $d = 0$. Hence

$$\begin{aligned} t_\rho^{\mathrm{eis}} &\simeq \ker\left(\mathrm{H}^1(\mathbb{Q}, W_\rho) \to \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)\right) \text{ and} \\ t_{\rho,0}^{\mathrm{eis}} &\simeq \ker\left(\mathrm{H}^1(\mathbb{Q}, W_\rho^0) \to \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)\right), \end{aligned} \tag{11}$$

where the maps come from the homomorphism $W_\rho \to \mathrm{Hom}_{\bar{\mathbb{Q}}_p}(V_\rho/F_\rho, V_\rho/F_\rho) = \bar{\mathbb{Q}}_p$ sending $\left[\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right]$ to $[d]$. As $\mathrm{H}^0(\mathbb{Q}, \rho) = \{0\}$, there are exact sequences in cohomology

$$\begin{aligned} \mathrm{H}^0(\mathbb{Q}, W_\rho) \xrightarrow{\sim} \mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p) \to \mathrm{H}^1(\mathbb{Q}, W_\rho') \to \mathrm{H}^1(\mathbb{Q}, W_\rho) \to \mathrm{H}^1(\mathbb{Q}, W_\rho/W_\rho'), \\ \mathrm{H}^0(\mathbb{Q}, \bar{\mathbb{Q}}_p) \xrightarrow{\sim} \mathrm{H}^1(\mathbb{Q}, \phi) \to \mathrm{H}^1(\mathbb{Q}, W_\rho^0) \to \mathrm{H}^1(\mathbb{Q}, W_\rho^0/W_\rho'^0). \end{aligned} \tag{12}$$

Here we have used that $W_\rho' \simeq \rho$ and $W_\rho'^0 \simeq \phi$, because by (7) one has $\rho\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)\rho^{-1} = \left(\begin{smallmatrix} a & b\phi - a\eta \\ 0 & 0 \end{smallmatrix}\right)$. It follows then from (11) that $t_\rho^{\mathrm{eis}} = \mathrm{H}^1(\mathbb{Q}, W_\rho')$ and $t_{\rho,0}^{\mathrm{eis}} = \{0\}$. □

To determine the dimensions of these cohomology groups we will need the following lemma.

Lemma 2.2. *One has* $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}, \rho) = 1$ *and* $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^2(\mathbb{Q}, \rho) = 0$.

Proof. The global Euler characteristic formula yields:

$$\dim \mathrm{H}^2(\mathbb{Q},\phi) = \dim \mathrm{H}^1(\mathbb{Q},\phi) - \dim \mathrm{H}^0(\mathbb{Q},\phi) + \dim \mathrm{H}^0(\mathbb{R},\phi) - \dim(\phi) = 1-0+0-1 = 0.$$

Since $W'_\rho \simeq \rho$ the exact sequence (12) implies that $\mathrm{H}^1(\mathbb{Q},\rho) \simeq \mathrm{H}^1(\mathbb{Q},\bar{\mathbb{Q}}_p)$ is 1-dimensional. Another application of Euler's global characteristic formula yields $\mathrm{H}^2(\mathbb{Q},\rho) = \{0\}$. □

Since $W^0_\rho \simeq \rho \simeq W'_\rho$ it follows then from Proposition 2.1 that

$$\dim t^{\mathrm{red}}_\rho = \dim t^{\mathrm{ord}}_\rho = 2,\ \dim t^{\mathrm{eis}}_\rho = \dim t^{\mathrm{red}}_{\rho,0} = \dim t^{\mathrm{ord}}_{\rho,0} = 1 \text{ and } t^{\mathrm{eis}}_{\rho,0} = \{0\}. \tag{13}$$

Remark 2.3. By the proof of Proposition 2.1, a deformation $[\rho_\epsilon] \in t^{\mathrm{red}}_\rho \smallsetminus t^{\mathrm{eis}}_\rho$ is represented by a lift $\rho_\epsilon = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ such that d is ramified at p, and yet ρ_ϵ admits a $\mathrm{G}_{\mathbb{Q}_p}$-filtration with unramified quotient. The non-uniqueness of the $\mathrm{G}_{\mathbb{Q}_p}$-stable line is due to the fact that ϕ is trivial on $\mathrm{G}_{\mathbb{Q}_p}$.

2.2. **An application of Baker–Brumer's Theorem.** The fixed field H of $\ker(\phi) \subset \mathrm{G}_{\mathbb{Q}}$ is a totally imaginary cyclic extension of $\mathbb{Q}$ of degree $2r \geqslant 2$, in which p splits completely. The embedding $\iota_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ determines a place v_0 of H and an embedding $\mathrm{G}_{\mathbb{Q}_p} = \mathrm{G}_{H_{v_0}} \subset \mathrm{G}_H$ yielding a canonical restriction map

$$\mathrm{res}_p : \mathrm{H}^1(H,\bar{\mathbb{Q}}_p) \to \mathrm{H}^1(\mathbb{Q}_p,\bar{\mathbb{Q}}_p). \tag{14}$$

Fixing a generator σ of $G = \mathrm{Gal}(H/\mathbb{Q})$ allows us to number the places in H above p as $v_i = v_0 \circ \sigma^i$, $0 \leqslant i \leqslant 2r-1$. Let $\mathcal{O}_H$ (resp. $\mathcal{O}_{v_i}$) be the ring of integers of H (resp. H_{v_i}).

Recall the standard choice of p-adic logarithm $\log_p$ sending p to 0. Denoting $\mathrm{ord}_p : \mathbb{Q}_p^\times \to \mathbb{Z}$ the valuation, we consider the $\bar{\mathbb{Q}}_p$-linear maps

$$\log_{v_0} : \mathcal{O}_H[\tfrac{1}{p}]^\times \otimes \bar{\mathbb{Q}}_p \longrightarrow \bar{\mathbb{Q}}_p \qquad \mathrm{ord}_{v_0} : \mathcal{O}_H[\tfrac{1}{p}]^\times \otimes \bar{\mathbb{Q}}_p \longrightarrow \bar{\mathbb{Q}}_p$$
$$u \otimes x \mapsto \log_p(\iota_p(u))x \qquad\qquad u \otimes x \mapsto \mathrm{ord}_p(\iota_p(u))x$$

Given any *odd* character ψ of G, the ψ^{-1}-eigenspace of $\mathcal{O}_H[\frac{1}{p}]^\times \otimes \bar{\mathbb{Q}}_p$ is a line, and we let u_ψ be a basis. Note that $\mathrm{ord}_{v_0}(u_\psi) \neq 0$ since otherwise, by ψ^{-1}-equivariance, one would have $\mathrm{ord}_{v_i}(u_\psi) = \mathrm{ord}_{v_0}(\sigma^{-i}(u_\psi)) = 0$ for all $0 \leqslant i \leqslant 2r-1$, which is impossible since the ψ^{-1}-eigenspace of $\mathcal{O}_H^\times \otimes \bar{\mathbb{Q}}_p$ is zero. Following [18, (7)] we define the $\mathscr{L}$-invariant of ψ as

$$\mathscr{L}(\psi) := -\frac{\log_{v_0}(u_\psi)}{\mathrm{ord}_{v_0}(u_\psi)} \in \bar{\mathbb{Q}}_p. \tag{15}$$

As well known, $\mathrm{H}^1(\mathbb{Q},\psi)$ is a line isomorphic to the ψ^{-1}-eigenspace of $\mathrm{H}^1(H,\bar{\mathbb{Q}}_p)$. Fix $[\eta_\psi] \in \mathrm{H}^1(\mathbb{Q},\psi)$ whose restriction to $\mathrm{I}_{\mathbb{Q}_p}$ corresponds to $\log_p$ (as for $\eta_{\mathbf{1}}$ defined in §1.1). Then

$$L_{\bar{\mathbb{Q}}} := \bar{\mathbb{Q}}\eta_{\mathbf{1}} \oplus \bigoplus_{\psi \text{ odd}} \bar{\mathbb{Q}}\eta_\psi$$

is a $\bar{\mathbb{Q}}$-linear subspace of $L_{\bar{\mathbb{Q}}} \otimes_{\bar{\mathbb{Q}}} \bar{\mathbb{Q}}_p = \mathrm{H}^1(H,\bar{\mathbb{Q}}_p)$.

Proposition 2.4. *The element $\eta_\psi - \eta_{\mathbf{1}}$ is unramified at p and $(\eta_\psi - \eta_{\mathbf{1}})(\mathrm{Frob}_p) = \mathscr{L}(\psi^{-1})$.*

Proof. There is an exact sequence of $\bar{\mathbb{Q}}_p[G]$-modules

$$0 \to \mathrm{Hom}(\mathrm{G}_H, \bar{\mathbb{Q}}_p) \to \bigoplus_{i=0}^{2r-1} \mathrm{Hom}(H_{v_i}^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_H[\tfrac{1}{p}]^\times, \bar{\mathbb{Q}}_p). \tag{16}$$

where $\xi : \mathrm{G}_H \to \bar{\mathbb{Q}}_p$ is sent to the collection of maps $\xi_i : H_{v_i}^\times \to \bar{\mathbb{Q}}_p$, $0 \leqslant i \leqslant 2r-1$, defined by taking the restriction to $H_{v_i}^\times \subset \widehat{H_{v_i}^\times} \simeq \mathrm{G}_{H_{v_i}}^{\mathrm{ab}}$. Then $(\eta_\psi - \eta_{\mathbf{1}})(\mathrm{Frob}_p) = (\eta_{\psi,0} - \eta_{\mathbf{1},0})(\varpi_0)$, where ϖ_0 denotes a uniformiser of H_{v_0}. Denoting by e the exponent of the Hilbert class group of H, there exists $x_0 \in \mathcal{O}_H[\frac{1}{p}]^\times$ whose valuation at v_0 is e, while it is 0 at all other finite places of H. We can write $x_0 = \varpi_0^e y$ with $y \in \mathcal{O}_{v_0}^\times$ and we have

$$(\eta_{\psi,0} - \eta_{\mathbf{1},0})(x_0) = (\eta_{\psi,0} - \eta_{\mathbf{1},0})(\varpi_0^e y) = e \cdot (\eta_{\psi,0} - \eta_{\mathbf{1},0})(\varpi_0) = e \cdot (\eta_\psi - \eta_{\mathbf{1}})(\mathrm{Frob}_p).$$

Since $x_0 \in \mathcal{O}_H[\frac{1}{p}]^\times$ and $\eta_\psi - \eta_{\mathbf{1}} \in \mathrm{Hom}(\mathrm{G}_H, \bar{\mathbb{Q}}_p)$, the exact sequence (16) implies that

$$(\eta_{\psi,0} - \eta_{\mathbf{1},0})(x_0) = -\sum_{i=1}^{2r-1} (\eta_{\psi,i} - \eta_{\mathbf{1},i})(x_0). \tag{17}$$

Since by definition η_ψ belongs to the ψ^{-1}-eigenspace for the G-action, it is entirely determined by $\eta_{\psi,0}$. More precisely one has $\eta_{\psi,i} = \psi(\sigma)^{-i}(\eta_{\psi,0} \circ \sigma^i)$ for all $0 \leqslant i \leqslant 2r-1$. Combining this with (17) and observing that $\sigma^i(x_0) \in \mathcal{O}_{v_0}^\times$ for every $1 \leqslant i \leqslant 2r-1$, we obtain

(18)

$$e \cdot (\eta_\psi - \eta_{\mathbf{1}})(\mathrm{Frob}_{v_0}) = (\eta_{\psi,0} - \eta_{\mathbf{1},0})(x_0) = -\sum_{i=1}^{2r-1} (\eta_{\psi,i} - \eta_{\mathbf{1},i})(x_0) =$$
$$= -\sum_{i=1}^{2r-1} (\psi(\sigma)^{-i}\eta_{\psi,0}(\sigma^i(x_0)) - \eta_{\mathbf{1},0}(\sigma^i(x_0))) = -\sum_{i=0}^{2r-1} (\psi(\sigma)^{-i} - 1) \log_p(\iota_p(\sigma^i(x_0)))$$

because the restrictions of η_ψ and of $\eta_{\mathbf{1}}$ to $\mathrm{I}_{H_{v_0}} = \mathrm{I}_{\mathbb{Q}_p}$ are given by $\log_p$. Observe first that

$$\sum_{i=0}^{2r-1} \log_p(\iota_p(\sigma^i(x_0)) = \log_p(\iota_p(\mathrm{N}_{H/\mathbb{Q}}(x_0)) \in \log_p(\iota_p(\pm p^{\mathbb{Z}})) = \{0\},$$

and $\sum_{i=0}^{2r-1} \psi(\sigma)^{-i} \log_p(\iota_p(\sigma^i(x_0))) = \log_{v_0}(u_{\psi^{-1}})$, where $u_{\psi^{-1}} = \sum_{i=0}^{2r-1} \sigma^i(x_0) \otimes \psi(\sigma)^{-i}$. As $u_{\psi^{-1}}$ belongs to the ψ-eigenspace of $\mathcal{O}_H[\frac{1}{p}]^\times \otimes \bar{\mathbb{Q}}_p$, and as $\mathrm{ord}_{v_0}(u_{\psi^{-1}}) = \mathrm{ord}_{v_0}(x_0 \otimes 1) = e$, one has $\log_{v_0}(u_{\psi^{-1}}) = -e \cdot \mathscr{L}(\psi^{-1})$ by definition (15). Combining this with (18) yields the claim. □

Proposition 2.5.

(i) The $\mathscr{L}(\psi)$ are linearly independent over $\bar{\mathbb{Q}}$, as ψ runs over all odd characters of G.

(ii) The restriction to $L_{\bar{\mathbb{Q}}}$ of the map res_p *defined in* (14) *is injective.*

Proof. (i) Suppose that $\sum_\psi m_\psi \mathscr{L}(\psi) = 0$ for some $m_\psi \in \bar{\mathbb{Q}}$. As in the proof of Proposition 2.4, we denote by e the exponent of the Hilbert class group of H, and fix an element $x_0 \in \mathcal{O}_H[\frac{1}{p}]^\times$ with valuation e at v_0 and 0 at all other finite places of H. It follows that

$$\sum_{\psi \text{ odd}} m_\psi \sum_{i=0}^{2r-1} (\psi(\sigma)^i - 1)\log_p(\iota_p(\sigma^i(x_0))) = 0.$$

Since the $i = 0$ summand vanishes, letting $m_{\mathbf{1}} = -\sum_{\psi \text{ odd}} m_\psi$ the formula can be written as

$$\sum_{i=1}^{2r-1} \log_p(\iota_p(\sigma^i(x_0))) \left(\sum_{\psi \text{ odd or } \psi=\mathbf{1}} m_\psi \psi(\sigma)^i \right) = 0$$

We claim that the values $\{\log_p(\iota_p(\sigma^i(x_0)))\}_{1\leqslant i\leqslant 2r-1}$ are linearly independent over $\mathbb{Q}$. To see this suppose $\log_p(\iota_p(x)) = 0$ for some element $x = \prod_{1\leqslant i\leqslant 2r-1} \sigma^i(x_0)^{n_i}$ with $n_i \in \mathbb{Z}$. As $\iota_p(x) \in \bar{\mathbb{Z}}_p^\times$ this implies that x is a root of unity in H, leading to $n_i = \mathrm{ord}_{v_i}(x) = 0$ for all $1 \leqslant i \leqslant 2r-1$. By Baker–Brumer's Theorem [8], the elements $\{\log_p(\iota_p(\sigma^i(x_0)))\}_{1\leqslant i\leqslant 2r-1}$ are therefore linearly independent over $\bar{\mathbb{Q}}$, leading to

$$\sum_{\psi \text{ odd or } \psi=\mathbf{1}} m_\psi \psi(\sigma)^i = 0 \tag{19}$$

for any $1 \leqslant i \leqslant 2r-1$. Moreover, as $m_{\mathbf{1}} = -\sum_{\psi \text{ odd}} m_\psi$, (19) holds for $i = 0$ as well.

