THE LANGLANDS-SHAHIDI L-FUNCTIONS FOR BRYLINSKI-DELIGNE EXTENSIONS

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Abstract. We firstly discuss properties of the L-groups for Brylinski-Deligne extensions of split reductive groups constructed by M. Weissman. Secondly, the Gindikin-Karpelevich formula for an arbitrary Brylinski-Deligne extension is computed and expressed in terms of naturally defined elements of the group. Following this, we show that the Gindikin-Karpelevich formula can be interpreted as Langlands-Shahidi type L-functions associated with the adjoint action of the L-group for the Levi covering subgroup on certain Lie algebras. As a consequence, the constant term of Eisenstein series for Brylinski-Deligne extensions could be expressed in terms of global (partial) Langlands-Shahidi type L-functions. These L-functions are shown to possess meromorphic continuation to the whole complex plane. In the end, we determine the residual spectra of Brylinski-Deligne covers of some semisimple rank one groups.

Contents

1. Introduction

1.1. **Covering groups and L-groups.** Recently, Brylinski and Deligne have developed quite a general theory of covering groups of algebraic nature in their influential paper [BD01]. In particular, they classified multiplicative $\mathbb{K}_2$-torsors $\overline{\mathbb{G}}$ (equivalently in another language, central extensions of $\mathbb{G}$ by $\mathbb{K}_2$) over an algebraic group $\mathbb{G}$ in the Zariski site of Spec(F). Such a central extension is written as

$$\mathbb{K}_2 \hookrightarrow \overline{\mathbb{G}} \twoheadrightarrow \mathbb{G},$$

which has kernel the sheaf $\mathbb{K}_2$ defined by Quillen. In fact, they actually work over general schemes and not necessarily Spec(F), but for our purpose we take this more restrictive consideration in this paper.

There are two features among others which make the Brylinski-Deligne extensions distinct. Firstly, the classification of the $\mathbb{K}_2$-torsors above is functorial in terms of combinatorial data. Thus, it could be viewed as a generalization of the classification of a connected reductive group by its root data. Secondly, the category is quite encompassing. Suppose F is a number field. From $\overline{\mathbb{G}}$, we obtain the local topological extension $\overline{G}_v := \overline{\mathbb{G}}(F_v)$ in

$$\mu_n \hookrightarrow \overline{G}_v \twoheadrightarrow \mathbb{G}(F_v)$$

2010 *Mathematics Subject Classification.* Primary 11F70; Secondary 22E50.

Key words and phrases. Gindikin-Karpelevich formula, Brylinski-Deligne extensions, L-groups, Langlands-Shahidi L-functions, Eisenstein series.

as well as the global extension

$$\mu_n \hookrightarrow \overline{G}(\mathbb{A}_F) \twoheadrightarrow \mathbb{G}(\mathbb{A}_F),$$

where we assume $\mu_n \subseteq F^\times$. Though topological covering groups which arise in this way do not exhaust all existing ones, such Brylinski-Deligne type does contain all classically interesting examples which are of concern to us. We could mention for example coverings for split and simply-connected groups and the Kazhdan-Patterson type extension for $\mathbb{GL}_n$.

In this paper, we concentrate on the case when $\mathbb{G}$ is a *split* but arbitrary connected reductive group. In the formation of Langlands functoriality, which has been established for several cases, a crucial role is played by the L-group ${}^L\mathbb{G}$ of $\mathbb{G}$. However, the construction of L-group classically is restricted only to linear reductive algebraic groups (cf. [Bor79]). Due to the algebraic nature of Brylinski-Deligne extensions, it is expected that the theory of automorphic forms and representations of such covering groups could be developed in line with the linear algebraic case. For this purpose, a global L-group ${}^L\overline{G}$ and its local analogue ${}^L\overline{G}_v$ are indispensable. The latter should fit in the exact sequence

$$\overline{G}^\vee \hookrightarrow {}^L\overline{G}_v \twoheadrightarrow \mathrm{W}_{F_v},$$

where $\overline{G}^\vee$ is the complex dual group of $\overline{G}_v$ and W_{F_v} the Weil group of F_v, see [Tat79].

There has been a series of works in this direction starting with P. McNamara and M. Weissman (cf. [McN12], [We09]-[We16-3]). In the geometric setting, one may refer to the work of Reich ([Re11]) and Finkelberg-Lysenko ([FiLy10]). Meanwhile, McNamara gave the definition of the root data of $\overline{G}^\vee$ in order to interpret the established Satake isomorphism for $\overline{G}_v$ in the number field case. The root data of $\overline{G}^\vee$ rely on the degree n and the root data of $\mathbb{G}$, modified using the combinatorics associated with $\overline{\mathbb{G}}$ in the Brylinski-Deligne classification. Therefore, it is independent of the place $v \in |F|$, and this justifies the absence of v in the notation $\overline{G}^\vee$ we use.

More importantly, the work of Weissman in [We12], [We13], [We16-2] and [We16-3] has supplied us the indispensable L-groups for any further development of the theory of automorphic forms on Brylinski-Deligne covers. The insight is that an L-parameter is just a splitting of ${}^L\overline{G}_v$ over W_{F_v}. Moreover, the key is that *even if* the group ${}^L\overline{G}_v$ as an extension

$$\overline{G}^\vee \hookrightarrow {}^L\overline{G}_v \twoheadrightarrow \mathrm{W}_{F_v}$$

is isomorphic to the direct product $\overline{G}^\vee \times \mathrm{W}_{F_v}$, it is *not canonically* so. This reflects the fact, locally for instance, that there is no canonical genuine representation of $\overline{G}_v$. In fact, one could show by examples (for instance certain degree two coverings of $\mathbb{PGL}_2(F_v)$) that in general ${}^L\overline{G}_v$ is only a semidirect product of $\overline{G}^\vee$ and W_{F_v}, see [GaG14].

In this paper, we concentrate on Brylinski-Deligne type coverings of a general split reductive group. By using the L-groups constructed by Weissman, we show that Langlands-Shahidi type partial L-functions appear naturally in the constant terms of Eisenstein series.

1.2. **Main results.** We now briefly explain the content and results of the paper.

In this paper, we assume only the necessary requirement $\mu_n \subseteq F^\times$ as opposed to $\mu_{2n} \subseteq F^\times$ in quite a part of the literature on covering groups. For notations and terminologies, we follow [GaG14]. In particular, we introduce an incarnation category, which is equivalent to the Brylinski-Delgine category of multiplicative $\mathbb{K}_2$-torsors over $\mathbb{G}$. After this, we recall briefly the construction of L-groups both locally and globally. It is important that the construction is functorial with respect to Levi covering subgroups of the ambient group.

Suppose $\mathbb{T}$ is a split maximal F-torus in $\mathbb{G}$. With the construction of L-group, one obtains a local Langlands correspondence for Brylinski-Deligne covering tori $\overline{T}_v$. More precisely, given an irreducible genuine representation of $\overline{T}_v$ (or equivalently a genuine character $\overline{\chi}$ of the center $Z(\overline{T}_v)$ of $\overline{T}_v$), there is a naturally associated splitting $\rho_{\overline{\chi}}$ of ${}^L\overline{T}_v$ over W_{F_v}. If $\overline{T}_v \subseteq \overline{G}_v$ is the covering torus of some Brylinski-Deligne covering group $\overline{G}_v$; then, coupled with the functorial map ${}^L\varphi$ from ${}^L\overline{T}_v$ to ${}^L\overline{G}_v$ mentioned above, one has a splitting ${}^L\varphi \circ \rho_{\overline{\chi}}$ of ${}^L\overline{G}_v$ over W_{F_v}.

Our next goal is to compute the unramified Gindikin-Karpelevich formula. The computation is carried out as in the linear algebraic case. Importantly, the Gindikin-Karpelevich formula is expressed in terms of naturally defined elements of the covering group $\overline{G}_v$. It is precisely this fact which enables us to give an interpretation in terms of local Langlands-Shahidi L-functions.

Consider the adjoint representation $Ad : {}^L\overline{G}_v \to GL(\overline{\mathfrak{g}}^\vee)$. We aim to express the Gindikin-Karpelevich formula for unramified principal series in terms of the composition $Ad \circ {}^L\varphi \circ \rho_{\overline{\chi}}$, i.e., as the local Artin L-function associated with it. For this purpose, Proposition 4.3 plays a pivotal role in this paper. This gives rise to the desired Theorem 7.8, where the Gindikin-Karpelevich formula is interpreted as local Langlands-Shahidi L-functions.

To obtain global L-functions, we recall in §8.1 the definition of automorphic (partial) L-function of an automorphic representation $\overline{\sigma}$ of $\overline{\mathbb{H}}(\mathbb{A}_F)$ of Brylinski-Deligne type associated with a finite dimensional representation $R : {}^L\overline{H} \to GL(V)$. In particular, we are interested in the case where $\overline{\mathbb{H}} = \overline{\mathbb{M}}$ is a Levi subgroup of $\overline{\mathbb{G}}$ and that R is the adjoint representation of ${}^L\overline{M}$ on a certain subspace $\overline{\mathfrak{u}}^\vee$ of the Lie algebra $\overline{\mathfrak{g}}^\vee$. By Satake isomorphism, the computation for the unramified representations could be reduced to the covering tori case.

In view of these, the constant term of Eisenstein series for induction from general parabolic subgroups can be expressed in terms of Langlands-Shahidi type L-functions, by combining the formula from the unramified places. We work out the case for maximal parabolic subgroups, and the general case is similar despite the complication in notations. The main result for this part is Theorem 8.1. We also show that every Langlands-Shahidi L-function for Brylinski-Deligne covering groups has meromorphic continuation to the whole complex plane, see Theorem 8.4.

As simple examples, we determine at the end of the paper the residual spectra of Brylinski-Deligne covers $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$ and $\overline{\mathbb{GL}}_2(\mathbb{A}_F)$ of an arbitrary degree. We also compute the partial L-functions appearing in the constant terms of Eisenstein series for induction from maximal parabolic subgroups of the double cover $\overline{\mathbb{Sp}}_4(\mathbb{A}_F)$. It is shown to agree with that given in [Gao12].

1.3. **Notations and Terminology.**

F: a number field, or a local field with finite residue field of characteristic p and cardinality q in the nonarchimedean case. Write $\mathcal{O}_F$ for the ring of integers of F.

I or I_v: the inertia group of the absolute Galois group of a local field.

Frob or Frob_v: the geometric Frobenius class of a local field.

$\mathbb{G}_{\mathrm{add}}$ and $\mathbb{G}_{\mathrm{mul}}$: the additive and multiplicative group over F respectively.

$\mathbb{T}$: a split torus with character group X and cocharacter group Y.

$(\mathbb{G}, \mathbb{T}, \mathbb{B}, \{\varphi_\alpha\})$: $\mathbb{G}$ is a general *split* reductive group over F with a fixed maximal split torus $\mathbb{T}$. It has root datum $(X, \Psi, Y, \Psi^\vee)$. We also fix a set of positive roots $\Psi^+ \subseteq \Psi$, and thus also a set of simple roots $\Delta \subseteq \Psi$. Let $\mathbb{B} = \mathbb{T}\mathbb{U}$ be the Borel subgroup of $\mathbb{G}$ associated with Ψ^+. Let $\mathbb{G}^{sc}$ be the simply connected cover of the derived subgroup $\mathbb{G}^{der} \subseteq \mathbb{G}$ with the natural map denoted by $\Phi : \mathbb{G}^{sc} \to \mathbb{G}$. Besides, we fix a Chevalley system of épinglage (pinning) for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$ (cf. [BrTi84, §3.2.1-2]). That is, we have an isomorphism $\mathbb{e}_\alpha : \mathbb{G}_{\mathrm{add}} \to \mathbb{U}_\alpha$ for each $\alpha \in \Psi$, where $\mathbb{U}_\alpha$ is the root subgroup associated with α. Moreover, for each $\alpha \in \Psi$, there is the induced morphism $\varphi_\alpha : \mathbb{SL}_2 \to \mathbb{G}$ which restricts to $\mathbb{e}_{\pm\alpha}$ on the upper and lower triangular subgroup of unipotent matrices of $\mathbb{SL}_2$.

Q: an integer-valued Weyl-invariant quadratic form on Y with associated symmetric bilinear form

$$B_Q(y_1, y_2) := Q(y_1 + y_2) - Q(y_1) - Q(y_2).$$

"character": a character of a group is a continuous homomorphism valued in $\mathbf{C}^\times$, while a *unitary* character refers to a character with absolute value 1.

"section" and "splitting": for an exact sequence $A \hookrightarrow B \twoheadrightarrow C$ of groups we call any map $s : C \to B$ a *section* if its post composition with the last projection map on C is the identity map on C. We call s a *splitting* if it is a homomorphism.

$\mathsf{Spl}(B, C)$: all splittings of B over C of the exact sequence above. It is a torsor over $\mathrm{Hom}(C, A)$ when the extension is central.

In general, notations will be explained at the first time they appear in the text.

2. The Brylinski-Deligne extensions

2.1. **The Brylinski-Deligne extensions.** Let F be a number field or its localization, and we will be more specific when the context requires so. Let $\mathbb{G}$ be a split reductive group over F with root data $(X, \Psi, Y, \Psi^\vee)$. We also fix a set of simple roots $\Delta \subseteq \Psi$.

In their seminal paper [BD01], Brylinski and Degline have given a classification of the central extensions $\overline{\mathbb{G}}$ of $\mathbb{G}$ by $\mathbb{K}_2$ in the category of sheaves of groups on the big Zariski site over $\mathrm{Spec}(F)$, which are written in the form $\mathbb{K}_2 \hookrightarrow \overline{\mathbb{G}} \twoheadrightarrow \mathbb{G}$. Denote by $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ the category of such central extensions. Any $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ gives an exact sequence of F'-rational points for any field extension F' of F:

$$\mathbb{K}_2(F') \longhookrightarrow \overline{\mathbb{G}}(F') \longtwoheadrightarrow \mathbb{G}(F').$$

The left exactness follows from the the fact that the extension $\overline{\mathbb{G}}$ is an extension of sheaves, while the right exactness at the last term is due to the vanishing of $\mathrm{H}^1_{\mathrm{Zar}}(F', \mathbb{K}_2)$, an analogue of Hilbert Theorem 90.

We will recall briefly the classification of such extensions for $\mathbb{G}$ being a torus, a semi-simple simply-connected group and a general reductive group in the sequel. For notations, we follow [GaG14, §2.1-2.5].

2.1.1. *Central extensions of tori.* Let $\mathbb{T}$ be a split torus with character group $X = X(\mathbb{T})$ and cocharacter group $Y = Y(\mathbb{T})$. The category of central extensions $\mathsf{CExt}(\mathbb{T}, \mathbb{K}_2)$ is equivalent to the category of pairs $(Q, \mathcal{E})$, where Q is a quadratic form on Y and $\mathcal{E}$ a central extension of Y by $F^\times$ whose commutator is given by

$$[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}.$$

Here B_Q is the symmetric bilinear form associated with Q, as given in §1.3. For any two pairs $(Q, \mathcal{E})$ and $(Q', \mathcal{E}')$, morphisms exists if and only if $Q = Q'$, in which case they are defined to be the isomorphisms between the two extensions $\mathcal{E}$ and $\mathcal{E}'$.

For $\overline{\mathbb{T}} \in \mathsf{CExt}(\mathbb{T}, \mathbb{K}_2)$ over F, the extension $\mathcal{E}$ is obtained as follows. First, taking the rational points of the Laurent field $F(\!(\tau)\!)$ gives $\mathbb{K}_2(F(\!(\tau)\!)) \hookrightarrow \overline{\mathbb{T}}(F(\!(\tau)\!)) \twoheadrightarrow \mathbb{T}(F(\!(\tau)\!))$. Pull-back by the map $Y \to \mathbb{T}(F(\!(\tau)\!))$ which sends $y \in Y$ to $y \otimes \tau \in \mathbb{T}(F(\!(\tau)\!))$, and then push-out by the tame symbol $\mathbb{K}_2(F(\!(\tau)\!)) \to F^\times$ to give the extension $\mathcal{E}$ over Y by $F^\times$. Here the tame symbol is defined to be

$$\{f, g\} \mapsto (-1)^{\mathrm{val}(f)\mathrm{val}(g)} \left(\frac{f^{\mathrm{val}(g)}}{g^{\mathrm{val}(f)}} \right)(0).$$

In particular, $\{a, \tau\} \in \mathbb{K}_2(F(\!(\tau)\!))$ is sent to a for all $a \in F^\times$.

For convenience, for any lifting $\overline{y \otimes \tau} \in \overline{\mathbb{T}}(F(\!(\tau)\!))$ of $y \otimes \tau \in \mathbb{T}(F(\!(\tau)\!))$, we write

$$\left[\overline{y \otimes \tau}\right] := \text{ the image of } \overline{y \otimes \tau} \text{ in } \mathcal{E}. \tag{1}$$

2.1.2. *Central extensions of semi-simple simply-connected groups.* Let $\mathbb{G}$ be a split semi-simple simply-connected group over F with root data $(X, \Psi, Y, \Psi^\vee)$. Let W be the Weyl-group of $\mathbb{G}$.

Theorem 2.1 ([BD01, Theorem 4.7])**.** *The category $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ is rigid, i.e., any two objects have at most one morphism between them. The set of isomorphism classes is classified by W-invariant integer-valued quadratic forms.*

A special case is when $\mathbb{G}$ is almost simple. In this case, any Weyl-invariant quadratic form is determined by $Q(\alpha^\vee)$, where $\alpha^\vee \in \Psi^\vee$ is the short coroot associated to a long root α. The fact that $Q(\alpha^\vee)$ for short coroot uniquely determines the quadratic form Q follows from the following easy fact.

Lemma 2.2 ([GaG14, Proposition 6.5])**.** *For any $\alpha^\vee \in \Psi^\vee$ and $y \in Y$,*

$$B_Q(\alpha^\vee, y) = Q(\alpha^\vee) \cdot \langle \alpha, y \rangle,$$

where $\langle -, - \rangle$ denotes the paring between X and Y.

Example 2.3. The classical metaplectic double cover arises from a Brylinski-Deligne central extension $\overline{\mathbb{Sp}}_{2r}$ over $\mathbb{Sp}_{2r}$ of this type. Let $\alpha_1^\vee, \alpha_2^\vee, ..., \alpha_r^\vee$ be the simple coroots of $\mathbb{Sp}_{2r}$ with $\alpha_1^\vee$ the unique short one. Let Q be the unique Weyl invariant quadratic form on Y with $Q(\alpha_1^\vee) = 1$, see also [BD01, p.g. 7-8]. This gives the desired $\overline{\mathbb{Sp}}_{2r}$ according to the classification theorem above.

As a result of Theorem 2.1, any Q gives rise to $\overline{\mathbb{G}}_Q \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$, unique up to unique isomorphism. By pull-back, one obtains $\overline{\mathbb{T}}_Q$ and thus the central extension $\mathcal{E}_Q$. We recall the rigidification of $\mathcal{E}_Q$, as in [GaG14, §2.4],

2.1.3. *The Brylinski-Deligne liftings.* We continue to assume $\mathbb{G}$ simply-connected. We also assume a fixed Chevalley system of eṕinglage for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$ (cf. §1.3). In particular, for each $\alpha \in \Psi$ with associated root subgroup $\mathbb{U}_\alpha$, there is a fixed isomorphism $\mathbb{e}_\alpha : \mathbb{G}_{\text{add}} \to \mathbb{U}_\alpha$. Also, there is the induced morphism $\varphi_\alpha : \mathbb{SL}_2 \to \mathbb{G}$.

Let $a \in \mathbb{G}_{\text{mul}}$, and consider $\mathbb{e}_+(a), \mathbb{e}_-(a), \mathbb{w}_o(a)$ of $\mathbb{SL}_2$ as follows:

$$\mathbb{e}_+(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \mathbb{e}_-(a) = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix},$$

$$\mathbb{w}_o(a) = \mathbb{e}_+(a)\mathbb{e}_-(a^{-1})\mathbb{e}_+(a) = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix}, \quad \mathbb{h}_o(a) = \mathbb{w}_o(a)\mathbb{w}_o(-1) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

For each α, consider the following elements in $\mathbb{G}$ (called Tits trijection in [BD01, §11])

$$\mathbb{e}_\alpha(a) = \varphi_\alpha(\mathbb{e}_+(a)), \quad \mathbb{e}_{-\alpha}(a^{-1}) = \varphi_\alpha(\mathbb{e}_-(a^{-1})), \quad \mathbb{w}_\alpha(a) := \varphi_\alpha(\mathbb{w}_o(a)).$$

We also write $\mathbb{h}_\alpha(a) := \varphi_\alpha(\mathbb{h}_o(a))$ and thus $\mathbb{h}_\alpha(a) = \mathbb{w}_\alpha(a)\mathbb{w}_\alpha(-1)$. Moreover, write $\mathbb{T}_\alpha = \{\mathbb{h}_\alpha(a) : a \in \mathbb{G}_{\text{mul}}\} \subseteq \mathbb{G}$.

Recall that any Brylinski-Deligne extension $\overline{\mathbb{G}}$ splits uniquely over the unipotent subgroup $\mathbb{U}_\alpha \subseteq \mathbb{G}$, see [BD01, §3]. For any $a \in \mathbb{G}_{\text{add}}$, denote by $\overline{\mathbb{e}}_\alpha(a)$ this unique splitting of $\mathbb{e}_\alpha(a)$. Consider the lifting $\overline{\mathbb{w}}_\alpha(a) \in \overline{\mathbb{G}}$ of the element $\mathbb{w}_\alpha(a) \in N(\mathbb{T})$ given by

$$\overline{\mathbb{w}}_\alpha(a) := \overline{\mathbb{e}}_\alpha(a) \cdot \overline{\mathbb{e}}_{-\alpha}(a^{-1}) \cdot \overline{\mathbb{e}}_\alpha(a).$$

For any $b \in \mathbb{G}_{\text{mul}}$, the Brylinski-Deligne lifting $\overline{\mathbb{h}}_\alpha^{[b]}(a)$ of $\mathbb{h}_\alpha(a)$ in $\overline{\mathbb{T}}_\alpha$ is then by definition (cf. [BD01, §11.1])

$$\overline{\mathbb{h}}_\alpha^{[b]}(a) := \overline{\mathbb{w}}_\alpha(ab) \cdot \overline{\mathbb{w}}_\alpha(b)^{-1}.$$

Two important properties are

$$\overline{\mathbb{h}}_\alpha^{[b]}(a) \cdot \overline{\mathbb{h}}_\alpha^{[b]}(c) = \overline{\mathbb{h}}_\alpha^{[b]}(ac) \cdot \{a, c\}^{Q(\alpha^\vee)}, \tag{2}$$

$$\overline{\mathbb{h}}_\alpha^{[db]}(a) = \overline{\mathbb{h}}_\alpha^{[b]}(a) \cdot \{d, a\}^{Q(\alpha^\vee)}. \tag{3}$$

This gives rise to an inherited lifting into $\mathcal{E}_Q$ of the one-dimension lattice $Y_\alpha \subseteq Y$ spanned by $\alpha^\vee \in \Psi^\vee$, in view of §2.1.1. Recall that for any lifting $\overline{y \otimes \tau} \in \overline{\mathbb{T}}(F(\!(\tau)\!))$ of $y \otimes \tau \in \mathbb{T}(F(\!(\tau)\!))$, we have denoted by $\left[\overline{y \otimes \tau}\right] \in \mathcal{E}_Q$ its image in $\mathcal{E}$, as in (1). In particular, for any $f \in F(\!(\tau)\!)^\times$ and $k \in \mathbf{Z}$, we consider

$$\left[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^k)\right] := \text{ the image of } \overline{\mathbb{h}}_\alpha^{[f]}(\tau^k) \in \overline{\mathbb{T}}(F(\!(\tau)\!)) \text{ in } \mathcal{E}_Q. \tag{4}$$

Definition 2.4. The map given by

$$\mathbb{T}_\alpha \to \overline{\mathbb{T}}_\alpha, \quad \mathbb{h}_\alpha(a) \mapsto \overline{\mathbb{h}}_\alpha^{[b]}(a),$$

is called the Brylinski-Deligne lifting. For every $k \in \mathbf{Z}$, the element $\overline{\mathbb{h}}_\alpha^{[f]}(k \cdot \alpha^\vee) := \left[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^k)\right] \in \mathcal{E}_Q$ is a lifting of $k\alpha^\vee \in Y_\alpha$. We also call this lifting of Y_α into $\mathcal{E}_Q$ the Brylinski-Deligne lifting.

2.1.4. *Rigidifying $\mathcal{E}_Q$.* As in [GaG14, §2.4], we could rigidify $\mathcal{E}_Q$ by equipping it with the set $\{\overline{\mathbb{h}}_\alpha^{[1]}(\alpha^\vee)\}_{\alpha\in\Delta}$: there is a unique automorphism of $\mathcal{E}_Q$ which fixes all these elements. For convenience, we identify $\mathcal{E}_Q$ with $\mathcal{E}_Q^{sc}$, the abstract group generated by $\{a\}_{a\in F^\times} \cup \{\gamma_\alpha\}_{\alpha^\vee\in\Delta^\vee}$ subject to the conditions:

- $F^\times$ is contained in the center of $\mathcal{E}_Q^{sc}$,
- $[\gamma_\alpha, \gamma_\beta] = (-1)^{B_Q(\alpha^\vee,\beta^\vee)}$ for any $\alpha^\vee, \beta^\vee \in \Delta^\vee$.

The unique isomorphism from $\mathcal{E}_Q$ to $\mathcal{E}_Q^{sc}$ is given by $\overline{\mathbb{h}}_\alpha^{[1]}(\alpha^\vee) \mapsto \gamma_\alpha$. From now we will use $\mathcal{E}^{sc}$ and $\mathcal{E}_Q^{sc}$ interchangeably.

2.1.5. *Central extensions of general split reductive groups.* Now we use $\mathbb{G}$ to denote a split reductive group $\mathbb{G}$, with a maximal split torus $\mathbb{T}$. Write $(X, \Psi, Y, \Psi^\vee)$ for its root data, where X and Y are the character and cocharacter groups of $\mathbb{T}$ respectively. Assume we have a fixed Chevalley system of epínglage for $(\mathbb{G}, \mathbb{T}, \mathbb{B})$.

Let $\Phi : \mathbb{G}^{sc} \twoheadrightarrow \mathbb{G}^{der} \hookrightarrow \mathbb{G}$ be the natural composition, where $\mathbb{G}^{sc}$ is the simply connected cover of the derived group $\mathbb{G}^{der}$ of $\mathbb{G}$. Let $\mathbb{T}^{sc} = \Phi^{-1}(\mathbb{T})$ be the maximal split torus of $\mathbb{G}^{sc}$ with cocharacter group $Y^{sc} \subseteq Y$. The restriction $Q|_{Y^{sc}}$ gives an element $\overline{\mathbb{G}}^{sc} \in \mathsf{CExt}(\mathbb{G}^{sc}, \mathbb{K}_2)$ by Theorem 2.1. It also gives the extension $\mathcal{E}^{sc}_{Q|_{Y^{sc}}}$ in §2.1.4. However, for simplicity of notations, we just write $\mathcal{E}_Q^{sc}$ with no confusion caused.

