COMPLETE CONSTANT MEAN CURVATURE HYPERSURFACES IN EUCLIDEAN SPACE OF DIMENSION FOUR OR HIGHER

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ABSTRACT. In this article we provide a general construction when \( n \geq 3 \) for immersed in Euclidean \((n+1)\)-space, complete, smooth, constant mean curvature hypersurfaces of finite topological type (in short CMC \( n \)-hypersurfaces). More precisely our construction converts certain graphs in Euclidean \((n+1)\)-space to CMC \( n \)-hypersurfaces with asymptotically Delaunay ends in two steps: First appropriate small perturbations of the given graph have their vertices replaced by round spherical regions and their edges and rays by Delaunay pieces so that a family of initial smooth hypersurfaces is constructed. One of the initial hypersurfaces is then perturbed to produce the desired CMC \( n \)-hypersurface which depends on the given family of perturbations of the graph and a small in absolute value parameter \( \tau \). This construction is very general because of the abundance of graphs which satisfy the required conditions and because it does not rely on symmetry requirements. For any given \( k \geq 2 \) and \( n \geq 3 \) it allows us to realize infinitely many topological types as CMC \( n \)-hypersurfaces in \( \mathbb{R}^{n+1} \) with \( k \) ends. Moreover for each case there is a plethora of examples reflecting the abundance of the available graphs. This is in sharp contrast with the known examples which in the best of our knowledge are all (generalized) cylindrical obtained by ODE methods and are compact or with two ends. Furthermore we construct embedded examples when \( k \geq 3 \) where the number of possible topological types for each \( k \) is finite but tends to \( \infty \) as \( k \to \infty \). Finally we remark that in ongoing work [5], we extend these results to construct infinitely many topological types of closed (immersed) examples for each \( n \geq 3 \). Moreover, for each \( n \geq 3 \) and \( k \geq 6 \), we construct in [5] infinitely many topological types of embedded complete examples with \( k \) ends.

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1. Introduction

The general framework.

Constant Mean Curvature (CMC) (hyper)surfaces in a Riemannian manifold can be described variationally as critical points of the induced intrinsic volume (or area in dimension two) functional, subject to an enclosed volume constraint. Alternatively they can be described as soap films (or fluid interfaces) in equilibrium under only the forces of surface tension and uniform enclosed pressure. In both cases the geometric condition is that the mean curvature \( H \) of the hypersurface is constant as the name suggests.

Of particular interest are the complete CMC (hyper)surfaces of finite topological type smoothly immersed in Euclidean spaces. The only classically known such examples were the round spheres, the cylinders, and the rotationally invariant surfaces in Euclidean three-space discovered by Delaunay in 1841 [11]. In the 1950’s two major results were proved characterizing the round two-spheres as the only closed CMC surfaces in Euclidean three-space: by Alexandrov under the assumption of embeddedness [3] and by Hopf under the
assumption of zero genus [24]. These results and their methods of proof had a profound
influence in mathematics. They also led to the celebrated conjecture (or question according
to some) by Hopf on whether the only immersed closed CMC surfaces in Euclidean three-space
are round spheres.

In 1986 Wente disproved the Hopf conjecture by constructing genus one closed immersed
examples [64]. Wente’s approach was based on finding and using solutions of the sinh-Gordon
equation. Wente’s success motivated further work based on his approach [65, 63, 2, 1, 57,
4, 14, 13], including systematic use of ODE and integrable system methods. Despite all this
progress in the genus one case, not much was accomplished with these methods in the case of
higher genus. Furthermore, contrary to some expectation, the celebrated Lawson conjecture
was recently settled by Brendle by following a different approach [7].

Soon after Wente’s groundbreaking result [64], most of the possible finite topological types
were realized as immersed (or Alexandrov embedded) CMC surfaces for the first time [36, 37]
by using a gluing Partial Differential Equations (PDE) approach. Historically, gluing PDE
methods had been applied extensively and with great success in Gauge Theories by Donaldson
[12], Taubes [60, 61, 62], and others. The particular kind of methods used in [36, 37] originated
from Schoen’s [59]. The Delaunay surfaces served as building blocks. The Hopf question for
high genus closed surfaces was settled in particular by providing examples of any genus
$g \geq 3$ [37]. In spite of its success, the use of Delaunay pieces as building blocks has the
limitation that it does not allow the construction of closed genus-two CMC examples. In
[38] a systematic and detailed refinement of the original gluing methodology made it possible
to construct genus-two (actually any genus $g \geq 2$) closed examples with the Wente tori as
building blocks. Since then, many other gluing problems have been successfully resolved by
using this refined approach. These results include gluing constructions for special Lagrangian
cones [21, 22, 20] and various gluing constructions for minimal surfaces [66, 44, 41, 40, 39,
42, 43].

It is worth pointing out that the constructions in [59, 36, 37] are quite general in two ways:
first, each construction is reduced to finding graphs satisfying some rather general conditions.
There is an abundance of such graphs and so a plethora of examples can be produced. Second,
no symmetry is required—although it can be imposed in special cases—and indeed most
examples constructed do not satisfy any symmetries. These constructions can serve then as
a prototype for general constructions in other geometric settings—see [40, 41, 35].

We briefly mention that much progress has been made in the case of embedded, or more
generally Alexandrov embedded, complete CMC surfaces of finite genus $g$ with $k$ ends. Meeks
[55] proved that such (noncompact) surfaces have at least two ends and all their ends are
cylindrically bounded. Motivated by [55, 36], Korevaar, Kusner, and Solomon [48] showed
that each end converges exponentially fast to a Delaunay surface and if there are only two
ends then the surface is Delaunay. Further progress in this direction was made in [46, 47] and
also in understanding the moduli space of these surfaces as for example in [49]. Moreover a
significant success was that in some cases of genus zero, complete classification results were
obtained with a satisfactory understanding of the surfaces involved [18, 19, 17].

We briefly also mention that various constructions extended the results of [36]: Große-
Brauckmann [16] used a conjugate surface construction to construct genus zero examples
with $k$ ends under maximal ($k$-fold dihedral) symmetry, including examples with large neck
size for the first time. Various gluing constructions based on non-degeneracy were developed
in certain cases [51, 58, 53, 54, 9, 34]. This allowed some new examples, in particular examples
with asymptotically cylindrical ends [53], with noncatenoidal necks used as nodes instead of spheres [51], and a modified construction (end-to-end gluing) of the closed CMC examples [58, 34]. Recently the construction and estimates in [36] were refined in [6] by applying the improved methodology of [38]. This way a large class of embedded examples was produced. [6] served also as an intermediate step for developing the high-dimensional constructions presented in this article.

Contrary to the case of Euclidean three-space, very little is currently known in the case of higher-dimensional Euclidean spaces: Rotationally invariant CMC hypersurfaces analogous to the ones found by Delaunay have been constructed [45]. In 1982 Hsiang [25] constructed immersed CMC hyperspheres that are not round, demonstrating this way that unlike Alexandrov’s theorem, the theorem of Hopf does not extend to higher dimensions; see also [26, 27, 28, 10] for more constructions which produce a few more topological types. Jleli has studied moduli spaces [30] and has developed an end-to-end gluing construction [29] which will provide new symmetric closed examples [31] when [33] appears. He also constructed examples bifurcating from the Delaunay-like ones [32].

Finally we briefly mention that constructions of CMC hypersurfaces have also been carried out in compact ambient manifolds under certain metric restrictions. Ye [67] provided the first such example, proving that there exists a foliation by CMC hyperspheres in a neighborhood of a non-degenerate critical point of the scalar curvature. Pacard and Xu [56] partially extended Ye’s result. Mazzeo and Pacard extended Ye’s result to geodesic tubes [52]. Further constructions of CMC surfaces (two-dimensional) condensing around geodesic intervals or rays were provided in [8], and for CMC hypersurfaces condensing around higher dimensional submanifolds in [50].

**Brief discussion of the results.**

In this article we extend the results of [36] to higher dimensions, that is to the construction of CMC $n$-dimensional hypersurfaces in Euclidean $(n+1)$-space for $n \geq 3$. Note that although the present proof and construction work for $n = 2$ with small appropriate modifications, we restrict our attention to $n > 2$ to simplify the presentation. For the same reason we restrict our attention to the construction of CMC hypersurfaces of finite topological type.

Our constructions, as in [36, 6], are based on a suitable family of graphs $\mathcal{F}$ which consists of small perturbations of a central graph $\Gamma$ (see 2.14). Our graphs have vertices, edges, rays, and nonzero weights assigned to the edges and the rays (see 2.1). $\Gamma$ is balanced in the sense that the resultant forces exerted on the vertices by the edges and rays vanish (see 2.6 and 2.9), and moreover its edges have even integer lengths. The other graphs in $\mathcal{F}$ have approximately prescribed resultant forces (unbalancing condition) and prescribed small changes of the lengths of the edges (flexibility condition).

Given $\mathcal{F}$ and a small nonzero $\tau$ a family of initial immersions is constructed, where the image of each such immersion is built around a properly chosen $\Gamma' \in \mathcal{F}$, and consists of unit spheres (with small geodesic balls removed) centered at the vertices of $\Gamma'$, and appropriately perturbed Delaunay pieces of parameter $\tau$ times the corresponding weight of $\Gamma'$. We have then the following (see 8.2 for a more precise statement).

**Theorem 1.1 (Main Theorem).** Given a family of graphs $\mathcal{F}$, there exists $T(\mathcal{F}) > 0$ such that for all $0 < |\tau| \leq T$, there exists a $\Gamma' \in \mathcal{F}$ and an immersion built around $\Gamma'$ as outlined above which admits a small graphical perturbation which has mean curvature $H \equiv 1$. Moreover the
immersion is an embedding if the central graph $\Gamma$ satisfies certain conditions (see 2.10) and $\tau > 0$.

Note that the conditions in 2.10 are the expected ones, that is they ensure that the various pieces stay away from each other and that the Delaunay pieces are embedded. It is easy then to realize infinitely many topological types as immersed complete CMC surfaces with $k$ ends, where any $k \geq 2$ can be given in advance. These constructions (when no symmetries are imposed) have $(k - 1)(n + 1) - \left(\binom{n+1}{2}\right) + \left(\binom{n+1-k}{2}\right)$ continuous parameters, reflecting thus the asymptotics of the $k$ Delaunay ends. Moreover there is further great variety in the immersions of a given number of ends and topological type reflected by the central graphs $\Gamma$ we can choose. Finally in the current paper we find embedded examples with $k$ ends, for $k \geq 3$, and we obtain finitely many topological types for each $k$, with the number of topological types tending to $\infty$ as $k \to \infty$.

We remark that in ongoing work [5], we extend these results (in a manner similar to [37] extending [36]) to construct infinitely many topological types of closed (immersed) examples for each $n \geq 3$. Moreover, for each $n \geq 3$ and $k \geq 6$, we construct in [5] infinitely many topological types of embedded complete examples with $k$ ends.

Outline of the approach.

The construction in this article is an extension to high dimensions of the constructions in [36, 6] with [6] serving also as an intermediate step in the development of this article. The main difficulties and their resolution in extending to high dimensions are the following:

1. A careful understanding of the geometry and analysis of the Delaunay hypersurfaces in high dimensions is needed, which to the best of our knowledge is new at least at this level of detail. In particular understanding their periods requires some work and is similar to work for special Legendrian submanifolds [23, 20].

2. In dimension two the Laplacian is conformally covariant and the conformal metric $h = \frac{1}{2} |A|^2 g$ makes the catenoidal necks in the Delaunay surfaces isometric to the actual spherical regions, compactifying them in the limit [36]. These simplifications fail in higher dimensions, so we had to redesign the linear theory by using on the (hyper)catenoidal necks, instead of the $h$ metric, appropriate weighted estimates, Fourier decompositions on the meridians, and some $L^2$ estimates.

3. We use the ideas of [36], modified for the high dimensions, to understand the linearized equation on the central spherical regions where the fusion with the Delaunay pieces occurs. We also use semi-localization, that is studying the linearized equation on the extended standard regions and combining the results.

4. Following the methodology in [38], where the geometric principle and the extended substitute kernel were introduced, we introduce dislocations, as suggested by the geometric principle, between the central spherical regions and the Delaunay pieces attached to them. This way we create extended substitute kernel, which is then used to achieve much faster decay (compared to [36]) away from the central spherical regions. To estimate the creation of extended substitute kernel in Proposition 7.23 we argue as in [38, Proposition 6.7].

5. Instead of monitoring the use of the substitute kernel at each step, we choose to use a balancing formula [48] on the final hypersurface to estimate the unbalancing error, thus obtaining better control.
Organization of the presentation.

Appendix A contains a thorough treatment of the geometry of Delaunay surfaces. Appendix B provides standard background on the quadratic error estimates. Finally, in Appendix C we study the Dirichlet problem on a flat annulus.

Section 2 contains a description of the family of graphs which provides the structure for the immersion of the initial surfaces. We discuss the unbalancing and flexibility conditions and we associate to each graph in the family two parameters \((\tilde{d}, \tilde{\ell})\) which give quantitative meaning to these conditions.

In Section 3, we describe the building blocks of the construction: spheres with balls removed and Delaunay pieces with perturbations near their boundaries. The Delaunay building blocks are not necessarily CMC near their boundaries; the estimates are controlled by the parameters describing the perturbation. We are careful to describe these building blocks independently of any reference to graphs and the building blocks depend only on general parameters.

In Section 4 we study the linear operator \(L_g\) on compact pieces of Delaunay surfaces. At this stage we choose a fixed large constant \(b \gg 1\) and a small \(T > 0\) depending on \(b\). For any \(0 < |\tau| < T\), we consider regions on a Delaunay immersion with parameter \(\tau\). The size of the regions considered depend upon \(\tau\) and \(b\). The choice of \(b, \tau\) together with our understanding of the geometry of Delaunay surfaces provide then good geometric estimates. Again, the statements and proofs of this section do not reference or rely on a graph or family of graphs.

In Section 5, we construct a family of initial surfaces which depend upon a parameter \(\tau\) and a pair of parameters \((d, \zeta)\). We presume a given family of graphs \(\mathcal{F}\). The parameter \(\tau\) satisfies \(0 < |\tau| < T\Gamma\) where \(T\Gamma\) depends upon \(T\) and the graph \(\Gamma\) only. The parameters \((d, \zeta)\) and \(\tau\) determine \((\tilde{d}, \tilde{\ell})\) and thus a graph \(\Gamma'\) in the family \(\mathcal{F}\). We build the initial surface by positioning and fusing building blocks at designated locations determined by \(\Gamma'\). The parameters describing the building blocks are encoded in \(\tau, (d, \zeta)\) and \(\Gamma'\).

In Section 6, we study the linearized operator on the family of initial surfaces. In particular we define the extended substitute kernel \(K\) (6.23, 6.26 and 6.30) and suitable global norms in 6.34. These are used to state and prove the main result of the section, Proposition 6.44, where we solve with estimates the linear problem modulo \(K\). We also provide estimates for the mean curvature on the initial hypersurfaces in 6.10.

In Section 7, we understand the creation of substitute kernel by balancing considerations in 7.4, we estimate the main contribution to the solution of the linearized problem in the mean curvature due to gluing—\(H_{\text{gluing}}[d, \zeta]\)—in 7.9, and we understand the creation of extended substitute kernel in 7.23. In Section 8, we provide global quadratic estimates in 8.1 and then state and prove the Main Theorem 8.2 using a fixed point theorem argument.

Preliminaries.

**Definition 1.2.** For \(k \in \mathbb{N} \cup \{0\}, \beta \in (0, 1)\), a domain \(\Omega\) in a Riemannian manifold, \(u \in C^{k,\beta}_{\text{loc}}(\Omega)\), and \(f, \rho : \Omega \to \mathbb{R}^+\) we define the norm

\[
\|u : C^{k,\beta}(\Omega, \rho, g, f)\| := \sup_{x \in \Omega} f(x)^{-1}\|u : C^{k,\beta}(B_x \cap \Omega, \rho^{-2}(x)g)\|.
\]

Here \(B_x\) is a geodesic ball centered at \(x\) with radius 1/10 in the metric \(\rho^{-2}(x)g\). For simplicity, when \(\rho = 1\) or \(f = 1\) we may omit them from the notation.

Note from the definition that

\[
\|\nabla u : C^{k-1,\beta}(\Omega, \rho, g, \rho^{-1}f)\| \leq \|u : C^{k,\beta}(\Omega, \rho, g, f)\|.
\]
and
\[ \|u_1 u_2 : C^{k,\beta}(\Omega, \rho, g, f_1 f_2)\| \leq C(k)\|u_1 : C^{k,\beta}(\Omega, \rho, g, f_1)\| \|u_2 : C^{k,\beta}(\Omega, \rho, g, f_2)\|. \]

**Definition 1.3.** If \( a, b > 0 \) and \( c > 1 \), then we write
\[ a \sim_c b \]
if \( a \leq cb \) and \( b \leq ca \).

Throughout this paper we make extensive use of cut-off functions, and thus we adopt the following notation: Let \( \Psi : \mathbb{R} \to [0, 1] \) be a smooth function such that
1. \( \Psi \) is non-decreasing
2. \( \Psi \equiv 1 \) on \([1, \infty)\) and \( \Psi \equiv 0 \) on \((-\infty, -1]\)
3. \( \Psi - 1/2 \) is an odd function.

For \( a, b \in \mathbb{R} \) with \( a \neq b \), let \( \psi[a, b] : \mathbb{R} \to [0, 1] \) be defined by \( \psi[a, b] = \Psi \circ L_{a,b} \) where \( L_{a,b} : \mathbb{R} \to \mathbb{R} \) is a linear function with \( L(a) = -3, L(b) = 3 \). Then \( \psi[a, b] \) has the following properties:
1. \( \psi[a, b] \) is weakly monotone.
2. \( \psi[a, b] = 1 \) on a neighborhood of \( b \) and \( \psi[a, b] = 0 \) on a neighborhood of \( a \).
3. \( \psi[a, b] + \psi[b, a] = 1 \) on \( \mathbb{R} \).

**Notation 1.4.** For \( X \) a subset of a Riemannian manifold \((M, g)\) we write \( d^M,g_X \) for the distance function from \( X \) in \((M, g)\). For \( \delta > 0 \) we define a tubular neighborhood of \( X \) by
\[ D^M,g_X(\delta) := \{ p \in M : d^M,g_X(p) < \delta \} . \]
In both cases we may omit \( M \) or \( g \) if understood from the context and if \( X \) is finite we may just enumerate its points.

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**2. Finite Graphs**

The gluing construction carried out in this article uses round spheres and pieces of Delaunay surfaces to build initial hypersurfaces which are then perturbed to become CMC hypersurfaces. The parameters of the Delaunay pieces and the positioning of the spheres and the Delaunay pieces are naturally encoded by graphs. In this article for simplicity we restrict ourselves to finite graphs which we discuss in this section. The initial graph we use should satisfy all of the relations one expects for a singular CMC surface and thus we impose a balancing restriction on each vertex and a restriction on the length of each edge. We first define the kind of graphs we will be using:

**Definition 2.1 (Graphs).** We define a finite graph in \( \mathbb{R}^{n+1} \) for some \( n > 2 \) to be a collection \( \{V(\Gamma), E(\Gamma), R(\Gamma), \tilde{\tau}\} \) such that...
(1) $V(\Gamma) \subset \mathbb{R}^{n+1}$ is a finite collection of vertices.
(2) $E(\Gamma)$ is a finite collection of edges in $\mathbb{R}^{n+1}$, each with its two endpoints in $V(\Gamma)$.
(3) $R(\Gamma)$ is a finite collection of rays in $\mathbb{R}^{n+1}$, each with its one endpoint in $V(\Gamma)$.
(4) $\hat{\tau} : E(\Gamma) \cup R(\Gamma) \to \mathbb{R}\{0\}$ is a function.

Notation 2.2. Given a finite graph $\Gamma$, the input of a function or vector valued function of $V(\Gamma), E(\Gamma), R(\Gamma)$ will be given by $[\cdot]$.

Definition 2.3 (Edge and Vertex Relations). Let $E_p$ denote the collection of edges and rays that have $p \in V(\Gamma)$ as an endpoint. We have then
\[ \bigcup_{p \in V(\Gamma)} E_p = E(\Gamma) \cup R(\Gamma). \]

We also define the set of attachments
\[ A(\Gamma) := \{ [p, e] \in V(\Gamma) \times (E(\Gamma) \cup R(\Gamma)) : e \in E_p \}. \]

Finally for each $[p, e] \in A(\Gamma)$ we denote the unit vector pointing away from $p$ and in the direction of $e$ by $v[p, e]$.

Definition 2.5. For a graph $\Gamma$, let $L(\Gamma)$ denote the space of functions from $E(\Gamma)$ to $\mathbb{R}$, let $D(\Gamma)$ denote the space of functions from $V(\Gamma)$ to $\mathbb{R}^{n+1}$, and let $Z(\Gamma)$ denote the space of functions from $A(\Gamma)$ to $\mathbb{R}^{n+1}$. Equip each of these spaces with the maximum norm.

Definition 2.6. We define $\hat{d}([\Gamma, \cdot]) = \hat{d} \in D(\Gamma)$ such that
\[ \hat{d}([\Gamma, p]) = \hat{d}[p] := \left( \frac{\omega_{n-1}}{n} \right) \left( \frac{n+1}{\omega_n} \right)^{1/2} \sum_{e \in E_p} \hat{\tau}[e]v[p, e] := \frac{\omega_{n-1}}{\omega_n^{1/2}} \sum_{e \in E_p} \hat{\tau}[e]v[p, e] \]
measures the deviation from balancing at the vertex $p$. Here $\bar{\omega}_{k-1} := \frac{\omega_{k-1}}{k}$ and $\omega_k$ denotes as usual the $k$-dimensional volume of $\mathbb{S}^k \subset \mathbb{R}^{k+1}$.

We let $l([\Gamma, \cdot]) = l \in L(\Gamma)$ such that for $e \in E(\Gamma)$, $2l[e]$ equals the length of $e$.

Remark 2.8. The constant $\bar{\omega}_{k-1}$ will arise because of various normalizations throughout the argument. Absorbing it into the definition of $\hat{d}$ will be convenient later.

Our construction will be based on a family of graphs that are perturbations of some fixed graph which we will call the central graph $\Gamma$ (see 2.9). The idea of the construction is to replace each edge or ray $e$ of $\Gamma$ by a Delaunay piece of parameter $\tau \hat{\tau}[e]$, where $\tau$ is a sufficiently small global parameter. (See Section 3 for a description of the Delaunay pieces.) The construction of the initial surfaces requires appropriate small perturbations of $\Gamma$ depending on $\tau$ and on other parameters. The central graph $\Gamma$ will be the limit of the graphs employed as $\tau \to 0$. In this limit our surfaces will tend to tangentially touching unit spheres. Correspondingly, the period of the Delaunay surfaces will tend to 2. Therefore $\Gamma$ has to satisfy the condition that its edges have even integer length. Moreover the balancing conditions satisfied by CMC surfaces (see 7.1, (A.4), (A.3)) imply the vanishing of $\hat{\tau}$ on $\Gamma$. These considerations motivate the following definition.

Definition 2.9. Let $\Gamma$ be a finite graph. If $\hat{d}[p] = 0$ for all $p \in V(\Gamma)$, we say $\Gamma$ is a balanced graph. We call $\Gamma$ a central graph if $\Gamma$ is balanced and $l[e] \in \mathbb{N}$ for all $e \in E(\Gamma)$.

Finally, we define central graphs that guarantee that our construction produces an embedded CMC hypersurface:
Definition 2.10 (Pre-embedded graphs). We say $\hat{\Gamma}$ is pre-embedded if it is a central graph with $\hat{\tau}: E(\hat{\Gamma}) \cup R(\hat{\Gamma}) \to \mathbb{R}^+$ and

1. For all $p \in V(\Gamma)$ and all $e_i \neq e_j \in E_p$, $\angle(v[p, e_i], v[p, e_j]) \geq \pi/3$, where $\angle(x, y)$ measures the angle between the two vectors $x, y$.

2. For all $e, e' \in E(\hat{\Gamma}) \cup R(\hat{\Gamma})$ that do not share any common endpoints, the Euclidean distance between $e, e'$ is greater than 2.

3. For any two rays $e \neq e' \in R(\hat{\Gamma})$, $1 - v[p, e] \cdot v[p', e'] > 0$.

For a pre-embedded $\hat{\Gamma}$ and sufficiently small $\tau$, each of the initial surfaces constructed from one of the possible perturbations of $\hat{\Gamma}$ is embedded. In the singular setting, when $\tau = 0$, the angle condition between edges and rays about a fixed vertex allows for a singular surface with unit spheres touching tangentially. We do not require a strict inequality for this condition since the change in the period for small $\tau$ (on the order $|\tau|^{1/2}$) dominates both the radius change and the changes we allow via unbalancing and dislocation (on the order $|\tau|$). The second item requires a strict inequality as the maximum radius of an embedded Delaunay surface is on the order $1 - \tau + O(\tau^2)$ but we allow for the edges to move with order $C|\tau\tau|$ where $C$ can be quite large. The final condition also requires a strict inequality. Indeed if the central graph $\hat{\Gamma}$ has two parallel rays pointing into the same half-plane, then the family of graphs on which we base our initial surfaces may include graphs with intersecting rays.

Deforming the graphs. Given a central graph $\hat{\Gamma}$, we will consider perturbations of this graph subject to parameters $\hat{\tilde{d}}, \hat{\ell}$. We need the perturbations to be smoothly dependent on the parameters and are thus interested in graphs $\hat{\Gamma}$ which can be deformed in this way.

Definition 2.11 (Isomorphic graphs). We define two graphs as isomorphic if there exists a one-to-one correspondence between the vertices, edges, and rays, such that corresponding rays and edges emanate from the corresponding vertices. For convenience we will often use the same letter to denote corresponding objects for isomorphic graphs. Using this correspondence, for $\hat{\Gamma}$ isomorphic to $\Gamma$, we identify $D(\hat{\Gamma}), L(\hat{\Gamma}), Z(\hat{\Gamma})$ with $D(\Gamma), L(\Gamma), Z(\Gamma)$ respectively.

We proceed to define the function $\ell$, which quantifies the length change of each edge for a perturbation of $\hat{\Gamma}$.

Definition 2.12. Given a graph $\Gamma_1$ isomorphic to a central graph $\Gamma$, we define $\ell[\Gamma_1, \cdot] \in L(\Gamma)$ such that (following 2.11) for all $e \in E(\Gamma) \approx E(\Gamma_1)$,

$$\ell[\Gamma_1, e] := l[\Gamma_1, e] - l[\Gamma, e],$$

and therefore the length of the edge of $\Gamma_1$ corresponding to $e \in E(\Gamma)$ is

$$2l[\Gamma_1, e] = 2l[\Gamma, e] + 2\ell[\Gamma_1, e].$$

Definition 2.14 (Families of graphs). We define a family of graphs $\mathcal{F}$ to be a collection of graphs parametrized by $(\hat{\tilde{d}}, \hat{\ell}) \in B_\mathcal{F}$ such that the following hold:

1. $\Gamma := \hat{\Gamma}(0, 0)$ is a central graph in the sense of 2.9 and $B_\mathcal{F}$ is a small ball about $(0, 0)$ in $D(\Gamma) \times L(\Gamma)$.

2. $\Gamma(\hat{\tilde{d}}, \hat{\ell})$ is isomorphic to $\hat{\Gamma}(0, 0)$ and depends smoothly on $(\hat{\tilde{d}}, \hat{\ell})$.

3. Following 2.11, $\hat{\tilde{d}}[\Gamma(\hat{\tilde{d}}, 0), \cdot] = \hat{\delta}[]$ (unbalancing condition).

4. Following 2.11, $\hat{\ell}[\Gamma(\hat{\tilde{d}}, \hat{\ell}), \cdot] = \hat{\ell}[]$ (flexibility condition).

5. $\hat{\tau}[\Gamma(\hat{\tilde{d}}, 0), \cdot] = \hat{\tau}[\Gamma(\hat{\tilde{d}}, \hat{\ell}), \cdot]$. 

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Note that by the above definition each $\Gamma(\tilde{d},0)$ with $\tilde{d} \neq 0$ is a modification of the central graph that is unbalanced as prescribed by $\tilde{d}$ while the lengths of the edges remain unchanged. Perturbing $\Gamma(\tilde{d},0)$ to $\Gamma(\tilde{d},\tilde{\ell})$ is achieved by changing the lengths of the edges as prescribed by $\tilde{\ell}$.

Definition 2.17. Throughout the paper, let $\{e_1, \ldots, e_{n+1}\}$ denote the standard orthonormal basis of $\mathbb{R}^{n+1}$.

We now choose a frame associated to each edge in the graph $\Gamma$ and use this frame to determine a frame on each edge for any graph in $\mathcal{F}$.

Definition 2.16. For $e \in E(\Gamma)$ we choose once for all one of its endpoints to call $p^+[e]$. We call then its other endpoint $p^-[e]$ and we define $\text{sgn}(p^+[e],e) := \pm 1$. For $e \in E(\Gamma) \cup R(\Gamma)$ we choose once and for all an ordered, positively oriented orthonormal frame $F_1[e] = \{v_1[e], \ldots, v_{n+1}[e]\}$, such that $v_1[e] = v[p,e]$, where $p$ is the endpoint of $e$ if $e \in R(\Gamma)$ and $p = p^+[e]$ if $e \in E(\Gamma)$. We have therefore when $[p,e] \in A(\Gamma)$ and $e \in E(\Gamma)$

$$v[p^+[e],e] = v_1[e] = -v[p^-[e],e]$$

and $\text{sgn}[p,e] = v[p,e] \cdot v_1[e]$.

Definition 2.17. Given two unit vectors $x, y \in \mathbb{R}^{n+1}$ such that $\angle(x, y) < \pi/2$, let $\mathcal{R}[x,y]$ denote the unique rotation defined in the following manner.

- If $x = y$, take $\mathcal{R}[x,y]$ to be the identity.
- If $x \neq y$, set $x \cdot y = \cos a$ and $y_y := \frac{x - x \cos a}{\sin a}$. We define $\mathcal{R}[x,y]$ to be the rotation in the plane given by $x, y$ that rotates $x$ to $y$, that is in closed form

$$\mathcal{R}[x,y] = I + \sin a \left( y_y x^T - x y_y^T \right) - (1 - \cos a) \left( x x^T + y_y y_y^T \right).$$

Lemma 2.18. The rotation $\mathcal{R}[x,y]$ depends smoothly on $x$ and $y$.

Proof. Simplifying the expression, using the definition of $y_y$, we observe that for $x \neq y$,

$$\mathcal{R}[x,y] = I + (y x^T - x y^T) - (1 - \cos a) \frac{1}{1 + \cos a} (y y^T - \cos a (y x^T + x y^T) + \cos^2 a x x^T).$$

This expression is clearly smooth in $x, y$. \qed

For $[p,e] \in A(\Gamma)$ and $[p',e']$ the corresponding attachment on an isomorphic graph, let

$$\angle(e,e') := \arccos(v[p,e] \cdot v[p',e']).$$

We use the rotation defined above to describe an orthonormal frame on the edges and rays of any graph in the family $\mathcal{F}$. By the smooth dependence on $\tilde{d}, \tilde{\ell}$, and the presumed smallness of their norms, $\angle(e,e') < \pi/2$ for $e \in E(\Gamma) \cup R(\Gamma)$ and $e'$ a corresponding edge or ray on any graph in the family. It follows that the rotation we need will always be well-defined.

Definition 2.19. For $\Gamma(\tilde{d},\tilde{\ell}) \in \mathcal{F}$ with $\mathcal{F}$ as in 2.14, given $e \in E(\Gamma) \cup R(\Gamma)$ we define an orthonormal frame $F_{\Gamma(\tilde{d},\tilde{\ell})}[e] = \{v_1[e;\tilde{d},\tilde{\ell}], \ldots, v_{n+1}[e;\tilde{d},\tilde{\ell}]\}$ uniquely by requiring the following:

1. $v_1[e;\tilde{d},\tilde{\ell}] = v[\Gamma(\tilde{d},\tilde{\ell}),p^+[e],e]$.
2. $v_i[e;\tilde{d},\tilde{\ell}] = \mathcal{R}[v_1[e],v_1[e;\tilde{d},\tilde{\ell}]](v_i[e])$ for $i = 2, \ldots, n+1$.

Remark 2.20. $F_{\Gamma(\tilde{d},\tilde{\ell})}[e]$ depends smoothly on $\tilde{d}, \tilde{\ell}$. 
3. The Building Blocks

The initial hypersurfaces we construct will be built out of appropriately fused pieces of spheres and perturbed Delaunay hypersurfaces. The positioning of these pieces and the parameter of each Delaunay piece is determined by the graphs of $\mathcal{F}$ and the parameters $d, \zeta$. The building blocks however can be described independently of any reference to the graphs of $\mathcal{F}$. To highlight this fact, we first develop the immersions of the building blocks to depend upon other general parameters not related to any graph. In Section 5 we use these immersions to produce a family of hypersurfaces from a family of graphs $\mathcal{F}$, where each hypersurface will depend on the central graph $\Gamma$ of $\mathcal{F}$ as well as the parameters $d, \zeta$.

**Spherical building blocks.** Let $Y_0 : \mathbb{R} \times S^{n-1} \to S^n \subset \mathbb{R}^{n+1}$ be as in A.6. Immediately we see that

$$g_0 = \text{sech}^2 dt^2 + g_{S^{n-1}}, \quad |A|^2 = n, \quad H \equiv 1.$$

**Definition 3.1.** Let $\delta'$ be a small positive constant which we will choose in 5.4 and define $a > 0$ by $\tanh(a + 1) = \cos(\delta')$. Note that $Y_0(\{a + 1\} \times S^{n-1}) = \partial D^S_{\{1,0\}}(\delta') \subset S^n$ (recall 1.4).

We determine now sphere diffeomorphisms that will be used to guarantee that the immersion is well-defined. First we define a rotation $\hat{R}[F,F']$ which maps $F$ to $F'$ for a given orthonormal frame $F$ and a perturbation $F'$ of $F$.

**Definition 3.2.** Let $F := \{x_1, \ldots, x_{n+1}\}, F' := \{y_1, \ldots, y_{n+1}\}$ be two orthonormal frames of $\mathbb{R}^{n+1}$ with the same orientation. We define $\hat{R}[F,F'] : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to be the unique rotation such that

$$\hat{R}[F,F'](x_i) = y_i.$$

We now define a map on $S^n$ that consists of $m$ local frame transformations and smoothly transits to the identity map away from these transformations. In application, the first vector in each frame will describe the positioning of an edge on a graph $\Gamma(d, \tilde{e}) \in \mathcal{F}$.

**Definition 3.3 (Spherical Building Blocks).** We assume given two sets of positively oriented ordered orthonormal frames $W = \{F_1, F_2, \ldots, F_m\}$ and $W' = \{F'_1, F'_2, \ldots, F'_m\}$, where, for $i,j \in \{1, \ldots, m\}$,

$$F_i = \{x_{1,i}, x_{2,i}, \ldots, x_{n+1,i}\}, \quad F'_i = \{y_{1,i}, \ldots, y_{n+1,i}\},$$

$$\angle(x_{1,j}, x_{1,i}) > 16\delta' \quad \forall i \neq j, \quad \angle(x_{1,j}, y_{1,i}) \leq (\delta')^2 \quad \forall i.$$

That is, the first vectors in each frame of $W$ are not close, while the first vectors in each pair of frames $F_i, F'_i$ are close. We define then a family of diffeomorphisms $\hat{Y}[W,W'] : S^n \to S^n \subset \mathbb{R}^{n+1}$, smoothly dependent on $W, W'$, by

$$\hat{Y}[W,W'](x) := \begin{cases} x & \text{for } x \in S^n \setminus \bigcup_{i} D^{S^n}_{x_{1,i}}(4\delta'), \\ \psi_W(x) x + (1 - \psi_W(x)) \hat{R}[F_i, F'_i](x) & \text{for } x \in D^{S^n}_{x_{1,i}}(4\delta') \setminus D^{S^n}_{x_{1,i}}(3\delta'), \\ \hat{R}[F_i, F'_i](x) & \text{for } x \in D^{S^n}_{x_{1,i}}(3\delta'), \end{cases}$$

where $\psi_W := \psi[3\delta', 4\delta'] \circ d^{S^n}_{\{x_{1,1}, x_{1,2}, \ldots, x_{1,n+1}\}}$ (recall 1.4).
Delauanay building blocks. We now describe a general immersion of an appropriately perturbed Delaunay piece. For a description of Delaunay immersions, see Section A. Throughout this subsection, let \( \alpha \) be the value defined in 3.1, let \( l \in \mathbb{Z}^+ \), and let \( p_\tau \) and \( \hat{p}_\tau \) be as in A.10 so that \( 2p_\tau \) is the domain period and \( 2\hat{p}_\tau \) the translational period of a Delaunay hypersurface of parameter \( \tau \). We presume throughout that \( 0 < T \ll 1 \) is a constant chosen sufficiently small to guarantee that all immersions are smooth and well-defined and that all error estimates will hold as stated. Finally, we let \( C \) denote a possibly large constant that is independent of \( T \).

**Definition 3.4.** Let \( \psi_{\text{dislocation}}^\pm, \psi_{\text{gluing}}^\pm : [a, 2p_\tau l - a] \to \mathbb{R} \) be cutoff functions such that:

- \( \psi_{\text{dislocation}}^+ = \psi[a + 2, a + 1] \),
- \( \psi_{\text{dislocation}}^- = \psi[2p_\tau l - (a + 2), 2p_\tau l - (a + 1)] \),
- \( \psi_{\text{gluing}}^+ = \psi[a + 3, a + 4] \),
- \( \psi_{\text{gluing}}^- = \psi[2p_\tau l - (a + 3), 2p_\tau l - (a + 4)] \).

With these cutoff functions, we define the building blocks. Notice that \( Y_0 \) is the embedding of \( S^n \) defined in (A.6) and \( Y \) is the Delaunay immersion defined in (A.1).

