OPTIMAL BOUNDS FOR THE VOLUMES OF KÄHLER-EINSTEIN FANO MANIFOLDS

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Abstract. We show that any n-dimensional Ding semistable Fano manifold X satisfies that the anti-canonical volume is less than or equal to the value $(n+1)^n$. Moreover, the equality holds if and only if X is isomorphic to the n-dimensional projective space. Together with a result of Berman, we get the optimal upper bound for the anti-canonical volumes of n-dimensional Kähler-Einstein Fano manifolds.

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1. Introduction

An n-dimensional smooth complex projective variety X is said to be a *Fano manifold* if the anti-canonical divisor $-K_X$ is ample. If $n \leq 3$, then the anti-canonical volume $((-K_X)^{\cdot n})$ is less than or equal to $(n+1)^n$, and the equality holds if and only if X is isomorphic to the projective space $\mathbb{P}^n$ by [Isk77, Isk78, MM81]. However, if $n \geq 4$, there exists an n-dimensional Fano manifold X such that $((-K_X)^{\cdot n}) > (n+1)^n$ holds (see [IP99, p. 128] for example). Recently, Berman and Berndtsson [BB11] conjectured that, if X admits *Kähler-Einstein metrics*, then the value $((-K_X)^{\cdot n})$ would be less than or equal to $(n+1)^n$. In fact, if X is toric, then the conjecture is true by [BB11, Theorem 1] and [NP14, Proposition 1.3]. Moreover, Berman and Berndtsson [BB12]

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proved the above conjecture under the assumption that X admits a $\mathbb{G}_m$-action with finite number of fixed points.

The purpose of this article is to refine the result [BB12] in full generality. The following is the main result in this article.

Theorem 1.1 (Main Theorem). *Let X be an n-dimensional Fano manifold admitting Kähler-Einstein metrics. If $((-K_X)^{\cdot n}) \geq (n+1)^n$, then $X \simeq \mathbb{P}^n$.*

The strategy to prove Theorem 1.1 is algebraic and is completely different from the argument in [BB12]. For a Fano manifold X, recall that, X admits Kähler-Einstein metrics if and only if the pair $(X, -K_X)$ is *K-polystable* (see [Tia97, Don05, CT08, Sto09, Mab08, Mab09, Bm16, CDS15a, CDS15b, CDS15c, Tia15]). In [Bm16], Berman proved the "only if" direction by viewing the slope of the Ding functional (see [Din88]) along a geodesic ray in the space of Kähler potentials. Berman also treated the case that X is a $\mathbb{Q}$*-Fano variety*, that is, a complex projective variety which is log terminal and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. In this article, we heavily use Berman's results [Bm16]. In Section 3 of this article, we introduce the notions of *Ding polystability* and *Ding semistability*. These notions are nothing but interpretations of Berman's formula for the slope of the Ding functional. The result in [Bm16, §3] shows that, if a $\mathbb{Q}$-Fano variety X admits Kähler-Einstein metrics, then X is Ding polystable (and also Ding semistable, see Theorem 3.2). A $\mathbb{Q}$-Fano variety X is said to be Ding semistable if the *Ding invariant* $\mathrm{Ding}(\mathcal{X}, \mathcal{L})$ satisfies that $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$ (see Section 3 in detail). The key idea for the proof of Theorem 1.1 is constructing specific test configurations of $(X, -rK_X)$ from any nonzero proper closed subscheme $Z \subset X$ and calculating those Ding invariants and taking the limit. The construction of test configurations is similar to the construction in [Fuj15a, Fuj15b]. We consider a *sequence* of test configurations. The following is one of the main consequence of the key idea.

Theorem 1.2 (=Theorem 4.10). *Let X be a $\mathbb{Q}$-Fano variety. Assume that X is Ding semistable. Take any nonempty proper closed subscheme $\emptyset \neq Z \subsetneq X$ corresponds to an ideal sheaf $0 \neq I_Z \subsetneq \mathcal{O}_X$. Let $\sigma\colon \hat{X} \to X$ be the blowup along Z, let $F \subset \hat{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\hat{X}}(-F) = I_Z \cdot \mathcal{O}_{\hat{X}}$. Then we have $\beta(Z) \geq 0$, where*

$$\beta(Z) := \mathrm{lct}(X; I_Z) \cdot \mathrm{vol}_X(-K_X) - \int_0^\infty \mathrm{vol}_{\hat{X}}\left(\sigma^*(-K_X) - xF\right) dx.$$

Note that, vol *is the volume function* (*see Definition 2.1*)*, and* $\mathrm{lct}(X; I_Z)$ *is the log canonical threshold of* I_Z *with respects to* X (*see Definition 2.6*)*.*

More generally, we construct a sequence of test configurations from *filtered linear series* in Section 4.2. From Theorem 1.2, we can immediately show the following corollary.

Corollary 1.3 (see Theorem 5.1)**.** *Let* X *be an* n*-dimensional* $\mathbb{Q}$*-Fano variety. Assume that* X *is Ding semistable. Then we have* $((-K_X)^{\cdot n}) \leq (n+1)^n$*.*

Remark 1.4. Recently, Berman, Boucksom and Jonsson showed in [BBJ15, Theorem 2.1] that Ding semistability of Fano manifolds is equivalent to K-semistability. Moreover, this is also true for $\mathbb{Q}$-Fano varieties. See [Fuj16, Corollary 3.4].

Theorem 1.1 is immediately obtained by Corollary 1.3 and a description of Seshadri constants (Theorem 2.3), together with the results [CMSB02] and [Keb02]. For detail, see Section 5.

The article is organized as follows. In Section 2, we recall the notions of the volume functions, Seshadri constants, log canonical thresholds and K-stability. We characterize Seshadri constants in terms of the volume function in Theorem 2.3. The theorem is important in order to characterize the projective space. In Section 3, we recall Berman's result [Bm16]. We introduce the notions of Ding invariants, Ding polystability and Ding semistability. Section 4 is the core of this article. In Section 4.1, we consider a general theory of the saturation of filtered linear series. In Section 4.2, we construct a sequence of semi test configurations from given filtered linear series. The construction is similar to the one in [Szé15]. Our construction enables us to calculate (a kind of) the limit of those Ding invariants via the saturation of the given filtration. See Corollary 4.9 in detail. In Section 4.3, motivated by the work of Ross and Thomas [RT07], we consider specific test configurations obtained by the natural filtered linear series coming from fixed closed subschemes. By taking the limit of those Ding invariants, we get Theorem 4.10. In Section 5, we prove Theorem 1.1. This is an immediate consequence of previous sections.

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Throughout this paper, we work in the category of algebraic (separated and of finite type) scheme over the complex number field $\mathbb{C}$. A *variety* means a reduced and irreducible algebraic scheme. For a projective surjective morphism $\alpha\colon \mathcal{X} \to C$ with $\mathcal{X}$ a normal variety and C a smooth curve, let $K_{\mathcal{X}/C} := K_{\mathcal{X}} - \alpha^* K_C$ be the relative canonical divisor. Moreover, for a closed point $t \in C$, let $\mathcal{X}_t$ be the scheme-theoretic fiber of α at $t \in C$. For a $\mathbb{Q}$-Fano variety X, ω is said to be a *Kähler-Einstein metric* on X if ω is a Kähler-Einstein metric on the smooth locus X^{sm} of X and the volume of ω on X^{sm} coincides with the value $((-K_X)^{\cdot n})$ (see [BBEGZ11, Bm16] for detail).

For any $c \in \mathbb{R}$, let $\lfloor c \rfloor \in \mathbb{Z}$ be the biggest integer which is not bigger than c and let $\lceil c \rceil \in \mathbb{Z}$ be the smallest integer which is not less than c.

2. Preliminaries

In this section, we recall some basic definitions and see those properties.

2.1. The volumes of divisors.

Definition 2.1 (see [Laz04a, Laz04b])**.** Let X be an n-dimensional projective variety. For a Cartier divisor L on X, we set

$$\mathrm{vol}_X(L) := \limsup_{k \to \infty} \frac{h^0(X, \mathcal{O}_X(kL))}{k^n/n!}.$$

We know that the limsup computing $\mathrm{vol}_X(L)$ is actually a limit (see [Laz04b, Example 11.4.7]). If L and L' are numerically equivalent, then $\mathrm{vol}_X(L) = \mathrm{vol}_X(L')$ (see [Laz04a, Proposition 2.2.41]). Moreover, we can extend uniquely to a continuous function

$$\mathrm{vol}_X\colon \mathrm{N}^1(X) \to \mathbb{R}_{\geq 0}$$

(see [Laz04a, Corollary 2.2.45]).

2.2. Seshadri constants, pseudo-effective thresholds.

Definition 2.2. Let X be a projective variety, L be an ample $\mathbb{Q}$-divisor on X, $\emptyset \neq Z \subsetneq X$ be a nonempty proper subscheme corresponds to an ideal sheaf $0 \neq I_Z \subsetneq \mathcal{O}_X$, $\sigma\colon \hat{X} \to X$ be the blowup along Z, and $F \subset \hat{X}$ be the Cartier divisor defined by the equation $\mathcal{O}_{\hat{X}}(-F) = I_Z \cdot \mathcal{O}_{\hat{X}}$.

(1) The *Seshadri constant* $\varepsilon_Z(L)$ *of* L *along* Z is defined by
$$\varepsilon_Z(L) := \sup\{x \in \mathbb{R}_{>0} \,|\, \sigma^*L - xF : \text{ ample}\}.$$

(2) The *pseudo-effective threshold* $\tau_Z(L)$ *of* L *along* Z is defined by
$$\tau_Z(L) := \sup\{x \in \mathbb{R}_{>0} \,|\, \sigma^*L - xF : \text{ big}\}.$$

If X is a $\mathbb{Q}$-Fano variety, then we write $\varepsilon_Z := \varepsilon_Z(-K_X)$ and $\tau_Z := \tau_Z(-K_X)$ for simplicity.