Theorem 2.5 ([BD01, Theorem 6.2])**.** *The category $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ is equivalent to the category specified by the triples $(Q, \mathcal{E}, \phi)$ with the following properties:*

- *Q is a Weyl invariant quadratic form on Y and $\mathcal{E}$ a central extension of Y by $F^\times$ such that the commutator is given by $[y_1, y_2] = (-1)^{B_Q(y_1,y_2)}$.*
- *ϕ is a morphism from $\mathcal{E}_Q^{sc}$ to $\mathcal{E}$ such that the following diagram commute:*

$$\begin{tikzcd}
F^\times \arrow[r, hook] \arrow[d, equal] & \mathcal{E}_Q^{sc} \arrow[r, two heads] \arrow[d, "\phi"] & Y^{sc} \arrow[d, hook] \\
F^\times \arrow[r, hook] & \mathcal{E} \arrow[r, two heads] & Y.
\end{tikzcd}$$

Homomorphisms between two triples $(Q_1, \mathcal{E}_1, \phi_1)$ and $(Q_2, \mathcal{E}_2, \phi_2)$ exist only for $Q_1 = Q_2$, in which case they are defined to be the homomorphisms between $\mathcal{E}_1$ and $\mathcal{E}_2$ which respect the above commutative diagram.

2.2. **Incarnation functor and an equivalent category.** In [GaG14], we defined a category $\mathsf{Bis}_{\mathbb{G}} = \bigsqcup_Q \mathsf{Bis}_{\mathbb{G}}^Q$ and an incarnation functor from $\mathsf{Bis}_{\mathbb{G}}$ to $\mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$, which actually gives an equivalence of Picard categories.

Definition 2.6 ([GaG14, §2.6])**.** The category $\mathsf{Bis}_{\mathbb{G}}^Q$ consists of pairs (D, η), where D is a $\mathbf{Z}$-valued bilinear (not necessarily symmetric) form on Y such that $D(y_1, y_2) + D(y_2, y_1) = B_Q(y_1, y_2)$ and $\eta : Y^{sc} \to F^\times$ a group homomorphism. In particular, $D(y, y) = Q(y)$.

We call D a bisector of Q. Morphisms of pairs (D_i, η_i) for $i = 1, 2$ consist of maps $H : Y \to F^\times$ (not necessarily a homomorphism) such that

(i) $(-1)^{D_2(y_1,y_2)-D_1(y_1,y_2)} = H(y_1 + y_2) \cdot H(y_1)^{-1} \cdot H(y_2)^{-1}$,
(ii) $\eta_2(\alpha^\vee)/\eta_1(\alpha^\vee) = H(\alpha^\vee)$ for all $\alpha^\vee \in \Delta^\vee$.

The composition of morphisms is given by multiplication, i.e., $H_1 \circ H_2(y) = H_1(y) \cdot H_2(y)$.

It is shown in [We14, Proposition 2.4] that for $(D_i, \eta_i), i = 1, 2$ associated with the same Q, there always exists H satisfying (i). Consequently, we see that up to isomorphism we could always fix a base D and allow η to be varied. More precisely, we have the following.

Example 2.7. Let D_1, D_2 be two bisectors of Q. Then, for any given η_1, (D_1, η_1) is isomorphic to (D_2, η_2) for some η_2. We explain how the η_2 can be obtained. Pick $H : Y \to F^\times$ such that the property (i) is satisfied with respect to D_1 and D_2. Define η_2 to be such that $\eta_2(\alpha^\vee)/\eta_1(\alpha^\vee) = H(\alpha^\vee)$ for all $\alpha^\vee \in \Delta^\vee$. Then $(D_1, \eta_1) \simeq (D_2, \eta_2)$ for η_2 obtained in this way.

However, on the other hand, we observe easily that not every (D_2, η) is isomorphic to $(D_1, \mathbb{1})$ for some D_1.

Example 2.8. Assume that $\mathbb{G}$ has a simply-connected derived group $\mathbb{G}^{\mathrm{der}}$. Then, Y/Y^{sc} is a free $\mathbf{Z}$-module. Let $(D,\eta) \in \mathsf{Bis}^Q_{\mathbb{G}}$; then $(D,\eta) \simeq (D,\mathbb{1})$. In fact, any $H \in \mathrm{Hom}(Y, F^\times)$ extending η will provide a morphism from $(D,\mathbb{1})$ to (D,η). Therefore it follows from Example 2.7 that any two $(D_i,\eta_i) \in \mathsf{Bis}^Q_{\mathbb{G}}, i = 1,2$ for such $\mathbb{G}$ are isomorphic.

2.2.1. *Equivalence with the Brylinski-Deligne category.* The incarnation functor from $\mathsf{Bis}^Q_{\mathbb{G}}$ to $\mathsf{CExt}(\mathbb{G},\mathbb{K}_2)$ is realized by defining a functor from $\mathsf{Bis}^Q_{\mathbb{G}}$ to the category of $\{(Q,\mathcal{E},\phi)\}$, in which the target object of (D,η) is denoted as $(Q,\mathcal{E}_D,\phi_{D,\eta})$. The extension $\mathcal{E}_D$ is described as $F^\times \times_D Y$ with group law given by

$$(a,y_1)\cdot(b,y_2) = \big(ab\cdot(-1)^{D(y_1,y_2)}, y_1+y_2\big).$$

The map $\phi_{D,\eta} : \mathcal{E}_Q^{sc} \to \mathcal{E}$ is the one uniquely determined by

$$\gamma_\alpha \mapsto (\eta(\alpha^\vee),\alpha^\vee) \text{ for all } \alpha^\vee \in \Delta^\vee.$$

With this definition, one can check easily that the incarnation functor $\mathsf{Inc}_{\mathbb{G}}$ establishes an equivalence of commutative Picard categories between $\mathsf{Bis}_{\mathbb{G}}$ and $\mathsf{CExt}(\mathbb{G},\mathbb{K}_2)$. Note that implicitly we have fixed a quasi-inverse functor from the category $\{(Q,\mathcal{E},\phi)\}$ to $\mathsf{CExt}(\mathbb{G},\mathbb{K}_2)$. We will work with $\overline{\mathbb{G}}$ which is incarnated by (D,η), and write it as $\overline{\mathbb{G}}_{D,\eta}$.

It is desirable to have a more precise description of $\overline{\mathbb{G}}_{D,\eta}$. First of all, we consider the case $\mathbb{G} = \mathbb{T}$ and a $\mathbb{K}_2$-torsor $\overline{\mathbb{T}}_D$ which is incarnated by D. Then, if $D = \sum_i x_1^i \otimes x_2^i \in X \otimes X$, we can write $\overline{\mathbb{T}}_D = \mathbb{K}_2 \times_D \mathbb{T}$ with the group law given by

$$(1,t_1)\cdot(1,t_2) = \Big(\prod_i \left\{x_1^i(t_1), x_2^i(t_2)\right\}, t_1t_2\Big). \tag{5}$$

In the case of a general reductive group $\mathbb{G}$, we will write $\Phi_{D,\eta} : \overline{\mathbb{G}}^{sc} \to \overline{\mathbb{G}}_{D,\eta}$ for the natural pull-back map, to emphasize the dependence of the target group expressed using (D,η). First, the covering torus of $\overline{\mathbb{G}}_{D,\eta}$ is incarnated by D as above, for which we have written as $\overline{\mathbb{T}}_D$. For any $a,b \in \mathbb{G}_{\mathrm{mul}}$, we can describe the element $\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha^{[b]}(a)\big)$ for $\alpha \in \Delta$ in terms of $\mathbb{K}_2 \times_D \mathbb{T}$, as follows.

We fix $\alpha \in \Delta$. First, for $b=1$, one has $\Phi_{D,\eta}(\overline{\mathbb{h}}_\alpha^{[1]}(a)) = \big(\{\eta(\alpha^\vee),a\}, \mathbb{h}_\alpha(a)\big)$, which is proved in [GaG14, Proposition 2.5]. For general $b \in F^\times$, it follows from the equality (3) that

$$\Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha^{[b]}(a)\big) = \Big(\Big\{b^{Q(\alpha^\vee)}\cdot\eta(\alpha^\vee), a\Big\}, \mathbb{h}_\alpha(a)\Big) \text{ for all } \alpha^\vee \in \Delta^\vee, \tag{6}$$

where the right hand side is written in terms of $\mathbb{K}_2 \times_D \mathbb{T}$.

2.3. **Local finite degree topological covers.** Assume first that F is a local field with residual characteristic p. Let $n \in \mathbf{N}_{\geq 1}$ be a natural number and assume $\mu_n \subseteq F$. The Hilbert symbol descends to a bilinear from on $F^\times/(F^\times)^n$ and factors through $\mathbb{K}_2(F)$. In the tame case, i.e. when $\gcd(n,p)=1$, we have the formula

$$(x,y)_n = \kappa(x,y)^{\frac{q-1}{n}}, \quad \kappa(x,y) = \overline{(-1)}^{\mathrm{val}(x)\mathrm{val}(y)}\overline{\left(\frac{x^{\mathrm{val}(y)}}{y^{\mathrm{val}(x)}}\right)} \in \mathbf{f},$$

where $\overline{(-)}$ denotes the reduction of $\mathcal{O}_F$ into the residue field $\mathbf{f}$ of F.

Let $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G},\mathbb{K}_2)$ be a Brylinski-Deligne central extension over F. It gives a central extension of F-groups. From the push-out by the n-th Hilbert symbol, we get a central extension $\overline{G}$ of $G = \mathbb{G}(F)$ by μ_n. We say that $\overline{G}$ is incarnated by (D,η) if $\overline{\mathbb{G}}$ is, and in this case write $\overline{G}_{D,\eta}$ for it.

Example 2.9. Let $\mathbb{G} = \mathbb{T}$ be a split torus. Then from (5), $\overline{T}$ can be described as $\mu_n \times_D T$ with the group law given by:

$$\begin{cases} \mu_n \text{ is central in } \overline{T}, \\ \big[(\zeta_1, y_1\otimes a),(\zeta_2,y_2\otimes b)\big] = (a,b)_n^{B_Q(y_1,y_2)}, \\ (1,y_1\otimes a)\cdot(1,y_2\otimes a) = \big((a,a)_n^{D(y_1,y_2)}, (y_1+y_2)\otimes a\big), \\ (1,y\otimes a)\cdot(1,y\otimes b) = \big((a,b)_n^{Q(y)}, y\otimes(ab)\big). \end{cases} \tag{7}$$

2.4. **Local splitting properties.**

2.4.1. *Splitting over a maximal compact group.* Continue to assume F a local field. The Bruhat-Tits building of $G = \mathbb{G}(F)$ over F has a hyperspecial point determined by the epínglage of $\mathbb{G}$, which gives an associated group scheme $\mathbf{G}$ over $\mathcal{O}_F$ with generic fibre $\mathbb{G}$ via the Bruhat-Tits theory, i.e., we have $\mathbb{G} = \mathbf{G} \times_{\mathcal{O}_F} F$.

Let $K = \mathbf{G}(\mathcal{O}_F)$, which is a maximal compact subgroup of G. Assuming n to be prime to the residue characteristic of F, the Hilbert symbol $(-,-)_n$ becomes a power of the tame symbol, and it gives a degree n central cover $\overline{G}$ of G. We are interested in the case when there exists a splitting s_K of K into $\overline{G}$.

Definition 2.10. The group $\overline{G}$ called s_K-unramified (or simply unramified) if $\gcd(n,p) = 1$ and there exists a splitting s_K of K into $\overline{G}$.

Note that the $\mathbb{K}_2$-torsor $\overline{\mathbb{G}}$ defined over F may not be the base change of some $\mathbf{K}_2$-torsor $\overline{\mathbf{G}}$ over $\mathrm{Spec}(\mathcal{O}_F)$. Otherwise suppose $\overline{\mathbb{G}} = \overline{\mathbf{G}} \times_{\mathcal{O}_F} F$, since the tame Hilbert symbol (if we assume $\gcd(n,p) = 1$) vanishes on $\mathbf{K}_2(\mathcal{O}_F)$, the existence of the splitting of K into $\overline{G}$ is then automatic (cf. [BD01, §10.7]).

In the language of incarnations, if one starts in general with $\overline{\mathbb{G}}_{D,\eta}$, it can be shown (cf. [GaG14, §3.7-3.9]) that one sufficient condition is that η takes values in the units $\mathcal{O}_F^\times$ of $F^\times$. In particular, the condition $\gcd(n,q) = 1$ is not sufficient to guarantee the splitting of K in general. In [GaG14], some simple examples are provided to show that indeed K may not split if η does not take values in the units.

2.4.2. *Unipotent splitting of* $\overline{G}$. We know that $\overline{\mathbb{G}}$ splits uniquely over any unipotent subgroup $\mathbb{U} \subseteq \mathbb{G}$ (cf. [BD01, §3]). Taking F-rational points, we obtain a unique splitting of $\mathbb{U}(F)$ into $\overline{G}$. In fact, these splittings on unipotent subgroups could be extended to a section of the set G_u of unipotent elements of G with certain properties. The conjugation action of $\overline{G}$ on G_u descends to G.

Proposition 2.11 ([Li12, Proposition 2.2.1])**.** *There exists uniquely a continuous set section $i_u : G_u \hookrightarrow \overline{G}$ such that*

- (i) *for all unipotent subgroup U of G, the restriction $i_u|_U$ is a homomorphism (i.e. a splitting), and*
- (ii) *i_u is G-conjugation invariant. That is, $i_u(g^{-1}ug) = g^{-1}i_u(u)g$ for all $g \in G$.*

Note in §2.1.3, we used the notation $\overline{\mathbb{e}}_\alpha(-)$ for the unique splitting of $\mathbb{U}_\alpha(F)$ into $\overline{\mathbb{G}}^{sc}(F)$. Consider the natural map $\Phi : \mathbb{G}^{sc}(F) \to \mathbb{G}(F)$, and also $\Phi_{D,\eta}$ for the induced pull-back map $\overline{\mathbb{G}}^{sc}(F) \to \overline{\mathbb{G}}(F)$. Write

$$e_\alpha(a) := \Phi(\mathbb{e}_\alpha(a)), \quad \overline{e}_\alpha(a) := \Phi_{D,\eta}(\overline{\mathbb{e}}_\alpha(a)).$$

Since Φ is injective on the unipotent elements, the map $e_\alpha(a) \mapsto \overline{e}_\alpha(a)$ is the splitting of $U_\alpha := \mathbb{U}_\alpha(F)$ into $\overline{G}$. As a consequence of above proposition,

Corollary 2.12. *For $\alpha \in \Psi$, the unique splitting the unipotent subgroup U_α given by $e_\alpha(a) \mapsto \overline{e}_\alpha(a)$ is just the restriction of i_u to U_α. Moreover, assume that $\overline{G}$ is s_K-unramified. Then,*

$$s_K|_{K\cap U_\alpha} = i_u|_{K\cap U_\alpha}.$$

Proof. We only need to show the second assertion. For this, note that $K \cap U_\alpha$ is a pro-p group. The two splittings $s_K|_{K\cap U_\alpha}$ and $i_u|_{K\cap U_\alpha}$ differ by a homomorphism from $K \cap U_\alpha$ to μ_n, which has to be trivial since n is prime to p. □

2.5. **Global finite degree topological covers.** Assume F a number field with $\mu_n \subseteq F^\times$. Write F_v for the completion of F with respect to any place $v \in |F|$. Let $\mathbb{A}_F$ be the adele ring of F. Let $\overline{\mathbb{G}}_{D,\eta} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ be a $\mathbb{K}_2$-torsor over F. In [BD01], it is shown that one has a degree n global covering group $\overline{\mathbb{G}}(\mathbb{A}_F)$ of $\mathbb{G}(\mathbb{A}_F)$ with naturally inherited data. For instance, $\overline{\mathbb{G}}(\mathbb{A}_F)$, in which each $\overline{G}_v$ is embedded naturally, splits over $\mathbb{G}(F)$. Also, for almost all v, there is a splitting s_{K_v} of the group $K_v := \mathbb{G}(\mathcal{O}_v)$ into $\overline{G}_v$. For more details, see [BD01, §10].

3. Dual groups and L-groups for topological Brylinski-Deligne extensions

The references for dual and L-groups are [FiLy10], [McN12], [We16-2] and [We16-3]. We will only recall here some facts important to us and refer to the original papers for details.

3.1. **The dual group $\overline{\mathbb{G}}^\vee$ à la Finkelberg-Lysenko-McNamara-Reich.** Let $\mathbb{G}$ be a split reductive group over F with root datum $(X, \Psi, Y, \Psi^\vee)$. Let $\Delta \subseteq \Psi$ be a fixed set of simple roots. A Brylinski-Deligne extension $\overline{\mathbb{G}}$ gives rise to local or global topological degree n covers, depending on whether F is local or global.

Let Q be the quadratic form associated with $\overline{\mathbb{G}}$. Write

$$Y_{Q,n} = \left\{ y \in Y |\ B_Q(y, y') \in n\mathbf{Z} \text{ for all } y' \in Y \right\},$$

which is a sublattice of Y and clearly contains nY. For any $\alpha \in \Psi$, we also write

$$n_\alpha = \frac{n}{\gcd(Q(\alpha^\vee), n)}, \quad \alpha^\vee_{Q,n} = n_\alpha \alpha^\vee, \quad \alpha_{Q,n} = n_\alpha^{-1}\alpha.$$

Let $Y^{sc}_{Q,n}$ be the sublattice of Y generated by $\alpha^\vee_{Q,n}, \alpha^\vee \in \Psi^\vee$. Now consider the quadruple given by

$$\begin{aligned} Y_1 &= Y_{Q,n}, \\ \Psi_1^\vee &= \left\{ \alpha^\vee_{Q,n} : \alpha^\vee \in \Psi^\vee \right\}, \\ X_1 &= \mathrm{Hom}(Y_1, \mathbf{Z}) \subseteq X \otimes \mathbf{Q}, \\ \Psi_1 &= \left\{ \alpha_{Q,n} : \alpha \in \Psi \right\}. \end{aligned}$$

By [McN12, §13.11], the quadruple $(Y_1, \Psi_1^\vee, X_1, \Psi_1)$ forms a root datum. Define the dual group $\overline{\mathbb{G}}^\vee$ to be the pinned split reductive group associated with this root datum. Let $\overline{G}^\vee = \overline{\mathbb{G}}^\vee(\mathbf{C})$. In particular, if $\mathbb{G} = \mathbb{T}$ is a torus, then $\overline{T}^\vee = X_1 \otimes \mathbf{C}$.

3.2. **Local L-group à la Weissman.** Let F be a number field or a local field. Let $(Q, \mathcal{E}, \phi)$ be the Brylinski-Deligne classification data associated with $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$; then the construction of $\overline{\mathbb{G}}^\vee$ uses the data n and B_Q alone. The construction of the L-group for covering groups is due to M. Weissman and utilizes the full data $(Q, \mathcal{E}, \phi)$. For the below, we recall briefly the definition of the L-group following [We16-2], [We16-3] and [GaG14]. From now, we fix an injective morphism $\epsilon : \mu_n \hookrightarrow \mathbf{C}^\times$, and thus always view μ_n as a subgroup of $\mathbf{C}^\times$ with respect to ϵ.

We assume F a local field. The construction of the extension of ${}^L\overline{G}$ relies on two abelian extensions E_i for $i = 1, 2$ by the center $Z(\overline{G}^\vee)$:

$$Z(\overline{G}^\vee) \hookrightarrow E_i \twoheadrightarrow F^\times.$$

Note $Z(\overline{G}^\vee) = \mathrm{Hom}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbf{C}^\times)$. The extension E_1 is given by the cocycle

$$F^\times \times F^\times \to \mathrm{Hom}(Y_{Q,n}/Y^{sc}_{Q,n}, \mathbf{C}^\times), \quad (a, b) \to \left(y \mapsto (a, b)_n^{Q(y)} \right). \tag{8}$$

Thus, we can write $E_1 = Z(\overline{G}) \times_Q F^\times$ with the group law given above. Note $(a, b)_n^{Q(y)} \in \mu_2$, as $n | 2Q(y)$ for $y \in Y_{Q,n}$, and therefore the cocycle actually takes values in $\mathrm{Hom}(Y_{Q,n}/Y^{sc}_{Q,n}, \mu_2)$.

For E_2, consider the extensions $\mathcal{E}$ and $\mathcal{E}_Q^{sc}$ associated to $\overline{\mathbb{G}}$ as from Theorem 2.5. For convenience, we write $F^\times/n = F^\times/(F^\times)^n$. Consider the pull-back of $\mathcal{E}_Q^{sc}$ by $Y^{sc}_{Q,n} \hookrightarrow Y^{sc}$, and then follow up by the push-out by the quotient map $F^\times \to F^\times/n$, we obtain a group $\mathcal{E}^{sc}_{Q,n}$ which lies in an exact sequence

$$F^\times/n \hookrightarrow \mathcal{E}^{sc}_{Q,n} \twoheadrightarrow Y^{sc}_{Q,n}.$$

Fix an arbitrary nonzero $f \in F(\!(\tau)\!)$. We define a splitting $\mathbf{s}$ of $\mathcal{E}^{sc}_{Q,n}$ over $Y^{sc}_{Q,n}$ as follows. For every element $\alpha^\vee_{Q,n} \in Y^{sc}_{Q,n}, \alpha \in \Psi$, consider its lifting in $\mathcal{E}^{sc}_{Q,n}$ given by

$$\mathbf{s} : \alpha^\vee_{Q,n} \mapsto \overline{\mathbb{h}}_\alpha^{[f]}(n_\alpha \alpha^\vee), \quad \alpha \in \Psi.$$

Recall from Definition 2.4 we have $\overline{\mathbb{h}}_\alpha^{[f]}(n_\alpha \alpha^\vee) = \left[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^{n_\alpha})\right] \in \mathcal{E}_Q^{sc}$, which clearly lies over $\alpha_{Q,n}^\vee \in Y_{Q,n}^{sc}$. By abuse of notation we also use it to denote its image in $\mathcal{E}_{Q,n}^{sc}$. Note $n_\alpha Q(\alpha^\vee) \in n\mathbf{Z}$. Because of the property (cf. (3))

$$\overline{\mathbb{h}}_\alpha^{[f]}(\tau^{n_\alpha}) = \overline{\mathbb{h}}_\alpha^{[1]}(\tau^{n_\alpha}) \cdot \{f, \tau^{n_\alpha}\}^{Q(\alpha^\vee)},$$

it follows that $\left[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^{n_\alpha})\right] \in \mathcal{E}_{Q,n}^{sc}$ is independent of f. Therefore, we may omit the superscript in both $\overline{\mathbb{h}}_\alpha^{[f]}(n_\alpha \alpha^\vee)$ and $\left[\overline{\mathbb{h}}_\alpha^{[f]}(\tau^{n_\alpha})\right]$, and instead just write $\overline{\mathbb{h}}_\alpha(n_\alpha \alpha^\vee)$, $\left[\overline{\mathbb{h}}_\alpha(\tau^{n_\alpha})\right]$ respectively. For computational convenience, one may take $f = 1$ and there is no loss of generality.

Proposition 3.1 ([We16-2, §3.3]). *The lifting $\mathbf{s}(\alpha_{Q,n}^\vee) = \overline{\mathbb{h}}_\alpha(n_\alpha \alpha^\vee)$ for all $\alpha \in \Psi$ above gives rise to a well-defined splitting (i.e. a homomorphism) of the sequence $F^\times/n \hookrightarrow \mathcal{E}_{Q,n}^{sc} \twoheadrightarrow Y_{Q,n}^{sc}$, which is still denoted by $\mathbf{s}$.*

We could obtain on the other hand a group $\mathcal{E}_{Q,n}$ by the pull-back of $\mathcal{E}$ via $Y_{Q,n} \hookrightarrow Y$, and then the push-out via the quotient map $F^\times \twoheadrightarrow F^\times/n$. It is an extension of $Y_{Q,n}$ by $F^\times/n$. There is the inherited map $\mathcal{E}_{Q,n}^{sc} \to \mathcal{E}_{Q,n}$ still denoted by $\phi_{D,\eta}$. That is, we have the following commutative diagram with a natural splitting $\mathbf{s}$ for the top row.

$$\begin{tikzcd}
F^\times/n \arrow[r, hook] \arrow[d, equal] & \mathcal{E}_{Q,n}^{sc} \arrow[r, two heads] \arrow[d, "\phi_{D,\eta}"] & Y_{Q,n}^{sc} \arrow[l, bend right, "\mathbf{s}"'] \arrow[d, hook] \\
F^\times/n \arrow[r, hook] & \mathcal{E}_{Q,n} \arrow[r, two heads] & Y_{Q,n}.
\end{tikzcd} \tag{9}$$

One obtains

$$F^\times/n \hookrightarrow \mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}) \twoheadrightarrow Y_{Q,n}/Y_{Q,n}^{sc}. \tag{10}$$

From this extension, after applying $\mathrm{Hom}(-, \mathbf{C}^\times)$, we further consider the pull-back by the map $h : F^\times \to \mathrm{Hom}(F^\times/n, \mathbf{C}^\times)$ given by the Hilbert symbol $a \mapsto h_a$ with $h_a(b) = (b, a)_n$. By definition E_2 is the pull-back of $\mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}), \mathbf{C}^\times)$ by h:

$$\begin{tikzcd}
Z(\overline{G}^\vee) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}), \mathbf{C}^\times) \arrow[r, two heads] & \mathrm{Hom}(F^\times/n, \mathbf{C}^\times) \\
Z(\overline{G}^\vee) \arrow[r, hook] & E_2 \arrow[u] \arrow[r, two heads] & F^\times. \arrow[u, "h"]
\end{tikzcd}$$

3.2.1. *The fundamental extension $E_{\overline{G}}$ and the group ${}^L\overline{G}$.* Consider the Baer sum $E_1 \oplus_B E_2$, which we denote by $E_{\overline{G}}$. We call $E_{\overline{G}}$ the fundamental extension associated with $\overline{G}$. Let W_F be the Weil group of F. Consider the map $\mathrm{Rec} : \mathrm{W}_F \to F^\times$, which is induced from the Artin recirpocity map $\mathrm{W}_F^{ab} \to F^\times$ sending the class of geometric Frobenius to the class of uniformizer (modulo $\mathcal{O}_F^\times$) of $F^\times$. We obtain the pull-back $\mathrm{Rec}^*(E_{\overline{G}})$:

$$Z(\overline{G}) \hookrightarrow \mathrm{Rec}^*(E_{\overline{G}}) \twoheadrightarrow \mathrm{W}_F.$$

Definition 3.2 ([We16-2]). Let $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ be a Brylinski-Deligne extension, and let $\overline{G} \in \mathsf{CExt}(G, \mu_n)$ be the resulting topological extension. Consider the natural inclusion $j^{\overline{G}^\vee} : Z(\overline{G}^\vee) \hookrightarrow \overline{G}^\vee$. The L-group ${}^L\overline{G}$ of $\overline{G}$ is defined to be the push-out $j_*^{\overline{G}}$ of $\mathrm{Rec}^*(E_{\overline{G}})$:

$${}^L\overline{G} := j_*^{\overline{G}^\vee} \circ \mathrm{Rec}^*(E_{\overline{G}}) = \frac{\overline{G}^\vee \times \mathrm{Rec}^*(E_{\overline{G}})}{\nabla Z(\overline{G}^\vee)},$$

where $\nabla : Z(\overline{G}^\vee) \hookrightarrow \overline{G}^\vee \times Z(\overline{G}^\vee)$ is the anti-diagonal embedding. Also, we may use equivalently the group $\mathrm{Rec}^* \circ j_*^{\overline{G}^\vee}(E_{\overline{G}})$ as the definition of ${}^L\overline{G}$. Therefore, by construction, the L-group is an extension

$$\overline{G}^\vee \hookrightarrow {}^L\overline{G} \twoheadrightarrow \mathrm{W}_F.$$