Let $\psi_1, \psi_2, \dots \psi_r$ be a numbering of the odd characters of G. The condition (19) can be rewritten as $(m_{\mathbf{1}}, m_{\psi_1}, \dots, m_{\psi_r}) \cdot M = (0, 0, \dots, 0)$, where

$$M = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \psi_1(\sigma) & \psi_1(\sigma)^2 & \dots & \psi_1(\sigma)^{2r-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \psi_r(\sigma) & \psi_r(\sigma)^2 & \dots & \psi_r(\sigma)^{2r-1} \end{pmatrix}$$

As $2r \geqslant r+1$, M contains the Vandermonde matrix of $(1, \psi_1(\sigma), \dots, \psi_r(\sigma))$ which as well-known is invertible, implying that $m_\psi = 0$ for every ψ.

(ii) It suffices to notice that the kernel of the restriction map $L_{\bar{\mathbb{Q}}_p} \to \mathrm{H}^1(\mathrm{I}_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p)$ is spanned by $\{(\eta_{\mathbf{1}} - \eta_\psi)\}_{\psi \text{ odd}}$. Combining Proposition 2.4 with (i) yields the desired result. □

2.3. **The tangent space for cuspidal deformations.** Let $t_{\mathrm{cusp}} = \mathcal{D}_{\mathrm{cusp}}(\bar{\mathbb{Q}}_p[\epsilon])$ and $t^0_{\mathrm{cusp}} = \mathcal{D}^0_{\mathrm{cusp}}(\bar{\mathbb{Q}}_p[\epsilon])$ be the tangent space and the relative tangent space to the functor $\mathcal{D}_{\mathrm{cusp}}$.

Proposition 2.6. *We have* $\dim_{\bar{\mathbb{Q}}_p} t_{\mathrm{cusp}} = 1$ *and* $\dim_{\bar{\mathbb{Q}}_p} t^0_{\mathrm{cusp}} = 0$.

Proof. By definition

$$t_{\mathrm{cusp}} = \left\{ (\rho_\epsilon, \rho'_\epsilon) \in t^{\mathrm{ord}}_\rho \times t^{\mathrm{ord}}_{\rho'} \mid \mathrm{tr}(\rho_\epsilon) = \mathrm{tr}(\rho'_\epsilon) \text{ and } \chi_\epsilon(\mathrm{Frob}_p) = \chi'_\epsilon(\mathrm{Frob}_p) \right\}.$$

By the proof of Proposition 2.1 an element $[\rho_\epsilon] \in t^{\mathrm{ord}}_\rho \simeq t^{\mathrm{red}}_\rho \simeq \mathrm{H}^1(\mathbb{Q}, W_\rho)$ can be written as:

$$\rho_\epsilon = \left(1 + \epsilon \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) \rho = \begin{pmatrix} \phi(1+\epsilon a) & \eta(1+\epsilon a) + b\epsilon \\ 0 & 1 + \epsilon d \end{pmatrix}$$

for $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \in \mathrm{H}^1(\mathbb{Q}, W_\rho)$. In particular $a, d \in \mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$. Recall the notation $\eta_{\mathbf{1}}$ for the generator of $\mathrm{H}^1(\mathbb{Q}, \bar{\mathbb{Q}}_p)$ whose restriction at $\mathrm{I}_{\mathbb{Q}_p}$ is $\log_p$. Writing $a = \lambda\eta_{\mathbf{1}}$, $d = \mu\eta_{\mathbf{1}}$ with $\lambda, \mu \in \bar{\mathbb{Q}}_p$ yields:

$$\mathrm{tr}(\rho_\epsilon) = 1 + \phi + \epsilon(\lambda\phi + \mu)\eta_{\mathbf{1}} \text{ and } \det(\rho_\epsilon) = \phi(1 + \epsilon(\lambda + \mu)\eta_{\mathbf{1}}). \tag{20}$$

By (10), the ordinary filtration F_{ρ_ϵ} has basis $e_1 + \epsilon\mu e_2$ and $\rho_\epsilon(\mathrm{G}_{\mathbb{Q}_p})$ acts on the quotient by the character $\chi_\epsilon = 1 + \epsilon(d - \mu\eta)$. It follows from Proposition 2.4 that

$$\chi_\epsilon(\mathrm{Frob}_p) = 1 + \epsilon\mu(\eta_{\mathbf{1}} - \eta)(\mathrm{Frob}_p) = 1 - \mu\mathscr{L}(\phi^{-1})\epsilon. \tag{21}$$

Since $\rho' = \begin{pmatrix} \mathbf{1} & \phi\eta' \\ 0 & \phi \end{pmatrix} = \begin{pmatrix} \phi^{-1} & \eta' \\ 0 & \mathbf{1} \end{pmatrix} \otimes \phi$ one can describe $t^{\mathrm{ord}}_{\rho'} \simeq t^{\mathrm{ord}}_{\rho' \otimes \phi^{-1}}$ by simply replacing ϕ by ϕ^{-1} and η by η' in the above description of t^{ord}_ρ. One then finds that:

$$\begin{aligned} \mathrm{tr}(\rho'_\epsilon) &= \phi(1 + \phi^{-1} + \epsilon(\lambda'\phi^{-1} + \mu')\eta_{\mathbf{1}}) = 1 + \phi + \epsilon(\lambda' + \mu'\phi)\eta_{\mathbf{1}}, \\ \chi'_\epsilon(\mathrm{Frob}_p) &= 1 + \epsilon\mu'(\eta_{\mathbf{1}} - \eta')(\mathrm{Frob}_p) = 1 - \mu'\mathscr{L}(\phi)\epsilon. \end{aligned} \tag{22}$$

From (20), (21) and (22) one sees that

$$(\rho_\epsilon, \rho'_\epsilon) \in t_{\mathrm{cusp}} \iff \lambda = \mu',\ \mu = \lambda' \text{ and } \mu\mathscr{L}(\phi^{-1}) = \lambda\mathscr{L}(\phi). \tag{23}$$

By Proposition 2.5, $\mathscr{L}(\phi)$ and $\mathscr{L}(\phi^{-1})$ are both non-zero, hence $\dim t_{\mathrm{cusp}} = 1$.

To compute the relative tangent space t^0_{cusp} it suffices to add to (23) the condition $\det \rho_\epsilon = \phi$, which is equivalent to $\lambda + \mu = 0$. By Proposition 2.5, $\mathscr{L}(\phi)$ and $\mathscr{L}(\phi^{-1})$ are linearly independent over $\bar{\mathbb{Q}}$ if ϕ is not quadratic, while when ϕ quadratic one has $\mathscr{L}(\phi) = \mathscr{L}(\phi^{-1}) \neq 0$. In either case $\mathscr{L}(\phi) + \mathscr{L}(\phi^{-1}) \neq 0$, hence the equation $\lambda + \mu = 0$ is linearly independent from (23), and $\dim_{\bar{\mathbb{Q}}_p} t^0_{\mathrm{cusp}} = 0$. □

Corollary 2.7. *We have* $t^{\mathrm{ord}}_\rho = t_{\mathrm{cusp}} \oplus t^{\mathrm{eis}}_\rho$.

Proof. As in the proof of Proposition 2.6 one can use λ and μ as coordinates on t^{ord}_ρ and by (23) the equation defining t_{cusp} is $\mu\mathscr{L}(\phi^{-1}) = \lambda\mathscr{L}(\phi)$. On the other hand by (11) the equation defining t^{eis}_ρ is $\mu = 0$, and we have seen that $\mathscr{L}(\phi) \neq 0$. □

2.4. **Iwasawa cohomology.** Let $\varepsilon_p : \mathrm{G}_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \simeq \mathbb{Z}_p^\times$ be the p-adic cyclotomic character and let $\omega_p : \mathrm{G}_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{2p})/\mathbb{Q}) \simeq (\mathbb{Z}/2p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$ be the p-adic Teichmüller character. Let $\nu = 2$ if $p = 2$ and $\nu = 1$ otherwise. The cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$ is the fixed field of

$$\varepsilon_p\omega_p^{-1} : \mathrm{G}_{\mathbb{Q}} \to 1 + p^\nu\mathbb{Z}_p.$$

The universal cyclotomic character $\chi_p : \mathrm{G}_{\mathbb{Q}} \to \Lambda^\times$ is obtained by composing $\varepsilon_p\omega_p^{-1}$ with

$$1 + p^\nu\mathbb{Z}_p \to \mathbb{Z}_p[\![1 + p^\nu\mathbb{Z}_p]\!]^\times \xrightarrow{\sim} \mathbb{Z}_p[\![X]\!]^\times \hookrightarrow \Lambda^\times, \tag{24}$$

where the isomorphism in the middle sends $1 + p^\nu$ to $1 + X$. One has:

$$\chi_p \equiv 1 - \frac{\eta_{\mathbf{1}}}{\log_p(1 + p^\nu)} X \pmod{X^2}, \tag{25}$$

where $\eta_{\mathbf{1}} : \mathrm{G}_{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ is the cyclotomic homomorphism defined in §1.1 sending Frob_ℓ to $-\log_p(\ell)$ for all $\ell \neq p$. It follows that

$$\frac{d}{dX}\Big|_{X=0} \chi_p = -\frac{\eta_{\mathbf{1}}}{\log_p(1+p^\nu)}. \tag{26}$$

Using Class Field Theory one can show that $\Phi = \phi\chi_p$ is the deformation of ϕ to its universal deformation ring Λ (see [4, §6]). For $n \in \mathbb{Z}_{\geqslant 1}$, we let $\Phi_n = \Phi \mod X^n$.

We will prove the existence of a non-torsion cohomology class in the Iwasawa cohomology group $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi)$, where $\mathrm{G}_{\mathbb{Q}}^{Np}$ is the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside Np and ∞. This will be used in §3.5 to show that there exists a surjection $\mathcal{R}_\rho^{\mathrm{eis}} \twoheadrightarrow \Lambda$.

Proposition 2.8. *(i) One has* $\mathrm{H}^2(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1}) = 0$. *Moreover* $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1})$ *is a free* Λ*-module of rank* 1 *and for all* $n \in \mathbb{Z}_{\geqslant 1}$ *the natural homomorphism*

$$\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1}) \otimes_\Lambda \Lambda/(X^n) \xrightarrow{\sim} \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_n^{\pm 1})$$

is an isomorphism.

(ii) For every $n \geqslant 1$*, the natural restriction map is injective:*

$$\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_n^{\pm 1}) \to \mathrm{H}^1(\mathbb{Q}_p, \Phi_n^{\pm 1}).$$

Proof. (i) The short exact sequence of $\bar{\mathbb{Q}}_p[\mathrm{G}_{\mathbb{Q}}^{Np}]$-modules

$$0 \to \Phi_{n-1}^{\pm 1} \xrightarrow{\cdot X} \Phi_n^{\pm 1} \to \phi^{\pm 1} \to 0$$

yields for $i \geqslant 1$ a long exact sequence in cohomology

$$\mathrm{H}^{i-1}(\mathrm{G}_{\mathbb{Q}}^{Np}, \phi^{\pm 1}) \to \mathrm{H}^i(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_{n-1}^{\pm 1}) \xrightarrow{\cdot X} \mathrm{H}^i(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_n^{\pm 1}) \to \mathrm{H}^i(\mathrm{G}_{\mathbb{Q}}^{Np}, \phi^{\pm 1}). \tag{27}$$

As $\mathrm{H}^2(\mathrm{G}_{\mathbb{Q}}^{Np}, \phi^{\pm 1}) = 0$ by the global Euler characteristic formula, one obtains $\mathrm{H}^2(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_n^{\pm 1}) = \{0\}$ by induction on n (note that $\Phi_1 = \phi$). As $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}, \phi^{\pm 1}) = 1$ and $\mathrm{H}^0(\mathbb{Q}, \phi^{\pm 1}) = \{0\}$, one similarly deduces that $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_n^{\pm 1}) = n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

The short exact sequence of $\Lambda[\mathrm{G}_{\mathbb{Q}}]$-modules

$$0 \to \Phi^{\pm 1} \xrightarrow{\cdot X^n} \Phi^{\pm 1} \to \Phi_n^{\pm 1} \to 0$$

yields a long exact sequence of Λ-modules in cohomology ([38, Proposition 3.5.1.3]):

$$\begin{aligned} 0 \to \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1}) \xrightarrow{\cdot X^n} \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1}) \to \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_n^{\pm 1}) \\ \to \mathrm{H}^2(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1}) \xrightarrow{\cdot X^n} \mathrm{H}^2(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1}) \to 0. \end{aligned} \tag{28}$$

According to [38, Proposition 4.2.3], $\mathrm{H}^i(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1})$ is a Λ-module of finite type for $i \in \{0, 1, 2\}$. Therefore, Nakayama's lemma applied to (28) for $n = 1$ implies that $\mathrm{H}^2(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1}) = \{0\}$ while $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1})$ is a cyclic Λ-module. Moreover (28) for an arbitrary n yields

$$\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1}) \otimes_\Lambda \Lambda/(X^n) \simeq \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_n^{\pm 1})$$

which has dimension n. Hence $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1})$ is a free Λ-module of rank 1.

(ii) Using (i), the long exact sequence (27) yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_{n-1}^{\pm 1}) & \xrightarrow{\cdot X} & \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_{n}^{\pm 1}) & \twoheadrightarrow & \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \phi^{\pm 1}) \\ & & \downarrow{\scriptstyle \mathrm{res}_{n-1}} & & \downarrow{\scriptstyle \mathrm{res}_{n}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{H}^0(\mathrm{G}_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p) & \xrightarrow{\delta_{n-1}} & \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}_p}, \Phi_{n-1}^{\pm 1}) & \xrightarrow{\cdot X} & \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}_p}, \Phi_{n}^{\pm 1}) & \twoheadrightarrow & \mathrm{H}^1(\mathrm{G}_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p), \end{array} \tag{29}$$

where the vertical arrows are the restriction maps. Note that the image of the connecting homomorphism δ_{n-1} is generated by the cohomology class of the cocycle $g \mapsto \frac{\chi_p^{\pm 1}(g)-1}{X} \pmod{X^{n-1}}$ which belongs to the X-torsion, because $g \mapsto \chi_p^{\pm 1}(g) - 1$ is a coboundary.