Theorem 2.3. *Let X be an n-dimensional projective variety with $n \geq 2$, L be an ample $\mathbb{Q}$-divisor on X, $p \in X$ be a smooth closed point, $\sigma\colon \hat{X} \to X$ be the blowup along p, and $F \subset \hat{X}$ be the exceptional divisor of σ.*

(1) *For any $x \in \mathbb{R}_{\geq 0}$, we have*
$$\mathrm{vol}_{\hat{X}}(\sigma^* L - xF) \geq ((\sigma^* L - xF)^{\cdot n}) = (L^{\cdot n}) - x^n.$$

(2) *Set $\Lambda_p(L) := \{x \in \mathbb{R}_{\geq 0} \,|\, \mathrm{vol}_{\hat{X}}(\sigma^* L - xF) = ((\sigma^* L - xF)^{\cdot n})\}$. Then we have*
$$\varepsilon_p(L) = \max\{x \in \mathbb{R}_{\geq 0} \,|\, y \in \Lambda_p(L) \text{ for all } y \in [0, x]\}.$$

Proof. Take any $k \in \mathbb{Z}_{>0}$ such that kL is Cartier. For any $j \in \mathbb{Z}_{>0}$, we have
$$\begin{aligned} h^0\left(jF, \sigma^*\mathcal{O}_X(kL)|_{jF}\right) &= \sum_{l=0}^{j-1}\binom{n-1+l}{n-1} = \binom{n-1+j}{n}, \\ h^i\left(jF, \sigma^*\mathcal{O}_X(kL)|_{jF}\right) &= 0 \ (\text{if } i > 0), \end{aligned}$$
since we have exact sequences
$$0 \to \mathcal{O}_{\mathbb{P}^{n-1}}(l) \to \sigma^*\mathcal{O}_X(kL)|_{(l+1)F} \to \sigma^*\mathcal{O}_X(kL)|_{lF} \to 0$$
for all $1 \leq l \leq j-1$.

(1) We can assume that $x \in \mathbb{Q}_{>0}$ since the function $\mathrm{vol}_{\hat{X}}(\sigma^* L - xF)$ is continuous. Take any sufficiently large $k \in \mathbb{Z}_{>0}$ with $kx \in \mathbb{Z}_{>0}$ and kL Cartier. Since
$$H^1\left(\hat{X}, \sigma^*\mathcal{O}_X(kL)\right) \simeq H^1\left(X, \mathcal{O}_X(kL)\right) = 0,$$
we get the following exact sequence:
$$0 \to H^0\left(\hat{X}, \mathcal{O}_{\hat{X}}(\sigma^*(kL) - kxF)\right) \to H^0\left(\hat{X}, \sigma^*\mathcal{O}_X(kL)\right)$$
$$\to H^0\left(kxF, \sigma^*\mathcal{O}_X(kL)|_{kxF}\right) \to H^1\left(\hat{X}, \mathcal{O}_{\hat{X}}(\sigma^*(kL) - kxF)\right) \to 0.$$
Thus we have
$$\begin{aligned} & h^0\left(\hat{X}, \mathcal{O}_{\hat{X}}(\sigma^*(kL) - kxF)\right) \\ \geq\ & h^0\left(X, \mathcal{O}_X(kL)\right) - \binom{n-1+kx}{n} = \frac{(L^{\cdot n}) - x^n}{n!}k^n + o(k^n). \end{aligned}$$

(2) Let a be the right-hand side of the equation in (2). For any nef divisor M, the volume of M is equal to the self intersection number. Thus the inequality $\varepsilon_p(L) \leq a$ is obvious. In particular, we have $a > 0$.

Take any $\varepsilon \in \mathbb{R}_{>0}$ such that $a-\varepsilon \in \mathbb{Q}_{>0}$. It is enough to show that $\sigma^*L-(a-\varepsilon)F$ is ample in order to show the inequality $\varepsilon_p(L)\geq a$. Fix $\delta \in \mathbb{Q}_{>0}$ such that $\delta < \varepsilon_p(L)$, that is, $\sigma^*L-\delta F$ is ample. Take any rational number t with

$$0\leq t<\min\left\{1, \frac{a-\varepsilon}{\delta}, \frac{\varepsilon}{a-\delta}\right\},$$

and set $x_t := (a-\varepsilon-t\delta)/(1-t)$. We note that $x_t \in (0,a)\cap\mathbb{Q}$. Moreover, we have

$$\sigma^*L-(a-\varepsilon)F-t(\sigma^*L-\delta F)=(1-t)(\sigma^*L-x_tF).$$

Take any sufficiently large $k\in\mathbb{Z}_{>0}$ with $kx_t\in\mathbb{Z}_{>0}$ and kL Cartier. Then, from the exact sequence

$$0\to\mathcal{O}_{\hat{X}}\left(\sigma^*(kL)-kx_tF\right)\to\mathcal{O}_{\hat{X}}\left(\sigma^*(kL)\right)\to\sigma^*\mathcal{O}_X(kL)|_{kx_tF}\to 0$$

and the previous arguments, we have

$$\begin{aligned}
&\limsup_k \frac{h^1\left(\hat{X},\mathcal{O}_{\hat{X}}(\sigma^*(kL)-kx_tF)\right)}{k^n/n!}\\
=\ &\limsup_k\Bigg(\frac{h^0\left(\hat{X},\mathcal{O}_{\hat{X}}(\sigma^*(kL)-kx_tF)\right)}{k^n/n!}\\
&+\frac{h^0\left(kx_tF,\sigma^*\mathcal{O}_X(kL)|_{kx_tF}\right)}{k^n/n!}-\frac{h^0(X,\mathcal{O}_X(kL))}{k^n/n!}\Bigg)\\
=\ &\mathrm{vol}_{\hat{X}}(\sigma^*L-x_tF)+x_t^n-(L^{\cdot n})=0
\end{aligned}$$

since $x_t\in\Lambda_p(L)$. Similarly, we have

$$h^i\left(\hat{X},\mathcal{O}_{\hat{X}}(\sigma^*(kL)-kx_tF)\right)=h^i\left(\hat{X},\mathcal{O}_{\hat{X}}(\sigma^*(kL))\right)=0$$

for any $i\geq 2$. Thus, by [dFKL07, §2.3 and Theorem 4.1], $\sigma^*L-(a-\varepsilon)F$ is ample. Therefore the assertion follows. $\square$

2.3. **Log canonical thresholds.**

Definition 2.4. (1) Let (Y,Δ) be a pair such that Y is a normal variety and Δ is a (possibly non-effective) $\mathbb{R}$-divisor on Y such that $K_Y+\Delta$ is $\mathbb{R}$-Cartier. The pair (Y,Δ) is said to be *sub log canonical* if $a(E,Y,\Delta)\geq -1$ holds for any proper birational morphism $\phi\colon\tilde{Y}\to Y$ with $\tilde{Y}$ normal and for any prime divisor E on $\tilde{Y}$, where $a(E,Y,\Delta):=\mathrm{ord}_E(K_{\tilde{Y}}-\phi^*(K_Y+\Delta))$.

(2) Let Y be a variety which is log terminal, $\mathfrak{a}_1, \dots, \mathfrak{a}_l \subset \mathcal{O}_Y$ be coherent nonzero ideal sheaves, and $c_1, \dots, c_l$ be (possibly negative) real numbers. The pair $(Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l})$ is said to be *sub log canonical* if $a(E, Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l}) \geq -1$ holds for any proper birational morphism $\phi\colon \tilde{Y} \to Y$ with $\tilde{Y}$ normal and for any prime divisor E on $\tilde{Y}$, where $a(E, Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l}) := \mathrm{ord}_E(K_{\tilde{Y}} - \phi^* K_Y) - \sum_{i=1}^{l} c_i \cdot \mathrm{ord}_E(\mathfrak{a}_i)$.

(3) Let Y be a variety which is log terminal, $r_0 \in \mathbb{Z}_{>0}$, $\{\mathfrak{a}_r\}_{r \geq r_0}$ be a graded family of coherent ideal sheaves on Y, that is, $\mathfrak{a}_r \cdot \mathfrak{a}_{r'} \subset \mathfrak{a}_{r+r'}$ holds for any r, $r' \geq r_0$, $\mathfrak{b}_1, \dots, \mathfrak{b}_l \subset \mathcal{O}_Y$ be coherent nonzero ideal sheaves, $c_1, \dots, c_l \in \mathbb{R}$ and $c \in \mathbb{R}_{>0}$. The pair $(Y, \mathfrak{a}_\bullet^{\cdot c} \cdot \mathfrak{b}_1^{\cdot c_1} \cdots \mathfrak{b}_l^{\cdot c_l})$ is said to be *sub log canonical* if $a(E, Y, \mathfrak{a}_\bullet^{\cdot c} \cdot \mathfrak{b}_1^{\cdot c_1} \cdots \mathfrak{b}_l^{\cdot c_l}) \geq -1$ holds for any proper birational morphism $\phi\colon \tilde{Y} \to Y$ with $\tilde{Y}$ normal and for any prime divisor E on $\tilde{Y}$, where $a(E, Y, \mathfrak{a}_\bullet^{\cdot c} \cdot \mathfrak{b}_1^{\cdot c_1} \cdots \mathfrak{b}_l^{\cdot c_l})$ is defined by the value

$$\mathrm{ord}_E(K_{\tilde{Y}} - \phi^* K_Y) - \sum_{i=1}^{l} c_i \cdot \mathrm{ord}_E(\mathfrak{b}_i) - \liminf_{r \to \infty} \frac{c \cdot \mathrm{ord}_E(\mathfrak{a}_r)}{r}.$$

Lemma 2.5. *Let Y be a variety which is log terminal, $r_0 \in \mathbb{Z}_{>0}$, $\{\mathfrak{a}_r\}_{r \geq r_0}$ be a graded family of coherent ideal sheaves on Y, $\mathfrak{b} \subset \mathcal{O}_Y$ be a coherent nonzero ideal sheaf, $c \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}$.*

(1) *Assume that there exists a sequence $\{a_r\}_{r \geq r_0}$ with $\lim_{r \to \infty} a_r = a$ and the pair $(Y, \mathfrak{a}_r^{\cdot (c/r)} \cdot \mathfrak{b}^{\cdot a_r})$ is sub log canonical for any sufficiently divisible $r \gg 0$. Then the pair $(Y, \mathfrak{a}_\bullet^{\cdot c} \cdot \mathfrak{b}^{\cdot a})$ is sub log canonical.*

(2) *Assume that there exists a coherent ideal sheaf $I \subset \mathcal{O}_Y$ such that $\mathfrak{a}_r \subset I^r$ for any $r \geq r_0$ and the pair $(Y, \mathfrak{a}_\bullet^{\cdot c} \cdot \mathfrak{b}^{\cdot a})$ is sub log canonical. Then the pair $(Y, I^{\cdot c} \cdot \mathfrak{b}^{\cdot a})$ is sub log canonical.*

Proof. Take any proper birational morphism $\phi\colon \tilde{Y} \to Y$ with $\tilde{Y}$ normal and a prime divisor E on $\tilde{Y}$. For any $r \geq r_0$ and $k \in \mathbb{Z}_{>0}$, we have

$$\frac{1}{kr} \mathrm{ord}_E(\mathfrak{a}_{kr}) \leq \frac{1}{kr} \mathrm{ord}_E(\mathfrak{a}_r^k) = \frac{1}{k} \mathrm{ord}_E(\mathfrak{a}_r).$$

Thus we have

$$\liminf_{r \to \infty} \frac{c \cdot \mathrm{ord}_E(\mathfrak{a}_r)}{r} = \liminf_{r \to \infty} \frac{c \cdot \mathrm{ord}_E(\mathfrak{a}_{kr})}{kr}$$

for any $k \in \mathbb{Z}_{>0}$.