3.2.2. *Functoriality for Levi subgroups.* Let $\mathbb{M} \subseteq \mathbb{G}$ be a Levi subgroup of $\mathbb{G}$. The group $\overline{G}$ gives rise to the Levi $\overline{M}$ by restriction of the Brylinski-Deligne data, and we obtain ${}^L\overline{G}$ and ${}^L\overline{M}$. It is shown in [We16-2, §5.7] (see also [GaG14, §5.5]) that there is a canonical morphism ${}^L\varphi$ extending the inclusion $\varphi^\vee : \overline{M}^\vee \hookrightarrow \overline{G}^\vee$ arising from construction such that the diagram commutes:

$$\begin{tikzcd}
\overline{G}^\vee \arrow[r, hook] & {}^L\overline{G} \arrow[r, two heads] & \mathrm{W}_F \\
\overline{M}^\vee \arrow[r, hook] \arrow[u, hook, "\varphi^\vee"] & {}^L\overline{M} \arrow[r, two heads] \arrow[u, hook, "{}^L\varphi"] & \mathrm{W}_F. \arrow[u, equal]
\end{tikzcd}$$

The case $\overline{M} = \overline{T}$ is crucial to us, and therefore we give more details. Note ${}^L\overline{T} = \mathrm{Rec}^*(E_{\overline{T}})$ since $Z(\overline{T}^\vee) = \overline{T}^\vee$. Let $j^{\overline{T}^\vee} : Z(\overline{G}^\vee) \hookrightarrow \overline{T}^\vee$ be the inclusion. It is easy to see that we have a canonical isomorphism

$$j_*^{\overline{T}^\vee}(E_{\overline{G}}) \simeq E_{\overline{T}},$$

which induces the map ${}^L\varphi$ from ${}^L\overline{T}$ to ${}^L\overline{G}$. Recall by definition $j_*^{\overline{T}^\vee}(E_{\overline{G}}) = (\overline{T}^\vee \times E_{\overline{G}})/\nabla Z(\overline{G}^\vee)$. We write $\mathcal{L}$ for the above canonical isomorphism

$$\mathcal{L} : \quad (\overline{T}^\vee \times E_{\overline{G}})/\nabla Z(\overline{G}^\vee) \to E_{\overline{T}}.$$

What is important for us is the explicit form of $\mathcal{L}$. From the construction we may identify

$$E_{2,\overline{T}} = \left\{([P_a], a) \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbf{C}^\times) \times F^\times : [P_a]|_{F^\times/n} = h_a\right\},$$

where as before $h_a(-) = (-, a)_n$ denotes the n-th Hilbert symbol. Here $[P_a]\big((b, y)\big) = h_a(b) \cdot P_a(y)$ is a character of $\mathcal{E}_{Q,n}$ with respect to a certain map $P_a : Y_{Q,n} \to \mathbf{C}^\times$ (not necessarily a homomorphism). Similarly $E_{2,\overline{G}}$ could be identified with the collection

$$\left\{([P_a], a) \in \mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}), \mathbf{C}^\times) \times F^\times : [P_a]|_{F^\times/n} = h_a\right\}.$$

Let $\bar{t}^\vee \in \overline{T}^\vee = \mathrm{Hom}(Y_{Q,n}, \mathbf{C}^\times)$. It gives rise to an element $[\bar{t}^\vee] \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbf{C}^\times)$ via inflation. Consider the E_2 extensions, we see that there is a canonical isomorphism which takes the explicit form:

$$(\overline{T}^\vee \times E_{2,\overline{G}})/\nabla Z(\overline{G}^\vee) \to E_{2,\overline{T}}, \quad \big(\bar{t}^\vee, ([P_a], a)\big) \mapsto \big([\bar{t}^\vee] \cdot [P_a], a\big).$$

Thus the desired isomorphism $\mathcal{L} : \overline{T}^\vee \times E_{\overline{G}}/\nabla Z(\overline{G}^\vee) \to E_{\overline{T}}$ is given by

$$\big(\bar{t}^\vee, ([P_a], a) \oplus_B (1, a)\big) \mapsto ([\bar{t}^\vee] \cdot [P_a], a) \oplus_B (1, a).$$

Conversely, it is important for our purpose of the Gindikin-Karpelevich formula to determine the element $\bar{t}_{P_a}^\vee \in \overline{T}^\vee$ associated with $\mathcal{L}^{-1}\big(([P_a], a) \oplus_B (1, a)\big)$, where $([P_a], a) \oplus_B (1, a) \in E_{\overline{T}}$. Moreover, it is easy to see that $\bar{t}_{P_a}^\vee$ could be only determined up to a class modulo $Z(\overline{G}^\vee)$. That is, only the image of $\bar{t}_{P_a}^\vee$ in the quotient map $\overline{T}^\vee \twoheadrightarrow \overline{T}^\vee/Z(\overline{G}^\vee)$ is uniquely determined.

For this purpose, consider the trivial extension $\overline{T}^\vee/Z(\overline{G}^\vee) \hookrightarrow \overline{T}^\vee/Z(\overline{G}^\vee) \times F^\times \twoheadrightarrow F^\times$, which has a canonical splitting s^{Tr} of the first map. There is a natural map q_* from $j_*^{\overline{T}^\vee}(E_{\overline{G}})$ to $(\overline{T}^\vee/Z(\overline{G}^\vee)) \times F^\times$ such that the following diagram commutes

$$\begin{tikzcd}
\overline{T}^\vee \arrow[r, hook] \arrow[d, "q"] & j_*^{\overline{T}^\vee}(E_{\overline{G}}) \arrow[r, two heads] \arrow[d, "q_*"] & F^\times \arrow[d, equal] \\
\overline{T}^\vee/Z(\overline{G}^\vee) \arrow[r, hook] & \overline{T}^\vee/Z(\overline{G}^\vee) \times F^\times \arrow[r, two heads] \arrow[l, bend left, "s^{\mathrm{Tr}}"] & F^\times.
\end{tikzcd}$$

Take the composition

$$\mathcal{C} = s^{\mathrm{Tr}} \circ q_* \circ \mathcal{L}^{-1} : \quad E_{\overline{T}} \to j_*^{\overline{T}^\vee}(E_{\overline{G}}) \to \overline{T}^\vee/Z(\overline{G}^\vee) \times F^\times \to \overline{T}^\vee/Z(\overline{G}^\vee).$$

Now we can describe explicitly the image of this map $\mathcal{C}$.

Lemma 3.3. *Let $\mathcal{P}_a = ([P_a], a) \oplus_B (1, a) \in E_{\overline{T}}$, and identify $Hom(Y_{Q,n}^{sc}, \mathbf{C}^\times) = \overline{T}^\vee / Z(\overline{G}^\vee)$. Then $\mathcal{C}(\mathcal{P}_a)(\alpha_{Q,n}^\vee) = [P_a] \circ \phi_{D,\eta}(\mathbf{s}(\alpha_{Q,n}^\vee))$ for all $\alpha \in \Psi$. That is, the image $\bar{t}_{P_a}^\vee$ of $\mathcal{L}^{-1}(\mathcal{P}_a)$ in $\overline{T}^\vee/Z(\overline{G}^\vee)$ is uniquely determined by*

$$\bar{t}_{P_a}^\vee(\alpha_{Q,n}^\vee) = [P_a] \circ \phi_{D,\eta}(\mathbf{s}(\alpha_{Q,n}^\vee)), \alpha \in \Psi. \tag{11}$$

Proof. Clearly $\bar{t}_{P_a}^\vee(\alpha_{Q,n}^\vee) = \mathcal{C}(\mathcal{P}_a)(\alpha_{Q,n}^\vee)$. Now write

$$\mathcal{L}^{-1}(\mathcal{P}_a) = (\bar{t}_{P_a}^\vee, ([P_a]/[\bar{t}_{P_a}^\vee], a) \oplus_B (1, a)) \in j_*^{\overline{T}^\vee}(E_{\overline{G}}),$$

where $[P_a]/[\bar{t}_{P_a}^\vee] \in \mathrm{Hom}(\mathcal{E}_{Q,n}/\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc}), \mathbf{C}^\times)$. In particular, it vanishes on $\phi_{D,\eta} \circ \mathbf{s}(Y_{Q,n}^{sc})$, and therefore

$$[\bar{t}_{P_a}^\vee] \circ \phi_{D,\eta}(\mathbf{s}(\alpha_{Q,n}^\vee)) = [P_a] \circ \phi_{D,\eta}(\mathbf{s}(\alpha_{Q,n}^\vee)), \ \alpha \in \Psi.$$

But we have $\bar{t}_{P_a}^\vee(\alpha_{Q,n}^\vee) = [\bar{t}_{P_a}^\vee] \circ \phi_{D,\eta}(\mathbf{s}(\alpha_{Q,n}^\vee))$ from the definition of $[\bar{t}_{P_a}^\vee]$. This gives the desired result and completes the proof. □

3.3. **Global L-group.** Let $\overline{\mathbb{G}} \in \mathsf{CExt}(\mathbb{G}, \mathbb{K}_2)$ be defined over a number field F. Assume $\mu_n \subseteq F^\times$, one obtains a global covering group $\mu_n \hookrightarrow \overline{\mathbb{G}}(\mathbb{A}_F) \twoheadrightarrow \mathbb{G}(\mathbb{A}_F)$ in §2.5. We will refrain from copying the definition of the global L-group extension ${}^L\overline{G}$, but refer the reader to [We16-2] and [GaG14] for details. However, we will only remark here that we have local and global compatibility. That is, there is a natural map of extensions from ${}^L\overline{G}_v$ to ${}^L\overline{G}$.

4. Local Langlands correspondence for covering tori

4.1. **Subgroups of $\overline{G}$.** Assume F a local field, unless stated otherwise. Let $\overline{G}$ be incarnated by (D, η), and let $\overline{T}$ be the covering torus of $\overline{G}$. Then the center $Z(\overline{T})$ allows for a simple description as follows.

Consider the inclusion $Y_{Q,n} \hookrightarrow Y$. Let $\mathbb{T}_{Q,n}$ be the algebraic torus associated with $Y_{Q,n}$ whose F-rational point is denoted by $T_{Q,n}$, i.e. $T_{Q,n} = \mathbb{T}_{Q,n}(F)$. Thus we have a natural map $i_{Q,n} : T_{Q,n} \to T$ whose image we denote by $T^\dagger$. Then by [We09, Proposition 4.1], the center $Z(\overline{T})$ is the preimage of $T^\dagger$ in $\overline{T}$. Now we define $\overline{T}_{Q,n} := i_{Q,n}^*(Z(\overline{T}))$ to be the pull-back of $Z(\overline{T})$ via $i_{Q,n}$:

$$\begin{tikzcd}
\mu_n \arrow[r, hook] & Z(\overline{T}) \arrow[r, two heads] & T^\dagger \\
\mu_n \arrow[u, equal] \arrow[r, hook] & \overline{T}_{Q,n} \arrow[r, two heads] \arrow[u, two heads, "\bar{i}_{Q,n}"] & T_{Q,n}. \arrow[u, two heads, "i_{Q,n}"]
\end{tikzcd}$$

That is, elements of $\overline{T}_{Q,n}$ are of the form $((\zeta, i_{Q,n}(t)), t) \in Z(\overline{T}) \times T_{Q,n}$. Since $\overline{T}$ is assumed to be incarnated by D, we can write $\overline{T} = \mu_n \times_D T$ and therefore also $Z(\overline{T}) = \mu_n \times_D T^\dagger$. From now, we will write (ζ, t) for $((\zeta, i_{Q,n}(t)), t) \in \overline{T}_{Q,n}$. The group law on $\overline{T}_{Q,n} = \mu_n \times_D T_{Q,n}$ inherited from $Z(\overline{T})$ is thus given by the analogue of (7).

4.2. **Local Langlands correspondence for covering tori and properties.** We are interested in splittings of ${}^L\overline{G}$ over W_F. For example, we will see that ${}^L\overline{T}$ always splits and we have a local Langlands correspondence between splittings of ${}^L\overline{T}$ and genuine characters of the center $Z(\overline{T})$.

Definition 4.1. An L-parameter is just a splitting ρ of ${}^L\overline{G}$ over W_F. Write $\mathsf{Spl}({}^L\overline{G}, \mathrm{W}_F)$ for the set of splittings, i.e. L-parameters.

Proposition 4.2 ([GaG14, §8])**.** *There is a natural injection as compositions*

$$Hom_\epsilon(Z(\overline{T}), \mathbf{C}^\times) \hookrightarrow \mathsf{Spl}(E_{\overline{T}}, F^\times) \xhookrightarrow{\simeq} \mathsf{Spl}({}^L\overline{T}, W_F),$$

where the first map is given explicitly by

$$\overline{\chi} \hookrightarrow \rho_{\overline{\chi}} \text{ with } \rho_{\overline{\chi}}(a) = ([\overline{\chi}(1, - \otimes a)], a) \oplus_B (1, a) \in E_{\overline{T}}.$$

Here $[P_a] := [\overline{\chi}(1, - \otimes a)]$ is the character of the group $\mathcal{E}_{Q,n}$ associated with the map $P_a := \overline{\chi}(1, - \otimes a) : Y_{Q,n} \to \mathbf{C}^\times$ given by $P_a(y) = \overline{\chi}(1, y \otimes a)$.

We call it the local Langlands correspondence for Brylinski-Deligne covering tori.

Now back to the case of general $\overline{G}$ with covering torus $\overline{T}$. Let $\overline{\chi} \in \mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times)$ be an arbitrary genuine character of $Z(\overline{T})$. For any $a \in F^\times$, it gives rise to an element $([P_a], a) \oplus_B (1, a) \in E_{\overline{T}}$ where $[P_a] \in \mathrm{Hom}(\mathcal{E}_{Q,n}, \mathbf{C}^\times)$ is given by

$$[P_a]\big((1, y)\big) := \overline{\chi}\big((1, y \otimes a)\big). \tag{12}$$

Consider the natural map $\Phi_{D,\eta} : \overline{T}^{sc} \to \overline{T}$. For any $a \in F^\times$ and any $\alpha \in \Psi$, let $\overline{\mathbb{h}}_\alpha^{[b]}(a) \in \overline{T}^{sc}$ be Brylinski-Deligne lifting of $\alpha^\vee(a) \in T^{sc}$. Consider the element $\Phi_{D,\eta}(\overline{\mathbb{h}}_\alpha^{[b]}(a^{n_\alpha})) \in Z(\overline{T})$, which a priori depends on $b \in F^\times$.

Proposition 4.3. *The element $\Phi_{D,\eta}(\overline{\mathbb{h}}_\alpha^{[b]}(a^{n_\alpha})) \in Z(\overline{T})$ is independent of $b \in F^\times$, and thus we could omit b for notational simplicity and just write $\overline{\mathbb{h}}_\alpha(a^{n_\alpha})$ for it. More importantly, let $\overline{\chi} \in Hom_\epsilon(Z(\overline{T}), \mathbf{C}^\times)$ be a genuine character. Then, for any $\alpha^\vee \in \Psi^\vee$ and any $a \in F^\times$, one has*

$$\overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) = [P_a] \circ \phi_{D,\eta}\big(\mathbf{s}(\alpha^\vee_{Q,n})\big), \tag{13}$$

where $[P_a]$ is associated with $\overline{\chi}$ by (12).

Proof. The independence of $b \in F^\times$ in the first assertion is a simple consequence of the equality (3).

For the desired equality (13), when $\alpha^\vee \in \Delta^\vee$ is a simple coroot, it is exactly the equality (9.7) in [GaG14, Proposition 9.5], the proof of which relies crucially on the equality (6). To show the general case, let $\gamma^\vee \in \Psi^\vee$. Inductively, we can assume $\gamma^\vee = s_{\alpha^\vee}(\beta^\vee)$ with $\alpha^\vee \in \Delta^\vee$ and that the equality hold for $\beta^\vee \in \Psi^\vee$, i.e. $\overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\beta(a^{n_\beta})\big) = [P_a] \circ \phi_{D,\eta}\big(\mathbf{s}(\beta^\vee_{Q,n})\big)$. We want to show the equality for $\gamma^\vee$.

Recall $\alpha_{Q,n} = n_\alpha^{-1} \cdot \alpha$. Since $\gamma^\vee = s_{\alpha^\vee}(\beta^\vee)$, we have $Q(\gamma^\vee) = Q(\beta^\vee)$ and $n_\gamma = n_\beta$. It follows

$$\gamma^\vee_{Q,n} = s_{\alpha^\vee_{Q,n}}(\beta^\vee_{Q,n}) = \beta^\vee_{Q,n} - \langle \alpha_{Q,n}, \beta^\vee_{Q,n}\rangle \cdot \alpha^\vee_{Q,n}$$

where

$$\langle \alpha_{Q,n}, \beta^\vee_{Q,n}\rangle \cdot \alpha^\vee_{Q,n} = n_\alpha \langle \alpha_{Q,n}, \beta^\vee_{Q,n}\rangle \cdot \alpha^\vee = n_\beta \langle \alpha, \beta^\vee\rangle \cdot \alpha^\vee.$$

For simplicity, we write $k_{\alpha,\beta} := \langle \alpha_{Q,n}, \beta^\vee_{Q,n}\rangle \in \mathbf{Z}$; then $n_\alpha \cdot k_{\alpha,\beta} = n_\beta \langle \alpha, \beta^\vee\rangle$.

Since $n | n_\alpha Q(\alpha^\vee)$, it is easy to check that $\overline{\mathbb{h}}_\alpha(a^{n_\alpha \cdot r}) = \overline{\mathbb{h}}_\alpha(a^{n_\alpha})^r$ for any $r \in \mathbf{Z}$. By [BD01, §11.3], we have the following equalities in $\overline{T}^{sc}$

$$\begin{aligned}
\overline{\mathbb{h}}_\gamma(a^{n_\gamma}) &= \overline{\mathbb{h}}_\beta(a^{n_\beta}) \cdot \overline{\mathbb{h}}_\alpha(a^{-n_\beta \langle \alpha, \beta^\vee\rangle}) \\
&= \overline{\mathbb{h}}_\beta(a^{n_\beta}) \cdot \overline{\mathbb{h}}_\alpha(a^{-n_\alpha \cdot k_{\alpha,\beta}}) \\
&= \overline{\mathbb{h}}_\beta(a^{n_\beta}) \cdot \overline{\mathbb{h}}_\alpha(a^{n_\alpha})^{-k_{\alpha,\beta}}.
\end{aligned}$$

On the other hand, $\mathbf{s}(\gamma^\vee_{Q,n}) = \mathbf{s}(\beta^\vee_{Q,n}) \cdot \mathbf{s}(\alpha^\vee_{Q,n})^{-k_{\alpha,\beta}} \in \mathcal{E}^{sc}_{Q,n}$. By the induction hypothesis on $\beta^\vee$ and the known equality for $\alpha^\vee \in \Delta^\vee$, we see $\overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\gamma(a^{n_\gamma})\big) = [P_a] \circ \phi_{D,\eta}\big(\mathbf{s}(\gamma^\vee_{Q,n})\big)$. That is, the assertion holds for $\gamma^\vee$, and the proof is completed. □

Recall the canonical homomorphism $\mathcal{C} : E_{\overline{T}} \to \overline{T}^\vee / Z(\overline{G}^\vee)$ defined right before Lemma 3.3. It gives an induced map $\mathcal{C}_* : \mathsf{Spl}(E_{\overline{T}}, F^\times) \to \mathrm{Hom}(F^\times, \overline{T}^\vee / Z(\overline{G}^\vee)$ by post composition with $\mathcal{C}$. Consider the composition

$$\mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times) \xhookrightarrow{\mathrm{LLC}} \mathsf{Spl}(E_{\overline{T}}, F^\times) \xrightarrow{\mathcal{C}_*} \mathrm{Hom}(F^\times, \overline{T}^\vee / Z(\overline{G}^\vee))$$

given by $\overline{\chi} \mapsto \rho_{\overline{\chi}} \mapsto \mathcal{C} \circ \rho_{\overline{\chi}}$.

The following result is of fundamental importance to the interpretation of the Gindikin-Karpelevich formula.

Corollary 4.4. *With notations as above, identify $\overline{T}^\vee / Z(\overline{G}^\vee)$ with $Hom(Y^{sc}_{Q,n}, \mathbf{C}^\times)$. Then for any $a \in F^\times$,*

$$\big(\mathcal{C} \circ \rho_{\overline{\chi}}(a)\big)(\alpha^\vee_{Q,n}) = \overline{\chi} \circ \Phi_{D,\eta}\big(\overline{\mathbb{h}}_\alpha(a^{n_\alpha})\big) \text{ for all } \alpha \in \Psi.$$

Proof. Just combine Lemma 3.3 and Proposition 4.3. □

5. Satake isomorphism and unramified representations

5.1. **Satake isomorphism.** Let $\overline{G} \in \mathsf{CExt}(\mathbb{G}(F), \mu_n)$ be a Brylinski-Deligne covering group incarnated by some $(D, \eta) \in \mathsf{Bis}_{\mathbb{G}}^{Q}$ over a local field F. Write ϖ for a chosen uniformizer of F. We are interested in ϵ-genuine representation of $\overline{G}$, i.e. with $\mu_n \subseteq \overline{G}$ acting by the fixed faithful character $\epsilon : \mu_n \hookrightarrow \mathbf{C}^\times$. We assume that the group $\overline{G}$ is unramified (see Definition 2.10) and fix a splitting s_K. We will identify K with its image $s_K(K)$ if no confusion arises.

An irreducible genuine representation $\sigma \in \mathrm{Irr}_\epsilon(\overline{G})$ is called unramified if and only if $\sigma^K \neq 0$, i.e., the space of K-fixed vectors is nonzero. The key to understanding unramified representation is the Satake isomorphism as in the linear algebraic group case.

Let $\mathcal{H}_\epsilon(\overline{G}, K)$ be the $\mathbf{C}$-algebra consisting of anti-genuine locally constant and compactly supported bi-K invariant functions on $\overline{G}$. Similarly, we can define the Hecke algebra for $\mathcal{H}_\epsilon(\overline{T}, K_T)$ for the covering torus $\overline{T}$ with respect to $K_T := T \cap K$. The Satake transform is given by

$$\mathcal{S} : \mathcal{H}_\epsilon(\overline{G}, K) \to \mathcal{H}_\epsilon(\overline{T}, K_T), \quad \mathcal{S}(f)(\bar{t}) := \delta(\bar{t})^{1/2} \int_U f(\bar{t}u) du.$$

Note that any function $f \in \mathcal{H}_\epsilon(\overline{T}, K_T)$ has support in the centralizer $C_{\overline{T}}(K_T)$ of K_T in $\overline{T}$. This follows from the chain of equalities $f(\bar{t}) = f(\bar{t}k) = [\bar{t}, k] \cdot f(k\bar{t}) = [\bar{t}, k] \cdot f(\bar{t})$. Here $\bar{t} \in \overline{T}$ and $k \in K_T$ are arbitrary, and $[-,-]$ is the commutator on $\overline{T}$. In fact, by [McN12, Lemma 1] or [We14, §3.2], we have $C_{\overline{T}}(K_T) = Z(\overline{T})K_T$ and it is a maximal abelian subgroup of $\overline{T}$ containing $Z(\overline{T})$. Thus, restriction of functions gives an isomorphism of algebras

$$\mathcal{V} : \mathcal{H}_\epsilon(\overline{T}, K_T) \to \mathcal{H}_\epsilon(Z(\overline{T})K_T, K_T),$$

where the right hand side consists of ϵ anti-genuine compactly supported functions on $Z(\overline{T})K_T$ invariant under K_T.

We may identify the Weyl group W with $(N(T) \cap K)/K_T$. Then W acts on $Z(\overline{T})K_T$, as it acts on $Z(\overline{T})$ and also K_T. It induces a well-defined action on $\mathcal{H}_\epsilon(\overline{T}, K_T)$ by the morphism $\mathcal{V}$ above. We call a character $\overline{\chi}$ of $Z(\overline{T})$ unramified if it is trivial on $T^\dagger \cap K_T$.

Theorem 5.1 ([McN12], [Li12], [We16-2])**.** *There are natural isomorphisms of algebras*

$$\mathcal{H}_\epsilon(\overline{G}, K) \xrightarrow{\mathcal{S}} \mathcal{H}_\epsilon(\overline{T}, K_T)^W \xrightarrow{\mathcal{V}} \mathcal{H}_\epsilon(Z(\overline{T}), T^\dagger \cap K_T)^W,$$

which give rise to natural bijections between the following isomorphism classes:

$$\begin{array}{c} \{\text{irred. unramified genuine representations of } \overline{G}\} \\ \Big\updownarrow \mathcal{S}^* \\ \{W\text{-orbits of irred. unramified genuine representations of } \overline{T}\} \\ \Big\updownarrow \mathcal{V}^* \\ \{W\text{-orbits of unramified genuine characters of } Z(\overline{T})\}. \end{array}$$

5.2. **Stone von-Neumann theorem and $i(\overline{\chi})$.** Given an unramified $\overline{\chi} \in \mathrm{Hom}_\epsilon(Z(\overline{T}), \mathbf{C}^\times)$, there is a natural extension $\overline{\chi}'$ to $C_{\overline{T}}(K_T) = Z(\overline{T})K_T$. We could consider the induced representation $\mathrm{Ind}_{Z(\overline{T})K_T}^{\overline{T}} \overline{\chi}'$ to $\overline{T}$. As $Z(\overline{T})K_T$ is a maximal abelian group containing $Z(\overline{T})$, by Stone von-Neumann theorem (cf. [We09, Theorem 3.1]), it is irreducible. The isomorphism class of $\mathrm{Ind}_{Z(\overline{T})K_T}^{\overline{T}}(\overline{\chi}')$ does not depend on the choice of the maximal abelian group containing $Z(\overline{T})$ and the extension $\overline{\chi}'$. Therefore, we may write $i(\overline{\chi})$ for the induced representation. This basically gives the correspondence $\mathcal{V}^*$ above.

5.3. **Unramified principal series representations of $\overline{G}$.** Given any $\overline{G}$, let $\overline{B} = \overline{T}N$ be the Borel covering subgroup of $\overline{G}$.

Definition 5.2. Consider any $i(\overline{\chi}) \in \mathrm{Irr}_\epsilon(\overline{T})$. We define the normalized induced representation $I_{\overline{B}}^{\overline{G}}(i(\overline{\chi}))$ as follows. First let

$$L(i(\overline{\chi})) = \left\{ f : \overline{G} \to i(\overline{\chi}) |\ f(\bar{b}\bar{g}) = \delta_{\overline{B}}^{1/2}(\bar{b}) \cdot i(\overline{\chi})(\bar{b}) f(\bar{g}) \right\}.$$

Then by definition, $I_{\overline{B}}^{\overline{G}}(i(\overline{\chi}))$ consists of the smooth vectors of $L(i(\overline{\chi}))$. Here $\delta_{\overline{B}}$ is the modular character of $\overline{B}$ and we view $i(\overline{\chi})$ as a representation of $\overline{B}$ via the quotient map $\overline{B} \twoheadrightarrow \overline{T}$.