**Definition 3.5.** Given \( \tau, l, a \) with \( 0 < |\tau| \leq T \) and \( \zeta^\pm \in \mathbb{R}^{n+1} \) with \( 0 \leq |\zeta^\pm| \leq C|\tau| \), we define two smooth immersions \( Y_{\text{edge}}[\tau, l, \zeta^+, \zeta^-] : [a, 2p_\tau l - a] \times S^{n-1} \to \mathbb{R}^{n+1} \) and \( Y_{\text{ray}}[\tau, \zeta^+] : [a, \infty) \times S^{n-1} \to \mathbb{R}^{n+1} \) such that, for \( x = (t, \Theta) \),

\[
Y_{\text{edge}}[\tau, l, \zeta^+, \zeta^-](x) = \psi_{\text{dislocation}}^+(t) \cdot (Y_0(x) + \zeta^+) + \psi_{\text{dislocation}}^-(t)(1 - \psi_{\text{gluing}}^+(t))Y_0(x) + \psi_{\text{gluing}}^+(t) \cdot \psi_{\text{gluing}}^-(t) \cdot Y_\tau(x) + \psi_{\text{dislocation}}^-(t)(1 - \psi_{\text{gluing}}^-)(t)Y_\tau(x) + \psi_{\text{dislocation}}^-(t) \cdot (Y_\tau^-(x) + \zeta^-)
\]

\[
Y_{\text{ray}}[\tau, \zeta^+](x) = \psi_{\text{dislocation}}^+(t)(Y_0(x) + \zeta^+) + (1 - \psi_{\text{dislocation}}^+(t))(1 - \psi_{\text{gluing}}^+(t))Y_0(x) + \psi_{\text{gluing}}^+(t) \cdot Y_\tau(x)
\]

where \( Y_\tau^-(x) = Y_0(t - 2p_\tau l, \Theta) + (2 + 2\hat{p}_\tau)e_1 \).

To aid the reader, we describe the geometry of the \( Y_{\text{edge}} \) immersion in some detail. For \( t \in [a, a + 1] \), the image is a geodesic hyperannulus sitting on a unit sphere with the sphere centered at \( \zeta^+ \). The annulus is centered at \( \zeta^+ + e_1 \) with inner radius \( \delta' \). When \( t \in [a + 1, a + 2] \), the immersion smoothly interpolates between the annular region on the dislocated sphere and an annular region centered at \( e_1 \) on a unit sphere centered at the origin. For \( t \in [a + 2, a + 3] \), the immersion remains on the unit sphere centered at the origin, while for \( t \in [a + 3, a + 4] \), the immersion smoothly transits between this sphere and a Delaunay piece with parameter \( \tau \). The same procedure happens toward the other end. First, the Delaunay piece transits back to a unit sphere centered at \( (Y_\tau(2p_\tau l, \Theta) \cdot e_1)e_1 \). This position represents the location of the end of a Delaunay piece with parameter \( \tau \) and \( l \) periods, with initial end at the origin. Finally, this sphere transits to a unit sphere centered at \( \zeta^- + (Y_\tau(2p_\tau l, \Theta) \cdot e_1)e_1 \), a dislocation of \( \zeta^- \) from the previously described sphere.

Of course, the \( Y_{\text{ray}} \) immersion has the same behavior as \( Y_{\text{edge}} \) near the origin. The only difference is that the Delaunay immersion continues out to infinity and there is no transiting back to a sphere.
Proposition 3.6. Let $g := Y_{\text{edge}}^*(g_{\mathbb{R}^n+1})$ or $g := Y_{\text{ray}}^*(g_{\mathbb{R}^n+1})$ as the situation dictates. For a fixed, large constant $x > a + 5$,

$$Y_{\text{edge}}[\tau, l, \zeta^+, \zeta^-] - Y_0 : C^k((a, x) \times S^{n-1}, g) \leq C(k, x)(|\zeta^+| + |\tau|),$$

$$Y_{\text{edge}}[\tau, l, \zeta^+, \zeta^-] - Y_0 : C^k((2p, l - x, 2p, l - a) \times S^{n-1}, g) \leq C(k, x)(|\zeta^-| + |\tau|),$$

and $Y_{\text{ray}}[\tau, \zeta^+] - Y_0 : C^k((a, x) \times S^{n-1}, g) \leq C(k, x)(|\zeta^+| + |\tau|)$.\[Proof.\] On the region where $t \in [a + 1, a + 2] \cup [2p, l - (a + 2), 2p, l - (a + 1)]$, the only difference between the immersions comes from the cutoff function applied to $\zeta^\pm$, where the $\pm$ is appropriate for the domain. Thus the $C^k$ estimates on these regions are immediate.

For the other regions, we first note that the immersion $Y_0$ defines $\tanh(s) = x_1, \sinh(s) = \rho_0(x_1)$ from A.16. Using an ODE comparison for $k(t)$ and $\tanh(t)$, we can appeal to A.16 to get the $C^k$ estimates for the remaining regions. $\square$

Definition 3.7. Let $H_X$ denote the mean curvature of the immersion $X : \Omega \subset \mathbb{R} \times S^{n-1} \rightarrow \mathbb{R}^{n+1}$. Define

$$H_{\text{dislocation}}[\tau, l, \zeta^+, \zeta^-], H_{\text{gluing}}[\tau, l, \zeta^+, \zeta^-] : [a, 2p, l - a] \times S^{n-1} \rightarrow \mathbb{R},$$

$$H_{\text{dislocation}}[\tau, \zeta^+], H_{\text{gluing}}[\tau, \zeta^+] : [a, \infty) \times S^{n-1} \rightarrow \mathbb{R}$$

by

$$\text{supp} \ (H_{\text{dislocation}}[\tau, l, \zeta^+, \zeta^-]) \subset ([a, a + 2] \cup [2p, l - (a + 2), 2p, l - a]) \times S^{n-1},$$

where $H_{\text{dislocation}}[\tau, l, \zeta^+, \zeta^-] := H_{\text{edge}}[\tau, l, \zeta^+, \zeta^-] - 1,$

$$\text{supp} \ (H_{\text{gluing}}[\tau, l, \zeta^+, \zeta^-]) \subset ([a + 3, a + 5] \cup [2p, l - (a + 5), 2p, l - (a + 3)]) \times S^{n-1},$$

where $H_{\text{gluing}}[\tau, l, \zeta^+, \zeta^-] := H_{\text{edge}}[\tau, l, \zeta^+, \zeta^-] - 1,$

$$\text{supp} \ (H_{\text{dislocation}}[\tau, \zeta^+]) \subset [a, a + 2] \times S^{n-1},$$

where $H_{\text{dislocation}}[\tau, \zeta^+] := H_{\text{ray}}[\tau, \zeta^+] - 1,$

$$\text{supp} \ (H_{\text{gluing}}[\tau, \zeta^+]) \subset [a + 3, a + 5] \times S^{n-1}, \text{ where } H_{\text{gluing}}[\tau, \zeta^+] := H_{\text{ray}}[\tau, \zeta^+] - 1.$$ From these definitions and 3.6 we immediately bound the error on the mean curvature.

Corollary 3.8.\[\|H_{\text{dislocation}}[\tau, l, \zeta^+, \zeta^-] : C^{0, \beta}([a, 2p, l - a] \times S^{n-1}, g)\| \leq C(\beta)(|\zeta^+| + |\zeta^-|)\]

$$\|H_{\text{gluing}}[\tau, l, \zeta^+, \zeta^-] : C^{0, \beta}([a, 2p, l - a] \times S^{n-1}, g)\| \leq C(\beta)|\tau|$$

$$\|H_{\text{dislocation}}[\tau, \zeta^+] : C^{0, \beta}([a, \infty) \times S^{n-1}, g)\| \leq C(\beta)|\zeta^+|$$

$$\|H_{\text{gluing}}[\tau, \zeta^+] : C^{0, \beta}([a, \infty) \times S^{n-1}, g)\| \leq C(\beta)|\tau|$$

Lemma 3.9. For $g$ as in 3.6, $N_X$ the unit normal of the immersion $X$, and $b \in (a + 3, p)$,

$$\int_{[a, b] \times S^{n-1}} H_{\text{dislocation}}[\tau, l, \zeta^+, \zeta^-] N_{\text{edge}}[\tau, l, \zeta^+, \zeta^-] \, dg = 0,$$

$$\int_{[2p, l - b, 2p, l - a] \times S^{n-1}} H_{\text{dislocation}}[\tau, l, \zeta^+, \zeta^-] N_{\text{edge}}[\tau, l, \zeta^+, \zeta^-] \, dg = 0,$$

$$\int_{[a, b] \times S^{n-1}} H_{\text{dislocation}}[\tau, \zeta^+] N_{\text{ray}}[\tau, \zeta^+] \, dg = 0.$$
Proof. We prove the result for the ray immersion as the others follow identically. For convenience we also remove the notation $[\tau, \zeta^+]$.

First recall that $H_{\text{dislocation}}$ is supported on $[a + 1, a + 2] \times \mathbb{S}^{n-1}$. Thus

$$n \int_{[a,b] \times \mathbb{S}^{n-1}} H_{\text{dislocation}} N_{Y_{\text{ray}}} \, dg =$$

$$= n \int_{[a+1/2,a+5/2] \times \mathbb{S}^{n-1}} H_{Y_0} N_{Y_{\text{ray}}} \, dg - n \int_{[a+1/2,a+5/2] \times \mathbb{S}^{n-1}} N_{Y_{\text{ray}}} \, dg.$$  

By the divergence theorem and since $Y_{\text{ray}} = Y_0 + \zeta^+$ on $[a, a + 1] \times \mathbb{S}^{n-1}$, $Y_{\text{ray}} = Y_0$ on $[a + 2, a + 3] \times \mathbb{S}^{n-1}$, the first term can be rewritten as

$$\int_{[a+1/2,a+5/2] \times \mathbb{S}^{n-1}} \sum_{i=1}^{n+1} \Delta_g x_i \, e_i \, dg = \int_{\partial([a+1/2,a+5/2] \times \mathbb{S}^{n-1})} \sum_{i=1}^{n+1} \left( \nabla_{\delta} x_i \cdot \eta_{Y_{\text{ray}}} \right) e_i \, d\sigma_g =$$

$$= \int_{\partial([a+1/2,a+5/2] \times \mathbb{S}^{n-1})} \sum_{i=1}^{n+1} \left( \nabla_{\delta} g_{\delta} x_i \cdot \eta_{Y_0} \right) e_i \, d\sigma_g = n \int_{[a+1/2,a+5/2] \times \mathbb{S}^{n-1}} H_{Y_0} N_{Y_0} \, dg_0$$

where $d\sigma_g$ is the induced metric on the boundary. By similar logic, we note that

$$\int_{[a+1/2,a+5/2] \times \mathbb{S}^{n-1}} N_{Y_{\text{ray}}} \, dg = \int_{[a+1/2,a+5/2] \times \mathbb{S}^{n-1}} N_{Y_0} \, dg_0$$

and thus

$$n \int_{[a,b] \times \mathbb{S}^{n-1}} H_{\text{dislocation}} N_{Y_{\text{ray}}} \, dg = n \int_{[a,b] \times \mathbb{S}^{n-1}} (H_{Y_0} - 1) N_{Y_0} \, dg_0 = 0$$

4. Linear theory on Delaunay hypersurfaces

In this section, we solve semi-local linear problems on Delaunay surfaces with small parameter. Throughout the paper we denote the linearized operator in the induced metric by $\mathcal{L}_g$. On a Delaunay immersion as described in Appendix A, by (A.11) and (A.13) the operator takes the form

$$\mathcal{L}_g := \Delta_g + |A_g|^2 = \frac{1}{r^2} \partial_t + \frac{n-2}{r^2} w' \partial_t + \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}} + n(1 + (n-1)r^2 r^{-2n}).$$

Assumption 4.2. Throughout this section, we will assume $b \gg 1$ is a fixed constant, chosen as large as necessary and depending only on $n$ and $\epsilon_1$, where $\epsilon_1$ is a small constant which depends on $\gamma \in (1, 2), \beta \in (0, 1)$. In particular, $b$ is independent of the constant $T > 0$, which will be chosen as small as needed, in terms of $b$. We also assume given $b \in \left(\frac{9}{10}b, \frac{11}{10}b\right)$. Unless otherwise stated we will denote by $C$ positive constants which depend on $b$ but not on $b, T$.

Definition 4.3. Given $0 < |\tau| < T$ and a Delaunay immersion $Y_{\tau} : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$ defined as in Appendix A, we define the following regions on the domain:

1. $A_{x,y} := [b + x, p_{\tau} - (b + y)] \times \mathbb{S}^{n-1}$
2. $C_{r}^{\text{out}} := \{b + x\} \times \mathbb{S}^{n-1}$
3. $C_{y}^{\text{in}} := \{p_{\tau} - (b + y)\} \times \mathbb{S}^{n-1}$
4. $S_{x}^{-} := [p_{\tau} - (b + x), p_{\tau} + (b + x)] \times \mathbb{S}^{n-1}$
5. $S_{x}^{+} := [b + x, 2p_{\tau} - (b + x)] \times \mathbb{S}^{n-1}$
6. $S_{x}^{\perp} := [2p_{\tau} - (b + x), 2p_{\tau} + (b + x)] \times \mathbb{S}^{n-1}$
(7) \( \tilde{S}_x^+ := [p_r + (b + x), 3p_r - (b + x)] \times S^{n-1} \)

Here \( 0 \leq x, y < p_r - b, \) where \( p_r - b > 0 \) is guaranteed by the smallness of \( T \) in terms of \( b \). When \( x = y = 0 \) we may drop the subscript.

Notice that for \( T \) small enough, by A.16, A.21 the immersion of the region \( S^+ \) has geometry roughly like \( S^n \) while the immersion of the region \( S^- \), after an appropriate rescaling, looks roughly like a catenoid. Following usual terminology we refer to these regions as standard regions and we refer to \( \tilde{S}^+, \tilde{S}^- \) as extended standard regions. The extended standard regions contain one standard region and two adjacent regions each with \( t \)-coordinate length \( p_r - 2b \).

We have labeled one such region \( \Lambda \) and we refer to \( \Lambda \) as a transition or intermediate region.

**The linearized equation on the transition region.** Let \( r_{\text{out}}, r_{\text{in}} \) denote the radius of the meridian spheres at \( C_{\text{out}}, C_{\text{in}} \) respectively, in the induced metric. That is

\[
 r_{\text{out}} = r(\tau) \quad \text{and} \quad r_{\text{in}} = r(p_r - b).
\]

We consider a flat metric on \( \Lambda \) given by

\[
 (4.4) \quad g_{\Lambda} := ds^2 + s^2 g_{S^{n-1}} \quad \text{where} \quad s : [b, p_r - b] \to \mathbb{R}^+ \quad \text{satisfies} \quad \begin{cases} ds \over dt = r(t) \\ s(p_r - b) = r_{\text{in}} \end{cases}
\]

**Lemma 4.5.** Let \( \gamma \in (1, 2), \beta \in (0, 1) \). Given \( 0 < \delta < \min\{\frac{1}{100}, \frac{1}{10n}\} \), there exists \( b \) large enough and \( T > 0 \) small enough depending on \( b \) such that for all \( 0 < |\tau| < T \), for \( \Lambda \) defined by \( \tau \) and \( b \) satisfying 4.2:

\[
 (4.6) \quad \|1 : C^{0,\beta}(\Lambda, r, g, r^{-2})\|_{\infty, \tau} \approx 10 \quad \|r : C^{0,\beta}(\Lambda, r, g)\|,
\]

\[
 \|r^{-2n} : C^{0,\beta}(\Lambda, r, g, r^{-2})\|_{\infty, \tau} \approx 10 \quad \|r^{-2n} : C^{0,\beta}(\Lambda, r, g)\|.
\]

Moreover, for \( s \) defined by (4.4),

\[
 (4.7) \quad \left| \frac{s}{r} - 1 \right| \leq 5\delta, \quad \left| \frac{ds}{dr} - 1 \right| \leq 4\delta, \quad \left| \frac{d^2 s}{dr^2} \right| \leq \frac{C(n)}{r} \delta.
\]

As a consequence, for any \( v \in C^{k,\beta}(\Lambda) \), for \( 0 \leq k \leq 2 \),

\[
 (4.8) \quad \|v : C^{k,\beta}(\Lambda, r, g, r^{-2})\|_{\infty, \tau} \approx 10 \quad \|v : C^{k,\beta}(\Lambda, s, g_A, s^{\gamma-2})\|,
\]

\[
 \|v : C^{k,\beta}(\Lambda, r, g, r^{-\gamma})\|_{\infty, \tau} \approx 10 \quad \|v : C^{k,\beta}(\Lambda, s, g_A, s^{-\gamma})\|.
\]

**Proof.** Notice that the geometry of \( Y_\tau \) near \( t = 0 \) and \( t = p_r \) (see A.16, A.21) implies that by picking \( b \) large, independent of \( T \) and \( T \) sufficiently small, for all \( 0 < |\tau| < T \) we have the bound \( r \in \left( |\tau|^{-\frac{1}{n-1}} / \delta, \delta \right) \) on \( \Lambda \).

To prove (4.6), consider a fixed \( (u, \Theta) \in \Lambda \) and note that \( r^{-2}(u)g = \frac{\tau^2}{r^2(u)} (dt^2 + g_{S^{n-1}}) \). Observe that as \( |w'| \in (1 - 3\delta^2, 1] \) by the choice of \( b, T \),

\[
 \frac{99}{100} |t - u| < (1 - 3\delta^2)|t - u| \leq |w(t) - w(u)| = \left| \int_u^t w'(t) dt \right|.
\]

Therefore, as \( \frac{r(u)}{r(t)} = e^{w(u)-w(t)} \), if \( |u - t| > \frac{1}{5} \) then the length of a curve connecting \( (u, \Theta), (t, \Theta) \) in the metric \( \frac{r^2}{r^2(u)} dt^2 \) is at least \( \frac{1}{10} \) as

\[
 \int_u^t e^{99|u-s|/100} ds \geq \int_u^t e^{99|u-t|/100} ds = \frac{100}{99} (e^{99|u-t|/100} - 1).
\]
It follows that a ball of radius 1/10 about \((u, \Theta)\) in the metric \(r^{-2}(u)g\), is contained in the cylinder \([u - 1/5, u + 1/5] \times S^{n-1}\). Now for \(m = 0\) or \(m = -2n\),

\[
\frac{r^{-m}(t)r^2(u)}{r^{2-m}(t)} = \frac{2n}{2n} e^{2u(t) - m(w(t))} = e^{2u(t) - 2w(t)}.
\]

The \(C^0\) equivalence in (4.6) follows as \((1 - 3\delta^2)|t - u| \leq |w(t) - w(u)| \leq |t - u|\) on \(\Lambda\) and \(|t - u| \leq \frac{1}{2}\) for every comparison in the weighted metric. To get the \(C^0,\beta\) equivalence, first observe that

\[
\frac{r^2(u)}{r^2(t)}(r^2(t))' = 2w'(t)r^2(u).
\]

The above is bounded by \(2\delta^2\) on \(\Lambda\) and the first equivalence holds. In the other case,

\[
\frac{d}{dt} r^{-2n}(t)r^2(u) = 2nr^{-2n-1}(t)r^2(u) = \frac{2n}{2n - 2} \cdot \frac{r^2(u)}{r^2(t)}
\]

so the same comparisons as in the \(C^0\) case give bounds on the ratio here, which implies the second equivalence in (4.6).

Recall that by (A.7), \(r'(t) = w'(t)\) for \(w = \log (|\tau|^{-1/2}r)\). Substituting into (A.8),

\[
\frac{dr}{dt} = r\sqrt{1 - (r + \tau r^{-n})^2}.
\]

Therefore,

\[
\frac{ds}{dr} = \left(1 - (r + \tau r^{-n})^2\right)^{-1/2}.
\]

As the maximum of the function \(|r + \tau r^{-n}|\), restricted to \(\Lambda\), occurs on \(\partial \Lambda\), by choosing \(T > 0\) perhaps smaller, when \(0 < |\tau| < T\) we can bound

\[
|\delta + \delta^{-1}r\tau| \leq 2\delta, \quad |\delta^{-1}|\tau|^{-1/2} + |\tau|^{-1}\delta^{-1}r\tau| \leq 2\delta.
\]

The derivative estimates then in (4.7) follow from (4.9) and the observation that

\[
\frac{d^2s}{dr^2} = \left|\frac{(r + r^{1-n}\tau)(1 + (1 - n)r^{-n}\tau)}{(1 - (r + r^{1-n}\tau r^{-n})^2)^{3/2}}\right| = \frac{1}{r} \cdot \left|\frac{(r + r^{1-n}\tau)(r + (1 - n)r^{1-n}\tau)}{(1 - (r + r^{1-n}\tau r^{-n})^2)^{3/2}}\right|.
\]

By the fact that \(\frac{ds}{dr} > 0\) and the estimate on \(ds/dr\) we conclude the proof of (4.7) by

\[
(1 - 4\delta)(r - r_{in}) \leq s(r) - r_{in} = \int_{r_{in}}^{r} \frac{ds}{dr} dr \leq (1 + 4\delta)(r - r_{in}).
\]

The \(C^0\) equivalence of the norms in (4.8) follows immediately from (4.7), and indeed the ratio of the weight functions will always contribute error ratios no worse than \((1 + 10\delta n) \leq 2\).

To prove equivalence up to higher derivatives, observe that for a fixed \((u, \Theta) \in \Lambda\),

\[
s^{-2}(u)g_A = \frac{1}{s^2(u)} \left( ds^2 + s^2 g_{S^{n-1}} \right) = \frac{r^2}{s^2(u)} dt^2 + \frac{s^2}{s^2(u)} g_{S^{n-1}}.
\]

Fix a point \((u, \Theta) \in \Lambda\) and consider the ball of radius 1/10 about this point with respect to the metric \(s^{-2}(u)g_A\). In the \(t\)-direction, the inequality

\[
\frac{r(t)}{r(u)} \left(1 - 5\delta\right) \leq \frac{r(t)}{s(u)} = \frac{r(t)}{r(u)} \frac{r(u)}{s(u)} \leq \frac{r(t)}{r(u)} \left(1 + 5\delta\right)
\]
implies that it is enough to consider $|t - u| \leq \frac{2}{5}$. Moreover, as
\[(4.10) \quad \frac{99}{100} \leq \frac{s(u)}{s(t)} \cdot \frac{r(t)}{r(u)} \leq \frac{100}{99},\]

$|t - u| < \frac{2}{5}$ is sufficient for the $S^{n-1}$ direction as well.

All derivatives purely in the $S^{n-1}$ direction are comparable in the norms as indicated by (4.10). So we consider only partial derivatives involving $t$. The $C^1$ comparison is straightforward since (presuming $\partial_t v \neq 0$)
\[
1 - 5\delta \leq \left( \frac{r(u)}{r(t)} \partial_t v \right) \cdot \left( \frac{s(u)}{r(t)} \partial_t v \right)^{-1} \leq 1 + 5\delta.
\]

Second derivatives in the $t$ direction then follow since
\[
\left( \frac{r^2(u) r'(t)}{r^3(t)} \right) \cdot \left( \frac{s^2(u) r'(t)}{r^3(t)} \right)^{-1}, \quad \left( \frac{r^2(u)}{r^2(t)} \right) \cdot \left( \frac{s^2(u)}{r^2(t)} \right)^{-1}
\]
satisfy equally good inequalities. The mixed partials and third derivatives again satisfy equal ratio estimates and the result follows. \qed

We define operators
\[(4.11) \quad L_{g_A} := \Delta_{g_A} = \partial_{ss} + \frac{n-1}{s} \partial_s + \frac{1}{s} \Delta_{S^{n-1}}, \quad L^\lambda_g := L_g + \lambda.
\]

We first demonstrate that in an appropriately weighted metric, for sufficiently small $\lambda$, the operator $L^\lambda_g$ is close to $L_{g_A}$:

**Lemma 4.12.** Let $\gamma \in (1, 2), \beta \in (0, 1)$. For $\epsilon_1 > 0$ there exists $b$ large enough depending on $\epsilon_1$ and $T > 0$ small enough depending on $b$ such that for all $0 < |\tau| < T$ the following holds: Consider $\Lambda$ defined by $\tau$ and $b$ satisfying 4.2. Let $0 \leq |\lambda| < (2r_{out})^{-1}$. Then for all $V \in C^{2,\beta}(\Lambda)$
\[
\|L^\lambda_g V - L_{g_A} V : C^{0,\beta}(\Lambda, r, g, r^{-n-\gamma})\| \leq \epsilon_1 \|V : C^{2,\beta}(\Lambda, r, g, r^{2-n-\gamma})\|,
\]
\[
\|L^\lambda_g V - L_{g_A} V : C^{0,\beta}(\Lambda, r, g, r^{-2})\| \leq \epsilon_1 \|V : C^{0,\beta}(\Lambda, r, g, r^{\gamma})\|.
\]

**Proof.** Choose $\delta > 0$ small enough so that $C(n)\delta < \epsilon_1/2$ where $C(n)$ is a fixed constant depending only on $n$. Decrease $\delta$ if necessary so that it also satisfies the hypotheses in 4.5. Then choose $b, T$ as in 4.5 for this $\delta$.

Applying (4.6) and recalling (A.13),
\[
\|A_g\|^2 + \lambda : C^{0,\beta}(\Lambda, r, g, r^{-2})\| = \|n(1 + (n - 1)\tau^2 r^{-2n}) + \lambda : C^{0,\beta}(\Lambda, r, g, r^{-2})\|
\leq 100\|n(r^2 + n(n-1)\tau^2 r^{-2n}) + \lambda r^2 : C^{0,\beta}(\Lambda, r, g)\| \leq C(n)\delta.
\]

By calculation,
\[
L_{g_A} - L^\lambda_g = \frac{1}{r} \left( \frac{n-1}{s} - \frac{n-1}{r} w' \right) \partial_t + \left( \frac{1}{s^2} - \frac{1}{r^2} \right) \Delta_{S^{n-1}} - n(1 + (n - 1)\tau^2 r^{-2n}) - \lambda.
\]

Given the constraints on $\delta$, the estimates of 4.5 and multiplicative properties of Hölder norms imply the result. \qed

**Definition 4.13.** We define $\hat{f}_0 : \mathbb{R} \times S^{n-1} \to \mathbb{R}$ (recall A.11) such that
\[
\hat{f}_0 := \nu_r \cdot e_1 = \frac{r'}{r} = \pm \sqrt{1 - (r + \tau r^{1-n})^2}.
\]
Here the sign for \( \hat{f}_0 \) depends on the domain of definition but note that \( \hat{f}_0 \) is odd about \( t = 0 \).

**Lemma 4.14.** The lowest eigenvalue for the Dirichlet problem for \( \mathcal{L}_g \) on \( \Lambda \) is bounded below by \( (2r_{\text{out}})^{-1} \).

**Proof.** First notice that \( \mathcal{L}_g \hat{f}_0 = 0 \) and \( \hat{f}_0(0) = \hat{f}_0(p_r) = 0 \). Moreover, by definition, \( \hat{f}_0 < 0 \) on \( (0, p_r) \times S^{n-1} \). Classical theory implies that on \( (0, p_r) \times S^{n-1} \), the lowest eigenvalue for the Dirichlet problem for \( \mathcal{L}_g \) is 0. Domain monotonicity then implies that on \( \Lambda \subset (0, p_r) \times S^{n-1} \), the lowest eigenvalue for the Dirichlet problem for \( \mathcal{L}_g \) is positive. Suppose \( \lambda_1 \) is the lowest eigenvalue for the Dirichlet problem on \( \Lambda \) and that \( 0 < \lambda_1 < (2r_{\text{out}})^{-1} \). For any \( 0 < \lambda < (2r_{\text{out}})^{-1} \), 4.12 applies to the operators \( \mathcal{L}_g^\lambda, \mathcal{L}_g^\lambda \). Let \( \tilde{V} \) satisfy \( \mathcal{L}_g^\lambda \tilde{V} = 0 \) on \( \Lambda \), \( \tilde{V}|_{C_{\text{out}}} = 1, \tilde{V}|_{C_{\text{in}}} = 0 \). By inspection, one determines the estimate

\[
\|\tilde{V} : C^{2,\beta}(\Lambda, r, g)\| \leq C(\beta).
\]

Using 4.12 with the weaker decay estimate \( r^{-2} \), we may iterate to produce \( V \) such that \( \mathcal{L}_g^\lambda V = 0 \) with the same boundary data and \( \|V : C^{2,\beta}(\Lambda, r, g)\| \leq C(\beta) \). Let \( f \) be the lowest eigenfunction for \( \mathcal{L}_g^{\lambda} \). Then \( \mathcal{L}_g f = -\lambda_1 f \) and \( f > 0 \) on \( \Lambda \) with \( f = 0 \) on \( \partial\Lambda \). Since \( f \neq 0 \), there exists \( C \) sufficiently large such that \( Cf > V \) on a domain \( \Omega \subset \Lambda \). Then \( \mathcal{L}_g(V - Cf) = -\lambda_1(V - Cf) \) and \( Cf - V > 0 \) on \( \Omega \subset \Lambda \). Domain monotonicity then implies \( \lambda_1 \) is not the lowest eigenvalue, giving a contradiction. \( \square \)

**Corollary 4.15.**

1. The Dirichlet problem on \( \Lambda \) for \( \mathcal{L}_g^\lambda \) with \( 0 \leq |\lambda| < (4r_{\text{out}})^{-1} \) and given \( C^{2,\beta} \) Dirichlet data has a unique solution.

2. For \( E \in C^{0,\beta}(\Lambda) \) there exists a unique \( \varphi \in C^{2,\beta}(\Lambda) \) such that \( \mathcal{L}_g^\lambda \varphi = E \) and \( \varphi|_{\partial\Lambda} = 0 \). Moreover

\[
\|\varphi : C^{2,\beta}(\Lambda, g)\| \leq C(\beta, \gamma)r_{\text{out}}\|E : C^{0,\beta}(\Lambda, g)\|.
\]

**Proof.** The first item follows immediately from the lemma and by noting that if \( |\lambda| < (4r_{\text{out}})^{-1} \) then the lowest eigenvalue for \( \mathcal{L}_g^\lambda \) is greater than \( (4r_{\text{out}})^{-1} \). The second follows from the Rayleigh quotient and standard techniques. \( \square \)

We now use 4.12 and C.1 to prove the decay estimates we desire. Note that C.1 gives the analogous decay estimates for solutions to \( \mathcal{L}_g^\lambda V = E \) on flat annuli.

**Definition 4.16.** For \( i = 1, \ldots, n \), let \( \phi_i \) denote the \( i \)-th component of the canonical immersion of \( S^{n-1} \) into \( \mathbb{R}^n \). For convenience going forward, let \( \phi_0 \equiv 1 \).

Note that \( \Delta_{S^{n-1}} \phi_i = -(n-1)\phi_i \) and \( \Delta_{S^{n-1}} \phi_0 = 0 \) and that the functions are \( L^2 \) orthogonal but we have chosen not to normalize them. Since we will be particularly interested in understanding the low harmonics of a function on the boundary of \( \Lambda \), we introduce the following notation.

**Definition 4.17.** Let \( \mathcal{H}_k[C] \) denote the finite dimensional space of spherical harmonics on the meridian sphere at \( C \) that includes all of those up to (and including) the \( k \)-th eigenspace. That is, \( \mathcal{H}_0[C] \) is the span of \( \{\phi_0\} \) and \( \mathcal{H}_1[C] \) is the span of \( \{\phi_0, \ldots, \phi_n\} \).

**Proposition 4.18.** Given \( \beta \in (0, 1) \) and \( \gamma \in (1, 2) \), there exists \( b \) large enough depending on \( \beta, \gamma \), and \( T > 0 \) small depending on \( b \) such that the following holds.

For \( 0 < |\tau| < T \) and \( b \) satisfying 4.2 and any \( |\lambda| < (4r_{\text{out}})^{-1} \), there are linear maps \( \mathcal{R}_{\Lambda, \lambda}^{\text{out}}, \mathcal{R}_{\Lambda, \lambda}^{\text{in}} : C^{0,\beta}(\Lambda) \rightarrow C^{2,\beta}(\Lambda) \) such that, given \( E \in C^{0,\beta}(\Lambda) \):

1. if \( V_{\text{out}} := \mathcal{R}_{\Lambda, \lambda}^{\text{out}}(E) \) then
\begin{itemize}
  \item \( L^\lambda V^\text{out} = E \) on \( \Lambda \).
  \item \( V^\text{out}|_{C^\text{out}} \in H_1[C^\text{out}] \) and vanishes on \( C^\text{in} \).
  \item \( \|V^\text{out} : C^{2,\beta}(\Lambda, r, g, r^2)\| \leq C(\beta, \gamma)\|E : C^{0,\beta}(\Lambda, r, g, r^{\gamma-2})\| \).
\end{itemize}

(ii) if \( V^\text{in} := \mathcal{R}^{\text{in}}_{\lambda, \Lambda}(E) \) then
\begin{itemize}
  \item \( L^\lambda V^\text{in} = E \) on \( \Lambda \).
  \item \( V^\text{in}|_{C^\text{in}} \in H_1[C^\text{in}] \) and vanishes on \( C^\text{out} \).
  \item \( \|V^\text{in} : C^{2,\beta}(\Lambda, r, g, r^{2-n-\gamma})\| \leq C(\beta, \gamma)\|E : C^{0,\beta}(\Lambda, r, g, r^{n-\gamma})\| \).
\end{itemize}

In either case, \( \mathcal{R}^{\text{out}}_{\lambda, \Lambda}, \mathcal{R}^{\text{in}}_{\lambda, \Lambda} \) both depend continuously on the choice of \( \tau, b \).

\textbf{Proof.} We prove the result for \( \mathcal{R}^{\text{out}}_{\lambda, \Lambda} \) as the other argument follows similarly. Let \( E \in C^{0,\beta}(\Lambda) \) where \( b, T \) of 4.12 are determined by choosing \( \epsilon_1 < 1/(20C(\beta, \gamma)) \). We now apply C.1 with \( s \) defined as a function of \( t \) as in (4.4) and the domain of definition equal to \( \Lambda \). Thus, there exists \( V_0 = \mathcal{R}^{\text{out}}_{\lambda, \Lambda}(E) \) such that
\begin{itemize}
  \item \( (1) ~ L^\lambda V_0 = E \).
  \item \( (2) ~ V_0|_{C^\text{out}} \in H_1[C^\text{out}] \) and vanishes on \( C^\text{in} \).
  \item \( (3) ~ \|V_0 : C^{2,\beta}(\Lambda, s, g_\Lambda, s^2)\| \leq C(\beta, \gamma)\|E : C^{0,\beta}(\Lambda, s, g_\Lambda, s^{\gamma-2})\| \).
\end{itemize}

4.8 and 4.12 together imply that
\[
\|L^\lambda V_0 - E : C^{0,\beta}(\Lambda, r, g, r^{\gamma-2})\| \leq 10\epsilon_1 C(\beta, \gamma)\|E : C^{0,\beta}(\Lambda, r, g, r^{\gamma-2})\|.
\]

We complete the proof by iteration. \( \square \)

In a similar fashion, we can prove the following corollary.

\textbf{Corollary 4.19.} Assuming \( \epsilon_1 \) of 4.12 is small enough in terms of \( \epsilon_2 \) and \( \beta \in (0,1) \), \( \gamma \in (1,2) \), for any \( 0 \leq |\lambda| < (4V^\text{out})^{-1} \), there are two linear maps:
\[
\begin{align*}
\mathcal{R}^{\text{out}}_{\partial, \Lambda} &: \{ u \in C^{2,\beta}(C^\text{out}) : u \text{ is } L^2(C^\text{out}, g_{S^{n-1}})\text{-orthogonal to } H_1[C^\text{out}] \} \to C^{2,\beta}(\Lambda), \\
\mathcal{R}^{\text{in}}_{\partial, \Lambda} &: \{ u \in C^{2,\beta}(C^\text{in}) : u \text{ is } L^2(C^\text{in}, g_{S^{n-1}})\text{-orthogonal to } H_1[C^\text{in}] \} \to C^{2,\beta}(\Lambda).
\end{align*}
\]

such that the following hold:
\begin{itemize}
  \item[(1)] If \( u \) is in the domain of \( \mathcal{R}^{\text{out}}_{\partial, \Lambda} \) and \( V^\text{out} := \mathcal{R}^{\text{out}}_{\partial, \Lambda}(u) \) then
    \begin{itemize}
      \item \( L^\lambda V^\text{out} = 0 \) on \( \Lambda \).
      \item \( V^\text{out}|_{C^\text{out}} - u \in H_1[C^\text{out}] \) and \( V^\text{out} \) vanishes on \( C^\text{in} \).
      \item \( \|V^\text{out}|_{C^\text{out}} - u : C^{2,\beta}(C^\text{out}, g_{S^{n-1}})\| \leq \epsilon_2\|u : C^{2,\beta}(C^\text{out}, g_{S^{n-1}})\| \).
      \item \( \|V^\text{out} : C^{2,\beta}(\Lambda, r, g, (r/\tau)^{\gamma})\| \leq C(\beta, \gamma)\|u : C^{2,\beta}(C^\text{out}, g_{S^{n-1}})\| \).
    \end{itemize}
  \item[(2)] If \( u \) is in the domain of \( \mathcal{R}^{\text{in}}_{\partial, \Lambda} \) and \( V^\text{in} := \mathcal{R}^{\text{in}}_{\partial, \Lambda}(u) \) then
    \begin{itemize}
      \item \( L^\lambda V^\text{in} = 0 \) on \( \Lambda \).
      \item \( V^\text{in}|_{C^\text{in}} - u \in H_1[C^\text{in}] \) and \( V^\text{in} \) vanishes on \( C^\text{out} \).
      \item \( \|V^\text{in}|_{C^\text{in}} - u : C^{2,\beta}(C^\text{in}, g_{S^{n-1}})\| \leq \epsilon_2\|u : C^{2,\beta}(C^\text{in}, g_{S^{n-1}})\| \).
      \item \( \|V^\text{in} : C^{2,\beta}(\Lambda, r, g, (r/\tau)^{n-2+\gamma})\| \leq C(\beta, \gamma)\|u : C^{2,\beta}(C^\text{in}, g_{S^{n-1}})\| \).
    \end{itemize}
\end{itemize}

In either case \( \mathcal{R}^{\text{out}}_{\partial, \Lambda}, \mathcal{R}^{\text{in}}_{\partial, \Lambda} \) depend continuously on \( \tau, b \).

\textbf{Proof.} Again, we prove the result only for \( \mathcal{R}^{\text{out}}_{\partial, \Lambda} \). We first note that as an immediate corollary to C.1, we may define a linear map
\[
\mathcal{R}^{\text{out}}_{\partial, \Lambda} : \{ u \in C^{2,\beta}(C^\text{out}) : u \text{ is } L^2(C^\text{out}, g_{S^{n-1}})\text{-orthogonal to } H_1[C^\text{out}] \} \to C^{2,\beta}(\Lambda)
\]
such that if \( u \) is in the domain of \( \mathcal{R}^{\text{out}}_{\partial, \Lambda} \) and \( \tilde{V}^\text{out} := \mathcal{R}^{\text{out}}_{\partial, \Lambda}(u) \) then
\[
\|\mathcal{R}^{\text{out}}_{\partial, \Lambda}|_{C^\text{out}} - u : C^{2,\beta}(C^\text{out}, g_{S^{n-1}})\| \leq \epsilon_2\|u : C^{2,\beta}(C^\text{out}, g_{S^{n-1}})\|.
\]
For any $\lambda$, we can understand the behavior of the low harmonics of any function defined on $\Lambda$.

