LOOSE CRYSTALLINE LIFTS AND OVERCONVERGENCE OF ÉTALE
(ϕ,τ)-MODULES

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Abstract. Let p be a prime, K a finite extension of Qp, and let G_K be the absolute Galois group of K. The category of étale (ϕ,τ)-modules is equivalent to the category of p-adic Galois representations of G_K. In this paper, we show that all étale (ϕ,τ)-modules are overconvergent; this answers a question of Caruso. Our result is an analogue of the classical overconvergence result of Cherbonnier and Colmez in the setting of étale (ϕ,Γ)-modules. However, our method is completely different from theirs. Indeed, we first show that all p-power-torsion representations admit loose crystalline lifts; this allows us to construct certain Kisin models in these torsion representations. We study the structure of these Kisin models, and use them to build an overconvergence basis.

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1. Introduction

1.1. Overview and main theorem. Let p be a prime, K a finite extension of Q_p. Let O_K be the ring of integers, k the residue field, and K_0 the maximal unramified subextension. Then the ring of Witt vectors W := W(k) is just the ring of integers of K_0 and K/K_0 is totally ramified. Write e := [K : K_0], f := [k : F_p] with F_p := Z/pZ. We fix an algebraic closure ̄K of K and set G_K := Gal(̄K/K).

Let ρ : G_K → GL_d(Z_p) be a continuous representation and let T be the ambient space. In [7], for each such T is associated an étale (ϕ,τ)-module, which is a triple ̂M = (M,ϕ_M,̂G) (actually,
we are using a variant of the definition in [7], see our Definition 2.1.5). By [7], the category of finite free $\mathbb{Z}_p$-representation of $G_K$ is equivalent to the category of finite free étale $(\varphi, \tau)$-modules. The functor from $T$ to $\hat{M}$ is via:

$$M(T) = (O_{\mathcal{E}_w} \otimes_{\mathbb{Z}_p} T)^{G_{\infty}} \quad \text{and} \quad \hat{M}(T) = (W(\text{Fr} R) \otimes_{\mathbb{Z}_p} T)^{H_{\infty}}.$$ 

See §2 for any unfamiliar terms and more details. Here $O_{\mathcal{E}_w}$ and $W(\text{Fr} R)$ are certain “period rings”.

For the $\mathbb{Z}_p$-representation $T$, we can also define its “overconvergent periods” via:

$$M^{1,r}(T) := \left( O^{1,r}_{\mathcal{E}_w} \otimes_{\mathbb{Z}_p} T \right)^{G_{\infty}} \quad \text{and} \quad \hat{M}^{1,r}(T) := (W(\text{Fr} R)^{1,r} \otimes_{\mathbb{Z}_p} T)^{H_{\infty}},$$

where $r \in \mathbb{R}^{>0}$, and $O^{1,r}_{\mathcal{E}_w}$ and $W(\text{Fr} R)^{1,r}$ are the “overconvergent period rings”. We say that $\hat{M} = (M, \varphi_M, \hat{G})$ is overconvergent if it can be recovered by its “overconvergent periods”, i.e., if for some $r \in \mathbb{R}^{>0}$, we have

$$M(T) = O_{\mathcal{E}} \otimes_{O^{1,r}_{\mathcal{E}_w}} M^{1,r}(T) \quad \text{and} \quad \hat{M}(T) = W(F_r) \otimes_{W(F_r)^{1,r}} \hat{M}^{1,r}(T).$$

Our main theorem is the following, which answers the question of Caruso [7, §4]:

1.1.1. **Theorem.** For any finite free $\mathbb{Z}_p$-representation $T$ of $G_K$, its associated $(\varphi, \tau)$-module is overconvergent.

1.1.2. **Remark.**

1. In the $(\varphi, \Gamma)$-module setting, the corresponding overconvergence theorem in [10] (see also [4, 20]) is valid without assuming the residue field of $K$ is finite (i.e., they only need to assume $k$ perfect). In this paper, finiteness of $k$ is needed for our loose crystalline lifting theorems; in particular, we will need to consider nontrivial finite unramified extensions of $K$ (cf. §3), which are not well-defined if $k$ is infinite.

2. By using the ideas in the current paper, we can give a reproof of the overconvergence of étale $(\varphi, \Gamma)$-modules (when $k$ is finite); see upcoming [15].

3. In [19, §6.2], Kedlaya and R. Liu formulated another approach to the overconvergence question for $(\varphi, \tau)$-modules. The approach does not assume the finiteness of $k$. As they pointed out in [19, Rem. 6.2.8], it would be very interesting to compare their approach with ours.

4. In Berger’s recent work [3] proving certain overconvergence result in the Lubin-Tate setting, he critically used the calculation (in the $F$-analytic setting) of “locally analytic vectors” in [5]. In fact, in loc. cit., the “locally analytic vectors” in the $(\varphi, \tau)$-module setting were also calculated (see [5, §4.4]). It would be interesting to see if the approach of [5, 3] can also be used to prove the overconvergence of $(\varphi, \tau)$-modules.

1.2. **Why overconvergence?** Our theorem is clearly an analogue of the result of Cherbonnier and Colmez [10], where they show that the $(\varphi, \Gamma)$-module associated to $T$ is overconvergent. Note that, the category of étale $(\varphi, \Gamma)$-modules, similarly as the category of étale $(\varphi, \tau)$-modules, is also equivalent to the category of all $p$-adic $G_K$-representations. Indeed, this similarity is what prompts Caruso to ask the question in [7, §4]. The overconvergence result of [10] plays a fundamental role in the application of $(\varphi, \Gamma)$-modules to study various problems, e.g., the proof of $p$-adic local monodromy theorem ([2]), and the study of $p$-adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$ (cf. [11] for the most recent advance), to name just a few. To study $p$-adic local Langlands correspondence for $\text{GL}_2(F)$ where $F/\mathbb{Q}_p$ is a finite extension, there has been very recent work to generalize [10] to the Lubin-Tate setting, culminating in Berger’s proof (see [3], also [13]).
Considering the very useful applications of overconvergent \((\varphi, \tau)\)-modules, it is natural to expect that our overconvergent \((\varphi, \tau)\)-modules should also generate interesting applications. In particular, one of the reasons that the overconvergence result of [10] is important, is because that it allows us to link \((\varphi, \Gamma)\)-modules to Fontaine modules (see [2]). Namely, overconvergence helps to link the category of all Galois representations to the category of geometric (i.e., semi-stable, crystalline) representations. However, in our current paper, we already use such a link to prove our overconvergence result. Namely, we use crystalline representations to “approximate” general Galois representations (see §1.3 for a sketch). We believe that our approach will shed some more light on the deeper meaning of overconvergence, and in particular, our overconvergent \((\varphi, \Gamma)\)-modules could potentially have certain advantages (than the overconvergent \((\varphi, \tau)\)-modules) in applications. We will report some of our progress in future papers.

1.3. Strategy of proof. It turns out that our method is very different from that used in [10]. We start with a conjecture of crystalline lifts which predicts that every mod \(\ell\)-adic representation admits a crystalline lift (with non-positive Hodge-Tate weights). We can not fully prove this conjecture. But we prove a weaker version (see Theorem 3.2.1) which states that for each mod \(\ell\)-adic representation, after restricting to a finite unramified extension, the residual representation does admit a crystalline lift. This allows us to construct (inductively on \(n\)) loose crystalline lifts \(\hat{L}_n\) of \(T_n := T/p^nT\). In fact, if we let \(h_n\) denote the minimal Hodge-Tate weight of \(\hat{L}_n\), then \(h_n = h_{n-1} + s\) where \(s\) is a constant positive number independent of \(n\), i.e., the growth of \(h_n\) is linear! We will see that the increasing rate of \(h_n\) is closely related to the overconvergence radius; indeed, it makes it possible to study the relation between (maximal) Kisin models inside \(M_n := M/p^nM\) for different \(n\). After carefully analyzing the structure of such Kisin models, we are able to construct an overconvergent basis inside the \((\varphi, \tau)\)-module, and use it to prove our final overconvergence theorem.

1.4. Structure of the paper. In §2, we collect some basic facts about various categories of modules (in particular, étale \((\varphi, \tau)\)-modules). We define what it means for an étale \((\varphi, \tau)\)-module to be overconvergent, and state our main theorem. In §3, we show that all \(p\)-power-torsion representations admit loose crystalline lifts. In §4, we study maximal liftable Kisin models, and in particular show that they are invariant under finite unramified base change. In §5, we study Kisin models in étale \(\varphi\)-modules corresponding to \(p^n\)-torsion representation of \(G_K\). Finally in §6, we prove our main theorem.

1.5. Notations.

1.5.1. Some notations in \(p\)-adic Hodge theory. We put \(R := \lim_{\xrightarrow{p} x} \mathcal{O}_K/p\mathcal{O}_K\), equipped with its natural coordinate-wise action of \(G_K\). There is a unique surjective projection map \(\theta : W(R) \to \hat{O}_K\) to the \(p\)-adic completion \(\hat{O}_K\) of \(\mathcal{O}_K\), which lifts the projection \(R \to \mathcal{O}_K/p\) onto the first factor in the inverse limit. We denote by \(A_{\text{crys}}\) the \(p\)-adic completion of the divided power envelope of \(W(R)\) with respect to \(\text{Ker}(\theta)\). As usual, we write \(B_{\text{crys}}^+ = A_{\text{crys}}[1/p]\) and \(B_{\text{dR}}^+\), the \(\text{Ker}(\theta)\)-adic completion of \(W(R)[1/p]\). For any subring \(A \subset B_{\text{dR}}^+\), we define filtration on \(A\) by \(\text{Fil}^iA = A \cap (\text{Ker}(\theta))^iB_{\text{dR}}^+\).

We fix a uniformizer \(\pi \in \mathcal{O}_K\) and the Eisenstein polynomial \(E(u) \in W[u]\) of \(\pi\) in this paper. Define \(\pi_n \in \overline{K}\) inductively such that \(\pi_0 = \pi\) and \((\pi_{n+1})^p = \pi_n\). Define \(\mu_n \in \overline{K}\) inductively such that \(\mu_1\) is a primitive \(p\)-th root of unity and \((\mu_{n+1})^p = \mu_n\). Set \(K_{\infty} := \cup_{n=1}^{\infty} K(\pi_n), K_{p\infty} := \cup_{n=1}^{\infty} K(\mu_n),\) and \(\hat{K} := \cup_{n=1}^{\infty} K(\pi_n, \mu_n)\). Let \(G_{\infty} := \text{Gal}(\overline{K}/K_{\infty}), G_{p\infty} := \text{Gal}(\overline{K}/K_{p\infty}), H_{\infty} := \text{Gal}(\overline{K}/\hat{K}),\)
$H_K := \text{Gal}(\hat{K}/K)$, and $\hat{G} := \text{Gal}(\hat{K}/K)$. Set $\varepsilon := (\mu_i)_{i \geq 0} \in R$ and $t := -\log([\varepsilon]) \in A_{\text{cris}}$ as usual. We use $\varepsilon_p$ to denote the $p$-adic cyclotomic character.

Recall that $\{\pi_n\}_{n \geq 0}$ defines an element $\pi \in R$, and let $[\pi] \in W(R)$ be the Techm"uller representative. Recall that $\mathcal{S} = W[u]$ with Frobenius extending the arithmetic Frobenius on $W(k)$ and $\varphi(u) = u^p$. We can embed the $W(k)$-algebra $W(k)[u]$ into $W(R) \subset A_{\text{cris}}$ by the map $u \mapsto [\pi]$. This embedding extends to the embedding $\mathcal{S} \hookrightarrow W(R)$ which is compatible with Frobenious endomorphisms.

We denote by $S$ the $p$-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. Write $S_{K_0} := S[1/p]$. There is a unique map (Frobenius) $\varphi_S : S \to S$ which extends the Frobenius on $\mathcal{S}$. Let $\text{Fil}^n S \subset S$ be the $p$-adic completion of the ideal generated by $\gamma_i(E(u)) := \frac{E(u)^i}{i!}$ with $i \geq n$. One can show that the embedding $W(k)[u] \to W(R)$ via $u \mapsto [\pi]$ extends to the embedding $S \hookrightarrow A_{\text{cris}}$ compatible with Frobenious $\varphi$ and filtration (note that $E([\pi])$ is a generator of $\text{Fil}^1 W(R)$).