For simplicity we may write $I(\overline{\chi})$ for $I_{\overline{B}}^{\overline{G}}(i(\overline{\chi}))$ and call it a principal series representation of $\overline{G}$. Since the modular character $\delta_{\overline{B}}$ factors through that of B, i.e. $\delta_{\overline{B}}(\bar{b}) = \delta_B(b)$ where $b \in B$ is the image of $\bar{b}$, we may use interchangeably both notations $\delta_{\overline{B}}$ and δ_B. Suppose $\overline{\chi}$ is unramified. Then, by the Satake isomorphism, the principal series $I(\overline{\chi})$ has a one-dimensional space of K-fixed vectors.

Thus, one can argue as in the linear case that the correspondence $\mathcal{S}^*$ in Theorem 5.1 is given by $i(\overline{\chi}) \mapsto$ the unramified component of $I(\overline{\chi})$.

6. Intertwining operators

6.1. **Notations and basic set-up.** We use the boldface $\mathbf{w}$ to denote an element of W. For the fixed pinning, there is the associated morphism $\varphi_\alpha : \mathbb{SL}_2 \to \mathbb{G}^{sc}$ for any coroot $\alpha^\vee \in \Psi^\vee$. In §2.1.3, we also defined the elements $\mathbb{e}_\alpha, \mathbb{e}_{-\alpha}, \mathbb{w}_\alpha, \mathbb{h}_\alpha \in \mathbb{G}^{sc}$ associated with φ_α. Clearly $\mathbb{w}_\alpha \in N(\mathbb{T})$. More importantly, for $\overline{\mathbb{G}}$ and its pull-back to $\overline{\mathbb{G}}^{sc}$ via $\mathbb{G}^{sc} \to \mathbb{G}$, there are natural liftings of these elements in $\overline{\mathbb{G}}^{sc}$, which are denoted by $\overline{\mathbb{e}}_\alpha, \overline{\mathbb{w}}_\alpha, \overline{\mathbb{h}}_\alpha^{[b]}$.

Recall we have used $\Phi_{D,\eta}$ to denote the natural pull-back map as in the following diagram:

$$\begin{tikzcd} \mu_n \arrow[r, hook] \arrow[d, equal] & \overline{G}^{sc} \arrow[r, two heads] \arrow[d, "\Phi_{D,\eta}"] & G^{sc} \arrow[d, "\Phi"] \\ \mu_n \arrow[r, hook] & \overline{G} \arrow[r, two heads] & G. \end{tikzcd}$$

From now, for any root $\alpha \in \Psi$ and any $a, b \in F^\times$, we will write $e_\alpha(a), w_\alpha(a)$ and $h_\alpha(a)$ for $\Phi(\mathbb{e}_\alpha(a)), \Phi(\mathbb{w}_\alpha(a))$ and $\Phi(\mathbb{h}_\alpha(a))$ respectively. These elements all lie in G. On the other hand, the elements $\overline{\mathbb{e}}_\alpha(a)$, $\overline{\mathbb{w}}_\alpha(a)$ and $\overline{\mathbb{h}}_\alpha^{[b]}(a)$ all lie in $\overline{G}^{sc}$. We thus introduce the following notation

$$\overline{e}_\alpha(a) := \Phi_{D,\eta}(\overline{\mathbb{e}}_\alpha(a)), \quad \overline{w}_\alpha(a) := \Phi_{D,\eta}(\overline{\mathbb{w}}_\alpha(a)), \quad \overline{h}_\alpha^{[b]}(a) := \Phi_{D,\eta}(\overline{\mathbb{h}}_\alpha^{[b]}(a)) \in \overline{G}. \tag{14}$$

As in Proposition 4.3, we will write $\overline{\mathbb{h}}_\alpha(a)$ and $\overline{h}_\alpha(a)$ for $\overline{\mathbb{h}}_\alpha^{[b]}(a) \in \overline{G}^{sc}$ and $\overline{h}_\alpha^{[b]}(a) \in \overline{G}$ respectively, whenever the latter are independent of the choice $b \in F^\times$. For example, we have used the notation $\overline{\mathbb{h}}_\alpha(a^{n_\alpha})$ in Proposition 4.3.

Furthermore, for convenience we will write $e_\alpha := e_\alpha(1)$, $w_\alpha := w_\alpha(1) \in G$. Similarly, write

$$\overline{e}_\alpha := \overline{e}_\alpha(1), \quad \overline{w}_\alpha := \overline{w}_\alpha(1), \quad \overline{h}_\alpha^{[b]} := \overline{h}_\alpha^{[b]}(1) \ \in \overline{G}.$$

In particular, $\overline{h}_\alpha^{[1]} = 1_{\overline{G}}$.

Following [Sav04], we consider the group $W^K \subseteq K$ generated by $w_\alpha(-1)$ for $\alpha \in \Psi$, which lies in the exact sequence

$$T^{sgn} \hookrightarrow W^K \twoheadrightarrow W, \qquad w_\alpha(-1) \mapsto \mathbf{w}_\alpha,$$

where $T^{sgn} \subseteq T$ is the finite group generated by $h_\alpha(-1)$ for $\alpha \in \Psi$. For application of global purpose, there is no loss of generality to choose representatives from W^K for the Weyl group W.

There is a preferred section of W^K over W. Let $\mathbf{w} \in W$, write $\mathbf{w} = \mathbf{w}_{\alpha_k} ... \mathbf{w}_{\alpha_2} ... \mathbf{w}_{\alpha_1}$ in the form of a minimum decomposition. We would choose the following as a representative of $\mathbf{w}$:

$$s_W(\mathbf{w}) := w_{\alpha_k} \cdot w_{\alpha_{k-1}} ... \cdot w_{\alpha_2} \cdot w_{\alpha_1} \in W^K.$$

One property of s_W is that the representative $s_W(\mathbf{w})$ is independent of the minimum decomposition of $\mathbf{w}$. Moreover, it is multiplicative (cf. [Hum75, §29.4]). That is, if $l(\mathbf{w}\mathbf{w}') = l(\mathbf{w}) + l(\mathbf{w}')$, then $s_W(\mathbf{w}\mathbf{w}') = s_W(\mathbf{w}) \cdot s_W(\mathbf{w}')$.

For simplicity, we may also write w for $s_W(\mathbf{w})$ and similarly w_{α_i} for $s_W(\mathbf{w}_{\alpha_i})$. The finite subgroup $T^{sgn} \subseteq T$ does not lie in $T^\dagger$, and this follows from computing the commutator $[h_\alpha(-1), y \otimes \varpi] = (-1, \varpi)_n^{B_Q(\alpha^\vee, y)}$ for any $y \in Y$. The obstruction is given by $(-1, \varpi)_n$. Equivalently, if we consider W^K as a subgroup of $\overline{G}$ via the splitting s_K of K, then its finite subgroup T^{sgn} viewed as a subgroup of $\overline{G}$ does not lie in the center $Z(\overline{T})$. Thus, the conjugation action of W^K on $\overline{T}$ does not descend to W. However, the action of W on the isomorphism class of representations of $\overline{T}$ is well-defined (cf. Corollary 5.1).

The splitting s_K agrees with the unipotent splitting i_u as in Corollary 2.12, and therefore

$$\begin{aligned} s_K(w_{\alpha_i}) =& s_K(e_{\alpha_i} \cdot e_{-\alpha_i} \cdot e_{\alpha_i}) \\ =& s_K(e_{\alpha_i}) \cdot s_K(e_{-\alpha_i}) \cdot s_K(e_{\alpha_i}) \\ =& i_u(e_{\alpha_i}) \cdot i_u(e_{-\alpha_i}) \cdot i_u(e_{\alpha_i}) \\ =& \overline{e}_{\alpha_i} \cdot \overline{e}_{-\alpha_i} \cdot \overline{e}_{\alpha_i} \\ =& \overline{w}_{\alpha_i} . \end{aligned}$$

It follows in general

$$\begin{aligned} s_K(w) &= s_K(w_{\alpha_k}) \cdot s_K(w_{\alpha_{k-1}}) \cdot ... \cdot s_K(w_{\alpha_2}) \cdot s_K(w_{\alpha_1}) \\ &= \overline{w}_{\alpha_k} \cdot \overline{w}_{\alpha_{k-1}} \cdot ... \cdot \overline{w}_{\alpha_2} \cdot \overline{w}_{\alpha_1}. \end{aligned}$$

The expression is independent of the minimum decomposition of $\mathbf{w}$. Therefore, we may also write $\overline{w}$ for $s_K(w) = s_K \circ s_W(\mathbf{w})$ without any ambiguity.

To summarize the notations, for every $\mathbf{w} \in W$, we have a well-defined representative $w \in W^K \subseteq K$ and $\overline{w} \in s_K(W^K) \subseteq \overline{W^K}$ (the preimage of W^K in $\overline{G}$):

$$\begin{tikzcd} & \mu_n \arrow[d, hook] & \\ & \overline{W^K} \arrow[d, two heads] & \\ T^{sgn} \arrow[r, hook] & W^K \arrow[u, bend left, "s_K"] \arrow[r, two heads] & W. \arrow[l, bend right, "s_W"'] \end{tikzcd}$$

6.1.1. *The canonical isomorphism* ${}^w i(\overline{\chi}) \simeq i({}^{\mathbf{w}}\overline{\chi})$. Let $i(\overline{\chi})$ be an irreducible representation of $\overline{T}$ as in §5.2. Let ${}^w i(\overline{\chi})$ be the irreducible representation of $\overline{T}$ with the same vector space as $i(\overline{\chi})$ and $\bar{t} \in \overline{T}$ acting by

$${}^w i(\overline{\chi})(\bar{t}) = i(\overline{\chi})(w^{-1}\bar{t}w).$$

On the other hand, one can consider ${}^{\mathbf{w}}\overline{\chi}$, given by the conjugation action of w on the genuine character $\overline{\chi}$ of $Z(\overline{T})$. Since it actually only depends on $\mathbf{w} \in W$ and not its representative in W^K, we can write ${}^{\mathbf{w}}\overline{\chi}$ with no ambiguity. As a simple consequence of the Stone von-Neumann theorem in [We09], we have the following:

Lemma 6.1. *The two representations are naturally isomorphic:* ${}^w i(\overline{\chi}) \simeq i({}^{\mathbf{w}}\overline{\chi})$. *If* $i(\overline{\chi})$ *is unramified with unramified character* $\overline{\chi}$, *then both* ${}^w i(\overline{\chi})$*and* $i({}^{\mathbf{w}}\overline{\chi})$ *are unramified. Since any isomorphism between the two spaces preserves unramified vectors, there is a canonical isomorphism which identifies the normalized unramified vectors of the two sides.*

In fact, it is not hard to check that the canonical isomorphism is given by

$$r_w : {}^w i(\overline{\chi}) \to i({}^{\mathbf{w}}\overline{\chi}), \quad \mathbb{f} \mapsto r_w(\mathbb{f})(\bar{t}) := \mathbb{f}(w^{-1}\bar{t}w). \tag{15}$$

Moreover, for another representative $w' \in W^K$ of $\mathbf{w}$, the following diagram commutes

$$\begin{tikzcd} & {}^w i(\overline{\chi}) \arrow[dl, "r_{w,w'}"'] \arrow[d, "r_w"] \\ {}^{w'} i(\overline{\chi}) \arrow[r, "r_{w'}"] & i({}^{\mathbf{w}}\overline{\chi}), \end{tikzcd} \tag{16}$$

where $r_{w,w'}$ is given by $r_{w,w'}(\mathbb{f})(\bar{t}) = \mathbb{f}(w^{-1}w' \cdot \bar{t} \cdot w'^{-1}w)$. Note that the unramified vectors $\mathbb{f}_{{}^{w}i(\overline{\chi})}$ and $\mathbb{f}_{{}^{w'}i(\overline{\chi})}$ are the same functions and equal to the unramified vector $\mathbb{f}_{i(\overline{\chi})} \in i(\overline{\chi})$. Moreover, the map $r_{w,w'}$ restricts to the identity map from $\mathbf{C} \cdot \mathbb{f}_{{}^{w}i(\overline{\chi})}$ to $\mathbf{C} \cdot \mathbb{f}_{{}^{w'}i(\overline{\chi})}$ (with both identified with $\mathbf{C} \cdot \mathbb{f}_{i(\overline{\chi})}$).

From now, we will fix the isomorphism ${}^{w}i(\overline{\chi}) \simeq i({}^{\mathbf{w}}\overline{\chi})$ by requiring that the normalized unramified functions on two sides correspond to each other as in the above lemma.

6.2. **Intertwining operators and cocycle relations.** As before, let $w = w_{\alpha_k} w_{\alpha_{k-1}} ... w_{\alpha_2} w_{\alpha_1}$ be the element of W^K representing $\mathbf{w} = \mathbf{w}_{\alpha_k} \mathbf{w}_{\alpha_{k-1}} ... \mathbf{w}_{\alpha_2} \mathbf{w}_{\alpha_1}$ in a minimum expansion. Let $\overline{w} = s_K(w) \in \overline{G}$ be the representative of $\mathbf{w}$, which is independent of the factorization of $\mathbf{w}$.

Consider the map

$$T\big(w, i(\overline{\chi})\big) : I(i(\overline{\chi})) \to I({}^{w}i(\overline{\chi})), \text{ given by } f \mapsto \int_{N^w} f(\overline{w}^{-1}\overline{u}g)du. \tag{17}$$

Here $N^w = N \cap wN^- w^{-1}$, where N is the unipotent radical of the Borel $\overline{B} = \overline{T}N$ and N^- the unipotent subgroup opposite to N.

Lemma 6.2. *The map $T\big(w, i(\overline{\chi})\big) : I(i(\overline{\chi})) \to I({}^{w}i(\overline{\chi}))$ is well-defined and nonzero. It intertwines the two unramified representations and sends unramified vector to unramified.*

Proof. The proof is routine and follows the same argument as in the linear case, see [Sha10, §4.1]. □

Let $f_{i(\overline{\chi})}$ and $f_{{}^{w}i(\overline{\chi})}$ be the normalized unramified vectors in $I(i(\overline{\chi}))$ and $I({}^{w}i(\overline{\chi}))$ respectively. Write $c(w, \overline{\chi}) \in \mathbf{C}$ for the coefficient such that

$$T\big(w, i(\overline{\chi})\big) f_{i(\overline{\chi})} = c(w, \overline{\chi}) f_{{}^{w}i(\overline{\chi})}.$$

The coefficient $c(w, \overline{\chi})$, which depends a priori on the preferred representative w of $\mathbf{w}$, does not in the following sense. Take any other representative $w' \in W^K$ of $\mathbf{w}$, one has the intertwining operator $T\big(w', i(\overline{\chi})\big) : I(i(\overline{\chi})) \to I({}^{w'}i(\overline{\chi}))$ and $T\big(w', i(\overline{\chi})\big) f_{i(\overline{\chi})} = c(w', \overline{\chi}) f_{{}^{w'}i(\overline{\chi})}$. We have the commutative diagram

$$\begin{tikzcd}
I(i(\overline{\chi})) \arrow[r, "T(w,i(\overline{\chi}))"] \arrow[d, "T(w',i(\overline{\chi}))"'] & I({}^{w}i(\overline{\chi})) \arrow[d, "r_w^*"] \arrow[dl, "r^*_{w,w'}"] \\
I({}^{w'}i(\overline{\chi})) \arrow[r, "r^*_{w'}"'] & I({}^{\mathbf{w}}\overline{\chi}),
\end{tikzcd} \tag{18}$$

where the maps from the right lower triangle are induced from those in (16). It is easy to see $r^*_{w,w'}(f_{{}^{w}i(\overline{\chi})}) = f_{{}^{w'}i(\overline{\chi})}$, and it follows

$$c(w, \overline{\chi}) = c(w', \overline{\chi}).$$

Therefore, we may write $c(\mathbf{w}, \overline{\chi})$ instead of $c(w, \overline{\chi})$. To take another view of $c(\mathbf{w}, \overline{\chi})$, we may consider the compositions from above diagram:

$$T(\mathbf{w}, \overline{\chi}) = r_w^* \circ T(w, i(\overline{\chi})) : \quad I(i(\overline{\chi})) \to I({}^{w}i(\overline{\chi})) \to I(i({}^{\mathbf{w}}\overline{\chi})). \tag{19}$$

It is justified from the diagram (18) that this map is independent of the choice of representative in W^K for the fixed $\mathbf{w} \in W$, hence the notation. In brief, we may write $T(\mathbf{w}, \overline{\chi}) : I(\overline{\chi}) \to I({}^{\mathbf{w}}\overline{\chi})$. Thus $T(\mathbf{w}, \overline{\chi})$ is the intertwining map between unramified spaces uniquely determined by $T(\mathbf{w}, \overline{\chi}) f_{i(\overline{\chi})} = c(\mathbf{w}, \overline{\chi}) f_{i({}^{\mathbf{w}}\overline{\chi})}$.

Keep notations as above, and write $\Psi_{\mathbf{w}} := \{\alpha \in \Psi^+ : \ \mathbf{w}(\alpha) \in \Psi^-\}$. Proceed and compute as in the linear case (cf. [CKM04, pg. 139-140]), one can show easily the following.

Proposition 6.3. *Let $\mathbf{w} = \mathbf{w}_{\alpha_k} ... \mathbf{w}_{\alpha_2} \mathbf{w}_{\alpha_1} \in W$ be an expansion of minimum length into simple reflections, and let $w = w_{\alpha_k} ... w_{\alpha_2} w_{\alpha_1}$ be defined as above. Then the intertwining operator factorizes as*

$$T\big(w, i(\overline{\chi})\big) = T\big(w_{\alpha_k}, {}^{w_{\alpha_{k-1}} w_{\alpha_{k-2}} ... w_{\alpha_1}} i(\overline{\chi})\big)\big) \circ ... \circ T\big(w_{\alpha_2}, {}^{w_{\alpha_1}} i(\overline{\chi})\big) \circ T\big(w_{\alpha_1}, i(\overline{\chi})\big).$$

Immediately it follows:

Corollary 6.4. *With notations above, one has*

$$c(\mathbf{w},\overline{\chi})=\prod_{m=1}^{k}c(\mathbf{w}_{\alpha_m},{}^{\mathbf{w}_{\alpha_{m-1}}\ldots\mathbf{w}_{\alpha_2}\mathbf{w}_{\alpha_1}}\overline{\chi}),\tag{20}$$

which will be referred to as the cocycle relation. Equivalently, we have

$$T(\mathbf{w},\overline{\chi})=T(\mathbf{w}_{\alpha_k},{}^{\mathbf{w}_{\alpha_{k-1}}\mathbf{w}_{\alpha_{k-2}}\ldots\mathbf{w}_{\alpha_1}}\overline{\chi})\circ\ldots\circ T(\mathbf{w}_{\alpha_2},{}^{\mathbf{w}_{\alpha_1}}\overline{\chi})\circ T(\mathbf{w}_{\alpha_1},\overline{\chi}).$$

7. The Gindikin-Karpelevich formula

7.1. **The Gindikin-Karpelevich formula.** Let $\overline{G}$ be an unramified central cover of Brylinski-Deligne type over a local field F. Choose a uniformizer ϖ of F. The Gindikin-Karpelevich formula is obtained in [McN11, Theorem 6.4] by using a crystal basis decomposition of the domain of integration. Recently, as a consequence of the Casselman-Shalika formula computed, the Gindikin-Karpelevich formula is obtained as in [McN16, Theorem 12.1]. However, we will compute directly below using a straightforward method as in the linear algebraic case and remove restrictions such as $\mu_{2n}\subseteq F^\times$. Moreover, the Gindikin-Karpelevich formula here is expressed in terms of naturally defined terms (i.e. the Brylinski-Deligne liftings) and could be considered as a refinement of that in [McN11] for example. This allows us to interpret it as local Langlands-Shahidi L-function later.

Lemma 7.1. *Let du be an additive measure of F with $du(\mathcal{O}_F)=1$. If $k\in\mathbf{Z}$, then*

$$\int_{\mathcal{O}_F^\times}(\varpi,u)_n^k\,du=\begin{cases}1-1/q & \text{if } n|k,\\ 0 & \text{otherwise.}\end{cases}\tag{21}$$

Proof. It follows from the observation that $u\mapsto(\varpi,u)_n^k$ is a trivial character if and only if $n|k$. □

Lemma 7.2. *For any root $\alpha\in\Psi$ and any $u\in F^\times$, the following relation holds:*

$$\overline{w}_\alpha^{-1}\overline{e}_\alpha(u)=\overline{h}_\alpha^{[1]}(u^{-1})\overline{e}_\alpha(-u)\overline{e}_{-\alpha}(-u^{-1})\in\overline{G}.$$

Proof. It suffices to show the equality

$$\overline{\mathbb{w}}_\alpha^{-1}\overline{\mathbb{e}}_\alpha(u)=\overline{\mathbb{h}}_\alpha^{[1]}(u^{-1})\overline{\mathbb{e}}_\alpha(-u)\overline{\mathbb{e}}_{-\alpha}(-u^{-1})\in\overline{G}^{sc},$$

from which we could apply the morphism $\Phi_{D,\eta}:\overline{G}^{sc}\to\overline{G}$ to obtain the desired result. Now, the right hand side is equal to

$$\begin{aligned}
&=\overline{\mathbb{h}}_\alpha^{[1]}(u^{-1})\cdot\overline{\mathbb{e}}_\alpha(-u)\overline{\mathbb{e}}_{-\alpha}(-u^{-1})\overline{\mathbb{e}}_\alpha(-u)\cdot\overline{\mathbb{e}}_\alpha(u)\\
&=\overline{\mathbb{h}}_\alpha^{[1]}(u^{-1})\overline{\mathbb{w}}_\alpha(-u)\cdot\overline{\mathbb{e}}_\alpha(u)\\
&=\overline{\mathbb{h}}_\alpha^{[1]}(u^{-1})\cdot\overline{\mathbb{w}}_\alpha(-u)\overline{\mathbb{w}}_\alpha(-1)\cdot\overline{\mathbb{w}}_\alpha(1)\overline{\mathbb{e}}_\alpha(u)\\
&=\overline{\mathbb{h}}_\alpha^{[1]}(u^{-1})\overline{\mathbb{h}}_\alpha^{[1]}(-u)\cdot\overline{\mathbb{w}}_\alpha(1)\overline{\mathbb{e}}_\alpha(u)\\
&=(u^{-1},-u)_n^{Q(\alpha^\vee)}\cdot\overline{\mathbb{h}}_\alpha^{[1]}(-1)\overline{\mathbb{w}}_\alpha(1)\overline{\mathbb{e}}_\alpha(u)\\
&=\overline{\mathbb{w}}_\alpha(-1)\overline{\mathbb{w}}_\alpha(-1)\overline{\mathbb{w}}_\alpha(1)\overline{\mathbb{e}}_\alpha(u)\\
&=\overline{\mathbb{w}}_\alpha(-1)\overline{\mathbb{e}}_\alpha(u).
\end{aligned}$$

Note $\overline{\mathbb{w}}_\alpha=\overline{\mathbb{w}}_\alpha(1)$ by definition, and therefore $\overline{\mathbb{w}}_\alpha^{-1}=\overline{\mathbb{w}}_\alpha(-1)$. The proof is completed. □

Recall the convention on notations from §6.1: we write $\overline{h}_\alpha(a^{n_\alpha})$ for $\overline{h}_\alpha^{[b]}(a^{n_\alpha})$ since the latter does not depend on $b\in F^\times$.

Proposition 7.3. *Let $i(\overline{\chi})$ be an unramified representation of $\overline{T}$ for some unramified character $\overline{\chi}$ of $Z(\overline{T})$. Let $\alpha\in\Delta$, consider the intertwining operator $T\big(w_\alpha,i(\overline{\chi})\big):I(i(\overline{\chi}))\to I({}^{w_\alpha}i(\overline{\chi}))$*

between unramified principal series of $\overline{G}$. Let $f_{i(\overline{\chi})}$ and $f_{^{w_\alpha}i(\overline{\chi})}$ be the normalized unramified vectors of $I(i(\overline{\chi}))$ and $I(^{w_\alpha}i(\overline{\chi}))$ respectively. Write $T\big(w_\alpha, i(\overline{\chi})\big)f_{i(\overline{\chi})} = c(\mathbf{w}_\alpha, \overline{\chi})f_{^{w_\alpha}i(\overline{\chi})}$. Then

$$c(\mathbf{w}_\alpha, \overline{\chi}) = \frac{1 - q^{-1}\overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))}{1 - \overline{\chi}(\overline{h}_\alpha(\varpi^{n_\alpha}))},$$

which is independent of the uniformizer ϖ chosen.

Proof. Write $\overline{\pi} := i(\overline{\chi})$. The unities $1_{\overline{G}}, 1_{\overline{T}}$ of all subgroups $\overline{G}, \overline{T}$ etc of $\overline{G}$ are all the same; but for convenience, we use the subscript to remind us the group on which the representation lives.

By the definition of $T\big(w_\alpha, i(\overline{\chi})\big)$, it suffices to compute

$$\begin{aligned}
&T\big(w_\alpha, i(\overline{\chi})\big)(f_{i(\overline{\chi})})(1_{\overline{G}})(1_{\overline{T}})\\
=&\int_{U_\alpha} f\big(\overline{w}_\alpha^{-1}\overline{e}_\alpha(u)\big)(1_{\overline{T}})du\\
=&\int_{0<|u|\le 1} f\big(\overline{w}_\alpha^{-1}\overline{e}_\alpha(u)\big)(1_{\overline{T}})du + \int_{|u|>1} f\big(\overline{w}_\alpha^{-1}\overline{e}_\alpha(u)\big)(1_{\overline{T}})du.
\end{aligned}$$

Here du is the additive Haar measure as in Lemma 7.1. For the first integral, $\overline{w}_\alpha^{-1}\overline{e}_\alpha(u) \in K \subseteq \overline{G}$ for all $0 < |u| \le 1$. Since f is K-invariant, the integrand is equal to $f(1_{\overline{G}})(1_{\overline{T}}) = 1$. Therefore, the first integral is equal to 1.