**Lemma 4.21.** The previous estimates immediately imply the result. □

4.12 of $\epsilon$ each $V$ with boundary conditions

**Proof.** By inspection $\tilde{V}_0[A,0,1]$, $\tilde{V}_0[A,0,1]$ satisfy the estimates

\[
\|\tilde{V}_0[A,0,1] : C^{2,\beta}(\Lambda, r, g)\| \leq C(\beta)\epsilon_1.
\]

\[
\|\tilde{V}_0[A,0,1] : C^{2,\beta}(\Lambda, r, g, (r_{in}/r)^{n-2})\| \leq C(\beta)\epsilon_1.
\]

\[
\|\tilde{V}_0[A,0,1] : C^{2,\beta}(\Lambda, r, g, (r_{in}/r)^{n-1})\| \leq C(\beta)\epsilon_1.
\]

By 4.12,

\[
\|\mathcal{L}_g^\lambda \tilde{V}_0[A,1,0] : C^{0,\beta}(\Lambda, r, g, r^{-2})\| \leq C(\beta)\epsilon_1,
\]

\[
\|\mathcal{L}_g^\lambda \tilde{V}_0[A,0,1] : C^{0,\beta}(\Lambda, r, g, r^{-n})\| \leq C(\beta)\epsilon_1.
\]

We now introduce Dirichlet solutions to $\mathcal{L}_g^\lambda$ for $\lambda$ in the specified region. These solutions will allow us to understand the behavior of the low harmonics of any function defined on $\Lambda$.

**Definition 4.20.** For any $0 \leq |\lambda| < (4r_{out})^{-1}$ and $i = 0, \ldots, n$, let $V_i^\lambda[A, a_1, a_2]$, $\tilde{V}_i[A, a_1, a_2]$ denote solutions to the Dirichlet problem given by

\[
\mathcal{L}_g^\lambda V_i^\lambda[A, a_1, a_2] = 0, \quad \mathcal{L}_g^\lambda \tilde{V}_i[A, a_1, a_2] = 0
\]

with boundary conditions

\[
V_0^\lambda[A, a_1, a_2] = \tilde{V}_0[A, a_1, a_2] = a_1 \text{ on } C_{out}^{in}
\]

\[
V_0^\lambda[A, a_1, a_2] = \tilde{V}_0[A, a_1, a_2] = a_2 \text{ on } C_{in}^{in}
\]

\[
V_i^\lambda[A, a_1, a_2] = \tilde{V}_i[A, a_1, a_2] = a_1 \phi_i \text{ on } C_{out}, \text{ for } i = 1, \ldots, n
\]

\[
V_i^\lambda[A, a_1, a_2] = \tilde{V}_i[A, a_1, a_2] = a_2 \phi_i \text{ on } C_{in}, \text{ for } i = 1, \ldots, n.
\]

Recall that $s(r_{in}) = r_{in} := s_{in}$ and set $s_{out} := s(r_{out})$. By (4.7), $|s_{out}/r_{out} - 1| \leq 4\delta$. We observe that, recall 4.16,

\[
\tilde{V}_0[A, 1, 0] = \frac{s^{2-n} - s_{in}^{2-n}}{s_{out}^{2-n} - s_{in}^{2-n}} s_{in}^{s_{out}^{1-n}} \phi_i, \quad \tilde{V}_0[A, 0, 1] = \frac{s^{2-n} - s_{out}^{2-n}}{s_{in}^{2-n} - s_{out}^{2-n}} s_{in}^{s_{out}^{1-n}} \phi_i.
\]

**Lemma 4.21.** For each $0 \leq |\lambda| < (4r_{out})^{-1}$, $V_0^\lambda$ is constant on each meridian sphere and each $V_i^\lambda$ is a multiple of $\phi_i$ on each meridian sphere. Moreover, there exists a choice as in 4.12 of $\epsilon_1 > 0$ small enough so the following hold:

1. $\|V_i^\lambda[A, 1, 0] - \tilde{V}_i[A, 1, 0] : C^{2,\beta}(\Lambda, r, g)\| \leq C(\beta)\epsilon_1$.
2. $\|V_i^\lambda[A, 0, 1] - \tilde{V}_i[A, 0, 1] : C^{2,\beta}(\Lambda, r, g, (r_{in}/r)^{n-2})\| \leq C(\beta)\epsilon_1$.
3. $\|V_i^\lambda[A, 1, 0] - \tilde{V}_i[A, 1, 0] : C^{2,\beta}(\Lambda, r, g, (r_{in}/r)^{n-1})\| \leq C(\beta)\epsilon_1$.
4. $\|V_i^\lambda[A, 0, 1] - \tilde{V}_i[A, 0, 1] : C^{2,\beta}(\Lambda, r, g, (r_{in}/r)^{n-1})\| \leq C(\beta)\epsilon_1$.

**Proof.** By inspection $\tilde{V}_0[A, 1, 0], \tilde{V}_0[A, 0, 1]$ satisfy the estimates
Using 4.18 applied to the operator $\mathcal{L}_g^\lambda$ (with $\gamma = 0$), let $\hat{V}_\text{out} = \mathcal{R}_\Lambda^\text{out}(\mathcal{L}_g^\lambda \tilde{V}_0[\Lambda, 1, 0])$ and $\hat{V}_\text{in} = \mathcal{R}_\Lambda^\text{in}(\mathcal{L}_g^\lambda \tilde{V}_0[\Lambda, 0, 1])$. Then
\[ \|\hat{V}_\text{out} : C^{2,\beta}(\Lambda, r, g)\| \leq C(\beta)e_1, \quad \|\hat{V}_\text{in} : C^{2,\beta}(\Lambda, r, g, r^{2-n})\| \leq C(\beta)r_n^{n-2}e_1. \]
Note that the boundary data is in $\mathcal{H}_0^\text{out}[C^\text{out}]$, $\mathcal{H}_0^\text{in}[C^\text{in}]$. Set
\[ V := \tilde{V}_0[\Lambda, A_{\text{out}}, A_{\text{in}}] - A_{\text{out}}\hat{V}_\text{out} - A_{\text{in}}\hat{V}_\text{in}, \]
where $A_{\text{out}}, A_{\text{in}}$ are chosen such that
\[ V|_{C^\text{out}} = 1, \quad V|_{C^\text{in}} = 0. \]
Then $\mathcal{L}_g^\lambda V = 0$ by construction and since the Dirichlet problem has a unique solution
\[ V_0^\lambda[\Lambda, 1, 0] = V. \]
By definition,
\[ \begin{cases} 1 = A_{\text{out}} - A_{\text{out}}\hat{V}_\text{out}(r_{\text{out}}) - A_{\text{in}}\hat{V}_\text{in}(r_{\text{out}}) \\ 0 = A_{\text{in}} - A_{\text{out}}\hat{V}_\text{out}(r_{\text{in}}) - A_{\text{in}}\hat{V}_\text{in}(r_{\text{in}}). \end{cases} \]
Inspection of the estimates implies that $|1 - A_{\text{out}}| \leq C(\beta)e_1$ and $|A_{\text{in}}| \leq C(\beta)e_1$. Item (1) then follows from the triangle inequality and all previous estimates.

For item (2), choose $A_{\text{out}}, A_{\text{in}}$ such that
\[ V := \tilde{V}_0[\Lambda, A_{\text{out}}, A_{\text{in}}] - A_{\text{out}}\hat{V}_\text{out} - A_{\text{in}}\hat{V}_\text{in}, \]
where $A_{\text{out}}, A_{\text{in}}$ are chosen such that
\[ V|_{C^\text{out}} = 0, \quad V|_{C^\text{in}} = 1. \]
As before, the choice of boundary data and uniqueness of Dirichlet solutions implies that
\[ V_0^\lambda[\Lambda, 0, 1] = V. \]
Note that in this case
\[ \begin{cases} 0 = A_{\text{out}} - A_{\text{out}}\hat{V}_\text{out}(r_{\text{out}}) - A_{\text{in}}\hat{V}_\text{in}(r_{\text{out}}) \\ 1 = A_{\text{in}} - A_{\text{out}}\hat{V}_\text{out}(r_{\text{in}}) - A_{\text{in}}\hat{V}_\text{in}(r_{\text{in}}). \end{cases} \]
Again, the estimates imply that $|A_{\text{out}}| \leq C(\beta)e_1(\frac{r_{\text{out}}}{r_{\text{in}}})^{n-2}$ and $|1 - A_{\text{in}}| \leq C(\beta)e_1$.

For the estimates on $\tilde{V}_i^\lambda[\Lambda, 1, 0], \tilde{V}_i^\lambda[\Lambda, 0, 1]$ we note that
\[ \|\mathcal{L}_g^\lambda \tilde{V}_i[\Lambda, 1, 0] : C^{0,\beta}(\Lambda, r, g, r^{-1})\| \leq C(\beta)e_1, \quad \|\mathcal{L}_g^\lambda \tilde{V}_i[\Lambda, 0, 1] : C^{0,\beta}(\Lambda, r, g, (r_{\text{in}}/r)^{n-1}r^{-2})\| \leq C(\beta)e_1. \]
For $\tilde{V}_i[1, 0] := \mathcal{R}_\Lambda^\text{out}(\mathcal{L}_g^\lambda \tilde{V}_i[\Lambda, 1, 0])$ and $\tilde{V}_i[0, 1] := \mathcal{R}_\Lambda^\text{in}(\mathcal{L}_g^\lambda \tilde{V}_i[\Lambda, 0, 1])$ we have the estimates
\[ \|\tilde{V}_i[1, 0] : C^{2,\beta}(\Lambda, r, g, r)\| \leq C(\beta)e_1, \quad \|\tilde{V}_i[0, 1] : C^{2,\beta}(\Lambda, r, g, (r_{\text{in}}/r)^{n-1})\| \leq C(\beta)e_1. \]
Note that the boundary data is in $\mathcal{H}_1^\text{out}[C^\text{out}], \mathcal{H}_1^\text{in}[C^\text{in}]$. Using these estimates with the same techniques previously outlined implies the result. \[ \square \]
Solving the linearized equation semi-locally on $\tilde{S}^+, \tilde{S}^-$. The goal of this subsection is to prove 4.27 and 4.28 which provide semi-local estimates on $\tilde{S}^+$ and $\tilde{S}^-$. In contrast to [6] we do not attempt to solve a Dirichlet problem with zero boundary data. Instead, we solve an ODE where solutions to the ODE are allowed to grow at a particular rate back toward the nearest central sphere.

Throughout the subsection we will decompose functions by their projections onto various spaces of the kernel of the operator $\Delta_{\Sigma_{g-1}}$. For this reason, we introduce the following notation.

**Definition 4.22.** Let $L_k$ denote the projection of the operator $\mathcal{L}_g$ onto the $k$-th space of eigenfunctions for the operator $\Delta_{\Sigma_{g-1}}$. That is,

$$L_k := \frac{1}{r^2} \partial_t^2 + \frac{n - 2}{r^2} \partial_t + \left[ n(1 + n - 1) - \frac{k}{r^2}(n - 2 + k) \right].$$

We will use the projected operators to decompose the local linear problems and determine separate estimates for the high and the low eigenvalues. For ease of notation, we introduce the following decomposition which we will use throughout this subsection.

**Definition 4.23.** For $j = 0, \ldots, n$, let $\phi_j$ be defined as in 4.16. For $j \geq n + 1$, choose $\phi_j$ such that $\{\phi_{n+1}, \phi_{n+2}, \ldots\}$ is an $L^2$ orthonormal basis for the remaining eigenspaces of $\Delta_{\Sigma_{g-1}}$. (Recall that $\{\phi_0, \ldots, \phi_n\}$ is an $L^2$-orthogonal basis for the lowest two eigenspaces of $\Delta_{\Sigma_{g-1}}$.)

Let $f \in C^{k, \beta}$ on $\tilde{S}^+$ or $\tilde{S}^-$. We define the decompositions

$$f(t, \Theta) = \sum_{i=0}^{\infty} f_i(t) \phi_i = f_0 + f_1 + f_{\text{high}} \quad \text{where} \quad f_1 := \sum_{i=1}^{n} f_i(t) \phi_i, \quad f_{\text{high}} := \sum_{i=n+1}^{\infty} f_i(t) \phi_i.$$

We first consider the linear problem for functions with no low harmonics.

**Lemma 4.24.** Let $\beta \in (0, 1), \gamma \in (1, 2)$. For $b$ chosen as in 4.12 and $T > 0$ satisfying the requirements of 4.12 and the inequality $T^{\frac{1}{n+1}} \leq 1/(2n^2)$, let $b$ satisfy 4.2. Then there exist linear maps $\mathcal{R}_{\text{high}}^\pm, \mathcal{R}_{\text{low}}^\pm$ where

$$\mathcal{R}_{\text{high}}^\pm : \{E^\pm \in C^{0, \beta}(\tilde{S}^\pm) : E^\pm = E^\pm_{\text{high}}, \text{supp } (E^\pm) \subset S^\pm_1\} \rightarrow C^{2, \beta}(\tilde{S}^\pm)$$

such that for $E^\pm$ in the domain of $\mathcal{R}_{\text{high}}^\pm$, and $f^\pm := \mathcal{R}_{\text{high}}^\pm(E^\pm)$,

1. $\mathcal{L}_g f^\pm = E^\pm.$
2. $f^\pm = f^\pm_{\text{high}}.$
3. $f^\pm = 0$ on $\partial\tilde{S}^\pm$.
4. $\|f^+ : C^{2, \beta}(\tilde{S}^+, r, g, r^\gamma)\| \leq C(b, \beta, \gamma)\|E^+ : C^{0, \beta}(S^+_1, r, g)\|.$
5. $\|f^- : C^{2, \beta}(\tilde{S}^-, r, g, (r_{in}/r)^{n-2+\gamma})\| \leq C(\beta, \gamma)\|E^- : C^{0, \beta}(S^-_1, r, g)\|.$

Finally, $\mathcal{R}_{\text{high}}^\pm$ depend continuously on $\tau$.

The proof will follow from the decay estimates determined on $\Lambda$ and the following lemma.

**Lemma 4.25.** For a fixed $n \in \mathbb{N}, n > 2$, consider $b$ chosen as in 4.12 and $T > 0$ satisfying the requirements of 4.12 and the inequality $T^{\frac{1}{n+1}} \leq 1/(2n^2)$. Then the following holds:

For any $0 < |\tau| < T$ and $b$ satisfying 4.2, let $\tilde{S}^\pm$ be the domain defined by $\tau$ and $b$ as in 4.3. Consider the two sets of functions $X^\pm := \{f \in L^2(\tilde{S}^\pm) : f = f_{\text{high}}, f|_{\partial\tilde{S}^\pm} = 0, \int_{\tilde{S}^\pm} f^2 = 1\}.$
Thus
\[
\inf_{f \in X} - \int_{\bar{S}^+} f \mathcal{L}_g f = \inf_{f \in X} \int_{\bar{S}^+} |\nabla f|^2 - |A|^2 f \geq 1.
\]

Proof. Let \( f = \sum_{i=n+1}^\infty f_i \phi_i \). Since \( i \geq n + 1 \),
\[
\int_{S^{n-1}} |\nabla_{S^{n-1}} \phi_i|^2 dg_{S^{n-1}} \geq 2n \int_{S^{n-1}} \phi_i^2 dg_{S^{n-1}} = 2n.
\]

Therefore, recalling (A.13),
\[
(4.26) \quad \int_{\bar{S}^+} |\nabla f|^2 - |A|^2 f^2 \, dx = \int_{\bar{S}^+} \frac{1}{r^2} \sum_i (f_i' \phi_i + f_i \nabla_{S^{n-1}} \phi_i)^2 - |A|^2 f^2 \, dx
\]
\[
\geq \int r^{n-2} \sum_i (f_i')^2 + nr^{n-2} \sum_i f_i^2 (2 - r^2 - (n - 1)r^{2-2n}) \, dt.
\]

On \( \bar{S}^+ \), \( r \in ([\tau]^{1-n} / \delta, 1 + O(1)) \). Therefore, by the bound on \( \delta > 0 \) imposed in 4.12, (4.26) is bounded below by

\[
\int r^{n-2} \sum_i f_i^2 \, dt \geq \frac{n}{2} \int r^n \sum_i f_i^2 \, dt \geq 1.
\]

The last inequality follows since \( \|f\|_{L^2(\bar{S}^+)} = 1 \).

It remains to show the estimate on \( \bar{S}^{-} \). We now demonstrate that the positive terms on the right hand side are sufficiently large to more than overcome the negative contribution. First observe that
\[
-2 \int r^{n-2} w' f_i f_i' \, dt = - \int r^{n-2} w' (f_i^2)' \, dt = \int (r^{n-2} w')' f_i^2 \, dt = \int r^{n-2} f_i^2 (w'' + (n - 2)w') \, dt.
\]

Now we use Cauchy-Schwarz and an absorbing inequality to note that
\[
-2 \int r^{n-2} w' f_i f_i' \, dt \leq 2 \left( \int r^{n-2} (w')^2 f_i^2 \, dt \cdot \int r^{n-2} (f_i')^2 \, dt \right)^{1/2}
\]
\[
\leq (n - 2) \int r^{n-2} (w')^2 f_i^2 \, dt + \frac{1}{n - 2} \int r^{n-2} (f_i')^2 \, dt.
\]

Combining the above and simplifying,
\[
(n - 2) \int r^{n-2} f_i^2 w'' \, dt \leq \int r^{n-2} (f_i')^2 \, dt.
\]

Thus, recalling (A.12)
\[
\int r^{n-2} (f_i')^2 + nr^{n-2} f_i^2 (2 - r^2 - (n - 1)r^{2-2n}) \, dt
\]
\[
\geq \int r^{n-2} f_i^2 ((n - 2)w'' + 2n - nr^2 - n(n - 1)r^{2-2n}) \, dt
\]
\[
= \int r^{n-2} f_i^2 (2n - 2(n - 1)r^2 + (n - 2)^2 r^{2-n} - 2(n - 1)r^2 r^{2-2n}) \, dt.
\]

We simplify the above expression by using (A.12) to note that
\[
(2n - 2)(w')^2 = (2n - 2) - 2(n - 1)r^2 - 4(n - 1)r^{2-n} - 2(n - 1)r^2 r^{2-2n}.
\]
Thus,
\[
\int r^{n-2}(f')^2 + nr^{n-2}f_i^2(2 - r^2 - (n-1)r^2r^{2-n})
\geq \int r^{n-2}f_i^2((2n-2)(w')^2 + 2 + n^2\tau r^{2-n})
\geq \int r^{n-2}f_i^2(2 + n^2\tau r^{2-n})
\]

Now, since \(r \geq |\tau|^{\frac{1}{n-1}}\), the hypothesis on \(T\) implies that
\[2 + n^2\tau r^{2-n} \geq 1.5 + \left(0.5 - n^2|\tau|^{1-\frac{n-2}{n-1}}\right) \geq 1.5 \geq r^2.\]

Immediately we observe that
\[-\int_{\tilde{S}} fL_gf \geq \int r^n \sum_i f_i^2
dt = 1. \] 

We can now complete the proof of 4.24.

**Proof.** Given \(E = E_{\text{high}}\), the existence of \(f\) satisfying items (1), (2), (3) follows from standard theory using the coercivity estimate provided. We determine the decay estimates in the following manner. First, the coercivity estimate implies that the solvability of high harmonics and good estimates on the behavior of \(f\) solutions, we solve the semi-local linearized problem by appealing to 4.21 to understand the proof.

Given \(\Lambda\), we force the solution to decay to the boundary at a prescribed rate. Suffice it to say that on \(\Lambda\), makes little sense. We presume that \(\tilde{S}\) close and do not explain the definitions of \(\Lambda^{\text{close}}, \Lambda^{\text{far}}\) until they are needed later. Suffice it to say that on \(\Lambda^{\text{close}}\) we allow our solution to grow toward the boundary but on \(\Lambda^{\text{far}}\) we force the solution to decay to the boundary at a prescribed rate.

**Lemma 4.27.** Given \(\beta \in (0,1), \gamma \in (1,2)\), for each \(S^+\) there exists a linear map
\[\mathcal{R}_{\tilde{S}^+} : \{E \in C^{0,\beta}(\tilde{S}^+) : E \text{ is supported on } S^+_1\} \rightarrow C^{2,\beta}(\tilde{S}^+, g)\]
such that the following hold for \(E\) in the domain of \(\mathcal{R}_{\tilde{S}^+}\) and \(\varphi = \mathcal{R}_{\tilde{S}^+}(E)\):

(1) \(L_g\varphi = E\) on \(\tilde{S}^+\).

(2) \(\|\varphi : C^{2,\beta}(\tilde{S}^+, r, g)\| \leq C(\underline{b}, \beta)\|E : C^{0,\beta}(S^+_1, r, g)\|\).
exists a unique $\phi$ estimate on $\Lambda$

Proof. Consider $E$ in the domain of $\mathcal{R}_{\tilde{S}^+}$ and decompose $E = E_0 + E_1 + E_{\text{high}}$. For $E_0$, there exists a unique $\varphi_0(t)$ such that $L_0\varphi_0 = E_0$ and $\varphi_0(2p_t + b + 1) = -\varphi_0(2p_t + b + 1) = 0$. Since $E_0 \equiv 0$ on $\tilde{S}^+ \setminus \Lambda^+_1$, $\varphi_0 \equiv 0$ on $\Lambda^+_1$. By standard ODE theory, we note that

$$\|\varphi_0 : C^2,\beta(S^+_1, r, g)\| \leq C(\beta)\|E_0 : C^0,\beta(S^+_1, r, g)\| \leq C(\beta)\|E_1 : C^0,\beta(S^+_1, r, g)\|.$$

where the final inequality uses A.16. At $t = 2p_t - (b + 1)$, determine the unique $a_0, b_0$ such that

$$\varphi_0(2p_t - (b + 1)) = a_0V_0[\Lambda^+1, 0](2p_t - (b + 1)) + b_0V_0[\Lambda^+1, 0, 1](2p_t - (b + 1))$$

where $V_0$ are the functions defined in 4.20. Then on $\Lambda_0^+ := \Lambda^+_{0,1}$

$$\varphi_0 = a_0V_0[\Lambda^+1, 0] + b_0V_0[\Lambda^+1, 0, 1].$$

Combining the estimates of 4.21 and the ones above imply

$$\|\varphi_0 : C^2,\beta(\Lambda_0^+1, r, g, (r_{in}/r)^{n-1})\| \leq C(\beta)\|E_0 : C^0,\beta(\Lambda^+_1, r, g)\|.$$}

For $E_1$ we proceed in a similar fashion and produce $\varphi_1$ such that $L_1\varphi_1 = E_1$, $\varphi_1 \equiv 0$ on $\Lambda^+_{1,0}$ and

$$\|\varphi_1 : C^2,\beta(S^+_1, r, g)\| \leq C(\beta)\|E_1 : C^0,\beta(S^+_1, r, g)\|.$$
5. The Initial Hypersurfaces

In this section we assume given a family of graphs $\mathcal{F}$—defined as in 2.14—and we construct families of initial immersions which depend on a parameter $\tau$ which determines an overall scaling for the weights. The first step in the construction is to describe an abstract surface $M$ based on the central graph $\Gamma$ of $\mathcal{F}$. At the same time we construct parametrizations for $M$ which depend on $\Gamma$ and $\tau$. We then define a family of immersions of $M$ into $\mathbb{R}^{n+1}$ which depends on $\tau$ and is parametrized by parameters $(d, \zeta)$. The construction of each initial immersion is based on one of the graphs of $\mathcal{F}$ chosen on the basis of $(d, \zeta)$ and $\tau$.

**Assumption 5.1.** In what follows $b \gg 1$ will be as in Section 4, large enough to invoke all of the results of that section, but independent of the small constant $T > 0$. In this section, we choose a small constant $T_{\Gamma} > 0$ which will depend on $T > 0$ (and thus on $b$) and on $\max_{e \in E(\Gamma) \cup R(\Gamma)} |\hat{\tau}[\Gamma(0,0),e]|$ but not on the structure of the graph $\Gamma$ or on the parameters $d, \zeta$. Note in particular that $b$ will be independent of $T_{\Gamma}$.

While we are free to decrease $T_{\Gamma}$ as necessary, we presume throughout this section that

\begin{equation}
\max_{e \in E(\Gamma) \cup R(\Gamma)} |\hat{\tau}[\Gamma(0,0),e]| T_{\Gamma} < T/2.
\end{equation}

Moreover, the constant $\tau$ will be chosen so that

\begin{equation}
0 < |\tau| < T/2.
\end{equation}

**The abstract surface $M$.** Given a flexible, central graph $\Gamma$ with the rescaled function $\tau_0$, we determine an abstract surface which will be mapped into $\mathbb{R}^{n+1}$ by translating and rotating the maps described in Section 3. We construct $M$ in the following manner, noting that $M$ depends only on $\Gamma$ and $\tau$ and not on $d, \zeta$.

**Definition 5.4.** We choose $\delta' > 0$, depending only on $\Gamma$, such that for each $p \in V(\Gamma)$ and all $e \neq e' \in E_p$ we have $|v[p,e] - v[p,e']| > 50\delta'$. Recall that by 3.1 this defines also a constant $a$ such that $\tanh(a + 1) = \cos(\delta')$.

**Definition 5.5.** For $p \in V(\Gamma)$ define

\begin{equation}
M[p] = S^n \setminus D_{V_p}(\delta')
\end{equation}


As the length of each edge domain depends upon the period and the number of periods, we set

\[ P[e] := 2p_{\tau_0[e]l[e]} \]

For $e \in E(\Gamma)$, let

\[ M[e] = [a, P[e] - a] \times S^{n-1} \]

while for $e \in R(\Gamma)$, let

\[ M[e] = [a, \infty) \times S^{n-1} \]

\[ 25 \]
Definition 5.7. For $e \in E(\Gamma) \cup R(\Gamma)$, let $R[e] : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denote the rotation such that
\[
R[e](e_i) = v_i[e]
\]
for $i = 1, \ldots, n + 1$, where here the $v_i[e]$ refer to the ordered orthonormal frame chosen in 2.16. (The existence of such a rotation follow precisely because we chose an ordered frame.)

Definition 5.8. Let
\[
M' = \left( \bigsqcup_{p \in V(\Gamma)} M[p] \right) \bigsqcup \left( \bigsqcup_{e \in E(\Gamma) \cup R(\Gamma)} M[e] \right)
\]
and let
\[
M = M' / \sim (5.9)
\]
where we make the following identifications:
For $[p, e] \in A(\Gamma)$ with $p = p^+[e]$ and $x \in M[e] \cap ([a, a + 1] \times S^{n-1})$,
\[
x \sim (R[e] \circ Y_0(x)) \cap M[p].
\]
For $[p, e] \in A(\Gamma)$ with $p = p^-[e]$ and $x \in M[e] \cap ([P[e] - (a + 1), P[e] - a] \times S^{n-1})$,
\[
x \sim (R[e] \circ Y_0(t - P[e], \Theta)) \cap M[p].
\]

Standard and transition regions. In enumerating the important regions of the graph, we frequently reference the triple $[p, e, \cdot]$ where the third component will be described below. For each $e \in E(\Gamma) \cup R(\Gamma)$, we enumerate the standard and transition regions along the Delaunay piece by counting upward as we move away from each central sphere. As in [6], we denote a region as standard if the limiting geometry as $\tau \to 0$ is well understood and a region as transition otherwise. See Section A for a more complete description of these regions. Recall that $2|e|$ denotes the length of an edge $e$. Thus, an edge $e$ will have $2|e| - 1$ standard regions and $2|e|$ transition regions. We make precise the following definition.

Definition 5.10. We define
\[
V_S(\Gamma) := \{[p, e, m] : e \in E(\Gamma), [p^+[e], e] \in A(\Gamma), m \in \{1, 2, \ldots, l[e]\}\}
\]
\[
\bigcup \{[p, e, m] : e \in E(\Gamma), [p^-[e], e] \in A(\Gamma), m \in \{1, 2, \ldots, l[e] - 1\}\}
\]
\[
\bigcup \{[p, e, m] : e \in R(\Gamma), [p, e] \in A(\Gamma), m \in \mathbb{N}\},
\]
\[
V^+_S(\Gamma) := \{[p, e, m] \in V_S(\Gamma) : m \text{ is even}\},
\]
\[
V^-_S(\Gamma) := \{[p, e, m] \in V_S(\Gamma) : m \text{ is odd}\},
\]
\[
V_\Lambda(\Gamma) := \{[p, e, m'] : e \in E(\Gamma), [p^{\pm}[e], e] \in A(\Gamma), m' \in \{1, 2, \ldots, l[e]\}\}
\]
\[
\bigcup \{[p, e, m'] : e \in R(\Gamma), [p, e] \in A(\Gamma), m' \in \mathbb{N}\}.
\]

We choose this notation so that the set $V_S(\Gamma)$ enumerates every standard region on an edge or ray exactly once. Moreover, the enumeration of the standard regions is such that it increases along $M[e]$ as one moves further away from the nearest boundary. $V^+_S(\Gamma)$ and $V^-_S(\Gamma)$ enumerate the spherical and catenoidal regions respectively. $V_\Lambda(\Gamma)$ enumerates every transition region exactly once. Notice that $V_S(\Gamma) \subset V_\Lambda(\Gamma)$ and $V_\Lambda(\Gamma) \setminus V_S(\Gamma) = \{[p^-[e], e, l[e]] : e \in E(\Gamma)\}$. 
We now define regions of particular importance. A verbal description of these regions follows.

Recall that \( a \) is determined by 5.4. The constant \( b \) determines the size of each standard and transition region. We use \( x, y \) in subscripts to modify the size of the regions and the boundary circles. For example, \( S[p] \subset S_x[p] \) while \( \bar{S}_x[p] \subset \bar{S}[p] \).

**Definition 5.11.** For \( p \in V(\Gamma) \), \([p, e, m] \in V_S(\Gamma), \) and \([p, e, m'] \in V_{\Lambda}(\Gamma), \) (recall 5.10), we define the following.

1. \( S_x[p] := M[p] \cup \{e|p=p^+[e]\} \left( M[e] \cap [a, b + x] \times \mathbb{S}^{n-1} \right) \)

2. \( \bar{S}_x[p] := M[p] \cup \{e|p=p^+[e]\} \left( M[e] \cap [a, P^e[e] - (b + x)] \times \mathbb{S}^{n-1} \right) \)

\( \bar{S}_x[p] \) forms a cylindrical surface that encloses the region \( S_x[p] \).
where positivity of $b$ is forthcoming sections as necessary. We let $0$ in \[(5.12)\]
standard region geometric limit as $\Gamma(\tilde{\xi})$.

The graph under an immersion as those from 4.3.

extend the definition here to include all meridian spheres that exhibit the same behavior.

We presume throughout that $d$ be out superscripts.

\[\Lambda_{x,y}[p^+[e], e, m'] := M[e] \cap \{m'[p_{ro[e]} - (b + x)], m'[p_{ro[e]} - (b + x)]\} \times S^{n-1}\]

\[\Lambda_{x,y}[p^-[e], e, m'] := M[e] \cap \{m'[p_{ro[e]} - (b + x)], m'[p_{ro[e]} - (b + y)]\} \times S^{n-1}\]

\[\Lambda_{x,y}[p^+[e], e, m'] := M[e] \cap \{(m' - 1)p_{ro[e]} + (b + x)\} \times S^{n-1} \text{ for } m' \text{ odd,}\]

\[\Lambda_{x,y}[p^+[e], e, m'] := M[e] \cap \{(m' - 1)p_{ro[e]} + (b + x)\} \times S^{n-1} \text{ for } m' \text{ even.}\]

\[\Lambda_{x,y}[p^+[e], e, m'] := M[e] \cap \{m'[p_{ro[e]} - (b + x)]\} \times S^{n-1} \text{ for } m' \text{ odd,}\]

\[\Lambda_{x,y}[p^+[e], e, m'] := M[e] \cap \{m'[p_{ro[e]} - (b + x)]\} \times S^{n-1} \text{ for } m' \text{ even,}\]

\[\Lambda_{x,y}[p^+[e], e, m'] := M[e] \cap \{(m' - 1)p_{ro[e]} + (b + x)\} \times S^{n-1} \text{ for } m' \text{ odd,}\]

\[\Lambda_{x,y}[p^+[e], e, m'] := M[e] \cap \{m'[p_{ro[e]} - (b + x)]\} \times S^{n-1} \text{ for } m' \text{ even.}\]

The constant $b > a + 5$ was initially determined in Section 4 but may be further increased in forthcoming sections as necessary. We let $0 < x, y < p_{ro[e]} - b$ where positivity of $p_{ro[e]} - b$ is guaranteed by the smallness of $\text{d}x_{\Gamma}$ in relation to $b$. We set the convention to drop the subscript $x$ when $x = 0$; i.e. $S[p] = S_0[p]$. Moreover, we denote $\Lambda_{x,x} = \Lambda_x$.

Notice that unlike in the case $n = 2$, not all of the regions $S[p, e, m]$ have the same geometric limit as $\tilde{\xi} \to 0$. With this notation, each $S[p, e, m] \subset M$ with $[p, e, m] \in V_\mathcal{S}(\Gamma)$ will correspond to a standard region or almost spherical region. For $[p, e, m] \in V_\mathcal{S}(\Gamma), S[p, e, m]$ corresponds to a standard region or almost catenoidal region. Each $\Lambda[p, e, m']$ will correspond to a transition or neck region. For $e \in E(\Gamma)$, the middle standard region on $M[e]$ bears the label $S[p^+[e], e, l[e]]$. Each $\tilde{S}[p, e, m]$ is an extended standard region and contains both the standard region and the two adjacent transition regions. The $\tilde{S}[p]$ are central extended standard regions and contain all adjacent transition regions, where adjacency is determined by $e \in E_p$.

Finally, note that the spheres $C_{\text{out}}, C_{\text{in}}$ are enumerated so that

\[\partial \Lambda_{x,y}[p, e, m'] = C_{\text{out}}[p, e, m'] \cup C_{\text{in}}[p, e, m'] \text{ for } m' \text{ odd,}\]

\[\partial \Lambda_{x,y}[p, e, m'] = C_{\text{in}}[p, e, m'] \cup C_{\text{out}}[p, e, m'] \text{ for } m' \text{ even.}\]

The superscripts out, in are used to match those that are used throughout Section 4. We extend the definition here to include all meridian spheres that exhibit the same behavior under an immersion as those from 4.3.

**The graph $\Gamma(\tilde{d}, \hat{\xi})$.** We use the parameters $d, \zeta$ to determine a graph in $\mathcal{F}$. Recall that by assumption $\Gamma$ is a central graph in a family $\mathcal{F}$.

We presume throughout that $d : V(\Gamma) \to \mathbb{R}^{n+1}$ where

\[(5.12)\]

\[|d[p]| \leq |\tilde{\xi}|^{1 + \frac{1}{n-1}} \text{ for all } p \in V(\Gamma).\]

Choose $\tilde{d} \in D(\Gamma)$ (recall 2.11, 2.5) such that

\[(5.13)\]

\[\tilde{d}[\cdot] = \frac{1}{\tilde{\xi}}d[\cdot].\]
Choose $\Gamma(\tilde{d}, 0) \in \mathcal{F}$ and let

$$\tau_d[e] := \tau^\dagger_\tau \Gamma(\tilde{d}, 0), e].$$

(5.14)

**Remark 5.15.** The smooth dependence of $\Gamma(\tilde{d}, 0)$ on $\tilde{d}$ implies that

$$0 < |\tau_d[e]| < |\tau_0[e]|(1 + C|\tau^\dagger_\tau)^{\frac{1}{n}} < T.$$

We now determine the value of the function $\tilde{\ell} \in L(\Gamma)$ (recall 2.11, 2.5) that will rely – for each $e$ – on $l[e], \tau_d[e]$, and two vectors $\zeta[p^\pm, e] \in \mathbb{R}^{n+1}$. The maps $\zeta[p^\pm, e]$ will effectively describe the dislocation of each attached Delaunay piece from its central sphere. Though rays are not in the domain of $\tilde{\ell}$, they can be dislocated from their vertex, and thus when describing $\zeta[p, e]$ we must include rays in the domain.

**Definition 5.16.** Let $\zeta \in Z(\Gamma)$ (recall 2.11, 2.5) such that

$$\zeta[p, e] = \sum_{i=0}^n \zeta_i[p, e]e_{i+1}.$$  

(5.17)

As we will see, the norm of $\zeta$ can be quite large compared to the norm of $d$. Throughout the paper, we allow

$$|\zeta| \leq C|\tau|$$

(5.18)

where $C$ is a large, universal constant that is independent of $\tau$.

Let $\tilde{l} \in L(\Gamma)$ such that

$$\tilde{l}[e] := (2 + 2\tilde{p}_r[e]) l[e].$$

(5.19)

Thus, a Delaunay piece with $l[e]$ periods and parameter $\tau_d[e]$ will have length – i.e. axial length – equal to $\tilde{l}[e]$. Recall (2.13) which informs our choice of $\tilde{l}$.

**Definition 5.20.** Choose $\tilde{\ell} \in L(\Gamma)$ such that

$$2(l[e] + \tilde{\ell}[e]) = \left| \zeta[p^+ [e], e] - \left( \zeta[p^- [e], e] + (\tilde{l}[e], 0) \right) \right|.$$  

(5.21)

**Figure 2.** In the figure, we let $\zeta^+, \zeta^-$ correspond to $\zeta[p^+ [e], e], \zeta[p^- [e], e]$ respectively. Also, notice that $Y^-_0$ is defined so that its center is at $(2 + 2\tilde{p}_r[e]) l[e]e_1.$
For clarity, we provide a systematic description of $\tilde{l}$. First, we position a segment of length $\tilde{l}[e]$ so that it sits on the positive $x_1$-axis with one end fixed at the origin. Then we dislocate the two ends of this segment corresponding to $\zeta[p^+[e], e]$ and $\zeta[p^-[e], e]$ where $\zeta[p^+[e], e]$ is the dislocation of the origin. We then measure the length of the segment connecting these two points. Finally, we compare that length with the length of the edge $e$ in the graph $\Gamma$.