(1) By assumption, for any sufficiently divisible $r \gg 0$,

$$-1 \leq \mathrm{ord}_E(K_{\tilde{Y}} - \phi^* K_Y) - \frac{c \cdot \mathrm{ord}_E(\mathfrak{a}_r)}{r} - a_r \cdot \mathrm{ord}_E(\mathfrak{b})$$

holds. By taking $\limsup_{r\to\infty}$, we have $-1 \le a(E, Y, \mathfrak{a}_\bullet^{\cdot c} \cdot \mathfrak{b}^{\cdot a})$.

(2) For any $r \ge r_0$, we have $cr^{-1} \cdot \mathrm{ord}_E(\mathfrak{a}_r) \ge c \cdot \mathrm{ord}_E(I)$. Thus we get the inequality $-1 \le a(E, Y, I^{\cdot c} \cdot \mathfrak{b}^{\cdot a})$. $\square$

Definition 2.6. (1) Let (Y, Δ) be a pair as in Definition 2.4 (1) and B be a nonzero effective $\mathbb{R}$-Cartier divisor on Y. The *log canonical threshold* $\mathrm{lct}(Y, \Delta; B)$ *of* B *with respects to* (Y, Δ) is defined by the following:

- If the pair $(Y, \Delta + cB)$ is not sub log canonical for any $c \in \mathbb{R}$, then we set $\mathrm{lct}(Y, \Delta; B) := -\infty$.
- Otherwise, we set

$$\mathrm{lct}(Y, \Delta; B) := \sup\{c \in \mathbb{R} \,|\, (Y, \Delta + cB) : \text{ sub log canonical}\}.$$

(2) Let $(Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l})$ be a pair as in Definition 2.4 (2) and $0 \neq \mathfrak{b} \subsetneq \mathcal{O}_Y$ be a coherent ideal sheaf. The *log canonical threshold* $\mathrm{lct}(Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l}; \mathfrak{b})$ *of* $\mathfrak{b}$ *with respects to* $(Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l})$ is defined by the following:

- If the pair $(Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l} \cdot \mathfrak{b}^{\cdot c})$ is not sub log canonical for any $c \in \mathbb{R}$, then we set $\mathrm{lct}(Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l}; \mathfrak{b}) := -\infty$.
- Otherwise, we set

$$\begin{aligned} & \mathrm{lct}(Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l}; \mathfrak{b}) \\ := \ & \sup\{c \in \mathbb{R} \,|\, (Y, \mathfrak{a}_1^{\cdot c_1} \cdots \mathfrak{a}_l^{\cdot c_l} \cdot \mathfrak{b}^{\cdot c}) : \text{ sub log canonical}\}. \end{aligned}$$

Moreover, if $l = 1$ and $\mathfrak{a}_1 = \mathcal{O}_Y$, then we write $\mathrm{lct}(Y; \mathfrak{b}) := \mathrm{lct}(Y, \mathfrak{a}_1^{\cdot c_1}; \mathfrak{b})$ for simplicity.

2.4. K-stability.

Definition 2.7 ([Tia97, Don02, RT07, Odk13, LX14])**.** Let X be an n-dimensional $\mathbb{Q}$-Fano variety.

(1) Let $r \in \mathbb{Z}_{>0}$ such that $-rK_X$ is Cartier. A *test configuration* (resp. a *semi test configuration*) $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ *of* $(X, -rK_X)$ consists of the following data:

- a variety $\mathcal{X}$ such that admitting $\mathbb{G}_m$-action and the morphism $\alpha\colon \mathcal{X} \to \mathbb{A}^1$ is $\mathbb{G}_m$-equivariant, where the action $\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1$ is given by $(a, t) \mapsto at$, and
- a $\mathbb{G}_m$-equivariant α-ample (resp. α-semiample) line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $(\mathcal{X}, \mathcal{L})|_{\alpha^{-1}(\mathbb{A}^1 \setminus \{0\})}$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X, \mathcal{O}_X(-rK_X)) \times (\mathbb{A}^1 \setminus \{0\})$ with the natural $\mathbb{G}_m$-action.

Moreover, if $\mathcal{X}$ is normal in addition, then we call the $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ a *normal test configuration* (resp. a *normal semi test configuration*) *of* $(X, -rK_X)$.

(2) Assume that $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is a normal semi test configuration of $(X, -rK_X)$. Let $\alpha\colon (\bar{\mathcal{X}}, \bar{\mathcal{L}}) \to \mathbb{P}^1$ be the natural equivariant compactification of $(\mathcal{X}, \mathcal{L}) \to \mathbb{A}^1$ induced by the compactification $\mathbb{A}^1 \subset \mathbb{P}^1$. The *Donaldson-Futaki invariant* $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ of $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is defined by

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := \frac{1}{(n+1)((-K_X)^{\cdot n})} \left(\frac{n}{r^{n+1}} (\bar{\mathcal{L}}^{\cdot n+1}) + \frac{n+1}{r^n} (\bar{\mathcal{L}}^{\cdot n} \cdot K_{\bar{\mathcal{X}}/\mathbb{P}^1}) \right).$$

(3)
- The pair $(X, -K_X)$ is called *K-semistable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$.
- The pair $(X, -K_X)$ is called *K-polystable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$, and the equality holds only if $\mathcal{X} \simeq X \times \mathbb{A}^1$.
- The pair $(X, -K_X)$ is called *K-stable* if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$, and the equality holds only if the pair $(\mathcal{X}, \mathcal{L})$ is trivial, that is, the pair $(\mathcal{X}, \mathcal{L})$ is $\mathbb{G}_m$-equivariantly isomorphic to the pair $(X \times \mathbb{A}^1, \mathcal{O}_{X \times \mathbb{A}^1}(-rK_{X \times \mathbb{A}^1/\mathbb{A}^1}))$ with the natural $\mathbb{G}_m$-action.

3. Ding polystability

We recall the theory in [Bm16, §3]. The author learned the theory from Odaka.

Definition 3.1 (see [Bm16, §3]). Let X be an n-dimensional $\mathbb{Q}$-Fano variety.

(1) Let $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ be a normal semi test configuration of $(X, -rK_X)$ and $(\bar{\mathcal{X}}, \bar{\mathcal{L}})/\mathbb{P}^1$ be its natural compactification as in Definition 2.7 (2).

(i) Let $D_{(\mathcal{X},\mathcal{L})}$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ such that the following conditions are satisfied:
- The support $\operatorname{Supp} D_{(\mathcal{X},\mathcal{L})}$ is contained in $\mathcal{X}_0$. (Note that $\mathcal{X}_0$ is the fiber of $\mathcal{X} \to \mathbb{A}^1$ at $0 \in \mathbb{A}^1$.)
- The divisor $-rD_{(\mathcal{X},\mathcal{L})}$ is a $\mathbb{Z}$-divisor corresponds to the divisorial sheaf $\bar{\mathcal{L}}(rK_{\bar{\mathcal{X}}/\mathbb{P}^1})$. (Thus the divisor $-r(K_{\bar{\mathcal{X}}/\mathbb{P}^1} + D_{(\mathcal{X},\mathcal{L})})$ is a Cartier divisor corresponds to $\bar{\mathcal{L}}$.)

Since the divisorial sheaf $\bar{\mathcal{L}}(rK_{\bar{\mathcal{X}}/\mathbb{P}^1})$ is trivial on $\bar{\mathcal{X}} \setminus \mathcal{X}_0$, the $D_{(\mathcal{X},\mathcal{L})}$ exists and is unique.

(ii) The *Ding invariant* $\mathrm{Ding}(\mathcal{X},\mathcal{L})$ of $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ is defined by

$$\mathrm{Ding}(\mathcal{X},\mathcal{L}) := \frac{-(\bar{\mathcal{L}}^{\cdot n+1})}{(n+1)r^{n+1}((-K_X)^{\cdot n})} - \left(1-\mathrm{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0)\right).$$

(2)
- X is called *Ding semistable* if $\mathrm{Ding}(\mathcal{X},\mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ of $(X, -rK_X)$.
- X is called *Ding polystable* if
  - X is Ding semistable, and
  - if $(\mathcal{X},\mathcal{L})$ is a normal test configuration of $(X,-rK_X)$ which satisfies that $\mathcal{L} \simeq \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\mathbb{A}^1})$, $\mathcal{X}_0$ is log terminal and $\mathrm{Ding}(\mathcal{X},\mathcal{L}) = 0$, then $\mathcal{X} \simeq X \times \mathbb{A}^1$.

The following is a theorem of Berman.

Theorem 3.2 ([Bm16])**.** *Let X be a $\mathbb{Q}$-Fano variety.*
(1) *If X admits Kähler-Einstein metrics, then X is Ding polystable.*
(2) *For any normal test configuration $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ of $(X,-rK_X)$, we have* $\mathrm{DF}(\mathcal{X},\mathcal{L}) \geq \mathrm{Ding}(\mathcal{X},\mathcal{L})$*. Moreover, the equality holds if and only if*
- $\mathcal{L} \simeq \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\mathbb{A}^1})$*, and*
- *the pair $(\mathcal{X},\mathcal{X}_0)$ is log canonical.*

Proof. We repeat the proof in [Bm16, §3] for the reader's convenience. Pick any normal test configuration $(\mathcal{X},\mathcal{L})/\mathbb{A}^1$ of $(X,-rK_X)$. Set $n := \dim X$ and

$$\begin{aligned} q(\mathcal{X},\mathcal{L}) &:= \mathrm{DF}(\mathcal{X},\mathcal{L}) - \mathrm{Ding}(\mathcal{X},\mathcal{L}) \\ &= 1 - \mathrm{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0) - \frac{(\bar{\mathcal{L}}^{\cdot n}\cdot D_{(\mathcal{X},\mathcal{L})})}{r^n((-K_X)^{\cdot n})}. \end{aligned}$$

(1) Let $\gamma\colon \mathcal{X}' \to \mathcal{X}$ be a $\mathbb{G}_m$-equivariant log resolution of the pair $(\mathcal{X},\mathcal{X}_0)$ and let $\gamma\colon \bar{\mathcal{X}}' \to \bar{\mathcal{X}}$ be its natural compactification. Since $\bar{\mathcal{X}} \setminus \mathcal{X}_0$ is log terminal, if we set $D^* := -K_{\bar{\mathcal{X}}'\setminus\mathcal{X}'_0} + \gamma^* K_{\bar{\mathcal{X}}\setminus\mathcal{X}_0}$ then any coefficient of D^* is strictly smaller than one. Let Δ' be the $\mathbb{Q}$-divisor on $\bar{\mathcal{X}}'$ such that the following conditions are satisfied:
- The support $\mathrm{Supp}(\Delta' + \bar{D}^*)$ is contained in $\mathcal{X}'_0$, where $\bar{D}^*$ is the closure of D^* in $\bar{\mathcal{X}}'$.
- The divisor $r\Delta'$ is a $\mathbb{Z}$-divisor and corresponds to $\gamma^*\bar{\mathcal{L}}(rK_{\bar{\mathcal{X}}'/\mathbb{P}^1})$.