We argue by induction on n. The injectivity of $\mathrm{res} = \mathrm{res}_1$ follows from Class Field Theory. Suppose that res_{n-1} is injective for some $n \geqslant 2$. It suffices then to show that

$$\mathrm{Im}(\mathrm{res}_{n-1}) \cap \mathrm{Im}(\delta_{n-1}) = \{0\}.$$

Let $[\eta_{\Phi^{\pm 1}}]$ be a generator of $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi^{\pm 1})$ as a free rank one Λ module, and let $[\eta_{\Phi_n^{\pm 1}}]$ denote its image in $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}}^{Np}, \Phi_n^{\pm 1})$. Since $\mathrm{Im}(\delta_{n-1})$ is X-torsion and res_{n-1} is injective, an element of the above intersection is a scalar multiple of $\mathrm{res}_{n-1}(X^{n-2}[\eta_{\Phi_{n-1}^{\pm 1}}])$. Moreover, as any non-trivial element of $\mathrm{Im}(\delta_{n-1})$ remains non-trivial when letting $X = 0$, the above intersection automatically vanishes for $n \geqslant 3$. Finally, for $n = 2$ in virtue of (25) one has to show that $\mathrm{res}([\eta_{\phi^{\pm 1}}])$ and $\mathrm{res}(\eta_{\mathbf{1}})$ generate distinct lines in $\mathrm{H}^1(\mathrm{G}_{\mathbb{Q}_p}, \bar{\mathbb{Q}}_p)$ which follows from the fact that they have the same restriction to the inertia group, while taking different values on the Frobenius by Proposition 2.4. □

3. p-adic families containing f

In this section, we show that the completed strict local ring $\mathcal{T}_{\mathrm{cusp}}$ of the cuspidal eigencurve $\mathscr{C}_{\mathrm{cusp}}$ at f is isomorphic to the deformation ring $\mathcal{R}_{\mathrm{cusp}}$ and conclude that the cuspidal eigencurve is étale over the weight space at f.

3.1. **Some basic facts on the eigencurve.** Let $X/\mathbb{Z}_p$ be the proper smooth modular curve of level (see [19, Chap. IV])

$$\Gamma = \begin{cases} \Gamma_1(N), & \text{if } N \geqslant 4, \\ \Gamma(3), & \text{if } N = 3. \end{cases}$$

Let $E \to X$ be the generalised elliptic curve endowed with the identity section $e : X \to E$, let $\omega = e^*(\Omega_{E/X})$ be the conormal sheaf and X^{rig} be the rigid analytic space attached to the generic fibre of X (note that by properness $X^{\mathrm{rig}}(\bar{\mathbb{Q}}_p) = X(\bar{\mathbb{Q}}_p)$). The analytification of ω is an invertible sheaf on X^{rig} and will be denoted again by ω.

For $v \in \mathbb{Q}_{\geqslant 0}$ let $X(v)$ denote the open locus of X^{rig} where the truncated valuation of the Hasse invariant is at most v (see [42, §3.1]); in particular $X(0)$ is the ordinary locus. We recall that the weight space $\mathcal{W}$ is the rigid analytic space over $\mathbb{Q}_p$ such that

$$\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{C}_p^\times).$$

For $\mathcal{U}$ a connected open admissible affinoid of $\mathcal{W}$ we let

$$\kappa_{\mathcal{U}} : \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}(\mathcal{U})^\times$$

be the universal character, $\omega_{\mathcal{U}}$ be the invertible sheaf on $X(v) \times \mathcal{U}$ constructed in [42, §5.1] under the assumption that either $v = 0$, or that both $v > 0$ and $\mathcal{U}$ are sufficiently small. By construction, for any weight $k \in \mathbb{Z}_{\geqslant 1} \cap \mathcal{U}$, the sheaf $\omega_{\mathcal{U}}$ specialises to the invertible sheaf $\omega^{\otimes k}$ on $X(v)$ (see [42, Proposition 3.3]).

Consider the invertible sheaf $\omega_{\mathcal{U}}(-D_{\mathcal{U}})$, where $D_{\mathcal{U}}$ denotes the cuspidal divisor of $X(v) \times \mathcal{U}$. Note that $D_{\mathcal{U}}$ does not depend on v as the cusps of X all belong to $X(0)$. For $N \geqslant 4$, the space of families of overconvergent forms, resp. cuspforms, having slope at most $s \in \mathbb{Q}_{\geqslant 0}$ is defined as:

$$M_{\mathcal{U}}^{\dagger, \leqslant s} = \varinjlim_{v>0} \mathrm{H}^0(X(v) \times \mathcal{U}, \omega_{\mathcal{U}})^{\leqslant s}, \text{ resp. } S_{\mathcal{U}}^{\dagger, \leqslant s} = \varinjlim_{v>0} \mathrm{H}^0(X(v) \times \mathcal{U}, \omega_{\mathcal{U}}(-D_{\mathcal{U}}))^{\leqslant s}.$$

The space $M_{\mathcal{U}}^{\dagger, \leqslant s}$, resp. $S_{\mathcal{U}}^{\dagger, \leqslant s}$, is a locally free module of finite type over the Banach algebra $\mathcal{O}(\mathcal{U})$, and is contained in the space of families of p-adic forms, resp. cuspforms:

$$M_{\mathcal{U}} = \mathrm{H}^0(X(0) \times \mathcal{U}, \omega_{\mathcal{U}}), \text{ resp. } S_{\mathcal{U}} = \mathrm{H}^0(X(0) \times \mathcal{U}, \omega_{\mathcal{U}}(-D_{\mathcal{U}})).$$

For $N = 3$ we define $M_{\mathcal{U}}$ as the $(\Gamma_1(3)/\Gamma)$-invariants in $\mathrm{H}^0(X(0) \times \mathcal{U}, \omega_{\mathcal{U}})$ and we similarly define $S_{\mathcal{U}}$, $M_{\mathcal{U}}^{\dagger, \leqslant s}$ and $S_{\mathcal{U}}^{\dagger, \leqslant s}$. We let $M_{\mathcal{U}}^{\mathrm{ord}} = M_{\mathcal{U}}^{\dagger, \leqslant 0}$ denote the space of ordinary families and by $S_{\mathcal{U}}^{\mathrm{ord}} = S_{\mathcal{U}}^{\dagger, \leqslant 0}$ its cuspidal subspace. The notation is justified by the fact that any p-adic form of slope 0 is necessarily overconvergent (see [42, Proposition 6.2]).

By construction the eigencurve $\mathscr{C}$ (resp. the cuspidal eigencurve $\mathscr{C}_{\mathrm{cusp}}$) is a rigid analytic space over $\mathbb{Q}_p$ admissibly covered by the affinoids attached to the $\mathcal{O}(\mathcal{U})$-algebras generated by the Hecke operators $T_\ell, \ell \nmid Np$ and U_p acting on $\mathrm{End}_{\mathcal{O}(\mathcal{U})}(M_{\mathcal{U}}^{\dagger, \leqslant s})$ (resp. $\mathrm{End}_{\mathcal{O}(\mathcal{U})}(S_{\mathcal{U}}^{\dagger, \leqslant s})$), where both $s \in \mathbb{Q}_{\geqslant 0}$ and the open admissible affinoid $\mathcal{U} \subset \mathcal{W}$ vary. Thus, we obtain a closed immersion $\mathscr{C}_{\mathrm{cusp}} \hookrightarrow \mathscr{C}$ of rigid curves and $\mathscr{C}$ is endowed with a weight map $\kappa : \mathscr{C} \to \mathcal{W}$ which is locally finite and flat. Moreover $\mathscr{C}$ is reduced as the Hecke operators act semi-simply on the generalised eigenspace of any classical eigenform which is regular at p and has non-critical slope (see [12, Proposition 3.9]). Similarly, one can obtain $\mathscr{C}_{\rho'}$ (resp. $\mathscr{C}_{\rho}$) by considering the $\mathcal{O}(\mathcal{U})$-banach Hecke modules generated by $S_{\mathcal{U}}^{\dagger, \leqslant s}$ and the Eisenstein family $\mathcal{E}_{\phi, \mathbf{1}}$ (resp. $\mathcal{E}_{\mathbf{1}, \phi}$).

There exists a ring homomorphism $\mathbb{Z}[T_\ell, U_p]_{\ell \nmid Np} \to \mathcal{O}_{\mathscr{C}}(\mathscr{C})$ allowing one to see T_ℓ and U_p as analytic functions on $\mathscr{C}$ bounded by 1. For any $k \in \mathcal{W}$, the points of $\kappa^{-1}(k) \subset \mathscr{C}$ are in bijection with the set of systems of eigenvalues for $\{T_\ell, U_p; \ell \nmid Np\}$ acting on the space of finite slope overconvergent eigenforms of weight k and tame level dividing N.

The locus of $\mathscr{C}$ (resp. $\mathscr{C}_{\mathrm{cusp}}$) where $|U_p|_p = 1$ is open and closed in $\mathscr{C}$ (resp. $\mathscr{C}_{\mathrm{cusp}}$), and is called the ordinary locus. As the classical weights are Zariski dense in $\mathcal{W}$, it follows from the classicality criterion for overconvergent forms (see [13]) that the classical points are Zariski dense in $\mathscr{C}$. Moreover it follows from [1, Corollary 2.6] that $\mathscr{C}_{\mathrm{cusp}}$ has a Zariski dense set of points corresponding to classical cuspforms of weight at least 2.

Since $\mathrm{G}_{\mathbb{Q}}$ is compact and $\mathcal{O}_{\mathscr{C}}(\mathscr{C})$ is a reduced ring, there exists (see [11, §7]) a unique continuous two-dimensional pseudo-character

$$\mathrm{G}_{\mathbb{Q}} \to \mathcal{O}_{\mathscr{C}}(\mathscr{C}) \tag{30}$$

sending Frob_ℓ to T_ℓ for all $\ell \nmid Np$, and whose specialisation at any classical point $g \in \mathscr{C}(\bar{\mathbb{Q}}_p)$ equals the trace of the Galois representation $\rho_g : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ constructed by Eichler-Shimura, Deligne and Deligne-Serre.

Let $X_{\mathrm{Iw}}/\mathbb{Z}_p$ be the proper flat semi-stable modular curve of level $\Gamma \cap \Gamma_0(p)$ endowed with a canonical morphism $X_{\mathrm{Iw}} \to X$ obtained by forgetting the Iwahori $\Gamma_0(p)$-level structure (see [19, pp.101, 144]). The theory of the canonical subgroup yields, for $v > 0$ sufficiently small, a section of the latter morphism

$$X(v) \to X_{\mathrm{Iw}}^{\times}(v), \tag{31}$$

where $X_{\mathrm{Iw}}^{\times}(v)$ is a neighbourhood of the connected (multiplicative) component of the ordinary locus in $X_{\mathrm{Iw}}^{\mathrm{rig}}$, containing the cusp ∞. Given a classical modular form of level $\Gamma \cap \Gamma_0(p)$, the pullback of its restriction to $X_{\mathrm{Iw}}^{\times}(v)$ along this section yields an overconvergent modular form having the same q-expansion at ∞. In particular, classical eigenforms of level $\Gamma_1(N) \cap \Gamma_0(p)$ give rise to points in $\mathscr{C}$. Furthermore, if the classical eigenform vanishes at all cusps of $X_{\mathrm{Iw}}^{\times}(v)$, *i.e.*, at all cusps lying in the $\Gamma_0(p)$-orbit of ∞, then the corresponding point belongs to $\mathscr{C}_{\mathrm{cusp}}$.

3.2. **Evaluation of ordinary families at cusps.** The set of connected components of the cuspidal divisor $D_{\mathcal{U}}$ of $X(0) \times \mathcal{U}$ is indexed by $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$, thus is fibered over $\Gamma_1(N) \backslash \mathbb{P}^1(\mathbb{Q})$.

Proposition 3.1. *Evaluation at the cusps gives the following exact sequence of $\mathcal{O}(\mathcal{U})$-modules:*

$$0 \to S_{\mathcal{U}} \to M_{\mathcal{U}} \xrightarrow{\mathrm{res}_{\mathcal{U}}} \prod_{[\delta] \in \Gamma_1(N) \backslash \mathbb{P}^1(\mathbb{Q})} \mathcal{O}(\mathcal{U}) \to 0. \tag{32}$$

Proof. We have an exact sequence of sheaves on $X(0) \times \mathcal{U}$:

$$0 \to \omega_{\mathcal{U}}(-D_{\mathcal{U}}) \to \omega_{\mathcal{U}} \to \omega_{\mathcal{U}}/\omega_{\mathcal{U}}(-D_{\mathcal{U}}) \to 0 \tag{33}$$

where the support of quotient sheaf $\omega_{\mathcal{U}}/\omega_{\mathcal{U}}(-D_{\mathcal{U}})$ is $D_{\mathcal{U}}$. Hence

$$\mathrm{H}^0(X(0) \times \mathcal{U}, \omega_{\mathcal{U}}/\omega_{\mathcal{U}}(-D_{\mathcal{U}})) = \mathrm{H}^0(D_{\mathcal{U}}, \omega_{\mathcal{U}}/\omega_{\mathcal{U}}(-D_{\mathcal{U}})) = \mathrm{H}^0(D_{\mathcal{U}}, \omega_{\mathcal{U}}) = \prod_{[\delta] \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} \mathcal{O}(\mathcal{U}).$$

Since $X(0) \times \mathcal{U}$ is an affinoid and $\omega_{\mathcal{U}}(-D_{\mathcal{U}})$ is a coherent sheaf (even invertible), one has

$$\mathrm{H}^1(X(0) \times \mathcal{U}, \omega_{\mathcal{U}}(-D_{\mathcal{U}})) = 0.$$

Applying the functor global sections $\mathrm{H}^0(X(0) \times \mathcal{U}, -)$ to (33), and further taking $(\Gamma_1(N)/\Gamma)$-invariants, yields the desired result. □

We recall that $M_{\mathcal{U}}^{\mathrm{ord}} = e^{\mathrm{ord}}(M_{\mathcal{U}})$ and $S_{\mathcal{U}}^{\mathrm{ord}} = e^{\mathrm{ord}}(S_{\mathcal{U}})$, where $e^{\mathrm{ord}} = \lim_{n \to \infty} U_p^{n!}$ denotes Hida's ordinary idempotent. Applying this idempotent to (32) yields the following result.

Corollary 3.2. *There exists a direct factor* $C_{\mathcal{U}}$ *of* $\prod_{[\delta]\in\Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q})} \mathcal{O}(\mathcal{U})$ *and an exact sequence of* $\mathcal{O}(\mathcal{U})$*-modules:*

$$0 \to S_{\mathcal{U}}^{\mathrm{ord}} \to M_{\mathcal{U}}^{\mathrm{ord}} \xrightarrow{\mathrm{res}_{\mathcal{U}}} C_{\mathcal{U}} \to 0. \tag{34}$$

Remark 3.3. Under the assumption that $p \geqslant 5$, Ohta gave in [39, 40] a different description of the residue map on the ordinary part and proved its surjectivity.