As a subring of $A_{\text{cris}}$, $S$ is not stable under the action of $G_K$, though $S$ is fixed by $G_\infty$. Define a subring inside $B_{\text{cris}}^+$:

$$\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{(i)} , f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\},$$

where $t^{(i)} = \frac{t^i}{p^{r(i)+\hat{q}(i)}}$ and $\hat{q}(i)$ satisfies $i = \hat{q}(i)(p-1)+r(i)$ with $0 \leq r(i) < p-1$. Define $\hat{\mathcal{R}} := W(R) \cap \mathcal{R}_{K_0}$. One can show that $\mathcal{R}_{K_0}$ and $\hat{\mathcal{R}}$ are stable under the $G_K$-action and the $G_K$-action factors through $\hat{G}$ (see [23, §2.2]). Let $I_+ R$ be the maximal ideal of $R$ and $I_+ \hat{\mathcal{R}} = W(I_+ R) \cap \hat{\mathcal{R}}$. By [23, Lem. 2.2.1], one has $\hat{\mathcal{R}}/I_+ \hat{\mathcal{R}} \simeq \mathcal{S}/u\mathcal{S} = W(k)$.

1.5.2. Fontaine modules and Hodge-Tate weights. When $V$ is a semi-stable representation of $G_K$, we let $D_{\text{st}}(V) := (B_{\text{st}} \otimes \mathbb{Q}_p V^\vee)^{G_K}$ where $V^\vee$ is the dual representation of $V$. The Hodge-Tate weights of $V$ are defined to be $i \in \mathbb{Z}$ such that $\text{gr}^i D_{\text{st}}(V) \neq 0$. For example, for the cyclotomic character $\varepsilon_p$, its Hodge-Tate weight is $\{1\}$.

Note here that we are using a contravariant Fontaine functor, which is the more usual convention in integral $p$-adic Hodge theory (e.g., as in [22]). Later in our current paper, we will construct several covariant functors for our convenience in this paper.

1.5.3. Some other notations. Throughout this paper, we reserve $\varphi$ to denote Frobenius operator. We sometimes add subscripts to indicate on which object Frobenius is defined. For example, $\varphi_{M}$ is the Frobenius defined on $M$. We always drop these subscripts if no confusion will arise. Let $A$ be a ring endowed with Frobenius $\varphi_A$ and $M$ a module over $A$. We always denote $\varphi^* M := A \otimes_{\varphi_A, A} M$. Note that if $M$ has a $\varphi_A$-semi-linear endomorphism $\varphi_M : M \to M$ then $1 \otimes \varphi_M : \varphi^* M \to M$ is an $A$-linear map. We also reserve $v$ to denote valuations which are normalized so that $v(p) = 1$. Finally $M_d(A)$ always denotes the ring of $d \times d$-matrices with entries in $A$ and $I_d$ denotes the $d \times d$-identity matrix.

Acknowledgement. It is a great pleasure to thank Laurent Berger, Xavier Caruso, Kiran Kedlaya and Ruochuan Liu for very useful conversations and correspondences. We would like to thank Toby Gee and David Savitt for allowing us to use results (mainly in our §3) from an unpublished draft [17] which is collaborated work with the second author. We also thank the anonymous referee(s) for helping to improve the exposition.
2. \((\varphi, \tau)\)-modules and \((\varphi, \hat{G})\)-modules

In this section, we first collect some basic facts on \((\text{integral and torsion})\) étale \(\varphi\)-modules, étale \((\varphi, \tau)\)-modules, Kisin modules, \((\varphi, \hat{G})\)-modules and their attached representations. In particular, we also provide a covariant theory which will be more convenient for our paper. Then, we define what it means for an étale \((\varphi, \tau)\)-module to be overconvergent, and state our main theorem.

2.1. Étale \(\varphi\)-modules and \((\varphi, \tau)\)-modules. Recall that \(\mathcal{O}_E\) is the \(p\)-adic completion of \(\mathfrak{S}[1/u]\). Our fixed embedding \(\mathfrak{S} \hookrightarrow W(R)\) determined by \(\mathfrak{p}\) uniquely extends to a \(\varphi\)-equivariant embedding \(\iota : \mathcal{O}_E \hookrightarrow W(FR)\) (here \(FR\) denotes the fraction field of \(R\)), and we identify \(\mathcal{O}_E\) with its image in \(W(FR)\). We note that \(\mathcal{O}_E\) is a complete discrete valuation ring with uniformizer \(p\) and residue field \(k((\mathfrak{p}))\). Let \(\mathcal{E}\) denote the fractional field of \(\mathcal{O}_E\), \(\mathcal{E}^u\) the maximal unramified extension of \(\mathcal{E}\) inside \(W(FR)[\frac{1}{p}]\) and \(\mathcal{O}_{\mathcal{E}^u}\) the ring of integers. Set \(\mathcal{O}_{\mathcal{E}^u}\) the \(p\)-adic completion of \(\mathcal{O}_{\mathcal{E}^u}\) and \(\mathcal{E}^u := W(R) \cap \mathcal{O}_{\mathcal{E}^u}\).

2.1.1. Definition. Let \(\mathcal{M}od_{\mathcal{O}_E^\varphi}\) denote the category of finite type \(\mathcal{O}_E\)-modules \(M\) equipped with a \(\varphi_{\mathcal{O}_E}\)-semi-linear endomorphism \(\varphi_M : M \to M\) such that \(1 \otimes \varphi : \varphi' M \to M\) is an isomorphism. Morphisms in this category are just \(\mathcal{O}_E\)-linear maps compatible with \(\varphi\)'s. We call objects in \(\mathcal{M}od_{\mathcal{O}_E^\varphi}\) étale \(\varphi\)-modules.

2.1.2. Let \(\text{Rep}_{\mathbb{Z}_p}(G_{\infty})\) (resp. \(\text{Rep}_{\mathbb{Z}_p}(G_K)\)) denote the category of finite type \(\mathbb{Z}_p\)-modules \(T\) with a continuous \(\mathbb{Z}_p\)-linear \(G_{\infty}\) (resp. \(G_K\))-action. For \(M\) in \(\mathcal{M}od_{\mathcal{O}_E^\varphi}\), define
\[
V(M) := (\mathcal{O}_{\mathcal{E}^u} \otimes_{\mathcal{O}_E} M)^{\varphi = 1}.
\]

For \(T\) in \(\text{Rep}_{\mathbb{Z}_p}(G_{\infty})\), define
\[
M(T) := (\mathcal{O}_{\mathcal{E}^u} \otimes_{\mathbb{Z}_p} T)^{G_{\infty}}.
\]

2.1.3. Theorem ([12, Prop. A 1.2.6]). The functors \(V\) and \(\overline{M}\) induces an exact tensor equivalence between the categories \(\mathcal{M}od_{\mathcal{O}_E^\varphi}\) and \(\text{Rep}_{\mathbb{Z}_p}(G_{\infty})\).

For later use, we also record the following lemma.

2.1.4. Lemma. Let \(M\) be an étale \(\varphi\)-module, then
\[
V(M) = (M \otimes_{\mathcal{O}_E} W(FR))^{\varphi = 1}.
\]

Proof. Set \(V'(M) := (M \otimes_{\mathcal{O}_E} W(FR))^{\varphi = 1}\). It is obvious that \(V(M) \subset V'(M)\) as \(\mathcal{O}_{\mathcal{E}^u} \subset W(FR)\). To show that \(V(M) = V'(M)\) we easily reduce to the case that \(M\) is killed by \(p^n\). By standard dévissage procedure, we can produce an exact sequence
\[
0 \to M' \to M \to M'' \to 0
\]
with \(M'\) killed by \(p^{n-1}\) and \(M''\) killed by \(p\). By diagram chasing, to show that \(V(M) \supset V'(M)\), it suffices to prove the case when \(n = 1\). Recall that \(\mathcal{O}_{\mathcal{E}^u}/p = k((u))^{\text{sep}}\) the separable closure of \(k((u))\). It is well-known that (indeed, by 2.1.3)
\[
V(M) \otimes_{\mathbb{F}_p} k((u))^{\text{sep}} \simeq M \otimes_{k((u))^{\text{sep}}} k((u))^{\text{sep}}
\]
compatible with \(\varphi\)-actions. So \(V(M) \otimes_{\mathbb{F}_p} FR \simeq M \otimes_{k((u))^{\text{sep}}} FR\) compatible with \(\varphi\)-actions. Taking \(\varphi\)-invariants on both sides, since \((FR)^{\varphi = 1} = \mathbb{F}_p\), we see that \(V(M) = V'(M)\).

Now let us recall the theory of \((\varphi, \tau)\)-modules as in [7]. Let \(H_{\infty} = \text{Gal}(\overline{K}/\overline{k})\) and \(F_{\tau} := (FR)^{H_{\infty}}\). As a subring of \(W(FR)\), \(W(F_{\tau})\) is stable on \(G_K\)-action and the action factors through \(\hat{G}\).
2.1.5. **Definition.** An étale $(\varphi, \tau)$-module is a triple $(M, \varphi_M, \hat{G})$ where

- $(M, \varphi_M)$ is an étale $\varphi$-module;
- $\hat{G}$ is a continuous $W(F_r)$-semi-linear $\hat{G}$-action on $\hat{M} := W(F_r) \otimes_{\mathcal{O}_x} M$, and $\hat{G}$ commutes with $\varphi_{\hat{M}}$ on $\hat{M}$, i.e., for any $g \in \hat{G}$, $g\varphi_{\hat{M}} = \varphi_{\hat{M}}g$;
- regarding $M$ as an $\mathcal{O}_x$-submodule in $\hat{M}$, then $M \subset \hat{M}^{H_K}$.

2.1.6. **Remark.** Note that our definition of étale $(\varphi, \tau)$-modules is slightly different from [7, Definition 1.7] used by Caruso (also note that Caruso simply calls them $(\varphi, \tau)$-modules without the term “étale”). When $\hat{G} \simeq G_{p^\infty} \rtimes H_K$ (e.g. when $p \neq 2$ assumed in [7]), since $G_{p^\infty} \simeq \mathbb{Z}_p(1)$, we can always pick a topological generator $\tau \in G_{p^\infty}$. Then the last two axioms in the above definition can be reinterpreted as axioms for $\tau$ as in [7, Definition 1.7]. In such situations, it is easy to show that our definition is equivalent to loc. cit.

Given an étale $(\varphi, \tau)$-modules $\hat{M} = (M, \varphi_M, \hat{G})$, we define

$$T^*(\hat{M}) := (W(Fr) \otimes_{\mathcal{O}_x} M)^{e=1} = \left( W(Fr) \otimes_{W(F_r)} \hat{M} \right)^{\varphi=1}.$$ 

Note that $W(Fr) \otimes_{\mathcal{O}_x} M$ has a $G_K$-action, which is induced from that on $\hat{M} = W(F_r) \otimes_{\mathcal{O}_x} M$. Since $G_K$-action commutes with $\varphi$, $T^*(\hat{M})$ is a representation of $G_K$.

2.1.7. **Proposition.** Notations as the above. Then

1. $T^*(\hat{M})|_{G_{\infty}} \simeq V(M)$.
2. The functor $T^*$ induces an equivalence between the category of $(\varphi, \tau)$-modules and the category $\text{Rep}_{\mathbb{Z}_p}(G_K)$.