Let

$$\mathcal{X}'_0 = \sum_{i\in I} m'_i E'_i, \quad \Delta' + \bar{D}^* = \sum_{i\in I} c'_i E'_i$$

be the irreducible decompositions. By construction, we have

$$\gamma^*(K_{\bar{\mathcal{X}}} + D_{(\mathcal{X},\mathcal{L})} + c\mathcal{X}_0) = K_{\bar{\mathcal{X}}'} - \Delta' + c\mathcal{X}'_0$$

for any $c \in \mathbb{R}$. Thus we have

$$\operatorname{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0) = \operatorname{lct}(\mathcal{X}', -\Delta'; \mathcal{X}'_0) = \min_{i\in I}\left\{\frac{1+c'_i}{m'_i}\right\}.$$

Moreover, we have

$$-(\bar{\mathcal{L}}^{\cdot n}\cdot D_{(\mathcal{X},\mathcal{L})}) = (\gamma^*\bar{\mathcal{L}}^{\cdot n}\cdot\Delta' + \bar{D}^*) = \sum_{i\in I} c'_i(\gamma^*\bar{\mathcal{L}}^{\cdot n}\cdot E'_i)$$

since $\gamma_*(\Delta' + \bar{D}^*) = -D_{(\mathcal{X},\mathcal{L})}$ holds. Therefore,

$$q(\mathcal{X},\mathcal{L}) = \max_{i\in I}\left\{\frac{m'_i - 1 - c'_i}{m'_i}\right\} + \frac{1}{r^n((-K_X)^{\cdot n})}\sum_{i\in I} c'_i(\gamma^*\bar{\mathcal{L}}^{\cdot n}\cdot E'_i)$$

holds. The equation is nothing but Formula (3.30) in [Bm16]. Hence, if X admits Kähler-Einstein metrics, then $\operatorname{Ding}(\mathcal{X},\mathcal{L}) \geq 0$ holds by [Bm16, Theorem 3.11 and Formula (3.2)] (see also [Bn09] and [BBGZ12, Formula (6.5)]). If we further assume that $\mathcal{L} \simeq \mathcal{O}_{\mathcal{X}}(-rK_{\mathcal{X}/\mathbb{A}^1})$, $\mathcal{X}_0$ is log terminal and $\operatorname{Ding}(\mathcal{X},\mathcal{L}) = 0$, then $D_{(\mathcal{X},\mathcal{L})} = c\mathcal{X}_0$ for some $c \in \mathbb{Q}$ and $\operatorname{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0) = 1 - c$. This implies that $\operatorname{DF}(\mathcal{X},\mathcal{L}) = \operatorname{Ding}(\mathcal{X},\mathcal{L})$ since $q(\mathcal{X},\mathcal{L}) = 0$ holds. Hence $\mathcal{X} \simeq X \times \mathbb{A}^1$ by [Bm16, Theorem 1.1] (more precisely, by [Bm16, Proposition 3.5]). Thus X is Ding polystable.

(2) (See [Bm16, Proof of Theorem 3.11].) Let

$$\mathcal{X}_0 = \sum_{i\in J} m_i E_i, \quad -D_{(\mathcal{X},\mathcal{L})} = \sum_{i\in J} c_i E_i$$

be the irreducible decompositions. Note that

$$q(\mathcal{X},\mathcal{L}) = \frac{1}{r^n((-K_X)^{\cdot n})}\left(\bar{\mathcal{L}}^{\cdot n}\cdot(1 - \operatorname{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0))\mathcal{X}_0 - D_{(\mathcal{X},\mathcal{L})}\right).$$

Since

$$1 - \operatorname{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0) \geq \max_{i\in J}\left\{\frac{m_i - 1 - c_i}{m_i}\right\},$$

we have

$$\begin{aligned}&(1 - \operatorname{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0))\mathcal{X}_0 - D_{(\mathcal{X},\mathcal{L})}\\ \geq\ & \sum_{i\in J}\left(\frac{m_i - 1 - c_i}{m_i}\cdot m_i + c_i\right)E_i = \sum_{i\in J}(m_i - 1)E_i \geq 0\end{aligned}$$

Since $\bar{\mathcal{L}}$ is α-ample, we get $q(\mathcal{X},\mathcal{L}) \geq 0$. Moreover, $q(\mathcal{X},\mathcal{L}) = 0$ holds if and only if $\mathcal{X}_0$ is reduced and $D_{(\mathcal{X},\mathcal{L})} = (1 - \operatorname{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0))\mathcal{X}_0$ holds. Thus we get the assertion. □

Remark 3.3. From Theorem 3.2 and [LX14, Corollary 1] (see [Bm16]), if a $\mathbb{Q}$-Fano variety X is Ding semistable (resp. Ding polystable), then the pair $(X, -K_X)$ is K-semistable (resp. K-polystable). Thus, by [CDS15a, CDS15b, CDS15c, Tia15], if X is a Fano manifold, then the following three conditions are equivalent:

- X admits Kähler-Einstein metrics.
- X is Ding polystable.
- $(X, -K_X)$ is K-polystable.

Lemma 3.4. *Let X be a $\mathbb{Q}$-Fano variety and $\gamma\colon (\mathcal{Y}, \gamma^*\mathcal{L}) \to (\mathcal{X}, \mathcal{L})$ be a $\mathbb{G}_m$-equivariant birational morphism between normal semi test configurations of $(X, -rK_X)$. Then $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) = \mathrm{Ding}(\mathcal{Y}, \gamma^*\mathcal{L})$ holds.*

Proof. Since $K_{\bar{\mathcal{Y}}} + D_{(\mathcal{Y},\gamma^*\mathcal{L})} = \gamma^*(K_{\bar{\mathcal{X}}} + D_{(\mathcal{X},\mathcal{L})})$, we have

$$\mathrm{lct}(\mathcal{X}, D_{(\mathcal{X},\mathcal{L})}; \mathcal{X}_0) = \mathrm{lct}(\mathcal{Y}, D_{(\mathcal{Y},\gamma^*\mathcal{L})}; \mathcal{Y}_0).$$

Thus the assertion follows immediately. □

Theorem 3.5. *Let X be an n-dimensional $\mathbb{Q}$-Fano variety which is Ding semistable, let r be a positive integer such that $-rK_X$ is Cartier, let $I_M \subset \cdots \subset I_1 \subset \mathcal{O}_X$ be a sequence of coherent ideal sheaves, let $\mathcal{I} := I_M + I_{M-1}t^1 + \cdots + I_1 t^{M-1} + (t^M) \subset \mathcal{O}_{X\times\mathbb{A}^1_t}$, let $\Pi\colon \mathcal{X} \to X \times \mathbb{A}^1$ be the blowup along $\mathcal{I}$, let $E \subset \mathcal{X}$ be the Cartier divisor defined by $\mathcal{O}_{\mathcal{X}}(-E) = \mathcal{I} \cdot \mathcal{O}_{\mathcal{X}}$, and let $\mathcal{L} := \Pi^*\mathcal{O}_{X\times\mathbb{A}^1}(-rK_{X\times\mathbb{A}^1/\mathbb{A}^1}) \otimes \mathcal{O}_{\mathcal{X}}(-E)$. Assume that $\mathcal{L}$ is semiample over $\mathbb{A}^1$. Then $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is naturally seen as a (possibly non-normal) semi test configuration of $(X, -rK_X)$. Under these conditions, the pair $(X \times \mathbb{A}^1_t, \mathcal{I}^{\cdot(1/r)} \cdot (t)^{\cdot d})$ must be sub log canonical, where*

$$d := 1 + \frac{(\bar{\mathcal{L}}^{\cdot n+1})}{(n+1)r^{n+1}((-K_X)^{\cdot n})}.$$

Moreover, we have the equality

$$(\bar{\mathcal{L}}^{\cdot n+1}) = -\lim_{k\to\infty} \frac{\dim\left(\frac{H^0(X\times\mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}(-krK_{X\times\mathbb{A}^1/\mathbb{A}^1}))}{H^0(X\times\mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}(-krK_{X\times\mathbb{A}^1/\mathbb{A}^1})\cdot\mathcal{I}^k)}\right)}{k^{n+1}/(n+1)!}.$$

Proof. Let $\nu\colon \mathcal{X}^\nu \to \mathcal{X}$ be the normalization. Then $\alpha\colon (\mathcal{X}^\nu, \nu^*\mathcal{L}) \to \mathbb{A}^1$ is a normal semi test configuration of $(X, -rK_X)$. Set

$$\mathcal{Y} := \mathrm{Proj} \bigoplus_{m\geq 0} \alpha_*(\nu^*\mathcal{L}^{\otimes m})$$

and let $\phi\colon \mathcal{X}^\nu \to \mathcal{Y}$ be the natural morphism. Then there exist a positive integer m and a line bundle $\mathcal{M}$ on $\mathcal{Y}$ with a $\mathbb{G}_m$-action such that $\phi^*\mathcal{M}$ is $\mathbb{G}_m$-equivariantly isomorphic to $\nu^*\mathcal{L}^{\otimes m}$ and $(\mathcal{Y}, \mathcal{M})/\mathbb{A}^1$ is a normal test configuration of $(X, -mrK_X)$. Since X is Ding semistable, we have $\mathrm{Ding}(\mathcal{Y}, \mathcal{M}) \geq 0$. On the other hand, by Lemma 3.4, we have $\mathrm{Ding}(\mathcal{Y}, \mathcal{M}) = \mathrm{Ding}(\mathcal{X}^\nu, \nu^*\mathcal{L}^{\otimes m})$. Thus we have $\mathrm{Ding}(\mathcal{X}^\nu, \nu^*\mathcal{L}) \geq 0$ since $\mathrm{Ding}(\mathcal{X}^\nu, \nu^*\mathcal{L}^{\otimes m}) = \mathrm{Ding}(\mathcal{X}^\nu, \nu^*\mathcal{L})$ holds. Note that

$$\begin{aligned} \mathcal{O}_{\bar{\mathcal{X}}^\nu}\left(r(K_{\bar{\mathcal{X}}^\nu/\mathbb{P}^1} + D_{(\mathcal{X}^\nu, \nu^*\mathcal{L})})\right) &\simeq \nu^*\bar{\mathcal{L}}^{\otimes(-1)} \\ &\simeq \nu^*\mathcal{O}_{\bar{\mathcal{X}}}\left(r(\Pi^* K_{X\times\mathbb{P}^1/\mathbb{P}^1} + (1/r)E)\right). \end{aligned}$$

Hence, for $c \in \mathbb{R}$, the pair $(\mathcal{X}^\nu, D_{(\mathcal{X}^\nu, \nu^*\mathcal{L})} + c\mathcal{X}^\nu_0)$ is sub log canonical if and only if the pair $(X \times \mathbb{A}^1, \mathcal{I}^{\cdot(1/r)} \cdot (t)^{\cdot c})$ is sub log canonical. Thus we have the equality

$$\mathrm{lct}(\mathcal{X}^\nu, D_{(\mathcal{X}^\nu, \nu^*\mathcal{L})}; \mathcal{X}^\nu_0) = \mathrm{lct}(X \times \mathbb{A}^1, \mathcal{I}^{\cdot(1/r)}; (t)).$$

This implies that the pair $(X \times \mathbb{A}^1, \mathcal{I}^{\cdot(1/r)} \cdot (t)^{\cdot d})$ is sub log canonical. The remaining part is trivial (see [Odk13, §3] for example). □

4. Ding semistability and filtered linear series

4.1. **The saturations of filtered linear series.** We recall the definitions in [BC11, §1]. (See also [WN12].)