For the second integral, we apply the previous lemma to get it equal to

$$\begin{aligned}
&\int_{|u|>1} f\big(\overline{h}_\alpha^{[1]}(u^{-1})\overline{e}_\alpha(-u)\overline{e}_{-\alpha}(-u^{-1})\big)(1_{\overline{T}})du\\
=&\int_{|u|>1} f\big(\overline{h}_\alpha^{[1]}(u^{-1})\overline{e}_\alpha(-u)\big)(1_{\overline{T}})du\\
=&\int_{|u|>1} \delta_B^{1/2}(h_\alpha(u^{-1}))\cdot\Big(\overline{\pi}\big(\overline{h}_\alpha^{[1]}(u^{-1})\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})du.
\end{aligned}$$

Note $\delta_B^{1/2}(h_\alpha(u)) = |u|_F$. Use the partition $\{u : |u| > 1\} = \bigcup_{k\ge 1}\varpi^{-k}\mathcal{O}_F^\times$, the second integral is then equal to

$$\begin{aligned}
&=\sum_{k\ge1}\int_{u\in\varpi^{-k}\mathcal{O}_F^\times} \delta_B^{1/2}(h_\alpha(u^{-1}))\cdot\Big(\overline{\pi}\big(\overline{h}_\alpha^{[1]}(u^{-1})\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})du\\
&=\sum_{k\ge1}\int_{u\in\mathcal{O}_F^\times} \delta_B^{1/2}(h_\alpha(\varpi^k u^{-1}))\cdot\Big(\overline{\pi}\big(\overline{h}_\alpha^{[1]}(\varpi^k u^{-1})\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})\cdot|\varpi^{-k}|_F du\\
&=\sum_{k\ge1}\int_{u\in\mathcal{O}_F^\times} |\varpi|_F^k\cdot(\varpi^k, u)_n^{Q(\alpha^\vee)}\cdot\Big(\overline{\pi}\big(\overline{h}_\alpha^{[1]}(\varpi^k)\big)\overline{\pi}\big(\overline{h}_\alpha^{[1]}(u^{-1})\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})\cdot|\varpi^{-k}|_F du\\
&=\sum_{k\ge1}\int_{u\in\mathcal{O}_F^\times} (\varpi, u)_n^{kQ(\alpha^\vee)}\Big(\overline{\pi}\big(\overline{h}_\alpha^{[1]}(\varpi^k)\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})du\\
&=\sum_{k\ge1, n_\alpha|k}\int_{u\in\mathcal{O}_F^\times} \Big(\pi\big(\overline{h}_\alpha^{[1]}(\varpi^k)\big)f(1_{\overline{G}})\Big)(1_{\overline{T}})du \quad \text{by Lemma 7.1}\\
&=\sum_{k\ge1, n_\alpha|k} \overline{\chi}\big(\overline{h}_\alpha(\varpi^k)\big)\cdot f(1_{\overline{G}})(1_{\overline{T}})\cdot(1-q^{-1})\\
&=(1-q^{-1})\frac{\overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)}{1-\overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)},
\end{aligned}$$

where the last equality is due to the fact that $\overline{\chi}\big(\overline{h}_\alpha(\varpi^{r\cdot n_\alpha})\big) = \overline{\chi}^r\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)$ for $r \in \mathbf{N}_{\ge1}$. Now combine the first and second integral, we obtain the desired formula.

It remains to show the independence of the chosen uniformizer. We have for $u \in \mathcal{O}_F^\times$,

$$\begin{aligned} \overline{h}_\alpha\big((\varpi u)^{n_\alpha}\big) =& \overline{h}_\alpha(\varpi^{n_\alpha}) \cdot \overline{h}_\alpha(u^{n_\alpha}) \cdot (\varpi, u)_n^{n_\alpha^2 Q(\alpha^\vee)} \\ =& \overline{h}_\alpha(\varpi^{n_\alpha}) \cdot \overline{h}_\alpha(u^{n_\alpha}). \end{aligned}$$

However, since $\overline{h}_\alpha(u^{n_\alpha}) \in Z(\overline{T}) \cap K_T \subseteq s_K(K)$, it follows that $\overline{\chi}$ takes the value 1 at it. The independence of $c(\mathbf{w}_\alpha, \overline{\chi})$ on the choice of uniformizer has been verified. □

Now we come back to general $\mathbf{w} \in W$ and consider the intertwining operator $T(\mathbf{w}, \overline{\chi})$.

Corollary 7.4. *Let $f_{i(\overline{\chi})}$ and $f_{i({}^{\mathbf{w}}\overline{\chi})}$ be the normalized unramified vectors in $I(\overline{\chi})$ and $I({}^{\mathbf{w}}\overline{\chi})$ respectively. Then, $T(\mathbf{w}, \overline{\chi}) f_{i(\overline{\chi})} = c(\mathbf{w}, \overline{\chi}) f_{i({}^{\mathbf{w}}\overline{\chi})}$ with*

$$c(\mathbf{w}, \overline{\chi}) = \prod_{\alpha \in \Psi_{\mathbf{w}}} \frac{1 - q^{-1}\overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)}{1 - \overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big)},$$

where $\Psi_{\mathbf{w}} = \{\alpha \in \Psi^+ : \ \mathbf{w}\alpha \in \Psi^-\}$.

Proof. Let $\mathbf{w}_k \mathbf{w}_{k-1} ... \mathbf{w}_2 \mathbf{w}_1$ be a minimum decomposition of $\mathbf{w}$, where $\mathbf{w}_i$ represents a simple reflection $\mathbf{w}_{\alpha_i}$ with $\alpha_i \in \Delta$. By the cocycle condition in Corollary 6.4, it suffices to compute the coefficient $c(\mathbf{w}_m, {}^{\mathbf{w}_{m-1}...\mathbf{w}_1}\overline{\chi})$ for $m = 1, 2, ..., k$. First note, since Q is Weyl-invariant, one has

$$n_{\mathbf{w}_1 ... \mathbf{w}_{m-1}\alpha_m} = n_{\alpha_m}.$$

Meanwhile, we have

$$\begin{aligned} & {}^{\mathbf{w}_{m-1}...\mathbf{w}_1}\overline{\chi}\big(\overline{h}_{\alpha_m}(\varpi^{n_{\alpha_m}})\big) \\ =& \overline{\chi}\big((w_{m-1}...w_1)^{-1} \cdot \overline{h}_{\alpha_m}(\varpi^{n_{\alpha_m}}) \cdot w_{m-1}...w_1)\big) \\ =& \overline{\chi}\big((w_1^{-1}...w_{m-1}^{-1} \cdot \overline{h}_{\alpha_m}(\varpi^{n_{\alpha_m}}) \cdot w_{m-1}...w_1)\big). \end{aligned}$$

To proceed, consider in general the element $w_\beta^{-1}\overline{h}_\alpha^{[b]}(a) w_\beta$ for $\alpha, \beta \in \Psi$ and $a, b \in F^\times$. One has

$$\begin{aligned} & w_\beta^{-1}\overline{h}_\alpha^{[b]}(a) w_\beta \\ =& w_\beta^{-1} \cdot \overline{w}_\alpha(ab)\overline{w}_\alpha(-b) \cdot w_\beta \\ =& w_\beta^{-1}\overline{w}_\alpha(ab) w_\beta \cdot w_\beta^{-1}\overline{w}_\alpha(-b) w_\beta. \end{aligned}$$

At the same time, for general $c \in F^\times$, the following equalities hold

$$\begin{aligned} & w_\beta^{-1}\overline{w}_\alpha(c) w_\beta \\ =& w_\beta^{-1} \cdot \overline{e}_\alpha(c)\overline{e}_{-\alpha}(c^{-1})\overline{e}_\alpha(c) \cdot w_\beta \\ =& \overline{e}_{\mathbf{w}_\beta(\alpha)}(\epsilon c) \cdot \overline{e}_{-\mathbf{w}_\beta(\alpha)}(\epsilon c^{-1}) \cdot \overline{e}_{\mathbf{w}_\beta(\alpha)}(\epsilon c), \\ =& \overline{w}_{\mathbf{w}_\beta(\alpha)}(\epsilon c), \end{aligned}$$

where the second last equality follows from the fact that the unipotent section is G-equivariant (cf. Proposition 2.11). Here $\epsilon \in \{\pm 1\}$ is a certain sign (depending on α, β) associated with the Chevalley system of épinglage, see [BrTi84, §3.2.2].

Now it follows

$$w_\beta^{-1}\overline{h}_\alpha^{[b]}(a) w_\beta = \overline{w}_{\mathbf{w}_\beta(\alpha)}(\epsilon ab) \cdot \overline{w}_{\mathbf{w}_\beta(\alpha)}(-\epsilon b) = \overline{h}_{\mathbf{w}_\beta(\alpha)}^{[\epsilon b]}(a).$$

In the case of $a = \varpi^{n_\alpha}$, the element $\overline{h}_{\mathbf{w}_\beta(\alpha)}^{[\epsilon b]}(a)$ is independent of ϵ and b. Computing inductively we obtain

$${}^{\mathbf{w}_{m-1}...\mathbf{w}_1}\overline{\chi}\big(\overline{h}_{\alpha_m}(\varpi^{n_{\alpha_m}})\big) = \overline{\chi}\big(\overline{h}_{\mathbf{w}_1...\mathbf{w}_{m-1}\alpha_m}(\varpi^{n_{\mathbf{w}_1...\mathbf{w}_{m-1}\alpha_m}})\big).$$

Lastly, we have the equality $\Psi_{\mathbf{w}} = \{\mathbf{w}_1 ... \mathbf{w}_{m-1}\alpha_m : m = 1, 2, ..., k\}$, from which the result follows from combining all $c(\mathbf{w}_m, {}^{\mathbf{w}_{m-1}...\mathbf{w}_1}\overline{\chi})$'s. □

Remark 7.5. The usage of the element $\overline{h}_\alpha(\varpi^{n_\alpha})$ (or in general $\overline{h}_\alpha^{[1]}(a)$) from the Brylinski-Deligne lifting enables us to remove the assumption $\mu_{2n} \subseteq F^\times$ as in [McN11] and [McN12]. In fact, the computation of the metaplectic Casselman-Shalika formula in [McN16] could be carried over using such naturally defined elements. It can be checked that in the case of double cover of $\mathbb{Sp}_{2r}(F)$, McNamara's formula [McN16, Theorem 13.1] recovers that of Szpruch in [Szp13] which does not rely on the assumption that F contains $2n$-th roots of unity, provided that we make use of these naturally defined elements as in Lemma 7.2: $\overline{w}_\alpha, \overline{e}_\alpha(u), \overline{h}_\alpha^{[1]}(u)$ etc.

Remark 7.6. Recall that for any root $\alpha \in \Psi$ the morphism $\varphi_\alpha : \mathbb{SL}_2 \to \mathbb{G}$ induces a covering group $\overline{SL}_2^\alpha$ from any given $\overline{G}$ of Brylinski-Deligne type. Let T and T_o be the tori of G and SL_2, and let $\overline{T}$ and $\overline{T}^\alpha$ be the covering tori of $\overline{G}$ and $\overline{SL}_2^\alpha$ respectively. Denote by $Z(\overline{T})$ and $Z(\overline{T}^\alpha)$ the centers of the two covering tori respectively. Then we have the following commutative diagram

$$\begin{tikzcd}
 & \overline{T} \arrow[r, two heads] & T \\
Z(\overline{T}) \arrow[ru] & \overline{T}^\alpha \arrow[u, "\overline{\varphi}_\alpha"] \arrow[r, two heads] & T_o \arrow[u, "\varphi_\alpha"] \\
\overline{\varphi}_\alpha^*(Z(\overline{T})) \arrow[u, "\overline{\varphi}_\alpha"] \arrow[ru] \arrow[r, hook] & Z(\overline{T}^\alpha) \arrow[u, hook]
\end{tikzcd},$$

where $\overline{\varphi}_\alpha^*(Z(\overline{T}))$ is the pull-back. By definition, $Z(\overline{T}^\alpha)$ and $\overline{\varphi}_\alpha^*(Z(\overline{T}))$ are closely related to $\alpha_{Q,n}^\vee/\gcd(2, n_\alpha)$ and $\alpha_{Q,n}^\vee$. More precisely, define

$$\mathbf{n} = \begin{cases} 1 & \text{if } \alpha_{Q,n}^\vee/\gcd(2,n_\alpha) \in Y_{Q,n} \\ 2 & \text{otherwise.} \end{cases}$$

Then it is not hard to see $Z(\overline{T}^\alpha)/\overline{\varphi}_\alpha^*(Z(\overline{T})) \simeq F^\times/\mathbf{n}$. That is, in general $\overline{\varphi}_\alpha^*(Z(\overline{T}))$ is not equal to the whole group $Z(\overline{T}^\alpha)$ and $Z(\overline{T}^\alpha)/\overline{\varphi}_\alpha^*(Z(\overline{T}))$ is a torsion 2 group. In fact, since we have assumed $\gcd(n,p) = 1$ in the unramified setting, $F^\times/\mathbf{n} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ if $\mathbf{n} = 2$.

This has the following implication. For any genuine character $\overline{\chi}$ on $Z(\overline{T})$ we may write $\overline{\chi}_\alpha := \overline{\chi} \circ \overline{\varphi}_\alpha$, which is a genuine character on $\overline{\varphi}_\alpha^*(Z(\overline{T}))$. In the unramified case, the rank one intertwining operator $T(\mathbf{w}_\alpha, \overline{\chi})$, or equivalently the scalar $c(\mathbf{w}_\alpha, \overline{\chi})$, can be determined from computing the following intertwining operator on $\overline{SL}_2^\alpha$:

$$T(\mathbf{w}_\alpha, \overline{\chi}) : \quad I\Big(\mathrm{Ind}_{\overline{\varphi}_\alpha^*(Z(\overline{T}))}^{Z(\overline{T}^\alpha)}(\overline{\chi}_\alpha)\Big) \to I\Big({}^{\mathbf{w}_\alpha}\big(\mathrm{Ind}_{\overline{\varphi}_\alpha^*(Z(\overline{T}))}^{Z(\overline{T}^\alpha)}(\overline{\chi}_\alpha)\big)\Big).$$

Note however, if we have $Z(\overline{T}^\alpha)/\overline{\varphi}_\alpha^*(Z(\overline{T})) \simeq (\mathbf{Z}/2\mathbf{Z})^2$, then $\mathrm{Ind}_{\overline{\varphi}_\alpha^*(Z(\overline{T}))}^{Z(\overline{T}^\alpha)}(\overline{\chi}_\alpha) = \bigoplus_{i=1}^4 \overline{\chi}_{\alpha,i}$ is a 4-dimensional representation of $Z(\overline{T}^\alpha)$. Thus, $T(\mathbf{w}_\alpha, \overline{\chi})$ is given by

$$T(\mathbf{w}_\alpha, \overline{\chi}) = \bigoplus_{i=1}^4 T(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i}) : \quad \bigoplus_{i=1}^4 I(\overline{\chi}_{\alpha,i}) \to \bigoplus_{i=1}^4 I({}^{\mathbf{w}_\alpha}\overline{\chi}_{\alpha,i}).$$

Write $T(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i})(f_{i(\overline{\chi}_{\alpha,i})}) = c(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i}) f_{i({}^{\mathbf{w}_\alpha}\overline{\chi}_{\alpha,i})}$. Then it follows from the computation in this section that

$$c(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i}) = c(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,j}) = c(\mathbf{w}_\alpha, \overline{\chi}) \text{ for any } i \text{ and } j,$$

since it is shown that $c(\mathbf{w}_\alpha, \overline{\chi}_{\alpha,i})$ depends only on $\overline{\chi}_{\alpha,i}$ restricted to $\overline{\varphi}_\alpha^*(Z(\overline{T}))$ and these characters are all equal there.

This phenomenon already occurs in the case of $\overline{\mathbb{GL}}_2$ with $(Q, \mathcal{E}, \phi)$, where Q is given by

$$Q : Y \to \mathbf{Z}, \quad (y_1, y_2) \mapsto -y_1 y_2.$$

Let $n = 2$ and one obtains a degree two cover $\overline{GL}_2$. Here we identity $Y = \mathbf{Z}^2$, with respect to which the element $(1, -1) \in \mathbf{Z}^2$ gives the coroot $\alpha^\vee$ of $\mathbb{SL}_2 \subseteq \mathbb{GL}_2$. So $Q(\alpha^\vee) = 1$ and $n_\alpha = 2$. One thus has the degree two covers $\overline{SL}_2$ and $\overline{GL}_2$ of SL_2 and GL_2 respectively. It is easy to check $\alpha_{Q,n}^\vee/\gcd(2, n_\alpha) = \alpha^\vee \notin Y_{Q,n}$, and therefore $Z(\overline{T}^\alpha)$ is not equal to $\overline{\varphi}_\alpha^*(Z(\overline{T}))$.

7.2. **The adjoint action and Gindikin-Karpelevich formula for principal series.** For $\overline{G}$ of Brylinski-Deligne type, consider the adjoint action of ${}^L\overline{G}$ on its Lie algebra, which is equal to the Lie algebra $\overline{\mathfrak{g}}^\vee$ of $\overline{G}^\vee$. It gives rise to the adjoint representation $Ad : {}^L\overline{G} \to GL(\overline{\mathfrak{g}}^\vee)$. By definition ${}^L\overline{G} = j_*^{\overline{G}^\vee} \circ \mathrm{Rec}^*(E_{\overline{G}})$, where $E_{\overline{G}}$ is the fundamental extension over $F^\times$ by $Z(\overline{G}^\vee)$, i.e.

$$ {}^L\overline{G} = \frac{\overline{G}^\vee \times \mathrm{Rec}^*(E_{\overline{G}})}{\nabla Z(\overline{G}^\vee)}. $$

This implies that the adjoint action Ad depends only on the component $\overline{G}^\vee$ of ${}^L\overline{G}$, since $Z(\overline{G}^\vee)$ acts trivially on $\overline{\mathfrak{g}}^\vee$. More precisely, define $\overline{G}_{ad}^\vee := \overline{G}^\vee / Z(\overline{G}^\vee)$ and consider the trivial extension $\overline{G}_{ad}^\vee \times W_F$ over W_F. Then there is a natural map q^* such that the diagram

$$\begin{tikzcd}
\overline{G}^\vee \arrow[r, hook] \arrow[d, two heads, "q"] & {}^L\overline{G} \arrow[r, two heads] \arrow[d, "q^*"] & \mathrm{W}_F \arrow[d, equal] \\
\overline{G}_{ad}^\vee \arrow[r, hook] & \overline{G}_{ad}^\vee \times W_F \arrow[r, hook] \arrow[l, bend left, "s^{\mathrm{Tr}}"] & \mathrm{W}_F
\end{tikzcd}$$

commutes. Here the bottom sequence is equipped with a canonical splitting s^{Tr} by projection onto the first component. The adjoint representation Ad factors through the usual complex adjoint representation $Ad^{\mathbf{C}}$ of $\overline{G}_{ad}^\vee$ on $\overline{\mathfrak{g}}^\vee$ with respect to the map $s^{\mathrm{Tr}} \circ q^*$:

$$ Ad = Ad^{\mathbf{C}} \circ \left(s^{\mathrm{Tr}} \circ q^* \right). \tag{22} $$

Now for any parabolic subgroup $\mathbb{P} = \mathbb{M}\mathbb{U}$ of $\mathbb{G}$ we obtain the dual group $\overline{M}$ and the L-group ${}^L\overline{M}$ as well. Recall that there exists a canonical map ${}^L\varphi : {}^L\overline{M} \to {}^L\overline{G}$ between the L-group extensions, which extends the natural map $\varphi^\vee : \overline{M}^\vee \to \overline{G}^\vee$. By restriction, this gives rise to a representation of ${}^L\overline{M}$ which factors through the complex representation $Ad^{\mathbf{C}} : \overline{M}^\vee / Z(\overline{G}^\vee) \to GL(\overline{\mathfrak{g}}^\vee)$.

Since $\overline{M}^\vee$ is a Levi subgroup of $\overline{G}^\vee$, we can define a complex unipotent $\overline{U}^\vee$ such that it is the unipotent radical of the parabolic subgroup $\overline{M}^\vee \overline{U}^\vee$ of $\overline{G}^\vee$. The group $\overline{M}^\vee / Z(\overline{G}^\vee)$ and therefore ${}^L\overline{M}$ act on the Lie algebra $\overline{\mathfrak{u}}^\vee = \mathrm{Lie}(\overline{U}^\vee)$, which is a invariant space of $\overline{\mathfrak{g}}^\vee$. That is, we have from (22) the following diagram

$$\begin{tikzcd}
{}^L\overline{M} \arrow[r, "Ad"] \arrow[d] & GL(\overline{\mathfrak{u}}^\vee) \\
\overline{M}^\vee / Z(\overline{G}^\vee), \arrow[ur, dashed, "Ad^{\mathbf{C}}"] &
\end{tikzcd} \tag{23}$$

where the vertical map is the composition $s^{\mathrm{Tr}} \circ q^* \circ {}^L\varphi$.

Now we specialize to the case where $\mathbb{P} = \mathbb{B}$ is the Borel subgroup of $\mathbb{G}$. To be consistent with previous notations, we write $\mathbb{B} = \mathbb{T}\mathbb{N}$. Then $\overline{\mathfrak{n}}^\vee = \mathrm{Lie}(\overline{N}^\vee)$ is generated by eigenvectors for Ad of the form $E_{\alpha_{Q,n}^\vee}, \alpha \in \Psi^+$. For any $\mathbf{w} \in W$, we are interested in the space

$$ \overline{\mathfrak{n}}_{\mathbf{w}}^\vee = \bigoplus_{\alpha \in \Psi_{\mathbf{w}}} \mathbf{C} \cdot E_{\alpha_{Q,n}^\vee} \subseteq \overline{\mathfrak{n}}^\vee, $$

where $\Psi_{\mathbf{w}} = \{ \alpha \in \Psi^+ : \ \mathbf{w}\alpha \in \Psi^- \}$.

The space $\mathbf{C} \cdot E_{\alpha_{Q,n}^\vee}$ is invariant under the adjoint action $Ad^{\mathbf{C}}$ of $\overline{T}^\vee / Z(\overline{G}^\vee)$ and this allows us to consider $Ad_{\mathbf{w}} := Ad|_{\overline{\mathfrak{n}}_{\mathbf{w}}^\vee}$, which has the decomposition

$$ Ad_{\mathbf{w}} = \bigoplus_{\alpha \in \Psi_{\mathbf{w}}} Ad_\alpha : \quad {}^L\overline{T} \longrightarrow GL(\overline{\mathfrak{n}}_{\mathbf{w}}^\vee), $$

where each Ad_α is the one-dimensional representation on $\mathbf{C} \cdot E_{\alpha_{Q,n}^\vee}$.

Let $I(\overline{\chi}) = I(i(\overline{\chi}))$ be an unramified principal series of $\overline{G}$. Let $\rho_{\overline{\chi}} : \mathrm{W}_F \to {}^L\overline{T}$ be the splitting of ${}^L\overline{T}$ over W_F associated with $\overline{\chi}$ by the local Langlands correspondence. We could identify $\rho_{\overline{\chi}}$ with a splitting $\rho_{\overline{\chi}} : F^\times \to E_{\overline{T}}$. Note that for an unramified character $\overline{\chi}$, the element $\rho_{\overline{\chi}}(\varpi) \in E_{\overline{T}}$ may still depend on the choice of the uniformizer ϖ. However, consider the canonical map $\mathcal{C}$ from $E_{\overline{T}}$ to $\overline{T}^\vee/Z(\overline{G}^\vee)$ in Lemma 3.3. Then the element $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi) \in \mathcal{C}(E_{\overline{T}})$ is independent of the uniformizer, as follows.

Proposition 7.7. *Let $\overline{\chi}$ be an unramified genuine character of $Z(\overline{T})$. Identify $\overline{T}^\vee/Z(\overline{G}^\vee)$ with $Hom(Y_{Q,n}^{sc}, \mathbf{C})$. Then for all $\alpha \in \Psi$, we have*

$$\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)(\alpha_{Q,n}^\vee) = \overline{\chi}\big(\overline{h}(\varpi^{n_\alpha})\big) \in \mathbf{C}.$$

In particular, $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)$ is independent of the uniformizer ϖ chosen.

Proof. The identity follows from Corollary 4.4, while the independence of the uniformizer holds by Proposition 7.3. □

Write $\overline{\pi} := i(\overline{\chi})$. Then the local Langlands L-function $L(s, \overline{\pi}, Ad_\alpha)$ is given by the Artin L-function associated with $Ad_\alpha \circ \rho_{\overline{\chi}} : \mathrm{W}_F \to {}^L\overline{T} \to GL(\overline{\mathfrak{n}}_\alpha^\vee)$. That is,

$$L(s, \overline{\pi}, Ad_\alpha) := \det\big(1 - q^{-s} \cdot Ad_\alpha \circ \rho_{\overline{\chi}}(\mathrm{Frob})|_{\overline{\mathfrak{n}}_\alpha^{\vee I}}\big)^{-1}.$$

Now fix $\mathbf{w} \in W$.

Theorem 7.8. *The eigenvalue of $\rho_{\overline{\chi}}(\varpi)$ for the representation Ad_α on the one-dimensional invariant space $\overline{\mathfrak{n}}_\alpha^\vee$ is given by*

$$Ad_\alpha(\rho_{\overline{\chi}}(\varpi))(E_{\alpha_{Q,n}^\vee}) = \overline{\chi}\big(\overline{h}_\alpha(\varpi^{n_\alpha})\big) \cdot E_{\alpha_{Q,n}^\vee}, \tag{24}$$

which is independent of the uniformizer chosen.

It follows that the Gindikin-Karpelevich formula for the intertwining operator $T(\mathbf{w}, \overline{\chi})$ acting on the unramified representation $I(\overline{\chi})$ can be rewritten as $T(\mathbf{w}, \overline{\chi}) f_{i(\overline{\chi})} = c(\mathbf{w}, \overline{\chi})(f_{i({}^{\mathbf{w}}\overline{\chi})})$ with

$$c(\mathbf{w}, \overline{\chi}) = \prod_{\alpha \in \Psi_{\mathbf{w}}} \frac{L(0, \overline{\pi}, Ad_\alpha)}{L(1, \overline{\pi}, Ad_\alpha)},$$

whenever the right hand side is well-defined.

Proof. It suffices to show the equality in (24) regarding the eigenvalue. Identify the splitting $\rho_{\overline{\chi}}$ as that of $E_{\overline{T}}$ over $F^\times$. Note the adjoint action of ${}^L\overline{T}$ naturally factor through that of $E_{\overline{T}}$.

Further more, the adjoint representation of $E_{\overline{T}}$ on $\overline{\mathfrak{n}}_\alpha^\vee$ also factors through $\overline{T}^\vee/Z(\overline{G}^\vee)$:

$$\begin{tikzcd} E_{\overline{T}} \arrow[r, "Ad_\alpha"] \arrow[d, "\mathcal{C}"] & GL(\overline{\mathfrak{n}}_\alpha^\vee) \\ \overline{T}^\vee/Z(\overline{G}^\vee). \arrow[ur, "Ad_\alpha^{\mathbf{C}}"'] & \end{tikzcd}$$

Therefore, with $\rho_{\overline{\chi}}$ viewed as a splitting of $E_{\overline{T}}$ over $F^\times$, we have the composition

$$Ad_\alpha^{\mathbf{C}} \circ \mathcal{C} \circ \rho_{\overline{\chi}} : \quad F^\times \to E_{\overline{T}} \to \overline{T}^\vee/Z(\overline{G}^\vee) \to GL(\overline{\mathfrak{n}}_\alpha^\vee).$$

Now it suffices to compute the eigenvalue of $Ad_\alpha^{\mathbf{C}} \circ \mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)$ on $E_{\alpha_{Q,n}^\vee}$, which is equal to $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)(\alpha_{Q,n}^\vee)$ since $E_{\alpha_{Q,n}^\vee}$ is the unipotent vector corresponding to the root $\alpha_{Q,n}^\vee$ of $\overline{G}^\vee$. Here we view $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi) \in \overline{T}^\vee/Z(\overline{G}^\vee) \simeq \mathrm{Hom}(Y_{Q,n}^{sc}, \mathbf{C}^\times)$.