**Proof.** (1) follows from Lemma 2.1.4 and the definition of étale $(\varphi, \tau)$-modules.

(2) For an étale $\varphi$-module $M$, we have a natural isomorphism

$$\mathcal{O}_{\mathcal{E}_w} \otimes_{\mathcal{O}_x} M \simeq \mathcal{O}_{\mathcal{E}_w} \otimes_{\mathbb{Z}_p} V(M)$$

which is compatible with $G_{\infty}$-action and $\varphi$-action on both sides. Tensoring $W(Fr)$ on both sides and using (1), we have

$$W(Fr) \otimes_{\mathcal{O}_x} M \simeq W(Fr) \otimes_{\mathbb{Z}_p} V(M) \simeq W(Fr) \otimes_{\mathbb{Z}_p} T^*(\hat{M})$$

By the construction of $T^*(\hat{M})$, it is clear the above isomorphism is compatible with $G_K$-actions on both sides, where the $G_K$-action on the left is induced from that on $\hat{M}$. In particular, we can recover $M$ and $\hat{M}$ via the formula:

$$M = (\mathcal{O}_{\mathcal{E}_w} \otimes_{\mathbb{Z}_p} V(M))^{G_{\infty}} \quad \text{and} \quad \hat{M} = \left( W(Fr) \otimes_{\mathbb{Z}_p} T^*(\hat{M}) \right)^{H_{\infty}}$$

Then we can easily show that the functor $T^*$ is fully faithful and essentially surjective. \hfill $\square$

2.2. **Kisin modules and $(\varphi, \hat{G})$-modules.**

2.2.1. **Definition.** For a nonnegative integer $r$, we write $\text{Mod}_{\mathfrak{S}}^{\varphi,r}$ for the category of finite-type $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a $\varphi_{\mathfrak{M}}$-semilinear endomorphism $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ satisfying

- the cokernel of the linearization $1 \otimes \varphi : \varphi^* \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$;
- the natural map $\mathfrak{M} \to \mathcal{O}_x \otimes_{\mathfrak{M}} \mathfrak{M}$ is injective.

Morphisms in $\text{Mod}_{\mathfrak{S}}^{\varphi,r}$ are $\varphi$-compatible $\mathfrak{S}$-module homomorphisms.
We call objects in \('\text{Mod}_{\mathfrak{S}}^{\varphi,\hat{T}}'\) Kisin module of \(E(u)\)-height \(r\). The category of finite free Kisin modules of \(E(u)\)-height \(r\), denoted \(\text{Mod}_{\mathfrak{S}}^{\varphi,\hat{T}}\), is the full subcategory of \('\text{Mod}_{\mathfrak{S}}^{\varphi,\hat{T}}'\) consisting of those objects which are finite free over \(\mathfrak{S}\). We call an object \(\mathcal{M} \in '\text{Mod}_{\mathfrak{S}}^{\varphi,\hat{T}}'\) a torsion Kisin module of \(E(u)\)-height \(r\) if \(\mathcal{M}\) is killed by \(p^n\) for some \(n\). Since \(E(u)\) is always fixed in this paper, we often drop \(E(u)\) from the above notions.

For any finite free Kisin module \(\mathcal{M} \in \text{Mod}_{\mathfrak{S}}^{\varphi,\hat{T}}\), we define

\[
T_{\mathfrak{S}}(\mathcal{M}) := \text{Hom}_{\mathfrak{S},\varphi}(\mathcal{M}, W(R)).
\]

See [22, §2.2] for more details on \(T_{\mathfrak{S}}\). In particular, \(T_{\mathfrak{S}}(\mathcal{M})\) is a finite free \(\mathbb{Z}_p\)-representation of \(G_{\infty}\) and \(\text{rank}_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathcal{M}) = \text{rank}_{\mathfrak{S}}(\mathcal{M})\).

Now let us review the theory of \((\varphi, \hat{G})\)-modules.

2.2.2. Definition. Following [23], a finite free (resp. torsion) \((\varphi, \hat{G})\)-module of height \(r\) is a triple \((\mathcal{M}, \varphi, \hat{G})\) where

1. \((\mathcal{M}, \varphi_{\mathcal{M}}) \in '\text{Mod}_{\mathfrak{S}}^{\varphi,\hat{T}}'\) is a finite free (resp. torsion) Kisin module of height \(r\);
2. \(\hat{G}\) is a continuous \(\hat{R}\)-semi-linear \(\hat{G}\)-action on \(\mathfrak{R} := \hat{R} \otimes_{\varphi,\mathfrak{S}} \mathcal{M}\);
3. \(\hat{G}\) commutes with \(\varphi_{\mathcal{M}}\) on \(\mathcal{M}\), i.e., for any \(g \in \hat{G}\), \(g \varphi_{\mathcal{M}} = \varphi_{\mathcal{M}} g\);
4. regard \(\mathcal{M}\) as a \(\varphi(\mathfrak{S})\)-submodule in \(\mathfrak{R}\), then \(\mathcal{M} \subset \mathfrak{R}^{H_K}\);
5. \(\hat{G}\) acts on \(W(k)\)-module \(M := \mathcal{M}/I_r \mathfrak{R} \mathcal{M} \cong \mathcal{M}/\varphi \mathcal{M}\) trivially.

Morphisms between \((\varphi, \hat{G})\)-modules are morphisms of Kisin modules that commute with \(\hat{G}\)-action on \(\mathfrak{M}\)'s.

2.2.3. Remark. When \(\mathcal{M}\) is a torsion or finite free Kisin module, then the natural map \(\mathcal{M} \to \hat{\mathcal{M}}\) induced by \(\mathcal{M} \mapsto 1 \otimes_{\varphi,\mathfrak{S}} \mathcal{M}\) is injective (see [9, Lem.3.1.2] and discussion above the lemma). So it makes sense to regard \(\mathcal{M}\) as a \(\varphi(\mathfrak{S})\)-submodule of \(\hat{\mathcal{M}}\) (item (4) in above definition). Also, the topology on \(\hat{\mathcal{M}}\) is induced by the (weak) topology of \(W(R)\) which is compatible with the \(p\)-adic topology of \(A_{\text{cris}}\).

2.2.4. For a finite free \((\varphi, \hat{G})\)-module \(\hat{\mathcal{M}} = (\mathcal{M}, \varphi, \hat{G})\), we can associate a \(\mathbb{Z}_p[G_K]\)-module:

\[
\hat{T}(\hat{\mathcal{M}}) := \text{Hom}_{\mathfrak{R},\varphi}(\hat{R} \otimes_{\varphi,\mathfrak{S}} \mathcal{M}, W(R)),
\]

where \(G_K\) acts on \(\hat{T}(\hat{\mathcal{M}})\) via \(g(f)(x) = g(f(g^{-1}(x)))\) for any \(g \in G_K\) and \(f \in \hat{T}(\mathcal{M})\).

If \(T\) is a finite free \(\mathbb{Z}_p\)-representation of \(G_{\infty}\) or \(G_K\) we denote \(T^\vee := \text{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)\) the dual representation of \(T\). We fix \(t \in W(R)\) so that \(\varphi(t) = c_0^{-1} E(u)t\) and \(t \neq 0 \mod p\), where \(c_0 = \frac{E(0)}{p}\). Such \(t\) exists and is unique up to multiplying \(\mathbb{Z}_p^\times\) (see [22, Example 2.3.5]). With these notations, we have the following:

2.2.5. Theorem ([23]).

1. \(\hat{T}\) induces an anti-equivalence between the category of finite free \((\varphi, \hat{G})\)-modules of height \(r\) and the category of \(G_K\)-stable \(\mathbb{Z}_p\)-lattices in semi-stable representations of \(G_K\) with Hodge-Tate weights in \(\{0, \ldots, r\}\).
2. \(\hat{T}\) induces a natural \(W(R)\)-linear injection

\[
\hat{\iota} : W(R) \otimes_{\varphi,\mathfrak{S}} \mathcal{M} \to \hat{T}(\hat{\mathcal{M}}) \otimes_{\mathbb{Z}_p} W(R),
\]
which is compatible with Frobenius and $G_K$-actions on both sides (here, we use $\hat{T}^\vee(M)$ to mean the dual of $\hat{T}(M)$). Moreover,

$$(\varphi(t))^*(\hat{T}^\vee(M) \otimes_{\mathbb{Z}_p} W(R)) \subset i(W(R) \otimes_{\varphi, \mathbb{E}} M).$$

(3) There exists a natural isomorphism $T_{\mathbb{E}}(M) \cong \hat{T}(M)$ of $\mathbb{Z}_p[\mathbb{G}_\infty]$-modules.

2.3. Covariant theory. Note that $T_{\mathbb{E}}$ and $\hat{T}$ are contravariant functors which is not convenient for us in many ways. In this section, we construct covariant variants for $T_{\mathbb{E}}$ and $\hat{T}$.

Let $M \in \text{Mod}_{\mathbb{E}}^r$ be a Kisin module of height $r$, we define

$$T_{\mathbb{E}}^*(M) := (M \otimes_{\mathbb{E}} W(FR))^\varphi=1.$$}

Since $\mathbb{E} \subset W(R)^{G_{\infty}}$, we see that $G_{\infty}$ acts on $T_{\mathbb{E}}^*(M)$.

2.3.1. Lemma. There is a natural isomorphism of $\mathbb{Z}_p[\mathbb{G}_\infty]$-modules

$$T_{\mathbb{E}}^*(M) \simeq V(O_{\mathbb{E}} \otimes_{\mathbb{E}} M)$$

Proof. Note that $M := O_{\mathbb{E}} \otimes_{\mathbb{E}} M$ is an étale $\varphi$-module, and the lemma follows from Lemma 2.1.4. □

Now let us turn to the situation of $(\varphi, \hat{G})$-modules. Given $\mathfrak{M} = (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ a $(\varphi, \hat{G})$-module, either finite free or torsion, we define

$$\hat{T}^*(\mathfrak{M}) := (W(FR) \otimes_{\varphi, \mathbb{E}} M)^\varphi=1.$$

Note that $W(FR) \otimes_{\varphi, \mathbb{E}} M$ has a $G_K$-action induced from that on $\mathfrak{M} := \hat{R} \otimes_{\varphi, \mathbb{E}} M$. So $\hat{T}^*(\mathfrak{M})$ is indeed a $\mathbb{Z}_p[\mathbb{G}_K]$-module.

Now we obtain a covariant version of Theorem 2.2.5:

2.3.2. Theorem. (1) $\hat{T}^*$ induces an equivalence between the category of finite free $(\varphi, \hat{G})$-modules of height $r$ and the category of $G_K$-stable $\mathbb{Z}_p$-lattices in semi-stable representations of $G_K$ with Hodge-Tate weights in $\{-r, \ldots, 0\}$.

(2) If $\mathfrak{M}$ is either finite free or torsion then there exists a natural isomorphism $T_{\mathbb{E}}^*(M) \cong \hat{T}^*(\mathfrak{M})$ of $\mathbb{Z}_p[\mathbb{G}_\infty]$-modules.

Proof. (1) It suffices to prove that $\hat{T}^*(\mathfrak{M})$ is isomorphic to $\check{T}^\vee(M)$, the dual of $\hat{T}(M)$. Tensor $W(FR)$ on both sides of (2.2.2). Since $\varphi(t)$ is a unit in $W(FR)$, we get an isomorphism

$$W(FR) \otimes_{\varphi, \mathbb{E}} M \simeq \hat{T}^\vee(M) \otimes_{\mathbb{Z}_p} W(FR)$$

which is compatible with $G_K$-actions and $\varphi$-actions on the both sides. By taking $\varphi$-invariants on both sides, we get $\hat{T}^*(\mathfrak{M}) \simeq \hat{T}^\vee(M)$.

(2) Since $\varphi : W(FR) \to W(FR)$ is a ring isomorphism, $W(FR) \otimes_{\mathbb{E}} M \to W(FR) \otimes_{\varphi, \mathbb{E}} M$ induced by $\varphi \otimes 1$ is an isomorphism which is compatible with Frobenius and $G_{\infty}$-actions. In particular, we have $T_{\mathbb{E}}^*(M) = (W(FR) \otimes_{\varphi, \mathbb{E}} M)^\varphi=1$. Combining the fact that $M$ is a $\varphi(\mathbb{E})$-submodule of $M^{G_K}$, we see that $\hat{T}^*(\mathfrak{M}) \simeq T_{\mathbb{E}}^*(M)$ as $\mathbb{Z}_p[G_{\infty}]$-modules. □

2.3.3. Convention. From now on throughout this paper, when we use semi-stable (indeed, most of the time, crystalline) representations, without further mentioning, we always mean representations with non-positive Hodge-Tate weights (so that we can attach $(\varphi, \hat{G})$-modules or Kisin modules via Theorem 2.3.2).
2.3.4. For any integer $s \geq 0$, let $\mathfrak{M}(\varepsilon_p^{-s})$ denote the Kisin module corresponding to $\varepsilon_p^{-s}$ via our Theorem 2.3.2. By [22, Example 2.3.5] (and our Theorem 2.3.2), this is a rank-1 $\mathfrak{S}$-module with a base $c$ so that $\varphi(c) = (c_0^{-1}E(u))^s c$ where $c_0 = E(0)/p$. For any Kisin module $\mathfrak{M}$, we denote $\mathfrak{M}(s) := \mathfrak{M} \otimes \mathfrak{M}(\varepsilon_p^{-s})$. So $\varphi_{\mathfrak{M}(s)} = (c_0^{-1}E(u))^s \varphi_{\mathfrak{M}}$.