Definition 4.1 (see [BC11, §1]). Let X be a projective variety, L be a big line bundle on X, $V_\bullet$ be the complete graded linear series of L, that is, $V_r := H^0(X, L^{\otimes r})$ for any $r \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{F}$ be a decreasing, left-continuous $\mathbb{R}$-filtration of the graded $\mathbb{C}$-algebra $V_\bullet$.

(1) $\mathcal{F}$ is said to be *multiplicative* if

$$\mathcal{F}^x V_r \otimes_{\mathbb{C}} \mathcal{F}^{x'} V_{r'} \to \mathcal{F}^{x+x'} V_{r+r'}$$

holds for any r, $r' \in \mathbb{Z}_{\geq 0}$ and x, $x' \in \mathbb{R}$.

(2) $\mathcal{F}$ is said to be *linearly bounded* if $e_{\min}(V_\bullet, \mathcal{F})$, $e_{\max}(V_\bullet, \mathcal{F}) \in \mathbb{R}$, where

$$\begin{aligned} e_{\min}(V_\bullet, \mathcal{F}) &:= \liminf_{r\to\infty}\left(\frac{\inf\{x \in \mathbb{R} \mid \mathcal{F}^x V_r \neq V_r\}}{r}\right), \\ e_{\max}(V_\bullet, \mathcal{F}) &:= \limsup_{r\to\infty}\left(\frac{\sup\{x \in \mathbb{R} \mid \mathcal{F}^x V_r \neq 0\}}{r}\right). \end{aligned}$$

(3) Assume that $\mathcal{F}$ is multiplicative. For any $x \in \mathbb{R}$, we set

$$\mathrm{vol}(\mathcal{F}V_\bullet^x) := \limsup_{r\to\infty} \frac{\dim \mathcal{F}^{rx} V_r}{r^n/n!},$$

where $n := \dim X$.

Definition 4.2. Let X be a projective variety, L be an ample line bundle on X, $V_\bullet$ be the complete graded linear series of L and $\mathcal{F}$ be a decreasing, left-continuous, multiplicative and linearly bounded $\mathbb{R}$-filtration of $V_\bullet$. For any $r \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$, we set

$$I^{\mathcal{F}}_{(r,x)} := I_{(r,x)} := \mathrm{Image}(\mathcal{F}^x V_r \otimes_{\mathbb{C}} L^{\otimes(-r)} \to \mathcal{O}_X),$$

where the homomorphism is the evaluation homomorphism. Moreover, we set $\bar{\mathcal{F}}^x V_r := H^0(X, L^{\otimes r} \cdot I_{(r,x)})$.

Proposition 4.3. *Let X, L, $V_\bullet$ and $\mathcal{F}$ be as in Definition 4.2.*

(1) *For any r, $r' \in \mathbb{Z}_{\geq 0}$ and x, $x' \in \mathbb{R}$, we have $I_{(r,x)} \cdot I_{(r',x')} \subset I_{(r+r',x+x')}$.*
(2) *For any $r \in \mathbb{Z}_{\geq 0}$ and $x \leq x'$, we have $I_{(r,x')} \subset I_{(r,x)}$.*
(3) *For any $r \in \mathbb{Z}_{\geq 0}$ and $x > r \cdot e_{\max}(V_\bullet, \mathcal{F})$, we have $\mathcal{F}^x V_r = 0$. In particular, $I_{(r,x)} = 0$ holds.*
(4) *For any $e_- < e_{\min}(V_\bullet, \mathcal{F})$, there exists $r_1 \in \mathbb{Z}_{>0}$ such that $\mathcal{F}^{re_-} V_r = V_r$ and $I_{(r,re_-)} = \mathcal{O}_X$ hold for any $r \geq r_1$.*
(5) *For any $r \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$, $\mathcal{F}^x V_r \subset \bar{\mathcal{F}}^x V_r$ holds. Moreover, the homomorphism $\bar{\mathcal{F}}^x V_r \otimes_{\mathbb{C}} \mathcal{O}_X \to L^{\otimes r} \cdot I_{(r,x)}$ is surjective.*
(6) *$\bar{\mathcal{F}}$ is also a decreasing, left-continuous, multiplicative and linearly bounded $\mathbb{R}$-filtration of $V_\bullet$. Moreover, we have*
$$e_{\min}(V_\bullet, \mathcal{F}) \leq e_{\min}(V_\bullet, \bar{\mathcal{F}}) \leq e_{\max}(V_\bullet, \bar{\mathcal{F}}) = e_{\max}(V_\bullet, \mathcal{F}).$$
Furthermore, for any $r \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$, we have $I^{\mathcal{F}}_{(r,x)} = I^{\bar{\mathcal{F}}}_{(r,x)}$.

Proof. (1) Follows from the diagram

$$\begin{tikzcd}
(\mathcal{F}^x V_r \otimes_{\mathbb{C}} \mathcal{F}^{x'} V_{r'}) \otimes_{\mathbb{C}} L^{\otimes(-(r+r'))} \arrow[r, two heads] \arrow[d] & I_{(r,x)} \cdot I_{(r',x')} \arrow[r, hook] & \mathcal{O}_X \arrow[d, equal] \\
\mathcal{F}^{x+x'} V_{r+r'} \otimes_{\mathbb{C}} L^{\otimes(-(r+r'))} \arrow[r, two heads] & I_{(r+r',x+x')} \arrow[r, hook] & \mathcal{O}_X.
\end{tikzcd}$$

(2) This is obvious since $\mathcal{F}$ is decreasing.

(3) (See [BC11, Lemma 1.4].) By the definition of $e_{\max}(V_\bullet, \mathcal{F})$, there exists $k \in \mathbb{Z}_{>0}$ such that $\mathcal{F}^{kx} V_{kr} = 0$. Thus $\mathcal{F}^x V_r = 0$ since $V_\bullet$ is an integral domain.

(4) By [Laz04a, Example 1.2.22], there exists $r_0 \in \mathbb{Z}_{>0}$ such that the homomorphisms

$$V_r \otimes_{\mathbb{C}} V_{r'} \to V_{r+r'},$$
$$V_r \otimes_{\mathbb{C}} L^{\otimes(-r)} \to \mathcal{O}_X$$

are surjective for all r, $r' \geq r_0$. By the choice of e_-, there exist distinct prime numbers p_1, p_2 with p_1, $p_2 \geq r_0$ such that $\mathcal{F}^{p_i e_-} V_{p_i} = V_{p_i}$ for

$i = 1, 2$. Set $r_1 := p_1 p_2$. For any $r \geq r_1$, there exist $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that $r = k_1 p_1 + k_2 p_2$ holds. Then $\mathcal{F}^{re_-} V_r$ contains the image of $(\mathcal{F}^{p_1 e_-} V_{p_1})^{\otimes k_1} \otimes_{\mathbb{C}} (\mathcal{F}^{p_2 e_-} V_{p_2})^{\otimes k_2} = V_{p_1}^{\otimes k_1} \otimes_{\mathbb{C}} V_{p_2}^{\otimes k_2}$. Thus $\mathcal{F}^{re_-} V_r = V_r$ and $I_{(r, re_-)} = \mathcal{O}_X$ hold.

(5) Consider the diagram

$$\begin{tikzcd} \mathcal{F}^x V_r \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r, two heads] \arrow[d, hook] & L^{\otimes r} \cdot I_{(r,x)} \arrow[d, hook] \\ H^0(X, L^{\otimes r}) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] & L^{\otimes r}. \end{tikzcd}$$

By taking H^0, we get $\mathcal{F}^x V_r \subset \bar{\mathcal{F}}^x V_r$. From the diagram

$$\begin{tikzcd} \mathcal{F}^x V_r \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r, two heads] \arrow[d, hook] & L^{\otimes r} \cdot I_{(r,x)} \arrow[d, equal] \\ \bar{\mathcal{F}}^x V_r \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] & L^{\otimes r} \cdot I_{(r,x)}, \end{tikzcd}$$

we get the assertion.

(6) From (2), $\bar{\mathcal{F}}$ is decreasing, and obviously left-continuous. From (1), $\bar{\mathcal{F}}$ is multiplicative. From (5), $e_{\min}(V_\bullet, \mathcal{F}) \leq e_{\min}(V_\bullet, \bar{\mathcal{F}})$ and $e_{\max}(V_\bullet, \mathcal{F}) \leq e_{\max}(V_\bullet, \bar{\mathcal{F}})$ hold. Moreover, from (3), $e_{\max}(V_\bullet, \mathcal{F}) \geq e_{\max}(V_\bullet, \bar{\mathcal{F}})$ holds. Thus $\bar{\mathcal{F}}$ is linearly bounded. Moreover, the condition $I_{(r,x)}^{\mathcal{F}} = I_{(r,x)}^{\bar{\mathcal{F}}}$ follows from (5). $\square$

Definition 4.4. Let X, L, $V_\bullet$, $\mathcal{F}$ be as in Definition 4.2.

(1) The filtration $\bar{\mathcal{F}}$ of $V_\bullet$ in Definition 4.2 is called the *saturation* of $\mathcal{F}$.
(2) If $\mathcal{F}^x V_r = \bar{\mathcal{F}}^x V_r$ for any $r \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{R}$, then we say that the filtration $\mathcal{F}$ is *saturated*. Note that, by Proposition 4.3, for any $\mathcal{F}$ in Definition 4.2, the saturation $\bar{\mathcal{F}}$ is saturated.