3.3. **Construction of an irreducible deformation of** ρ**.** Consider the p-stabilised Eisenstein series f as in the introduction. By [16, Proposition 1.3] (or Proposition 4.7 further below) the weight 1 eigenform f vanishes at all cusps of X_{Iw} lying in the $\Gamma_0(p)$-orbit of ∞, hence as explained above defines a point f in $\mathscr{C}_{\mathrm{cusp}}$, which lies in the ordinary locus as $U_p(f) = 1$. Recall that $\mathcal{T}$ (resp. $\mathcal{T}_{\mathrm{cusp}}$) denotes the completed strict local ring of $\mathscr{C}$ (resp. $\mathscr{C}_{\mathrm{cusp}}$) at f and $\mathfrak{m}_{\mathcal{T}}$ (resp. $\mathfrak{m}_{\mathcal{T}_{\mathrm{cusp}}}$) its maximal ideal.

Proposition 3.4. *Let* $\chi_{\mathcal{T}_{\mathrm{cusp}}} : \mathrm{G}_{\mathbb{Q}_p} \to \mathcal{T}_{\mathrm{cusp}}^{\times}$ *be the unramified character sending* Frob_p *to* U_p*.*

(i) *There exists an irreducible deformation* $\rho_{\mathcal{T}_{\mathrm{cusp}}} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{T}_{\mathrm{cusp}})$ *of* ρ *such that* $\mathrm{tr}(\rho_{\mathcal{T}_{\mathrm{cusp}}})(\mathrm{Frob}_\ell) = T_\ell$ *for all* $\ell \nmid Np$*, and* $\rho_{\mathcal{T}_{\mathrm{cusp}}|\mathrm{G}_{\mathbb{Q}_p}} = \left(\begin{smallmatrix} * & * \\ 0 & \chi_{\mathcal{T}_{\mathrm{cusp}}}\end{smallmatrix}\right)$*.*

(ii) *There exists an irreducible deformation* $\rho'_{\mathcal{T}_{\mathrm{cusp}}} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{T}_{\mathrm{cusp}})$ *of* ρ' *such that* $\det(\rho'_{\mathcal{T}_{\mathrm{cusp}}}) = \det(\rho_{\mathcal{T}_{\mathrm{cusp}}})$*,* $\mathrm{tr}(\rho'_{\mathcal{T}_{\mathrm{cusp}}}) = \mathrm{tr}(\rho_{\mathcal{T}_{\mathrm{cusp}}})$ *and* $\rho'_{\mathcal{T}_{\mathrm{cusp}}|\mathrm{G}_{\mathbb{Q}_p}} = \left(\begin{smallmatrix} * & * \\ 0 & \chi_{\mathcal{T}_{\mathrm{cusp}}}\end{smallmatrix}\right)$*.*

Proof. Since $\mathcal{T}_{\mathrm{cusp}}$ is reduced, it injects in its total quotient field L. By [48, Theorem 4] the 2-dimensional pseudo-character $\mathrm{G}_{\mathbb{Q}} \to \mathcal{O}_{\mathscr{C}}(\mathscr{C}) \to \mathcal{T}_{\mathrm{cusp}} \to L$ lifting $\phi + \mathbf{1}$ (see (30)) gives rise to a p-ordinary representation $\rho_L : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(L)$ such that $\rho_L(\tau) = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ and $\mathrm{tr}(\rho_L)(\mathrm{Frob}_\ell) = T_\ell$ for $\ell \nmid Np$. Moreover, each component of ρ_L is absolutely irreducible, as the classical cuspidal points of $\mathscr{C}_{\mathrm{cusp}}$ form a Zariski dense subset (see §3.1). As $\dim_{\bar{\mathbb{Q}}_p} \mathrm{H}^1(\mathbb{Q}, \phi) = 1$, [2, Corollary 2] implies that there exists a conjugate of $\rho_L : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(L)$, by a diagonal matrix, taking values in $\mathrm{GL}_2(\mathcal{T}_{\mathrm{cusp}})$ and reducing to ρ modulo $\mathfrak{m}_{\mathcal{T}_{\mathrm{cusp}}}$. We denote this representation $\rho_{\mathcal{T}_{\mathrm{cusp}}}$. The same argument applied to ϕ^{-1} instead of ϕ yields a representation $\rho'_{\mathcal{T}_{\mathrm{cusp}}}$ whose trace and determinant agree with those of $\rho_{\mathcal{T}_{\mathrm{cusp}}}$, because they can be compared in L, where they are equal by definition.

In order to prove the statement about the restrictions to $\mathrm{G}_{\mathbb{Q}_p}$, we write an exact sequence of $\mathcal{T}_{\mathrm{cusp}}$-modules analogous to the one in [4, (20)] and adapt the argument as follows. The fact that $\rho_{|\mathrm{I}_{\mathbb{Q}_p}}$ (resp. $\rho'_{|\mathrm{I}_{\mathbb{Q}_p}}$) has infinite image and admits a unique $\mathrm{I}_{\mathbb{Q}_p}$-stable line, shows that the last term of that exact sequence is a monogenic $\mathcal{T}_{\mathrm{cusp}}$-module, hence it is free, because it is generically free of rank 1 and $\mathcal{T}_{\mathrm{cusp}}$ is reduced. □

3.4. **Etaleness of** $\mathscr{C}_{\mathrm{cusp}}$ **over** $\mathcal{W}$ **at** f**.** In this section we give a proof of Theorem A(i).

Let Λ be the completed strict local ring of $\mathcal{W}$ at $\kappa(f)$. The weight map κ induces a finite flat homomorphism $\kappa^{\#} : \Lambda \to \mathcal{T}_{\mathrm{cusp}}$ of reduced complete local rings. The local ring at f of the fibre $\kappa^{-1}(\kappa(f))$ is an Artinian local $\bar{\mathbb{Q}}_p$-algebra given by

$\mathcal{T}^0_{\mathrm{cusp}} = \mathcal{T}_{\mathrm{cusp}}/\mathfrak{m}_\Lambda \mathcal{T}_{\mathrm{cusp}}$. We will prove in this section that $\mathcal{T}_{\mathrm{cusp}}$ is étale over Λ, or equivalently that $\mathcal{T}^0_{\mathrm{cusp}} \simeq \bar{\mathbb{Q}}_p$.

According to Proposition 3.4 we have $(\rho_{\mathcal{T}_{\mathrm{cusp}}}, \rho'_{\mathcal{T}_{\mathrm{cusp}}}) \in \mathcal{D}_{\mathrm{cusp}}(\mathcal{T}_{\mathrm{cusp}})$ yielding by functoriality a homomorphism of local rings

$$\varphi_{\mathrm{cusp}} : \mathcal{R}_{\mathrm{cusp}} \to \mathcal{T}_{\mathrm{cusp}}. \tag{35}$$

As Langlands' correspondence relates the determinant to the central character, the homomorphism (35) is Λ-linear (see for example [4, Proposition 6.11]). Reducing modulo $\mathfrak{m}_\Lambda$ yields:

$$\varphi^0_{\mathrm{cusp}} : \mathcal{R}^0_{\mathrm{cusp}} \to \mathcal{T}^0_{\mathrm{cusp}}.$$

Theorem 3.5. *(i) Both φ_{cusp} and $\varphi^0_{\mathrm{cusp}}$ are isomorphisms.*

(ii) The homomorphism $\kappa^\# : \Lambda \to \mathcal{T}_{\mathrm{cusp}}$ is an isomorphisms.

Proof. (i) The local ring $\mathcal{T}_{\mathrm{cusp}}$ is topologically generated over Λ by U_p and T_ℓ for $\ell \nmid Np$. A direct consequence of Proposition 3.4 is that all those elements belong to the image of φ_{cusp} proving its surjectivity. Since by Proposition 2.6 the dimension of the tangent space of $\mathcal{R}_{\mathrm{cusp}}$ is 1 and the local ring $\mathcal{T}_{\mathrm{cusp}}$ is equidimensional of dimension 1, the surjective homomorphism φ_{cusp} is necessarily an isomorphism of regular complete local rings. Reducing modulo $\mathfrak{m}_\Lambda$ proves that $\varphi^0_{\mathrm{cusp}}$ is an isomorphism as well.

(ii) By Proposition 2.6 the tangent space of $\mathcal{R}^0_{\mathrm{cusp}}$ is trivial, hence $\mathcal{T}^0_{\mathrm{cusp}}$ has trivial tangent space as well, *i.e.* $\mathcal{T}^0_{\mathrm{cusp}} \simeq \bar{\mathbb{Q}}_p$. Hence $\Lambda \to \mathcal{T}_{\mathrm{cusp}}$ is unramified and therefore it is an isomorphism, because both Λ and $\mathcal{T}_{\mathrm{cusp}}$ are complete local rings with same residue field. □

3.5. **Eisenstein components containing f.** The eigencurve $\mathscr{C}$ has two irreducible components corresponding to Eisenstein families containing f. In this subsection, we relate the completed strict local rings of these components at f to universal deformation rings.

By Proposition 2.8, one can attach to $\mathcal{E}_{\mathbf{1},\phi}$ a $\mathrm{G}_{\mathbb{Q}}$-reducible deformation of ρ:

$$\rho_{\mathrm{eis}} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\Lambda), \text{ such that } \mathrm{tr}(\rho_{\mathrm{eis}}) = \phi\chi_p + \mathbf{1} \text{ and } \det(\rho_{\mathrm{eis}}) = \phi\chi_p. \tag{36}$$

The irreducible component of $\mathscr{C}$ corresponding to the Eisenstein family $\mathcal{E}_{\mathbf{1},\phi}$ is étale over the weight space, hence the completed strict local ring $\mathcal{T}^{\mathrm{eis}}_\rho$ of this component at f is isomorphic to Λ.

Since ρ_{eis} admits a trivial (hence unramified at p) rank 1 quotient, it defines an element of $\mathcal{D}^{\mathrm{eis}}_\rho(\Lambda)$ (see Definition 1.4), *i.e.* a $\bar{\mathbb{Q}}_p$-algebra homomorphism

$$\mathcal{R}^{\mathrm{eis}}_\rho \to \Lambda. \tag{37}$$

Lemma 3.6. *The map (37) is an isomorphism of Λ-algebras.*

Proof. We explained in §1.2 that the Λ-algebra structure of $\mathcal{R}^{\mathrm{ord}}_\rho$ comes from the determinant, and the same is true for its quotient $\mathcal{R}^{\mathrm{eis}}_\rho$. As $\det \rho_{\mathrm{eis}} = \phi\chi_p$ it follows that (37) is a Λ-algebra homomorphism, in particular it is surjective. By Lemma 2.1 the tangent space of $\mathcal{R}^{\mathrm{eis}}_\rho$ is 1-dimensional, hence (37) is an isomorphism. □

Let $\mathcal{T}_{\rho'}^{\mathrm{eis}}$ be the completed strict local ring of the irreducible component of $\mathscr{C}$ corresponding to the Eisenstein family $\mathcal{E}_{\phi,\mathbf{1}}$. Similarly to Definition 1.4 let $\mathcal{D}_{\rho'}^{\mathrm{eis}}$ be the sub functor of $\mathcal{D}_{\rho'}^{\mathrm{ord}}$ consisting of deformations which are reducible and ordinary for the same filtration. An argument as in Lemmas 1.5 and 3.6 with $\rho' \otimes \phi^{-1}$ (resp. ϕ^{-1}) replaced by ρ (resp. ϕ) shows that $\mathcal{D}_{\rho'}^{\mathrm{eis}}$ is pro-representable by a regular ring $\mathcal{R}_{\rho'}^{\mathrm{eis}} \xrightarrow{\sim} \Lambda \xrightarrow{\sim} \mathcal{T}_{\rho'}^{\mathrm{eis}}$, and that the structural homomorphism $\kappa^{\#} : \Lambda \xrightarrow{\sim} \mathcal{T}_{\rho'}^{\mathrm{eis}}$ is an isomorphism.

4. The Ferrero–Greenberg Theorem and modularity

We use the notations from §2.4. For any odd Dirichlet character $\psi \neq \omega_p^{-1}$, the Kubota–Leopoldt p-adic L-function $L_p(\psi\omega_p, s)$ is analytic in $s \in \mathbb{Z}_p$ and characterised by the interpolation property that for all integers $k \geqslant 1$ such that $\omega_p^{k-1} = \mathbf{1}$ one has:

$$L_p(\psi\omega_p, 1-k) = (1 - \psi(p)p^{k-1})L(\psi, 1-k). \tag{38}$$

As $\phi(p) = 1$, it follows that $L_p(\phi\omega_p, s)$ has a trivial zero at $s = 0$. As well known (see [47, Theorem 7.10]), there exists an element $\zeta_\phi(X) \in \mathbb{Z}_p[\![X]\!] \simeq \mathbb{Z}_p[\![1 + p^\nu \mathbb{Z}_p]\!]$ such that for $k \in \mathbb{Z}_{\geqslant 1}$:

$$\zeta_\phi((1+p^\nu)^{k-1} - 1) = L_p(\phi\omega_p, 1-k). \tag{39}$$

4.1. **The Eisenstein ideal.** In this section we study the congruences between the Eisenstein family $\mathcal{E}_{\mathbf{1},\phi}$ and the unique cuspidal family $\mathcal{F}$ specialising to f. To achieve our goal we define an appropriate quotient of the completed strict local ring $\mathcal{T}$ of $\mathscr{C}$ at f which has two generic points, one corresponding to $\mathcal{F}$ and the other to $\mathcal{E}_{\mathbf{1},\phi}$. We let $\mathcal{T}_\rho^{\mathrm{ord}}$ be the image of the abstract Hecke algebra $\Lambda[U_p, T_\ell]_{\ell \nmid pN}$ in $\mathcal{T}_{\mathrm{cusp}} \times_{\bar{\mathbb{Q}}_p} \Lambda^{\mathrm{eis}}$, where $\Lambda^{\mathrm{eis}} = \Lambda$ is the Eisenstein Hecke algebra corresponding to $\mathcal{E}_{\mathbf{1},\phi}$. By definition $\mathcal{T}_\rho^{\mathrm{ord}}$ is a reduced local quotient of $\mathcal{T}$, and is in fact a Λ-sub-algebra of $\mathcal{T}_{\mathrm{cusp}} \times_{\bar{\mathbb{Q}}_p} \Lambda^{\mathrm{eis}}$ surjecting to each one of the factors.