**Lemma 5.22.** For $\tilde{l}$ defined as in (5.21), we may decrease $T_\Gamma > 0$ so that for all $0 < |\tau| < T_\Gamma$, there exists $C > 0$ depending on $\Gamma$ but independent of $\tau$, such that for all $e \in E(\Gamma)$,

\begin{equation}
|\tilde{l}[e]| \leq C|\tau|^{\frac{1}{n-1}}.
\end{equation}

**Proof.** We immediately get the bounds

\begin{equation}
\tilde{l}[e] - 2|\zeta| \leq \left| \zeta[p^+[e], e] - \left( \zeta[p^-[e], e] + \tilde{l}[e], 0 \right) \right| \leq \tilde{l}[e] + 2|\zeta|.
\end{equation}

Thus,

\begin{equation}
\frac{\tilde{l}[e] - 2|\zeta|}{2} - \tilde{l}[e] \leq \tilde{l}[e] + 2|\zeta| - \tilde{l}[e].
\end{equation}

The definition of $\tilde{l}[e]$, the bound on $\zeta$ given by (5.18), and the estimates of (A.30) then immediately imply the result. \qed

**Lemma 5.24.** For a central graph $\Gamma$ with an associated family $F$, we may decrease $T_\Gamma > 0$ so that for all $0 < |\tau| < T_\Gamma$ and $\tilde{d}, \zeta$ as in (5.12), (5.18), there exists $\Gamma(\tilde{d}, \hat{e}) \in F$ with $\tilde{d}, \hat{e}$ given by (5.13) and (5.21) respectively, and a constant $C > 0$ depending on $\Gamma$ but independent of $\tau$ such that

\begin{enumerate}
\item
\begin{equation}
\frac{\tau_d[e]}{\tau_0[e]} \in \left( 1 - C|\tau|^{\frac{1}{n-1}}, 1 + C|\tau|^{\frac{1}{n-1}} \right),
\end{equation}
\end{enumerate}

and

\begin{enumerate}
\item
\begin{equation}
\left| 1 - \frac{P_{\tau_d[e]}}{P_{\tau_0[e]}} \right| \leq -C \frac{|\tau|^{\frac{1}{n-1}}}{\log(|\tau_0[e]|)} \leq -C \frac{|\tau|^{\frac{1}{n-1}}}{\log(C|\tau|)}.
\end{equation}
\end{enumerate}

**Proof.** The smooth dependence of $\Gamma(\tilde{d}, \hat{e})$ on $(\tilde{d}, \hat{e})$ and (5.12) together imply (5.25) and (5.27).

To see (5.26), note that by (5.25) and A.22, there exists $\tau'$ between $\tau_d[e], \tau_0[e]$ such that

\begin{equation}
\left| 1 - \frac{P_{\tau_d[e]}}{P_{\tau_0[e]}} \right| = \frac{\tau_d[e] - \tau_0[e]}{P_{\tau_0[e]}},
\end{equation}

\begin{equation}
\left| 1 - \frac{\tau_d[e]}{\tau_0[e]} \right| \leq C \left| \frac{\tau_d[e]}{\tau_0[e]} \right| \leq -C \frac{|\tau|^{\frac{1}{n-1}}}{\log |\tau_0[e]|}.
\end{equation}

Finally, to see (5.28) let $\theta[e] := \angle(e, e')$. At worst,

\begin{equation}
\sin \theta[e] \leq \frac{2|\zeta|}{\sqrt{\tilde{l}[e]^2 + 4|\zeta|^2}} \leq C|\zeta|.
\end{equation}

Thus, $\theta[e] \leq C|\tau|$. \qed
Remark 5.29. Since $\tau_0[e]/\tau = \hat{\tau} [\Gamma(0,0), e]$, the finiteness of the graph $\Gamma$ and (5.25) imply that there exists $C$ depending only on $\Gamma$ such that $|\tau_0[e]| \sim C \cdot |\tau|$. This gives us the freedom to replace any bounds in $|\tau_0[e]|^{\pm 1}$ by $C|\tau|^{\pm 1}$, reducing notation and bookkeeping.

The smooth immersion. The immersion we describe is an appropriate positioning of the building blocks described in Section 3. Notice that the building blocks depend upon $\Gamma$ and the parameters $d, \zeta$ and on $\tau$, but the immersions describing the building blocks are determined prior to any positioning. For each $e \in E(\Gamma)$, the positioning of the associated Delaunay building block will depend upon a rotation that takes an orthonormal frame of the edge connecting the vectors $\tilde{l}[e] + \zeta [p^- [e], e]$ and $\zeta [p^+[e], e]$ to the orthonormal frame of the corresponding edge $e' \in E(\Gamma(d, \ell))$. We first prove that this rotation is well defined and determine the estimates we will need.

Proposition 5.30. For $\zeta$ as defined in 5.16 and each $e \in E(\Gamma)$ there exists a unique orthonormal frame $F_\zeta[e] = \{ e_1[e], \ldots, e_{n+1}[e] \}$, depending smoothly on $\zeta$, such that

1. $e_1[e]$ is the unit vector parallel to $\zeta [p^- [e], e] + (\tilde{l}[e], 0) - \zeta [p^+[e], e]$ such that $e_1[e] \cdot e_1 > 0$.
2. For $i = 2, \ldots, n + 1$, $e_i[e] = R[\zeta[e_1[e] - e_1]](v) = R[\zeta[e_1[e]]](v^T) + v^T$.

3. For $v \in \mathbb{R}^{n+1}$,

$$|v - R[e_1[e], e_1[e]](v)| \leq C|\zeta| |v|.$$ 

Proof. The first two items are by definition. If $e_1 = e_1[e]$ then the third item is immediately true as the rotation is the identity matrix. Now suppose $e_1 \neq e_1[e]$. By 2.17, for $v = v^T + v^\perp$ where $v^T$ is the projection onto the 2-plane spanned by $e_1, e_1[e]$, $R[e_1[e], e_1[e]](v) = R[e_1, e_1[e]](v^T) + v^T$.

Writing $v^T = a_1 e_1 + a_2 \left( \frac{e_1[e] - e_1 \cos \theta[e]}{\sin \theta[e]} \right)$, the definition of the rotation implies that

$$R[e_1, e_1[e]](v^T) - v^T = \sin \theta[e] (a_1 z - a_2 e_1) + (1 - \cos \theta[e]) v^T$$

where $\theta[e]$ is the smallest angle between $e_1, e_1[e]$. Recall, in the proof of (5.28), we observed that $\sin \theta[e] \leq C|\zeta|$. Therefore, $\cos \theta[e] \geq 1 - C|\zeta|^2$. It follows that

$$|R[e_1, e_1[e]](v^T) - v^T| \leq C|\zeta| |v^T|.$$

□

Definition 5.32. For $e \in R(\Gamma)$ we simply let $e_i[e] := e_i$.

Using the frame previously defined, we describe the rigid motion that will position each Delaunay building block.

Definition 5.33. For each $e \in E(\Gamma) \cup R(\Gamma)$ with $e'$ denote the corresponding edge on the graph $\Gamma(d, \ell)$, let $R[e; d, \zeta]$ denote the rotation in $\mathbb{R}^{n+1}$ such that $R[e; d, \zeta](e_1[e]) = v_i[e; d, \ell]$ for $i = 1, \ldots, n + 1$ (recall 2.19). Let $T[e; d, \zeta]$ denote the translation in $\mathbb{R}^{n+1}$ such that $T[e; d, \zeta](R[e; d, \zeta](e_1[e], e)) = p^+[e']$. Letting $U[e; d, \zeta] = T[e; d, \zeta] \circ R[e; d, \zeta]$ we see that for all $c_i \in \mathbb{R}$,

$$U[e; d, \zeta] (\zeta [p^+[e], e] + c_i e_1[e]) = p^+[e'] + c_i v_i[e; d, \ell].$$

At each $p' \in V(\Gamma(d, \ell))$, we position a spherical building block. The rigid motion required for positioning these building blocks is simply a translation. The immersion of the building block associated with $p'$ depends upon a diffeomorphism determined by the frames $F_\Gamma[e]$ and the frames $F_\xi[e]$, for $e \in E_p$ where $p \in V(\Gamma)$ corresponds to $p'$.
For each \( p \in V(\Gamma) \), let \( \{e_1, \ldots, e_{|E_p|}\} \) be an ordering of the edges and rays that have \( p \) as an endpoint. For \( i = 1, \ldots, |E_p| \), let

\[
F_i[p] = \{v[p, e_i], v_2[e_i], \ldots, v_{n+1}[e_i]\}.
\]

Notice that \( F_i[p] \) is a set of vectors where the first vector represents the direction the edge or ray \( e \) emanates from \( p \) in the graph \( \Gamma \) and the next \( n \) vectors complete the orthonormal frame \( F_{\Gamma[e_i]} \) given in 2.16. Recalling 2.15, let

\[
F_{\zeta,i}[p] = \{\text{sgn}[p, e_i]R[e_i; d, \zeta](e_1), \ldots, R[e_i; d, \zeta](e_{n+1})\}.
\]

This set of vectors almost corresponds to rotating the elements of the standard frame in \( \mathbb{R}^{n+1} \) by \( R[e_i; d, \zeta] \). The only change from the rotation is on the first element, which will differ from the rotation by a minus sign if \( p = p^-[e_i] \). For the reader, it may be useful to note that in general \( R[e_i; d, \zeta](e_1) \neq v_1[e_i; \tilde{d}, \tilde{\ell}] \). See Figure 3.

**Figure 3.** A rough idea of the immersion of one edge. Notice that the transformation \( U[e; d, \zeta] \) sends the dislocated spheres to the vertices of the graph. The bold segment in the bottom picture corresponds to the positioning of the edge on the graph \( \Gamma(\tilde{d}, \tilde{\ell}) \). The Delauney piece has axis parallel to \( R[e; d, \zeta]e_1 = v_1[e; \tilde{d}, 0] \), which is parallel to the corresponding edge on the graph \( \Gamma(\tilde{d}, 0) \).

These sets of vectors will determine the diffeomorphisms describing the spherical building blocks. The geodesic disks removed from each \( M[p] \) will be repositioned under the diffeomorphism. The centers of the repositioned disks do not correspond to the vectors \( v[\Gamma(\tilde{d}, \tilde{\ell}), p, e] \).
Moreover, of 3.8, we have the following characterization of the global mean curvature error function.

\( W[p] := \{ F_1[p], \ldots, F_{|E_p|}[p] \}, \ W'[p] := \{ F_{|E_p|+1}[p], \ldots, F_{|E_p+1|}[p] \}. \)

**Definition 5.36.** Let \( t_d : \bigsqcup_{e \in E(\Gamma) \cup R(\Gamma)} M[e] \to \mathbb{R} \) such that for \( e \in E(\Gamma) \),

\[
(5.37) \quad t_d|_{M[e]}(t, \Theta) := \psi(a + 3, a + 2)(t) \cdot t + \psi(P[e] - (a + 3), P[e] - (a + 2))(t) \cdot t + \\
+ \psi(a + 2, a + 3)(t) \cdot \psi(P[e] - (a + 2), P[e] - (a + 3))(t) \cdot \left( \frac{P_{\tau_d[e]}[t]}{P_{\tau_0[e]}[t]} \right)
\]

and for \( e \in R(\Gamma) \),

\[
(5.38) \quad t_d|_{M[e]}(t, \Theta) := \psi(a + 3, a + 2)(t) \cdot t + \psi(a + 2, a + 3)(t) \cdot \left( \frac{P_{\tau_d[e]}[t]}{P_{\tau_0[e]}[t]} \right).
\]

Note that \( t_0(t, \Theta) = t \).

**Definition 5.39.** Let \( Y_{d, \zeta} : M \to \mathbb{R}^{n+1} \) be defined so that, recall 3.3, 3.5, 5.33,

\[
Y_{d, \zeta}(x) := \begin{cases} 
\frac{p'}{W[p] \cdot W'[p]}(x) & x \in M[p] \\
U[e; d, \zeta] \circ Y_{\text{edge}}[\tau_d[e], l[e], \zeta[p^+[e], e], \zeta[p^-[e], e]](t_d(x), \Theta) & x = (t, \Theta) \in M[e], e \in E(\Gamma) \\
U[e; d, \zeta] \circ Y_{\text{ray}}[\tau_d[e], \zeta[p^+[e], e]](t_d(x), \Theta) & x = (t, \Theta) \in M[e], e \in R(\Gamma)
\end{cases}
\]

where \( p' \in V(\Gamma(d, \tilde{\Theta}) \) is the vertex corresponding to \( p \).

Let \( H_{d, \zeta} \in C^\infty(M) \) denote the mean curvature of \( Y_{d, \zeta}(M) \).

Notice that a Delaunay building block will only be positioned parallel to the associated edge of \( \Gamma(d, \tilde{\Theta}) \) if \( \zeta[p^+[e], e] = \zeta[p^-[e], e] \) as in that case \( e_1[e] = e_1 \).

**Definition 5.40.** Recalling 3.7, define \( H_{\text{dislocation}}[d, \zeta], H_{\text{gluing}}[d, \zeta] : M \to \mathbb{R} \) in the following manner:

\[
H_{\text{dislocation}}[d, \zeta](x) := \begin{cases} 
H_{\text{dislocation}}[\tau_d[e], l[e], \zeta^+[p^+[e], e], \zeta^-[p^-[e], e]](t_d(x), \Theta) & x = (t, \Theta) \in M[e], e \in E(\Gamma), \\
0 & \text{otherwise,}
\end{cases}
\]

\[
H_{\text{gluing}}[d, \zeta](x) := \begin{cases} 
H_{\text{gluing}}[\tau_d[e], l[e], \zeta^+[p^+[e], e], \zeta^-[p^-[e], e]](t_d(x), \Theta) & x = (t, \Theta) \in M[e], e \in E(\Gamma), \\
0 & \text{otherwise.}
\end{cases}
\]

As an immediate consequence of the immersion and the definitions, and using the estimates of 3.8, we have the following characterization of the global mean curvature error function.

**Corollary 5.41.** All of the functions described above are smooth. Moreover the smooth function \( H_{\text{error}}[d, \zeta] := H_{d, \zeta} - 1 : M \to \mathbb{R} \) can be decomposed as

\[
H_{\text{error}}[d, \zeta] = H_{\text{dislocation}}[d, \zeta] + H_{\text{gluing}}[d, \zeta].
\]

Moreover,
Lemma 6.7. On \(4, 6.5\) by (5.26) \(b\) \(\mathbf{r}\) \(6.4\), we may further decrease \(T\) as \(b\) is independent of \(T\). The constant \(\tau\) will always satisfy 
\(0 < |\tau| < T\) and \(d, \zeta\) will satisfy (5.12), (5.18) respectively for this fixed \(\tau\). The immersion \(Y_{d, \zeta}\) will as described in 5.39.

**Definition 6.2.** Let \(T_d : \bigsqcup_{e \in E(\Gamma) \cup R(\Gamma)} M[e] \to \mathbb{R}\) such that
\[
T_d|_{M[e]} := r_{\tau_d[e]} \circ t_{d|M[e]} \quad \text{(recall 5.36)}. \tag{6.3}
\]
Moreover we define \(b = b[e; d] := \frac{p_{\tau_d[e]}[b]}{\tau_0[e]} = t_{d|b, \Theta}\) with \(b\) as in 4.2 and \(\forall \Theta \in \mathbb{S}^{n-1}.\) Note that by (5.26) \(b\) is then as in 4.2. Finally we define
\[
T_{\text{out}}[e; d] := T_d(b, \Theta) = r_{\tau_d[e]}(b), \quad T_{\text{in}}[e; d] := T_d(p_{\tau_0[e]} - b, \Theta) = r_{\tau_d[e]}(p_{\tau_d[e]} - b). \tag{6.4}
\]
We point out that \(b[e; d]\) does not influence the immersion \(Y_{d, \zeta}\) in any way, as should be clear since the definition appears after the definition of \(Y_{d, \zeta}\). We are compelled to define \(b[e; d]\) to close the linear iterative process in 6.44. If estimates used \(b\) rather than \(b[e; d]\), the core of the standard regions would “drift” under the reparametrization of \(M[e]\) by \(t_d\) and thus weighted estimates could not be well controlled as one moved away from the central spheres.

**Remark 6.5.** On \(M[e] \cap ([a + 4, P[e] - (a + 4)] \times \mathbb{S}^{n-1})\) for \(e \in E(\Gamma)\) and on \(M[e] \cap ([a + 4, \infty) \times S^{n-1})\) for \(e \in R(\Gamma),\)
\[
g = Y_{d, \zeta}^*(g_{\mathbb{R}^{n+1}}) = \sum_{a,k}^2 (dt_d^2 + g_{\mathbb{S}^{n-1}}). \tag{6.6}
\]

**Lemma 6.7.** On \(\bigsqcup_{e \in E(\Gamma) \cup R(\Gamma)} M[e],\)
\[
\frac{T_d}{\tau_0} \sim C(b) \frac{T_0}{\tau_0}. \tag{6.7}
\]

**Proof.** First observe that by A.14 and (5.25), for any \(e \in E(\Gamma) \cup R(\Gamma),\)
\[
\frac{T_d(p_{\tau_0[e]}, \Theta)}{\tau_0(p_{\tau_0[e]}, \Theta)} = \left(\frac{|\tau_d[e]|}{|\tau_0[e]|}\right)^{\frac{1}{n+1}} \left(1 + O(|\tau|^{\frac{1}{n+1}})\right) = 1 + O(|\tau|^{\frac{1}{n+1}}), \tag{6.8}
\]
and
\[
\frac{T_d(2p_{\tau_0[e]}, \Theta)}{\tau_0(2p_{\tau_0[e]}, \Theta)} = 1 + O(|\tau|). \tag{6.9}
\]
Therefore, by the uniform geometry on each \(S_{[p, e, m]},\) the previous estimates imply that for all \(x \in S[p, e, m],\)
\[
\frac{T_d(x)}{\tau_0(x)} \sim C(b) 1. \tag{6.10}
\]
We will improve this estimate at \( x = (\mathbf{b}, \Theta) \) and use this improvement as the starting point to produce the equivalence on \( \Lambda[p, e, m] \). By the triangle inequality, A.16 and (5.26),

\[
\begin{align*}
|\mathcal{L}_0(\mathbf{b}, \Theta) - \mathcal{L}_d(\mathbf{b}, \Theta)| & \leq |r_{\tau_0[e]}(\mathbf{b}) - \text{sech}(\mathbf{b}) + \text{sech} \left( \mathbf{b} \cdot \frac{\mathbf{P}_{\tau_d[e]}}{\mathbf{P}_{\tau_0[e]}} \right) - r_{\tau_d[e]} \left( \mathbf{b} \cdot \frac{\mathbf{P}_{\tau_d[e]}}{\mathbf{P}_{\tau_0[e]}} \right) | \\
& \quad + \left| \text{sech}(\mathbf{b}) - \text{sech} \left( \mathbf{b} \cdot \frac{\mathbf{P}_{\tau_d[e]}}{\mathbf{P}_{\tau_0[e]}} \right) \right| \\
& \leq C(\mathbf{b}) \left( |\mathcal{L}_d| - \frac{|\mathcal{L}_d|^{\frac{1}{n-1}}}{\log |\tau_0[e]|} \right).
\end{align*}
\]

Thus, we may decrease \( T_\Gamma \) so that

\[
\left| \frac{r_{\tau_d}(\mathbf{b}, \Theta)}{\mathcal{L}_0(\mathbf{b}, \Theta)} - 1 \right| \leq |\mathcal{L}_d|^{\frac{1}{n-1}}.
\]

Let

\[
f(x) := \log \frac{\mathcal{L}_d(x)}{\mathcal{L}_0(x)}
\]

and observe that

\[
|f(\mathbf{b}, \Theta)| \leq 2|\mathcal{L}_d|^{\frac{1}{n-1}}.
\]

Going forward, we will assume always that \( |f| < \frac{1}{10} \) so that we are free to Taylor expand at will. Then, on any \( \Lambda[p, e, m] \), letting \( u_{\tau^*}(t, \Theta) := r_{\tau^*}(\mathbf{P}_{\tau^*}t/\mathbf{P}_{\tau_0[e]}) + \tau^* r_{\tau^*}^{1-n}(\mathbf{P}_{\tau^*}t/\mathbf{P}_{\tau_0[e]}), \)

\[
\frac{d}{dt} f(t, \Theta) = \frac{d}{dt} \left( t_d(t, \Theta), \Theta \right) \cdot \frac{\mathbf{P}_{\tau_d[e]}}{\mathbf{P}_{\tau_0[e]}} - \frac{d}{dt} \left( t_0(t, \Theta) \right)
\]

\[
= \sqrt{1 - u_{\tau_d[e]}^2(t, \Theta)} - \sqrt{1 - u_{\tau_0[e]}^2(t, \Theta)} + \sqrt{1 - u_{\tau_d[e]}^2(t, \Theta)} \left( \frac{\mathbf{P}_{\tau_d[e]}}{\mathbf{P}_{\tau_0[e]}} - 1 \right)
\]

\[
= - \frac{u_{\tau^*}(t, \Theta)}{\sqrt{1 - u_{\tau_d[e]}^2(t, \Theta)}} (u_{\tau_d[e]}(t, \Theta) - u_{\tau_0[e]}(t, \Theta)) + \sqrt{1 - u_{\tau_d[e]}^2(t, \Theta)} \left( \frac{\mathbf{P}_{\tau_d[e]}}{\mathbf{P}_{\tau_0[e]}} - 1 \right)
\]

for some \( \tau^* \) between \( \tau_d[e], \tau_0[e] \). As \( \mathcal{L}_d = e^f \mathcal{L}_0 \),

\[
u_{\tau_d[e]} - u_{\tau_0[e]} = (e^{f-1}) \mathcal{L}_0 + (\tau_d[e] - \tau_0[e]) \mathcal{L}_0^{1-n} + \tau_d[e] (t)(1-n)(f \cdot h) \mathcal{L}_0^{1-n}.
\]

Since we are presuming \( |f| \) is small,

\[
u_{\tau_d[e]} - u_{\tau_0[e]} = (f \cdot h) \mathcal{L}_0 + \left( \frac{\tau_d[e]}{\tau_0[e]} - 1 \right) \mathcal{L}_0^{1-n} + \tau_d[e] (1-n)(f \cdot h) \mathcal{L}_0^{1-n}
\]

where \( |h| \leq 1 + 2|f| \). Thus,

\[
\frac{d}{dt} \tau = fA + B
\]

where

\[
A := - \frac{u_{\tau^*}}{\sqrt{1 - u_{\tau_d[e]}^2}} \left( h \mathcal{L}_0 + \tau_d[e] (1-n) \mathcal{L}_0^{1-n} \right),
\]

\[
B := - \frac{u_{\tau^*}}{\sqrt{1 - u_{\tau_d[e]}^2}} \left( \frac{\tau_d[e]}{\tau_0[e]} - 1 \right) \mathcal{L}_0^{1-n} + \sqrt{1 - u_{\tau_d[e]}^2} \left( \frac{\mathbf{P}_{\tau_d[e]}}{\mathbf{P}_{\tau_0[e]}} - 1 \right).
\]
Since we are presuming that \(|f|\) is small, \(|u_r| \leq C(\mathcal{L}_0 + |\tau_0[e]|)^{\frac{1}{n}}\). Moreover, since \(\frac{d}{dx} \mathcal{L}_0 \sim C(\mathcal{L}_0)\) \(r_0\) on \(\Lambda[p, e, m]\) as long as \(|f| < 1/10\) we may estimate for \(r_0(x) \in [\tau_{in}[e; 0], \tau_{out}[e; 0]]\),

\[
\left| \int_{\mathcal{L}_0} A(t, \Theta) dt \right| \leq C(\mathcal{L}_0) \int_{\tau_{out}[e; 0]} r^2 (r + |z|r^{1-n} + |z|^2 r^{1-2n}) dr \\
\leq C(\mathcal{L}_0, n) \left( r^2 + |z|r^{2-n} + |z|^2 r^{2-2n} \right) \left| \tau_{out}[e; 0] \right| \\
\leq C(\mathcal{L}_0, n).
\]

Moreover, for all \(r(x) \in [\tau_{in}[e; 0], \tau_{out}[e; 0]]\),

\[
\left| \int_{\mathcal{L}_0} B(t, \Theta) dt \right| \leq C(\mathcal{L}_0) \int_{\tau_{out}[e; 0]} (1 + |z|r^{-n}) |z|^{\frac{1}{n}} - \frac{|z|^{\frac{1}{n-1}}}{r \log |\tau_0[e]|} dr \\
\leq C(\mathcal{L}_0) |z|^{\frac{1}{n-1}}.
\]

It follows that for \(x = (t', \Theta)\),

\[
f(x) = \exp \left( \int_{\mathcal{L}_0} A(t) dt \right) \left( f(\mathcal{L}_0, \Theta) + \int_{\mathcal{L}_0} B(t) dt \right).
\]

Thus, as long as \(|f| < \frac{1}{10}\), the previous estimates imply that

\[
|f(x)| \leq C(\mathcal{L}_0, n)|z|^{\frac{1}{n-1}}.
\]

Since the result holds for \(x = (\mathcal{L}_0, \Theta)\), it will continue to hold for all \(x \in \Lambda[p, e, m]\). This implies the \(C^0\) equivalence.

Because the problem proves more tractable when considering norms that allow for the natural scaling, we will record the initial error estimates with respect to this scaling.

**Definition 6.8.** We define the function \(\rho_d : M \to \mathbb{R}\) such that

\[
\rho_d(x) = \begin{cases} \\
1 & \text{if } x \in M[p], p \in V(\Gamma) \\
(\psi[a + 4, \mathcal{L}_0](t) \cdot \psi[\mathcal{L}_0[e] - (a + 4), \mathcal{L}_0[e] - \mathcal{L}_0](x) \\
\psi[b, a + 4](t) + \psi[\mathcal{L}_0[e] - b, \mathcal{L}_0[e] - (a + 4)](t) & \text{if } x = (t, \Theta) \in M[e], e \in E(\Gamma) \\
\psi[a + 4, \mathcal{L}_0](t) \cdot \rho_{\mathcal{L}_0}(x) + \psi[b, a + 4](t) & \text{if } x = (t, \Theta) \in M[e], e \in R(\Gamma)
\end{cases}
\]

Observe that \(\rho_d\) is a smooth function that behaves like \(\mathcal{L}_0\) on each \(M[e]\) and is 1 on each central sphere. The cutoff function smooths out the transition between them.

Because the error was previously determined in the standard Hölder norm, we now record the error induced by gluing in the chosen weighted metric. Note that we could estimate the same way \(H_{\text{dislocation}}[d, \zeta] : C^{0, \beta}(M, \rho_d, g) \leq C(\beta, \mathcal{L}_0, \zeta)\), but this is not needed since we provide a more detailed understanding of \(H_{\text{dislocation}}[d, \zeta]\) in 7.23.

**Proposition 6.10.** \(\|H_{\text{gluing}}[d, \zeta] : C^{0, \beta}(M, \rho_d, g) \| \leq C(\beta, \mathcal{L}_0)\|\zeta\|\).

**Proof.** First recall that \(H_{\text{gluing}}[d, \zeta]\) is supported on \(\cup p S[p]\). Thus, for all \(x \in \text{supp}(H_{\text{gluing}}[d, \zeta])\), \(\rho_d \sim C(\mathcal{L}_0) 1\). The bound then follows immediately from 5.41. \(\square\)

**Proposition 6.11.** There exists \(c_1(\mathcal{L}_0, k, n) > 0\) such that

\[
\|\rho_d^{\pm 1} : C^k(M, \rho_d, g, \rho_d^{\pm 1})\| \leq c_1(\mathcal{L}_0, k, n).
\]
Proof. First note that the uniform geometry of $\Omega = S^p_b[p], S^+_{b}[p, e, m], S^-_{b}[p, e, m]$ in the $g$ metric immediately implies the estimate

$$\|\rho^\pm_d : C^k(\Omega, \rho_d; g, \rho^\pm_d)\| \leq C(b, k).$$

Now for any fixed $x \in \Lambda[p, e, m]$, consider the function $\hat{\rho}(y) := \rho_d(y)/\rho_d(x)$. Then,

$$\hat{\rho}(y) = e^{w_{\rho_d[e]}(t_d(y)) - w_{\rho_d[e]}(t_d(x))}.$$

Because of the local nature of the estimate and since $\frac{d}{dt_d}w_{\rho_d[e]} \circ t_d \in [1 - 3\delta, 1]$, we are interested in $y$ such that $|t_d(y) - t_d(x)| \leq \frac{1}{5}$. (Recall the proof of (4.6) and note that $|t_d(y) - t_0(y)| \leq C|z|^{-1}$ by A.22 and (5.26).) Thus

$$|w_{\rho_d[e]}(t_d(y)) - w_{\rho_d[e]}(t_d(x))| = \left| \int_{t_d(x)}^{t_d(y)} \frac{d w_{\rho_d[e]}}{d t_d} \right| \leq |t_d(y) - t_d(x)| \leq \frac{1}{5}.$$

This implies the $C^0$ estimates. Now note that

$$\frac{\partial }{\partial t_d} \hat{\rho} = \frac{d}{dt_d} w_{\rho_d[e]} \cdot \hat{\rho}, \quad \frac{\partial^2 }{\partial t_d^2} \hat{\rho} = \frac{d^2}{dt_d^2} w_{\rho_d[e]} \cdot \hat{\rho} + \left( \frac{d}{dt_d} w_{\rho_d[e]} \right)^2 \hat{\rho}.$$

Using the estimates in Appendix A, we recall that

$$-\frac{d^2}{dt_d^2} w_{\rho_d[e]} = \mathcal{L}_d \left( 2 - n \right) \tau_d[e] \mathcal{L}_d^{-n} + \left( 1 - n \right) \tau_d[e] \mathcal{L}_d^{2-2n}.$$

Moreover, $\mathcal{L}_d \geq \tau_d[e] \geq C(b)|z|^{-1} + O(|z|^{-2})$. Taken together we see that on $\Lambda[p, e, m]$,

$$\left| (2 - n) \tau_d[e] \mathcal{L}_d^{-n} + (1 - n) \tau_d[e] \mathcal{L}_d^{2-2n} \right| \leq C(b, n).$$

It follows from the previous analysis and these new estimates that

$$\|\rho^\pm_d : C^2(M, \rho_d; g, \rho^\pm_d)\| \leq C(b, n).$$

As $w_{\rho_d[e]}$ satisfies a second order ODE, any higher derivatives in $w_{\rho_d[e]}$ can be written in terms of the function and its first and second derivatives. Since $\frac{\partial }{\partial t_d} \hat{\rho}$ can be written in terms of $\frac{\partial }{\partial t_d} w_{\rho_d[e]}$ and $\hat{\rho}$ where $m = 0, \ldots, k - 2$, the uniform bounds for $\rho_d$ in $C^k$ follow immediately. For $\rho_d^{-1}$ we only need to note that the denominator will contain the power $\hat{\rho}^{-k-1}$ which will also be controlled in terms of some constant $C(k)$. \hfill \Box

**Solving the semi-linear problem on $\vec{S}[p]$.** The goal of this subsection is to prove 6.31. We wish to solve a linearized problem with zero boundary data and fast decay toward all boundary components. These requirements force us to proceed as in the lower dimensional version [6] and introduce the extended substitute kernel. We prove that a modified version of the linearized problem is solvable by what are now standard methods (see for example [21]).

We introduce maps $\vec{Y}[p]$ on $\vec{S}[p]$ which are useful parametrizations of $\mathbb{S}^n$. A comparison between these maps and $Y_d,e$ will help us understand the possible obstructions to solving the linearized problem on these central spheres by considering the linearized operator in the induced metric of $\vec{Y}$ (which corresponds to the metric on the round sphere).
Definition 6.13. Let $\tilde{Y}[p] : \tilde{S}[p] \to S^n \subset \mathbb{R}^{n+1}$ such that (recall 5.7)
\begin{align}
(6.14)
\tilde{Y}[p](x) = \begin{cases} 
\tilde{Y}[W, W](x) & \text{if } x \in M[p], \\
R[e] \circ Y_0(x) & \text{if } p = p^+[e], x \in M[e] \cap ([a, p_{\tau[e]} - b] \times S^{n-1}), \\
R[e] \circ Y_0(t - P[e], \Theta) & \text{if } p = p^-[e], x = (t, \Theta), \\
& x \in M[e] \cap ([P[e] - (p_{\tau[e]} - b), P[e] - a] \times S^{n-1}).
\end{cases}
\end{align}

Let $\tilde{g} := \tilde{Y}[p]^*(g_{\mathbb{R}^{n+1}})$.

Proposition 6.15. Let $p \in V(\Gamma)$ and let $p'$ be the corresponding vertex in the graph $\Gamma(\tilde{d}, \tilde{\ell})$. Then
\begin{align}
\| (Y_d, \xi - p') - \tilde{Y}[p] : C^k(S_x[p], \tilde{g}) \| \leq C(k, x) \left( |\tau|^{\frac{1}{n-1}} + |\xi| \right) \leq C(k, x)|\tau|^{\frac{1}{n-1}}.
\end{align}

Proof. Note that $(\tilde{Y}[p] + p)|_{M[p]} = Y_{0,0}|_{M[p]}$. Since the mapping $Y_{d, \xi}|_{M[p]}$ depends smoothly on $d, \xi$ and approaches $Y_{0,0}$ as $|\tau| \to 0$, the result on $M[p]$ follows immediately.

Recall 3.5, 5.33, 5.36. For $e \in E(\Gamma)$, $x \in M[e] \cap \tilde{S}[p^+[e]]$, we determine the $C^k$ estimate by considering the norms of the two immersions
\begin{align}
R[e; d, \xi] \circ Y_1(y) := R[e; d, \xi] \circ (Y_{edge}[\tau_d[e], l, \xi[p^+[e], e], \xi[p^-[e], e]](t_d(y), \Theta) - Y_0(y)),
\end{align}
and applying the triangle inequality.

First, 3.5, 5.36, (A.6) imply that, for $y = (t, \Theta)$,
\begin{align}
\| Y_1 : C^0(M[e] \cap [a, a+3] \times S^{n-1}, \tilde{g}) \| \leq C(|\xi| + |\tanh(t) - \tanh(t_d(y))| + |\sech(t) - \sech(t_d(y))|).
\end{align}

By (5.26),
\begin{align}
| \tanh(t) - \tanh(t_d(y)) | + | \sech(t) - \sech(t_d(y)) | \leq -C \frac{|\tau|^{\frac{1}{n-1}}}{\log(C|\tau|)}.
\end{align}

The $C^k$ estimates on this region then follow from the definitions and further applications of (5.26). On $[a + 4, x] \times S^{n-1}$, for $y = (t, \Theta)$,
\begin{align}
Y_1(y) := (k_d(t_d(y)) - \tanh(t), r_d(t_d(y)) - \sech(t)\Theta).
\end{align}

(5.26) and A.16 then imply the $C^k$ bounds on this region. The immersion on $[a+2, a+5] \times S^{n-1}$ is simply a smooth transition between the immersions at $t = a + 2$ and $t = a + 5$. Thus, the $C^k$ estimates hold here since the transition function and its derivatives are well controlled.

For the second immersion, note that $\| Y_0 : C^k(S_x[p], \tilde{g}) \| \leq C(x, k)$. Moreover, by (5.31),
\begin{align}
\text{the smooth dependence of } F \text{ on } \tilde{d}, \tilde{\ell}, \text{ and}
\end{align}
\begin{align}
[(R[e; d, \xi] - R[e])e_i] \leq |(R[e; d, \xi] - R[e])e_i| + |R[e; d, \xi]e_i] - R[e]e_i|
\end{align}
\begin{align}
= |(I - R[e_1, e_1])e_i] + |v_i[e; \tilde{d}, \tilde{\ell}] - v_i[e]|, \text{ recall 2.19}
\end{align}
\begin{align}
\leq C \| \xi \| + C(|\tau|^{\frac{1}{n-1}} + |\xi|) \text{ by (5.27), (5.28)}
\end{align}
\begin{align}
\leq C \left( |\tau|^{\frac{1}{n-1}} + |\xi| \right).
\end{align}

Identical arguments hold for $e \in R(\Gamma)$ and the immersion $Y_{ray}$ replacing $Y_{edge}$. When $p = p^-[e]$, the only modification in the proof comes from orientation of $v[p, e] = -v_1[e]$. The
We now consider the nature of the approximate kernel of $\mathcal{L}_g$ on $\bar{S}[p]$. By approximate kernel, we mean the span of eigenfunctions of $\mathcal{L}_g$ with small eigenvalue. Following standard methodology we will use the methods of Appendix B in [36] and compare each $\bar{S}[p]$ in the induced metric with an appropriate embedding of the round sphere. The maps $\bar{Y}[p]$ will be used for the comparison.

We also find it helpful to define scaled Jacobi functions, induced by translation vector fields. Notice that for $d = 0, \zeta = 0$, these functions behave on $\bar{Y}[p](S[p])$ like an orthonormal basis of eigenfunctions for the lowest two eigenspaces of the operator $\Delta_{S^n} + n$.

**Definition 6.16.** Recalling 2.6, let $\hat{F}_i[d, \zeta] : M \to \mathbb{R}$ for $i = 0, \ldots, n$ be defined by

$$\hat{F}_i[d, \zeta](x) := \frac{N_{d, \zeta}(x) \cdot e_{i+1}}{\|N_{d, \zeta} \cdot e_{i+1}\|_{L^2(S^n)}} = \tilde{\omega}_{n-1}^{-1} N_{d, \zeta}(x) \cdot e_{i+1}.$$  

Here $N_{d, \zeta}$ is the unit normal to the immersion $Y_{d, \zeta}$.

Before determining the approximate kernel, we prove a technical lemma. This lemma provides supremum bounds for the eigenfunctions with low eigenvalue.

**Lemma 6.18.** For each $p \in V(\Gamma)$ any eigenfunction $f$ of the Dirichlet problem for $\mathcal{L}_g$ on $\bar{S}[p]$ with eigenvalue $0 \leq |\lambda| < (4r_{out})^{-1}$ satisfies the estimate

$$\|f : C^{2, \beta}(\bar{S}[p], \rho_d, g)\| \leq C(\beta)\|f : L^2(\bar{S}[p], g)\|.$$  

**Proof.** Suppose $\mathcal{L}_g^\lambda f = 0$ on $\bar{S}[p]$ for some $f$. We first note that the uniform geometry on $S[p]_5$ (even as $r \to 0$) implies the boundedness on $S[p]$. Next, we note that on each $\Lambda[p, e, 1]$ adjoining $S[p]$ the Dirichlet problem for $\mathcal{L}_g^\lambda$ has a unique solution. At $C^1_{out}[p, e, 0]$, decompose $f = f_0 + f_1 + f_{high}$ following 4.23. Determine $a_i$ such that $(f_0 + f_1)|_{C^1_{out}} = \sum_{i=0}^n a_i V_i^\lambda|_{\Lambda[p, e, 1] | 0 = 0}$, then the equality holds on all of $\Lambda[p, e, 1] \setminus S[p]$ and the bound follows by 4.21. For $f_{high}$ on $\Lambda[p, e, 1]$, the bounds follow immediately from the estimates of 4.19, with $C^1_{out}$ replace by $C^2_{out}$.