4.2. **Test configurations from filtered linear series.** In this section, we fix

- an n-dimensional $\mathbb{Q}$-Fano variety X which is Ding semistable,
- $r_0 \in \mathbb{Z}_{>0}$ such that $-r_0 K_X$ is Cartier,
- $L := \mathcal{O}_X(-r_0 K_X)$,
- the complete graded linear series $V_\bullet$ of L,
- a decreasing, left-continuous, multiplicative, linearly bounded $\mathbb{R}$-filtration $\mathcal{F}$ of $V_\bullet$, and
- $e_+, e_- \in \mathbb{Z}$ with $e_+ > e_{\max}(V_\bullet, \mathcal{F})$ and $e_- < e_{\min}(V_\bullet, \mathcal{F})$.

Set $e := e_+ - e_-$. Fix $r_1 \in \mathbb{Z}_{>0}$ as in Proposition 4.3 (4). For any $r \geq r_1$, we set

$$\mathcal{I}_r := I_{(r,re_+)} + I_{(r,re_+-1)}t^1 + \cdots + I_{(r,re_-+1)}t^{re-1} + (t^{re}) \subset \mathcal{O}_{X\times\mathbb{A}^1_t}.$$

By Proposition 4.3, $\{\mathcal{I}_r\}_{r\geq r_1}$ is a graded family of coherent ideal sheaves. For any $r \geq r_1$, $k \in \mathbb{Z}_{\geq 0}$ and $j \in [kre_-, kre_+] \cap \mathbb{Z}$, we set

$$J_{(k;r,j)} := \sum_{\substack{j_1+\cdots+j_k=j,\\ j_1,\dots,j_k\in[re_-,re_+]\cap\mathbb{Z}}} I_{(r,j_1)}\cdots I_{(r,j_k)}.$$

By construction,

$$\mathcal{I}_r^k = J_{(k;r,kre_+)} + J_{(k;r,kre_+-1)}t^1 + \cdots + J_{(k;r,kre_-+1)}t^{kre-1} + (t^{kre})$$

holds. Moreover, by Proposition 4.3 (5), $J_{(k;r,j)}$ is the image of the homomorphism

$$W_{(k;r,j)} \otimes_{\mathbb{C}} L^{\otimes(-kr)} \to \mathcal{O}_X,$$

where $W_{(k;r,j)}$ is defined by the image of the homomorphism

$$\bigoplus_{\substack{j_1+\cdots+j_k=j,\\ j_1,\dots,j_k\in[re_-,re_+]\cap\mathbb{Z}}} \bar{\mathcal{F}}^{j_1}V_r \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bar{\mathcal{F}}^{j_k}V_r \to \bar{\mathcal{F}}^{j}V_{kr}.$$

Lemma 4.5. *For any $r \geq r_1$, $k \in \mathbb{Z}_{\geq 0}$ and $j \in [kre_-, kre_+] \cap \mathbb{Z}$, we have the following:*

(1) $W_{(k;r,j)} \subset H^0(X, L^{\otimes kr}\cdot J_{(k;r,j)}) \subset \bar{\mathcal{F}}^jV_{kr}$ *holds.*
(2) *The homomorphism*

$$H^0(X, L^{\otimes kr}\cdot J_{(k;r,j)}) \otimes_{\mathbb{C}} \mathcal{O}_X \to L^{\otimes kr}\cdot J_{(k;r,j)}$$

is surjective.

Proof. From the homomorphism

$$W_{(k;r,j)} \otimes_{\mathbb{C}} \mathcal{O}_X \to L^{\otimes kr}\cdot J_{(k;r,j)},$$

we get the inclusion $W_{(k;r,j)} \subset H^0(X, L^{\otimes kr}\cdot J_{(k;r,j)})$. Furthermore, from the diagram

$$\begin{tikzcd} W_{(k;r,j)} \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r, two heads] \arrow[d, hook] & L^{\otimes kr}\cdot J_{(k;r,j)} \arrow[d, equal] \\ H^0(X, L^{\otimes kr}\cdot J_{(k;r,j)}) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] & L^{\otimes kr}\cdot J_{(k;r,j)}, \end{tikzcd}$$

we have proved (2). Moreover, from the diagram

$$\begin{tikzcd} W_{(k;r,j)} \otimes_{\mathbb{C}} L^{\otimes(-kr)} \arrow[r, two heads] \arrow[d, hook] & J_{(k;r,j)} \arrow[r, hook] & \mathcal{O}_X \arrow[d, equal] \\ \bar{\mathcal{F}}^j V_{kr} \otimes_{\mathbb{C}} L^{\otimes(-kr)} \arrow[r, two heads] & I_{(kr,j)} \arrow[r, hook] & \mathcal{O}_X, \end{tikzcd}$$

we have $J_{(k;r,j)} \subset I_{(kr,j)}$. Thus we have proved (1). $\square$

For any $r \geq r_1$, let

- $\Pi_r \colon \mathcal{X}_r \to X \times \mathbb{A}^1$ be the blowup along $\mathcal{I}_r$,
- $E_r \subset \mathcal{X}_r$ be the Cartier divisor defined by $\mathcal{O}_{\mathcal{X}_r}(-E_r) = \mathcal{I}_r \cdot \mathcal{O}_{\mathcal{X}_r}$, and
- $\mathcal{L}_r := \Pi_r^* \mathcal{O}_{X\times\mathbb{A}^1}(-rr_0 K_{X\times\mathbb{A}^1/\mathbb{A}^1}) \otimes \mathcal{O}_{\mathcal{X}_r}(-E_r)$.

Lemma 4.6. *$\mathcal{L}_r$ is semiample over $\mathbb{A}^1$. Thus $(\mathcal{X}_r, \mathcal{L}_r)/\mathbb{A}^1$ is a semi test configuration of $(X, -rr_0K_X)$.*

Proof. (See also [Fuj15a, Lemma 3.4].) Let $\alpha \colon \mathcal{X}_r \to \mathbb{A}^1$ and $p_2 \colon X \times \mathbb{A}^1 \to \mathbb{A}^1$ be the natural morphisms. For any $k \in \mathbb{Z}_{\geq 0}$, by Lemma 4.5 (2), we have

$$\begin{aligned}
& H^0(X\times\mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}(-krr_0K_{X\times\mathbb{A}^1/\mathbb{A}^1}) \cdot \mathcal{I}_r^k) \otimes_{\mathbb{C}[t]} \mathcal{O}_{X\times\mathbb{A}^1} \\
= & \left(\sum_{j=0}^{kre-1} t^j \cdot H^0(X, L^{\otimes kr} \cdot J_{(k;r,kre_+-j)}) \right. \\
& \left. + \sum_{j\geq kre} t^j \cdot H^0(X, L^{\otimes kr}) \right) \otimes_{\mathbb{C}[t]} \mathcal{O}_{X\times\mathbb{A}^1} \\
\twoheadrightarrow & \sum_{j=0}^{kre-1} t^j \cdot L^{\otimes kr} \cdot J_{(k;r,kre_+-j)} + \sum_{j\geq kre} t^j \cdot L^{\otimes kr} \\
= & \mathcal{O}_{X\times\mathbb{A}^1}(-krr_0K_{X\times\mathbb{A}^1/\mathbb{A}^1}) \cdot \mathcal{I}_r^k.
\end{aligned}$$

Therefore, by [Laz04a, Lemma 5.4.24], for any $k \gg 0$, we have

$$\begin{aligned}
& \alpha^*\alpha_*\mathcal{L}_r^{\otimes k} \\
= & \Pi_r^*(p_2)^*(p_2)_*\left(\mathcal{O}_{X\times\mathbb{A}^1}(-krr_0K_{X\times\mathbb{A}^1/\mathbb{A}^1}) \cdot \mathcal{I}_r^k\right) \\
= & \Pi_r^*\left(H^0(X\times\mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}(-krr_0K_{X\times\mathbb{A}^1/\mathbb{A}^1}) \cdot \mathcal{I}_r^k) \otimes_{\mathbb{C}[t]} \mathcal{O}_{X\times\mathbb{A}^1}\right) \\
\twoheadrightarrow & \Pi_r^*\left(\mathcal{O}_{X\times\mathbb{A}^1}(-krr_0K_{X\times\mathbb{A}^1/\mathbb{A}^1}) \cdot \mathcal{I}_r^k\right) \\
\twoheadrightarrow & \Pi_r^*\mathcal{O}_{X\times\mathbb{A}^1}(-krr_0K_{X\times\mathbb{A}^1/\mathbb{A}^1}) \otimes \mathcal{O}_{\mathcal{X}_r}(-kE_r) = \mathcal{L}_r^{\otimes k}.
\end{aligned}$$

Thus $\mathcal{L}_r$ is semiample over $\mathbb{A}^1$. $\square$

Thus, by Theorem 3.5, the pair $(X\times\mathbb{A}^1, \mathcal{I}_r^{\cdot(1/(rr_0))}\cdot(t)^{\cdot d_r})$ is sub log canonical, where

$$d_r := 1+\frac{(\bar{\mathcal{L}}_r^{\cdot n+1})}{(n+1)r^{n+1}r_0^{n+1}((-K_X)^{\cdot n})}.$$

Set

$$w_r(k) := -\dim\left(\frac{H^0(X\times\mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}(-krr_0K_{X\times\mathbb{A}^1/\mathbb{A}^1}))}{H^0(X\times\mathbb{A}^1, \mathcal{O}_{X\times\mathbb{A}^1}(-krr_0K_{X\times\mathbb{A}^1/\mathbb{A}^1})\cdot\mathcal{I}_r^k)}\right).$$

Then

$$(\bar{\mathcal{L}}_r^{\cdot n+1}) = \lim_{k\to\infty}\frac{w_r(k)}{k^{n+1}/(n+1)!}$$

holds by Theorem 3.5. We set

$$\begin{aligned} v_r(k) &:= \sum_{j=kre_-+1}^{kre_+} h^0(X, L^{\otimes kr}\cdot J_{(k;r,j)}),\\ A_r &:= \lim_{k\to\infty}\frac{v_r(k)}{k^{n+1}r^{n+1}r_0^{n+1}/n!}. \end{aligned}$$

Since $w_r(k) = -kre\cdot h^0(X, L^{\otimes kr}) + v_r(k)$, the limit in the definition of A_r actually exists. Note that $d_r = 1 - e/r_0 + A_r/((-K_X)^{\cdot n})$.