2.4. Kisin models and $(\varphi, \hat{G})$-modules.

2.4.1. Definition ([21]).

1. Given an étale $\varphi$-module $M$ in $\text{Mod}^\varphi_{\mathfrak{O}_p}$. If $\mathfrak{M} \in \text{Mod}^\varphi_{\mathfrak{O}_p}$ is a Kisin module so that $\mathfrak{M} = \mathfrak{O}_\varphi \otimes_\mathfrak{S} \mathfrak{M}$ then $\mathfrak{M}$ is called a Kisin model of $M$, or simply a model of $M$.

2. Given $\hat{M} := (M, \varphi_M, \hat{G}_M)$ a torsion (resp. finite free) $(\varphi, \tau)$-module. A torsion (resp. finite free) $(\varphi, \hat{G})$-module $\mathfrak{M} := (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ is called a model of $\hat{M}$ if $\mathfrak{M}$ is a model of $M$ and the isomorphism

$$W(F_r) \otimes_{\varphi, \mathfrak{O}_\varphi} \mathfrak{M} \simeq W(F_r) \otimes_{\varphi, W(F_r)} \hat{M}$$

induced by $\mathfrak{O}_\varphi \otimes \mathfrak{M} \simeq M$ is compatible with $\hat{G}$-actions on both sides.

2.4.2. Lemma. (1) If $\mathfrak{M}$ is a model of $M$, then $T^*_\mathfrak{O}(\mathfrak{M}) \simeq V(M)$ as $\mathbb{Z}_p[G_\infty]$-modules.

(2) If $\hat{M}$ is a model of $\hat{M}$ then $\hat{T}^*(\mathfrak{M}) \simeq T^*(\hat{M})$ as $\mathbb{Z}_p[G_K]$-modules.

Proof. (1) is Lemma 2.3.1.

For (2), by the definition of $(\varphi, \hat{G})$-model, we get an isomorphism

$$W(F_r) \otimes_{\varphi, \mathfrak{O}_\varphi} \mathfrak{M} \simeq W(F_r) \otimes_{\varphi, \mathfrak{O}_\varphi} M$$

which is compatible with $\varphi$-actions and $G_K$-actions on both sides. By the proof of Theorem 2.3.2, we see that the isomorphism $W(F_r) \otimes_{\mathfrak{O}_\varphi} M \to W(F_r) \otimes_{\varphi, \mathfrak{O}_\varphi} M$ induced by $\varphi \otimes 1$ is also compatible with Frobenius and $G_\infty$-actions. By taking Frobenius invariants on these two isomorphisms, we see that $T^*(\mathfrak{M}) \simeq T^*(\hat{M})$. □

Clearly, the most natural models of étale $(\varphi, \tau)$-modules come from lattices in semi-stable representations.

2.4.3. Lemma. Suppose $T$ is a $G_K$-stable $\mathbb{Z}_p$-lattice in a semi-stable representation of $G_K$ with Hodge-Tate weights in $\{-r, \ldots, 0\}$. Let $\hat{M} := (M, \varphi_M, \hat{G}_M)$ be the $(\varphi, \tau)$-module associated to $T$ via Proposition 2.1.7. Let $\mathfrak{M} := (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G})$ be the $(\varphi, \hat{G})$-module associated to $T$ via Theorem 2.3.2. Then $\mathfrak{M}$ is a model of $\hat{M}$.

Proof. Easy. □

Now suppose that $T_n$ is a $p$-power torsion representation of $G_K$, and $M_n$ the étale $\varphi$-module associated to $T_n|_{G_\infty}$. A natural source of Kisin models of $M_n$ comes from the following. Suppose that we have a surjective map of $G_K$-representations $f : L \to T_n$ where $L$ is a semi-stable finite free $\mathbb{Z}_p$-representation, then it induces the surjective map (which we still denote by $f$) $f : \mathcal{L} \to M_n$, where $\mathcal{L}$ is the étale $\varphi$-module associated to $L|_{G_\infty}$. If $\mathcal{L}$ is the Kisin module associated to $L$, then by Lemma 2.4.3 above, $\mathcal{L}$ is a Kisin model of $L$. And so $f(\mathcal{L})$ is clearly a Kisin model of $M_n$.

2.4.4. Example. Let $T_1 = \mathbb{F}_p$ be the trivial $G_K$-representation and $M_1$ denote the corresponding trivial étale $\varphi$-module. Then $\mathfrak{M}_1 = k[[u]]$ is a model of $M_1$ which is realized by the surjection $\mathbb{Z}_p \to \mathbb{F}_p$. We also have another surjection $\mathbb{Z}_p(1-p) \to \mathbb{F}_p$, which realizes another model $\mathfrak{M}_1(p-1) \subset M_1$. It is easy to check that $\mathfrak{M}_1(p-1) \subset \mathfrak{M}_1$. 
2.5. Overconvergence and the main result. For any \( x \in W(\text{Fr}R) \), we can write \( x = \sum_{i=0}^{\infty} p^i [x_i] \) with \( x_i \in \text{Fr}R \). Denote \( v_R(\cdot) \) the valuation on \( \text{Fr}R \) normalized by \( v_R(\pi) = 1 \). For any \( r \in \mathbb{R}^{>0} \), set
\[
W(\text{Fr}R)^{\dagger, r} := \left\{ x = \sum_{i=0}^{\infty} p^i [x_i] \in W(\text{Fr}R) \mid i + r v_R(x_i) \to +\infty \right\}.
\]
It turns out that \( W(\text{Fr}R)^{\dagger, r} \) is a ring, stable under \( G_K \)-action but not Frobenius, namely \( \varphi(W(\text{Fr}R)^{\dagger, r}) = W(\text{Fr}R)^{\dagger, r/p} \). See [10, §II.1] for more details. For any subring \( A \subset W(\text{Fr}R) \), denote
\[
A^{\dagger, r} := A \cap W(\text{Fr}R)^{\dagger, r}.
\]
Recall that for a finite free \( \mathbb{Z}_p \)-representation \( T \) of \( G_K \), we can associate the \( (\varphi, \tau) \)-module via:
\[
M(T) = (\mathcal{O}_{\mathbb{F}_w} \otimes_{\mathbb{Z}_p} T)^{G_{\infty}} \quad \text{and} \quad \hat{M}(T) = (W(\text{Fr}R) \otimes_{\mathbb{Z}_p} T)^{H_{\infty}}.
\]
Now for any \( r > 0 \), we define
\[
M^{\dagger, r}(T) := \left( \mathcal{O}_{\mathbb{F}_w}^{\dagger, r} \otimes_{\mathbb{Z}_p} T \right)^{G_{\infty}} \quad \text{and} \quad \hat{M}^{\dagger, r}(T) := (W(\text{Fr}R)^{\dagger, r} \otimes_{\mathbb{Z}_p} T)^{H_{\infty}}.
\]

2.5.1. Definition. For a finite free \( \mathbb{Z}_p \)-representation \( T \) of \( G_K \), its associated \( (\varphi, \tau) \)-module is called overconvergent if there exists \( r > 0 \) such that
\[
M(T) = \mathcal{O}_E \otimes_{\mathcal{O}_{\mathbb{F}_w}^{\dagger, r}} M^{\dagger, r}(T) \quad \text{and} \quad \hat{M}(T) = W(F_r) \otimes_{W(K_r)} \hat{M}^{\dagger, r}(T).
\]

The main theorem of our paper is the following:

2.5.2. Theorem. For any finite free \( \mathbb{Z}_p \)-representation \( T \) of \( G_K \), its associated étale \( (\varphi, \tau) \)-module is overconvergent.

The main input to prove the above theorem is the following, which says that there exists an “overconvergent basis”, with respect to which all entries of matrices for \( \varphi \) and \( \hat{G} \) are overconvergent elements. In fact, we can make these matrix entries to fall in more precise rings.

2.5.3. Theorem. Let \( \rho : G_K \to \text{GL}_d(\mathbb{Z}_p) \) be a continuous representation and \( \hat{M} = (M, \varphi_M, \hat{G}) \) the associated étale \( (\varphi, \tau) \)-module. Then there exists an \( \mathcal{O}_E \)-basis of \( M \), and a constant \( \alpha = \alpha(p, f, e, d) \) depending only on \( p, f, e \) and \( d \), such that with respect to this basis,
\[
\begin{align*}
\bullet & \text{ the matrix of } \varphi_M \text{ is in } M_d(\mathfrak{S}(\frac{1}{\alpha})), \\
\bullet & \text{ the matrix of } \tau \text{ is in } M_d(W(R)[\frac{1}{\alpha}] ) \text{ for any } \tau \in \hat{G}.
\end{align*}
\]

3. Loose crystalline lifts of torsion representations

In this section, we show that all \( p \)-power-torsion representations admit loose crystalline lifts. The proof is via an inductive method. We first show that for a mod \( p \) representation of \( G_K \), if we restrict it to a certain subgroup \( G_K' \) where \( K'/K \) is finite unramified, then it admits strict crystalline lift.

3.1. Definition of strict and loose lifts. Let \( E \) be a finite extension of \( \mathbb{Q}_p \), \( \mathcal{O}_E \) its ring of integers, \( \varpi_E \) a uniformizer, \( \mathfrak{m}_E \) the maximal ideal and \( k_E = \mathcal{O}_E/\mathfrak{m}_E \) the residue field.

3.1.1. Definition. (1) For \( \bar{\rho} : G_K \to \text{GL}_d(k_E) \) a torsion representation, we say that a continuous representation \( r : G_K \to \text{GL}_d(\mathcal{O}_E) \) is a strict \( \mathcal{O}_E \)-lift of \( \bar{\rho} \) if \( r(\text{mod} \mathfrak{m}_E) \simeq \bar{\rho} \).
(2) Suppose that $\bar{\rho}$ is a finite $\mathcal{O}_E$-module with a continuous (discrete) $G_K$-action and killed by some $p^n$. Then we say that $\bar{\rho}$ admits a loose $\mathcal{O}_E$-lift $r$ if $r$ is a finite free $\mathcal{O}_E$-representation of $G_K$ and there exists a surjective morphism $f : r \twoheadrightarrow \bar{\rho}$ as $\mathcal{O}_E[G_K]$-modules.

3.1.2. **Definition.** For a $p$-torsion representation of the form $\bar{\rho} : G_K \rightarrow \text{GL}_d(k_E)$, we say that “$E$ is big enough” (for $\bar{\rho}$) if each direct summand of $\bar{\rho}^n$ has strict crystalline $\mathcal{O}_E$-lift with non-positive Hodge-Tate weights.

3.1.3. **Lemma.**

1. Suppose $\bar{\rho} : G_K \rightarrow \text{GL}_d(\mathbb{F}_p)$ is absolutely irreducible (i.e., $\bar{\rho} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ is irreducible), then $\bar{\rho} \otimes_{\mathbb{F}_p} k_d$ admits crystalline strict $W(k_d)$-lifts with Hodge-Tate weights in $[-(p^{ld} - 2), 0]$. Here, $k_d$ is the degree $d$ extension of $k$, and $W(k_d)$ is the ring of Witt vectors.

2. Let $\bar{\rho} : G_K \rightarrow \text{GL}_d(k_E)$ be a $p$-torsion representation. If $E$ contains $K_d := W(k_d)[1/p]$, then $E$ is big enough for $\bar{\rho}$.