The proof is thus completed in view of the equality $\mathcal{C} \circ \rho_{\overline{\chi}}(\varpi)(\alpha_{Q,n}^\vee) = \overline{\chi}\big(\overline{h}(\varpi^{n_\alpha})\big)$ and the independence statement from the preceding proposition. □

Remark 7.9. When $\mathbb{G}$ is a quasi-split unitary group that splits over an unramified extension E of F, the computation in [McN16, Theorem 12.1] gives the rank one Gindikin-Karpelevich coefficient as

$$\frac{\big(1 - (-1)^{n_\alpha} \cdot q^{-1}\overline{h}_\alpha(\varpi^{n_\alpha})\big)\big(1 + (-1)^{n_\alpha} \cdot q^{-2}\overline{h}_\alpha(\varpi^{n_\alpha})\big)}{1 - \overline{h}_\alpha(\varpi^{n_\alpha})^2},$$

where $\overline{h}_\alpha(\varpi^{n_\alpha})^2 = \overline{h}_\alpha(\varpi^{2n_\alpha})$. We believe that a proper understanding of the L-group construction for Brylinski-Deligne covers of quasi-split $\mathbb{G}$ in [We16-2] will enable us to express the Gindikin-Karpelevich coefficient obtained as Langlands-Shahidi type L-function as well. We leave the investigation of this to a future work.

7.3. **The Gindikin-Karpelevich formula for induction from maximal parabolic subgroup.** In this subsection, we consider parabolic induction from a maximal parabolic subgroup of $\overline{G}$. Let $\mathbb{G}$ be a reductive group split over F with root datum $(X, \Psi, Y, \Psi^\vee)$. As before, we have fixed a set of simple roots $\Delta \subseteq \Psi^+$ with $\Psi = \Psi^+ \sqcup \Psi^-$. Consider any simple root $\beta \in \Delta$, and let $\mathbb{P} = \mathbb{M}\mathbb{U}$ be the maximal parabolic of $\mathbb{G}$ associated with $\Delta \backslash \{\beta\}$. Let $2\rho_P$ be the sum of positive roots in $\mathbb{U}$, define $\omega_P = \langle \rho_P, \beta^\vee \rangle^{-1} \cdot \rho_P$, where $\langle -, - \rangle$ is the pairing between X and Y. Then $\omega_P \in X \otimes \mathbf{Q}$ is the fundamental weight associated with β.

From the μ_n-extension $\overline{G}$, we obtain a μ_n extension $\overline{M}$ of $M = \mathbb{M}(F)$. Since the construction of μ_n extension $\overline{G}$ is functorial, the extension $\mu_n \hookrightarrow \overline{M} \twoheadrightarrow M$ over the Levi subgroup M is obtained from the Brylinski-Deligne data inherited from those associated with $\overline{\mathbb{G}}$. Let $\overline{\pi}$ be an irreducible unramified genuine representation of $\overline{M}$. Consider the normalized induced representation $I_{\overline{P}}^{\overline{G}}(\overline{\pi} \otimes \mathbb{1}_U)$. For simplicity we may just write $I(\overline{\pi})$ for it.

Since $\overline{\pi}$ is unramified, by the Satake isomorphism in Corollary 5.1 there is a ϵ-genuine character $\overline{\chi}$ of $Z(\overline{T})$ such that $\overline{\pi} \hookrightarrow I_{\overline{B}_M}^{\overline{M}}(\overline{\chi})$, where $\overline{B}_M = \overline{T} N_M$ is the Borel subgroup of $\overline{M}$ whose Levi factor is just $\overline{T}$. Then the representation $\mathrm{Ind}_{\overline{P}}^{\overline{G}}\big(I_{\overline{B}_M}^{\overline{M}}(\overline{\chi})\big) \otimes \mathbb{1}_U$, by induction in stages, is just the unramified principal series $I(\overline{\chi}) = I(i(\overline{\chi}))$ of $\overline{G}$ introduced before. Moreover, we have $I(\overline{\pi}) \hookrightarrow I(\overline{\chi})$.

It is known (cf. [CKM04, pg. 122]) that there exists a unique $\mathbf{w} \in W$ such that

$$\mathbf{w}(\Delta \backslash \{\beta\}) \subseteq \Delta \text{ and } \mathbf{w}(\beta) \in \Psi^-.$$

In fact, $\mathbf{w}(\omega_P) = -\omega_P$. From now on, we fix this $\mathbf{w}$ whenever we consider intertwining operators for induction from a maximal parabolic subgroup. Let $w = s_W(\mathbf{w}) \in W^K$ and $\overline{w} = s_K(w)$ be the representatives of $\mathbf{w}$ defined in §6.1. We are interested in the intertwining operator

$$T(w, \overline{\pi}) : I(\overline{\pi}) \to I({}^w\overline{\pi}), \quad f \mapsto T(w, \overline{\pi})f(\overline{g}) = \int_{U^w} f(\overline{w}^{-1}\overline{u}\ \overline{g})du,$$

where $U^w = N \cap wU^-w^{-1}$ with U^- the unipotent radical opposed to U.

Let $f_{\overline{\pi}}, f_{{}^w\overline{\pi}}$ be the normalized unramified vectors of $I(\overline{\pi}), I({}^w\overline{\pi})$ respectively. We view them as vectors in the unramified principal series $I(i(\overline{\chi}))$ and $I({}^w i(\overline{\chi}))$, normalized such that $f_{\overline{\pi}}(1_{\overline{G}})(1_{\overline{T}}) = f_{{}^w\overline{\pi}}(1_{\overline{G}})(1_{\overline{T}}) = 1$. We want to compute the constant $c(\mathbf{w}, \overline{\pi})$ that appears in $T(w, \overline{\pi})f_{\overline{\pi}} = c(\mathbf{w}, \overline{\pi})f_{{}^w\overline{\pi}}$. As in the linear case, we could use $T(w, i(\overline{\chi})) : I(i(\overline{\chi})) \to I({}^w i(\overline{\chi}))$ instead for this purpose. This fact, coupled with the computation of the Gindikin-Karpelevich formula in Proposition 7.3, gives

$$c(\mathbf{w}, \overline{\pi}) = \prod_{\alpha \in \Psi_{\mathbf{w}}} \frac{L(0, Ad_\alpha \circ \rho_{\overline{\chi}})}{L(1, Ad_\alpha \circ \rho_{\overline{\chi}})},$$

where each $Ad_\alpha \circ \rho_{\overline{\chi}} : \mathrm{W}_F \to \mathbf{C}^\times$ is a character. To give it an interpretation as Langlands-Shahidi type L-functions, we consider the adjoint action $Ad_{\overline{\mathfrak{u}}^\vee} : {}^L\overline{M} \to GL(\overline{\mathfrak{u}}^\vee)$, where $\overline{\mathfrak{u}}^\vee$ is the Lie algebra of $\overline{U}^\vee$ such that $\overline{M}^\vee \overline{U}^\vee$ is a parabolic subgroup of $\overline{G}^\vee$. It factors through $Ad_{\overline{\mathfrak{u}}^\vee}^{\mathbf{C}}$:

$$\begin{tikzcd} {}^L\overline{M} \arrow[r, "Ad_{\overline{\mathfrak{u}}^\vee}"] \arrow[d] & GL(\overline{\mathfrak{u}}^\vee) \\ \overline{M}^\vee / Z(\overline{G}^\vee). \arrow[ur, "Ad_{\overline{\mathfrak{u}}^\vee}^{\mathbf{C}}"'] & \end{tikzcd}$$

Therefore, irreducible subspaces of $\overline{\mathfrak{u}}^\vee$ for $Ad_{\overline{\mathfrak{u}}^\vee}$ are in correspondence with irreducible subspaces with respect to $Ad_{\overline{\mathfrak{u}}^\vee}^{\mathbf{C}}$, which are familiar (cf. [Lan71]). More precisely, we consider the decomposition of $Ad_{\overline{\mathfrak{u}}^\vee}$ into irreducibles $Ad_{\overline{\mathfrak{u}}^\vee} = \bigoplus_{i=1}^m Ad_i$. Let $V_i \subseteq \overline{\mathfrak{u}}^\vee$ be the irreducible space for Ad_i. Then, as observed by Langlands, V_i is given by

$$V_i = \bigoplus_{\langle \omega_P/n_\beta, \alpha_{Q,n}^\vee \rangle = i} \mathbf{C} \cdot E_{\alpha_{Q,n}^\vee}.$$

Moreover the following equality holds:

$$\Psi_{\mathbf{w}} = \bigsqcup_{i=1}^{m} \left\{ \alpha \in \Psi^+ : \langle \omega_P/n_\beta, \alpha_{Q,n}^\vee \rangle = i \right\}.$$

Recall that the local Artin L-function $L(s, \overline{\pi}, Ad_i)$ is by definition equal to $L(s, Ad_i \circ \rho_{\overline{\chi}})$ associated with $Ad_i \circ \rho_{\overline{\chi}}$, i.e. $L(s, \overline{\pi}, Ad_i) := \det\left(1 - q^{-s} Ad_i \circ \rho_{\overline{\chi}}(\mathrm{Frob})|_{V_i^I}\right)^{-1}$, where we also write $\rho_{\overline{\chi}}$ for the composition $\mathrm{W}_F \to {}^L\overline{T} \to {}^L\overline{M}$. For unramified $\overline{\pi}$, if we identify $\rho_{\overline{\chi}}$ with a splitting of $E_{\overline{T}}$ over $F^\times$, then the inertia group I acts trivially on V_i by Proposition 7.3. It follows

$$L(s, \overline{\pi}, Ad_i) = \det\left(1 - q^{-s} Ad_i \circ \rho_{\overline{\chi}}(\varpi)|_{V_i}\right)^{-1}.$$

In view of Theorem 7.4, we could summarize our discussion above as:

Theorem 7.10. *The Gindikin-Karpelevich formula takes the form $T(w, \overline{\pi}) f_{\overline{\pi}} = c(\mathbf{w}, \overline{\pi}) f_{{}^w\overline{\pi}}$ with*

$$c(\mathbf{w}, \overline{\pi}) = \prod_{i=1}^{m} \frac{L(0, \overline{\pi}, Ad_i)}{L(1, \overline{\pi}, Ad_i)},$$

where in this case we have the equality

$$L(s, \overline{\pi}, Ad_i) = \prod_{\substack{\alpha \in \Psi^+ \\ \langle \omega_P/n_\beta, \alpha_{Q,n}^\vee \rangle = i}} L(s, Ad_\alpha \circ \rho_{\overline{\chi}}).$$

8. The Langlands-Shahidi L-functions for Brylinski-Deligne extensions

The aim of this section is to compute the constant term of Eisenstein series, and to write the coefficient of global intertwining operators in terms of certain Langlands-Shahidi type L-functions. The main tool is the theory of Eisenstein series, and the computation of its constant terms in terms of intertwining operators as given in [MW95]. Since the global harmonic analysis is developed systematically in the book by Moeglin-Waldspurger ([MW95]), we will just give a brief review below and refer to the book for properties of Eisenstein series, e.g. meromorphic continuation and functional equations etc. We also refer to the book for a detailed introduction on automorphic forms on $\overline{\mathbb{G}}(\mathbb{A}_F)$ and the spectral decomposition of $L^2(\mathbb{G}(F)\backslash\overline{\mathbb{G}}(\mathbb{A}_F))$.

Now let F be a number field, and ${}^L\overline{G}$ the global L-group for a global Brylinski-Deligne cover $\overline{\mathbb{G}}(\mathbb{A}_F)$.

8.1. **Automorphic L-function.** Let $\overline{\sigma} = \bigotimes_v \overline{\sigma}_v$ be a genuine automorphic representation of $\overline{\mathbb{G}}(\mathbb{A}_F)$. Then for almost all v, $\overline{\sigma}_v$ is unramified and $\overline{\sigma}_v \hookrightarrow I(\overline{\chi}_v)$ for some $\overline{\chi}_v \in \mathrm{Irr}_\epsilon(Z(\overline{T}_v))$. It gives rise to a splitting of the local L-group ${}^L\overline{G}_v$ as the composition

$$\rho_v : \quad \mathrm{W}_{F_v} \to {}^L\overline{T}_v \hookrightarrow {}^L\overline{G}_v,$$

where the first map is given by $\rho_{\overline{\chi}_v}$ from the LLC in Proposition 4.2. Recall that for all $v \in |F|$, there is the canonical map ${}^L\overline{G}_v \to {}^L\overline{G}$. Let $R : {}^L\overline{G} \to GL(V)$ be any finite dimensional representation. For any v, let $R_v : {}^L\overline{G}_v \to {}^L\overline{G} \to GL(V)$ be the post composition with R. Let $S \subseteq |F|$ be a finite set of places of F containing the archimedean ones, and we assume that $\overline{\sigma}_v$ is unramified for $v \notin S$. Then the global partial L-function of $\overline{\sigma}$ with respect to R is defined to be

$$L^S(s, \overline{\sigma}, R) = \prod_{v \notin S} L(s, \overline{\sigma}_v, R),$$

where $L(s, \overline{\sigma}_v, R)$ is the local Artin L-function associated with the unramified representation

$$\rho_{v,R} : \mathrm{W}_{F_v} \xrightarrow{\rho_v} {}^L\overline{G}_v \xrightarrow{R_v} GL(V),$$

and is given by $L(s,\overline{\sigma}_v,R) := \det\big(1 - q_v^{-s}\cdot \rho_{v,R}(\mathrm{Frob}_v)|_{V^{I_v}}\big)^{-1}$. The function $L^S(s,\overline{\sigma},R)$ converges uniformly for $\mathrm{Re}(s)$ sufficiently large, and the proof of this is essentially the same as that of Langlands' in [Lan71], see [GaG14, Theorem 13.1] for the details.

Since ${}^L\overline{G}$ is a disconnected reductive complex Lie group, it is not easy to give its irreducible representations. However, there is a natural family of representations which are of interest to us, namely the adjoint type L-functions. As noted in §7.2, any representation of $\overline{G}^\vee_{ad}$ pulls back to a representation of ${}^L\overline{G}$. In particular, the adjoint representation Ad of $\overline{G}^\vee_{ad}$ gives the adjoint representations of ${}^L\overline{G}$ on the Lie algebra $\overline{\mathfrak{g}}^\vee$. More generally, for $\mathbb{P} = \mathbb{M}\mathbb{U}$ a parabolic subgroup of $\mathbb{G}$, the the adjoint representation of $\overline{M}^\vee/Z(\overline{G}^\vee)$ on the Lie algebra $\overline{\mathfrak{u}}^\vee$ can be pulled back to give a representation of ${}^L\overline{M}$.

Moreover, from ${}^L\overline{G}_v \to {}^L\overline{G}$ the adjoint representation Ad of ${}^L\overline{G}$ can be pulled back to ${}^L\overline{G}_v$ to give the adjoint representation of the local L-group. One thus obtains the global Langlands-Shahidi type L-functions by combining those from local adjoint representations.

8.2. **Eisenstein series and its constant terms.** For simplicity, we concentrate on the maximal parabolic case, while the general case follows from similar treatment despite the complication in notations.

Follow notations in §7.3, let $\mathbb{P} = \mathbb{M}\mathbb{U}$ be a maximal parabolic subgroup of $\mathbb{G}$ associated with $\Delta\backslash\{\beta\}$. Let $2\rho_P$ be the sum of positive roots in $\mathbb{U}$ and ω_P be the fundamental weight corresponding to β. Consider the character group $X^*(\mathbb{M})$ of $\mathbb{M}$, and also the real and complex vector space

$$X^*(\mathbb{M})_{\mathbf{R}} = X^*(\mathbb{M}) \otimes_{\mathbf{Z}} \mathbf{R}, \quad X^*(\mathbb{M})_{\mathbf{C}} = X^*(\mathbb{M}) \otimes_{\mathbf{Z}} \mathbf{C}.$$

Any $\nu_o \in X^*(\mathbb{M})$ could be viewed as a character on $\mathbb{M}(\mathbb{A}_F)$ valued in $\mathbb{A}_F^\times$. Further composition with the valuation of $\mathbb{A}_F^\times$ gives us a character of $\mathbb{M}(\mathbb{A}_F)$ valued in $\mathbf{C}^\times$. Similarly, for any $\nu = \nu_o \otimes s \in X^*(\mathbb{M})_{\mathbf{C}}$, we denote by $\boldsymbol{\delta}^\nu$ the following character of $\mathbb{M}(\mathbb{A}_F)$:

$$\boldsymbol{\delta}^\nu : \mathbb{M}(\mathbb{A}_F) \to \mathbf{C}, \quad m \mapsto |\nu_o(m)|^s_{\mathbb{A}_F}.$$

The relation between $\boldsymbol{\delta}$ and the modular character δ_P is $\boldsymbol{\delta}^{\rho_P\otimes 1} = \delta_P^{1/2}$. In the case of maximal parabolic subgroup, $X^*(\mathbb{M}/Z(\mathbb{G})) \otimes \mathbf{C}$ is of dimension one over $\mathbf{C}$ with $\omega_P \otimes 1$ or $\rho_P \otimes 1$ as a basis vector. Henceforth, we will write

$$\boldsymbol{\delta}^s := \boldsymbol{\delta}^{\omega_P \otimes s}, \quad s \in \mathbf{C}.$$

For example for $\mathbb{SL}_2$ with positive root β, $\rho_P = \beta/2$ and $\omega_P = \rho_P$. Then $\boldsymbol{\delta}^s = \delta_P^{s/2}$, with δ_P the modular character of the Borel subgroup P.

Let $\overline{\pi}$ be a genuine cuspidal automorphic representation of $\overline{\mathbb{M}}(\mathbb{A}_F)$, i.e., $\overline{\pi}$ occurs as a direct summand $V_{\overline{\pi}}$ in the decomposition of $L^2_{\mathrm{cusp}}(\mathbb{M}(F)\backslash\overline{\mathbb{M}}(\mathbb{A}_F))$. Here $\overline{\pi}$ is a unitary representation. We take $\boldsymbol{\delta}^s$ to be a character of the covering $\overline{\mathbb{M}}(\mathbb{A}_F)$ by the inflation via the surjection $\overline{\mathbb{M}}(\mathbb{A}_F) \twoheadrightarrow \mathbb{M}(\mathbb{A}_F)$. Now we consider the induction $I(s,\overline{\pi}) := Ind_{\overline{P}}^{\overline{G}}(\boldsymbol{\delta}^s\overline{\pi}) \otimes \mathbb{1}$. We have the tensor product decomposition $I(s,\overline{\pi}) = \bigotimes_v I(s,\overline{\pi}_v)$, where $I(s,\overline{\pi}_v)$ is unramified for almost all v. Let $\phi_s \in I(s,\overline{\pi})$ be a flat section, define the Eisenstein series

$$E(s,\overline{\pi},\phi_s,\overline{g}) = \sum_{\gamma\in\mathbb{P}(F)\backslash\mathbb{G}(F)} \phi_s(\gamma\overline{g}).$$

We may also write $E(s,\overline{\pi},\phi,\overline{g})$ for $E(s,\overline{\pi},\phi_s,\overline{g})$.

Pick the representative $w \in \mathbb{G}(F)$ as in §6.1, which implies $w \in \mathbb{G}(F_v)$ for all v. That is, the embedding $\mathbb{G}(F) \hookrightarrow \mathbb{G}(\mathbb{A}_F)$ gives $w \mapsto (w_v)_v$ such that $w_v = w \in \mathbb{G}(F_v)$ for all v. Moreover, the splitting $\mathbb{G}(F) \hookrightarrow \overline{\mathbb{G}}(\mathbb{A}_F)$ gives $w \mapsto (\widetilde{w}_v)_v$ with $\widetilde{w}_v \in \overline{\mathbb{G}}(F_v)$ for all v. Since w is generated by unipotent elements; for almost all v, the element $\widetilde{w}_v$ is in fact equal to the lifting of $w_v \in \mathbb{G}(F_v)$ by s_{K_v}. That is, for almost all v we have $w \in W^{K_v}$ and $\widetilde{w}_v = \overline{w}_v \in s_{K_v}(W^{K_v})$.

Recall we have the unique $\mathbf{w} \in W$ such that $\mathbf{w}(\Delta\backslash\{\beta\}) \subseteq \Delta$ and $\mathbf{w}(\beta) \in \Psi^-$. The parabolic $\mathbb{P}$ is called self-associated if $\mathbf{w}(\Delta\backslash\{\beta\}) = \Delta\backslash\{\beta\}$. In general, consider the parabolic subgroup associated with $\mathbf{w}(\Delta\backslash\{\beta\}) \subseteq \Delta$ whose Levi subgroup is then given by $\mathbb{M}' := w\mathbb{M}w^{-1}$. As in [MW95, §II.1.6], denote by ${}^w\overline{\pi}$ the representation on $\overline{\mathbb{M}'}(\mathbb{A}_F)$. Then we have the global intertwining operator

$$T(w, s, \overline{\pi}) = \bigotimes_v T(\widetilde{w}_v, s, \overline{\pi}_v) : \quad I(s, \overline{\pi}) \to I(-s, {}^w\overline{\pi}).$$

By definition, the constant term of $E(s, \overline{\pi}, \phi, \overline{g})$ along a general parabolic subgroup $\mathbb{P}_1 = \mathbb{M}_1\mathbb{U}_1$ is defined by

$$E_{P_1}(s, \overline{\pi}, \phi, \overline{g}) = \int_{\mathbb{U}_1(F)\backslash\mathbb{U}_1(\mathbb{A}_F)} E(s, \overline{\pi}, \phi, u\overline{g})du.$$

For the global integration we use the Tamagawa measure normalized such that $F\backslash\mathbb{A}_F$ has measure 1. From [MW95, §II.1.7], we have the following

- If $\mathbb{P}_1$ is neither equal to $\mathbb{P}$ nor the parabolic $\mathbb{P}'$ associated with $\mathbf{w}(\Delta\setminus\{\beta\}) \subseteq \Delta$, then $E_{P_1}(s, \overline{\pi}, \phi, \overline{g}) = 0$.
- If $\mathbb{P}$ is self-associated, i.e. $\mathbb{P} = \mathbb{P}'$, then

$$E_P(s, \overline{\pi}, \phi, \overline{g}) = E_{P'}(s, \overline{\pi}, \phi, \overline{g}) = \phi_s(\overline{g}) + T(w, s, \overline{\pi})\phi_s(\overline{g}).$$

- If $\mathbb{P}$ is not self-associated, i.e. $\mathbb{P} \neq \mathbb{P}'$, then the two cases of interest are given by

$$E_P(s, \overline{\pi}, \phi, \overline{g}) = \phi_s(\overline{g}), \quad E_{P'}(s, \overline{\pi}, \phi, \overline{g}) = \phi_s(\overline{g}) + T(w, s, \overline{\pi})\phi_s(\overline{g}).$$

The intertwining operator has a tensor product decomposition

$$T(w, s, \overline{\pi}) = \bigotimes_v T(\widetilde{w}_v, s, \overline{\pi}_v) : \quad \bigotimes_v I(s, \overline{\pi}_v) \to \bigotimes_v I(s, {}^{\widetilde{w}_v}\overline{\pi}_v).$$

Choose the finite set $S \subseteq |F|$ big enough, such that for all $v \notin S$ we have $\widetilde{w}_v = \overline{w}_v$ and the operator $T(\overline{w}_v, s, \overline{\pi}_v)$ intertwines between unramified representations. We computed in the previous section the coefficient $c(\overline{w}_v, s, \overline{\pi}_v)$ such that $T(\overline{w}_v, s, \overline{\pi}_v)f_{\overline{\pi}_v} = c(\overline{w}_v, s, \overline{\pi}_v)f_{\overline{w}_v\overline{\pi}_v}$. By applying the results in Theorem 7.10 to $\boldsymbol{\delta}_v^s \otimes \overline{\pi}_v$ we get:

Theorem 8.1. *Let $f = \bigotimes_{v\notin S} f_{\overline{\pi}_v} \otimes \bigotimes_{v\in S} f_v \in \overline{\sigma}$. The global intertwining operator $T(w, s, \overline{\pi})$ is then given by*

$$T(w, s, \overline{\pi})f = \prod_{i=1}^m \frac{L^S(n_\beta i \cdot s, \overline{\pi}, Ad_i)}{L^S(1 + n_\beta i \cdot s, \overline{\pi}, Ad_i)} \bigotimes_{v\notin S} f_{\overline{w}_v\overline{\pi}_v} \otimes \bigotimes_{v\in S} T(\widetilde{w}_v, s, \overline{\pi}_v)f_v.$$

Here $L^S(s, \overline{\pi}, Ad_i)$ is the automorphic partial L-function (cf. §8.1) associated with the adjoint representation $Ad_{\overline{\mathfrak{u}}^\vee} = \bigoplus_{i=1}^m Ad_i$ of ${}^L\overline{M}$ on $\overline{\mathfrak{u}}^\vee$. More explicitly, it is given by

$$L^S(s, \overline{\pi}, Ad_i) = \prod_{v\notin S} L(s, \overline{\pi}_v, Ad_i),$$

where the local L-function $L(s, \overline{\pi}_v, Ad_i)$ is the one determined in Theorem 7.10.

Proof. Observe that the adjoint representation $Ad_{\overline{\mathfrak{u}}^\vee}$ of the global ${}^L\overline{M}$ on $\overline{\mathfrak{u}}^\vee$ restricts to a representation of the local ${}^L\overline{M}_v$, which is just the adjoint representation of ${}^L\overline{M}_v$ on $\overline{\mathfrak{u}}^\vee$. They both factor through the complex adjoint representation $Ad^{\mathbf{C}}_{\overline{\mathfrak{u}}^\vee}$ of $\overline{M}^\vee$. We will use $Ad = \bigoplus_{i=1}^m Ad_i$ to represent both local and global situations, and no confusion will arise.

Thus, in view of Theorem 7.10 it suffices to show that the equality $L(0, \boldsymbol{\delta}_v^s \otimes \overline{\pi}_v, Ad_i) = L(n_\beta i \cdot s, \overline{\pi}_v, Ad_i)$ holds for almost all v. Consider the character

$$\chi_{\boldsymbol{\delta}_v^s} : T_v^\dagger \longrightarrow T_v \longrightarrow \mathbb{M}_v \xrightarrow{\boldsymbol{\delta}_v^s} \mathbf{C}^\times.$$

We then have $I(s,\overline{\chi}) \simeq I(\overline{\chi}\otimes\chi_{\boldsymbol{\delta}^s})$ and thus $I(s,\overline{\pi}) \hookrightarrow I(\overline{\chi}\otimes\chi_{\boldsymbol{\delta}^s})$. Therefore, it follows that for $v \notin S$ one has

$$\begin{aligned} L(0,\boldsymbol{\delta}_v^s\otimes\overline{\pi}_v, Ad_i) &= \prod_{\substack{\alpha\in\Psi^+\\ \langle\omega_P/n_\beta,\alpha_{Q,n}^\vee\rangle=i}} L(0, Ad_\alpha\circ\rho_{\overline{\chi}_v\otimes\chi_{\boldsymbol{\delta}_v^s}}) \\ &= \prod_{\substack{\alpha\in\Psi^+\\ \langle\omega_P/n_\beta,\alpha_{Q,n}^\vee\rangle=i}} \frac{1}{1-\overline{\chi}_v\otimes\chi_{\boldsymbol{\delta}_v^s}\left(\overline{h}_\alpha(\varpi_v^{n_\alpha})\right)} \\ &= \prod_{\substack{\alpha\in\Psi^+\\ \langle\omega_P/n_\beta,\alpha_{Q,n}^\vee\rangle=i}} \frac{1}{1-\overline{\chi}_v\left(\overline{h}_\alpha(\varpi_v^{n_\alpha})\right)\cdot|\varpi_v^{\langle\omega_P,\alpha_{Q,n}^\vee\rangle}|_F^s} \\ &= \prod_{\substack{\alpha\in\Psi^+\\ \langle\omega_P/n_\beta,\alpha_{Q,n}^\vee\rangle=i}} L(n_\beta i\cdot s, Ad_\alpha\circ\rho_{\overline{\chi}_v}), \end{aligned}$$

which is clearly equal to $L(n_\beta i\cdot s,\overline{\pi}_v, Ad_i)$. The proof is completed. □

8.3. **Meromorphic continuation of $L^S(s,\overline{\pi},Ad_i)$.** In this subsection, we show that every Langlands-Shahidi L-function for a Brylinski-Deligne covering group has meromorphic continuation to the whole complex plane.