**Lemma 6.19** (The approximate kernel on $\bar{S}[p]$). For $b$ as in 5.1 and for given $\epsilon > 0$ there exists a small $T_{r} > 0$ such that for each $0 < |\tau| < T_{r}$ and for each $p \in V(\Gamma)$ the Dirichlet problem for $\mathcal{L}_g$ on $\bar{S}[p]$ has exactly $n + 1$ eigenvalues in $[-\epsilon, \epsilon]$ and no other eigenvalues in $[-1, 1]$. Moreover the approximate kernel for $\bar{S}[p]$ (defined as the span of these eigenfunctions) has an orthonormal basis $\{f_0[p], \ldots, f_n[p]\}$ where each $f_i[p]$ is in $C^\infty_0(S[p])$, depends continuously on the parameters of the construction, and satisfies

$$\|f_i[p] - \hat{F}_i[d, \zeta] : C^{2, \beta}(S[p], g)\| < \epsilon.$$  

**Proof.** We prove the proposition by some modifications of the results of [36], Appendix B, which determine relationships between eigenfunctions and eigenvalues of the Laplace operator for two Riemannian manifolds that are shown to be close in some reasonable sense. (Throughout the proof, all references to Appendix B or enumerations with B are references to the paper [36].)

In the lower dimensional setting in [6, 36], the linear problem was solved in a conformal metric so that $\mathcal{L}_h = \Delta_h + c$ for some constant $c$. Therefore, it was enough to consider the
Laplace operator and compare eigenvalues and eigenfunctions there. In our current setting, we are not free to choose such a conformal metric and the potential is not constant in the metric \( g \). Therefore, we will adapt the ideas of Appendix B in [36] to include our non-constant potential.

Let \((S[p], g)\) and \((S^n, g_{S^n})\) be the two manifolds under consideration. Note that these manifolds satisfy assumptions (1) and (2) of B.1.4. Also note that assumption (3) was needed to provide supremum bounds for the eigenfunctions. We observe that for each \( S[p] \), 6.18 implies such a bound exists for eigenfunctions with eigenvalue \( \leq (4 \max_{e \in E_p} L_{out}[e; d])^{-1} \). This bound is \( > 2(n + 1) \) and so it will be sufficient for our purposes.

We follow the convention that \( \lambda \) is an eigenvalue for the operator \( \mathcal{L} \) if there exists \( f \) such that \( \mathcal{L} f = -\lambda f \). Then the operator \( \mathcal{L}_{g_{S^n}} := \Delta_{S^n} + n \) has lowest eigenvalues \(-n, 0, n + 2\) and the eigenvalue 0 has multiplicity \( n + 1 \). Thus, the only eigenvalue for \( \mathcal{L}_{g_{S^n}} \) in the interval \([-1, 1]\) is zero with multiplicity \( n + 1 \). Indeed, an orthonormal basis of the kernel of \( \mathcal{L}_{g_{S^n}} \) is given by the functions

\[
\hat{f}_{i, S^n} := \frac{N_{S^n} \cdot e_{i+1}}{\| N_{S^n} \cdot e_{i+1} \| L^2(S^n)}, \quad \text{for } i = 0, \ldots, n,
\]

where \( N_{S^n} \) is the inward normal to the unit hypersphere.

We now construct two functions \( F_1, F_2 \) that will satisfy the assumptions B.1.4 and one additional assumption. Let \( \tilde{Y}[p] : S[p] \rightarrow S^n \) be as defined in 6.13 and recall that \( \tilde{g} := \tilde{Y}^*(g_{S^n}) \).

Let \( \psi : S[p] \rightarrow [0, 1] \) be a smooth cutoff function on \( S[p] \) such that \( \psi \equiv 1 \) on \( S[p] \) and on each adjacent \( \Lambda[p, e, 1] \), \( \psi \equiv 1 \) on \( \Lambda[p, e, 1] \cap ([b, d_1] \times S^{n-1}) \), \( \psi \equiv 0 \) on \( \Lambda[p, e, 1] \cap ([d_2, p_{r_0(e)}] - b \times S^{n-1}) \) for \( d_1, d_2 \) where \( d_1 < d_2 \) are chosen so that sech \( (d_1) = \epsilon \) and sech \( (d_2) = \epsilon/2 \) for some \( \epsilon > 0 \) to be determined. If \( \tilde{\psi}(t, \Theta) := \frac{2}{\epsilon} \text{sech}(t) - 1 \) on \( [d_1, d_2] \times S^{n-1} \) then \( |\nabla_{\tilde{g}} \tilde{\psi}| \leq 4 \epsilon^{-1} \) and elsewhere the gradient vanishes. Moreover, by 6.7 and A.16 \( |\nabla_{\tilde{g}} \tilde{\psi}| \sim_{C(\tilde{b})} |\nabla_{\tilde{g}} \psi| \).

Let \( F_1 : C_0^\infty(S[p]) \rightarrow C_0^\infty(S^n) \) such that for \( f \in C_0^\infty(S[p]) \),

\[
F_1(f) \circ \tilde{Y}[p] := \tilde{\psi} f.
\]

Let \( F_2 : C_0^\infty(S^n) \rightarrow C_0^\infty(S[p]) \) such that for \( f \in C_0^\infty(S^n) \),

\[
F_2(f) := \overline{\tilde{\psi}} f \circ \tilde{Y}[p].
\]

For any \( \epsilon > 0 \) there exists \( \epsilon \) sufficiently small so that the requirements of B.1.6 are met. In addition, we demonstrate that

\[
\int_{M_i} |A_i|^2 F_i(f)^2 dg_i - \int_{M_{i'}} |A_{i'}|^2 f^2 dg_{i'} \leq \epsilon \| f \|^2_\infty \text{ for } i \neq i'; i, i' \in \{1, 2\}.
\]

Here \( (M_i, g_i), (M_{i'}, g_{i'}) \) correspond to the two manifolds and metrics of interest and \( A_i, A_{i'} \) correspond to the second fundamental form on the appropriate manifold. We require an estimate like (6.21) since the Rayleigh quotient now includes such a term in the numerator.

We demonstrate (6.21) and a few of the estimates in B.1.6 and leave the rest to the reader as they can be easily verified. Note that the first inequality in B.1.6 should read

\[
\| F_i f \|_\infty \leq 2 \| f \|_\infty
\]
and this is immediately verified by the definitions. Further, since \( n \geq 3 \), using 6.15 with \( x = d_2 \) implies that
\[
\int_{\mathcal{S}[p]} |\nabla \tilde{g}|^2 dg \leq C(b)(1 + C(d_2)|\tau|^{-\frac{1}{n-1}}) \int_{\mathcal{S}[p]} |\nabla \tilde{\psi}|^2 dg \leq C(n, b)(1 + C(d_2)|\tau|^{-\frac{1}{n-1}}) \epsilon^{-2} \leq C\epsilon.
\]
Again using 6.15, by the definition of the \( F_i \),
\[
\int_{\mathcal{M}[p]} f g \, dg = (1 + O(|\tau|^{-\frac{1}{n-1}})) \int_{\mathcal{S}[p] \cap \mathcal{Y}[p](\mathcal{M}[p])} F_1(f) F_1(g) \, dg_{\mathcal{S}[n]},
\]
(6.22)
\[
\int_{\mathcal{M}[p]} F_2(f) F_2(g) \, dg = (1 + O(|\tau|^{-\frac{1}{n-1}})) \int_{\mathcal{S}[p] \cap \mathcal{Y}[p](\mathcal{M}[p])} f g \, dg_{\mathcal{S}[n]}.
\]
Therefore, to demonstrate that orthogonality is almost preserved, we only need consider the behavior on each \( \Lambda[p, e, 1] \). In that case, one can verify that
\[
\left| \int_{\Lambda[p, e, 1]} F_2(f) F_2(g) \, dg - \int_{\mathcal{S}[p] \cap \mathcal{Y}[p](\Lambda[p, e, 1])} f g \, dg_{\mathcal{S}[n]} \right| \leq (1 + O(|\tau|^{-\frac{1}{n-1}})) \int_{\Lambda[p, e, 1]} (f \circ \tilde{Y}[p]) (g \circ \tilde{Y}[p]) (1 - \tilde{\psi}) \, dg \leq C\epsilon n \parallel f \parallel_{\infty} \parallel g \parallel_{\infty}.
\]
Other estimates in B.1.6 proceed similarly.

By (6.22) and the fact that \( |A_g| = n \) on \( \mathcal{M}[p] \), (6.21) holds on the domain \( \mathcal{M}[p] \). So we consider each \( \Lambda[p, e, 1] \). Of critical importance is the fact that while \( |A_g|^2 \) becomes unbounded as \( \tau \to 0 \), we may choose \( \epsilon \) small enough so that \( \int_{[d_1, \mathcal{P}_0[e] - b] \times \mathbb{S}^{n-1}} |A_g|^2 \, dg \) is as small as we like. To see this, first recall that \( \frac{|dw_{cd}^2|}{d\tau} \leq 1 \) and thus by (5.26), \( \frac{1}{\mathcal{L}_d} \frac{\partial \tau}{\partial t} \leq 1 + |\tau|^{\frac{1}{2(n-1)}} \).
Therefore, we may make the change of variables
\[
\int_{[d_1, \mathcal{P}_0[e] - b] \times \mathbb{S}^{n-1}} |A_g|^2 \, dg = n \int_{d_1} \mathcal{P}_0[e]^{-\frac{1}{2}} \int_{\mathbb{S}^{n-1}} (1 + (n - 1)\tau_d[e]^2 \mathcal{L}_d^{-2n}) \mathcal{L}_d^n \, dt \, dg_{\mathbb{S}^{n-1}} \leq \leq n(1 + |\tau|^{\frac{1}{2(n-1)}}) \omega_{n-1} \int_{\mathcal{L}_d(d_1)} (1 + (n - 1)\tau_d[e]^2 r^{-2n}) r^{n-1} \, dr \leq \leq C(n) \left( \epsilon^n + (n - 1)\tau_d[e]^2 \mathcal{L}_d[e; d]^{-n} \right) \leq C(n) \epsilon^n
\]
since \( \tau_d[e]^2 \mathcal{L}_d[e; d]^{-n} = O(|\tau|^{\frac{n}{n-1}}) \). Therefore, given \( \epsilon > 0 \) we may increase \( d_1 \) if necessary (decreasing \( \epsilon \)) so that \( \int_{[d_1, \mathcal{P}_0[e] - b] \times \mathbb{S}^{n-1}} |A_g|^2 \, dg \leq \epsilon/2 \). Thus,
\[
\left| \int_{\mathcal{S}[p] \cap \mathcal{Y}[p](\Lambda[p, e, 1])} n F_1(f)^2 \, dg_{\mathcal{S}[n]} - \int_{\Lambda[p, e, 1]} |A_g|^2 \, fg \, dg \right| \leq O(|\tau|^{-\frac{1}{n-1}}) \int_{[0, d_1] \times \mathbb{S}^{n-1}} f^2 \, dg + \int_{[d_1, \mathcal{P}_0[e] - b] \times \mathbb{S}^{n-1}} |A_g|^2 \, f^2 \, dg \leq \epsilon \parallel f \parallel_{\infty}^2.
\]
On the other hand,

$$\left| \int_{\Lambda[p,e,1]} F_2(f)^2 |A_g|^2 \,dg - \int_{S^n \cap \widetilde{Y}[p](\Lambda[p,e,1])} n f^2 \,dg_{S^n} \right| = O(|\mathcal{Z}|^{n-1}) \int_{S^n \cap \widetilde{Y}[p]([\tilde{\nu},d_1] \times S^{n-1})} n f^2 \,dg_{S^n} + \int_{S^n \cap \widetilde{Y}[p]([d_1,\nu_{0|\nu}]-\mathcal{S}) \times S^{n-1})} n f^2 \,dg_{S^n} \leq C(n)\xi^n \|f\|_{\infty}^2 \leq \epsilon \|f\|_{\infty}^2.$$ 

With the addition of the estimate (6.21), B.2.2 holds for eigenfunctions and eigenvalues of \( \mathcal{L}_g \). Perhaps the most difficult estimate to confirm in this new setting is B.2.2 (4). Using the Rayleigh quotient, we have that (for \( \| \cdot \|^2 \) signifying the squared \( L^2 \) norm)

$$\| |A| f'_n\|^2 + \|f'_n\|^2 \geq \frac{\delta - C\epsilon}{\lambda_n}.$$ 

When \( |A|^2 \equiv n \) (i.e. the manifold is \( S^n \)), we immediately get the required lower bound on \( \|f'_n\|^2 \). On the other hand, if the manifold is \( \tilde{S} \), for each \( \Lambda = \Lambda[p,e,1] \),

$$\int_{\Lambda} |A_g|^2 f^2 \,dg \leq C(n) \int_{\Lambda} f^2 \,dg + \epsilon \|f\|_{\infty}^2.$$ 

Therefore, the lower bound holds in this case as well. All other applications of the Rayleigh quotient to the proof in B.2.2 are more obvious and do not need the small integrability condition of \( |A_g|^2 \) along \( \Lambda \).

Now we may apply the results of Appendix B, appropriately modified, to find an orthonormal basis of \( n + 1 \) eigenfunctions on \( \tilde{S} \) with small eigenvalue that are \( L^2 \) close to those described in (6.20). We get the desired \( C^{2,\beta} \) estimate in the following manner. Because of the uniform geometry of \( S_6 \) for \( T_\Gamma \) sufficiently small, we can use standard linear theory on the interior to increase the \( L^2 \) norms of Appendix B to \( C^{2,\beta} \) norms. Moreover, on \( S_{d_1} \),

$$F_2(\tilde{f}_i,S^n) = f_i,S^n \circ \tilde{Y}.$$ 

By the definition of the immersions and 3.6 and 6.15, for any \( \epsilon > 0 \) we can choose \( T_\Gamma \) sufficiently small so that

$$\|F_2(\tilde{f}_i,S^n) - \tilde{F}_i[d,\zeta] : C^{2,\beta}(S_6, g)\| < \epsilon/2.$$ 

To make the dependence continuous, we let \( f_i \) denote the normalized \( L^2(\tilde{S}, g) \) projection of \( \tilde{F}_i[d, \zeta] \) onto the span of \( F_2(\tilde{f}_i,S^n) \).

Following the general methodology, we introduce the extended substitute kernel. Notice that we have already solved the semi-local linearized problem everywhere except \( S[p] \). Thus, the extended substitute kernel is a much smaller space of functions than for previous constructions of similar type.

Let \( p \in V(\Gamma) \). We fix \( e'_i[p] \) for \( i = 1, \ldots, n + 1 \) depending only on \( \Gamma \) and such that (recall 5.4)

$$|e'_i[p] - e_i| < 5\delta', \quad \text{and} \quad \forall e \in E_p(\Gamma), \quad |e'_i[p] - v[p,e]| > 9\delta',$$

which by the smallness of the parameters implies that \( e'_i[p] \in S[p] \subset M \). We have then
Definition 6.23 (The substitute kernel $\mathcal{K}[p]$). We define $\mathcal{K}[p]$ to be the span of (recall 6.17)

\[ \left\{ \psi[2\delta', \delta'] \circ d_{\epsilon_i[p], \ldots, \epsilon_{n+1}[p]} F_i[d, \zeta] \right\}_{i=0}^{n} \subset C^\infty(M). \]

We also define a basis $\{w_i[p]\}_{i=0}^{n}$ of $\mathcal{K}[p]$ by

\[ \int_{S[p]} w_i[p] F_j[d, \zeta] \, dg = \delta_{ij}. \] (6.24)

Lemma 6.25. For each $p$, the following hold:

1. $w_i[p]$ is supported on $S[p]$.
2. $\|w_i[p] : C^{2, \beta}(M, g)\| \leq C$.
3. For $E \in C^{0, \beta}(S[p], g)$ there is a unique $\bar{w} \in \mathcal{K}[p]$ such that $E + \bar{w}$ is $L^2(\nabla[p], g)$ orthogonal to the approximate kernel on $\bar{S}[p]$. Moreover, if $E$ is supported on $S_1[p]$, then

\[ \|\bar{w} : C^{2, \beta}(M, g)\| \leq C(\beta) \|E : C^{0, \beta}(S_1[p], g)\|. \]

Definition 6.26 (The extended substitute kernel $\mathcal{K}(p, e)$). For $i = 0, \ldots, n$, $[p, e] \in A(\Gamma)$, let $v_i[p, e] : M \to \mathbb{R}$ such that (recall 4.20)

\[ v_i[p, e](x) := \begin{cases} \bar{c}_i[p, e] V_i[\Lambda[p, e, 1], 1, 0] \psi[b, b+1] \circ \tau_d(x), & x \in \Lambda[p, e, 1] \\ 0, & x \not\in \Lambda[p, e, 1] \end{cases}, \]

where the $\bar{c}_i[p, e]$ are normalized constants so that (recall 4.16)

\[ \|\bar{c}_i[p, e]\| \leq C(\beta) 1. \] (6.29)

Definition 6.30 (The global extended substitute kernel $\mathcal{K}$). We define the extended substitute kernel

\[ \mathcal{K} := \mathcal{K}_V \oplus \mathcal{K}_A, \quad \text{where} \quad \mathcal{K}_V := \bigoplus_{p \in V(\Gamma)} \mathcal{K}[p], \quad \mathcal{K}_A := \bigoplus_{[p, e] \in A(\Gamma)} \mathcal{K}[p, e]. \]

We now demonstrate how to solve a modified linear problem on $\nabla[p]$ with good estimates.

Lemma 6.31. Let $\mathcal{K}_A[p] := \bigoplus_{e \in E_p} \mathcal{K}[p, e]$. Given $\beta \in (0, 1)$, $\gamma \in (1, 2)$, there is a linear map

\[ \mathcal{R}[\nabla[p]] : \{ E \in C^{0, \beta}(\nabla[p]) : E \text{ is supported on } S_1[p] \} \to C^{2, \beta}(\nabla[p]) \oplus \mathcal{K}[p] \oplus \mathcal{K}_A[p], \]

such that the following hold for $E$ in the domain of $\mathcal{R}[\nabla[p]]$ above and $(\varphi, w_v, w_a) = \mathcal{R}[\nabla[p]](E)$:

1. $\mathcal{L}_g \varphi = E + w_v + w_a$ on $\nabla[p]$.
2. $\varphi$ vanishes on $\partial \nabla[p]$.
3. $\|w_v, w_a : C^{2, \beta}(\nabla[p], g)\| + \|\varphi : C^{2, \beta}(\nabla[p], \rho_d, g)\| \leq C(\beta) \|E : C^{0, \beta}(S_1[p], \rho_d, g)\|$.
4. $\|\varphi : C^{2, \beta}(\Lambda[p, e, 1], \rho_d, g)\| \leq C(\beta, \gamma) \|E : C^{0, \beta}(S_1[p], \rho_d, g)\|$ for all $e \in E_p$.
5. $\mathcal{R}[\nabla[p]]$ depends continuously on $d, \zeta$. 43
Proof. 6.25 and classical theory together imply there exists \( w_v \in K[p] \) and \( \tilde{\varphi} \in C^{2,\beta}(\tilde{S}[p]) \) such that \( \mathcal{L}_g \tilde{\varphi} = E + w_v \) and \( \tilde{\varphi}_{|\partial \tilde{S}[p]} = 0 \). For each \( \Lambda[p,e,1] \), \( e \in E_p \), we modify \( \tilde{\varphi} \) using the elements \( v_i[p,e] \). Let \( \tilde{\varphi}_e^T \) denote the projection of \( \tilde{\varphi} \) onto \( H_1(C_1^{\text{out}}[p,e,0]) \). Let \( \tilde{\varphi}_e = \tilde{\varphi}_e^T \) on \( C_1^{\text{out}}[p,e,0] \) and let \( V_{\tilde{\varphi}_e} := R_{\partial}^{\text{out}}(\tilde{\varphi}_e^T|C_1^{\text{out}}[p,e,0]) \). Notice that \( (\tilde{\varphi} - V_{\tilde{\varphi}_e})_{|C_1^{\text{out}}[p,e,0]} \in \mathcal{H}_1(C_1^{\text{out}}[p,e,0]) \) and we denote

\[
(\tilde{\varphi} - V_{\tilde{\varphi}_e})_{|C_1^{\text{out}}[p,e,0]} = \sum_{i=0}^{n} \alpha_i \phi_i_{|C_1^{\text{out}}[p,e,0]} = \sum_{i=0}^{n} \alpha_i v_i[p,e]_{|C_1^{\text{out}}[p,e,0]}.
\]

Standard theory implies that \( ||\tilde{\varphi} : C^{2,\beta}(S_2, \rho_d, g)|| \leq C(0,\beta)||E : C^{0,\beta}(S_1, \rho_d, g)|| \). Coupling this with 4.19 implies that \( |\alpha_i| \leq C(b,\beta)||E : C^{0,\beta}(S_1, \rho_d, g)|| \). Set

\[
\varphi = \tilde{\varphi} - \sum_{e \in E_p} \sum_{i=0}^{n} \alpha_i v_i[p,e], \quad w_a = -\mathcal{L}_g \left( \sum_{e \in E_p} \sum_{i=0}^{n} \alpha_i v_i[p,e] \right).
\]

By construction \( \varphi_e^T = \varphi_e \) and thus on each \( \Lambda[p,e,1] \), \( R_{\partial}^{\text{out}}(\varphi_e^T) = \varphi \). 4.19 then provides the necessary decay. \( \square \)

Solving the linearized equation globally. We will solve the global problem in a manner analogous to [6]. The hypotheses of the semi-local lemmas require that the inhomogeneous term on each extended standard region is supported on the enclosed standard region. Thus, to solve the global problem we will first use a partition of unity defined on \( M \) to allow us to consider the inhomogeneous problem on separate regions that allow for solvability and good estimates. After solving on each region separately, we patch the solutions back together. Obviously, the partitioning and patching introduces error. We demonstrate that the error estimates are sufficiently small to iterate away.

We first introduce the cutoff functions we require.

Definition 6.32. For \( d \) satisfying (5.12), we define uniquely smooth functions \( \psi_{S[p]}[d], \psi_{S[p,e,m]}[d], \psi_{S[p,e,m']}[d], \psi_{\Lambda[p,e,m']}[d] \) such that

(i) \( \psi_{S[p]}[d] = \psi_{S[p]}[d] = 1 \) on \( S[p] \),

\[
\psi_{S[p]}[d] = \begin{cases} 
\psi[b+1,b] \circ t_d|_{M[e]} & \text{if } p = p^+[e] \\
\psi[P[e] - (b+1), P[e] - b] \circ t_d|_{M[e]} & \text{if } p = p^-[e],
\end{cases}
\]

\[
\psi_{S[p]}[d] = \begin{cases} 
\psi[P_{\rho_0[e]} - b, P_{\rho_0[e]} - (b+1)] \circ t_d|_{M[e]} & \text{if } p = p^+[e] \\
\psi[P[e] - P_{\rho_0[e]} - b, P[e] - P_{\rho_0[e]} - (b+1)] \circ t_d|_{M[e]} & \text{if } p = p^-[e],
\end{cases}
\]

and the functions are 0 elsewhere.

(ii) \( \psi_{\Lambda[p^+[e],e,m']}[d] = \psi[(m' - 1)P_{\rho_0[e]} + b, (m' - 1)P_{\rho_0[e]} + (b+1)] \circ t_d|_{M[e]} \),

\[
\psi_{\Lambda[p^-[e],e,m']}[d] = \psi[P[e] - (m'P_{\rho_0[e]} - b), P[e] - (m'P_{\rho_0[e]} - (b+1))] \circ t_d|_{M[e]},
\]

and the functions are 0 elsewhere.

(iii) For \( m < l[e] \), \( \psi_{S[p,e,m]}[d] = (1 - \psi_{\Lambda[p,e,m]}[d])(1 - \psi_{\Lambda[p,e,m+1]}[d]) \) on \( S_1[p,e,m] \) and is 0 elsewhere.

(iv) For \( m = l[e] \), \( \psi_{S[p,e,l[e]]}[d] = (1 - \psi_{\Lambda[p^+[e],e,l[e]}[d])(1 - \psi_{\Lambda[p^-[e],e,l[e]}[d]) \) on \( S_1[p,e,m] \) and is 0 elsewhere.
Definition 6.33. Let \( u[p] \in C^{k,\beta}(\tilde{S}[p]) \), \( u[p, e, m] \in C^{k,\beta}(\tilde{S}[p, e, m]) \), \( p \in V(\Gamma), [p, e, m] \in V_S(\Gamma) \), be functions that are zero in a neighborhood of \( \partial\tilde{S}[p], \partial\tilde{S}[p, e, m] \). We define \( U = U(\{ u[p], u[p, e, m] \}) \in C^{k,\beta}(M) \) to be the unique function such that

(i) \( U|_{S[p]} = u[p], U|_{S[p, e, m]} = u[p, e, m] \).

(ii) \( U|_{\Lambda[p, e, 1]} = u[p] + u[p, e, 1] \).

(iii) For \( m' < l[e] \), \( U|_{\Lambda[p, e, m']} = u[p, e, m' - 1] + u[p, e, m'] \).

(iv) For \( U|_{\Lambda[p+1, e, l[e]]} = u[p^+[e], e, l[e] - 1] + u[p^+[e], e, l[e]] \) \( \text{ while } \)
\[
U|_{\Lambda[p^-[e], e, l[e]]} = u[p^-[e], e, l[e] - 1] + u[p^+[e], e, l[e]].
\]

Finally, we define the global norm that we will use. In order to close the fixed point argument, the global norm we define must be uniformly equivalent for all immersions \( Y_{d,\zeta} \) that may arise. After defining the global norm, we establish this equivalence in 6.42. Before we prove this equivalence and before precisely defining the global norm, we give some indication as to why we choose to define the norm in this particular manner.

Given any \( d, \zeta \) satisfying (5.12), (5.18), it will be straightforward to show that the semi-local norms we used in the semi-local settings are uniformly equivalent to the norm given by \( d = 0, \zeta = 0 \). Therefore, on the semi-local level we are free to use those norms already given. On the other hand, the global norm will need to incorporate a decaying weight function. If this weight function is given entirely in terms of \( d, \zeta \) then the ratio between two norms for \( d \neq 0 \) will blow up along a ray of \( M \). Therefore, it is convenient to use a decay function that depends upon the immersion given by \( d = 0, \zeta = 0 \). To clearly distinguish the semi-local norms and the decay, we therefore define the global norm by taking the supremum of semi-local norms on overlapping regions.

Definition 6.34 (The global norms). For \( k \in \mathbb{N}, \beta \in (0, 1), \gamma \in (1, 2) \), and \( u \in C^{k,\beta}(M) \), we define \( \| u \|_{k,\beta,\gamma,d,\zeta} \) to be the supremum of the following semi-norms (when they are finite)

(i) \( \| u : C^{k,\beta}(S_1[p], \rho_d, g) \| \) for each \( p \in V(\Gamma) \),

(ii) \( t_0\| u : C^{k,\beta}(S^+[p, e, m], \rho_d, g) \| \) for each \( [p, e, m] \in V^+_S(\Gamma) \),

(iii) \( t_0\| u : C^{k,\beta}(S^-[p, e, m], \rho_d, g, f_d^{-2}) \| \) for each \( [p, e, m] \in V^-_S(\Gamma) \),

where \( f_d : \cup_{[p, e, m]} S^-[p, e, m] \to \mathbb{R} \) is such that, for \( m \neq l[e] \),

\[
(6.35) \quad f_d(x) = \begin{cases} 
\sum_{d}^\cdot(x)\gamma, & x \in \Lambda[p, e, m], \\
\sum_{in}^\cdot[e; d]\gamma, & x \in \sum_{in}^\cdot[p, e, m], \\
\sum_{in}^\cdot[e; d]^{\gamma}(\sum_{in}^\cdot[e; d]/\sum_{d}(x))^{n-2+\gamma}, & x \in \Lambda[p, e, m + 1].
\end{cases}
\]
and when $m = l[e]$ define

\begin{equation}
(6.36) \quad f_d(x) = \begin{cases} 
\tau_d(x)^\gamma, & x \in \Lambda[p^+[e], e, l[e]] \\
\tau_m[e; d]^\gamma, & x \in S^-[p, e, l[e]] \\
\tau_d(x)^\gamma, & x \in \Lambda[p^-[e], e, l[e]]
\end{cases}
\end{equation}

Also, observe that

\begin{equation}
(6.37) \quad t_0[e] := \tau_m[e; 0]^{2\gamma+n-2} \sim_C |\tau_0[e]|^{1+(2\gamma-1)/(n-1)}.
\end{equation}

For $w = w_v + w_a \in K$ such that

\[
w_v = \sum_{i=0}^n \sum_{p \in V(\Gamma)} \mu_i[p] w_i[p], \quad w_a = \sum_{i=0}^n \sum_{p, e \in A(\Gamma)} \mu_i[p, e] w_i[p, e],
\]

we define the norms on $K_V, K_A$ such that

\begin{equation}
(6.38) \quad |w_v|_V^2 := \max_{p \in V(\Gamma)} \left( \sum_{i=0}^n (\mu_i[p])^2 \right), \quad |w_a|_A^2 := \max_{p, e \in A(\Gamma)} \left( \sum_{i=0}^n (\mu_i[p, e])^2 \right).
\end{equation}

Notice that since $\rho_a \sim_C 1$ on $S_1[p], S^+_1[p, e, m]$, the semi-norms on these regions in the definition above are uniformly equivalent to those taken with respect to the unscaled metric $g$.

**Remark 6.39.** The total decay for the function $f_d$ moving through one $\tilde{S}^-$ region is:

\[ t_d[e] = \tau_m[e; d]^{2\gamma+n-2} \sim_C |\tau_0[e]|^{1+(2\gamma-1)/(n-1)}. \]

The global decay factor given by $t_0[e]$ does not correspond to this value. However, since

\[ \frac{t_d[e]}{t_0[e]} \sim_C \left( \frac{\tau_d[e]}{\tau_0[e]} \right)^{1+(2\gamma-1)/(n-1)} \sim_2 1, \]

where the second relation follows by (5.25), we have that

\[ t_0[e] \sim_C t_d[e] \]

on each $\tilde{S}^-[p, e, m]$.

**Remark 6.40.** While the definition of norms for $w$ might appear unnatural, the choice is motivated by the nature of the construction. Because the functions $w_i[p], w_i[p, e, j]$ have uniform $C^{2,\beta}$ bounds and are supported on $S_1[p]$, for elements of $K_V, K_A$.

\[ \| : \|_{2, \beta; \gamma; d, \zeta} \sim_C \| |_V, \| |_A \]

for elements of $K_V, K_A$.

**Remark 6.41.** Observe that the exponent of $t_0[e]$ chosen in the definition corresponds – in absolute value – to the number of extended standard regions $\tilde{S}^-$ between the region on which the norm is being determined and the closest central sphere. (Recall 5.10.)

**Lemma 6.42.** There exists $\tilde{C}(\beta) > 0$, independent of $T_{l}$, such that if $T_{l}$ is sufficiently small then for any $0 < |z| < T_{l}$ and $d, \zeta$ satisfying (5.12), (5.18) the following holds:

If $u : M \to \mathbb{R}$ such that $\|u\|_{2, \beta; \gamma; d, \zeta} < \infty$, then

\begin{equation}
(6.43) \quad \|u\|_{2, \beta; \gamma; d, \zeta} \sim_{\tilde{C}(\beta)} \|u\|_{2, \beta; \gamma; 0, 0}.
\end{equation}
Proof. The definition of the global norm allows us to consider the equivalence on the semi-local norms.

Let \( g_0 := Y_{0,0}^*(g_{\mathbb{R}^n+1}) \). By 6.15

\[
\|(Y_{0,0} - p) - (Y_{d,\xi} - p') : C^k(S_d[p], \rho_0, g_0)\| \leq C(k,\xi) |x|^{-\frac{1}{k+1}}
\]

for each \( p \in V(\Gamma) \) and corresponding \( p' \in V(\Gamma(d, \xi)) \). Thus,

\[
\|u : C^{k,\beta}(S_d[p], \rho_d, g)\| \sim_{C(k,\xi)} \|u : C^{k,\beta}(S_d[p], \rho_0, g_0)\|
\]

Now consider the comparison for each \( e \in E(\Gamma) \) and \( x \in M[e] \cap [\overline{b}, \overline{P}[e] - \overline{b}] \times S^{n-1} \). On these regions, we consider \( C^{k,\beta} \) norms on balls of radius 1/10 with respect to the metrics

\[
\rho_0(x)^{-2}g_0 = \left( \frac{r_0}{\rho_0(x)} \right)^2 (dt_0^2 + g_{S^{n-1}}), \quad \rho_d(x)^{-2}g = \left( \frac{r_d}{\rho_d(x)} \right)^2 (dt_d^2 + g_{S^{n-1}}).
\]

The equivalence of the weights and the metrics is immediately given by 5.26 and 6.7. The argument for \( e \in R(\Gamma) \) is identical so the proof is complete. \(\Box\)

We are now ready to state and prove the main proposition of this section. The strategy is as follows. We first presume that the inhomogeneous term is supported on the standard regions. Using the semi-local lemmas, we solve the problem on each extended standard region. We then patch together cutoffs of these semi-local solutions, which introduces error that can be removed by iteration.

For the more general case, we first partition the inhomogeneous term and use 4.18 to solve the problem on each \( \Lambda \). We then show that the problem remaining can be reduced to the first case.

**Proposition 6.44.** For each \( d, \xi \) satisfying (5.12), (5.18) respectively, there exists a linear map \( \mathcal{R}_{d,\xi} : C^{0,\beta}(M) \to C^{2,\beta}(M) \oplus K_V \oplus K_A \) such that for \( E \in C^{0,\beta}(M) \) and \( (\varphi, w_v, w_a) = \mathcal{R}_{d,\xi}(E) \) the following hold:

1. \( \mathcal{L}_d \varphi = E + w_v + w_a \) on \( M \).
2. \( \|\varphi\|_{2,\beta,\gamma;d,\xi} + |w_v|_V + |w_a|_A \leq C(\beta,|\gamma|)\|E\|_{0,\beta,\gamma;d,\xi} \)
3. \( \mathcal{R}_{d,\xi} \) depends continuously on \( d, \xi \).

**Proof.** We first presume that \( \text{supp } (E) \subset (\cup_{\Gamma} S_1[p] \cup \cup_{\Gamma} S_1[p, e, m]) \). Let \( \varphi[p, e, m] := \mathcal{R}_{S_1[p,e,m]}(E|S_1[p,e,m]) \) where \( \mathcal{R}_{S_1[p,e,m]} \) denotes the linear map from 4.27 or 4.28 as appropriate. We will directly apply the results of Section 4 using the decay and metric dilation in terms of \( t_d \) rather than \( r \) to account for the coordinate change induced by the map \( t_d \).

Let \( (\varphi[p], w_v[p], w_a[p]) := \mathcal{R}_{S_1[p]}(E|S_1[p]) \), defined by 6.31. Let

\[
\mathcal{R}_0^0 E := \bigcup \left\{ \psi_{S_1[p]}[d]\varphi[p], \psi_{S_1[p,e,m]}[d]\varphi[p, e, m] \right\} \in C^{2,\beta}(M),
\]

\[
\mathcal{W}_v^0 E := \sum_{p \in V(\Gamma)} w_v[p] \in K_V,
\]

\[
\mathcal{W}_a^0 E := \sum_{p \in V(\Gamma)} w_a[p] \in K_A,
\]

\[
\mathcal{E} E := \bigcup \left\{ [[\psi_{S_1[p]}[d], t_d]]\varphi[p], [[\psi_{S_1[p,e,m]}[d], t_d]]\varphi[p, e, m] \right\} \in C^{0,\beta}(M),
\]

where here \( [[\cdot,\cdot]] \) denotes the commutator. That is, \( [[\psi_{S_1[p]}[d], t_d]]\varphi[p] = \psi_{S_1[p]}[d]t_d\varphi[p] - t_d(\psi_{S_1[p]}[d]\varphi[p]) \) and the like for \( \varphi[p, e, m] \).
One easily checks that, as the support of $E$ implies $\psi_{\overline{S}[d]} \varphi[p] = \mathcal{L}_g \varphi[p]$ and the like for $\varphi[p, e, m],$

$$\mathcal{L}_g \mathcal{R}^0 E + \mathcal{E} E = E + \mathcal{W}_b^0 E + \mathcal{W}_a^0 E \quad \text{on $M$.}$$

Notice that by construction, on regions where $\psi_{\overline{S}[d]}$ is not constant, $|\partial^k_{d} \psi_{\overline{S}[d]}| \leq C$. We will use frequently without repeated reference the fact that

$$\|\varphi : C^{2,\beta}(\Omega \cap \text{supp (} |\nabla \psi|, \rho_d, g, f_d)) \| \leq C\|\psi : C^{2,\beta}(\Omega \cap \text{supp (} |\nabla \psi|, \rho_d, g)\| \leq C\|\varphi : C^{2,\beta}(\Omega, \rho_d, g, f_d)\|.$$

Using the estimates of 4.27, 4.28, 6.31 and the inequalities above, we quickly verify that

$$\|\mathcal{R}^0 E\|_{2,\beta,\gamma;d,\xi} \leq C(\beta, \gamma)\|E\|_{0,\beta,\gamma;d,\xi},$$

$$\|\mathcal{W}^0 E\|_{2,\beta,\gamma;d,\xi} \leq C(\beta, \gamma)\|E\|_{0,\beta,\gamma;d,\xi},$$

$$\|\mathcal{W}^0 E\|_{2,\beta,\gamma;d,\xi} \leq C(\beta, \gamma)\|E\|_{0,\beta,\gamma;d,\xi}.$$
On the other adjoining transition region we note that for \( m < l[e] \),
\[
 t_0[e]^{-\frac{m+1}{2}} \left\| \mathcal{E} E : C^{0,\beta}\left( \Lambda[p,e,m+1] \cap S_1[p,e,m], \rho_d, g, \xi_{\text{in}}[e; d]^{\beta} \left( \frac{\xi_{\text{in}}[e; d]}{L_d} \right)^{n-2+\gamma} L_d^{-2} \right) \right\| 
\]
\[
\leq C(h, \beta, \gamma) t_0[e]^{-\frac{m+1}{2}} \xi_{\text{in}}[e; d]^{-\gamma}.
\]
\[
\left\| \varphi[p,e,m+1] : C^{2,\beta}\left( S_1[p,e,m], \rho_d, g, \left( \frac{\xi_{\text{in}}[e; d]}{L_d} \right)^{n-1} \right) \right\| 
\]
\[
\leq C(h, \beta, \gamma) t_0[e]^{-\frac{m+1}{2}} \left\| \varphi[p,e,m+1] : C^{0,\beta}\left( S[p,e,m+1], \rho_d, g \right) \right\| 
\]
by 4.27 (4)
\[
\leq C(h, \beta, \gamma) \xi_{\text{in}}[e; d]^{-\gamma-1} \left\| E \right\|_{0,\beta;\gamma; d; \xi},
\]
where the second inequality follows by 4.28 (3). And finally
\[
 t_0[e]^{-\frac{m}{2}} \left\| \mathcal{E} E : C^{0,\beta}\left( \Lambda[p,e,m+1] \cap S_1[p,e,m], \rho_d, g, \xi_{\text{in}}[e; d]^{-\gamma} \left( \frac{\xi_{\text{in}}[e; d]}{L_d} \right)^{n-2+\gamma} L_d^{-2} \right) \right\| 
\]
\[
\leq C(h, \beta, \gamma) t_0[e]^{-\frac{m}{2}} \left\| \varphi[p,e,m+1] : C^{2,\beta}\left( S_1[p,e,m], \rho_d, g, \left( \frac{\xi_{\text{in}}[e; d]}{L_d} \right)^{n-2+\gamma} \right) \right\| 
\]
\[
\leq C(h, \beta, \gamma, \gamma') t_0[e]^{-\frac{m}{2}} \xi_{\text{in}}[e; d]^{-\gamma-1} \left\| E \right\|_{0,\beta;\gamma; d; \xi},
\]
where the second inequality follows by 4.28 (4). For \( T_1 > 0 \) sufficiently small,
\[
\max_{e \in E(G) \cup R(G)} \left\{ C(h, \beta, \gamma) \xi_{\text{in}}[e; d]^{-\gamma-1} + C(h, \beta, \gamma, \gamma') \xi_{\text{in}}[e; d]^{-\gamma-1} \right\} < \frac{1}{2}.
\]
This implies (6.46). As \( \supp(\mathcal{E} E) \subset \supp(E) \) we can apply the same procedure and produce \( R_1 E, W^1_E, W'^1_E, \mathcal{E}^1 E \) such that
\[
\mathcal{L}_g R_1 E + \mathcal{E}^1 E = \mathcal{E} E + W^1_E + W'^1_E
\]
with
\[
\left\| R_1 E \right\|_{2,\beta;\gamma; d; \xi} + \left\| W^1_E \right\|_{2,\beta;\gamma; d; \xi} + \left\| W'^1_E \right\|_{2,\beta;\gamma; d; \xi} + \left\| \mathcal{E}^1 E \right\|_{0,\beta;\gamma; d; \xi} \leq C(\beta, \gamma)_{\frac{1}{2}} \left\| E \right\|_{0,\beta;\gamma; d; \xi}.
\]
We continue by induction and produce, for all \( k \in \mathbb{Z}^+ \),
\[
\mathcal{L}_g R^k E + \mathcal{E}^k E = \mathcal{E}^{k-1} E + W^k_E + W'^k_E \text{ on } M.
\]
Set
\[ \varphi := \sum_{k=0}^{\infty} R_k E, \quad w_v := \sum_{k=0}^{\infty} W_k^v E, \quad w_a := \sum_{k=0}^{\infty} W_k^a E. \]

The estimates imply that all three of the series converge and we have proven the proposition in the first case.