Lemma 4.7 (cf. [BC11, Theorem 1.14])**.** *We have*

$$\lim_{r\to\infty} A_r = \frac{1}{r_0^{n+1}}\int_{e_-}^{e_+}\mathrm{vol}(\bar{\mathcal{F}}V_\bullet^x)dx.$$

Proof. Take any $r\geq r_1$. For $k\in\mathbb{Z}_{\geq 0}$, set

$$W_{r,k} := \mathrm{Image}(V_r^{\otimes k}\to V_{kr}) = V_{kr}.$$

Moreover, we consider the $\mathbb{R}$-filtration $\mathcal{G}$ of the complete graded linear series $W_{r,\bullet}$ of $L^{\otimes rr_0}$, where $\mathcal{G}^xW_{r,k}$ is defined by the image of the homomorphism

$$\sum_{\substack{x_1+\cdots+x_k=x,\\ x_1,\dots,x_k\in\mathbb{R}}}\bar{\mathcal{F}}^{x_1}V_r\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}\bar{\mathcal{F}}^{x_k}V_r\to\bar{\mathcal{F}}^xV_{kr}.$$

Claim 4.8. (1) *$\mathcal{G}$ is a decreasing, left-continuous, multiplicative, linearly bounded $\mathbb{R}$-filtration of $W_{r,\bullet}$.*

(2) *We have*

$$re_-\leq e_{\min}(W_{r,\bullet},\mathcal{G})\leq e_{\max}(W_{r,\bullet},\mathcal{G})\leq re_+.$$

(3) *For any $k\in\mathbb{Z}_{\geq 0}$ and $j\in[kre_-, kre_+]\cap\mathbb{Z}$, we have*

$$\mathcal{G}^{j+k-1}W_{r,k}\subset H^0(X, L^{\otimes kr}\cdot J_{(k;r,j)})\subset\bar{\mathcal{F}}^jV_{kr}.$$

Proof of Claim 4.8. (1) We check that $\mathcal{G}$ is left-continuous. For any $i \in [1, \dim V_r] \cap \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, we set

$$\begin{aligned} e_{r,i} &:= \sup\{x \in \mathbb{R} \mid \dim \bar{\mathcal{F}}^x V_r \geq i\}, \\ \mathbb{E}_{r,k} &:= \left\{ \sum_{i=1}^{k} e_{r,j_i} \,\middle|\, j_1, \dots, j_k \in [1, \dim V_r] \cap \mathbb{Z} \right\} \subset \mathbb{R}. \end{aligned}$$

Moreover, we set $e_{r,0} := +\infty$ and $e_{r,\dim V_r + 1} := -\infty$ for convenience. Take any $x \in \mathbb{R}$. Then $x \notin \{e' + \varepsilon' \mid e' \in \mathbb{E}_{r,k},\ \varepsilon' \in (0, \varepsilon]\}$ holds for any $0 < \varepsilon \ll 1$. Take such ε. It is enough to show $\mathcal{G}^{x-\varepsilon} W_{r,k} \subset \mathcal{G}^x W_{r,k}$ for proving that $\mathcal{G}$ is left-continuous. Pick any $x'_1, \dots, x'_k \in \mathbb{R}$ with $x'_1 + \cdots + x'_k = x - \varepsilon$. For any $1 \leq i \leq k$, there exists a unique $0 \leq j_i \leq \dim V_r$ such that $x'_i \in (e_{r,j_i+1}, e_{r,j_i}]$. By the choice of ε, we have $\sum_{i=1}^{k} (e_{r,j_i} - x'_i) \geq \varepsilon$. Thus there exist $x_1, \dots, x_k \in \mathbb{R}$ such that $x_1 + \cdots + x_k = x$ and $x_i \in (e_{r,j_i+1}, e_{r,j_i}]$ for any $1 \leq i \leq k$. Since $\bar{\mathcal{F}}^{x'_1} V_r \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bar{\mathcal{F}}^{x'_k} V_r = \bar{\mathcal{F}}^{x_1} V_r \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bar{\mathcal{F}}^{x_k} V_r$, we get $\mathcal{G}^{x-\varepsilon} W_{r,k} \subset \mathcal{G}^x W_{r,k}$. The remaining assertions are trivial.

(2) Pick any $k \in \mathbb{Z}_{\geq 0}$. For any $x < kre_-$, we have $\bar{\mathcal{F}}^{x/k} V_r = V_r$. Thus $\mathcal{G}^x W_{r,k} = W_{r,k}$ and this implies that $re_- \leq e_{\min}(W_{r,\bullet}, \mathcal{G})$. For any $x > kre_+$ and for any $x_1, \dots, x_k \in \mathbb{R}$ with $x_1 + \cdots + x_k = x$, there exists $1 \leq i \leq k$ such that $x_i > re_+$. Thus $\bar{\mathcal{F}}^{x_1} V_r \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bar{\mathcal{F}}^{x_k} V_r = 0$ and this implies that $e_{\max}(W_{r,\bullet}, \mathcal{G}) \leq re_+$.

(3) By Lemma 4.5 (1), it is enough to show that $\mathcal{G}^{j+k-1} W_{r,k} \subset W_{(k;r,j)}$. Take any $x_1, \dots, x_k \in \mathbb{R}$ with $x_1 + \cdots + x_k = j + k - 1$. Then $\lfloor x_1 \rfloor + \cdots + \lfloor x_k \rfloor \geq j$. Thus the image of $\bar{\mathcal{F}}^{x_1} V_r \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \bar{\mathcal{F}}^{x_k} V_r$ is contained in $W_{(k;r,j)}$. $\square$

By Claim 4.8 (3), we get

$$\begin{aligned} \int_{e_- + 1/r}^{e_+ + 1/r} \frac{\dim \mathcal{G}^{krx} W_{r,k}}{k^n r^n / n!} dx &\leq \sum_{j = kre_- + 1}^{kre_+} \frac{h^0(X, L^{\otimes kr} \cdot J_{(k;r,j)})}{k^{n+1} r^{n+1} / n!} \\ &\leq \int_{e_-}^{e_+} \frac{\dim \bar{\mathcal{F}}^{krx} V_{kr}}{k^n r^n / n!} dx. \end{aligned}$$

We note that both $\dim \bar{\mathcal{F}}^{krx} V_{kr}$ and $\dim \mathcal{G}^{krx+k} W_{r,k}$ are Lebesgue measurable on $x \in [e_-, e_+]$ since both are monotone decreasing functions. For any $x \in [e_-, e_+] \setminus \{e_{\max}(V_\bullet, \bar{\mathcal{F}})\}$, the limit

$$\lim_{k \to \infty} \frac{\dim \bar{\mathcal{F}}^{kx} V_k}{k^n / n!}$$

exists by [BC11, Lemma 1.6], [LM09, Theorem 2.13] and Proposition 4.3 (3). Hence, for any $r \geq r_1$, we have

$$\lim_{k\to\infty}\frac{\dim \bar{\mathcal{F}}^{krx}V_{kr}}{k^n r^n/n!} = \mathrm{vol}(\bar{\mathcal{F}}V_\bullet^x).$$

From dominated convergence, we have

$$\lim_{k\to\infty}\int_{e_-}^{e_+}\frac{\dim \bar{\mathcal{F}}^{krx}V_{kr}}{k^n r^n/n!}dx = \int_{e_-}^{e_+}\mathrm{vol}(\bar{\mathcal{F}}V_\bullet^x)dx.$$

By the same argument, the limit

$$\lim_{k\to\infty}\frac{\dim \mathcal{G}^{krx}W_{r,k}}{k^n r^n/n!}$$

exists for any $x \in [e_-, e_+] \setminus \{r^{-1}\cdot e_{\max}(W_{r,\bullet}, \mathcal{G})\}$ and

$$\lim_{k\to\infty}\int_{e_-+1/r}^{e_++1/r}\frac{\dim \mathcal{G}^{krx}W_{r,k}}{k^n r^n/n!}dx = \int_{e_-}^{e_+}\frac{\mathrm{vol}(\mathcal{G}W_{r,\bullet}^{rx})}{r^n}dx - \frac{r_0^n((-K_X)^{\cdot n})}{r}$$

holds since we have

$$\int_{e_-+1/r}^{e_++1/r}\frac{\dim \mathcal{G}^{krx}W_{r,k}}{k^n r^n/n!}dx = \int_{e_-}^{e_+}\frac{\dim \mathcal{G}^{krx}W_{r,k}}{k^n r^n/n!}dx - \frac{h^0(X, L^{\otimes kr})}{k^n r^{n+1}/n!}.$$

Thus we get

$$\int_{e_-}^{e_+}\frac{\mathrm{vol}(\mathcal{G}W_{r,\bullet}^{rx})}{r^n}dx - \frac{r_0^n((-K_X)^{\cdot n})}{r} \leq r_0^{n+1}A_r \leq \int_{e_-}^{e_+}\mathrm{vol}(\bar{\mathcal{F}}V_\bullet^x)dx.$$

By [BC11, Lemma 1.6] and [LM09, Theorem 3.5], for any $x \in [e_-, e_+] \setminus \{e_{\max}(V_\bullet, \bar{\mathcal{F}})\}$, we have

$$\lim_{r\to\infty}\frac{\mathrm{vol}(\mathcal{G}W_{r,\bullet}^{rx})}{r^n} = \mathrm{vol}(\bar{\mathcal{F}}V_\bullet^x).$$

Again by dominated convergence, we have

$$\lim_{r\to\infty}\int_{e_-}^{e_+}\frac{\mathrm{vol}(\mathcal{G}W_{r,\bullet}^{rx})}{r^n}dx = \int_{e_-}^{e_+}\mathrm{vol}(\bar{\mathcal{F}}V_\bullet^x)dx.$$

Therefore the limit $\lim_{r\to\infty}A_r$ exists and is equal to the right-hand side of Lemma 4.7. □

By Lemmas 2.5 (1) and 4.7, the pair $(X\times\mathbb{A}^1, \mathcal{I}_\bullet^{\cdot(1/r_0)}\cdot(t)^{\cdot d_\infty})$ is sub log canonical, where

$$d_\infty := 1 - \frac{e}{r_0} + \frac{1}{r_0^{n+1}((-K_X)^{\cdot n})}\int_{e_-}^{e_+}\mathrm{vol}(\bar{\mathcal{F}}V_\bullet^x)dx.$$

Consequently, we have proved the following:

Corollary 4.9. *Let* X, r_0, L, $V_\bullet$, $\mathcal{F}$, e_+, e_- *be as in the beginning of Section 4.2. Then the pair* $(X\times\mathbb{A}^1, \mathcal{I}_\bullet^{\cdot(1/r_0)}\cdot(t)^{\cdot d_\infty})$ *is sub log canonical, where*

$$\begin{aligned}\mathcal{I}_r &= I^{\mathcal{F}}_{(r,re_+)}+I^{\mathcal{F}}_{(r,re_+-1)}t^1+\cdots+I^{\mathcal{F}}_{(r,re_-+1)}t^{r(e_+-e_-)-1}+(t^{r(e_+-e_-)}),\\ d_\infty &= 1-\frac{e_+-e_-}{r_0}+\frac{1}{r_0^{n+1}((-K_X)^{\cdot n})}\int_{e_-}^{e_+}\mathrm{vol}(\bar{\mathcal{F}}V_\bullet^x)dx.\end{aligned}$$