**Proof.** Both statements follow from [25, Prop. 2.1.2] (note that there is a minor error in the statement of loc. cit., namely, his $\bar{\rho}$ should have image in $\text{GL}_d(k_d)$, not $\text{GL}_d(k)$). Indeed, for (2), without loss of generality, we can assume that $\bar{\rho} : G_K \rightarrow \text{GL}_d(k_E)$ is absolutely irreducible. Then in the notation of loc. cit., the tame character $\omega'_{\bar{\rho}}$ always lands in $k_d$ (before embedding to $\overline{\mathbb{F}_p}$), and so admits strict crystalline $W(k_d)$-lifts (and thus $\mathcal{O}_E$-lifts); the unramified character $\bar{\mu} : G_{K_d} \rightarrow k_E^\times$ certainly can be lifted.

3.1.4. **Remark.** So in our Definition 3.1.2, any $E$ that contains $K_d := W(k_d)[1/p]$ is big enough. However, in our writing, we will stick with $E$ and $k_E$ (rather than $K_d$ and $k_d$) with an eye for future application. In particular, in future applications, we might need to have $E$ to contain the Galois closure of $K_d$.

3.1.5. **Remark.**

1. For $\bar{\rho} : G_K \rightarrow \text{GL}_d(k_E)$ where $E$ is big enough, we tend to believe that strict crystalline $\mathcal{O}_E$-lift always exists. See the introduction of [16]. In particular, we know this is true for $d \leq 3$ by [25, Prop. 2.5.7].

2. For $\bar{\rho} : G_K \rightarrow \text{GL}_d(\mathbb{F}_p)$, we can certainly consider strict $\mathbb{Z}_p$-lifts, but in general, even if $\bar{\rho}$ is irreducible (or even absolutely irreducible), we do not know if such strict (crystalline) $\mathbb{Z}_p$-lifts exist at all.

3. Suppose that $\bar{\rho}$ is a torsion $\mathbb{Z}_p$-representation, so we can talk about loose $\mathbb{Z}_p$-lifts. Clearly,

- If a torsion $\mathbb{Z}_p$-representation $\bar{\rho}$ admits loose (crystalline) $\mathbb{Z}_p$-lifts, then $\bar{\rho} \otimes_{\mathbb{Z}_p} k_E$ also admits loose (crystalline) $\mathcal{O}_E$-lifts.

- If a torsion $\mathcal{O}_E$-representation $\bar{\rho}$ admits loose (crystalline) $\mathcal{O}_E$-lifts, then if we regard $\bar{\rho}$ as a torsion $\mathbb{Z}_p$-representation, it automatically admits loose (crystalline) $\mathbb{Z}_p$-lifts.

So in general, when we say a torsion representation $\bar{\rho}$ admits loose (crystalline) lifts, it does not matter if we are regarding $\bar{\rho}$ as a $\mathbb{Z}_p$-module or an $\mathcal{O}_E$-module.

The main theorem in this section is that, for $\bar{\rho}$ a torsion Galois representation, crystalline loose lifts always exist. The proof uses an induction on $n$ where $p^n$ kills $\bar{\rho}$. The $n = 1$ case will be (easily) deduced from the following Theorem 3.2.1.

3.2. **Potentially strict lift for $p$-torsion representation.**

3.2.1. **Theorem.** Suppose $\bar{\rho} : G_K \rightarrow \text{GL}_d(k_E)$ is a Galois representation where $E$ is big enough. Then there exists a finite unramified extension $K'$ of $K$, so that $\bar{\rho}|_{G_{K'}}$ admits a strict crystalline $\mathcal{O}_E$-lift.
Proof. Step 1. We will first show that there exists a lift, and make it to be crystalline in the final step. By Definition 3.1.2, we can suppose that \( \bar{\rho} \) is a successive extension of a sequence of irreducible representations \( V_i \). And for each \( i \), we let \( V_i \) denote a fixed strict crystalline \( \mathcal{O}_E \)-lift (with non-positive Hodge-Tate weights) of \( V_i \). It will suffice to prove the following:

- **(Statement A):** There exists a finite unramified extension \( K'/K \) such that \( \bar{\rho}|_{G_{K'}} \) can be strictly lift to a representation \( L' \), which is a successive extension of \( \chi_i V_i \), where \( \chi_i \) are characters satisfying \( \chi_i \equiv 1 \mod \varpi \). Here \( \varpi = \varpi_E \).

**Step 2.** By an obvious induction argument, (Statement A) is reduced to the following:

- **(Statement B):** Suppose that \( \bar{\rho} \) sits in a short exact sequence of \( G_K \)-representations: \( 0 \to \bar{U} \to \bar{V} \to 0 \), where \( U, V \) are \( \mathcal{O}_E \)-finite free \( G_K \)-representations. Then there exists a finite unramified extension \( K'/K \), and a \( G_{K'} \)-representation \( L' \) over \( \mathcal{O}_E \) sitting in the short exact sequence \( 0 \to \bar{U}|_{G_{K'}} \to \bar{V}|_{G_{K'}} \to \chi V|_{G_{K'}} \to 0 \) such that \( \bar{L} = \bar{\rho}|_{G_{K'}} \).

Note that in order to use (Statement B) to prove (Statement A), we could have required either \( \bar{U} \) or \( \bar{V} \) to be irreducible; but actually we do not need this assumption.

As an extension, \( \bar{\rho} \) corresponds to an element \( \bar{c} \) in \( H^1(G_K, \text{Hom}_{\mathcal{O}_E}(\bar{V}, \bar{U})) \), and \( \rho \) (if it exists) corresponds to a element in \( H^1(G_{K'}, \text{Hom}_{\mathcal{O}_E}(\chi V, U)) \). First note the exact sequence

\[
0 \to \text{Hom}_{\mathcal{O}_E}(\chi V, U) \overset{\Delta}{\to} \text{Hom}_{\mathcal{O}_E}(\chi V, U) \to \text{Hom}_{\mathcal{O}_E}(\bar{V}, \bar{U}) \to 0.
\]

induces the exact sequence

\[
H^1(G_K, \text{Hom}_{\mathcal{O}_E}(\chi V, U)) \to H^1(G_K, \text{Hom}_{\mathcal{O}_E}(\bar{V}, \bar{U})) \overset{\delta}{\to} H^2(G_K, \text{Hom}_{\mathcal{O}_E}(\chi V, U))[\varpi] \to 0.
\]

In general, the strategy is to select \( \chi \) to make \( \delta(\bar{c}) = 0 \). But we do not know if such \( \chi \) always exists. After restricting everything to \( G_{K'} \), we get the following commutative diagram:

\[
\begin{array}{ccc}
H^1(G_K, \text{Hom}_{\mathcal{O}_E}(\chi V, U)) & \xrightarrow{\text{res}} & H^1(G_K, \text{Hom}_{\mathcal{O}_E}(\bar{V}, \bar{U})) \\
\downarrow{\text{res}} & & \downarrow{\text{res}} \\
H^1(G_{K'}, \text{Hom}_{\mathcal{O}_E}(\chi V, U)) & \xrightarrow{\delta} & H^2(G_{K'}, \text{Hom}_{\mathcal{O}_E}(\chi V, U))[\varpi] \\
\end{array}
\]

So it suffices to choose a finite unramified extension \( K'/K \), and a character \( \chi \) so that \( (\text{res}\circ\delta)(\bar{c}) = 0 \).

In fact, we will show that \( \text{res} : H^2(G_K, \text{Hom}_{\mathcal{O}_E}(\chi V, U))[\varpi] \to H^2(G_{K'}, \text{Hom}_{\mathcal{O}_E}(\chi V, U))[\varpi] \) can be made into the zero map.

Now let \( M := \text{Hom}_{\mathcal{O}_E}(\chi V, U) \) and \( M' := \text{Hom}_{\mathcal{O}_E}(U, \chi V)(1) \). By Tate duality in Lemma 3.2.3, we are reduced to show (namely, the following implies Statement B):

- **(Statement C):** We can choose a finite unramified extension \( K'/K \), and a character \( \chi : G_K \to \mathcal{O}_E^* \) such that the corestriction map \( \text{cor} : (M' \otimes E/\mathcal{O}_E)^{G_K'/\varpi} \to (M' \otimes E/\mathcal{O}_E)^{G_K}/\varpi \) is the zero map.

**Step 3** To prove (Statement C), we first let \( \chi = \varepsilon_p^{-a} \) where \( a \gg 0 \) and \( p-1 \mid a \). We can choose \( a \) large enough so that \( (M')^{1K} = \{0\} \) (just by making all Hodge-Tate weights of \( M' \) negative). This implies that \( (M' \otimes E/\mathcal{O}_E)^{1K} \) is a finite set, and so we can choose \( K''/K \) finite and unramified such that \( (M' \otimes E/\mathcal{O}_E)^{G_K'/\varpi} = (M' \otimes E/\mathcal{O}_E)^{G_K}/\varpi \).

Let \( K'/K'' \) be the unramified extension with \( [K' : K''] = p \), and we verify Statement C in this situation. Note that the corestriction map

\[
\text{cor} : (M' \otimes E/\mathcal{O}_E)^{G_{K'}/\varpi} \to (M' \otimes E/\mathcal{O}_E)^{G_{K}}/\varpi
\]
is defined by the trace map. Let \( x \in (M^\vee \otimes E/\mathcal{O}_E)^{G_{K'}} \), then \( \text{cor}(x) = \sum_{g \in \text{Gal}(K'/K)} g(x) \). Since we have
\[
(M^\vee \otimes E/\mathcal{O}_E)^{G_{K'}} = (M^\vee \otimes E/\mathcal{O}_E)^{G_{K''}} = (M^\vee \otimes E/\mathcal{O}_E)^{I_K},
\]
so \( g(x) = x \) for any \( g \in \text{Gal}(K'/K'') \). Hence, \( \text{cor}(x) = \rho \sum_{g \in \text{Gal}(K''/K)} g(x) \). Since \( x \) is also in \( (M^\vee \otimes E/\mathcal{O}_E)^{G_{K''}} \), we see that \( \sum_{g \in \text{Gal}(K''/K)} g(x) \) is in \( (M^\vee \otimes E/\mathcal{O}_E)^{G_{K}} \), and so \( \text{cor}(x) = 0 \). This completes the proof of Statement C.

**Step 4** To make \( L' \) to be crystalline, it suffices to choose \( \chi = \varepsilon_p^{-a} \) where \( a >> 0 \) and \( p - 1 \mid a \). When \( a \) is sufficiently large, e.g., when \( -a + 1 \) is smaller than all Hodge-Tate weights of \( U \), then \( L' \) is crystalline by [26, Prop 1.24, Prop 1.26].

### 3.2.2. Corollary
With notations as in Theorem 3.2.1, then the strict crystalline lift of \( \bar{\rho}|_{G_{K'}} \) can be constructed such that its Hodge-Tate weights are in the range \( [-p^{d_2} - 2], 0 \].