We now move to the general case. First apply 4.18 to each \( \Lambda[p, e, m'] \) such that
\[ V^\text{out}[p, e, m'] := R_{\Lambda}^\text{out} E |_{\Lambda[p, e, m']} \text{ for } m' \text{ odd}, \]
\[ V^\text{in}[p, e, m'] := R_{\Lambda}^\text{in} E |_{\Lambda[p, e, m']} \text{ for } m' \text{ even}. \]

By the proposition,
\[ V^\text{out}[p, e, m'] |_{C^0[p, e, m']} \in H^1[C^0[p, e, m']], V^\text{out}[p, e, m'] |_{C^2[p, e, m']} \equiv 0 \text{ for } m' \text{ odd, while} \]
\[ V^\text{in}[p, e, m'] |_{C^0[p, e, m']} \in H^1[C^0[p, e, m']], V^\text{in}[p, e, m'] |_{C^2[p, e, m']} \equiv 0 \text{ for } m' \text{ even.} \]

Let
\[ \tilde{E} := U \{ \psi_{S[p]}[d] E, \psi_{S[p, e, m]}[d] E \} + U \{ 0, [\psi_{\Lambda[p, e, m']}[d], L_g] V[p, e, m'] \} \in C^{0, \beta}(M). \]

Note that \( \tilde{E} \subset (\cup_{V(\Gamma)} S[p] \cup \cup_{S(\Gamma)} S[p, e, m]) \) so we can apply the initial argument of the proof with \( \tilde{E} \) in place of \( E \). Thus there exist \( (\tilde{\varphi}, w_v, w_a) \in C^{2, \beta}(M) \times K_V \times K_A \) such that \( L_g \tilde{\varphi} = \tilde{E} + w_v + w_a \) on \( M \). Set
\[ \varphi := \tilde{\varphi} + U \{ 0, \psi_{\Lambda[p, e, m']}[d] V[p, e, m'] \} . \]

Then by the definition of the cutoff functions, \( L_g \varphi = E + w_v + w_a \). Moreover, the estimates from 4.18 and the work done above imply \( (\varphi, w_v, w_a) \) satisfy the necessary estimates. \( \square \)

7. The Geometric Principle

The goal of this section is to understand the creation of extended substitute kernel induced by the unbalancing which is controlled by \( d \) and by the dislocations which are controlled by \( \zeta \). Note that we treat the \( K_V \) part in a different way to the \( K_A \) part: For the former we have lemma 7.4 in which we compare the \( K_A \) part of the mean curvature in the “final” surfaces to the \( d \) parameter itself by appealing to the balancing formula of [48]. This way we do not need to monitor the amount of substitute kernel \( K_V \) created by unbalancing on the initial surfaces and the errors introduced at each step. Instead we proceed with the construction and in the very end we appeal to lemma 7.4. On the other hand for the extended part \( K_A \) of the extended substitute kernel (see proposition 7.26) we estimate the extended substitute kernel content of the mean curvature induced by the dislocations on the initial surfaces and then monitor the error terms introduced at each step of the construction. We could have followed the latter approach for the \( K_V \) part also, and even avoid appealing to the balancing formula, but the current approach is more concise and precise.

Throughout this section, let \( \gamma \in (1, 2), \beta \in (0, 1), \gamma' \in (\gamma, 2), \) and \( \beta' \in (0, \beta) \). Any constant depending on \( h, \beta, \gamma, \beta', \gamma', n \) we simply denote as \( C \).
Estimates on the substitute kernel. Throughout this subsection, we consider “super-extended” central standard regions. For each \( p \in V(\Gamma) \), these regions will be determined by immersions of the domain

\[
S^+[p] := M[p] \bigsqcup \bigcup_{\{e | p = p^+[e]\}} (M[e] \cap [a, p_{\tau_0[e]}] \times \mathbb{S}^{n-1})
\]

\[
\bigsqcup \bigcup_{\{e | p = p^-[e]\}} (M[e] \cap [p[e] - p_{\tau_0[e]}, p[e] - a] \times \mathbb{S}^{n-1}).
\]

**Lemma 7.1.** Let \( d, \zeta \) satisfy (5.12), (5.18) respectively. Then, (recall 2.6, 2.16, 5.33, 5.40)

\[
\int_{S^+[p]} H_{\text{gluing}[d, \zeta]} N_{d, \zeta} \, dg = \tilde{\omega}_n \sum_{e \in E_p} \tau_d[e] \text{sgn}[p, e] R[e; d, \zeta] \mathbf{e}_1.
\]

**Proof.** This is an easy calculation involving the force vector. Let \( \partial S^+[p] := \bigcup_{e \in E_p} \tilde{\Gamma}_e \) and let \( K_e \subset \mathbb{R}^{n+1} \) be hypersurfaces such that \( \partial K_e = \tilde{\Gamma}_e \). Then, (recalling 3.9 and 5.41)

\[
n \int_{S^+[p]} H_{\text{gluing}[d, \zeta]} N_{d, \zeta} \, dg = n \int_{S^+[p]} H_{\text{error}[d, \zeta]} N_{d, \zeta} \, dg
\]

\[
= \int_{S^+[p]} \sum_{i=1}^{n+1} (\Delta g x_i) \mathbf{e}_i - n \int_{S^+[p]} N_{d, \zeta}
\]

\[
= \sum_{e \in E_p} \left( \int_{\tilde{\Gamma}_e} \sum_{i=1}^{n+1} \nabla_g x_i \cdot \eta_e \mathbf{e}_i - n \int_{K_e} \nu_e \right)
\]

\[
= \sum_{e \in E_p} \left( \int_{\Gamma_e} \eta_e - n \int_{K_e} \nu_e \right),
\]

Here \( \eta_e \) is the conormal to \( \tilde{\Gamma}_e, \) \( \nu_e \) is normal to \( K_e \) with the appropriate orientation. Applying (A.4) implies the result. \( \square \)

**Definition 7.2.** Given \( d, \zeta \) satisfying (5.12), (5.18), let \( Y_{d, \zeta} \) denote the corresponding immersion of \( M \). For \( \| f \|_{2, \beta, \gamma; d, \zeta} < \infty \), consider the immersion \( (Y_{d, \zeta})_f : M \to \mathbb{R}^{n+1} \). Let \( S[p] := (Y_{d, \zeta})_f(S^+[p]) \) and let \( d[(Y_{d, \zeta})_f, \cdot] \in D(\Gamma) \) such that (recall 2.6)

\[
d[(Y_{d, \zeta})_f, p] := \tilde{\omega}_n^{-\frac{3}{2}} \int_{S^+[p]} (H_S[p] - 1) N S[p] \, dg_S[p].
\]

**Lemma 7.4.** Given \( d, \zeta \) satisfying (5.12), (5.18), let \( Y_{d, \zeta} \) denote the corresponding immersion of \( M \). If \( \| f \|_{2, \beta, \gamma; d, \zeta} < CC|\tau| \), then for \( d[(Y_{d, \zeta})_f, \cdot] \) defined by (7.3) we have \( \forall p \in V(\Gamma) \)

\[
| d[(Y_{d, \zeta})_f, p] - d[p] | \leq CC|\tau|^{1+(a-3+\gamma)/(a-1)}.
\]

**Proof.** We will use the notation of 7.1, but the domains \( K_e \) will refer now to the parametrizing domain rather than the immersion itself.

More specifically, for \( p \in V(\Gamma) \) and each \( e \in E_p \), let \( K_e \) denote the domain \([0, r_{\tau_d[e]}^\min] \times \mathbb{S}^{n-1} \) and let \( g_e := ds^2 + s^2 g_{\mathbb{S}^{n-1}} \) denote the standard polar metric on \( K_e \). Let \( \nu_e := \text{sgn}[p, e] R[e; d, \zeta] \mathbf{e}_1 \). Let \( \tilde{\Gamma}_e := \partial K_e \) and note that the metric \( g_e \) restricted to \( \tilde{\Gamma}_e \) takes the form \( \sigma_e := (r_{\tau_d[e]}^\min)^{-1} g_{\mathbb{S}^{n-1}} \). Let \( \eta_e := \nu_e \).
Then we may consider each \( \nu_e \) as the outward pointing normal of an immersion of \( K_e \) into \( \mathbb{R}^{n+1} \) such that these immersions are the “caps” of the immersion of \( S^+[p] \) by the map \( Y_{d,\zeta} \). Moreover, each \( \eta_e \) represents the conormal of the immersion of \( S^+[p] \) by \( Y_{d,\zeta} \) along \( \Gamma_e \).

Using (A.4) and the calculation in the proof of 7.1, we observe that

\[
\frac{\Theta_n - 1}{\Theta_n} \int_{S^+[p]} (H_{d,\zeta} - 1) N_{d,\zeta} \, dg = \frac{\Theta_n - 1}{\Theta_n} \sum_{e \in E_p} \left( (r_{\tau_{d,\zeta}}) n - n \int_{\Gamma_e} \eta_e \, dg_{\zeta_{n-1}} - n \int_{K_e} \nu_e \, dg_e \right)
\]

By definition, (recall 5.33)

\[
d[Y_{d,\zeta}, p] - d[p] = \frac{\Theta_n - 1}{\Theta_n} \sum_{e \in E_p} \tau_d[e] \, sgn[p, e] \left( R[e; d, \zeta] \nu_1[e] + R[e; d, \zeta] \nu_1[e] - v_1[e; \tilde{d}, 0] \right)
\]

Using (5.28) and 5.30 we see

\[
|d[Y_{d,\zeta}, p] - d[p]| \leq C \zeta^2 \|
\]

We will get the full estimate by comparing \( d[Y_{d,\zeta}, p] \) and \( d[(Y_{d,\zeta})_f, p] \). Note that the first quantity depends upon \( d, \zeta, \Gamma \) while the second depends again on these and also on \( f \).

Again using the proof of 7.1,

\[
d[(Y_{d,\zeta})_f, p] = \frac{\Theta_n - 1}{\Theta_n} \sum_{e \in E_p} \left( \int_{\Gamma_e} \eta_{e,f} \, d\sigma_{e,f} - n \int_{K_e} \nu_{e,f} \, dg_{e,f} \right)
\]

Here \( \eta_{e,f}, \sigma_{e,f} \) represent the modified conormal and metric, respectively, for the immersion of \( \Gamma_e \) using the map \( (Y_{d,\zeta})_f \). And \( dg_{e,f} \) represents the immersion of \( K_e \) with respect to the same map. Note that since \( N_{d,\zeta} \cdot \nu_e = 0 \) and \( f \) is a normal graph over \( Y_{d,\zeta} \), the normal to the immersion of \( K_e \) by the map \( (Y_{d,\zeta})_f \) is the same as the normal of the immersion by \( Y_{d,\zeta} \).

That is, \( \nu_{e,f} = \nu_e \).

We compare \( d[(Y_{d,\zeta})_f, p], d[Y_{d,\zeta}, p] \) by components. For a fixed \( e, \)

\[
\left| \int_{K_e} \nu_{e,f} \, dg_{e,f} - \int_{K_e} \nu_e \, dg_e \right| = \left| \int_{K_e} \nu_{e,f} \, dg_{e,f} - \int_{K_e} \nu_e \, dg_e \right| \leq \left| \text{Vol}_{(Y_{d,\zeta})_f}(K_e) - \text{Vol}_{(Y_{d,\zeta})}(K_e) \right|
\]

where \( \text{Vol}_{\Omega}(\Omega) \) represents the volume of \( \Omega \) with respect to the metric \( g \).

By definition

\[
\text{Vol}_{(Y_{d,\zeta})}(K_e) = (r_{\tau_{d,e}})^n \omega_{n-1}.
\]

On the other hand, we calculate

\[
\text{Vol}_{(Y_{d,\zeta})_f}(K_e) \leq \int_{0}^{r_{\tau_{d,e}}} |f|_{C^0(\Gamma_e)} \int_{\sigma_{n-1}} \int r_{n-1} \, d\Theta dr \leq ||f||_{C^0(\Gamma_e)} + r_{\tau_{d,e}}^n \omega_{n-1}.
\]

Since \( \|f\|_{2,3,\gamma} \leq C \zeta \| \) implies that \( |f|_{C^0(\Gamma_e)} \leq C \zeta \| \) \| \text{in} [e; d] \|^\gamma, \)

\[
\int_{K_e} \nu_{e,f} \, dg_{e,f} - \int_{K_e} \nu_e \, dg_e \leq C |f|_{C^0(\Gamma_e)} (r_{\tau_{d,e}})^n \leq C \zeta \| \text{in} [e; d] \|^\gamma (r_{\tau_{d,e}})^n \leq C \zeta^{2+\gamma/(n-1)}.
\]

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The other estimate is a bit more delicate. We will use the triangle inequality and consider
\[
\left| \int_{\tilde{\Gamma}_e} \eta_e d\sigma_e - \int_{\tilde{\Gamma}_e} \eta_{e,f} d\sigma_{e,f} \right| \leq \left| \int_{\tilde{\Gamma}_e} \eta_e d\sigma_e - \int_{\tilde{\Gamma}_e} \eta_{e,f} d\sigma_{e,f} \right| + \int_{\tilde{\Gamma}_e} |\eta_{e,f} - \eta_e| d\sigma_e.
\]
Note that along \( \tilde{\Gamma}_e, \frac{\partial}{\partial t} Y_d \zeta \) is parallel to \( \frac{\partial}{\partial t} N_d \zeta \). Therefore, \( \angle(\eta_e, \eta_{e,f}) \) is maximized if \( f = 0 \) on \( \tilde{\Gamma}_e \). In that case,
\[
\eta_{e,f} \cdot \eta_e = \frac{\frac{\partial}{\partial t} Y_d \zeta + (\frac{\partial}{\partial t} f) N_d \zeta}{\sqrt{\left( \frac{\partial_{\min}}{\tau_{d[1]}} \right)^2 + \left( \frac{\partial f}{\tau_{d[1]}} \right)^2}} \cdot (1, 0) = \frac{\frac{\partial_{\min}}{\tau_{d[1]}}}{\sqrt{\left( \frac{\partial_{\min}}{\tau_{d[1]}} \right)^2 + \left( \frac{\partial f}{\tau_{d[1]}} \right)^2}}.
\]
By the definition of the global norm, \( |f|_{C^1(\Gamma_e)} \leq C|\Gamma| |\mathbb{R}_n ; e| \gamma^{-1} \leq C|\Gamma| \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{\gamma^{-1}} \). Thus,
\[
|\eta_{e,f} - \eta_e| = \sqrt{2} \sqrt{1 - \eta_{e,f} \cdot \eta_e} \leq C|\Gamma| \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{\gamma^{-2}}.
\]
and therefore
\[
\int_{\tilde{\Gamma}_e} |\eta_{e,f} - \eta_e| d\sigma_e \leq C|\Gamma| \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{\gamma^{-2}} \text{Vol}_{\sigma_\gamma} (\tilde{\Gamma}_e) \leq C|\Gamma| \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{n-3+\gamma} \leq C|\Gamma|^{2+2(\gamma-2)/(n-1)}.
\]
For the last estimate, we note that
\[
\sigma_{e,f} = \sum_{i=1}^{n} \left( r_e^2 + f_i^2 + f^2 - 2rf_i \right) d\Theta_i^2 = \sum_{i=1}^{n} \left( (r - f)^2 + f_i^2 \right) d\Theta_i^2
\]
where \( \sum_{i=1}^{n} d\Theta_i^2 \) represents the standard metric on \( \mathbb{S}^{n-1} \). Thus
\[
\int_{\tilde{\Gamma}_e} d\sigma_{e,f} \leq \omega_{n-1} \left( \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{2+2(\gamma-2)/(n-1)} \right)^{\gamma^{-2}} \leq \omega_{n-1} \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{\gamma^{-2}} + C|\Gamma| \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{n-3+\gamma}.
\]
Here we used the estimate \((a^2 + b^2)^{1/2} \leq a + b \) for \( a, b \geq 0 \) and the fact that the worst remaining term that appears in the expansion has the form of the second term above.

Applying this estimate reveals that
\[
\left| \int_{\tilde{\Gamma}_e} \eta_e d\sigma_e - \int_{\tilde{\Gamma}_e} \eta_{e,f} d\sigma_{e,f} \right| \leq |\text{Vol}_{\sigma_{e,f}} (\tilde{\Gamma}_e) - \text{Vol}_{\sigma_e}(\tilde{\Gamma}_e)| \leq C|\Gamma| \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{n-3+\gamma} \leq C|\Gamma|^{2+2(\gamma-2)/(n-1)}.
\]
Combining the estimates of (5.5), (7.6), (7.7), and (7.8) implies the result. \( \square \)

**Proposition 7.9** (The \( H_{\text{gluing}}[d, \zeta] \) part of the mean curvature). For \( d, \zeta \) satisfying (5.12), (5.18) respectively, we have the following:

For each \( p \in V(\Gamma) \), there exist \( \varphi_{\text{gluing}}[p] \in C^2(\tilde{S}[p]), \mu'_i[p], \) and \( \mu'_i[p, e] \), where \( i = 0, \ldots, n \), such that

\[
\text{Vol}(\Gamma) \leq C|\Gamma| \left( \frac{\tau_{\min}}{\tau_{d[1]}} \right)^{n-3+\gamma} \leq C|\Gamma|^{2+2(\gamma-2)/(n-1)}.
\]
(1) \( L_y \varphi_{\text{gluing}}[p] + H_{\text{gluing}}[d, \zeta] = \sum_{i=0}^{n} \mu'_i[p] w_i[p] + \sum_{e \in E_p} \sum_{i=0}^{n} \mu'_i[p, e] w_i[p, e] \) on \( \tilde{S}[p] \).

(2) \( \varphi_{\text{gluing}}[p] = 0 \) on \( \partial \tilde{S}[p] \).

(3) \( \| \varphi_{\text{gluing}}[p] : C^{2, \beta}(\tilde{S}[p], g) \| \leq C|\zeta| \).

(4) \( \| \varphi_{\text{gluing}}[p] : C^{2, \beta}(A[p, e, 1], \rho_d, g, \mathcal{L}_2^y) \| \leq C|\zeta| \) for all \( e \in E_p \).

(5) \( |\mu'_i[p, e]| \leq C|\zeta| \).

(6) Each of the \( \varphi_{\text{gluing}}, \mu'_i[p], \mu'_i[p, e] \) are all unique by their construction and depend continuously on \( d, \zeta \).

Proof. Determine \( \mu'_i[p] \) such that for \( j = 0, \ldots, n, \)

\[
\int_{\tilde{S}[p]} \left( \sum_{i=0}^{n} \mu'_i[p] w_i[p] - H_{\text{gluing}}[d, \zeta] \right) f_j[p] \, dg = 0.
\]

The estimates follow immediately from 6.31 and the bound on \( H_{\text{gluing}}[d, \zeta] \) given in 6.10. \( \square \)

Prescribing the extended substitute kernel.

**Lemma 7.10** (Local estimates for the quadratic terms). Let \( Y_{d, \zeta} : M \to \mathbb{R}^{n+1} \) be the immersion of 5.39. Let \( x \in M \) and \( D \subset M \) be a disk of radius 1/10 in the metric \( \rho_d^{-2}(x)g \), centered at \( x \). Let \( c_1 > 0 \) denote the constant found in 6.11.

If \( v \in C^{2, \beta}(D, \rho_d^{-2}(x)g) \) satisfies

\[
\| v : C^{2, \beta}(D, \rho_d^{-2}(x)g) \| \leq \rho_d(x)\epsilon(c_1)
\]

for \( \epsilon(c_1) \) given by B.3, then

\[
\rho_d(x)\| (Y_{d, \zeta})_v - Y_{d, \zeta} - v N_{d, \zeta} : C^{1, \beta}(D, \rho_d^{-2}(x)g) \| \leq C(c_1)\| v : C^{2, \beta}(D, \rho_d^{-2}(x)g) \|^2
\]

\[
\rho_d^2(x)\| H_v - H_{d, \zeta} - \mathcal{L}_y v : C^{0, \beta}(D, \rho_d^{-2}(x)g) \| \leq C(c_1)\rho_d^{-1}(x)\| v : C^{2, \beta}(D, \rho_d^{-2}(x)g) \|^2.
\]

Proof. We wish to apply B.3 and to that end, we rescale the target by \( \rho_d^{-1}(x) \).

We now proceed with the proof. By (6.12), the conditions of (B.1) are satisfied and the hypothesis of B.3 is satisfied for the rescaled function \( \rho_d^{-1}(x)v \). Under rescaling, \( H_{\rho_d^{-1}(x)v} = \rho_d(x)H_v \) and \( \mathcal{L}_{(\rho_d^{-1}(x)Y_{d, \zeta})^{-1}(\rho_d^{-2}(x)g)} = \rho_d^2(x)\mathcal{L}_y \). Thus, B.3 implies

\[
\| \rho_d(x)(H_v - H_{d, \zeta} - \mathcal{L}_y v) : C^{0, \beta}(D, \rho_d^{-2}(x)g) \| \leq C(c_1)\| \rho_d^{-1}(x)v : C^{2, \beta}(D, \rho_d^{-2}(x)g) \|^2.
\]

Simplifying implies the second estimate. The first estimate follows directly from the scaling of the target and the function. \( \square \)

For ease of presentation, we define rotations of the components of the normal vector.

**Definition 7.11.** Let \( \tilde{f}_i : \bigsqcup_{e \in E(\Gamma) \cup R(\Gamma)} M[e] \to \mathbb{R} \) for \( i = 0, \ldots, n \) such that for \( x \in M[e], \)

\[
\tilde{f}_i(x) := (R[e; d, \zeta]^{-1}N_{d, \zeta}(x)) \cdot e_{i+1}.
\]

Note that \( \mathcal{L}_y \tilde{f}_i = 0 \).

For \( e \in E(\Gamma) \cup R(\Gamma) \), let \( f^+[e] : M[e] \cap ([a, a+3] \times S^{n-1}) \to \mathbb{R} \) and for \( e \in E(\Gamma) \) let \( f^-[e] : M[e] \cap ([p[e] - (a+3), p[e] - a] \times S^{n-1}) \to \mathbb{R} \) such that on these regions

\[
(7.12) \quad \underline{U}[e; d, \zeta]^{-1} \circ Y_{d, \zeta} \circ D^+ = (Y_0 + \zeta[p^+[e], e]) f^+[e]
\]

\[
(7.13) \quad \underline{U}[e; d, \zeta]^{-1} \circ Y_{d, \zeta} \circ D^- = (Y_0 + \zeta[p^-[e], e]) f^-[e]
\]
where $D^\pm$ are small perturbations of the identity map. We prove estimates for $f^+[e]$ and note that the same estimates hold for $f^-[e]$, once we account for appropriate changes to the domain.

**Lemma 7.14.** For $f^+[e]$ as above, we have the following:

1. $f^+[e] = 0$ on $M[e] \cap ([a, a + 1] \times S^{n-1})$.
2. $\|f^+[e] : C^{2,\beta}(M[e] \cap ([a, a + 3] \times S^{n-1}), g)\| \leq C|\zeta|$.
3. For $\tilde{f}_i$ described above,
   \[
   \|f^+[e](x) - \sum_{i=0}^n \zeta_i[p^+[e], e]\tilde{f}_i(x) : C^{1,\beta}(M[e] \cap ([a + 2, a + 3] \times S^{n-1}), g)\| \leq C|\zeta|^2.
   \]
4. $\|\mathcal{L}_g f^+[e] - H_{\text{dislocation}}[d, \zeta] : C^{0,\beta}(M[e] \cap ([a, a + 3] \times S^{n-1}), g)\| \leq C|\zeta| |x|^{\frac{1}{n+1}}$.

**Remark 7.15.** Note that we could have stated the previous lemma to fit with the global norm on $S_1[p]$ but since $\rho_d \sim C|\zeta|$ on $S_1[p]$ the norm bounds given above can be used to bound the global norm.

**Proof.** Items (1) and (2) follow immediately from the definition of $f^+[e]$ and the behavior of the immersion $Y_{d, \zeta}$ on each of the domains. For items (3) and (4), we note that item (2) and the uniform estimates on $\rho_d$ allow us to invoke 7.10, which we do with $Y_0 + \zeta[p^+[e], e]$ in place of $Y_{d, \zeta}$. Then item (3) follows from the definition of the functions $\tilde{f}_i$ and the linear error estimate of 7.10 since $U[e; d, \zeta]^{-1} \circ Y_{d, \zeta} = R[e; d, \zeta]^{-1}N_{d, \zeta} = Y_0$ for $t \in [a + 2, a + 3]$. Item (4) follows from the quadratic error estimate of 7.10 applied to $Y_0 + \zeta[p^+[e], e]$ and by recalling 6.15 to compare $\mathcal{L}_g f^+[e]$ and $\mathcal{L}_{Y_0} f^+[e]$. \hfill \square

**Definition 7.16.** Let $S^x[p] \subset M$ such that

\[
S^x[p] := M[p] \bigsqcup \left\{ [a, a + 1] \times S^{n-1} \right\}_{e \in \{p^+[e] \}} \bigsqcup \left\{ [a, a + 3] \times S^{n-1} \right\}_{e \in \{p^-[e] \}}
\]

with the appropriate regions identified as in (5.9). For each $p \in V(\Gamma)$, let

\[
f[p] : S^{n+3}[p] \to \mathbb{R}
\]

such that

\[
f[p](x) = \begin{cases} 
0, & \text{if } x \in M[p], \\
f^+[e](x), & \text{if } p = p^+[e], x \in M[e] \cap [a, a + 3] \times S^{n-1}, \\
f^-[e](x), & \text{if } p = p^-[e], x \in M[e] \cap [a, a + 3] \times S^{n-1}.
\end{cases}
\]

The definition of $f[p]$ immediately implies the following corollary.

**Corollary 7.17.**

1. $f[p] = 0$ on $S^{n+1}[p]$.
2. $\|\mathcal{L}_g f[p] - H_{\text{dislocation}}[d, \zeta] : C^{0,\beta}(S^{n+3}[p], g)\| \leq C|\zeta| |x|^{\frac{1}{n+1}}$.
3. If $p = p^+[e]$, then
   \[
   \|f[p] - \sum_{i=0}^n \zeta_i[p^+[e], e]\tilde{f}_i(x) : C^{1,\beta}(M[e] \cap ([a + 2, a + 3] \times S^{n-1}), g)\| \leq C|\zeta|^2.
   \]
(4) if \( p = p^-[e] \),
\[
\| f[p] - \sum_{i=0}^{n} \zeta_i [p^+[e], e] \tilde{f}_i(x) : C^{1,\beta}(M[e] \cap ([P[e] - (a + 3), P[e] - (a + 2)] \times S^{n-1}), g) \| \leq C |\zeta|^2.
\]

We use these functions to prescribe the dislocation on each central sphere. For convenience, we normalize the functions \( \tilde{f}_i \) on the meridian circle \( C_1^{out}[p,e,0] \).

**Definition 7.18.** For \( [p,e] \in A(\Gamma) \), choose \( c_i'[p,e] \) such that for each \( i = 0, \ldots, n \), on \( C_1^{out}[p,e,0] \), (recall 4.16 and 7.11)
\[
c_i'[p,e] \tilde{f}_i = \phi_i.
\]

**Remark 7.19.** Notice that while \( c_i'[p,e] \) depends on \( d \), these values are independent of \( \zeta \) since \( \tilde{f}_i \) is independent of \( \zeta \) at \( (\hat{b} + 1, \Theta) \). Moreover, by the asymptotic geometric behavior at \( \hat{b} + 1 \) (recall A.16),
\[
|c_i'[p,e]| \sim_{C(\hat{b})} 1.
\]

**Assumption 7.20.** We now choose a constant \( c' \geq 1 \), independent of \( d, \zeta \) and of \( \tau \) but depending on \( \hat{b} \) such that for all \( d \) satisfying (5.12) and corresponding \( c_i'[p,e] \),
\[
c' \geq \max_{i=0, \ldots, n, \ [p,e] \in A(\Gamma)} |c_i'[p,e]|.
\]

Recalling (6.27) and 6.26, the normalization we choose implies that
\[
c_i'[p,e] \tilde{f}_i - v_i[p,e] |_{C_1^{out}[p,e,0]} = 0.
\]
This normalization will be convenient for estimating item (3) in the proposition below.

**Proposition 7.23** (Prescribing the \( K_A \) part of the mean curvature). Let \( d, \zeta \) satisfy (5.12), (5.18), respectively. For each \( p \in V(\Gamma) \) there exist \( \phi_{\text{dislocation}}[p] \in C^{2,\beta}([S[p]) \), \( \mu_\eta'[p] \), and \( \mu_\eta''[p,e] \), where \( i = 0, \ldots, n \), such that
\[
\begin{align*}
(1) & \quad L_g \phi_{\text{dislocation}}[p] + \text{H}_{\text{dislocation}}[d, \zeta] = \sum_{i=0}^{n} \left( \mu_\eta'[p] w_i[p] + \sum_{e \in E[p]} \mu_\eta''[p,e] w_i[p,e] \right) \text{ on } \tilde{S}[p]. \\
(2) & \quad \phi_{\text{dislocation}}[p] = 0 \text{ on } \partial \tilde{S}[p]. \\
(3) & \quad |\mu_\eta'[p]| + |\zeta_i[p,e]/c_i'[p,e] - \mu_\eta'[p,e]| \leq C |\zeta| \frac{1}{|z|^{n-1}} \text{ for all } i = 0, \ldots, n. \\
(4) & \quad \| \phi_{\text{dislocation}}[p] : C^{2,\beta}(\tilde{S}[p], g) \| \leq C |\zeta|. \\
(5) & \quad \| \phi_{\text{dislocation}}[p] : C^{2,\beta}(\Lambda[p,e,1], \rho_d, g, L^2 \zeta_\tau) \| \leq C |\zeta| \frac{1}{|z|^{n-1}} \text{ for all } e \in E[p]. \\
(6) & \quad \phi_{\text{dislocation}}[p], \mu_\eta'[p], \mu_\eta''[p,e] \text{ are all unique by their construction and depend continuously on the parameters of the construction.}
\end{align*}
\]

**Proof.** Let \( f[p] \) represent the function from 7.16. On \( M[e] \cap S[p] \), for \( p = p^+[e] \) set \( \tilde{\psi}_e := \psi[a + 3, a + 2](t) \) and for \( p = p^-[e] \) set \( \tilde{\psi}_e := \psi[P[e] - (a + 3), P[e] - (a + 2)](t) \). On each \( \Lambda[p,e,1] \), find \( V_e \) such that \( L_g V_e = 0, \ V_e = -\sum_{i=0}^{n} \zeta_i[p,e] \tilde{f}_i \) on \( C^{in}[p,e,1] \) and \( V_e = 0 \) on \( C^{out}[p,e,0] \). We construct \( \phi'_{\text{dislocation}}[p] \in C^{2,\beta}(\tilde{S}[p]) \) in the following way. Let
\[
\phi'_{\text{dislocation}}[p] = \begin{cases}
  f[p] & \text{on } M[p], \\
  \tilde{\psi}_e f[p] + (1 - \tilde{\psi}_e) \sum_{i=0}^{n} \zeta_i[p,e] \tilde{f}_i & \text{on } M[e] \cap S[p], [p,e] \in A(\Gamma), \\
  \sum_{i=0}^{n} \zeta_i[p,e] \tilde{f}_i + (1 - \tilde{\psi}_S[p,d]) V_e & \text{on } \Lambda[p,e,1].
\end{cases}
\]

The construction of \( V_e \) and the estimates of 4.21 imply that
\[
\| V_e : C^{2,\beta}(\text{[b, b + 2] } \times S^{n-1} \cap M[e], g) \| \leq C |\zeta| (x_n[e;d])^{n-2}.
\]

Noting the estimates provided by 7.17, we have the following for \( \phi'_{\text{dislocation}}[p] \):
(1) \(L_g \phi'_{\text{dislocation}}[p]\) is supported on \(S_1[p]\) and \(\phi'_{\text{dislocation}}[p] = 0\) on \(\partial \bar{S}[p]\).
(2) For each \(e \in E_p\),
\[
\|L_g \phi'_{\text{dislocation}}[p] - H_{\text{dislocation}}[d, \zeta] : C^{0, \beta}(S_1[p] \cap M[e], g) \leq C \|\zeta\|(|z|^{-1} + (\delta_{\text{in}}[e; d])^{n-2}).
\]

On \(C^1_1[p, e, 0]\), \(\phi'_{\text{dislocation}}[p] = V_e + \sum_{i=0}^n \zeta_i[p, e] \tilde{f}_i\). To modify \(\phi'_{\text{dislocation}}[p]\) and prescribe the fast decay, we follow the argument of 6.31. In this case, \(\tilde{f}_i \in \mathcal{H}_1[C^1_1[p, e, 0]]\) so we first find \(R_{\partial}^\perp(\perp V_e | C^1_1[p, e, 0])\). Recall that \(V_e = V_e - V_e^T\) where \(V_e^T\) denotes the projection of \(V_e\) onto \(\mathcal{H}_1[C^1_1[p, e, 0]]\). Thus, we find coefficients \(\mu_i[p, e]\) such that (recall 6.26, 7.18)
\[
\sum_{i=0}^n \mu_i[p, e] v_i[p, e] = \phi'_{\text{dislocation}}[p] - R_{\partial}^\perp(\perp V_e | C^1_1[p, e, 0])
\]
\[
= V_e + \sum_{i=0}^n \zeta_i[p, e] \tilde{f}_i - R_{\partial}^\perp(\perp V_e | C^1_1[p, e, 0]).
\]
Since this implies that on \(C^1_1[p, e, 0]\),
\[
- \sum_{i=0}^n \zeta_i[p, e] \tilde{f}_i + \sum_{i=0}^n \mu_i[p, e] v_i[p, e] = V_e^T + V_e^\perp - R_{\partial}^\perp(\perp V_e | C^1_1[p, e, 0])
\]
we can appeal to the estimates of 4.19, exploiting the normalization given in (7.22), to conclude that
\[
\left| \frac{\zeta_i[p, e]}{\tilde{f}_i[p, e]} - \mu_i[p, e] \right| \leq C \|V_e : C^{2, \beta}([b, b + 2] \times S_1[p] \cap M[e], g) \leq C \|\zeta\| (\tau_{\text{in}}[e; d])^{n-2},
\]
\[
\|\phi'_{\text{dislocation}}[p] - \sum_{i=0}^n \mu_i[p, e] v_i[p, e] : C^{2, \beta}(\Lambda[p, e, 1], \rho_d, g, \tau_{\text{in}}) \| \leq C \|\zeta\| (\tau_{\text{in}}[e; d])^{n-2}.
\]
Using 6.31 with \(E := L_g \phi'_{\text{dislocation}}[p] - H_{\text{dislocation}}[d, \zeta]\), let
\[
(\phi''_{\text{dislocation}}[p], w_{\text{dislocation}}) := R_{\bar{S}[p]}(E)
\]
where
\[
w_{\text{dislocation}} = \sum_i \mu_i''[p] w_i[p] + \sum_{e \in E_p} \sum_i \mu_i'''[p, e] w_i[p, e].
\]
Then
\[
L_g \phi''_{\text{dislocation}}[p] = L_g \phi'_{\text{dislocation}}[p] - H_{\text{dislocation}}[d, \zeta] + w_{\text{dislocation}} \text{ on } \bar{S}[p], \text{ and}
\]
\[
|\mu_i''[p]|, |\mu_i'''[p, e]| \leq C \|\zeta\| |z|^{-1}.
\]

Set
\[
\phi_{\text{dislocation}}[p] = \phi''_{\text{dislocation}}[p] - \phi'_{\text{dislocation}}[p] + \sum_{e \in E_p} \sum_i \mu_i[p, e] v_i[p, e]
\]
and
\[
\mu_i''[p, e] = \mu_i'''[p, e] + \mu[p, e].
\]
We complete the proof by appealing to all of the estimates above and those of 6.31. \(\Box\)
Prescribing the extended substitute kernel globally. Choose \(d \in D(\Gamma)\) satisfying (5.12) such that

\[
d[p] = \sum_{i=0}^{n} d_i[p] e_{i+1}
\]

and choose \(\zeta \in Z(\Gamma)\) satisfying (5.18) such that

\[
\zeta[p, e] = \sum_{i=0}^{n} \zeta_i[p, e] e_{i+1}.
\]

Using 6.44, determine Corollary 7.26. For \(\zeta\) defined as in (5.12) such that Prescribing the extended substitute kernel globally.