4.3. **Ding semistability along subschemes.**

Theorem 4.10. *Let* X *be an* n*-dimensional* $\mathbb{Q}$*-Fano variety. Assume that* X *is Ding semistable. Take any nonempty proper closed subscheme* $\emptyset\neq Z\subsetneq X$ *corresponds to an ideal sheaf* $0\neq I_Z\subsetneq\mathcal{O}_X$*. Let* $\sigma\colon\hat{X}\to X$ *be the blowup along* Z*, let* $F\subset\hat{X}$ *be the Cartier divisor defined by the equation* $\mathcal{O}_{\hat{X}}(-F)=I_Z\cdot\mathcal{O}_{\hat{X}}$*. Then we have* $\beta(Z)\geq 0$*, where*

$$\beta(Z):=\mathrm{lct}(X;I_Z)\cdot((-K_X)^{\cdot n})-\int_0^\infty\mathrm{vol}_{\hat{X}}\left(\sigma^*(-K_X)-xF\right)dx.$$

Proof. Fix $r_0\in\mathbb{Z}_{>0}$ with $-r_0K_X$ Cartier and set $L:=\mathcal{O}_X(-r_0K_X)$. Let $V_\bullet$ be the complete graded linear series of L. Consider the $\mathbb{R}$-filtration $\mathcal{F}$ of $V_\bullet$ defined by

$$\mathcal{F}^xV_r:=\begin{cases}H^0(X,L^{\otimes r}\cdot I_Z^{\lceil x\rceil}) & \text{if } x\in\mathbb{R}_{\geq 0},\\ V_r & \text{otherwise.}\end{cases}$$

Then $\mathcal{F}$ is a decreasing, left-continuous, multiplicative and linearly bounded $\mathbb{R}$-filtration of $V_\bullet$. In fact, we can immediately check that $e_{\min}(V_\bullet,\mathcal{F})=0$ and $e_{\max}(V_\bullet,\mathcal{F})=r_0\tau_Z$. We note that the filtration $\mathcal{F}$ is saturated. Indeed, the homomorphism

$$\mathcal{F}^xV_r\otimes_{\mathbb{C}}L^{\otimes(-r)}\twoheadrightarrow I_{(r,x)}$$

induces the inclusion $I_{(r,x)}\subset I_Z^{\lceil x\rceil}$ for any $x\in\mathbb{R}_{\geq 0}$. Thus $\bar{\mathcal{F}}^xV_r=H^0(X,L^{\otimes r}\cdot I_{(r,x)})\subset\mathcal{F}^xV_r$.

Fix e_+, $e_-\in\mathbb{Z}$ with $e_+>r_0\tau_Z$ and $e_-<0$. By Corollary 4.9, the pair $(X\times\mathbb{A}^1,\mathcal{I}_\bullet^{\cdot(1/r_0)}\cdot(t)^{\cdot d_\infty})$ is sub log canonical, where

$$\begin{aligned}\mathcal{I}_r &= I_{(r,re_+)}+I_{(r,re_+-1)}t^1+\cdots+I_{(r,re_-+1)}t^{r(e_+-e_-)-1}+(t^{r(e_+-e_-)}),\\ d_\infty &= 1-\frac{e_+-e_-}{r_0}+\frac{1}{r_0^{n+1}((-K_X)^{\cdot n})}\int_{e_-}^{e_+}\mathrm{vol}(\mathcal{F}V_\bullet^x)dx.\end{aligned}$$

Note that

$$\mathrm{vol}(\mathcal{F}V_\bullet^x)=\begin{cases}r_0^n\,\mathrm{vol}_{\hat{X}}(\sigma^*(-K_X)-(x/r_0)F) & \text{if } x\in\mathbb{R}_{\geq 0},\\ r_0^n((-K_X)^{\cdot n}) & \text{otherwise.}\end{cases}$$

Thus $d_\infty = 1 - \tau + S$ holds, where $\tau := e_+/r_0$ and

$$S := \frac{1}{((-K_X)^{\cdot n})}\int_0^\infty \mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)dx.$$

Moreover, for any $r \gg 0$,

$$\mathcal{I}_r \subset I_Z^{re_+} + I_Z^{re_+-1}t^1 + \cdots + I_Z^1 t^{re_+-1} + (t^{re_+}) = (I_Z + (t))^{re_+}.$$

By Lemma 2.5 (2), the pair $(X \times \mathbb{A}^1, (I_Z + (t))^{\cdot\tau} \cdot (t)^{\cdot d_\infty})$ is sub log canonical.

Let $\theta\colon \mathcal{Y} \to X \times \mathbb{A}^1$ be a common log resolution of $X \times \mathbb{A}^1$, $I_Z + (t)$ and (t), that is, $\mathcal{Y}$ is smooth, $(I_Z+(t))\cdot\mathcal{O}_{\mathcal{Y}} =: \mathcal{O}_{\mathcal{Y}}(-F_1)$, $(t)\cdot\mathcal{O}_{\mathcal{Y}} =: \mathcal{O}_{\mathcal{Y}}(-F_2)$ satisfy that $\mathrm{Exc}(\theta)$, $\mathrm{Exc}(\theta) + F_1 + F_2$ are divisors with simple normal crossing supports. For any c_1, $c_2 \in \mathbb{R}$, we set

$$\begin{aligned} &\mathcal{J}\left(X \times \mathbb{A}^1, (I_Z + (t))^{\cdot c_1} \cdot (t)^{\cdot c_2}\right) \\ := \quad &\theta_*\mathcal{O}_{\mathcal{Y}}\left(\lceil K_{\mathcal{Y}} - \theta^* K_{X\times\mathbb{A}^1} - c_1F_1 - c_2F_2\rceil\right), \end{aligned}$$

where $\lceil K_{\mathcal{Y}} - \theta^* K_{X\times\mathbb{A}^1} - c_1F_1 - c_2F_2\rceil$ is the smallest $\mathbb{Z}$-divisor which contains $K_{\mathcal{Y}} - \theta^* K_{X\times\mathbb{A}^1} - c_1F_1 - c_2F_2$. If c_1, $c_2 \in \mathbb{R}_{\geq 0}$, then this is nothing but the multiplier ideal sheaf of the pair $(X \times \mathbb{A}^1, (I_Z + (t))^{\cdot c_1} \cdot (t)^{\cdot c_2})$ (see [Laz04b, §9] or [Tak06]). Take any $0 < \varepsilon \ll 1$. Then we have

$$\mathcal{O}_{X\times\mathbb{A}^1} \subset \mathcal{J}\left(X \times \mathbb{A}^1, (I_Z + (t))^{\cdot(1-\varepsilon)\tau} \cdot (t)^{\cdot(1-\varepsilon)d_\infty}\right)$$

since $X \times \mathbb{A}^1$ is log terminal. Pick any positive integer N with $(1-\varepsilon)d_\infty + N > 0$. By the definition of $\mathcal{J}\left(X \times \mathbb{A}^1, (I_Z + (t))^{\cdot c_1} \cdot (t)^{\cdot c_2}\right)$, we have

$$(t^N) \subset \mathcal{J}\left(X \times \mathbb{A}^1, (I_Z + (t))^{\cdot(1-\varepsilon)\tau} \cdot (t)^{\cdot(1-\varepsilon)d_\infty + N}\right) \subset \mathcal{O}_{X\times\mathbb{A}^1}.$$

By [Tak06, Theorem 3.2] and [Laz04b, Remark 9.5.23], we have

$$\begin{aligned} &\mathcal{J}\left(X \times \mathbb{A}^1, (I_Z + (t))^{\cdot(1-\varepsilon)\tau} \cdot (t)^{\cdot(1-\varepsilon)d_\infty + N}\right) \\ = &\sum_{0\leq\tau'\leq(1-\varepsilon)\tau} \mathcal{J}\left(X \times \mathbb{A}^1, I_Z^{\cdot\tau'} \cdot (t)^{\cdot(1-\varepsilon)(d_\infty+\tau)-\tau'+N}\right) \\ = &\sum_{0\leq\tau'\leq(1-\varepsilon)\tau} \mathcal{J}(X, I_Z^{\cdot\tau'}) \cdot \left(t^{\lfloor(1-\varepsilon)(d_\infty+\tau)-\tau'\rfloor+N}\right), \end{aligned}$$

where $\mathcal{J}(X, I_Z^{\cdot\tau'})$ is the multiplier ideal sheaf of the pair $(X, I_Z^{\cdot\tau'})$. This implies that

$$\mathcal{O}_X = \sum_{\tau' > S - \varepsilon(1+S)} \mathcal{J}(X, I_Z^{\cdot\tau'})$$

since $(1-\varepsilon)(d_\infty+\tau) - 1 = S - \varepsilon(1+S)$. Therefore we get the inequality $\mathrm{lct}(X; I_Z) \geq S$. $\square$

Remark 4.11. Assume that X is smooth. If Z is a reduced divisor with (X, Z) log canonical (resp. Z is a smooth subvariety with [Fuj15b, Assumption 3.1]), then the value $\beta(Z)$ is equal to the value $\eta(Z)$ in [Fuj15a, Definition 1.1] (resp. in [Fuj15b, Remark 3.10]).

5. Proofs

Theorem 5.1. *Let X be an n-dimensional $\mathbb{Q}$-Fano variety which is Ding semistable. Then we have $((-K_X)^{\cdot n}) \le (n+1)^n$. Moreover, if we further assume that X is smooth and $((-K_X)^{\cdot n}) = (n+1)^n$, then X is isomorphic to the projective space $\mathbb{P}^n$.*

Proof. We can assume that $n \ge 2$. Take any smooth closed point $p \in X$. Let $\sigma\colon \hat{X} \to X$ be the blowup along p and let F be the exceptional divisor of σ. By Theorem 4.10, we have

$$n \cdot ((-K_X)^{\cdot n}) \ge \int_0^\infty \mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)dx.$$

On the other hand, by Theorem 2.3 (1), we have

$$\begin{aligned} \int_0^\infty \mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xF)dx &\ge \int_0^{\sqrt[n]{((-K_X)^{\cdot n})}} (((-K_X)^{\cdot n}) - x^n)dx \\ &= \sqrt[n]{((-K_X)^{\cdot n})} \cdot \frac{n}{n+1}((-K_X)^{\cdot n}). \end{aligned}$$

Hence we get the inequality $(n+1)^n \ge ((-K_X)^{\cdot n})$. Assume that $(n+1)^n = ((-K_X)^{\cdot n})$. Then

$$\mathrm{vol}_{\hat{X}}(\sigma^*(-K_X) - xF) = (n+1)^n - x^n$$

for all $x \in [0, n+1]$. Thus, by Theorem 2.3 (2), we have $\varepsilon_p = n+1$. If X is smooth, this implies that $X \simeq \mathbb{P}^n$ by [CMSB02] and [Keb02] (see also [BS09]). $\square$

Proof of Theorem 1.1. This is an immediate consequence of Theorems 3.2 and 5.1. $\square$

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