Since Λ is a discrete valuation ring, Hida's congruence module yoga gives an integer $e \geqslant 1$, such that $\mathcal{T}_\rho^{\mathrm{ord}} = \mathcal{T}_{\mathrm{cusp}} \times_{\bar{\mathbb{Q}}_p[X]/(X^e)} \Lambda^{\mathrm{eis}}$. Indeed, if one considers the natural projections

$$\pi_{\mathrm{cusp}} : \mathcal{T}_\rho^{\mathrm{ord}} \twoheadrightarrow \mathcal{T}_{\mathrm{cusp}} \text{ and } \pi_{\mathrm{eis}} : \mathcal{T}_\rho^{\mathrm{ord}} \twoheadrightarrow \Lambda^{\mathrm{eis}}$$

and let $J_{\mathrm{eis}} = \pi_{\mathrm{cusp}}(\ker(\pi_{\mathrm{eis}}))$ and $(X^e) = \pi_{\mathrm{eis}}(\ker(\pi_{\mathrm{cusp}}))$, one obtains a Cartesian diagram

$$\begin{array}{ccccc} & & \ker(\pi_{\mathrm{cusp}}) & \xrightarrow{\sim} & (X^e) \\ & & \downarrow & & \downarrow \\ \ker(\pi_{\mathrm{eis}}) & \hookrightarrow & \mathcal{T}_\rho^{\mathrm{ord}} & \xrightarrow{\pi_{\mathrm{eis}}} & \Lambda^{\mathrm{eis}} \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\pi_{\mathrm{cusp}}} & & \downarrow \\ J_{\mathrm{eis}} & \hookrightarrow & \mathcal{T}_{\mathrm{cusp}} & \twoheadrightarrow & \mathcal{T}_{\mathrm{cusp}}/J_{\mathrm{eis}} \xrightarrow{\sim} \Lambda^{\mathrm{eis}}/(X^e), \end{array} \tag{40}$$

justifying the Λ-algebras isomorphisms $\mathcal{T}_{\mathrm{cusp}}/J_{\mathrm{eis}} \xrightarrow{\sim} \Lambda^{\mathrm{eis}}/(X^e)$ and

$$\mathcal{T}_\rho^{\mathrm{ord}} = \left\{(a,b) \in \mathcal{T}_{\mathrm{cusp}} \times \Lambda^{\mathrm{eis}} \mid (a \mod J_{\mathrm{eis}}) = (b \mod X^e)\right\}.$$

Lemma 4.1. *One has* $\mathrm{Ann}(\ker(\pi_{\mathrm{eis}})) = \ker(\pi_{\mathrm{cusp}})$.

Proof. The above description of $\mathcal{T}_{\mathrm{cusp}}$ as fibre product implies that $\ker(\pi_{\mathrm{cusp}}) \subset \mathrm{Ann}(\ker(\pi_{\mathrm{eis}}))$. Suppose that $(a,b) \in \mathrm{Ann}(\ker(\pi_{\mathrm{eis}})) \subset \mathcal{T}_{\mathrm{cusp}} \times \Lambda^{\mathrm{eis}}$. If $a \neq 0$, then there would exist a cuspidal family $\mathcal{G}$ such that $a \cdot \mathcal{G} \neq 0$. The Eisenstein ideal $\ker(\pi_{\mathrm{eis}})$ would then annihilate the cuspidal family $a \cdot \mathcal{G}$, meaning that $\mathcal{T}_{\rho}^{\mathrm{ord}}$ would act on both $a \cdot \mathcal{G}$ and on $\mathcal{E}_{\mathbf{1},\phi}$ through its quotient Λ^{eis}. Hence $a \cdot \mathcal{G}$ is a Λ-adic eigenform having the same eigenvalues as $\mathcal{E}_{\mathbf{1},\phi}$. This is impossible as cuspidal eigenforms have irreducible Galois representations. □

Proposition 4.2. *One has* $e = 1$ *and* $\mathcal{T}_{\rho}^{\mathrm{ord}} \xrightarrow{\sim} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda$.

Proof. By Theorem 3.5 one knows that $\mathcal{T}_{\mathrm{cusp}} \xrightarrow{\sim} \Lambda$. Moreover the image of U_p in Λ^{eis} is 1, whereas by the proof of Proposition 2.6 its image in $\mathcal{T}_{\mathrm{cusp}}/(X^2) \simeq \mathcal{R}_{\mathrm{cusp}}/(X^2)$ equals $(1 - \mathscr{L}(\phi)X)$ with $\mathscr{L}(\phi) \neq 0$, where X is a topological generator of $\mathcal{T}_{\mathrm{cusp}}$ lifting the element of t_{cusp} corresponding to $\mu = 1$. □

4.2. **The full eigencurve and a duality.** Let $\mathscr{C}^{\mathrm{full}}$ be the p-adic eigencurve over $\mathcal{W}$ of tame level N constructed using the Hecke operators T_ℓ for $\ell \nmid Np$ and U_ℓ for $\ell \mid Np$. It is endowed with a locally finite surjective morphism $\mathscr{C}^{\mathrm{full}} \to \mathscr{C}$. Using the relations between abstract Hecke operators and the fact that the diamond operators at all $\ell \nmid Np$ belong to $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$, one sees that $T_n \in \mathcal{O}_{\mathscr{C}^{\mathrm{full}}}(\mathscr{C}^{\mathrm{full}})$ for all $n \geqslant 1$. There is a natural bijection between $\mathscr{C}^{\mathrm{full}}(\bar{\mathbb{Q}}_p)$ and the set of systems of eigenvalues of overconvergent eigenforms with finite slope, tame level dividing N and weight in $\mathcal{W}(\bar{\mathbb{Q}}_p)$, sending g to $\{(T_\ell(g))_{\ell \nmid Np}, (U_\ell(g))_{\ell \mid Np}\}$. Let $\mathcal{T}^{\mathrm{full}}$ be the completed strict local ring of $\mathscr{C}^{\mathrm{full}}$ at f. Let $\mathscr{C}^{\mathrm{full}}_{\mathrm{cusp}}$ be the closed analytic subspace of $\mathscr{C}^{\mathrm{full}}$ corresponding to cuspidal overconvergent modular forms and let $\mathcal{T}^{\mathrm{full}}_{\mathrm{cusp}}$ be the completed strict local ring of $\mathscr{C}^{\mathrm{full}}_{\mathrm{cusp}}$ at f.

Remark 4.3. By construction, the ordinary locus of $\mathscr{C}^{\mathrm{full}}$ (resp. $\mathscr{C}^{\mathrm{full}}_{\mathrm{cusp}}$) is isomorphic to the rigid analytic space attached to the maximal spectrum of the generic fibre of the p-ordinary Hecke algebra (resp. p-ordinary cuspidal Hecke algebra) of tame level N constructed by Hida [27]. As Galois orbits of cuspidal Hida families are in bijection with irreducible components of the ordinary locus of $\mathscr{C}^{\mathrm{full}}_{\mathrm{cusp}}$, Theorem 3.5 combined with Proposition 4.4 below shows that there exists a unique, up to Galois conjugacy, cuspidal Hida family specialising to f (see [20] for more details).

Proposition 4.4. *The natural injection* $\mathcal{T}_{\mathrm{cusp}} \hookrightarrow \mathcal{T}^{\mathrm{full}}_{\mathrm{cusp}}$ *is an isomorphism. Moreover, for any prime* ℓ *dividing* N, $\rho_{\mathcal{T}_{\mathrm{cusp}}}(\mathrm{I}_{\mathbb{Q}_\ell})$ *is finite, the* $\mathcal{T}_{\mathrm{cusp}}$*-module* $(\rho_{\mathcal{T}_{\mathrm{cusp}}})^{\mathrm{I}_{\mathbb{Q}_\ell}}$ *is free of rank* 1 *and* Frob_ℓ *acts on it as* U_ℓ.

Proof. By [3, Lemma 4.3.7] there exists an open admissible affinoid neighbourhood $\mathcal{V}$ of f in $\mathscr{C}_{\mathrm{cusp}}$ endowed with a representation $\rho_{\mathcal{V}} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}(\mathcal{V}))$ extending $\rho_{\mathcal{T}_{\mathrm{cusp}}}$. Assuming $\mathcal{V}$ to be connected which one always can, [3, Lemma 7.8.17] guarantees that the restriction of the pseudo-character $\mathrm{tr}(\rho_{\mathcal{V}})$ to $\mathrm{I}_{\mathbb{Q}_\ell}$ is constant and equal to $\phi + \mathbf{1}$. As $\phi_{|\mathrm{I}_{\mathbb{Q}_\ell}} \neq \mathbf{1}$, using the description of the Weil-Deligne representation attached to the Steinberg representation, for any classical point g in $\mathcal{V}$ the monodromy operator of Weil-Deligne representation of $\rho_{g|\mathrm{G}_{\mathbb{Q}_\ell}}$ vanishes. It follows that the monodromy operator of the Weil-Deligne representation attached to $\rho_{\mathcal{V}|\mathrm{G}_{\mathbb{Q}_\ell}}$ in [3, Lemma 7.8.14] vanishes as well, and the restriction of $\rho_{\mathcal{V}}$ to $\mathrm{I}_{\mathbb{Q}_\ell}$ is constant equal to $\phi \oplus \mathbf{1}$. In particular $\rho_{\mathcal{T}_{\mathrm{cusp}}}(\mathrm{I}_{\mathbb{Q}_\ell})$ is finite and the $\mathrm{I}_{\mathbb{Q}_\ell}$-invariants of $\rho_{\mathcal{T}_{\mathrm{cusp}}}$ are a rank 1 direct summand. It remains to show that Frob_ℓ acts $(\rho_{\mathcal{V}})^{\mathrm{I}_{\mathbb{Q}_\ell}}$ via U_ℓ, and thus $U_\ell \in \mathcal{T}_{\mathrm{cusp}}$. By local-global compatibility at ℓ, any classical point g in $\mathcal{V}$ is new at ℓ and $\rho_g(\mathrm{Frob}_\ell)$ acts on the line $(\rho_g)^{\mathrm{I}_{\mathbb{Q}_\ell}}$ by the U_ℓ-eigenvalue of g (see also [4, Proposition 7.1]). One then concludes using Zariski density. □

Let $S^\dagger_{\mathfrak{m}_f}$ (resp. $M^\dagger_{\mathfrak{m}_f}$) be the $\mathcal{T}_{\mathrm{cusp}}$-module (resp. $\mathcal{T}$-module) obtained by localising and completing $S^{\mathrm{ord}}_{\mathcal{U}}$ at the maximal ideal $\mathfrak{m}_f$ of the abstract Hecke algebra $\Lambda[U_p, T_\ell]_{\ell \nmid pN}$ corresponding to the system of Hecke eigenvalues of f. In view of Proposition 4.4, the works of Hida [28, §2] and Coleman [14, Proposition B.5.6] yield a natural isomorphism of Λ-modules:

$$S^\dagger_{\mathfrak{m}_f} \xrightarrow{\sim} \mathrm{Hom}_\Lambda(\mathcal{T}_{\mathrm{cusp}}, \Lambda), \quad \mathcal{G} \mapsto (T \mapsto a_1(T \cdot \mathcal{G})). \tag{41}$$

Corollary 4.5. *The generalised cuspidal eigenspace $S^\dagger[\![f]\!]$ equals $\bar{\mathbb{Q}}_p f$, and $S^\dagger_{\mathfrak{m}_f} = \Lambda \cdot \mathcal{F}$ for a unique normalised cuspidal eigenfamily $\mathcal{F}$.*

Proof. As $\mathcal{T}_{\mathrm{cusp}}$ is canonically isomorphic to Λ by Theorem 3.5, the identity map in the right hand side of (41) corresponds to a normalised cuspidal eigenfamily $\mathcal{F}$, which is a basis of $S^\dagger_{\mathfrak{m}_f}$ over Λ. By definition $S^\dagger[\![f]\!]$ is isomorphic to the $\mathcal{T}/\mathfrak{m}_\Lambda\mathcal{T}$-module $S^\dagger_{\mathfrak{m}_f}/\mathfrak{m}_\Lambda S^\dagger_{\mathfrak{m}_f}$, hence is 1-dimensional and spanned by f. □

Remark 4.6. Note that $S^\dagger[\![f]\!]$ is spanned by f which is classical and cuspidal-overconvergent, even though it is not cuspidal as a classical form. This result was first conjectured in [16, Hypothesis (C')] for the following arithmetic application. Let E be an elliptic curve over $\mathbb{Q}$ and g be a classical weight 1 form. When the analytic rank of E over the splitting field of $\rho_f \otimes \rho_g$ is 2, the elliptic Stark conjecture of [16] relates the values of some p-adic iterated integrals to formal group logarithms of global points of E. The classicality of $S^\dagger[\![f]\!]$ plays a crucial role in the formulation of the conjecture, as the definition of the p-adic iterated integrals relies on that assumption. The scenario in which f and g are both irregular Eisenstein series is particularly appealing because the splitting field is then cyclotomic. Numerical evidence towards this conjecture are given in §7 of *loc. cit.* and recently Rotger–Casazza [10] established a result in that direction, under the hypotheses (C) and (C') from *loc. cit.* which can now be waived thanks to Corollary 4.5.

4.3. **On the constant term of Eisenstein families.** The aim of this section is to show that the Kubota–Leopoldt p-adic L-function $L_p(\phi\omega_p, s)$ has a simple trivial zero at $s = 0$, or equivalently that $\zeta_\phi(X) \in \Lambda$ vanishes at order 1 at $X = 0$. From the interpolation property (38) we know that $\zeta_\phi(0) = 0$, so it suffices to prove that the order of vanishing is at most 1. To achieve this we first relate $\zeta_\phi(X)$ to the constant term of the Eisenstein family $\mathcal{E}_{\mathbf{1},\phi}$.

Let $\mathcal{U}$ be an open admissible affinoid of $\mathcal{W}$ containing the weight $\kappa(f)$ of f. By localising and completing (34) at the maximal ideal corresponding to $\kappa(f) \in \mathcal{U}$, we obtain an exact sequence of flat Λ-modules:

$$0 \to S^{\mathrm{ord}}_\Lambda \to M^{\mathrm{ord}}_\Lambda \xrightarrow{\mathrm{res}_\Lambda} C_\Lambda \to 0, \tag{42}$$

where C_Λ is a direct factor of $\prod_{[\delta]\in\Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q})} \Lambda$.

Let $A_\delta(\mathbf{1},\phi)$, resp. $A_\delta(\phi,\mathbf{1})$, be the constant terms of $\mathcal{E}_{\mathbf{1},\phi}$, resp. $\mathcal{E}_{\phi,\mathbf{1}}$, at $[\delta]\in\Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q})$.

Proposition 4.7.

(i) One has $A_\infty(\mathbf{1},\phi)=\frac{1}{2}\zeta_\phi$, $A_0(\mathbf{1},\phi)=0$ *and* $A_\delta(\mathbf{1},\phi)\in\Lambda\cdot\zeta_\phi$ *for all* δ.
(ii) One has $A_\infty(\phi,\mathbf{1})=0$, $A_0(\phi,\mathbf{1})\in\Lambda^\times\zeta_{\phi^{-1}}$ *and* $A_\delta(\phi,\mathbf{1})\in\Lambda\cdot\zeta_{\phi^{-1}}$ *for all* δ.