**Proof.** We prove by induction on the number of Jordan-Hölder factors of \( \bar{\rho} \). When \( \bar{\rho} \) is irreducible, this is Lemma 3.1.3. When \( \bar{\rho} \) sits in a short exact sequence as in (Statement B) above, then via induction hypothesis, the Hodge-Tate weights of \( U \) (resp. \( V \)) are in the range \( [-p^{d_1}, 0] \) (resp. \( [-(p^{d_2} - 2), 0] \)), where \( d_1, d_2 \) are the \( kE \)-dimensions of \( U, V \), and \( d_1 + d_2 = d \). By the argument in Step 4 of the proof above, we can choose \( a \) to be \( p^{d_1} + p - 2 \) (which is divisible by \( p - 1 \)), then it is easy to see that the minimal Hodge-Tate weight of \( L' \) is \( \geq -(p^{d_2} - 2) - (p^{d_1} + p - 2) \geq -(p^{d_2} - 2) \).}

### 3.2.3. Lemma
Let \( A \) be a finite \( \mathcal{O}_E \)-module killed by \( p \)-power with a continuous \( \mathcal{O}_E \)-linear \( G_K \)-action and \( L/K \) a finite extension. Write \( A^\vee := \text{Hom}_{\mathcal{O}_E}(A, E/\mathcal{O}_E)(1) \) the Tate dual of \( A \). Then the corestriction map
\[
\text{cor} : H^0(G_L, A^\vee) \to H^0(G_K, A^\vee)
\]
is the Tate dual of the map
\[
\text{res} : H^2(G_K, A) \to H^2(G_L, A).
\]

**Proof.** This is well-known for experts but we include the proof here for completeness. Suppose that \( A \) is killed by \( p^m \). The Tate duality is induced by cup product \( \cup : H^2(G_K, A) \times H^0(G_K, A^\vee) \to H^2(G_K, \mu_{p^m}(\overline{\mathbb{Q}_p})) \simeq \mathbb{Z}/p^m\mathbb{Z} \). Indeed we have the following commutative diagram:
\[
\begin{array}{ccc}
H^2(G_K, A) & \times & H^0(G_K, A^\vee) \\
\downarrow \text{res} & & \downarrow \text{cor} \\
H^2(G_L, A) & \times & H^0(G_L, A^\vee)
\end{array}
\]
That is, for any \( x \in H^2(G_K, A) \) and any \( y \in H^0(G_L, A^\vee) \), we have
\[
\text{cor}(\text{res}(x) \cup y) = x \cup \text{cor}(y).
\]
This is proved, for example, [1, Prop 9 (iv)]. Now suppose there is a commutative diagram:
\[
\begin{array}{ccc}
H^2(G_K, \mu_{p^m}(\overline{\mathbb{Q}_p})) & \simeq & \mathbb{Z}/p^m\mathbb{Z} \\
\downarrow \text{cor} & & \downarrow \\
H^2(G_L, \mu_{p^m}(\overline{\mathbb{Q}_p})) & \simeq & \mathbb{Z}/p^m\mathbb{Z}
\end{array}
\]
Note that we can identify \( H^0(G_K, A^\vee) \) with \( \text{Hom}_{\mathbb{Z}/p^m\mathbb{Z}}(H^2(G_K, A), \mathbb{Z}/p^m\mathbb{Z}) \) via duality induced by cup product. More precisely, any \( y \in H^0(G_K, A^\vee) \) is identified with the map \( f_y : H^2(G_K, A) \to \mathbb{Z}/p^m\mathbb{Z} \).
exists a \( \mathbb{Z}/p^n\mathbb{Z}; x \mapsto x \cup y \). We also identify \( H^0(G_L, A^\vee) \) with \( \text{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(H^2(G_L, A), \mathbb{Z}/p^n\mathbb{Z}) \) in a similar way. Now we see that the map \( \text{res} : H^2(G_K, A) \to H^2(G_L, A) \) induces a dual map \( \text{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(H^2(G_K, A), \mathbb{Z}/p^n\mathbb{Z}) \to \text{Hom}_{\mathbb{Z}/p^n\mathbb{Z}}(H^2(G_K, A), \mathbb{Z}/p^n\mathbb{Z}) \) by \( f \mapsto f \circ \text{res} \). Then for \( x \in H^2(G_K, A) \), we have \( (f_y \circ \text{res})(x) = \text{res}(x) \cup y \), which is just \( x \cup \text{cor}(y) \) by (3.2.1) and diagram (3.2.2). This means the dual map of \( \text{res} \) is \( f_y \mapsto f_{\text{cor}(y)} \). This proves the lemma.

Now it suffices to prove that diagram (3.2.2) is commutative. First recall that from the main theorem of local class field theory, we have the following commutative diagram

\[
\begin{array}{ccc}
H^2(G_K, \mathbb{Q}_p^\times) & \xrightarrow{\text{res}} & H^2(G_L, \mathbb{Q}_p^\times) \\
\downarrow \text{inv}_K & & \downarrow \text{inv}_L \\
\mathbb{Q}/\mathbb{Z} & \xrightarrow{[L:K]} & \mathbb{Q}/\mathbb{Z}
\end{array}
\]

and (here we use Hilbert 90)

\[
\begin{array}{ccc}
0 & \xrightarrow{\sim} & H^2(G_K, \mu_{p^n}(\mathbb{Q}_p)) \\
\downarrow \sim & & \downarrow \text{inv}_K \\
\mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Q}/\mathbb{Z} \\
\downarrow \text{cor} & & \downarrow \text{inv}_K \\
0 & & H^2(G_K, \mathbb{Q}_p^\times)
\end{array}
\]

So in particular, the isomorphism \( H^2(G_K, \mu_{p^n}(\mathbb{Q}_p)) \cong \mathbb{Z}/p^n\mathbb{Z} \) induced by \( \text{inv}_K \). Since \( \text{cor} \circ \text{res} = [L : K] \), we have the following diagram

\[
\begin{array}{ccc}
H^2(G_L, \mathbb{Q}_p^\times) & \xrightarrow{\text{cor}} & H^2(G_K, \mathbb{Q}_p^\times) \\
\downarrow \text{inv}_L & & \downarrow \text{inv}_K \\
\mathbb{Q}/\mathbb{Z} & \xrightarrow{1} & \mathbb{Q}/\mathbb{Z}
\end{array}
\]

Combing this diagram and previous diagram, we have proved diagram (3.2.2). \( \square \)

### 3.3. Loose crystalline lifts.

#### 3.3.1. Theorem. Suppose \( \bar{\rho} : G_K \to \text{GL}_d(\mathbb{F}_p) \) is a Galois representation. Suppose \( E \) is big enough for \( \bar{\rho} \otimes \mathbb{F}_p k_E \). Then there exists loose crystalline lifts for \( \bar{\rho} \), which can be made to be finite free over \( \mathcal{O}_E \).

**Proof.** It suffices to show that \( \bar{\rho} \otimes \mathbb{F}_p k_E \) admits loose crystalline \( \mathcal{O}_E \)-lifts. By Theorem 3.2.1, there exists a \( K' \) so that \( \bar{\rho} \otimes \mathbb{F}_p k_E|_{G_{K'}} \) admits a strict crystalline \( \mathcal{O}_E \)-lift \( \rho' \). Let \( L = \text{Ind}_{G_{K'}}^{G_K} \rho' \), then \( L/\varpi_E L = \text{Ind}_{G_{K'}}^{G_K} \circ \text{Res}_{G_{K'}}^{G_K}(\bar{\rho} \otimes \mathbb{F}_p k_E) \), which maps surjectively onto \( \bar{\rho} \otimes \mathbb{F}_p k_E \) (see Lemma 4.2.1). \( \square \)

#### 3.3.2. Theorem. For \( n \in \mathbb{Z}^+ \), suppose \( T_n \) is a \( p^n \)-torsion representation of \( G_K \) (\( T_n \) is not necessarily free over \( \mathbb{Z}/p^n\mathbb{Z} \)). Then there exists loose crystalline lifts for \( T_n \). If we let \( T_1 := T_n/pT_n \) and suppose \( E \) is big enough for \( T_1 \otimes \mathbb{F}_p k_E \), then we can always make the loose crystalline lift to be finite free over \( \mathcal{O}_E \).

#### 3.3.3. Convention. In the proof of Theorem 3.3.2 and §5.2, we will use a lot of representation as well as their associated modules (étale \( \varphi \)-modules or Kisin modules). We will adopt the following
convention on notations of these representations and their associated modules (only in Theorem 3.3.2 and §5.2).

(1) We use a letter with $n$ on subscript to mean a torsion object killed by certain $p$-power. For example, we can write $A_n$ as a torsion representation, and $\mathcal{A}_n$ as its associated étale $\varphi$-module. $A_n$ is not necessarily killed by $p^n$ (although sometimes it is); the subscript $n$ could also mean the induction step. Note that there does not necessarily exist a finite free object having $A_n$ as its “reduction”.

(2) We use a letter with a tilde to mean a finite free object. For example, we can write $\tilde{B}_n$ as a finite free crystalline $\mathbb{Z}_p$-representation, $\tilde{\mathcal{B}}_n$ as its associated étale $\varphi$-module, and $\tilde{\mathcal{B}}_n$ as its associated Kisin module. Note here that the subscript $n$ usually indicate the induction step.

(3) We use a letter with breve accent to mean an object having both free part and torsion part (actually, we only use this convention once). For example, we can write $\tilde{W}_n$ as a representation with both $\mathbb{Z}_p$-free part and $p$-power torsion part, and $\tilde{W}_n$ as its associated étale $\varphi$-module.

Note that we only adopt this notation system with representations and modules, although it is compatible with usual notations on rings. For example, throughout the paper, we use notations like $\mathfrak{S}_n$, $\mathcal{O}_{E_n}$ to mean reduction modulo $p^n$ of $\mathfrak{S}$, $\mathcal{O}_E$.

Proof of Theorem 3.3.2. We prove by induction on $n$. When $n = 1$, this is Theorem 3.3.1. Suppose the statement is true for $n - 1$. Denote $T_{n-1} := T_n/p^{n-1}T_n$. By induction hypothesis, we can suppose $f_{n-1} : \tilde{L}_{n-1} \twoheadrightarrow T_{n-1}$ is a loose crystalline lift such that $\tilde{L}_{n-1}$ is finite free over $\mathcal{O}_E$.

Let $\tilde{W}_n$ be the cartesian product of $\tilde{L}_{n-1} \twoheadrightarrow T_{n-1}$ and $T_n \twoheadrightarrow T_{n-1}$. We have the following diagram of short exact sequences (of $\mathbb{Z}_p[G_K]$-modules).

$$
\begin{array}{cccccc}
0 & \rightarrow & p^{n-1}T_n & \rightarrow & \tilde{W}_n & \rightarrow & \tilde{L}_{n-1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & p^{n-1}T_n & \rightarrow & T_n & \rightarrow & T_{n-1} & \rightarrow & 0
\end{array}
$$

(3.3.1)

It is obvious that we have $p\tilde{L}_{n-1} = p\tilde{W}_n$, which has a section to $\tilde{W}_n$. And we have the exact sequence

$$
0 \rightarrow p\tilde{L}_{n-1} \rightarrow \tilde{W}_n \rightarrow Z_n \rightarrow 0.
$$

(3.3.2)

Here $Z_n := \tilde{W}_n/p\tilde{W}_n$, and it sits in the following exact sequence of $G_K$-representations over $\mathbb{F}_p$:

$$
0 \rightarrow p^{n-1}T_n \rightarrow Z_n \rightarrow \tilde{L}_{n-1}/p\tilde{L}_{n-1} \rightarrow 0.
$$

(3.3.3)

Let us point out that at this point, there is a relatively shorter way to create a loose crystalline lift for $T_n$, see Remark 3.3.4. However, we need to create a special kind of loose crystalline lift, whose construction will be critically used later (see the proof of Proposition 5.2.1).