Setting \(\mu_i[p] := \mu_i[p] + \mu''_i[p, e], \mu_i[p, e] := \mu_i[p, e] + \mu''_i[p, e]\) where these coefficients come from 7.9, 7.23, define

\[
\begin{align*}
\mathcal{w}'_u &:= \sum_{i=0}^{n} \sum_{p \in V(\Gamma)} \mu_i[p] w_i[p] \in \mathcal{K}_V, \\
\mathcal{w}'_d &:= \sum_{i=0}^{n} \sum_{[p, e] \in A(\Gamma)} \mu_i[p, e] w_i[p, e] \in \mathcal{K}_A.
\end{align*}
\]

Using 6.44, determine

\[
(\Phi'_d,\mathcal{w}',\mathcal{w}'_d) := \mathcal{R}_M \left( -\mathcal{L}_g \Phi'_d + \mathcal{w}'_d - H_{\text{error}}[d, \zeta] \right).
\]

Now set

\[
\Phi_d,\zeta := \Phi'_d + \Phi'_{d,\zeta} \text{ and } (\mathcal{w}_d)_v := \mathcal{w}'_v + \mathcal{w}'_d \text{ and } (\mathcal{w}_z)_a := \mathcal{w}'_a + \mathcal{w}'_d.
\]

**Corollary 7.26.** For \(d, \zeta\) chosen as in (7.24), (7.25) respectively, the functions \(\Phi_d,\zeta, (\mathcal{w}_d)_v, \text{ and } (\mathcal{w}_z)_a\) depend continuously on \(d, \zeta\) and satisfy (by decreasing \(T\) if necessary):

1. \(\mathcal{L}_g \Phi_d,\zeta + H_{\text{error}}[d, \zeta] = (\mathcal{w}_d)_v + (\mathcal{w}_z)_a\) on \(M\) (recall 5.41).
2. \(\|\Phi_d,\zeta\|_{2,\beta,\gamma;\Gamma} \leq C(|\zeta| + |\zeta|) \leq CC_{M}|\zeta|\).
3. \(|(\mathcal{w}_z)_a - (\mathcal{w}_z)_a|_A \leq C|\zeta|\), where (recall 6.23 and 6.26)

\[
(\mathcal{w}_z)_a := \sum_{i=0}^{n} \sum_{[p, e] \in A(\Gamma)} \zeta_i[p, e] w_i[p, e].
\]

**Proof.** By construction, item (1) is immediately satisfied. Moreover, the estimates for \(\phi_{\text{gluing}}[p]\) and \(\phi_{\text{dislocation}}[p]\) imply that

\[
|\Phi_d,\zeta|_{2,\beta,\gamma;\Gamma} \leq C(|\zeta| + |\zeta|).
\]

To determine estimates for \(\Phi_d,\zeta\) note that for \(E := -\mathcal{L}_g \Phi_d,\zeta + \mathcal{w}_d + \mathcal{w}_d - H_{\text{error}}[d, \zeta]\),

\[
E|_{\mathcal{S}}[p] = \mathcal{L}_g \left( (1 - \psi_{\mathcal{S}}[p]) (\phi_{\text{gluing}}[p] + \phi_{\text{dislocation}}[p]) \right) = E[p].
\]

and

\[
\text{supp}(E) \subset \cup_{[p, e] \in A(\Gamma)} \left( A[p, e, 1] \cap [-2, 2] \right) \frac{1}{2} \times S^{n-1}.
\]

Using the same strategy that produced the estimate (6.46), we note that

\[
\|E\|_{0,\beta,\gamma;\Gamma} \leq \max_{p \in V(\Gamma)} \|E[p] : C^{0,\beta}(S_1[p, e, 1] \cap A[p, e, 1], \rho, g, \gamma_d^{e-2})\| \leq C \max_{e \in E_p} \gamma_d^{e-\gamma} (|\zeta| + |\zeta|).
\]

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Therefore, the estimates from 6.44 imply
$$\|\Phi_d\zeta\|_{2,\beta,\gamma;d,\zeta} + \|w_d''\|_{2,\beta,\gamma;d,\zeta} + \|w_d''\|_{2,\beta,\gamma;d,\zeta} \leq C \max_{e \in E(\Gamma) \cup R(\Gamma)} (\min[e; d])^{\gamma - \gamma} (|\zeta| + |\zeta|).$$

Finally, the estimates on $\mu_i[p, e]$ from 7.9, 7.23 imply $|w_d'' - (w_\zeta)_{d}| \leq C (|\zeta| + |\zeta| |\zeta|^{-1})^{-1}$. □

8. The Main Theorem

Proposition 8.1 (Quadratic estimates). There exists $T_\Gamma > 0$ sufficiently small such that for all $0 < |\zeta| < T_\Gamma$, $\alpha \in (0, 1)$, and $v \in C^{2,\beta}_\text{loc}(M)$ such that $\|v\|_{2,\beta,\gamma;d,\zeta} \leq |\zeta|^{-1/2}$, we have (recall 6.34 and 6.42)
$$\|H_v - H_{d,\zeta} - L_g v\|_{0,\beta,\gamma;d,\zeta} \leq |\zeta|^{1-\gamma} \|v\|_{2,\beta,\gamma;d,\zeta}.\overline{C}.$$ 

Proof. The estimate $\|v\|_{2,\beta,\gamma;d,\zeta} \leq |\zeta|^{1-\gamma/2}$ coupled with the definition of $\rho_d$ implies that for every $x \in M$, $v$ satisfies the hypothesis of 7.10 on the disk of radius $1/10$ in the metric $\rho_d^{-2}(x)g$, centered at $x$. We refer to it as $D$. The conclusion of the same lemma implies that
$$\|H_v - H_{d,\zeta} - L_g v : C^{0,\beta}(D, \rho_d, g, f_d, \rho_d^{-2})\| \leq (C)_{\Gamma}(x)\rho_d^{-1}(x) \|v : C^{2,\beta}(D, \rho_d, g, f_d)\|^2,$$
where, for $D \subset S_1[p], S^+\Gamma[p, e, m]$, we presume that $f_d \equiv 1$. The definition of the global norm and the fact that $\rho_d \sim C|\zeta|^{-1}$ on $S_1[p], S^+\Gamma[p, e, m]$ implies that it is enough to show that the right-hand side of the inequality is bounded above by $|\zeta|^{1-\gamma} \|v\|_{2,\beta,\gamma;d,\zeta}^2/\overline{C}$.

For each $D \subset S_1[p]$ or $D \subset S^+\Gamma[p, e, m]$, $f_d(x)\rho_d^{-1}(x) \leq C$ and thus
$$C(\Gamma)_{\Gamma}(x)\rho_d^{-1}(x) \|v : C^{2,\beta}(\Gamma, \rho_d, g, f_d)\|^2 \leq C\|v : C^{2,\beta}(D, \rho_d, g, f_d)\|^2 \leq |\zeta|^{1-\gamma} \|v : C^{2,\beta}(D, \rho_d, g, f_d)\|^2/\overline{C},$$
for sufficiently small $\zeta$. Since the weighting $\rho_0[\zeta]^{-1/2} > 1$, it follows that $\|v : C^{2,\beta}(\Gamma, \rho_d, g, f_d)\| < \|v\|_{2,\beta,\gamma;d,\zeta}$ for any $D \subset S^+\Gamma[p, e, m]$.

On each $M[\zeta]$, the definitions imply that $f_d\rho_d^{-1}$ is maximized at $p_{\rho_0}[\zeta]$. By decreasing $T_\Gamma$ if necessary, depending on $b$, we have
$$\max_{e \in E(\Gamma) \cup R(\Gamma)} (f_d(p_{\rho_0}[\zeta])\rho_d^{-1}(p_{\rho_0}[\zeta]) = \max_{e \in E(\Gamma) \cup R(\Gamma)} (\min[e; d])^{\gamma - \gamma} (r_{\rho_0}[\zeta])^{-1} \leq C(b)|\zeta|^{-1} \leq |\zeta|^{1-\gamma}/\overline{C}.$$ 

The result again follows immediately by the definition of the global norms. □

Before proceeding to the main theorem, we recall for the reader the process that determines $Y_{d,\zeta}$. The finite central graph $\Gamma[0, 0]$ is always in the background and we do not make explicit reference to its use. As a first step, we choose $\zeta < T_\Gamma$ and then $d, \zeta$ satisfying (5.12), (5.18) respectively. Together $d, \zeta$ determine $\tilde{d}$ (5.13) and $\tau_d[e]$ (5.14). Next $\tau_d[e]$ and $\zeta$ determine the immersions of the building blocks $Y_{\text{edge}}, Y_{\text{ray}}$ 3.5 and also determine $\ell$ (5.21). The quantities $\tilde{d}, \ell$ determine a graph $\Gamma[\tilde{d}, \ell] \in \mathcal{F}$, 2.14. The graph $\Gamma[\tilde{d}, \ell]$ and $\zeta$ determine the building blocks $\hat{Y}[\cdot, \cdot], 3.3, (5.35)$. Finally, the graph $\Gamma[\tilde{d}, \ell]$ positions the building blocks and finishes the construction of $Y_{d,\zeta}, 5.39$.

Theorem 8.2. Let $\Gamma$ be a finite central graph with an associated family $\mathcal{F}$. Then there exist $C, b$ sufficiently large and $T_\Gamma > 0$ sufficiently small so that for all $0 < |\zeta| < T_\Gamma$:

There exist $d, \zeta$ satisfying (5.12), (5.18) and a function $f \in C^{2,\beta}_\text{loc}(M)$ such that $(Y_{d,\zeta})_f : M \to \mathbb{R}^{n+1}$ is an immersed surface with CMC equal to 1 and $\|f\|_{2,\beta,\gamma;d,\zeta} \leq C(|\zeta|$ (recall 6.43). Moreover, if $\Gamma$ is pre-embedded then $(Y_{d,\zeta})_f$ is embedded for $\zeta > 0$. 

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In the statement, $M$ is the abstract surface based on the graph $\Gamma$ and the parameter $\tau$. $Y_{d,\zeta}$ is the immersion described above the theorem statement. Finally, $(Y_{d,\zeta})_f$ is the normal graph over $Y_{d,\zeta}$ by $f$, as defined in Appendix B.

**Proof.** Choose $\beta \gg 1$ as in 4.2. Recall that $\tilde{C}, c'$ (6.43, 7.21) and all $C$ appearing in the statement of 7.26 depend only on $\beta$. Choose $C$ independent of $T_\Gamma$ so that $\tilde{C} \geq 4c'C + c'/C$. Choose $T_\Gamma > 0$ as in 6.1. We again point out that $T_\Gamma$ does not depend upon the structure of $\Gamma$ but only on the function $\tilde{\tau}$ and on various geometric quantities. Moreover, $\beta$ is independent of $T_\Gamma$. Fix $\alpha \in (0,1)$ and $\gamma \in (1,2)$. Reduce $T_\Gamma$ if necessary so that $CC\tilde{C} \leq T_\Gamma^{-\alpha/2}$ and $CC\tilde{C}T_\Gamma^{-\alpha/2} \leq 1$. For any $0 < |\tau| < T_\Gamma$, we define $B$ to be the set

$$\{u \in C^{2,\beta}_{\text{loc}}(M) : \|u\|_{2,\beta,\gamma;0,0} \leq |\tau|\} \times \{d \in D(\Gamma) : |d| \leq |\tau|^{1+\frac{1}{\alpha}}\} \times \{\zeta \in Z(\Gamma) : |\zeta| \leq C|\tau|\}.$$  

We define $\mathcal{J} : B \to B$ in the following manner. For $(u, d, \zeta) \in B$, find $\Gamma(\tilde{d}, \tilde{\ell}) \in \mathcal{F}$ and determine $Y_{d,\zeta}$ in the manner outlined in Section 5. Determine $\Phi_{d,\zeta}$ and $(w_d)_v, (w_\zeta)_a$ by 7.26 and define a function $\tilde{u} = \Phi_{d,\zeta} - u$. Then,

$$L_gu = (1 - H_{d,\zeta}) + (w_d)_v + (w_\zeta)_a - L_gu,$$

where $\Phi_{d,\zeta}$ is the immersion described above the theorem statement. Finally, $(Y_{d,\zeta})_f$ is the normal graph over $Y_{d,\zeta}$ by $f$, as defined in Appendix B.

Using 6.44, with $H_\bar{u}$ denoting the mean curvature of the surface $(Y_{d,\zeta})_\bar{u}$, define $(u', w', w'_a) = \mathcal{R}_{d,\zeta}(H_\bar{u} - H_{d,\zeta} - L_gu)$, then

$$L_gu' = H_\bar{u} - 1 + L_gu + w'_v - (w_d)_v + w'_a - (w_\zeta)_a$$

and by 8.1,

$$\|u'\|_{2,\beta,\gamma;0,0} + |w'_v|_V + |w'_a|_A \leq |\tau|^\alpha - 1 \|\tilde{u}\|_{2,\beta,\gamma;0,0}^2/\tilde{C} \leq |\tau|/\tilde{C}.\]

By (6.43)

$$\|u'\|_{2,\beta,\gamma;0,0} \leq |\tau|.$$

Define $\mu_a \in Z(\Gamma)$ such that (recall 7.18)

$$\mu_a[p,e] := \sum_{i=0}^n \xi'_i[p,e] \mu_i[p,e] \epsilon_{i+1}$$

where the coefficients $\mu_i[p,e]$ are determined to satisfy

$$\sum_{i=0}^n \mu_i[p,e] w_i[p,e] = (w'_v + (w_\zeta)_a - (w_\zeta)_a)\big|_{S[p] \cap \Lambda[p,e,0]}$$

and the definition of $(w_\zeta)_a$ is given in 7.26. The estimates from 7.26 and (8.3) imply that (recall 7.21)

$$|\mu_a| \leq c' \left( |\tau|/\tilde{C} + C|\tau| \right) \leq C|\tau|.$$  

Define $\mu_v \in D(\Gamma)$ such that $\mu_v[p] := d[(Y_{d,\zeta})_\bar{u}, p]$ (recall (7.3)). By 7.4, for each $p \in V(\Gamma),$  

$$|d[p] - \mu_v[p]| \leq C\tilde{C}|\tau|^{1+(n-3+\gamma)/(n-1)} \leq C|\tau|^{1+\frac{1}{n+1} + \frac{2-1}{n-1}} \leq |\tau|^{1+\frac{1}{n+1}}.$$  

We use the procedure above to define the map

$$\mathcal{J}(u, d, \zeta) = (u', d - \mu_v, \mu_a).$$
Then by (8.4), (8.5), (8.6), \( \mathcal{J}(u, d, \zeta) \in \mathbb{B} \) and the map \( \mathcal{J} : B \to \mathbb{B} \) is well defined. Moreover, for some \( \beta' \in (0, \beta) \), \( \mathbb{B} \) is a compact, convex subset of \( C^{2,\beta'}_{\text{loc}}(M) \times D(\Gamma) \times Z(\Gamma) \) and one can easily check that \( \mathcal{J} \) is continuous in the induced topology. Thus, Schauder’s fixed point theorem [15, Theorem 11.1], implies there exists a fixed point \( (u', d', \mu'_0) \in \mathbb{B} \).

By inspection, at a fixed point one has that
\[
1 - H_{u'} = w'_v - (w_{d'})_v
\]
and
\[
d'([Y_{d',\zeta}]_{u'}, p) \equiv 0 \text{ for all } p \in V(\Gamma).
\]

Recall that the function \( w'_v - (w_{d'})_v \) is supported on the interior of \( \bigcup_{p \in V(\Gamma)} S[p] \). For a fixed \( p \in V(\Gamma) \),
\[
(w'_v - (w_{d'})_v) \big|_{S[p]} = \sum_{i=0}^{n} \lambda_i w_i[p]
\]
for some \( \lambda_i \in \mathbb{R} \). Let \( N_{d',\zeta,u'} \) denote the normal to the immersion \( (Y_{d',\zeta})_{u'}(M) \) and let \( \tilde{F}_i := \tilde{w}^{-1/2} N_{d',\zeta,u'} \cdot e_{i+1} \). The definition of the global norm implies that \( \|u' : C^{2,\beta}(U[p], g)\| \leq CC|\Sigma| \) and thus on \( U[p] \),
\[
|N_{d',\zeta} - N_{d',\zeta,u'}| \leq CC|\Sigma|, \quad |	ilde{F}_i - \tilde{F}_i| \leq CC|\Sigma|.
\]
Using then (6.17) and 6.24 we have
\[
\int_{U[p]} w_i[p] \tilde{F}_i \geq \frac{1}{2}, \quad \left| \int_{U[p]} w_i[p] \tilde{F}_j \right| \leq CC|\Sigma| \text{ for } i \neq j.
\]
Consider the \((n+1) \times (n+1)\) dimensional matrix \( \mathcal{M} \) where \( \mathcal{M}_{ij} = \int_{U[p]} w_j[p] \tilde{F}_i, i, j = 0, \ldots, n \).

The previous calculations demonstrate that \( \mathcal{M} \) is invertible. The definition for \( d'([Y_{d',\zeta}]_{u'}, \cdot) \) along with (8.7) and (8.8) together imply that
\[
\int_{U[p]} \left( \sum_{j=0}^{n} \lambda_j w_j[p] \right) \tilde{F}_i = 0 \text{ for all } i = 0, \ldots, n.
\]

Since \( \mathcal{M} \) is invertible, this implies that \( \lambda_j = 0 \) for all \( j = 0, \ldots, n \) and thus \( w'_v - (w_{d'})_v \equiv 0 \).

By (8.7), \( 1 - H_{u'} \equiv 0 \) and thus the immersion \( (Y_{d',\zeta})_{u'} : M \to \mathbb{R}^{n+1} \) has mean curvature identically 1.

Embeddedness follows when \( \Gamma \) is pre-embedded and \( \tau > 0 \) as in this case \( Y_{d',\zeta}(M) \) is embedded and \( \|u'\|_{2,\beta,\gamma;Y_{d',\zeta}} \leq \tilde{C}|\Sigma| \).

We do not provide an extensive list of examples but instead point out that those provided in [6, Section 2.2] and the finite topology examples in [36, Section 4] can easily be modified for the higher dimensional setting. For example [36, Example 4.1] remains valid and again produces infinitely many topological types with two ends. Moreover it is not hard to construct more examples by modifying those graphs to take advantage of the extra dimensions. In the embedded case also a finite number of topological types can easily be realized with \( k \) ends, with the number of the topological types tending to \( \infty \) as \( k \to \infty \). Finally an easy parameter count demonstrates that there are
\[
(k - 1)(n + 1) - \binom{n + 1}{2} + \binom{n + 1 - k}{2}.
\]
continuous parameters in these constructions in the absence of symmetry. Here the first summand reflects that we have \( k \) \(-1\) Delaunay ends whose direction and \( \tau \) parameter can be arbitrarily assigned, and the rest correct for the trivial changes induced by rotations.

**Appendix A. Delaunay hypersurfaces**

The Delaunay surfaces are CMC surfaces in \( \mathbb{R}^3 \) discovered by Delaunay in 1841 [11]. By analogy we call Delaunay (hyper)surfaces the \( O(n) \)-invariant CMC hypersurfaces in the \((n+1)\)-dimensional Euclidean space \( \mathbb{R}^{n+1} \). These are well known to form a one-parameter family. We derive them now in order to fix the notation and review their properties. We call the parameter of the family \( \tau \in \mathbb{R} \), and by the \( O(n) \) symmetry we can describe them by an immersion \( Y_{\tau} : \mathbb{R} \times S^{n-1} \to \mathbb{R}^{n+1} \) of the form

\[
Y_{\tau}(t, \Theta) = (k_r(t), r_r(t) \Theta) = (k(t), r(t) \Theta),
\]

where \( r_r = r : \mathbb{R} \to \mathbb{R}_+ \) and \( k_r = k : \mathbb{R} \to \mathbb{R} \) depend on the parameter \( \tau \) (although sometimes we omit \( \tau \) for simplicity), and \( \Theta \) are the standard coordinates in \( \mathbb{R}^n \) restricted to the unit sphere \( S^{n-1} \subset \mathbb{R}^n \). In order to determine the two unknown functions \( r \) and \( k \) we need two equations: First, we choose to impose the requirement that \( Y_{\tau} \) is conformal. Since by (A.1) \( Y_{\tau}^* g = ((k')^2 + (r')^2) dt^2 + r^2 g_{S^{n-1}} \), this is equivalent to the equation

\[
(r')^2 + (k')^2 = r^2.
\]

The second equation can be provided by the condition \( H = 1 \) which amounts to a second order ODE. Instead we use a first integral of this equation as follows.

**The force as a parameter.** Observe first that for an \( n-1 \) chain \( \Gamma \subset Y_{\tau}(\mathbb{R} \times S^{n-1}) \) with \( \Gamma = \partial K, K \subset \mathbb{R}^{n+1} \), the flux is defined as

\[
\text{Force}(\Gamma) = \int_{\Gamma} \eta - n \int_K \nu.
\]

Here \( \eta \) is the unit conormal to \( \Gamma \) in the surface and \( \nu \) is the unit normal to \( K \) with the appropriate orientation with respect to \( \eta \). Let \( \Gamma = \{Y_{\tau}(t_0, \Theta) : \Theta \in S^{n-1}\} \) for a fixed \( t_0 \). We clearly have then in (A.3) by using (A.2)

\[
\eta = \frac{\partial Y_{\tau}(t_0, \Theta)}{\|\partial Y_{\tau}(t_0, \Theta)\|} = r^{-1}(t_0) (k'(t_0), r'(t_0) \Theta), \quad \nu = (1, 0).
\]

We conclude that the flux is given by

\[
\text{Force}(\Gamma) = (r^{n-2} k' - r^n) \omega_{n-1} e_1 =: \tau \omega_{n-1} e_1,
\]

where the second equation is our definition of the parameter \( \tau \) and \( \omega_{n-1} \) denotes the volume of \( S^{n-1} \). By combining then (A.4) with (A.2) we have the equations

\[
k' = r^2 (1 + \tau r^{-n}), \quad r^4 (1 + \tau r^{-n})^2 + (r')^2 = r^2.
\]

**The family of the Delaunay hypersurfaces and their geometry.** The second equation in (A.5) easily implies that \( \tau \in (-\infty, n-n(n-1)^{n-1}) \). The boundary value \( \tau = n-n(n-1)^{n-1} \) clearly corresponds to the (only) solution \( r \equiv (n-1)/n \) and so in this case the image of \( Y_{\tau} \) is the \( n \)-cylinder \( \mathbb{R} \times S^{n-1}( \frac{n-1}{n} ) \). In the case \( \tau = 0 \) (A.5) has a unique solution up to translations given by

\[
k(t) = \tanh t, \quad r(t) = \sech t, \quad Y_0(t, \Theta) = (\tanh t, \sech t \Theta).
\]
$Y_0$ provides clearly a parametrization of $\mathbb{S}^{n-1} \setminus \{(\pm 1, 0)\}$. To study now the case $\tau \neq 0$ we introduce a function $w : \mathbb{R} \to \mathbb{R}$ defined by
\begin{equation}
(A.7) \quad r(t) = |\tau|^{1/n} e^{w(t)}.
\end{equation}
By rewriting then (A.5) we have the equations
\begin{equation}
(A.8) \quad \begin{cases}
(w')^2 = 1 - (r + \tau r^{1-n})^2 = 1 - \tau^2/(n) (e^w \pm e(1-n)w)^2, \\
w(0) = w_{\text{max}} > 0, \quad w'(0) = 0,
\end{cases}
\end{equation}
\begin{equation}
(A.9) \quad \begin{cases}
k' = r^2(1 + \tau r^{-n}) = \tau^2/(n) (e^{2w} \pm e^{2-n}w), \\
k(0) = 0,
\end{cases}
\end{equation}
where $\pm$ is the sign of $\tau$ and we assigned without loss of generality appropriate initial conditions. By analyzing the potential in (A.8) it is easy to see that the maximum value $w_{\text{max}}$ of $w$ is always positive as recorded.

By the nature of the equation and based on the initial conditions we chose we have ensured uniqueness of the solutions up to trivial changes. For $\tau \in (-\infty, n^{-n}(n-1)^{-n})$, $w$ is not constant and with periodic period we will designate by $2p_\tau$.

**Definition A.10 (Periods of $Y_\tau$).** We define the domain period of $Y_\tau$ to be the period of $w$, $2p_\tau$, and the translational period of $Y_\tau$ to be $2p_\tau := k_\tau(2p_\tau) - 2$. We have therefore
\begin{equation}
Y_\tau(t + 2p_\tau, \Theta) \equiv Y_\tau(t, \Theta) + (2 + 2p_\tau)e_1.
\end{equation}

Clearly $w$ has a maximum at $t = 0$ and a minimum at $t = p_\tau$ and it is an even function about both of those values of $t$. We have then for any $\tau \in (-\infty, 0) \cup (0, n^{-n}(n-1)^{-n})$ an immersion $Y_\tau : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1}$ which by construction is rotationally symmetric about the $x_1$ axis and its embeddedness is determined by the sign of $k'$: In fact, one quickly sees that for admissible $\tau > 0$ the surface is embedded while for $\tau < 0$ it is not. The image of $Y_\tau$ is foliated by spheres that sit in hyperplanes orthogonal to the $x_1$ axis, with maximum radius at $t = 2mp_\tau$ and minimum radius at $t = (2m - 1)p_\tau$ for $m \in \mathbb{Z}$.

By an easy calculation the Gauss map and first and second fundamental forms of $Y_\tau$ are given by
\begin{equation}
(A.11) \quad \nu_\tau(t, \Theta) = (w', -(k'/r)\Theta),
\end{equation}
\begin{equation}
Y_\tau g = ((k')^2 + (r')^2)dt^2 + r^2g_{\mathbb{S}^{n-1}} = r^2(dt^2 + g_{\mathbb{S}^{n-1}}),
\end{equation}
\begin{equation}
A = k'g_{(n-1)} + (k''w' - k'(w'' + (w')^2))dt^2.
\end{equation}

We simplify now the expression $(k''w' - k'(w'' + (w')^2))$ so that we may quickly determine essential geometric quantities. First observe that for $r_1 := r + \tau r^{1-n}$ and $r_2 := r + (1-n)\tau r^{1-n}$,
\begin{equation}
(A.12) \quad (w')^2 = 1 - r_1^2, \quad w'' = -r_1 r_2, \quad k' = r r_1, \quad k'' = r w'(r_1 + r_2).
\end{equation}
Thus
\begin{equation}
\frac{k'' w' - k'(w'' + (w')^2)}{r_1 r_2} = \frac{r_1 + r_2)(1 - r_1^2)}{r_1(r_1 r_2 + 1 - r_1^2)} = r r_2.
\end{equation}
To determine $H$ observe that
\begin{equation}
\frac{nH}{r^2} = \frac{(n-1)rr_1 + rr_2}{r^2} = \frac{r^2 + \tau r^{1-n})rr_1^2}{r^2} = \frac{r + \tau r^{1-n})r_1^2 - n \tau r^{1-n}} = n.
\end{equation}
This confirms that $H \equiv 1$ as expected. For later use, we now determine $|A|^2$. A simple calculation gives
\begin{equation}
(A.13) \quad |A|^2 = r^{-2}
\end{equation}
The limiting behavior as $\tau \to 0$. We first determine the asymptotic behavior of the maximum and minimum values $r^\text{max}_\tau$ and $r^\text{min}_\tau$ of $r_\tau$ which is as in A.1 and A.7.

**Lemma A.14.** For $|\tau| \neq 0$ small enough in absolute terms we have

$$
 r^\text{max}_\tau := r_\tau(0) = 1 - \tau + O(\tau^2), \quad r^\text{min}_\tau := r_\tau(p_\tau) = |\tau|^{\frac{1}{n-1}} + O(|\tau|^{2/(n-1)}).
$$

**Proof.** Let $f(r) = r + \tau r^{1-n}$. Note that $r_\tau$ has critical points where $w' = 0$ and thus at critical points for $r = r_\tau,$

$$
 1 = (r + \tau r^{1-n})^2 = f^2(r),
$$

so we are interested in values of $r$ such that $f(r) = \pm 1$. (Only $f(r^\text{min}_\tau) = 1$ when $\tau < 0$. All three of the other critical values satisfy $f(r) = 1$.)

To aid in the proof for the expansion near $r = 1$, we recall the Taylor expansion

$$(1 - x)^{1-n} = 1 + c_1 x + c_2 x^2 + c_3 x^3 + O(x^4)$$

where we ignore the exact value of the coefficients as they will not be needed in the proof.

Since $f(1) = 1 + \tau \neq 1$, we consider $r = 1 - \alpha \tau^\alpha$ for $\alpha > 0$. Then

$$
 f(1 - \tau^\alpha) = 1 - \tau^\alpha + \tau + c' r^{1+\alpha} + O(\tau^{1+2\alpha}).
$$

The only term that can cancel $\tau$ is $-c\tau^\alpha$ and thus $c = \alpha = 1$ is required. Now consider $r = 1 - \tau(1 + c\tau^\alpha)$ for $\alpha > 0$. Then using the expansion,

$$
 f(1 - \tau(1 + c\tau^\alpha)) = 1 - \tau(1 + c\tau^\alpha) + \tau + c' \tau(1 + c\tau^\alpha) + c'' \tau^2(1 + c\tau^\alpha)^2 + O(\tau^3))
$$

It follows that $\alpha = 1$ and $c = c'$ and the asymptotics for $r^\text{max}_\tau$ hold.

Now consider the expansion near $r = 0$. We require a power of $|\tau|$ so that $|\tau|^{|1-n\alpha|} \approx \pm 1$, depending on the sign of $\tau$. It follows that $\alpha = \frac{1}{n-1}$. Now consider $r = |\tau|^{\frac{1}{n-1}}(1 - c|\tau|^\alpha)$ for $\alpha > 0$. In that case, exploiting again the Taylor expansion,

$$
 f(|\tau|^{\frac{1}{n-1}}(1 - c|\tau|^\alpha) = |\tau|^{\frac{1}{n-1}}(1 - c|\tau|^\alpha) + \tau |\tau|^\alpha(1 + c^\prime |\tau|^\alpha + O(|\tau|^{2\alpha}))
$$

(Note that the fourth term might be absorbed into the big O term, depending on the value of $\alpha$.) Examining the powers, it is clear that $\alpha = \frac{1}{n-1}$ and the result for $r^\text{min}_\tau$ follows. □

Let $\rho_\tau : [-k(2p_\tau), k(2p_\tau)] \to \mathbb{R}$ be the function such that $\rho_\tau(x_1) = r_\tau(k(t))$ where $k(t_1) = x_1$. Then one can describe this fundamental piece of the surface as the rotation of $p_\tau$ about the $x_1$-axis and an easy computation of the mean curvature implies that

$$
 nH = \frac{n-1}{\rho \sqrt{1 + \rho^2}} - \frac{\rho''}{(1 + \rho^2)^{\frac{3}{2}}},
$$

Note that $\rho_\tau(0) = 1 + O(\tau)$ and $\rho'_\tau(0) = 0$. Thus, (A.15) implies $\rho_\tau$ satisfies a second order ODE – for $\rho \neq 0$ – with these initial conditions. We can immediately conclude the following lemma:

**Lemma A.16.** Let $\rho_0(x_1) = \sqrt{1 - x_1^2}$ defined on $[-1, 1]$. For any $\epsilon \in (0, 1)$, there exists $\tau_\epsilon > 0$ such that for $0 < |\tau| < \tau_\epsilon$, $\rho_\tau$ restricted to the interval $[-1 + \epsilon, 1 - \epsilon]$ depends smoothly on $\tau$ and

$$
 \|\rho_0 - \rho_\tau : C^k([-1 + \epsilon, 1 - \epsilon])\| \leq C(\epsilon, k)|\tau|.
$$
Remark A.17. Given the definition of the norms, it is straightforward to see that the previous lemma implies that, given any $b \gg 1$ there exists $T > 0$ such that for all $0 < |r| \leq T$,

$$\|u : C^{k,b}([-b,b] \times S^{n-1}, g_r)\| \sim_{C(b,k,b)} \|u : C^{k,b}([-b,b] \times S^{n-1}, g_0)\|.$$ 

We also consider the geometry of the necks. To that end, we denote the conformal parametrization of the catenoid $Y_C : \mathbb{R} \times S^{n-1} \to \mathbb{R}^{n+1}$:

$$Y_C(t, \Theta) := (k_C(t), r_C(t)\Theta)$$

where $(k_C')^2 + (r_C')^2 = r_C^2$. Note that since $H \equiv 0$, the force vector is determined only by $\text{Force}(\Gamma) := \int_\Gamma \eta$. Observe that at $r_C = 1$, $\text{Force}(\Gamma) = \omega_{n-1} e_1$. For any $\Gamma$, an explicit calculation implies that

$$\text{Force}(\Gamma) = \frac{1}{r_C} \int_\Gamma (k_C', r_C') r_C^{n-1} dg_{S^{n-1}} = k_C' r_C^{n-2} \omega_{n-1} e_1.$$ 

Therefore $k_C' = r_C^{-2-n}$ and setting $r_C(t) = e^{uw_C(t)}$, we have that $(w_C')^2 = 1 - r^{2-2n}$.

Denote the unit normal $\nu_C(t, \Theta) = (w_C', -(k_C'/r_C)\Theta)$.

Of particular interest will be the Jacobi field given by the dilation vector field. We denote it

$$f_C(t, \Theta) := Y_C \cdot \nu_C = w_C' k_C - k_C'.$$

We determine estimates for $k_C$ by observing first that

$$\frac{dk_C}{dr_C} = \frac{1}{\sqrt{r_C^{2n-2} - 1}}.$$ 

Using the above, the asymptotics of the immersion (as $r_C \to \infty$) are then given by a Gamma function. That is

$$k_C(r_C) = \frac{\sqrt{\pi} \Gamma(\frac{3n-2}{2n-1})}{(n-2) \Gamma(\frac{3n-4}{2n-1})} - \frac{1}{n-2} r_C^{2-n} + O(r_C^{4-3n})$$

where (using Maple)

$$T_n := \int_1^\infty \frac{dr}{\sqrt{r^{2n-2} - 1}} = \frac{\sqrt{\pi} \Gamma(\frac{3n-4}{2n-1})}{(n-2) \Gamma(\frac{3n-2}{2n-1})}.$$ 

Therefore, for large $r$,

$$f_C(r) = -r^{2-n} + \sqrt{1 - r^{2-2n}} \left( T_n - \frac{1}{n-2} r^{2-n} + O(r^{4-3n}) \right) = T_n - \frac{n-1}{n-2} r^{2-n} + O(r^{2-2n}).$$

Lemma A.21. Given any $b \gg 1$, there exists $T > 0$ such that the following holds:

For $0 < |\tau| < T$, define the immersion

$$Y_C^\tau(t, \Theta) := \begin{cases} \frac{1}{r^{\min}} Y_r(t + p_\tau, \Theta) & \text{if } \tau > 0 \\ \frac{1}{r^{\min}} Y_r(-t + p_\tau, \Theta) & \text{if } \tau < 0. \end{cases}$$

Then

$$\|Y_C^\tau - Y_C : C^k([-b,b] \times S^{n-1}, Y^*_{Cg_{R^{n+1}}})\| \leq C(k, b)|\tau|^{-\frac{1}{n-1}}.$$ 

An analogous result can be found in [23, equation 6.32].
Proof. For $R > 0$ and $|\beta|$ small, consider the immersion given by $Y_\beta(t, \Theta) = (k_\beta(t), r_\beta(t)\Theta)$ of a domain in $\mathbb{R} \times S^{n-1}$, where $Y_\beta(t, \Theta)$ is determined by

$$
\begin{align*}
(r_\beta')^2 + (k_\beta')^2 &= r_\beta^2 \\
r_\beta^{n-2}k_\beta' &= 1 + \beta(r_\beta^n - 1) \\
r_\beta(0) &= 1, \quad k_\beta(0) = 0
\end{align*}
$$

The family of immersions induced by this system varies smoothly in $\beta$. When $\beta = 0$, the system of ODEs gives the conformal embedding of the unit catenoid. Therefore, for any $R > 0$ there exists $\epsilon > 0$ such that for all $\beta \in [-\epsilon, \epsilon]$, $Y_\beta$ is a smooth immersion on $[-2R, 2R] \times S^{n-1}$.

We now consider the immersion for $\beta \neq 0$, calculating the mean curvature using the same method as done after (A.11). In this case,

$$A_\beta = k'_\beta g_{S^{n-1}} + \left( k''_\beta \left( \frac{r'_\beta}{r_\beta} \right) - k'_\beta \left( \left( \frac{r'_\beta}{r_\beta} \right)' + \left( \frac{r'_\beta}{r_\beta} \right)^2 \right) \right) dt^2$$

and with $r_{1, \beta} := \beta r_\beta + (1 - \beta)r_\beta^{1-n}$, $r_{2, \beta} := \beta r_\beta + (1 - \beta)(1 - n)r_\beta^{1-n}$, we can simplify the $dt^2$ term exactly as before. Thus,

$$nH = \frac{(n - 1)r_\beta r_{1, \beta} + r_\beta r_{2, \beta}}{r_\beta^2} = n\beta.$$

It follows that $Y_\beta$ is a rotationally symmetric CMC immersion with mean curvature equal to $\beta$ and $r_\beta(0) = 1$. Therefore, the immersion described by $|\beta|Y_\beta$ is a rotationally symmetric CMC surface with mean curvature $\pm 1$, where the sign is determined by the sign of $\beta$. Moreover, the radius of the meridian circle $|\beta|Y_\beta(0, \Theta)$ equals $|\beta|$. Thus $|\beta| = r_\beta^{1_\beta}$. For $\beta > 0$ the proof is complete as it follows immediately that $Y_\beta(t, \Theta) = \frac{1}{\beta}Y_\tau(t + p_\tau, \Theta)$ on the domain of interest. For $\beta < 0$, we need only observe that mean curvature $-1$ corresponds to a change of direction for the unit normal. The change of direction corresponds to the parameter change $t \mapsto -t$, giving a sign change on $k'_\beta$, in the definition of $Y_\tau$. $\square$

The change of parameter calculations.