Proof. (i) We will establish the Proposition via a computation of the constant term of the specialisations of the Eisenstein families $\mathcal{E}_{\mathbf{1},\phi}$ and $\mathcal{E}_{\phi,\mathbf{1}}$ at all classical weights $k\geqslant 2$ such that $\omega_p^{k-1}=\mathbf{1}$. Using the notations from §3.1, the set $\Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q})$ is in bijection with the $\Gamma_1(N)$-orbits of cusps of X. Similarly, the sets $(\Gamma_1(N)\cap\Gamma_0(p))\backslash\mathbb{P}^1(\mathbb{Q})$ and $(\Gamma_1(N)\cap\Gamma_0(p))\backslash(\Gamma_0(p)\infty)$ are in bijection with the $\Gamma_1(N)\cap\Gamma_0(p)$-orbits of cusps in X_{Iw} and in the multiplicative locus $X_{\mathrm{Iw}}^\times(0)$, respectively. On the level of cusps the canonical section (31) is simply given by the inverse of the natural bijection

$$(\Gamma\cap\Gamma_0(p))\backslash(\Gamma_0(p)\infty)\xrightarrow{\sim}\Gamma\backslash\mathbb{P}^1(\mathbb{Q}) \tag{43}$$

allowing us to represent any $\Gamma_1(N)$-orbit of cusps of X with an element $\delta=\left[\begin{smallmatrix}a\\c\end{smallmatrix}\right]\in\Gamma_0(p)\infty$ such that the integers a and c are relatively prime and $p\mid c$.

For every $g\in M_k(\Gamma_1(N),\phi)$, we define $g^{(p)}(z)=(g-g_{|\iota})(z)=g(z)-p^{k-1}g(pz)\in M_k(\Gamma,\phi)$, where $\iota=\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right)$. The modular form g (resp. $g^{(p)}$) can be evaluated at cusps of X (resp. X_{Iw}); moreover by $\Gamma_1(N)$-invariance (resp. $(\Gamma_1(N)\cap\Gamma_0(p))$-invariance) of g (resp. $g^{(p)}$) the value is well-defined on $\Gamma_1(N)$-orbits (resp. $(\Gamma_1(N)\cap\Gamma_0(p))$-orbits) of cusps. We first compute the q-expansion of $g^{(p)}$ at the cusp δ, in terms of q-expansions of g. Choose a matrix $\gamma_\delta=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$. The q-expansion of $g^{(p)}$ at the cusp δ is then given by

$$g^{(p)}_{|\gamma_\delta}(z)=g_{|\gamma_\delta}(z)-g_{|\iota\gamma_\delta}(z)=g_{|\gamma_\delta}(z)-g_{|\gamma_{p\delta}\iota}(z)=g_{|\gamma_\delta}(z)-p^{k-1}g_{|\gamma_{p\delta}}(pz),$$

where $\gamma_{p\delta}=\begin{pmatrix}a&bp\\cp^{-1}&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$. Letting A_δ (resp. $A_\delta^{(p)}$) denote the constant term of g (resp. $g^{(p)}$) at the cusp $\delta\in\mathbb{P}^1(\mathbb{Q})$ (resp. $\delta\in\Gamma_0(p)\infty$) we have

$$A_\delta^{(p)}=A_\delta-p^{k-1}A_{p\delta}.$$

Let now g be the weight k Eisenstein series $E_k(\mathbf{1},\phi)\in M_k(\Gamma_1(N),\phi)$. Then $g^{(p)}$ in the above notation is the ordinary p-stabilisation of g, and it is also the weight k specialisation of the Eisenstein family $\mathcal{E}_{\mathbf{1},\phi}$. By [41, Proposition 1.1], for $\delta=\left[\begin{smallmatrix}a\\c\end{smallmatrix}\right]$ we have

$$A_\delta=0\text{ , if }N\nmid c,\text{ and}\qquad A_\delta=\frac{\phi^{-1}(|a|)}{2}L(\phi,1-k)\text{ , if }N\mid c.$$

Thus, $A_\delta^{(p)}=0$ if $N\nmid c$, whereas if $N\mid c$ then using (38) and $\phi(p)=1$ one finds

$$A_\delta^{(p)}=A_\delta-p^{k-1}A_{p\delta}=(1-p^{k-1})\frac{\phi^{-1}(|a|)}{2}L(\phi,1-k)=\frac{\phi^{-1}(|a|)}{2}L_p(\phi\omega_p,1-k).$$

As the weights $k \geqslant 2$ such that $\omega_p^{k-1} = \mathbf{1}$ are Zariski dense in the connected component of $\mathcal{W}$ containing $\kappa(f)$, we deduce that $A_\delta(\mathbf{1}, \phi)$ equals $\frac{\phi^{-1}(|a|)}{2}\zeta_\phi(X)$ if $N \mid c$, and vanishes otherwise. In particular, $A_\infty(\mathbf{1}, \phi) = \frac{1}{2}\zeta_\phi$ and $A_0(\mathbf{1}, \phi) = 0$.

(ii) Let $g = E_k(\phi, \mathbf{1}) \in M_k(\Gamma_1(N), \phi)$ and let δ be as in (i). By [41, Proposition 1.1]

$$A_\delta = 0, \text{ if } (N, c) \neq 1, \text{ and } \qquad A_\delta = -\frac{\tau(\phi)}{2N^k}\phi(|c|)L(\phi^{-1}, 1-k), \text{ if } (N, c) = 1,$$

where $\tau(\phi)$ denotes the Gauss sum of ϕ. Thus, if $(c, N) \neq 1$ then $A_\delta^{(p)}$ vanishes, whereas if $(c, N) = 1$ then one finds

$$\begin{aligned} A_\delta^{(p)} = A_\delta - p^{k-1}A_{p\delta} &= -\frac{\tau(\phi)}{2N^k}\phi(|c|)(1 - p^{k-1})L(\phi^{-1}, 1-k) \\ &= -\frac{\tau(\phi)}{2N^k}\phi(|c|)L_p(\phi^{-1}\omega_p, 1-k). \end{aligned}$$

The form $g^{(p)}$ is the weight k ordinary specialisation of $\mathcal{E}_{\phi,\mathbf{1}}$. Since $(p, N) = 1$ and $\omega_p^{k-1} = \mathbf{1}$, N^k is the weight k specialisation of an element in $\Lambda^\times$, while $L_p(\phi^{-1}\omega_p, 1-k)$ is the weight k specialisation of $\zeta_{\phi^{-1}}(X)$. The claim then easily follows. □

We state a generalisation of a result of Wiles [49, Theorem 4.1] and Ohta [40, Corollary A.2.4] to the case of a trivial zero, and give an alternative proof of a famous result of Ferrero and Greenberg [24].

Proposition 4.8. *There exists an isomorphism of Λ-algebras $\mathcal{T}_{\mathrm{cusp}}/J_{\mathrm{eis}} \xrightarrow{\sim} \Lambda/(\zeta_\phi(X))$. The Kubota–Leopoldt p-adic L-function $\zeta_\phi(X)$ has a simple zero at $X = 0$.*

Proof. It follows from Proposition 4.7 and from (42) that $\mathcal{E}_{\mathbf{1},\phi} \mod (\zeta_\phi(X))$ is a cuspidal family. Using (41) and the freeness of $\mathcal{T}_{\mathrm{cusp}}$ over Λ one has

$$S^\dagger_{\mathfrak{m}_f} \otimes_\Lambda (\Lambda/(\zeta_\phi(X))) = \mathrm{Hom}_\Lambda(\mathcal{T}_{\mathrm{cusp}}, \Lambda/(\zeta_\phi(X))).$$

As $\mathcal{E}_{\mathbf{1},\phi}$ is an eigenfamily, the map $\mathcal{T}_{\mathrm{cusp}} \to \Lambda/(\zeta_\phi(X))$ resulting from the above equation yields (via Lemma 3.6) a Λ-algebra homomorphism $\mathcal{T}_{\mathrm{cusp}} \to \Lambda^{\mathrm{eis}}/(\zeta_\phi(X))$. As e defined above (40) is the largest integer such that the projection $\mathcal{T}_\rho^{\mathrm{ord}} \to \Lambda^{\mathrm{eis}}/(X^e)$ factors through $\mathcal{T}_{\mathrm{cusp}}$, it follows that $\zeta_\phi(X)$ divides X^e. The claims in the Proposition then follow from the fact that $e = 1$ by Proposition 4.2. □

4.4. **An application of Wiles' numerical criterion.** The goal of this subsection is to prove that there exists an isomorphism $\varphi : \mathcal{R}_\rho^{\mathrm{ord}} \xrightarrow{\sim} \mathcal{T}_\rho^{\mathrm{ord}}$ of complete intersections. Lemma 1.5 and Proposition 3.4 yield the following homomorphism of Λ-algebras:

$$\varphi : \mathcal{R}_\rho^{\mathrm{ord}} \to \mathcal{R}_{\mathrm{cusp}} \times_{\bar{\mathbb{Q}}_p} \mathcal{R}_\rho^{\mathrm{eis}} \to \mathcal{T}_{\mathrm{cusp}} \times_{\bar{\mathbb{Q}}_p} \Lambda = \mathcal{T}_\rho^{\mathrm{ord}}. \tag{44}$$

which is surjective by the same argument as in the proof of Theorem 3.5. To prove its injectivity we appeal to a variant of Wiles' numerical criterion due to Lenstra [33].

Theorem 4.9. *Let $\varphi : R \to T$ be a surjective homomorphism of local Λ-algebras. Suppose that T is finite and flat as Λ-module and let $\pi : T \to \Lambda$ be a Λ-algebra homomorphism. Let $J = \ker(\pi \circ \varphi)$ and assume that $\eta_T = \pi(\mathrm{Ann}(\ker \pi)) \neq 0$. Then*

$$\mathrm{length}_\Lambda(J/J^2) \geqslant \mathrm{length}_\Lambda(\Lambda/\eta_T)$$

and the equality holds if and only if φ is an isomorphism and T is a complete intersection.

We wish to apply the above criterion to (44) and $\pi_{\mathrm{eis}} : \mathcal{T}_\rho^{\mathrm{ord}} \to \Lambda$. By Proposition 4.2

$$\eta_T = \pi_{\mathrm{eis}}(\mathrm{Ann}(\ker(\pi_{\mathrm{eis}}))) = \pi_{\mathrm{eis}}(\ker(\pi_{\mathrm{cusp}})) = (X^e) = (X).$$

hence the Λ-module Λ/η_T has length 1. In order to apply the numerical criterion, we must compute the length of J/J^2 over Λ, where $J = \ker(\pi_{\mathrm{eis}} \circ \varphi)$.

Proposition 4.10. *The Λ-module J/J^2 is torsion of length at most 1.*

Proof. Note first that because $\mathcal{R}_\rho^{\mathrm{ord}}$ is Noetherian, J/J^2 is a module of finite type over $\mathcal{R}_\rho^{\mathrm{ord}}/J \simeq \Lambda$. The claim is equivalent to showing that for all $n \geqslant 1$ one has:

$$\mathrm{length}_\Lambda(\mathrm{Hom}(J/J^2, \Lambda/(X^n))) \leqslant 1. \tag{45}$$

To show this we will employ a Galois cohomology argument. Let us fix a lift $\rho_{\mathcal{R}} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \mathrm{G}_\mathbb{Q} \to \mathrm{GL}_2(\mathcal{R}_\rho^{\mathrm{ord}})$ representing the universal ordinary deformation in an ordinary basis. By Lemmas 1.5 and 3.6 the ideal J of $\mathcal{R}_\rho^{\mathrm{ord}}$ is generated by the set of $C(g)$ for $g \in \mathrm{G}_\mathbb{Q}$. Since

$$\rho_{\mathcal{R}} \otimes \mathcal{R}_\rho^{\mathrm{ord}}/J \simeq \rho_\Lambda \simeq \left(\begin{smallmatrix} \Phi & * \\ 0 & \mathbf{1} \end{smallmatrix}\right)$$

in the ordinary basis, a direct computation shows that the function

$$\bar{C} : \mathrm{G}_\mathbb{Q} \to J/J^2,\ g \to \Phi^{-1}(g)C(g) \mod J^2 \tag{46}$$

belongs to $\ker\Big(Z^1(\mathrm{G}_\mathbb{Q}^{Np}, \Phi^{-1} \otimes_\Lambda (J/J^2)) \to Z^1(\mathbb{Q}_p, \Phi^{-1} \otimes_\Lambda (J/J^2))\Big)$. As the image of $\bar{C}$ contains a set of generators of J/J^2 as a module over $\mathcal{R}_\rho^{\mathrm{ord}}/J \simeq \Lambda$, the natural map $h \mapsto h \circ \bar{C}$

$$\mathrm{Hom}_\Lambda(J/J^2, \Lambda/(X^n)) \hookrightarrow \ker\Big(Z^1(\mathrm{G}_\mathbb{Q}^{Np}, \Phi_n^{-1}) \to Z^1(\mathbb{Q}_p, \Phi_n^{-1})\Big)$$

is injective for all $n \geqslant 1$. Moreover by Proposition 2.8(ii) one has:

$$\ker\Big(Z^1(\mathrm{G}_\mathbb{Q}^{Np}, \Phi_n^{-1}) \to Z^1(\mathbb{Q}_p, \Phi_n^{-1})\Big) = \ker\Big(B^1(\mathrm{G}_\mathbb{Q}^{Np}, \Phi_n^{-1}) \to B^1(\mathbb{Q}_p, \Phi_n^{-1})\Big)$$

Since $\Phi_{1|\mathrm{G}_{\mathbb{Q}_p}} = \phi_{|\mathrm{G}_{\mathbb{Q}_p}} = \mathbf{1}$, while $\Phi_{2|\mathrm{G}_{\mathbb{Q}_p}} \neq \mathbf{1}$, the latter is given by the length 1 Λ-module $(X^{n-1})/(X^n) \simeq \bar{\mathbb{Q}}_p$. This proves (45), hence the Proposition. □

Theorem 4.11. *$\varphi : \mathcal{R}_\rho^{\mathrm{ord}} \twoheadrightarrow \mathcal{T}_\rho^{\mathrm{ord}}$ is an isomorphism of complete local intersections over Λ.*

Proof. The claim is a direct consequence of Theorem 4.9 in view of Proposition 4.10. □

5. Local structure of the eigencurve at f

In this section we will first complete the proof of Theorem A, then we will prove Theorem D using the methods of [15] and [17]. We recall that we have two Eisenstein families $\mathcal{E}_{\mathbf{1},\phi}$ and $\mathcal{E}_{\phi,\mathbf{1}}$, as well as a cuspidal family $\mathcal{F}$ (see Corollary 4.5) containing f. We also recall that $\Lambda = \bar{\mathbb{Q}}_p[\![X]\!]$ denotes the universal deformation ring of $\phi = \det(\rho)$.