Since $T_1 \rightarrow p^{n-1}T_n$ (when $T_n$ is finite free over $\mathbb{Z}/p^n\mathbb{Z}$, this surjection is bijective), so $E$ is also big enough for $p^{n-1}T_n \otimes_{\mathbb{F}_p} k_E$. By Theorem 3.2.1, there exists $K'/K$ finite unramified such that there exists a strict crystalline $\mathcal{O}_E$-lift $\tilde{N}'_n \rightarrow p^{n-1}T_n \otimes_{\mathbb{F}_p} k_E[G_{K'}]$. Also, $\tilde{L}_{n-1}$ is a strict crystalline $\mathcal{O}_E$-lift of $\tilde{L}_{n-1}/\varpi E \tilde{L}_{n-1}$. By (Statement B) in the proof of Theorem 3.2.1, there exists $K''/K'$
finite unramified and \( s \gg 0 \), such that we have the following diagram of short exact sequences of \( G_{K'} \)-representations:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{N}'_n(:= \tilde{N}'_n|_{G_{K'}}) & \longrightarrow & \tilde{Z}'_n & \longrightarrow & \tilde{L}_{n-1}(-s) & \longrightarrow & 0 \\
\end{array}
\] (3.3.4)

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & p^{n-1}T_n \otimes_{\mathbb{F}_p} k_E & \longrightarrow & Z_n \otimes_{\mathbb{F}_p} k_E & \longrightarrow & \tilde{L}_{n-1}/p\tilde{L}_{n-1} \otimes_{\mathbb{F}_p} k_E & \longrightarrow & 0 \\
0 & \longrightarrow & p^{n-1}T_n & \longrightarrow & Z_n & \longrightarrow & \tilde{L}_{n-1}/p\tilde{L}_{n-1} & \longrightarrow & 0,
\end{array}
\]

where the first row are strict crystalline \( \mathcal{O}_E \)-lifts of the second row, and the maps from the second row to the third row are compatible projections to chosen \( \mathbb{F}_p \)-direct summands. Note that here for \( s \), by Corollary 3.2.2, we only need to have \((p - 1)|s \) and \( s \geq \text{f}d \) where \( d = \dim_{\mathbb{F}_p} T_1 \geq \dim_{\mathbb{F}_p} p^{n-1}T_n \), i.e., we can simply let

\[
(3.3.5) \quad s := p^{\text{f}d} + p - 2.
\]

Define \( \tilde{L}'_n := \tilde{Z}'_n \times_{Z_n} \tilde{W}_n \), where \( \tilde{Z}'_n \rightarrow Z_n \) comes from diagram (3.3.4). The \( G_{K'} \)-representation \( \tilde{L}'_n \) sits in the following diagram of short exact sequences:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & p\tilde{L}_{n-1}|_{G_{K'}} & \longrightarrow & \tilde{L}'_n & \longrightarrow & \tilde{Z}'_n & \longrightarrow & 0 \\
0 & \longrightarrow & p\tilde{L}_{n-1}|_{G_{K'}} & \longrightarrow & \tilde{W}_n|_{G_{K'}} & \longrightarrow & Z_n|_{G_{K'}} & \longrightarrow & 0
\end{array}
\] (3.3.6)

Clearly \( \tilde{L}'_n \) has to be \( \mathcal{O}_E \)-finite free, because it is extension by two \( \mathcal{O}_E \)-finite free modules. Note that \( \tilde{L}'_n[\frac{1}{p}] = Z_n[\frac{1}{p}] \times_{Z_n[\frac{1}{p}]} \tilde{W}_n[\frac{1}{p}] \cong Z_n[\frac{1}{p}] \oplus \tilde{L}_{n-1}[\frac{1}{p}] \), so \( \tilde{L}'_n \) is crystalline. Clearly, \( \tilde{L}'_n \) maps surjectively onto \( \tilde{W}_n|_{G_{K'}} \), and so onto \( T_n|_{G_{K'}} \).

Let \( \tilde{L}_n := \text{Ind}_{G_{K'}}^{G_K} \tilde{L}'_n \); it is a crystalline \( G_K \)-representation, and it maps surjectively to \( T_n \). \( \square \)

3.3.4. Remark. As we point out earlier, after equation (3.3.3), we could give a shorter proof of the above theorem. Indeed, we could directly use a loose \( G_K \)-crystalline lift \( \tilde{Z}^{(2)}_n \) of \( Z_n \), and take \( \tilde{L}^{(2)}_n \) as the pull back in the following.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & p\tilde{L}_{n-1} & \longrightarrow & \tilde{L}^{(2)}_n & \longrightarrow & \tilde{Z}^{(2)}_n \otimes_{\mathbb{Z}_p} \mathcal{O}_E & \longrightarrow & 0 \\
0 & \longrightarrow & p\tilde{L}_{n-1} & \longrightarrow & \tilde{W}_n & \longrightarrow & Z_n & \longrightarrow & 0
\end{array}
\] (3.3.7)

Similarly, \( \tilde{L}^{(2)}_n \) is \( \mathcal{O}_E \)-finite free and crystalline, and it is a loose crystalline lift of \( T_n \).

3.3.5. Remark. (1) For any \( n \geq 1 \), let \(-h_n\) be the minimal Hodge-Tate weight of \( \tilde{L}_n \). Then from the construction in diagrams (3.3.4) and (3.3.6), it is easy to see that \( h_n = h_{n-1} + s \). So we have \( h_n = h_1 + (n - 1)s \), \( \forall n \geq 1 \), i.e., \( h_n \) grows linearly! We will also see later (e.g., in the proof of Lemma 5.2.6), that the “growth rate” \( s \) plays a role in overconvergence.
(2) By Corollary 3.2.2 and Equation (3.3.5), we have \( h_n \leq n(p^d + p - 2) - p \).
(3) Suppose we have a finite free \( \mathbb{Z}_p \)-representation \( T \), and let \( T_n = T/p^nT \). As we see, the “crystalline weight” \( h_n \) of \( T_n \) grows linearly. It is intriguing to ask if \( h_n \) can grow even more slowly. By the main result in [14], the linear growth is indeed the best we can hope for; in fact, if \( h_n \) grows in a certain log-growth as in loc. cit., then \( T \) will be forced to be crystalline (or semi-stable, in the more general setting of loc. cit.).

4. Maximal Kisin models

In this section, we study liftable Kisin models in torsion étale \( \varphi \)-modules, and show that they admit a (unique) maximal object. We study the finite free \( \mathcal{S}_1 \)-pieces of these maximal models. We also show that these maximal models are “invariant” under finite unramified base change.

4.1. Maximal Kisin models and devissage. For \( n \in \mathbb{Z}^+ \), suppose \( T_n \) is a \( p^n \)-torsion representation of \( G_K \) (\( T_n \) is not necessarily free over \( \mathbb{Z}/p^n\mathbb{Z} \)). Let \( M_n \) be the corresponding étale \( \varphi \)-module. Recall that in the torsion case, a Kisin module \( M \) is a Kisin model of \( \mathcal{S} \) in \( F \) respectively. Then the lift \( M \) can be realized by the surjection \( f : L \rightarrow T_n \).

4.1.1. Definition. A Kisin model \( \mathcal{M} \) is called loosely liftable (in short, liftable), if there exists a \( G_K \)-stable \( \mathbb{Z}_p \)-lattice \( L \) inside a crystalline representation and surjective map \( f : L \rightarrow T_n \), such that the corresponding map of étale \( \varphi \)-modules \( f : L \rightarrow M_n \) satisfies \( f(\mathcal{M}) = \mathcal{M} \), where \( L \) (resp. \( \mathcal{M} \)) is the étale \( \varphi \)-module (resp. Kisin module) for \( L \) (see the discussion before Example 2.4.4). In this case, we say that \( \mathcal{M} \) can be realized by the surjection \( f : L \rightarrow T_n \).

4.1.2. Lemma. (1) For any \( M_n \), a liftable model \( \mathcal{M} \subset M_n \) exists.
(2) The set of liftable models inside \( M_n \) has a unique maximal object (which we will denote as \( \mathcal{M}_{(n)} \)).

Proof. Item (1) is direct corollary of Theorem 3.3.2 (see also the paragraph after Lemma 2.4.3).

For Item (2), denote the set of all liftable models inside \( M_n \) as \( LF^\infty_\varphi(M_n) \). Indeed, this notation imitates the notations in [8, Def. 3.2.1], where \( F^\infty_\varphi(M_n) \) denotes the set of all Kisin models (not necessarily liftable). Let us emphasize here that in [8, Def. 3.2.1], \( r \) is allowed to be \( \infty \) (cf. [8, §2.1]). Clearly, \( LF^\infty_\varphi(M_n) \) (which is a subset of \( F^\infty_\varphi(M_n) \)) is partially ordered.

Let \( \mathcal{M} \) and \( \mathcal{M}' \) be two liftable Kisin models of \( M_n \), realized by \( f : L \rightarrow T_n \) and \( f' : L' \rightarrow T_n \) respectively. Then the lift \( g : L \oplus L' \rightarrow T_n \) defined by \( g(x,y) = f(x) + f'(y) \) realizes \( \mathcal{M} + \mathcal{M}' \). So the set \( LF^\infty_\varphi(M_n) \) admits finite supremum.

For a chosen \( \mathcal{M} \in LF^\infty_\varphi(M_n) \), any ascending chain in \( LF^\infty_\varphi(M_n) \) of the form \( \mathcal{M} \subset \mathcal{M}' \subset \mathcal{M}'' \cdots \subset M_n \) must stabilize after finite steps, since by [8, Cor. 3.2.6], it stabilizes after finite steps as a chain in \( F^\infty_\varphi(M_n) \). This implies that \( LF^\infty_\varphi(M_n) \) has a unique maximal object. \( \square \)

4.1.3. Some notations. In this subsubsection, we introduce some notations which will be used later.

Let \( \mathcal{M} \in \mathcal{I}\text{-Mod}_\varphi^\dagger \) be a torsion Kisin module such that \( M := \mathcal{M}[\frac{1}{p}] \) is a finite free \( \mathcal{S}_1 \)-module (i.e., the torsion \( G_\infty \)-representation associated to \( M \) is finite free over \( \mathbb{Z}/p^n\mathbb{Z} \)). In general, \( \mathcal{M} \) does not have to be \( \mathcal{S}_1 \)-free. By we can always use the following technique to dévissage \( \mathcal{M} \) to finite \( \mathcal{S}_1 \)-free pieces.

For \( i \leq n \), let \( M_i := M/p^iM \), and let \( q_i : M \rightarrow M_i \) be the natural projection. Then \( q_i(\mathcal{M}) \subset M_i \) is a Kisin model of \( M_i \). Obviously, \( q_i(\mathcal{M}) \) is the image of the natural map \( \mathcal{M}/p^i\mathcal{M} \rightarrow M_i \). Following
the discussion above [22, Lem. 4.2.4], for each \(0 \leq i < j \leq n\), we define \(\mathcal{M}^{i,j} := \ker(p^i\mathcal{M} \xrightarrow{p^{i-1}} p^i\mathcal{M}).\)

By the natural isomorphism \(p^{n-i}M/p^{n-j}M \cong M_{i-j}\), we have \(p^{n-i}\mathcal{M} \cong q_i(\mathcal{M})\) and \(p^i\mathcal{M}^{i,j} \cong \mathcal{M}^{i+1,j}\). Hence we have

\[p^{n-1}\mathcal{M} = \mathcal{M}^{n-1,n} \subset \mathcal{M}^{n-2,n-1} \subset \cdots \subset \mathcal{M}^{0,1} \subset M_1.\]

Indeed, it is not hard to see that \(\mathcal{M}^{i+1,j} = p^i\mathcal{M} \cap \ker(p) = p^i\mathcal{M} \cap p^{n-1}M.\)

Note that \(q_i(\mathcal{M}), \ker(q_i|_{\mathcal{M}})\) and \(\mathcal{M}^{i+1}\) are all objects in \(\text{Mod}_{\mathcal{O}}^e\).

### 4.1.4. Some more notations.

In this subsubsection, we introduce some more notations which will be used later.

Let \(T\) be a \(\mathbb{Z}_p\)-finite free \(G_K\)-representation, and set \(T_n := T/p^nT\). Let \(M\) be the finite free étale \(\varphi\)-module corresponding to \(T\), and set \(M_n := M/p^nM\).

Denote the natural projection \(q_{j,i} : M_j \to M_i\) for \(i < j\) induced by modulo \(p^i\). Recall that we use \(\mathcal{M}_{(j)}\) to denote the maximal liftable Kisin model of \(M_j\). Set \(\mathcal{M}_{(j,i)} = q_{j,i}(\mathcal{M}_{(j)})\). It is easy to see that \(\mathcal{M}_{(j,i)}\) is liftable. Indeed, if \(f : L \to T_j\) realizes \(\mathcal{M}_{(j)}\), then \(f' : L \to T_j \to T_i\) realizes \(\mathcal{M}_{(j,i)}\). So \(\mathcal{M}_{(j,i)} \subset \mathcal{M}_{(j)}\) by construction.

For \(i < j\), we denote \(\iota_{i,j} : M_i \to M_j\) the injective map where for \(x \in M_i\), we choose any lift \(\hat{x} \in M_j\), and let \(\iota_{i,j}(x) = p^j\hat{x}\). This is clearly well-defined, and we will use it to identify \(M_i\) with \(\iota_{i,j}(M_i) = M_j[p^j] = p^{i-j}M_j\) (recall that the notation \(M_j[p^j]\) denotes the \(p^j\)-torsion elements). The maps \(\iota_{i,j}\) are clearly transitive; namely, \(\iota_{i,j} \circ \iota_{j,k} = \iota_{i,k}\). Also, the composite \(M_i \xrightarrow{\iota_{i,j}} M_j \xrightarrow{q_{j,i}} M_i\) is precisely the map \(M_i \xrightarrow{\times p^{i-j}} M_j\).