To proceed, we first consider complex connected linear reductive algebraic groups exclusively, and recall the induction lemma of the Langlands-Shahidi method. For this purpose, let $G^{\mathbf{C}}$ be such a group with root datum $(X,\Phi,Y,\Phi^\vee)$. We may choose a set of positive roots $\Phi^+\subseteq\Phi$ and therefore simple roots $\Gamma\subseteq\Phi$. Let $M^{\mathbf{C}}N^{\mathbf{C}}\subseteq G^{\mathbf{C}}$ be a maximal parabolic subgroup of $G^{\mathbf{C}}$, associated with a subset $\Gamma\backslash\{\gamma\}$ of Γ. Let $(X,\Phi_{M^{\mathbf{C}}},Y,\Phi^\vee_{M^{\mathbf{C}}})$ be the root datum of $M^{\mathbf{C}}$ with $\Phi_{M^{\mathbf{C}}}\subseteq\Phi$, $\Phi^\vee_{M^{\mathbf{C}}}\subseteq\Phi^\vee$.

Consider the representation arising from the adjoint action of $G^{\mathbf{C}}$ on Lie($G^{\mathbf{C}}$):

$$\bigoplus_{i=1}^m Ad_i^{\mathbf{C}} : M^{\mathbf{C}} \longrightarrow GL(\mathrm{Lie}(N^{\mathbf{C}})),$$

where the decomposition is into irreducible components $Ad_i^{\mathbf{C}}$'s. As usual, the index i here denotes the eigenvalue of the fundamental weight associated with γ. We have the following induction lemma of Shahidi (cf. [Sha90, Proposition 4.2]), with refinement given by Arthur in [Art99].

Proposition 8.2. *For any $i\geq 2$ above, there exists a connected linear reductive group $G_i^{\mathbf{C}}\subseteq G^{\mathbf{C}}$ with maximal parabolic subgroup $M^{\mathbf{C}}N_i^{\mathbf{C}}$ such that:*

- *The adjoint representation of $M^{\mathbf{C}}$ on Lie($N_i^{\mathbf{C}}$) decomposes into irreducible components*

$$\bigoplus_{j=1}^{m_i} Ad_j^{[i],\mathbf{C}} : M^{\mathbf{C}} \longrightarrow GL(Lie(N_i^{\mathbf{C}}))$$

  with $m_i<m$.
- *More importantly, $Ad_i^{\mathbf{C}} = Ad_1^{[i],\mathbf{C}}$.*

In fact, $G_i^{\mathbf{C}} = Z_{G^{\mathbf{C}}}(s_i)$ is the centralizer of some semisimple element s_i in the center $Z(M^{\mathbf{C}})$ of $M^{\mathbf{C}}$. We recall the construction of s_i as follows (cf. [Art99, pg.1141]). First, let $\alpha\in\Phi^+$ be the unique positive root of $Z(M^{\mathbf{C}})^0$, the connected component of $Z(M^{\mathbf{C}})$, in $G^{\mathbf{C}}$. Then, the set

$$\left\{\alpha_{[i]} := i\cdot\alpha \mid 1\leq i\leq m\right\}$$

will be the set of all the positive roots of $Z(M^{\mathbf{C}})^0$ in $G^{\mathbf{C}}$. Meanwhile,

$$\ker(\alpha_{[i]})/\ker(\alpha)\simeq\langle\zeta_i\rangle$$

is an order i cyclic subgroup of $Z(M^{\mathbf{C}})^0$ generated by a primitive i-th root ζ_i. Take any $s_i\in\mathrm{Ker}(\alpha_{[i]})$ which maps to ζ_i in this isomorphism.

By definition,

$$G_i^{\mathbf{C}} = Z_{G^{\mathbf{C}}}(s_i).$$

It contains $M^{\mathbf{C}}$ as the Levi subgroup of a maximal parabolic subgroup. Meanwhile, one has a description of the root datum of $G_i^{\mathbf{C}}$ as follows. Consider $\Phi_i := \{\alpha \in \Phi \mid \alpha(s_i) = 1\}$. Then $\Phi_{M^{\mathbf{C}}} \subseteq \Phi_i \subseteq \Phi$. The root datum for G_i is then given by

$$(X, \Phi_i, Y, \Phi_i^\vee),$$

where $\Phi_i^\vee$ is the subset of Φ, naturally associated to Φ_i.

Now back to the setting of §8.2, we have a Brylinski-Deligne extension $\mathbb{G}$ incarnated by (D, η). We resume the notations in §7.3. In particular, write $(X, \Psi, \Delta, Y, \Psi^\vee, \Delta^\vee)$ for the root data of $\mathbb{G}$. Recall that D is a bisector of B_Q and $\eta : Y^{sc} \to F^\times$ a morphism defined on the coroot lattice Y^{sc} of $\mathbb{G}$. Denote by $(X, \Psi_{\mathbb{M}}, Y, \Psi_{\mathbb{M}}^\vee)$ the root data for the Levi subgroup $\mathbb{M}$. The Brylinski-Deligne extension $\overline{\mathbb{M}}$ by restriction is incarnated by $(D, \eta_{\mathbb{M}})$ where $\eta_{\mathbb{M}} : Y_{\mathbb{M}}^{sc} \to F^\times$ is the restriction of η on the coroot lattice $Y_{\mathbb{M}}^{sc}$ of M.

One has the Langlands-Shahidi L-functions $L^S(s, \overline{\pi}, Ad_i)$ arising from

$$\bigoplus_{i=1}^m Ad_i : {}^L\overline{M} \longrightarrow GL(\mathrm{Lie}(\overline{N}^\vee)),$$

where $\overline{M}^\vee \overline{N}^\vee \subseteq \overline{G}^\vee$ is a maximal parabolic subgroup for some unipotent subgroup $\overline{N}^\vee$.

Proposition 8.3. *There exists a Brylinski-Deligne extension $\overline{\mathbb{G}}_i$ (as an endoscopic group of $\overline{\mathbb{G}}$) satisfying the following properties:*

- *$\mathbb{G}_i$ is a linear reductive group, which contains $\mathbb{M}\mathbb{N}_i$ as a maximal parabolic subgroup with the same Levi factor $\mathbb{M}$. The restriction of $\overline{\mathbb{G}}_i$ gives rise to exactly $\overline{\mathbb{M}}$.*
- *Consider the representation*

$$\bigoplus_{j=1}^{m_i} Ad_j^{[i]} : {}^L\overline{M} \longrightarrow GL(Lie(\overline{N}_i^\vee))$$

decomposed into irreducible components, where $\overline{M}^\vee \overline{N}_i^\vee \subseteq \overline{G}_i^\vee$ is a maximal parabolic subgroup of $\overline{G}_i^\vee$ for a certain unipotent subgroup $\overline{N}_i^\vee$. One has $m_i < m$, and $Ad_i = Ad_1^{[i]}$. Consequently,

$$L^S(s, \overline{\pi}, Ad_i) = L^S(s, \overline{\pi}, Ad_1^{[i]}).$$

Proof. Consider the triple $(\overline{M}^\vee, \overline{G}^\vee, Ad_i^{\mathbf{C}})$, where $Ad_i^{\mathbf{C}}$ is the i-th irreducible component of adjoint representation of $\overline{M}^\vee$ on $\mathrm{Lie}(\overline{N}^\vee)$. By applying Proposition 8.2, there is a complex linear reductive group $G_i^{\mathbf{C}}$, which contains $\overline{M}^\vee N_i^{\mathbf{C}}$ as a maximal parabolic subgroup. Moreover, from the irreducible decomposition $\bigoplus_{j=1}^{m_i} Ad_j^{[i],\mathbf{C}} : \overline{M}^\vee \to GL(\mathrm{Lie}(N_i^{\mathbf{C}}))$, one has $Ad_i^{\mathbf{C}} = Ad_1^{[i],\mathbf{C}}$.

The roots of $G_i^{\mathbf{C}}$ form a subset of the roots for $\overline{G}^\vee$. Recall the root datum for $\overline{G}^\vee$ is given by

$$(Y_{Q,n},\ \{n_\alpha \alpha^\vee\}_{\alpha\in\Psi},\ X_{Q,n},\ \{n_\alpha^{-1}\alpha\}_{\alpha\in\Psi}).$$

Consider

$$\Psi_0^\vee := \{\alpha^\vee \in \Psi^\vee \mid n_\alpha \alpha^\vee \text{ is a root for } G_i^{\mathbf{C}}\}$$

and let $\Psi_0 \subseteq \Psi$ be the set naturally associated to $\Psi_0^\vee$.

Let $\mathbb{G}_i$ be a split reductive group over F with root data $(X, \Psi_0, Y, \Psi_0^\vee)$. It contains a maximal split torus which is canonically identified with the given maximal split torus $\mathbb{T}$ of $\mathbb{G}$. The group $\mathbb{G}_i$ also contains a standard Levi subgroup isomorphic to $\mathbb{M}$, and therefore we make the identification. Since $\Psi_0^\vee \subseteq \Psi^\vee$, let η_i be the restriction of η to the coroot lattice of $\mathbb{G}_i$ generated by $\Psi_0^\vee$. Let $\overline{\mathbb{G}}_i$ be the Brylinski-Deligne extension associated to (D, η_i). It is easy to see

$$\overline{G}_i^\vee = G_i^{\mathbf{C}}.$$

Clearly, the restriction of $\overline{\mathbb{G}}_i$ gives $\overline{\mathbb{M}}$, since η_i restricts to $\eta_{\mathbb{M}}$. Moreover, one can check easily that there is a natural commutative diagram of the fundamental extensions for the L-groups:

$$\begin{tikzcd} Z(\overline{G}^\vee) \arrow[r, hook] \arrow[d, hook] & E_{\overline{G}} \arrow[r, two heads] \arrow[d, hook] & \mathbb{A}_F^\times \arrow[d, equal] \\ Z(\overline{G}_i^\vee) \arrow[r, hook] & E_{\overline{G}_i} \arrow[r, two heads] & \mathbb{A}_F^\times. \end{tikzcd}$$

Thus, there is an L-group homomorphism ${}^L\overline{G}_i \hookrightarrow {}^L\overline{G}$, which extends $\overline{G}_i^\vee \hookrightarrow \overline{G}^\vee$. Meanwhile, locally there is a natural L-group homomorphism and we have local-global compatibility.

Now it suffices to show the last assertion. The representations $Ad_i, Ad_1^{[i]}$ factor through the vertical quotients as in

$$\begin{tikzcd}
{}^L\overline{M} \arrow[r, "Ad_i"] \arrow[d, two heads] & GL(Ad_i) \\
\overline{M}^\vee/Z(\overline{G}^\vee) \arrow[ur, "Ad_i^{\mathbf{C}}"] \arrow[d, two heads] & \\
\overline{M}^\vee/\mathrm{Ker}(Ad_i^{\mathbf{C}}) \arrow[uur, "{[Ad_i^{\mathbf{C}}]}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
{}^L\overline{M} \arrow[r, "Ad_1^{[i]}"] \arrow[d, two heads] & GL(Ad_1^{[i]}) \\
\overline{M}^\vee/Z(\overline{G}_i^\vee) \arrow[ur, "Ad_1^{[i],\mathbf{C}}"] \arrow[d, two heads] & \\
\overline{M}^\vee/\mathrm{Ker}(Ad_1^{[i],\mathbf{C}}) \arrow[uur, "{[Ad_1^{[i],\mathbf{C}}]}"'] &
\end{tikzcd}.$$

The equality $Ad_i^{\mathbf{C}} = Ad_1^{[i],\mathbf{C}}$ implies $\mathrm{Ker}(Ad_i^{\mathbf{C}}) = \mathrm{Ker}(Ad_1^{[i],\mathbf{C}})$ and also $[Ad_i^{\mathbf{C}}] = [Ad_1^{[i],\mathbf{C}}]$. It follows $Ad_i = Ad_1^{[i]}$, and therefore $L^S(s, \overline{\pi}, Ad_i) = L^S(s, \overline{\pi}, Ad_1^{[i]})$. $\square$

As in the linear algebraic case, one can apply induction to obtain:

Theorem 8.4. *Every Langlands-Shahidi (partial) L-function $L^S(s, \overline{\pi}, Ad_i)$ for Brylinski-Deligne covering groups has meromorphic continuation to the whole complex plane.*

8.4. **The residual spectrum for $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$.** In this part, we determine completely the residual spectrum for $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$ associated with an arbitrary $n \in \mathbf{N}_{\geq 1}$ and quadratic form Q on $Y = Y^{sc}$ of $\mathbb{SL}_2$.

Fix n and assume $\mu_n \subseteq F^\times$. Let $\alpha^\vee$ be the positive coroot; then Q is uniquely determined by $Q(\alpha^\vee) \in \mathbf{Z}$. We can readily compute the complex dual group $\overline{SL}_2^\vee$ and the L-group ${}^L\overline{SL}_2$. There are two cases accordingly.

1. (1) $\gcd(n_\alpha, 2) = 1$, then $\overline{T}^\vee \simeq \mathbf{C}^\times$ and $\overline{SL}_2^\vee = \mathbb{PGL}_2(\mathbf{C})$. By construction there are canonical isomorphisms ${}^L\overline{T} \simeq \mathbf{C}^\times \times \mathrm{W}_F$ and ${}^L\overline{SL}_2 \simeq \mathbb{PGL}_2(\mathbf{C}) \times \mathrm{W}_F$.
2. (2) $\gcd(n_\alpha, 2) = 2$, then $\overline{SL}_2^\vee = \mathbb{SL}_2(\mathbf{C})$. In this case, the L-group ${}^L\overline{SL}_2$ is isomorphic to $\mathbb{SL}_2(\mathbf{C}) \times \mathrm{W}_F$, but not canonically so (cf. [GaG14, §6-7]).

We obtain the global degree n covering $\mu_n \hookrightarrow \overline{\mathbb{SL}}_2(\mathbb{A}_F) \twoheadrightarrow \mathbb{SL}_2(\mathbb{A}_F)$. Let $\mathbb{P} = \mathbb{B} = \mathbb{TN}$ be the parabolic (Borel) subgroup of $\mathbb{SL}_2$. Let $\overline{\pi}$ be a genuine irreducible automorphic representation of $\overline{\mathbb{T}}(\mathbb{A}_F)$. By [We14, Thm. 4.5 & 4.15], the group $\mathbb{T}(F)Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ is a maximal abelian subgroup of $\overline{\mathbb{T}}(\mathbb{A}_F)$ and we have

$$\overline{\pi} \simeq \mathrm{Ind}_{\mathbb{T}(F)Z(\overline{\mathbb{T}}(\mathbb{A}_F))}^{\overline{\mathbb{T}}(\mathbb{A}_F)} \overline{\chi}, \tag{25}$$

where $\overline{\chi}$ is a global genuine unitary character of $\mathbb{T}(F)Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ that is trivial on $\mathbb{T}(F)$. We can also view $\overline{\chi} = \bigotimes_v \overline{\chi}_v$ as a genuine character on $Z(\overline{\mathbb{T}}(\mathbb{A}_F)) = \prod'_v Z(\overline{T}_v)$ trivial on $\mathbb{T}(F) \cap Z(\overline{\mathbb{T}}(\mathbb{A}_F))$.

We have $\overline{\pi} = \bigotimes_v \overline{\pi}_v$, where $\overline{\pi}_v = i(\overline{\chi}_v)$ by the Stone von-Neumann theorem (cf.§5.2). For almost all v, $\overline{\pi}_v$ is unramified with unramified character $\overline{\chi}_v$. We could compute the Satake parameter $\rho_{\overline{\chi}_v}(\mathrm{Frob}_v) \in {}^L\overline{SL}_2$ and there are two cases accordingly.

1. (1) $\gcd(n_\alpha, 2) = 1$, then the Satake parameter is given by

$$\rho_{\overline{\chi}_v}(\mathrm{Frob}_v) = \left(\begin{bmatrix} \overline{\chi_v}(\overline{h}(\varpi_v^{n_\alpha})) & 0 \\ 0 & 1 \end{bmatrix}, \mathrm{Frob}_v \right) \in {}^L\overline{T}_v \subseteq \mathbb{PGL}_2(\mathbf{C}) \times \mathrm{W}_{F_v}.$$

2. (2) $\gcd(n_\alpha, 2) = 2$. Identify $\rho_{\overline{\chi}_v}$ with a splitting of $E_{\overline{T}_v}$ over F_v, and the Satake parameter is determined by $\rho_{\overline{\chi}_v}(\varpi_v) \in E_{\overline{T}_v}$. Meanwhile, we may identify $E_{\overline{T}_v}$ with $(\overline{T}_v^\vee \times E_{\overline{G}_v})/\nabla Z(\overline{SL}_2^\vee)$, with respect to which we write $\rho_{\overline{\chi}_v}(\varpi_v) = (s_v, e_v) \in \overline{T}_v^\vee \times E_{\overline{G}_v}/\nabla Z(\overline{SL}_2^\vee)$. It is possible to obtain explicit form of s_v and e_v. However, for our

purpose, it suffices to note that the element (not uniquely determined, only so in the quotient of $\mathbb{SL}_2(\mathbf{C})$ modulo its center)

$$s_v = \begin{bmatrix} z_v & 0 \\ 0 & z_v^{-1} \end{bmatrix} \in \mathbb{SL}_2(\mathbf{C})$$

always has the property that $z_v^2 = \overline{\chi}_v\big(\overline{h}(\varpi_v^{n_\alpha})\big)$.

The fundamental weight for $\mathbb{P}$ is $\omega_P = \alpha/2$. We consider the induced representation $\overline{\sigma} = I(s, \overline{\pi})$ and form the Eisenstein series $E(s, \overline{\pi}, \phi, \overline{g})$ as before. Clearly, in our case $\mathbb{P}$ is self-associated, and thus the pole of $E(s, \overline{\pi}, \phi, \overline{g})$ agrees with that of $E_P(s, \overline{\pi}, \phi, \overline{g})$, which is given by

$$E_P(s, \overline{\pi}, \phi, \overline{g}) = \phi_s(\overline{g}) + T(w, s, \overline{\pi})\phi_s(\overline{g}).$$

Let $f = \bigotimes_{v \notin S} f_{\overline{\pi}_v} \otimes \bigotimes_{v \in S} f_v \in \overline{\sigma}$. By Theorem 8.1, we have

$$T(w, s, \overline{\pi})f = \frac{L^S(n_\alpha s, \overline{\pi}, Ad)}{L^S(1 + n_\alpha s, \overline{\pi}, Ad)} \bigotimes_{v \notin S} f_{\overline{w}_v \overline{\pi}_v} \otimes \bigotimes_{v \in S} T(\widetilde{w}_v, s, \overline{\pi}_v) f_v,$$

where

$$L^S(n_\alpha s, \overline{\pi}, Ad) = \prod_{v \notin S} L(n_\alpha s, \overline{\pi}_v, Ad) = \prod_{v \notin S} \frac{1}{1 - q_v^{-n_\alpha s} \overline{\chi}(\overline{h}_\alpha(\varpi_v^{n_\alpha}))}.$$

To state it in another way, consider the diagram below

$$\begin{tikzcd}
\mu_n \arrow[r, hook] & Z(\overline{\mathbb{T}}(\mathbb{A}_F)) \arrow[r, two heads] & \mathbb{T}^\dagger(\mathbb{A}_F) \\
& & \mathbb{T}^\ddagger(\mathbb{A}_F) \arrow[u, hook] \arrow[ul, dashed, "\mathbf{s}_{\mathbb{A}_F}"'] \\
& & \mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F). \arrow[u, two heads, "(-)^{n_\alpha}"']
\end{tikzcd}$$

First we explain the groups and maps in the diagram. The group $\mathbb{T}^\dagger(\mathbb{A}_F)$ (respectively $\mathbb{T}^\ddagger(\mathbb{A}_F)$) is the image of $\mathbb{T}_{Q,n}(\mathbb{A}_F)$ (resp. $\mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F)$) in $\mathbb{T}(\mathbb{A}_F)$ induced from the embedding $i_{Q,n} : Y_{Q,n} \hookrightarrow Y$ (resp. $i^{sc}_{Q,n} : Y^{sc}_{Q,n} \hookrightarrow Y$) of the rank-one lattices. If we identify $\mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F)$ and $\mathbb{T}(\mathbb{A}_F)$ both with $\mathbb{A}_F^\times$, then the induced map $\mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F) \to \mathbb{T}(\mathbb{A}_F)$ is simply the n_α-power, whence the notation $(-)^{n_\alpha}$ in the diagram. Moreover, the extension $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ of $\mathbb{T}^\dagger(\mathbb{A}_F)$ splits over $\mathbb{T}^\ddagger(\mathbb{A}_F) \subseteq \mathbb{T}^\dagger(\mathbb{A}_F)$ with the splitting is given by

$$\mathbf{s}_{\mathbb{A}_F} : \quad \big((a_v)_v\big)^{n_\alpha} \mapsto \big(\overline{h}_\alpha(a_v^{n_\alpha})\big)_v \tag{26}$$

for any $(a_v)_v \in \mathbb{A}_F^\times \simeq \mathbb{T}^{sc}_{Q,n}(\mathbb{A}_F)$.

Using this splitting, let $\chi^{sc} = \bigotimes_v \chi_v^{sc}$ be the following composition

$$\chi^{sc} = \overline{\chi} \circ s_{\mathbb{A}_F} \circ (-)^{n_\alpha} : \quad \mathbb{A}_F^\times \twoheadrightarrow \mathbb{T}^\ddagger(\mathbb{A}_F) \hookrightarrow Z(\overline{\mathbb{T}}(\mathbb{A}_F)) \to \mathbf{C}^\times. \tag{27}$$

Then $\chi^{sc} = \bigotimes \chi_v^{sc}$ is a unitary Hecke character and it follows $L(n_\alpha s, \overline{\pi}_v, Ad) = L(n_\alpha s, \chi_v^{sc})$ for $v \notin S$. Fix an additive character $\psi = \bigotimes_v \psi_v : \mathbb{A}_F \to \mathbf{C}$. In order to analyze $T(\widetilde{w}_v, s, \overline{\pi}_v)$ for $v \in S$, we make the normalization

$$r(\widetilde{w}_v, s, \overline{\pi}_v) = \frac{L(n_\alpha s, \chi_v^{sc})}{L(n_\alpha s + 1, \chi_v^{sc})\varepsilon(n_\alpha s, \chi_v^{sc}, \psi_v)}$$
$$T(\widetilde{w}_v, s, \overline{\pi}_v) f_v = r(\widetilde{w}_v, s, \overline{\pi}_v) \cdot N(\widetilde{w}_v, s, \overline{\pi}_v).$$

The ε-factor here is harmless for considering the pole of the (un-)normalized operators. However, we have included it since the cocycle relation for general normalized intertwining operators depends sensitively on it (cf. [Sha90]).

To determine the residual spectrum, we need

Lemma 8.5. *For all $v \in S$, the normalized operator $N(\widetilde{w}_v, s, \overline{\pi}_v)$ is holomorphic and nonvanishing for $Re(s) > 0$.*

Proof. It is easy to check the nonvanishing of $T(\widetilde{w}_v, s, \overline{\pi}_v)$ and $L(s, \chi_v^{sc})$ for $\mathrm{Re}(s) > 0$. Moreover, since χ_v^{sc} is unitary, the local $L(s, \chi_v^{sc})$ contains no poles. The gives the desired result. $\square$

Lemma 8.6. *For all $v \in |F|$, the images of $N(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$ and $T(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$ are both irreducible and nonzero.*

Proof. The normalizing factor $L(1+n_\alpha s, \chi_v^{sc}) L(n_\alpha s, \chi_v^{sc})^{-1} \varepsilon(n_\alpha s, \chi_v^{sc}, \psi_v)^{-1}$ has no pole or zero at $s = 1/n_\alpha$, and therefore it suffices to prove the lemma for $T(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$. However, since $s = 1/n_\alpha > 0$, it follows from the Langlands classification theorem (cf. [BaJa13, Theorem 4.1]) that the image of $T(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$ is irreducible and equal to the Langlands quotient of $I(s, \overline{\pi}_v)$. $\square$

In fact, one can show that $s = 1/n_\alpha$ is a reducibility point for the local induced representation, though we do not need this fact here. Now denote by $\mathcal{J}(1/n_\alpha, \overline{\pi}_v)$ the irreducible images of $N(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$. If $\overline{\chi}$ is such that $\chi^{sc} = \mathbb{1}$, then there is a simple pole of $E_P(s, \overline{\pi}, \phi, \overline{g})$ at $s = 1/n_\alpha$ which arises from the Hecke L-series $L(n_\alpha s, \chi^{sc})$. Under this condition,

$$\mathrm{Res}_{s=1/n_\alpha} E_P(s, \overline{\pi}, \phi, \overline{g}) = \bigotimes_v \mathcal{J}(1/n_\alpha, \overline{\pi}_v).$$

Note that taking constant terms commutes with taking residues:

$$\begin{tikzcd}
I(s,\overline{\pi}) \arrow[r, "\phi_s \mapsto \mathrm{Res}_{s_o} E(\phi_s)"] \arrow[dr, "\phi_s \mapsto \mathrm{Res}_{s_o} E_P(\phi_s)"'] & \mathcal{A}^2(\mathbb{SL}_2(F)\backslash \overline{\mathbb{SL}}_2(\mathbb{A}_F)) \arrow[d, "\text{take const. term}"] \\
& \mathcal{A}(\mathbb{N}(\mathbb{A}_F)\mathbb{T}(F)\backslash \overline{\mathbb{SL}}_2(\mathbb{A}_F)).
\end{tikzcd}$$

Moreover, the right vertical map is injective on the image of the top horizontal map (cf. [MW95, pg. 45]). Thus, we may identify $\mathrm{Res}_{s=1/n_\alpha} E(s, \overline{\pi}, \phi, \overline{g})$ with $\mathrm{Res}_{s=1/n_\alpha} E_P(s, \overline{\pi}, \phi, \overline{g})$ as abstract representations of $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$. Since $\mathbf{w}(s \cdot \omega_P) = \mathbf{w}(\alpha/2n_\alpha) = -\alpha/2n_\alpha$, it follows from the Langlands' criterion (cf. [MW95, §I.4]) that these residues are square integrable.

Let $\mathfrak{A}$ be the collection of unitary genuine characters $\overline{\chi} : Z(\overline{\mathbb{T}}(\mathbb{A}_F)) \to \mathbf{C}^\times$ such that

(i) $\overline{\chi}(\overline{g}) = 1$ for all $\overline{g} \in \mathbb{T}(F) \cap Z(\overline{\mathbb{T}}(\mathbb{A}_F))$;
(ii) $\chi^{sc} = \mathbb{1}$.