5.1. **Failure of Gorensteinness of $\mathscr{C}$ at f.** By Theorem 3.5 and Lemma 3.6 the structural homomorphisms $\Lambda \xrightarrow{\sim} \mathcal{T}_{\mathrm{cusp}}$, $\Lambda \xrightarrow{\sim} \mathcal{T}_{\rho}^{\mathrm{eis}}$ and $\Lambda \xrightarrow{\sim} \mathcal{T}_{\rho'}^{\mathrm{eis}}$ are isomorphisms. Since the eigencurve $\mathscr{C}$ is reduced, it results from the above discussion a canonical inclusion of local Λ-algebras:

$$\pi = (\pi_{\mathrm{cusp}}, \pi_{\rho}^{\mathrm{eis}}, \pi_{\rho'}^{\mathrm{eis}}) : \mathcal{T} \hookrightarrow \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda, \tag{47}$$

where $\mathcal{T}$ denotes the completed strict local ring of $\mathscr{C}$ at f.

Consider the sub-algebra $\mathcal{T}'$ of $\mathcal{T}$ generated over Λ by T_ℓ, $\ell \nmid Np$.

Theorem 5.1. *The image of $\mathcal{T}'$ under the natural inclusion* (47) *is given by*

(48)
$$\left\{(a,b,c) \in \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \,\middle|\, \left(\mathscr{L}(\phi^{-1}) + \mathscr{L}(\phi)\right) a'(0) = \mathscr{L}(\phi^{-1}) b'(0) + \mathscr{L}(\phi) c'(0)\right\}.$$

Moreover, π defined in (47) *is an isomorphism, $X\mathcal{T}$ is an ideal of $\mathcal{T}'$, and*

$$\mathcal{T}/X\mathcal{T} = (\mathcal{T}'/X\mathcal{T})[U_p]/(U_p - 1)^2.$$

Proof. We first show that $\pi(\mathcal{T}') \supset (X^2 \cdot \Lambda)^3$, by showing that $\pi(\mathcal{T}')$ surjects on each of the 3 double products $\Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda$ in (47), and by observing that $(0,b,X)\cdot(a,0,X) = (0,0,X^2)$, etc. The surjectivity of the homomorphism $\mathcal{R}^{\mathrm{ps}} \to \mathcal{R}_{\rho}^{\mathrm{ord}}$ established in the proof of Lemma 1.7, implies the surjectivity of the composed map $\mathcal{T}' \to \mathcal{T} \to \mathcal{T}_{\rho}^{\mathrm{ord}}$, *i.e.* $(\pi_{\mathrm{cusp}}, \pi_{\rho}^{\mathrm{eis}})$ is surjective, and similarly $(\pi_{\mathrm{cusp}}, \pi_{\rho'}^{\mathrm{eis}})$ is surjective as well. As for the surjectivity of $(\pi_{\rho}^{\mathrm{eis}}, \pi_{\rho'}^{\mathrm{eis}})$ it suffices to show that the image of

$$\mathrm{tr}(\rho_{\mathrm{eis}}) - \mathrm{tr}(\rho'_{\mathrm{eis}}) = (\phi - \mathbf{1}) \cdot (\chi_p - \mathbf{1}) : \mathrm{G}_{\mathbb{Q}} \to \Lambda,$$

contains an element of valuation 1, where ρ_{eis} is the Galois deformation of ρ attached to $\mathcal{E}_{\mathbf{1},\phi}$ (see (36)), whereas ρ'_{eis} is the Galois deformation of ρ' attached to $\mathcal{E}_{\phi,\mathbf{1}}$. This follows easily from the fact that the abelian extensions H and $\mathbb{Q}_\infty$ of $\mathbb{Q}$ (which are the fixed fields of $\ker(\phi)$ and $\ker(\chi_p)$, respectively) are linearly disjoint for ramification reasons. Hence $\pi(\mathcal{T}') \supset (X^2 \cdot \Lambda)^3$.

It follows that $\pi(\mathcal{T}')$ and $\pi(\mathcal{T})$ are uniquely determined by their images in $\bar{\mathbb{Q}}_p[\epsilon]^3 = (\Lambda/(X^2))^3$, which we will now determine using the tangent space computations from §2.

By Chebotarev's density Theorem $\mathcal{T}'$ is generated over Λ by the trace of $\rho_{\mathcal{T}} = (\rho_{\mathcal{T}_{\mathrm{cusp}}}, \rho_{\mathrm{eis}}, \rho'_{\mathrm{eis}})$, hence the image of $\pi(\mathcal{T}')$ in $\bar{\mathbb{Q}}_p[\epsilon]^3$ is generated by 1 and the image of the map $\xi : \mathrm{G}_{\mathbb{Q}} \to \bar{\mathbb{Q}}_p^3$ uniquely determined by

$$(\mathrm{tr}(\rho_{\mathcal{T},\epsilon}) - \mathrm{tr}(\rho)) = (\det(\rho_{\mathcal{T},\epsilon}) - \det(\rho)) \cdot \xi.$$

Using formulas (20) and (23) for $\rho_{\mathcal{T}_{\mathrm{cusp}}} \mod X^2$, together with the formula (36) for ρ_{eis} yields:

$$\xi = \left(\frac{\mathscr{L}(\phi^{-1}) + \mathscr{L}(\phi)\cdot\phi^{-1}}{\mathscr{L}(\phi) + \mathscr{L}(\phi^{-1})}, \mathbf{1}, \phi^{-1} \right), \tag{49}$$

which completes the proof of (48). By (21) the image of $U_p - 1$ in $\bar{\mathbb{Q}}_p[\epsilon]^3$ belongs to $\bar{\mathbb{Q}}_p^\times(\epsilon, 0, 0)$, hence π is an isomorphism. As $\pi(X \cdot \mathfrak{m}_{\mathcal{T}}) = (X^2 \cdot \Lambda)^3$, one sees that $X\mathcal{T}$ is an ideal of $\mathcal{T}'$, and the kernel of the natural surjective homomorphism $(\mathcal{T}'/X\mathcal{T})[U_p] \twoheadrightarrow \mathcal{T}/X\mathcal{T}$ is generated by $(U_p - 1)^2$. □

Remark 5.2. When ϕ is quadratic, the relation in (48) is given by the congruence

$$2a_\ell(\mathcal{F}) \equiv a_\ell(\mathcal{E}_{\mathbf{1},\phi}) + a_\ell(\mathcal{E}_{\phi,\mathbf{1}}) \pmod{X^2}$$

for all primes $\ell \nmid Np$. This particular setting is a fruitful ground for arithmetic applications such as the proof by Bertolini, Darmon and Venerucci [5] of a conjecture of Perrin-Riou relating the position of the Kato class in the ϕ-isotypic component of the Mordell-Weil group of an elliptic curve E satisfying $L(E, \phi, 1) = 0$ to a global point. Their work exploits the connection between that class and the generalised Kato class attached to ϕ seen as a genus character of its fixed imaginary quadratic field, for which they prove a formula mirroring the above relation.

Remark 5.3. Using an algorithm based on methods of [31], A. Lauder identified the linear relation (49) for ϕ an odd sextic character of conductor 21, and $p = 13$ for which $\phi(13) = 1$, which were used in [16, §7.2] to provide a numerical evidence for elliptic Stark points over cyclotomic fields.

As a corollary of Theorem 5.1 we investigate some ring theoretic properties of the completed strict local ring $\mathcal{T}$ of $\mathscr{C}$ at f. Examples of non-Gorenstein Hecke algebras abound in positive characteristic (see for example [46]), but seem to be less common in characteristic zero.

Corollary 5.4. *(i) The ring $\mathcal{T}$ is Cohen–Macaulay, but not Gorenstein.*

(ii) The local ring at f of the fibre $\kappa^{-1}(\kappa(f))$ in $\mathscr{C}$ has dimension 3 over $\bar{\mathbb{Q}}_p$.

(iii) The ring $\mathcal{T}' \simeq \Lambda[Y]/\big(Y(Y + \mathscr{L}(\phi)X)(Y - \mathscr{L}(\phi^{-1})X)\big)$ is complete intersection.

Proof. (i) The depth of $\mathcal{T}$ cannot exceed its Krull dimension which is 1. Since $X = (X, X, X) \in \mathcal{T}$ is a regular element, the depth of $\mathcal{T}$ is 1 and, in particular the local ring $\mathcal{T}$ is Cohen–Macaulay. By definition, $\mathcal{T}$ is Gorenstein if and only if its Artinian quotient $\mathcal{T}/X\mathcal{T}$ is Gorenstein which is equivalent to its socle $\mathrm{Hom}_{\mathcal{T}}(\bar{\mathbb{Q}}_p, \mathcal{T}/X\mathcal{T})$ having dimension 1 (see [22, §21.2, §21.3]). Since

$$\mathcal{T} \simeq \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \times_{\bar{\mathbb{Q}}_p} \Lambda \simeq \bar{\mathbb{Q}}_p[\![X_1, X_2, X_3]\!]/(X_1X_2, X_1X_3, X_2X_3),$$

where $X_1 = (X, 0, 0)$, $X_2 = (0, X, 0)$ and $X_3 = (0, 0, X)$, it follows that

$$\mathcal{T}/X\mathcal{T} \simeq \bar{\mathbb{Q}}_p[\![X_1, X_2]\!]/(X_1^2, X_1X_2, X_2^2),$$

hence its socle is 2-dimensional.

(ii) It follows from (i) that $\mathcal{T}/\mathfrak{m}_\Lambda\mathcal{T} = \mathcal{T}/X\mathcal{T}$ is a $\bar{\mathbb{Q}}_p$-vector space of dimension 3.

(iii) By (48), $\mathcal{T}'$ is equidimensional of dimension 1 and has 2-dimensional tangent space generated by X and $Y = (0, -\mathscr{L}(\phi)X, \mathscr{L}(\phi^{-1})X)$. It follows that $\mathcal{T}'$ is a quotient of the factorial ring $\bar{\mathbb{Q}}_p[\![X,Y]\!]$ by the element $Y(Y+\mathscr{L}(\phi)X)(Y-\mathscr{L}(\phi^{-1})X)$, in particular, $\mathcal{T}'$ is a complete intersection. □

5.2. **Duality for non-cuspidal Hida families.** In Corollary 4.5 we showed that $S^{\dagger}_{\mathfrak{m}_f} = \Lambda\cdot\mathcal{F}$. We will now exhibit a basis of the free rank 3 Λ-module $M^{\dagger}_{\mathfrak{m}_f}$ (see Corollary 5.4(ii)).

Proposition 5.5. *(i) The localisation of C_Λ at $\mathfrak{m}_f$ is free of rank 2 over Λ.*

(ii) The elements $E_{\mathbf{1},\phi} = (\mathcal{F}-\mathcal{E}_{\mathbf{1},\phi})/X$ and $E_{\phi,\mathbf{1}} = (\mathcal{F}-\mathcal{E}_{\phi,\mathbf{1}})/X$ are in $M^{\dagger}_{\mathfrak{m}_f}$ and generate a complement of $S^{\dagger}_{\mathfrak{m}_f}$, i.e. $M^{\dagger}_{\mathfrak{m}_f} = \Lambda\cdot\mathcal{F}\oplus\Lambda\cdot E_{\mathbf{1},\phi}\oplus\Lambda\cdot E_{\phi,\mathbf{1}}$.

Proof. (i) The freeness follows from (42) and the rank is 2 as it is given by the number of Eisenstein families containing f.

(ii) Since all three families $\mathcal{F}$, $\mathcal{E}_{\mathbf{1},\phi}$ and $\mathcal{E}_{\phi,\mathbf{1}}$ have coefficients in Λ and specialise to f in weight $\kappa(f)$, it follows that both $E_{\mathbf{1},\phi}$ and $E_{\phi,\mathbf{1}}$ belong to $M^{\dagger}_{\mathfrak{m}_f}$. By (42) it suffices to show that $\mathrm{res}_\Lambda(E_{\mathbf{1},\phi})$ and $\mathrm{res}_\Lambda(E_{\phi,\mathbf{1}})$ form a basis of the localisation of C_Λ at $\mathfrak{m}_f$. This follows from Propositions 4.7 and 4.8, according to which the constant term of $E_{\mathbf{1},\phi}$ at the cusp ∞ belongs to $\Lambda^\times$ and vanishes at 0, while $E_{\phi,\mathbf{1}}$ vanishes at ∞, but belongs to $\Lambda^\times$ at 0. □

Proposition 5.6. *One has $\mathcal{T}^{\mathrm{full}}\simeq\mathcal{T}$ and the natural Λ-bilinear pairing:*

$$\mathcal{T}\times M^{\dagger}_{\mathfrak{m}_f}\to\Lambda,\quad (T,\mathcal{G})\mapsto a_1(T\cdot\mathcal{G})$$

is a perfect duality. Its restriction to $\mathcal{T}^{\mathrm{ord}}_\rho\times(\Lambda\cdot\mathcal{F}\oplus\Lambda\cdot E_{\mathbf{1},\phi})$ is a perfect duality as well.

Proof. One has $\mathcal{T}^{\mathrm{full}}\simeq\mathcal{T}$, as $\mathcal{T}^{\mathrm{full}}_{\mathrm{cusp}}\simeq\mathcal{T}_{\mathrm{cusp}}$ by Proposition 4.4, and for all $\ell\mid N$ one has $U_\ell(\mathcal{E}_{\mathbf{1},\phi})=1$ and $U_\ell(\mathcal{E}_{\phi,\mathbf{1}})=\chi_p(\mathrm{Frob}_\ell)\in 1+\mathfrak{m}_\Lambda$ (see (25)). The non-degeneracy of the pairing follows from the q-expansion principle, as $a_n(\mathcal{G})=a_1(T_n\cdot\mathcal{G})=0$ for all $n\in\mathbb{Z}_{\geqslant 1}$. Via the isomorphisms

$$\mathcal{T}\simeq\mathcal{T}_{\mathrm{cusp}}\times_{\bar{\mathbb{Q}}_p}\mathcal{T}^{\mathrm{eis}}_\rho\times_{\bar{\mathbb{Q}}_p}\mathcal{T}^{\mathrm{eis}}_{\rho'}\simeq\Lambda\times_{\bar{\mathbb{Q}}_p}\Lambda\times_{\bar{\mathbb{Q}}_p}\Lambda,$$

established in Theorem 5.1, one deduces that the matrix of the pairing in the bases $(\mathcal{F},E_{\mathbf{1},\phi},E_{\phi,\mathbf{1}})$ and $((1,1,1),(0,X,0),(0,0,X))$ is the identity, hence the pairing is a perfect duality. □

Remark 5.7. As $\mathcal{T}^{\mathrm{ord}}_\rho$ is Gorenstein and finite flat over the regular ring Λ, it follows that the $\mathcal{T}^{\mathrm{ord}}_\rho$-module $\mathrm{Hom}_\Lambda(\mathcal{T}^{\mathrm{ord}}_\rho,\Lambda)\simeq M^{\dagger}_{\mathfrak{m}_f,\rho}$ is free of rank 1. On the other hand, $\mathcal{T}$ being not Gorenstein, the $\mathcal{T}$-module $\mathrm{Hom}_\Lambda(\mathcal{T},\Lambda)\simeq M^{\dagger}_{\mathfrak{m}_f}$ is not free of rank 1.