Lemma A.22. We have the following asymptotics as $\tau \to 0$:

$$\lim_{\tau \to 0} \frac{P_\tau}{\log |\tau|} = -\frac{1}{n-1}, \quad \lim_{\tau \to 0} |\tau| \frac{dP_\tau}{d\tau} = -\frac{1}{n-1}.$$

Proof. First observe that we now consider the setting where $\tau > 0$ as there are only minor changes necessary when $\tau < 0$ that produce the same asymptotics.

We begin with the formulation

$$P_\tau = \int_{r(0)}^{r_{\tau}^{\max}} \frac{dt}{dr} dr = \int_{r_{\tau}^{\min}}^{r_{\tau}^{\max}} \frac{dr}{r\sqrt{1 - (r + \tau r_\tau^{1-n})^2}}.$$

Given the estimates for $r_{\tau}^{\max}, r_{\tau}^{\min}$, one can easily check that for any $\epsilon > 0$ there exists $0 < \delta \ll 1$ such that for all $r \in (r_{\tau}^{\min}/\delta, r_{\tau}^{\max}/\delta)$, the function $(r + \tau r_\tau^{1-n}) \in (-\epsilon, \epsilon)$. Fixing an $\epsilon > 0$ we choose an appropriate $\delta > 0$ and subdivide the integral into three parts so that

$$P_\tau = I_+ + I_\Lambda + I_-,$$

where

$$I_+ := \int_{r_{\tau}^{\max}/\delta}^{r_{\tau}^{\max}} \frac{dr}{r\sqrt{1 - (r + \tau r_\tau^{1-n})^2}}, \quad I_- := \int_{r_{\tau}^{\min}/\delta}^{r_{\tau}^{\min}} \frac{dr}{r\sqrt{1 - (r + \tau r_\tau^{1-n})^2}}.$$

(A.23)
To estimate the contribution of $I_+$ in (A.23), consider the change of variables $u = -1 + r/r_{\tau}^{\max}$. Then
\[
I_+ = \int_{\delta^{-1}}^{0} \frac{du}{(u + 1)\sqrt{1 - (r_{\tau}^{\max}(u + 1) + \tau(r_{\tau}^{\max}(u + 1))^{1-n})^2}}.
\]
We expand
\[
r_{\tau}^{\max}(u + 1) + \tau(r_{\tau}^{\max}(u + 1))^{1-n} = r_{\tau}^{\max} + \tau r_{\tau}^{\max} 1 - n + r_{\tau}^{\max} u + \tau(r_{\tau}^{\max})^{1-n}(1 + u)^{1-n} - 1.
\]
Thus
\[
(r_{\tau}^{\max}(u + 1) + \tau(r_{\tau}^{\max}(u + 1))^{1-n})^2 = 1 + (r_{\tau}^{\max})^2 u^2 + 2(r_{\tau}^{\max})^2 u + 2\tau(r_{\tau}^{\max})^{2-n} u(1 + u)^{1-n} + \tau^2(r_{\tau}^{\max})^{2-2n}((1 + u)^{2-2n} - 1).
\]
The square root term thus simplifies to
\[
f^{\max}_\tau(u) := r_{\tau}^{\max} \sqrt{-(u^2 + 2u + 2\tau(r_{\tau}^{\max})^{-n} u(1 + u)^{1-n} + \tau^2(r_{\tau}^{\max})^{-2n}((1 + u)^{2-2n} - 1))}.
\]
Let
\[
F^{\max}_\tau(u) := \frac{1}{(u + 1)f^{\max}_\tau(u)}.
\]
When $\tau$ is sufficiently small, there exists $C_{\delta} > 0$ independent of $\tau$ such that for $u \in [\delta - 1, 0)$
\[
|F^{\max}_\tau(u)| \leq \frac{C_{\delta}}{\sqrt{|u|}} \quad \text{and} \quad \left| \frac{dF^{\max}_\tau}{d\tau}(u) \right| \leq \frac{C_{\delta}}{\sqrt{|u|}}.
\]
Then by the dominated convergence theorem,
\[
(A.24) \quad \left| \frac{d}{d\tau} I_+ \right| \leq \int_{\delta^{-1}}^{0} \left| \frac{dF^{\max}_\tau}{d\tau}(u) \right| du \leq C_{\delta}.
\]
To estimate the contribution from the second integral in (A.23), recall that we have chosen $\delta$ so that $(r + \tau r^{1-n}) \in (-\epsilon, \epsilon)$. Therefore,
\[
I_\Lambda = (1 + O(\epsilon)) \int_{r_{\tau}^{\min}/\delta}^{r_{\tau}^{\max}/\delta} dr \frac{r}{r^{n}}.
\]
By inspection
\[
\int_{r_{\tau}^{\min}/\delta}^{r_{\tau}^{\max}/\delta} \frac{dr}{r} = -\log |\tau|^{1/2} + 2 \log \delta + O(|\tau|^{1/2}).
\]
Note also that
\[
\left| \frac{d}{d\tau} \frac{1}{\sqrt{1 - (r + \tau r^{1-n})^2}} \right| = \left| \frac{r^{-n}(r + \tau r^{1-n})}{r(1 - (r + \tau r^{1-n})^2)^{3/2}} \right| \leq \epsilon r^{-n}.
\]
We calculate the derivative
\[
(A.25) \quad \frac{d}{d\tau} I_\Lambda = (1 + O(\epsilon)) \left( \frac{d}{d\tau} \frac{r_{\tau}^{\max}}{r_{\tau}^{\min}} - \frac{d}{d\tau} \frac{r_{\tau}^{\min}}{r_{\tau}^{\max}} + \epsilon \int_{r_{\tau}^{\min}/\delta}^{r_{\tau}^{\max}/\delta} r^{-n} dr \right) = -\frac{1}{n - 1} |\tau|^{-1} (1 + O(\epsilon)).
\]
Finally, to determine the contribution made by the third integral in (A.23), make the change of variables $u = -1 + r/r_{\tau}^{\min}$ so that
\[
I_- = \int_{0}^{1+1/\delta} \frac{du}{(u + 1)\sqrt{1 - (r_{\tau}^{\min}(u + 1) + \tau(r_{\tau}^{\min}(u + 1))^{1-n})^2}}.
\]
We observe that the square root can be simplified to

\[
\begin{align*}
    f_{\tau}^{\min}(u) = |\tau|^{(r_{\tau}^{\min})^{1-n}} \sqrt{1 - (1 + u)^{2-2n} - 2\tau^{-1}(r_{\tau}^{\min})^{n}u(1 + u) - |\tau|^{-2}(r_{\tau}^{\min})^{2n}(u^2 + 2u)}
\end{align*}
\]

Let \( F_{\tau}^{\min} := \frac{1}{(u+1)f_{\tau}^{\min}(u)} \). To determine the derivative of \( F_{\tau}^{\min} \), recall that \( r_{\tau}^{\min} + \tau(r_{\tau}^{\min})^{1-n} = \pm 1 \). Therefore

\[
    \frac{d}{d\tau} f_{\tau}^{\min} = \frac{(r_{\tau}^{\min})^{1-n}}{\tau(n-1)(r_{\tau}^{\min})^{-n} - 1} = \frac{r_{\tau}^{\min} - 1}{n-1} \left( 1 + O(|\tau|^{\frac{1}{n-1}}) \right)
\]

and thus, using \( \pm, \mp \) to distinguish the two cases for the sign of \( \tau \),

\[
    \frac{d}{d\tau} (|\tau|^{-1}(r_{\tau}^{\min})^{n-1}) = \mp|\tau|^{-2}(r_{\tau}^{\min})^{n-1} + |\tau|^{-1}(n-1)(r_{\tau}^{\min})^{n-2} \pm \frac{r_{\tau}^{\min}|\tau|^{-1}}{n-1} \left( 1 + O(|\tau|^{\frac{1}{n-1}}) \right)
\]

\[
    = |\tau|^{-2}(r_{\tau}^{\min})^{n-1}O(|\tau|^{\frac{1}{n-1}})
\]

\[
    = |\tau|^{-1}O(|\tau|^{\frac{1}{n-1}}).
\]

Using similar techniques, we compute

\[
    |\tau|^{-1}(r_{\tau}^{\min})^{n-1} \frac{d}{d\tau} (r_{\tau}^{\min})^{n} = |\tau|^{-1} \left( \frac{1}{n-1} + O(|\tau|^{\frac{1}{n-1}}) \right),
\]

and

\[
    |\tau|^{-1}(r_{\tau}^{\min})^{n-1} \frac{d}{d\tau} (|\tau|^{-2}(r_{\tau}^{\min})^{2n}) = |\tau|^{-1} \left( \frac{n+1}{n-1} + O(|\tau|^{\frac{2}{n-1}}) \right).
\]

Therefore

\[
    \frac{dF_{\tau}^{\min}}{d\tau} = |\tau|^{-1}O(|\tau|^{\frac{1}{n-1}}) \left( |\tau|(r_{\tau}^{\min})^{1-n} + \frac{2u}{(u+1)f_{\tau}^{\min}(u)} + \frac{u^2 + 2u}{(n-1)(f_{\tau}^{\min}(u))^3} \right).
\]

For \( 0 \leq u \leq \frac{1}{2} \), there exists \( C > 0 \), independent of \( \tau \) and of \( \delta \), so that

\[
    |F_{\tau}^{\min}(u)| \leq \frac{C}{\sqrt{u}} \text{ and } |\tau| |\frac{dF_{\tau}^{\min}}{d\tau}| \leq \frac{O(|\tau|^{\frac{1}{n-1}})}{\sqrt{u}}.
\]

For \( \frac{1}{2} \leq u \leq \frac{1}{\delta} - 1 \), there exists \( C > 0 \), independent of \( \tau \) and \( \delta \), so that

\[
    |F_{\tau}^{\min}(u)| \leq C \text{ and } |\tau| |\frac{dF_{\tau}^{\min}}{d\tau}| \leq CuO(|\tau|^{\frac{1}{n-1}}).
\]

Therefore,

\[
    (A.26) \quad \left| \frac{d}{d\tau} \int_{0}^{\frac{1}{\delta} - 1} F_{\tau}^{\min} \right| \leq -C \frac{1}{\delta^2 |\tau|} O(|\tau|^{\frac{1}{n-1}}).
\]

Combining (A.24), (A.25), and (A.26), we observe that

\[
    \frac{dF_{\tau}}{d\tau} = |\tau|^{-1} \left( \frac{1}{n-1} + C\delta^{-2}O(|\tau|^{\frac{1}{n-1}}) + O(\epsilon) + C_\delta \epsilon \right).
\]

Noting that \( \epsilon > 0 \) can be chosen arbitrarily small, taking \( \tau \to 0 \) implies the result. \( \Box \)
Lemma A.27. Let $Y_\tau$ be a Delaunay immersion with axis along the $x_1$-axis. For $\phi \in C^\infty(M)$, let $(Y_\tau)_\phi$ be a CMC graph over $Y_\tau$ possessing the same axis of symmetry as $Y_\tau$. The force through the meridian $\{t\} \times S^{n-1}$ of $(Y_\tau)_\phi$ is given by

\begin{equation}
\text{Force}(\Gamma) = \omega_{n-1} \left( \tau + r^{n-2} \left( \phi' \hat{f}_0 - \phi \hat{f}'_0 \right) + \text{h.o.t.} \right) e_1
\end{equation}

Here h.o.t. stands for terms that are quadratic and higher in $\phi$ and its derivatives and $\hat{f}_0 := \nu_\tau \cdot e_1 = \frac{\nu_\tau}{\tau}$.

Proof. Recall that

$$(Y_\tau)_\phi(x) := Y_\tau(x) + \phi(x)N_{Y_\tau}(x)$$

for $\phi \in C^\infty(M)$. Let $\Gamma \subset (Y_\tau)_\phi$ be an $n-1$ chain with a fixed value for $x_1$ and let $K$ be the $n$ sphere in this hyperplane such that $\partial K = \Gamma$. The radius of $\Gamma$ is $r - \phi r_1$. To compute $\text{Force}(\Gamma)$, observe that for $r_2 := r'/w'$,

$$\partial_t((Y_\tau)_\phi) = \left( k' + \phi' \hat{f}_0 + \phi \hat{f}'_0, \left( r' - k' \phi' + k \hat{f}_0 \right) / r \right) \Theta,$$

$$\{\partial_t((Y_\tau)_\phi)\}^2 = r^2 - 2\phi r r_2 + \text{h.o.t.}$$

Thus

$$\text{Force}(\Gamma) = \int_\Gamma \eta - n \int_K N_K$$

$$= \left( \left( r_1 + \phi' \hat{f}_0 + \phi \hat{f}'_0 + \phi r_2 + \text{h.o.t.} \right) \left( r - \phi r_1 \right)^{n-1} - \left( r - \phi r_1 \right)^n \right) \omega_{n-1} e_1$$

$$= \left( r - \phi r_1 \right)^{n-1} \left( r + \tau r^{1-n} + \phi' \hat{f}_0 + \phi \hat{f}'_0 + \phi r_2 - r + \phi r_1 + \text{h.o.t.} \right) \omega_{n-1} e_1$$

$$= \left( \tau + r^{n-2} \left( \phi' \hat{f}_0 - \phi \hat{f}'_0 \right) + \text{h.o.t.} \right) \omega_{n-1} e_1$$

$\square$

Lemma A.29. We have the following asymptotics as $\tau \to 0$ (recall A.19):

\begin{equation}
\lim_{\tau \to 0} |\tau|^{1-n} \hat{p}_\tau = T_n; \quad \lim_{\tau \to 0} |\tau|^{n-2} \frac{d\hat{p}_\tau}{d\tau} = \frac{T_n}{n-1}.
\end{equation}

Proof. We first define motions on the domain cylinders and the ambient $\mathbb{R}^{n+1}$. Let $T_x, R_x : \mathbb{R} \times S^{n-1} \to \mathbb{R} \times S^{n-1}$ such that $T_x(t, \Theta) = (t + x, \Theta)$ and $R_x(t, \Theta) = (2x - t, \Theta)$. Thus, $T_x$ is a translation by $x$ and $R_x$ a reflection about $(x, \Theta)$ on the cylindrical domain. Let $\hat{T}_y, \hat{R}_y : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $\hat{T}_y(p) := p + y e_1$ and $\hat{R}_y(\sum_{i=1}^{n+1} a_i e_i) := (2y - a_1)e_1 + \sum_{i=2}^{n+1} a_i e_i$. Thus $\hat{T}_y$ is a translation by $y e_1$ and $\hat{R}_y$ is a reflection of the first component about $y e_1$. Consider an admissible $\tau$, a $\sigma$ near $\tau$, and three immersions $X := Y_\tau, Y := Y_\sigma$ and $Z := \hat{T}_{p_\tau - p_\sigma} \circ Y \circ T_{p_\tau - p_\sigma}$ defined on $[-2p_\tau, 2p_\tau] \times S^{n-1}$ by

$$\hat{R}_0 \circ X = X \circ R_0; \quad \hat{R}_0 \circ Y = Y \circ R_0; \quad \hat{R}_1 \circ Z = Z \circ R_0.$$

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For $\sigma$ sufficiently close to $\tau$ there exist smooth functions $\phi_{\sigma}, \psi_{\sigma}$ and smooth diffeomorphisms $D_Y, D_Z$, of the identity map on $[-2p_{\tau}, 2p_{\tau}] \times S^{n-1}$, such that

$$Y = X_{\phi_{\sigma}} \circ D_Y \quad Z = X_{\psi_{\sigma}} \circ D_Z.$$  

By the symmetries, $\phi_{\sigma}'(0) = 0, \psi_{\sigma}'(p_{\tau}) = 0$. Also note that $\phi_{\sigma}, \psi_{\sigma}, D_Y, D_Z$ all depend on and are smooth in $\sigma$. We now calculation the linearization of the normal part of each of these variations. Expanding the immersion for $Z$, we observe that

$$Z = X_{\psi_{\sigma}} \circ D_Z = X \circ D_Z + (\psi_{\sigma} \circ D_Z)(\nu_X \circ D_Z).$$

Note that $D_Z : \mathbb{R}^2 \times S^{n-1} \rightarrow \mathbb{R}^2 \times S^{n-1}$ where $(\sigma, t, \Theta) \mapsto (\sigma, T_1(\sigma, t, \Theta), T_2(\sigma, t, \Theta))$ so

$$\frac{\partial}{\partial \sigma}(\psi_{\sigma} \circ D_Z) = \frac{\partial \psi_{\sigma}}{\partial \sigma} + \frac{\partial \psi_{\sigma}}{\partial T_1} \frac{\partial T_1}{\partial \sigma} + \frac{\partial \psi_{\sigma}}{\partial T_2} \frac{\partial T_2}{\partial \sigma}.$$  

Evaluating this at $\sigma = \tau$, note that $\psi_{\tau} \equiv 0$ so $\frac{\partial \psi_{\sigma}}{\partial T_1} = \frac{\partial \psi_{\sigma}}{\partial T_2} = 0$ for $\sigma = \tau$ and thus

$$\frac{\partial}{\partial \sigma}|_{\sigma=\tau} (\psi_{\sigma} \circ D_Z) = \frac{\partial \psi_{\sigma}}{\partial \sigma}|_{\sigma=\tau}.$$  

Further, observe that both $\frac{\partial}{\partial \sigma}(X \circ D_Z), \frac{\partial}{\partial \sigma}(\nu_X \circ D_Z)$ are in the tangent space of $X$ when $\sigma = \tau$ and thus

$$\frac{\partial Z}{\partial \sigma}|_{\sigma=\tau} \cdot \nu_X = \frac{\partial \psi_{\sigma}}{\partial \sigma}|_{\sigma=\tau}.$$

Performing a similar calculation for $Y$, we define

$$\hat{\phi}_{\sigma} := \frac{\partial \phi_{\sigma}}{\partial \sigma}|_{\sigma=\tau} = \frac{\partial Y}{\partial \sigma}|_{\sigma=\tau} \cdot \nu_X; \quad \hat{\psi}_{\sigma} := \frac{\partial \psi_{\sigma}}{\partial \sigma}|_{\sigma=\tau} = \frac{\partial Z}{\partial \sigma}|_{\sigma=\tau} \cdot \nu_X.$$

We now determine an equation for $\frac{\partial \phi_{\sigma}}{\partial \sigma}|_{\sigma=\tau}$ using the second immersion of $Z$. Since $Y$, and thus $Z$, depends smoothly on $\sigma$, we rewrite the immersion as

$$Z(\sigma, t, \Theta) = (\hat{p}_{\tau} - \hat{p}_{\sigma})e_1 + Y \circ T_{P_{\sigma} - P_{\tau}}(\sigma, t, \Theta).$$

Since

$$Y \circ T_{P_{\sigma} - P_{\tau}}(\sigma, t, \Theta) = Y(\sigma, P_{\sigma} - P_{\tau} + t, \Theta) := Y(T_1(\sigma, t, \Theta), T_2(\sigma, t, \Theta), T_3(\sigma, t, \Theta)),$$

we calculate

$$\frac{\partial}{\partial \sigma} Y \circ T_{P_{\sigma} - P_{\tau}} = \frac{\partial Y}{\partial T_1} \frac{\partial T_1}{\partial \sigma} + \frac{\partial Y}{\partial T_2} \frac{\partial T_2}{\partial \sigma} + \frac{\partial Y}{\partial T_3} \frac{\partial T_3}{\partial \sigma}.$$  

The second two terms above are in the tangent space of $X$ when $\sigma = \tau$. Moreover, $\frac{\partial Y}{\partial T_1} \frac{\partial T_1}{\partial \sigma} = \frac{\partial Y}{\partial T_2}$. Therefore, comparing the normal components of the two linearizations of $Z$ we observe that

$$(A.31) \quad \hat{\psi}_{\sigma}(t) = \hat{\phi}_{\sigma}(t) - \frac{d\hat{\phi}_{\sigma}}{d\sigma}|_{\sigma=\tau} \cdot e_1 \cdot \nu_X = \hat{\phi}_{\sigma}(t) - \frac{d\hat{\phi}_{\sigma}}{d\sigma}|_{\sigma=\tau} \hat{f}_0(t).$$

Consider the force calculation of (A.28) where $\hat{\phi}$ is replaced by either $\hat{\phi}_{\sigma}$ or $\hat{\psi}_{\sigma}$. Note that the functions $r, w$ are independent of $\sigma$. If we differentiate with respect to $\sigma$ and evaluate at $\sigma = \tau$ we observe that

$$r^{n-2}(\hat{\phi}_{\sigma}' \hat{f}_0 - \hat{\phi}_{\sigma}' \hat{f}_0') = 1 \quad \text{and} \quad r^{n-2}(\hat{\psi}_{\sigma}' \hat{f}_0 - \hat{\psi}_{\sigma}' \hat{f}_0') = 1.$$
Therefore, for any region \([x, y]\) where \(\hat{f}_0(t) \neq 0\), we determine that
\[
\left(\frac{\hat{\phi}}{\hat{f}_0}\right)' = \left(\frac{\hat{\psi}_0}{\hat{f}_0}\right)' = \frac{r^{2-n}}{\hat{f}_0^2}
\]
and thus
\[
\frac{\dot{\phi}_0}{\hat{f}_0} = \frac{\dot{\phi}_0}{\hat{f}_0}(x) + \int_x^t \frac{r^{2-n}(s)}{\hat{f}_0^2(s)} ds,
\quad \frac{\dot{\psi}_0}{\hat{f}_0} = \frac{\dot{\psi}_0}{\hat{f}_0}(y) - \int_t^y \frac{r^{2-n}(s)}{\hat{f}_0^2(s)} ds.
\]
Therefore, substituting into (A.31), for any region \([x, y]\) where \(\hat{f}_0(t) \neq 0\),
\[
(A.32) \quad \frac{d\hat{p}_\sigma}{ds}\bigg|_{\sigma=\tau} = \int_x^y \frac{r^{2-n}(s)}{\hat{f}_0^2(s)} ds + \frac{\dot{\phi}_0}{\hat{f}_0}(x) - \frac{\dot{\psi}_0}{\hat{f}_0}(y).
\]

By a change of variables, we rewrite (recalling \(\hat{p}_0 < 0\) on the domain of integration),
\[
\int_x^y \frac{r^{2-n}(s)}{\hat{f}_0^2(s)} ds = \int_{r(x)}^{r(y)} \frac{r^{2-n}}{|\hat{f}_0|^3(r)} dr = \int_{r(x)}^{r(y)} \frac{r^{1-n}}{|\hat{f}_0|^3(r)} dr.
\]
We now choose \(x, y\) so as to clearly estimate all of the terms in (A.32).

Recall that \(\hat{f}_0 = -\sqrt{1 - (r + \tau r^{1-n})^2}\). Therefore, as in the proof of the asymptotics for \(p_r\), for any \(\epsilon > 0\), we can choose \(0 < \delta \ll 1\) such that \(|\hat{f}_0|^{-3}(r) \in [1-\epsilon, 1]\) for all \(r \in [r^{\min}_\tau/\delta, r^{\max}_\tau\delta]\).

Choose, \(0 < x < y < p_r\) so that \(r(y) = r^{\min}_\tau/\delta, r(x) = r^{\max}_\tau\delta\). Then
\[
1 - \epsilon \left(\frac{r^{\min}_\tau}{\delta}\right)^{(2-n)} \leq \int_{r^{\min}_\tau/\delta}^{r^{\max}_\tau/\delta} \frac{r^{1-n}}{|\hat{f}_0|^3(r)} dr \leq \frac{1}{n - 2} \left(\frac{r^{\min}_\tau}{\delta}\right)^{(2-n)} + O(1).
\]

Observe that the construction of \(\hat{\phi}\) implies \(\hat{\phi}(0) = 0, |\hat{\phi}(0)| \leq C\) independent of \(\tau\). Moreover, \(\hat{\phi}(t)\) satisfies \(\hat{\phi}' \hat{f}_0 - \hat{\phi}'' \hat{f}_0 = r^{2-n}\) and on \([0, x]\) the coefficients of this ODE are uniformly bounded independent of \(\tau\). Thus, for \(\tau\) sufficiently small, \(\hat{\phi}(x)/\hat{f}_0(x) = O(1)\).

Finally, we consider the value of \(\hat{\psi}_\sigma(y)/\hat{f}_0(y)\). Note that for any \(R > 0\), Lemma A.21 implies that for \(\tau\) sufficiently small, on \([p_r - R, p_r + R] \times S^{n-1}\), \(\hat{\psi}_\sigma\) behaves like a multiple of the dilation Jacobi field on the unit catenoid. Indeed,
\[
\hat{\psi}_\sigma(t) = c f_C(t - p_r)(1 + O(|\tau|^{-1/2}))
\]
for some constant \(c\). By calculation, \(\hat{\psi}_\sigma(p_r) = \pm \frac{1}{n-1} |\tau|^{2-n} (1 + O(|\tau|^{-1/2}))\). The sign on this term is positive for \(\tau > 0\) since \(\psi_\sigma(p_r) \approx r^{\min}_\tau - r^{\min}_\sigma\) as the normal points inward. For \(\tau < 0\), the normal points outward and \(\dot{\psi}_\sigma\) and thus \(\hat{\psi}_\sigma\) changes sign.

Since \(f_C(0) = 1\), we observe that \(c = -\frac{1}{n-1} |\tau|^{2-n}\). (The sign is negative since when \(\tau > 0\) the normals for \(Y_\tau, Y_C\) agree but when \(\tau < 0\) the normals point in opposite directions). By substitution, using (A.20),
\[
f_C(y - p_r) = T_n(1 + O(\delta^{2n-2})) - \frac{n - 1}{n - 2} \delta^{n-2}
\]
It follows that
\[
\hat{\psi}_\sigma(y) = -\frac{1}{n-1} |\tau|^{2-n} T_n(1 + O(\delta^{2n-2})) + \frac{1}{n - 2} |\tau|^{2-n} \delta^{n-2}
\]
Inserting the estimates into (A.32), we observe that
\[
\frac{d\hat{p}_\sigma}{d\sigma}\bigg|_{\sigma=\tau} = \frac{1}{n-1}\left|\tau\right|^{\frac{2-n}{n-1}}T_n(1 + o(\delta)) + O(1).
\]

\[\square\]

Appendix B. Quadratic Estimates

For completeness, we include here a proposition we will need. The proposition is analogous to the ones in the appendices of [44, 21]. We have adapted it here for our purposes, though the proof is identical to that in [44]. Let \(X : D \to U\) be an immersion of a disk of radius \(1/10\) in \(\mathbb{R}^n\) into an open cube \(U \subset \mathbb{R}^{n+1}\) equipped with a metric \(g\). Assume \(\text{dist}_g(X(D), \partial U) > 1\) and there exists \(c_1 > 0\) such that:

\[
(B.1) \quad \|\partial X : C^{2,\beta}(D, g_0)\| \leq c_1, \quad \|g_{ij}, g^{ij} : C^{4,\beta}(U, g_0)\| \leq c_1, \quad g_0 \leq c_1 X^*g,
\]

where here \(\partial X\) represents the partial derivatives of the coordinates of \(X\), \(g^{ij}\) are the components of the inverse of the metric \(g\), and \(g_0\) denotes the standard Euclidean metric on \(D\) or \(U\) respectively. We note that (B.1) can be arranged by an appropriate magnification of the target, which we will exploit in order to make use of the following proposition.

Let \(\nu : D \to \mathbb{R}^{n+1}\) be the unit normal for the immersion \(X\) in the \(g\) metric. Given a function \(\phi : D \to \mathbb{R}\) which is sufficiently small, we define \(X_\phi : D \to U\) by

\[
(B.2) \quad X_\phi(p) := \exp_X(p)\phi(p)\nu(p)
\]

where here \(\exp\) is the exponential map with respect to the \(g\) metric. Then the following holds:

**Proposition B.3.** There exists a constant \(\epsilon(c_1) > 0\) such that if \(X\) is an immersion satisfying (B.1) and the function \(\phi : D \to \mathbb{R}\) satisfies

\[
\|\phi : C^{2,\beta}(D, g_0)\| \leq \epsilon(c_1)
\]

then \(X_\phi : D \to U\) is a well defined immersion by (B.2) and satisfies

\[
\|X_\phi - X - \phi \nu : C^{1,\beta}(D, g_0)\| \leq C(c_1)\|\phi : C^{2,\beta}(D, g_0)\|^2
\]

and

\[
\|H_\phi - H - L_X \nu \phi : C^{0,\beta}(D, g_0)\| \leq C(c_1)\|\phi : C^{2,\beta}(D, g_0)\|^2.
\]

Here \(H = \text{tr}_gA\) is the mean curvature of \(X\), defined with respect to the metric \(X^*g\) where \(A\) is the second fundamental form, \(H_\phi\) the mean curvature of \(X_\phi\), and \(L_X \nu \phi := \Delta_\phi + |A|^2\).

**Proof.** That the linear terms are as stated is well known and follows by a straightforward calculation we omit. The nonlinear terms are given by expressions of monomials consisting of contractions of derivatives of \(X, g_{ij}, g^{ij}\), the exponential map, and \(\phi\). This implies both the existence results and the estimate on the nonlinearity. \(\square\)

Appendix C. An easy result for flat annuli

Let \(\Lambda := [s_{\text{in}}, s_{\text{out}}] \times S^{n-1}\) and \(g_A := ds^2 + s^2g_{S^{n-1}}\). Let \(C^{\text{in}} := \{s_{\text{in}}\} \times S^{n-1}\) and \(C^{\text{out}} := \{s_{\text{out}}\} \times S^{n-1}\). The following result is well known, though we could not find a reference. We sketch the steps of the proof and leave a few technicalities to the reader.

**Proposition C.1.** Given \(\beta \in (0, 1)\) and \(\gamma \in (1, 2)\) there exist linear maps \(\mathcal{R}_A^{\text{out}}, \mathcal{R}_A^{\text{in}} : C^{0,\beta}(\Lambda, g_A) \to C^{2,\beta}(\Lambda, g_A)\) such that if \(E \in C^{0,\beta}(\Lambda, g_A)\) then either one of the following can occur:
(i) if \( V_{\text{out}} = R_{\text{out}}^A(E) \) then
- \( \mathcal{L}_{g_A} V_{\text{out}} = E \) on \( \Lambda \).
- \( V_{\text{out}}|_{\mathcal{C}_{\text{out}}} \in H_1[\mathcal{C}_{\text{out}}] \) and vanishes on \( C_{\text{in}} \).
- \( \| V_{\text{out}} : C^{2,\beta}(\Lambda, s, g_A, s') \| \leq C(\beta, \gamma) \| E : C^{0,\beta}(\Lambda, s, g_A, s^{\gamma-2}) \| \).

(ii) if \( V_{\text{in}} = R_{\text{in}}^A(E) \) then
- \( \mathcal{L}_{g_A} V_{\text{in}} = E \) on \( \Lambda \).
- \( V_{\text{in}}|_{\mathcal{C}_{\text{in}}} \in H_1[\mathcal{C}_{\text{in}}] \) and vanishes on \( C_{\text{out}} \).
- \( \| V_{\text{in}} : C^{2,\beta}(\Lambda, s, g_A, s^{2-n-\gamma}) \| \leq C(\beta, \gamma) \| E : C^{0,\beta}(\Lambda, s, g_A, s^{-n-\gamma}) \| \).

Recall that \( H_1[\mathcal{C}_{\text{in}}], H_1[\mathcal{C}_{\text{out}}], \phi_i \) are defined in 4.17.

**Proof.** First observe that \( \mathcal{L}_{g_A} = \Delta_{g_A} = \partial_{ss} + \frac{2-1}{s} \partial_s + \frac{1}{s} \Delta_{g,n-1} \). Recall 4.16. Let \( E := E_0 + E_1 + E_2 \) where \( E_0(s, \Theta) = E_0(s) \), \( E_1 = \sum_{i=1}^{n} E_i(s) \phi_i(\Theta) \), and \( E_2 = E - E_0 - E_1 \).

Let \( r_k = 2^{-k} \) and let \( A_k := B_{r_k} \setminus B_{r_{k+1}} \subset \mathbb{R}^n \) and \( \tilde{A}_k := A_{k-1} \cup A_k \cup A_{k+1} \). Let \( \{ \psi_k \}_{k \in \mathbb{N}} \) be a partition of unity of \( B_1 \) such that \( \psi_k \equiv 1 \) on \( A_k \) and \( \psi_k = 0 \) on \( \mathbb{R}^n \setminus \tilde{A}_k \). Finally, let \( E^k := \psi_k E \) and \( E^k_i = \psi_k E_i \) for \( i = 1, 2, 3 \).

We begin by considering \( E_2 \). Let \( u_k \) satisfy
\[
\begin{cases}
\mathcal{L}_{g_A} u_k = E^k_2 & \text{in } \Lambda, \\
u_k = 0 & \text{on } \partial \Lambda
\end{cases}
\]

Then since, for a fixed \( c \in \mathbb{R} \), \( \mathcal{L}_{g_A} u_k = c^{-2} E^2_2 \), by applying De Giorgi-Nash-Moser techniques and then Schauder theory we determine
\[
\| u_k : C^{2,\alpha}(\tilde{A}_k, s, g_A) \| \leq C(b) r^2_k \| E_k : C^{0,\alpha}(\tilde{A}_k, s, g_A) \|.
\]

As \( \mathcal{L}_{g_A} u_k = 0 \) on \( \Lambda \setminus \tilde{A}_k \) we observe that on each “tail”, the worst decay for \( u_k \) comes from the terms of the form \( a_k^{\text{in}} s^{2} + b_k^{\text{in}} s^{-n} \). The Dirichlet conditions imply that
\[
a_k^{\text{in}} = -b_k^{\text{in}} (s_{\text{in}})^{-n-2} \quad \text{and} \quad a_k^{\text{out}} = -b_k^{\text{out}} (s_{\text{out}})^{-n-2}.
\]

Thus, referring to each tail as \( u_k^{\text{in, out}} \), we note that
\[
u_k^{\text{in}} = b_k^{\text{in}} (s^{n} - (s_{\text{in}})^{-n-2} s^2) \quad \text{and} \quad u_k^{\text{out}} = b_k^{\text{out}} (s^{n} - (s_{\text{out}})^{-n-2} s^2).
\]

Moreover, the estimates imply
\[
|b_k^{\text{in}}| \leq C s_{\text{in}}^{n+2} \frac{r_{\text{in}}^{n+2}}{r^{n+2} s_{\text{in}}^{n+2}},
\]
\[
|b_k^{\text{out}}| \leq C r_{\text{out}}^{n+2} \frac{r_{\text{out}}^{n+2}}{r^{n+2} s_{\text{out}}^{n+2}}.
\]
Let $u_2 := \sum_\ell u_\ell$. Then $L_{gA} u_2 = E_2$. Finally, we consider the estimates. On any fixed dyadic $A_k$, note that $u_2 = \sum_{m<k} u_\text{in}^m + \sum_{m\geq k} u_\text{out}^m$. Thus,

$$
\|u_2 : C^{2,\beta}(A_k, s, gA, s^{2-n}-\gamma)\| \leq C r_k^{n+\gamma-2} \left( \sum_{m > r_k} \| E_m : C^{0,\beta}(A_m, s, gA)\| \right) + \sum_{m \leq r_k} \| E_m : C^{0,\beta}(A_m, s, gA)\|^{\gamma+n+2} + \sum_{m > r_k} \| E_m : C^{0,\beta}(A_m, s, gA)\|^{1-n+2} + \sum_{m \leq r_k} \| E_m : C^{0,\beta}(A_m, s, gA)\|^{n+2}.
$$

The estimate with decay $s^\gamma$ works similarly.

We now consider the low harmonic terms. Let $v_{0,0,\ell}$ denote the solution to the initial value problem $\partial_s f + \frac{n-1}{s} \partial_s f = E_\ell^f$ where $f(2^{-\ell}) = f(2^{-\ell}) = 0$ at $s = 2^{-\ell-1}$, choose $u_{0,0,\ell}(s) = \partial_s u_{0,0,\ell}(s) = 0$. Then $u_{0,0,\ell} = a_0 + b_0 s^{2-n}$ on $\Lambda \setminus B_\ell$ where $|a_0| \leq C 2^{-2\ell} \| E_\ell^f : C^{0,\alpha}(A, s, gA)\|$, $|b_0| \leq C 2^{-n\ell} \| E_\ell^f : C^{0,\alpha}(A, s, gA)\|$. Now consider $\nu_{0,\ell} := \sum_{\ell=1}^{\infty} v_{0,0,\ell}$. First, $L_{gA} \nu_{0,\ell} = E_0$ on $A$ and $u_{0,\ell} = 0$ on $C_\text{in}, u_{0,\ell} \in C_\text{out}$. Moreover, consider any dyadic annulus $A_k \subset A$. Then on $A_k$, $u_{0,\ell} = \sum_{r_\ell < r_k} u_{0,\ell}$. We estimate

$$
\|u_{0,\ell} : (A_k, s, gA, s^{\gamma})\| \leq C \frac{r_\ell^{2\gamma-2}}{r_k^{\gamma}} \| E_\ell : C^{0,\beta}(A_\ell, s, gA)\| \leq C \frac{r_\ell^{2\gamma-2}}{r_k^{\gamma}} \| E : C^{0,\beta}(A, s, gA, s^{\gamma-2})\|.
$$

For $u_{0,\ell}^\text{in}$ we perform the identical construction but now prescribe data so that each $u_{0,0,\ell}^\text{in}$, so that $u_{0,0,\ell}^\text{in} = 0$ on $\mathbb{R}^n \setminus B_{2^{-\ell+1}}$.

On any $A_k$, $u_{0,\ell}^\text{in} = \sum_{r_\ell > r_k} u_{0,0,\ell}^\text{in}$. Now

$$
\|u_{0,\ell}^\text{in} : (A_k, s, gA, s^{2-n}-\gamma)\| \leq C \frac{r_\ell^{2-n-\gamma}}{r_k^{2-n-\gamma}} \| E_\ell : C^{0,\beta}(A_\ell, s, gA)\| \leq C \frac{r_\ell^{2-n-\gamma}}{r_k^{2-n-\gamma}} \| E : C^{0,\beta}(A, s, gA, s^{2-n})\| \leq C \frac{r_\ell^{2-n-\gamma}}{r_k^{2-n-\gamma}} \| E : C^{0,\beta}(A, s, gA, s^{2-n})\|.
$$

For $u_{1,\ell}^\text{out, in}$ we apply similar techniques as for $u_0$ to produce the necessary estimates. Taken together, these imply the result for $V^{\text{in}} := u_0^\text{in} + u_1^\text{in} + u_2$ and $V^{\text{out}} := v_0^\text{out} + v_1^\text{out} + u_2$. 

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