Let $\overline{\pi} = i(\overline{\chi})$ be the globally induced representation of $\overline{\mathbb{T}}(\mathbb{A}_F)$ as in (25). Write $\mathcal{J}(1/n_\alpha, \overline{\pi}) = \bigotimes_v \mathcal{J}(1/n_\alpha, \overline{\pi}_v)$. Let $L^2_{res}(\mathbb{SL}_2(F)\backslash \overline{\mathbb{SL}}_2(\mathbb{A}_F))$ denote the residual spectrum.

Theorem 8.7. *The representation $\mathcal{J}(1/n_\alpha, \overline{\pi})$ occurs in the residual spectrum of $\overline{\mathbb{SL}}_2(\mathbb{A}_F)$. In fact, we have the decomposition*

$$L^2_{res}(\mathbb{SL}_2(F)\backslash \overline{\mathbb{SL}}_2(\mathbb{A}_F)) = \bigoplus_{\substack{\overline{\pi} = i(\overline{\chi}) \\ \overline{\chi} \in \mathfrak{A}}} \mathcal{J}(1/n_\alpha, \overline{\pi}).$$

In view of the existence of global Weyl-group invariant characters as discussed in [GaG14, §12.3], we could have an alternative description of the condition $\mathfrak{A}$ and thus also the residual spectrum. For simplicity, we restrict ourselves to consider the case when $n | 2Q(\alpha^\vee)$, under which assumption n_α could be equal to either 1 or 2. This includes the linear case when $n = 1$ and the classical metaplectic double cover of $\mathbb{SL}_2(\mathbb{A}_F)$ when $n = 2, Q(\alpha^\vee) = 1$.

With the above assumption we see that $Y_{Q,n} = Y$ and thus $\overline{\mathbb{T}}(\mathbb{A}_F)$ is abelian, i.e. $Z(\overline{\mathbb{T}}(\mathbb{A}_F)) = \overline{\mathbb{T}}(\mathbb{A}_F)$. Therefore the first condition on $\overline{\chi} \in \mathfrak{A}$ is equivalent to

(i)' $\overline{\chi}$ is a unitary genuine character on $\mathbb{T}(F)\backslash \overline{\mathbb{T}}(\mathbb{A}_F)$.

For the second condition (ii), we fix an additive character $\psi = \bigotimes_v \psi_v : \mathbb{A}_F \to \mathbf{C}^\times$. Then we obtain a Weyl invariant genuine character $\overline{\chi}_\psi = \bigotimes \overline{\chi}_{\psi_v} : \overline{\mathbb{T}}(\mathbb{A}_F) \to \mathbf{C}^\times$. In our case, the local Weyl invariant genuine character $\overline{\chi}_{\psi_v}$ is determined by (cf. [GaG14, §7])

$$\overline{\chi}_{\psi_v} : \overline{T}_v \to \mathbf{C}^\times, \quad (1, \alpha^\vee \otimes a_v) \mapsto \gamma_{\psi_v}(a_v)^{-2(n_\alpha - 1)Q(\alpha^\vee)/n},$$

where $\overline{T}_v = \overline{\mathbb{T}}(F_v)$ and γ_{ψ_v} is the Weil index. In fact, $\overline{\chi}_\psi$ is an automorphic character, i.e. trivial on $\mathbb{T}(F)$. Using $\overline{\chi}_\psi$, any unitary genuine character $\overline{\chi}$ can be written as

$$\overline{\chi} = \overline{\chi}_\psi \cdot \chi \text{ for some unitary } \chi \in \mathrm{Hom}(\mathbb{T}(F)\backslash\mathbb{T}(\mathbb{A}_F), \mathbf{C}^\times).$$

If we identify $\mathbb{T}(\mathbb{A}_F)$ with $\mathbb{A}_F^\times$, then we could write $\chi : F^\times\backslash\mathbb{A}_F^\times \to \mathbf{C}^\times$, which is simply a unitary Hecke-character.

Keep notations as before, in the local setting, the splitting of $\overline{T}_v$ over $T_v^\ddagger$ in (26) is given by

$$T_v^\ddagger \to \overline{T}_v, \quad \alpha_{Q,n}^\vee \otimes a_v \in T_v^\ddagger \mapsto \overline{h}_\alpha(a_v^{n_\alpha}) \in \overline{T}_v.$$

Note that by the defining property of $\overline{\chi}_{\psi_v}$, it is trivial on $\overline{h}_\alpha(a_v^{n_\alpha})$ (cf. [GaG14, §7]). Therefore for all $a_v \in F_v^\times$,

$$\begin{aligned}\chi_v^{sc}(a_v) &= \overline{\chi}_v\big(\overline{h}_\alpha(a_v^{n_\alpha})\big)\\ &= \overline{\chi}_{\psi_v}\big(\overline{h}_\alpha(a_v^{n_\alpha})\big)\cdot\chi_v\big(h_\alpha(a_v^{n_\alpha})\big)\\ &= \chi_v\big(h_\alpha(a_v^{n_\alpha})\big)\\ &= \chi_v^{n_\alpha}(a_v)\end{aligned}$$

Thus globally $\chi^{sc} = \chi^{n_\alpha}$. The second condition for $\overline{\chi} \in \mathfrak{A}$ is then equivalent to

$$\text{(ii)'} \quad \chi^{n_\alpha} = \mathbb{1}.$$

To summarize,

Theorem 8.8. *Suppose $n|2Q(\alpha^\vee)$. Keep notations as above, and denote by $\mathfrak{A}'$ characters χ of $F^\times\backslash\mathbb{A}_F^\times = \mathbb{T}(F)\backslash\mathbb{T}(\mathbb{A}_F)$ satisfying $\chi^{n_\alpha} = \mathbb{1}$. Then we have the decomposition of the residual spectrum*

$$L^2_{res}(\mathbb{SL}_2(F)\backslash\overline{\mathbb{SL}}_2(\mathbb{A}_F)) = \bigoplus_{\chi\in\mathfrak{A}'} \mathcal{J}(1/n_\alpha, \overline{\chi}_\psi \otimes \chi).$$

8.5. **Interlude: Plancherel measure and reducibility points of $\overline{\mathbb{SL}}_2(F_v)$.** To the best of our knowledge, the reducibility points and the nature of the components of the principal series representations $I(s,\overline{\chi}_v)$ are still not completely understood. However, for both linear algebraic groups and central covers, it is already shown by Savin (cf. [Sav]) that for unitary irregular character, the reducibility is governed by the pole of the Plancherel measure. This coupled with Casselman's criterion for the regular cases determine the reducibility point for $I(s,\overline{\chi}_v)$ in principle.

On the other hand, we will show below that regarding the reducibility point, the problem on $\overline{\mathbb{SL}}_2(F_v)$ can be reduced to the linear case. We would like to thank G. Savin for the idea in this part.

In this subsection, to ease notations we omit the subindex v. For example, we use F to denote a local field with residue characteristic p and $\overline{\chi}$ to denote a genuine character etc. Assume $\mu_n \subseteq F$. Without loss of generality, we fix $Q(\alpha^\vee) = 1$ for the discussion below, where $\alpha^\vee$ is the coroot of $\mathbb{G} := \mathbb{SL}_2$. As indicated by the L-group, we expect that the representation theory of the n-fold cover $\overline{\mathbb{G}}(F)$ is related to $\mathbb{PGL}_2(F)$ if n is even and to $\mathbb{SL}_2(F)$ if n is odd.

We assume that the local representation $I(s,\overline{\chi})$ is *not* unramified, otherwise the reduction to the linear case is trivial by the unramified Gindikin-Karpelevich formula.

The idea is to globalize the problem. By Krasner's lemma, there exists a number field $\mathbf{F}/\mathbf{Q}$ such that p is inert in $\mathbf{F}$ and $\mathbf{F}_{v_o} = F$, where v_o is the unique place of $\mathbf{F}$ lying over p. We obtain the degree n cover $\overline{\mathbb{T}}(\mathbb{A}_\mathbf{F})$ over the torus $\mathbb{T}(\mathbb{A}_\mathbf{F}) \subseteq \mathbb{G}(\mathbb{A}_\mathbf{F})$. The character $\overline{\chi}$ can be globalized to be a character of $\overline{\boldsymbol{\chi}}$ of $\mathbb{T}(\mathbf{F})\cdot Z(\overline{\mathbb{T}}(\mathbb{A}_\mathbf{F}))$ trivial on $\mathbb{T}(\mathbf{F})$, i.e. $\overline{\boldsymbol{\chi}}_{v_o} = \overline{\chi}$. Equivalently, $\overline{\boldsymbol{\chi}}$ is an automorphic genuine character of $Z(\overline{\mathbb{T}}(\mathbb{A}_\mathbf{F}))$ trivial on $\mathbb{T}(\mathbf{F}) \cap Z(\overline{\mathbb{T}}(\mathbb{A}_\mathbf{F}))$. As in (27), we obtain the linear character $\boldsymbol{\chi}^{sc} : \mathbb{A}_\mathbf{F}^\times \to \mathbf{C}^\times$ associated with $\overline{\boldsymbol{\chi}}$.

Now define

$$\mathbb{H} = \begin{cases}\mathbb{PGL}_2 & \text{if } n \text{ is even},\\ \mathbb{SL}_2 & \text{if } n \text{ is odd}.\end{cases}$$

Consider the global induced representation $I_{\overline{\mathbb{G}}}(s, i(\overline{\chi}))$ of $\overline{\mathbb{SL}}_2(\mathbb{A}_{\mathbf{F}})$. We have the global functions equation of intertwining operators

$$T_{\overline{\mathbb{G}}}(w, -s, {}^w i(\overline{\chi})) \circ T_{\overline{\mathbb{G}}}(w, s, i(\overline{\chi})) = \mathrm{id}. \tag{28}$$

By definition, the Plancherel measure for each place v is the meromorphic function in s such that

$$\mu_{\overline{\mathbb{G}},v}(s, \overline{\chi}) \cdot \mathrm{id} = T_{\overline{\mathbb{G}}}(w, -s, {}^w i(\overline{\chi}_v)) \circ T_{\overline{\mathbb{G}}}(w, s, i(\overline{\chi}_v)).$$

Thus by the identity (28) we have

$$\prod_v \mu_{\overline{\mathbb{G}},v}(s, \overline{\chi}) = 1.$$

Note that the left hand side is $\prod_{v \in S} \mu_{\overline{\mathbb{G}},v}(s, \overline{\chi}) \cdot \prod_{v \notin S} \mu_{\overline{\mathbb{G}},v}(s, \overline{\chi})$. By our computation of the Gindikin-Karpelevich formula, it follows

$$\prod_{v \notin S} \mu_{\overline{\mathbb{G}},v}(s, \overline{\chi}) = \frac{L^S(ns, \chi^{sc})}{L^S(1+ns, \chi^{sc})} \cdot \frac{L^S(-ns, {}^w\chi^{sc})}{L^S(1-ns, {}^w\chi^{sc})},$$

where outside S the representations are unramified. In particular, $v_o \in S$.

Let $\mathbb{T}_{\mathbb{H}}$ be the torus of $\mathbb{H}$. We view χ^{sc} as a linear character of $\mathbb{T}_{\mathbb{H}}(\mathbb{A}_{\mathbf{F}}) \simeq \mathbb{A}_{\mathbf{F}}^\times$. Similar consideration as above applies. Write $n' = n/\gcd(2, n)$. Consider the globally induced representation $I_{\mathbb{H}}(n's, \chi^{sc})$ and the identity

$$T_{\mathbb{H}}(w, -n's, {}^w\chi^{sc}) \circ T_{\mathbb{H}}(w, n's, \chi^{sc}) = \mathrm{id}.$$

It gives

$$\begin{aligned} 1 &= \prod_v \mu_{\mathbb{H},v}(n's, \chi^{sc}) \\ &= \prod_{v \in S} \mu_{\mathbb{H},v}(n's, \chi^{sc}) \cdot \prod_{v \notin S} \mu_{\mathbb{H},v}(n's, \chi^{sc}) \end{aligned}$$

By the Gindikin-Karpelevich formula,

$$\mu_{\mathbb{H},v}(n's, \chi^{sc}) = \mu_{\overline{\mathbb{G}},v}(s, \overline{\chi}) \text{ for } v \notin S.$$

Therefore, we obtain

$$\prod_{v \in S} \mu_{\mathbb{H},v}(n's, \chi^{sc}) = \prod_{v \in S} \mu_{\overline{\mathbb{G}},v}(s, \overline{\chi}).$$

For each $v \in S$, both $\mu_{\mathbb{H},v}(n's, \chi^{sc})$ and $\mu_{\overline{\mathbb{G}},v}(s, \overline{\chi})$ are rational functions in q_v^{-s}. This implies that any poles or zeros of the two meromorphic functions lie in a family of period $2\pi i/\log q_v$. In particular, the poles and zeros of $\mu_{\overline{\mathbb{G}},v_i}(s, \overline{\chi})$ for $v_i, i = 1, 2$ over different primes could be distinguished. The same applies to $\mu_{\mathbb{H},v}(n's, \chi^{sc})$. Since there is only one prime v_o lying over p, we can deduce that $\mu_{\overline{\mathbb{G}},v_o}(s, \overline{\chi})$ and $\mu_{\mathbb{H},v_o}(n's, \chi^{sc})$ have the same zeros and poles. Thus the reduction of the problem on reducibility points to the linear case is proved.

8.6. **The residual spectrum of $\overline{\mathbb{GL}}_2(\mathbb{A}_F)$.** Since the derived group of $\mathbb{GL}_2$ is $\mathbb{SL}_2$ which is simply-connected, any $(D, \eta) \in \mathsf{Bis}^Q_{\mathbb{GL}_2}$ is isomorphic to a fair $(D, \mathbb{1})$ (cf. Example 2.7). Therefore, without loss of generality we work with $\overline{\mathbb{GL}}_2$ which is incarnated by $(D, \mathbb{1})$. The result is independent of the bisector D of Q.

Let $\{e_1, e_2\}$ be the standard $\mathbf{Z}$-basis of the cocharacter group Y of $\mathbb{GL}_2$ such that the coroot is $\alpha^\vee = e_1 - e_2$. Any Weyl-invariant bilinear form B_Q is uniquely determined by the two numbers $\mathbf{p} := Q(e_1), \mathbf{q} := B_Q(e_1, e_2) \in \mathbf{Z}$. The matrix $B(e_i, e_j), i, j = 1, 2$ is given by

$$B_Q(e_i, e_j) = \begin{bmatrix} 2\mathbf{p} & \mathbf{q} \\ \mathbf{q} & 2\mathbf{p} \end{bmatrix}.$$

It follows $Q(\alpha^\vee) = 2\mathbf{p} - \mathbf{q}$. Fix a natural number $n \in \mathbf{N}_{\geq 1}$, and thus by definition

$$n_\alpha = \frac{n}{\gcd(n, 2\mathbf{p} - \mathbf{q})}.$$

Define $Y_{Q,n}$ and $Y_{Q,n}^{sc}$ as before with $Y_{Q,n}^{sc}$ generated by $\alpha_{Q,n}^\vee$.

Note that in general the complex dual group $\overline{GL}_2^\vee$ may not be equal to $\mathbb{GL}_2(\mathbf{C})$. For instance, consider the case where $\mathbf{q} = 2\mathbf{p}$ and $n = 2\mathbf{q}$. Then, the complex dual group is given by $\overline{GL}_2^\vee = \mathbb{GL}_2(\mathbf{C})/\mu_2$.

It is not difficult to check that there exists a complementary one-dimensional lattice $Y_{Q,n}^o$ such that

$$Y_{Q,n} = Y_{Q,n}^{sc} \oplus Y_{Q,n}^o.$$

However, in general the quadratic form Q may not be decomposable with respect to this direct sum. Denote by $\mathbb{T}_{Q,n}^{sc}$ and $\mathbb{T}_{Q,n}^o$ the tori corresponding to $Y_{Q,n}^{sc}$ and $Y_{Q,n}^o$ respectively. Then, in general $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ is not isomorphic to the image of $\overline{\mathbb{T}}_{Q,n}^{sc}(\mathbb{A}_F) \times \overline{\mathbb{T}}_{Q,n}^o(\mathbb{A}_F)/\nabla\mu_n$ in $\overline{\mathbb{T}}(\mathbb{A}_F)$.

Nevertheless, we could still proceed as follows. Let $\overline{\chi}$ be a genuine character of $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$. Clearly the fundamental weight ω_P associated with α is equal to $\rho_P = \alpha/2$. Identify s with $\omega_P \otimes s \in X^*(\mathbb{T})_{\mathbf{C}}$ as in §7.3. Write $\overline{\pi} = i(\overline{\chi})$. Define the Eisenstein series $E(s, \overline{\pi}, \phi, \overline{g})$ for the representation $I(s, \overline{\pi})$. From Theorem 8.1 the residue of the Eisenstein series is determined by

$$T(w, s, \overline{\pi})f = \frac{L^S(n_\alpha s, \overline{\pi}, Ad)}{L^S(1 + n_\alpha s, \overline{\pi}, Ad)} \bigotimes_{v \notin S} f_{\overline{w}_v \overline{\pi}_v} \otimes \bigotimes_{v \in S} T(\widetilde{w}_v, s, \overline{\pi}_v) f_v,$$

where $f = \bigotimes_{v \notin S} f_{\overline{\pi}_v} \otimes \bigotimes_{v \in S} f_v$. More explicitly,

$$L^S(s, \overline{\pi}, Ad) = \prod_{v \notin S} \frac{1}{1 - q_v^{-s} \cdot \overline{\chi}_v^{sc}(\overline{h}_\alpha(\varpi_v^{n_\alpha}))}.$$

As in the $\overline{\mathbb{SL}}_2$ case, let χ^{sc} be the linear character (cf. (27)):

$$\chi^{sc} = \bigotimes\nolimits_v \chi_v^{sc} : \quad \mathbb{A}_F^\times \xrightarrow{(-)^{n_\alpha}} \mathbb{T}^\ddagger(\mathbb{A}_F) \xhookrightarrow{\mathbf{s}_{\mathbb{A}_F}} Z(\overline{\mathbb{T}}(\mathbb{A}_F)) \xrightarrow{\overline{\chi}^{sc}} \mathbf{C}^\times,$$

where as before we identify $\mathbb{T}_{Q,n}^{sc}(\mathbb{A}_F)$ with $\mathbb{A}_F^\times$ and use $\mathbb{T}^\ddagger(\mathbb{A}_F)$ to denote its image in $\mathbb{T}(\mathbb{A}_F)$. It follows

$$T(w, s, \overline{\pi})f = \frac{L^S(n_\alpha s, \chi^{sc})}{L^S(1 + n_\alpha s, \chi^{sc})} \bigotimes_{v \notin S} f_{\overline{w}_v \overline{\pi}_v} \otimes \bigotimes_{v \in S} T(\widetilde{w}_v, s, \overline{\pi}_v) f_v.$$

To determine the residues of $E(s, \overline{\pi}, \phi, \overline{g})$, we follow the proof for covers of $\mathbb{SL}_2$ exactly, and details may be omitted here. In particular, we denote by $\mathcal{J}(1/n_\alpha, \overline{\pi}_v)$ the irreducible and nonzero image of the normalized operator $N(\widetilde{w}_v, 1/n_\alpha, \overline{\pi}_v)$. Write $\mathcal{J}(1/n_\alpha, \overline{\pi}) = \bigotimes_v \mathcal{J}(1/n_\alpha, \overline{\pi}_v)$.

Finally, let $\mathfrak{B}$ be the collection of characters $\overline{\chi}$ of $Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ trivial on $\mathbb{T}(F) \cap Z(\overline{\mathbb{T}}(\mathbb{A}_F))$ such that χ^{sc} defined above is trivial.

Theorem 8.9. *The residual spectrum $L^2_{res}(\mathbb{GL}_2(F)\backslash\overline{\mathbb{GL}}_2(\mathbb{A}_F))$ has a decomposition of the form*

$$L^2_{res}(\mathbb{GL}_2(F)\backslash\overline{\mathbb{GL}}_2(\mathbb{A}_F)) = \bigoplus_{\substack{\overline{\pi}=i(\overline{\chi}) \\ \overline{\chi}\in\mathfrak{B}}} \mathcal{J}(1/n_\alpha, \overline{\pi}).$$

8.7. **The residual spectrum of $\overline{\mathbb{Sp}}_4(\mathbb{A}_F)$.** Let $\Delta = \{\alpha_1, \alpha_2\}$ be two simple roots of $\mathbb{Sp}_4$ with α_1 the the long root. Let Q be the Weyl-invariant quadratic form on $Y = Y^{sc}$ uniquely determined by $Q(\alpha_1^\vee) = 1$. Let $n = 2$, then we obtain the classical metaplectic group

$$\mu_2 \hookrightarrow \overline{\mathbb{Sp}}_4(\mathbb{A}_F) \twoheadrightarrow \mathbb{Sp}_4(\mathbb{A}_F).$$

The residual spectrum $L^2_{res}(\mathbb{Sp}_4(F)\backslash\overline{\mathbb{Sp}}_4(\mathbb{A}_F))$ is completely determined in [Gao12], and therefore we will not give any elaborate discussion here.

However, as an example, we will show that the partial L-functions appearing in the constant terms of Eisenstein series induced from the two maximal parabolic subgroups as in [Gao12] agree with the ones given by Theorem 8.1.

Let $\mathbb{P}_j = \mathbb{M}_j\mathbb{U}_j$ be the maximal parabolic subgroups generated by α_j. We may call $\mathbb{P}_2$ and $\mathbb{P}_1$ the Siegel and non-Siegel parabolic subgroups respectively. To avoid confusion, for $i = 1, 2$ we still write $\alpha_{j,Q,n}^\vee := 2\alpha_j^\vee/\gcd(2, Q(\alpha_j^\vee))$ for $\alpha_{j,Q,2}^\vee$.

In this case, the complex dual group is $\overline{Sp}_4^\vee = \mathbb{Sp}_4(\mathbf{C})$. The complex dual group $\overline{M}_j^\vee$ is contained in some parabolic $\overline{P}_j^\vee = \overline{M}_j^\vee\overline{U}_j^\vee$ generated by the two simple roots $\alpha_{j,Q,n}^\vee$ of $\overline{Sp}_4^\vee$

respectively, with $\alpha^\vee_{1,Q,n}$ being the long root of $\overline{Sp}^\vee_4$. That is, $\overline{P}^\vee_1$ is the non-Siegel parabolic subgroup of $\overline{Sp}^\vee_4$, while $\overline{P}^\vee_2$ the Siegel parabolic subgroup.

8.7.1. *The* $\mathbb{P}_1$ *case.* Write $\mathbb{M}_1 = \mathbb{GL}_1 \times \mathbb{Sp}_2$, we have

$$\overline{\mathbb{M}}_1(\mathbb{A}_F) \simeq \overline{\mathbb{GL}}_1(\mathbb{A}_F) \times \overline{\mathbb{Sp}}_2(\mathbb{A}_F)/\nabla\mu_2.$$

Any genuine cuspidal representation of $\overline{\mathbb{M}}_1(\mathbb{A}_F)$ could be identified with $\overline{\chi} \boxtimes \overline{\pi}$. Here $\overline{\chi}$ is a genuine automorphic character of $\overline{\mathbb{GL}}_1(\mathbb{A}_F)$ and $\overline{\pi}$ a cuspidal representation of the degree two cover $\overline{\mathbb{Sp}}_2(\mathbb{A}_F)$. By using certain global Weyl-invariant character $\overline{\chi}_\psi$ defined as in [GaG14, §7], one has $\overline{\chi} = \overline{\chi}_\psi \otimes \chi$ where χ is a unitary Hecke character.

Let $I(s, \overline{\chi}\boxtimes\overline{\pi})$ be the induced representation defined before, where we identify s with $(\alpha_1/2 + \alpha_2) \otimes s \in X^*(\mathbb{M}_1)_{\mathbf{C}}$. The Weyl group element of interest is $\mathbf{w} = \mathbf{w}_{\alpha_2}\mathbf{w}_{\alpha_1}\mathbf{w}_{\alpha_2}$.

We have $n_{\alpha_2} = 1$. By Theorem 8.1, the partial L-functions which appear in the constant term of Eisenstein series in this case is given by

$$\prod_{i=1}^{m=2} \frac{L^S(n_{\alpha_2} i \cdot s, \overline{\chi} \boxtimes \overline{\pi}, Ad_i)}{L^S(1 + n_{\alpha_2} i \cdot s, \overline{\chi} \boxtimes \overline{\pi}, Ad_i)} = \frac{L^S(s, \chi \times \overline{\pi})}{L^S(1+s, \chi \times \overline{\pi}))} \cdot \frac{L^S(2s, \chi^2)}{L^S(1+2s, \chi^2)}.$$

Here the Rankin-Selberg product $L^S(s, \chi \times \overline{\pi})$, or more precisely its local counterpart, is given in [Szp10, §7]. It agrees with [Gao12, §4.2].

8.7.2. *The* $\mathbb{P}_2$ *case.* For $j = 2$, the Siegel parabolic subgroup $\mathbb{P}_2$ has Levi subgroup $\mathbb{M}_2 \simeq \mathbb{GL}_2$. Therefore,

$$\overline{\mathbb{M}}_2(\mathbb{A}_F) \simeq \overline{\mathbb{GL}}_2(\mathbb{A}_F).$$

Using an additive character ψ of $\mathbb{A}_F$, there is a genuine character $\overline{\mathbb{GL}}_2(\mathbb{A}_F)$ which is also denoted by $\overline{\chi}_\psi$ by abuse of notation. Any cuspidal representation $\overline{\pi}$ of $\overline{\mathbb{GL}}_2(\mathbb{A}_F)$ could be written as $\overline{\pi} = \pi \otimes \overline{\chi}_\psi$, where π is a cuspidal representation of $\mathbb{GL}_2(\mathbb{A}_F)$.

Identify s with $(\alpha_1 + \alpha_2) \otimes s \in X^*(\mathbb{M}_2)_{\mathbf{C}}$. Let $I(s, \overline{\pi})$ be the induced representation. We will consider the intertwining operator for $\mathbf{w} = \mathbf{w}_{\alpha_1}\mathbf{w}_{\alpha_2}\mathbf{w}_{\alpha_1}$.

Note $n_{\alpha_1} = 2$. By Theorem 8.1, the partial L-function that appears in the constant term of Eisenstein series in this case is given by

$$\prod_{i=1}^{m=1} \frac{L^S(n_{\alpha_1} i \cdot s, \overline{\pi}, Ad_i)}{L^S(1 + n_{\alpha_1} i \cdot s, \overline{\pi}, Ad_i)} = \frac{L^S(2s, \pi, \mathrm{Sym}^2)}{L^S(1+2s, \pi, \mathrm{Sym}^2)},$$

which also agrees with [Gao12, §3.2].

9. Acknowledgment

This paper forms a part of my dissertation, and I would like to thank Professor Wee Teck Gan for his guidance and numerous discussions on various topics. Thanks are also due to Professor Martin Weissman for many helpful discussions, and especially for generously sharing his letter to P. Deligne [We12] and his preprint [We16-2], without which this work would not be possible. Meanwhile, I would like to thank Professors Gordan Savin and Hung Yean Loke for very helpful comments which help improve the exposition of the paper. In particular, thanks are due to Professor Savin for making his manuscript [Sav] available. Finally, I would like to thank the referee for careful reading and very detailed and helpful comments for an earlier version of the paper.

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