5.3. **Non-classical overconvergent weight 1 modular forms.** Recall the universal cyclotomic character $\chi_p:\mathrm{G}_{\mathbb{Q}}\to\Lambda^\times$ from §2.4, where $\Lambda=\mathbb{Q}_p[\![X]\!]$ is isomorphic to the completed strict local ring of the weight space $\mathcal{W}$ at $\kappa(f)$. We will next compute infinitesimally the q-expansion of the unique cuspidal family $\mathcal{F}=\sum_{n\geqslant 1}a_n(\mathcal{F})q^n\in\Lambda[\![q]\!]$ containing f (see Corollary 4.5).

Proposition 5.8. *One has*

$$\frac{d}{dX}\Big|_{X=0} a_p(\mathcal{F}) = \frac{\mathscr{L}(\phi)\mathscr{L}(\phi^{-1})}{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}, \text{ and} \tag{50}$$

$$\frac{d}{dX}\Big|_{X=0} a_\ell(\mathcal{F}) = \frac{(\phi(\ell)\mathscr{L}(\phi^{-1})+\mathscr{L}(\phi))\log_p(\ell)}{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}, \text{ for every prime } \ell \neq p. \tag{51}$$

Proof. Let $\ell \nmid N$ be a prime. Reading the first component of (49) yields:

$$\frac{d}{dX}\Big|_{X=0} a_\ell(\mathcal{F}) = \frac{d}{dX}\Big|_{X=0} \mathrm{tr}(\rho_{\mathcal{T}_{\mathrm{cusp}}})(\mathrm{Frob}_\ell) = \frac{\mathscr{L}(\phi^{-1})+\mathscr{L}(\phi)\phi^{-1}(\ell)}{\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1})} \frac{d}{dX}\Big|_{X=0} \det(\rho_{\mathcal{T}_{\mathrm{cusp}}})(\mathrm{Frob}_\ell).$$

As $\det(\rho_{\mathcal{T}_{\mathrm{cusp}}}) = \phi\chi_p$ and $\frac{d}{dX}\big|_{X=0} \chi_p(\mathrm{Frob}_\ell) = \frac{\log_p(\ell)}{\log_p(1+p^\nu)}$ by (26), we obtain (51) for $\ell \nmid N$.

In order to compute $\frac{d}{dX}\big|_{X=0} a_p(\mathcal{F}) = \frac{d}{dX}\big|_{X=0} \chi_{\mathcal{T}_{\mathrm{cusp}}}(\mathrm{Frob}_p)$ we go back to the proof of Proposition 2.6. Comparing $\det(\rho_\epsilon) = \phi(1+\epsilon(\lambda+\mu)\eta_{\mathbf{1}})$ from (20) with (26) gives $\lambda+\mu = -\frac{1}{\log_p(1+p^\nu)}$. Combining this with $\mu\mathscr{L}(\phi^{-1}) = \lambda\mathscr{L}(\phi)$ from (23) we obtain

$$\lambda = -\frac{\mathscr{L}(\phi^{-1})}{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}, \quad \mu = -\frac{\mathscr{L}(\phi)}{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}. \tag{52}$$

By (21) one finds $\frac{d}{dX}\big|_{X=0} \chi_{\mathcal{T}_{\mathrm{cusp}}}(\mathrm{Frob}_p) = -\mu\mathscr{L}(\phi^{-1}) = \frac{\mathscr{L}(\phi)\mathscr{L}(\phi^{-1})}{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}$ as claimed. It remains to compute the derivative of $a_\ell(\mathcal{F})$ for primes $\ell|N$. From Proposition 4.4, $a_\ell(\mathcal{F})$ is given by the action of Frob_ℓ on the $(\rho_{\mathcal{T}_{\mathrm{cusp}}})^{\mathrm{I}_{\mathbb{Q}_\ell}}$. By the proof of Proposition 2.6 and (52) we get

$$\frac{d}{dX}\Big|_{X=0} a_\ell(\mathcal{F}) = \mu\eta_{\mathbf{1}}(\mathrm{Frob}_\ell) = \frac{\mathscr{L}(\phi)\log_p(\ell)}{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)},$$

yielding (51) also for $\ell \mid N$, as in this case $\phi(\ell) = 0$. □

Let $M^{\mathrm{ord}}_{\kappa(f)}$ be the space of ordinary overconvergent p-adic modular forms of weight 1 and central character ϕ. The eigenform f corresponds to a maximal ideal $\mathfrak{m}_f$ of the Hecke algebra $\mathcal{T}_{\kappa(f)}$ acting on $M^{\mathrm{ord}}_{\kappa(f)}$. For $i \geqslant 1$, let $M^\dagger[\mathfrak{m}_f^i]$ denote the subspace of $M^{\mathrm{ord}}_{\kappa(f)}$ annihilated by $\mathfrak{m}_f^i$.

Proof of Theorem D. By definition the generalised eigenspace at f is given by the $\mathcal{T}/\mathfrak{m}_\Lambda\mathcal{T}$-module $M^\dagger_{\mathfrak{m}_f}/\mathfrak{m}_\Lambda M^\dagger_{\mathfrak{m}_f}$. By Theorem 5.1 $\mathfrak{m}_f^2\mathcal{T} \subset \mathfrak{m}_\Lambda\mathcal{T}$, hence $M^\dagger[\![f]\!] = M^\dagger[\mathfrak{m}_f^2]$. Note that by Proposition 5.6 one already knows that $\dim_{\bar{\mathbb{Q}}_p} M^\dagger[\mathfrak{m}_f^2] = \dim_{\bar{\mathbb{Q}}_p}(\mathcal{T}/\mathfrak{m}_f^2\mathcal{T}) = 3$. By Proposition 5.5, the specialisations of the families $E_{\phi,\mathbf{1}}(X)$ and $E_{\mathbf{1},\phi}(X)$ at $X = 0$ span a complement of $S^\dagger[\![f]\!]$ in $M^\dagger[\![f]\!]$. Consider

$$\begin{aligned} f^\dagger_{\mathbf{1},\phi} &= \frac{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}{\mathscr{L}(\phi)} E_{\mathbf{1},\phi}(0) = \frac{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}{\mathscr{L}(\phi)} \frac{d}{dX}\Big|_{X=0}\left(\mathcal{F}-\mathcal{E}_{\mathbf{1},\phi}\right), \\ f^\dagger_{\phi,\mathbf{1}} &= \frac{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}{\mathscr{L}(\phi^{-1})} E_{\phi,\mathbf{1}}(0) = \frac{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\log_p(1+p^\nu)}{\mathscr{L}(\phi^{-1})} \frac{d}{dX}\Big|_{X=0}\left(\mathcal{F}-\mathcal{E}_{\phi,\mathbf{1}}\right). \end{aligned} \tag{53}$$

Since $a_p(\mathcal{E}_{\mathbf{1},\phi}) = a_p(\mathcal{E}_{\phi,\mathbf{1}}) = 1$ one finds that $\frac{d}{dX}\big|_{X=0} a_p(\mathcal{E}_{\mathbf{1},\phi}) = \frac{d}{dX}\big|_{X=0} a_p(\mathcal{E}_{\phi,\mathbf{1}}) = 0$. Further $a_\ell(\mathcal{E}_{\mathbf{1},\phi}) = 1+\phi(\ell)\chi_p(\ell)$ and $a_\ell(\mathcal{E}_{\phi,\mathbf{1}}) = \phi(\ell)+\chi_p(\ell)$, together with (26) yields that

$$\frac{d}{dX}\Big|_{X=0} a_\ell(\mathcal{E}_{\mathbf{1},\phi}) = \phi(\ell)\frac{\log_p(\ell)}{\log_p(1+p^\nu)}, \text{ and } \frac{d}{dX}\Big|_{X=0} a_\ell(\mathcal{E}_{\phi,\mathbf{1}}) = \frac{\log_p(\ell)}{\log_p(1+p^\nu)}.$$

Combining these formulas with those given in Proposition 5.8 we obtain the desired formulas for the non-constant coefficients of $f^\dagger_{\mathbf{1},\phi}$ and $f^\dagger_{\phi,\mathbf{1}}$ defined in (53):

$$a_p(f^\dagger_{\mathbf{1},\phi}) = \mathscr{L}(\phi^{-1}),\ a_p(f^\dagger_{\phi,\mathbf{1}}) = \mathscr{L}(\phi) \text{ and } a_\ell(f^\dagger_{\mathbf{1},\phi}) = (1-\phi(\ell))\log_p(\ell) = -a_\ell(f^\dagger_{\phi,\mathbf{1}}).$$

In order to compute the remaining positive coefficients of $f^\dagger = f^\dagger_{\mathbf{1},\phi}$ or $f^\dagger_{\phi,\mathbf{1}}$, we observe that since $\mathcal{E}_{\mathbf{1},\phi}$, $\mathcal{E}_{\phi,\mathbf{1}}$ and $\mathcal{F}$ are normalised eigenform for all Hecke operators $(T_n)_{n\geqslant 1}$ (see Proposition 4.4 for $\mathcal{F}$), the classical relations between abstract Hecke operators imply:

$$(54)\quad \begin{aligned} &a_{mn}(f^\dagger) = a_m(f)a_n(f^\dagger) + a_n(f)a_m(f^\dagger), \text{ for } (n,m)=1,\\ &a_{\ell^r}(f^\dagger) = ra_\ell(f)^{r-1}a_\ell(f^\dagger) = ra_\ell(f^\dagger) \text{ for all primes } \ell \mid Np, r\geqslant 1, \text{ and}\\ &a_{\ell^r}(f^\dagger) = a_\ell(f)a_{\ell^{r-1}}(f^\dagger) + a_{\ell^{r-1}}(f)a_\ell(f^\dagger) - \phi(\ell)a_{\ell^{r-2}}(f^\dagger) \text{ for } \ell\nmid Np, r\geqslant 2.\end{aligned}$$

As $a_{\ell^r}(f) = \sum_{i=0}^r \phi(\ell)^i$ for all primes $\ell \neq p$, an induction on r yields

$$a_{\ell^r}(f^\dagger) = a_\ell(f^\dagger)\textstyle\sum_{i=0}^r (i+1)(r-i)\phi(\ell)^i, \text{ hence}$$

$$a_{\ell^r}(f^\dagger_{\mathbf{1},\phi}) = \textstyle\sum_{i=0}^r (r-2i)\phi(\ell^i)\log_p(\ell) = -a_{\ell^r}(f^\dagger_{\phi,\mathbf{1}}), \text{ for all primes } \ell\neq p.$$

Combining this with (54) yields the desired formulas for $a_n(f^\dagger_{\mathbf{1},\phi})$ and $a_n(f^\dagger_{\phi,\mathbf{1}})$ for all $n\geqslant 1$.

We have $a_0(f^\dagger_{\phi,\mathbf{1}}) = 0$ because $a_0(\mathcal{E}_{\phi,\mathbf{1}})$ is identically zero. To compute $a_0(f^\dagger_{\mathbf{1},\phi})$ we notice that by the above computations one has $a_n\left(\frac{f^\dagger_{\mathbf{1},\phi}+f^\dagger_{\phi,\mathbf{1}}}{\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1})}\right) = a_n(E_1(\mathbf{1},\phi) - f)$ for all $n\geqslant 1$, where $E_1(\mathbf{1},\phi)$ is the classical Eisenstein series defined in (1). We then deduce from Proposition 5.6 that $a_0\left(\frac{f^\dagger_{\mathbf{1},\phi}+f^\dagger_{\phi,\mathbf{1}}}{\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1})}\right) = a_0(E_1(\mathbf{1},\phi)-f)$ as well, hence

$$a_0(f^\dagger_{\mathbf{1},\phi}) = (\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))\frac{L(\phi,0)}{2}. \tag{55}$$

□

Corollary 5.9. $L_p'(\phi\omega_p,0) = \mathscr{L}(\phi)L(\phi,0)$.

Proof. We will now compute $a_0(f^\dagger_{\mathbf{1},\phi})$ by a different method and obtain the formula for the derivative of the Kubota–Leopoldt p-adic L-function (see [18, (8)]). By Proposition 4.7 one has $a_0(\mathcal{E}_{\mathbf{1},\phi}) = \frac{\zeta_\phi}{2}$. Since by (39) one has

$$\zeta_\phi((1+p^\nu)^{s-1}-1) = L_p(\phi\omega_p, 1-s),$$

taking derivatives yields $\zeta_\phi'(0) = -\frac{L_p'(\phi\omega_p,0)}{\log_p(1+p^\nu)}$. Using (53) one finds:

$$\frac{\mathscr{L}(\phi)}{(\mathscr{L}(\phi)+\mathscr{L}(\phi^{-1}))}a_0(f^\dagger_{\mathbf{1},\phi}) = -\log_p(1+p^\nu)\frac{d}{dX}\Big|_{X=0} a_0(\mathcal{E}_{\mathbf{1},\phi}) = -\log_p(1+p^\nu)\frac{\zeta_\phi'(0)}{2} = \frac{L_p'(\phi\omega_p,0)}{2}.$$

From (55) and the latter we obtain $L_p'(\phi\omega_p,0) = \mathscr{L}(\phi)L(\phi,0)$. □

Remark 5.10. In this closing remark we let $p = 5$, $N = 11$ and we denote by Δ the 5-Sylow subgroup of $(\mathbb{Z}/11\mathbb{Z})^\times$. Mazur observed in [34] that the weight 2 level $\Gamma_0(11)$ Eisenstein series is congruent modulo 5 to the unique weight 2 cuspform g of level $\Gamma_0(11)$. Merel [37] gave a numerical criterion for this uniqueness in terms of the non-vanishing of a tame derivative, in the sense of Mazur-Tate, of the zeta element $\zeta_\Delta \in (\mathbb{Z}/5\mathbb{Z})[\Delta]$ specialising to $L(-1,\chi)$ for any character χ of Δ (considered as even Dirichlet character of conductor 11). To draw a parallel with our current work, we consider the ordinary 5-stabilisation of g which is congruent modulo 5 to each of the two 11-stabilisations of the unique weight 2 Eisenstein series of level $\Gamma_0(5)$. The Hecke $\mathbb{Z}_5$-algebra of level $\Gamma_0(55)$ localised at the corresponding maximal ideal has structure analogous to that of the Λ-algebra $\mathcal{T}$ described in Theorem A(ii); in particular, it is not Gorenstein and its cuspidal quotient is free of rank one over $\mathbb{Z}_5$. The analogy goes further, as the tame derivative $\zeta'_\Delta = \sum_{\delta\in\Delta} a_\delta \log(\delta)[\delta]$ evaluated at the $\chi = 1$ does not vanish and its value differs from $\zeta(-1) = \frac{1}{12}$ by a 'tame $\mathscr{L}$-invariant' given by $\log(4)$, where $\log : \Delta \xrightarrow{\sim} \mathbb{Z}/5\mathbb{Z}$ is a choice of discrete logarithm as in [32].

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