### 4.1.5. Lemma.

Use notations in 4.1.4. In particular, for \(i < j\), we identify \(M_i\) with \(M_j[p^j]\). Then \(\mathcal{M}_{(j)}[p^j] = \mathcal{M}_{(i)}\) as Kisin models of \(M_i\).

**Proof.** Suppose \(f : L \to T_j\) realizes \(\mathcal{M}_{(j)}\). Let \(g : L \to T_j \to p^iT_j\) be the composite map (where the second map is the \(\times p^i\) map), and let \(K := \ker g\). We have the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & K & \to & L & \to & p^iT_j & \to & 0 \\
& & \downarrow{f} & \downarrow{=} & & & \downarrow{=} & & \\
0 & \to & T_j[p^j] & \to & T_j & \to & p^iT_j & \to & 0 \\
\end{array}
\]

where both rows are short exact sequences of \(G_K\)-representations. The above diagram induces the following diagram of Kisin modules:

\[
\begin{array}{ccc}
0 & \to & K & \to & L & \to & p^i\mathcal{M}_{(j)} & \to & 0 \\
& & \downarrow{f} & \downarrow{=} & & & \downarrow{=} & & \\
0 & \to & \mathcal{M}_{(j)} & \to & p^i\mathcal{M}_{(j)} & \to & 0 \\
\end{array}
\]

Now, the top row of (4.1.2) is short exact by [9, Thm. 3.1.3(3), Lem. 3.1.4] (or see the nice summary in [18, Thm. 5.2]). This implies that \(f(K) = \mathcal{M}_{(j)}[p^j]\), and so \(\mathcal{M}_{(j)}[p^j]\) is a liftable model. By maximality of \(\mathcal{M}_{(j)}\), we have \(\mathcal{M}_{(j)}[p^j] \subset \mathcal{M}_{(i)}\).
For the other direction, it is clear that $\mathcal{M}_{(i) + M_{(j)}}$ is a liftable model in $M_j$. So we have $\mathcal{M}_{(i)} \subseteq \mathcal{M}_{(j)}$, and so $\mathcal{M}_{(i)} \subseteq \mathcal{M}_{(j)}[p']$. \hfill \square

4.1.6. Corollary. With notations in 4.1.4, we have $\mathcal{M}_{(j)}^{i-1,i} = \mathcal{M}_{(i,1)}$ for $j \geq i$.

Proof. By definition, we have $\mathcal{M}_{(j)}^{i-1,i} = p^{i-1}(\mathcal{M}_{(j)}[p'])$. This is equal to $p^{i-1}\mathcal{M}_{(i)} (= \mathcal{M}_{(i,1)})$ by the above lemma. \hfill \square

4.2. Étale descent of Kisin modules and Kisin models. Since the constructions in the previous section involve restriction and induction of representations over finite unramified extensions, in this subsection, we discuss how these operations affect the corresponding Kisin modules. Since we are only dealing with unramified field extension, the situation here is not too difficult.

Let $K'$ be a finite unramified extension of $K$ with residue field $k'$, $K'_0 = W(k')[\frac{1}{p}]$ and $\Gamma := \text{Gal}(K'/K) \simeq \text{Gal}(K'_0/K_0) \simeq \text{Gal}(k'/k)$. Set $\mathcal{S}' = W(k') \otimes_{W(k)} \mathcal{S}$. Then $\Gamma$ acts on $\mathcal{S}'$ and $(\mathcal{S}')^\Gamma = \mathcal{S}$.

Set $K'_\infty := K'K_\infty$, and since $K' \cap K_\infty = K$, we have $G'_{\infty} := \text{Gal}(K'/K'_\infty) = G_K' \cap G_\infty$, and $G_\infty/G'_{\infty} \simeq \Gamma$. For each element $\gamma \in \Gamma$, we fix a lift in $G_\infty$, which we still denote as $\gamma$. In the remainder of this subsection, without further notice, we will use $\gamma$ to mean its lift in $G_\infty$. Since $G_\infty$ acts on $u = [\frac{1}{\gamma}]$ trivially, the $\Gamma$ action we mentioned in the previous paragraph is now the same as the induced action from $G_\infty$ on $\mathcal{S} \subset W(R)$.

Since we will use induction and restriction of representations a lot, we include the following easy lemma.

4.2.1. Lemma. Let $G$ be a topological group, $H \subseteq G$ a closed normal subgroup of finite index. We use $\text{Ind}$ and $\text{Res}$ to denote the functors $\text{Ind}_H^G$ and $\text{Res}_H^G$ respectively.

Suppose $T$ is a representation of $H$ and $\rho_T : H \to \text{GL}(T)$ denote the action $H$ on $T$. For any $\gamma \in G/H$, let $T^\gamma$ be the representation of $H$ acting on $T$ so that $\rho_T(h) = \rho_T(\gamma^{-1}h\gamma)$.

(1) The functor $\text{Ind}$ is both left and right adjoint of $\text{Res}$.
(2) If $V$ is a $G$-representation, then $\text{Ind} \circ \text{Res} V$ naturally maps surjectively onto $V$.
(3) If $W$ is a $H$-representation, then $\text{Res} \circ \text{Ind} W = \oplus \gamma \in G/H \text{Ind}^G_H W^\gamma$.
(4) If $V$ is a $G$-representation, then $\text{Res} \circ \text{Ind} \circ \text{Res} V = \oplus \gamma \in G/H \text{Res} V$.
(5) Suppose we have $G = G_K, H = G_K'$. If $L$ is a crystalline $G_{K'}$-representation, then so is $L^\gamma$.

Proof. (1) is because $H \subseteq G$ is of finite index. (2) is easy corollary of (1). (3) is via Mackey decomposition. (4) is easy corollary of (3). To prove (5), note that by [27, Lem. 2.2.9], $\text{Ind}_{G_K'}^{G_K} L$ is crystalline, and so $\text{Res}_{G_K'}^{G_K} (\text{Ind}_{G_K'}^{G_K} L) = \oplus \gamma \in \Gamma L^\gamma$ is a crystalline $G_{K'}$-representation, hence $L^\gamma$ is also crystalline. \hfill \square

4.2.2. Definition. Suppose $\mathcal{M}$ is a Kisin module over $\mathcal{S}'$. We say that $\mathcal{M}$ admits a descent data if $\Gamma$ acts on $\mathcal{M}$ semi-linearly and the action commutes with $\varphi_{\mathcal{M}}$.

Obviously, if $\mathcal{M}$ is a Kisin module over $\mathcal{S}$, then $\mathcal{M}' := \mathcal{S}' \otimes_{\mathcal{S}} \mathcal{M}$ naturally admits a descent data, and so we can define a functor $\mathcal{M} \rightsquigarrow \mathcal{M}'$.

4.2.3. Proposition. (1) The functor $\mathcal{M} \rightsquigarrow \mathcal{M}' := \mathcal{S}' \otimes_{\mathcal{S}} \mathcal{M}$ induces an equivalence between the category of torsion Kisin modules over $\mathcal{S}$ and the category of torsion Kisin modules over $\mathcal{S}'$. 
\[ \mathcal{S}' \text{ with descent data. The quasi-inverse of the functor is given by } \mathcal{M}' \twoheadrightarrow (\mathcal{M}')^\Gamma. \] We will say \( \mathcal{M}' \) descends to \( \mathcal{M} \).

(2) \( \mathcal{M} \) is finite \( \mathcal{S}_n \)-free if and only if \( \mathcal{M}' \) is \( \mathcal{S}'_n \)-finite free.

(3) \( T^*_G(\mathcal{M})|_{G_{\infty}} \simeq T^*_G(\mathcal{M}') \).

\textbf{Proof.} (1) is a standard consequence of étale descent (see [6, §6.2 Example B]) as \( \mathcal{S}' \) is finite étale over \( \mathcal{S} \). For statement (2), we only need to prove the “if” direction. We thank an anonymous referee for the following concise proof. Since \( \mathcal{M}' = \mathcal{S}' \otimes \mathcal{S} \), we have \( \mathcal{M}'/(u,p)\mathcal{M}' \simeq k' \otimes_k \mathcal{M}/(u,p)\mathcal{M} \).

Lifting any \( k \)-basis of \( \mathcal{M}/(u,p)\mathcal{M} \) (suppose the \( k \)-dimension is \( d \)) to \( \mathcal{M} \) gives a surjective map \( f : (\mathcal{S}_n)^{\otimes d} \twoheadrightarrow \mathcal{M} \). Consider the map \( 1 \otimes f : W(k') \otimes_{W(k)} (\mathcal{S}_n)^{\otimes d} = (\mathcal{S}_n')^{\otimes d} \twoheadrightarrow W(k') \otimes_{W(k)} \mathcal{M} = \mathcal{M}' \).

1 \otimes f is an isomorphism modulo \((u,p)\) and hence an isomorphism itself. Since \( W(k') \) is faithfully flat over \( W(k) \), \( f \) is an isomorphism. Statement (3) is clear by the definition of \( T^*_G \) via (2.3.1). \( \square \)

\section{4.2.4. Corollary.} Suppose that \( \mathcal{M}' \) is a finite free Kisin module over \( \mathcal{S}' \) with descent data. Then \( \mathcal{M}' \) descends to a finite free Kisin module \( \mathcal{M} \) over \( \mathcal{S} \).

\section{4.2.5. Corollary.} Suppose that \( \mathcal{T} \) is a torsion representation of \( G_K \) and \( M \) is the corresponding étale \( \varphi \)-module. Then \( \Gamma \) acts on \( M' := W(k') \otimes_{W(k)} M \) semilinearly. Suppose \( \mathcal{M} \subset M' \) is a Kisin model such that \( \gamma(\mathcal{M}') \subset \mathcal{M}' \), \( \forall \gamma \in \Gamma \). Then \( \mathcal{M}' \) descends to a Kisin model \( \mathcal{M} \subset M \).

\section{4.2.6. Definition.} Let \( \mathcal{M}' \) be a Kisin module over \( \mathcal{S}' \). For any \( \gamma \in \Gamma \), set \( \mathcal{M}'_{\gamma} := \mathcal{S}' \otimes_{\gamma, \mathcal{S}'} \mathcal{M}' \).

\section{4.2.7. Lemma.} \( T^*_G(\mathcal{M}'_{\gamma}) \simeq (T^*_G(\mathcal{M}'))^\gamma \) as \( G_{\infty} \)-representations.

\textbf{Proof.} From the definition of \( T^*_G \), we can define a map from \( T^*_G(\mathcal{M}')_{\gamma} \) to \( (T^*_G(\mathcal{M}''))^\gamma \) so that \( \sum x_i \otimes m_i \mapsto \sum \gamma^{-1}(x_i) \otimes m_i \) with \( x_i \in W(F_r R) \) and \( m_i \in \mathcal{M}' \). One can easily check that the map is an isomorphism of \( G_{\infty} \)-representations. \( \square \)

\section{4.2.8.} Suppose \( T_n \) is a \( p^n \)-torsion representation of \( G_K \), \( M_n \) the corresponding étale \( \varphi \)-module. Then it is easy to see that \( M'_n := W(k') \otimes_{W(k)} M_n \) is the corresponding étale \( \varphi \)-module for \( T_n|_{G_K'} \).

\section{4.2.9.} \textbf{Lemma.} With notations in 4.2.8. Suppose \( f : L \rightarrow T_n|_{G_K'} \) is a surjection of \( G_{K'} \)-representations where \( L \) is crystalline. Let \( \mathcal{L} \) be the Kisin module corresponding to \( L \), and denote \( \mathcal{M}' := f(\mathcal{L}) \). If \( \mathcal{M}' \) is \( \Gamma \)-stable as in Corollary 4.2.5, namely, \( \mathcal{M}' \) descends to a \( \mathcal{S} \)-module \( \mathcal{M} \), then \( \mathcal{M} \) is a liftable Kisin model of \( M_n \).

\textbf{Proof.} Since \( f : L \rightarrow T_n|_{G_K'} \), so we have \( \text{Ind}_{G_K'}^{G_K} L \rightarrow \text{Ind}_{G_K'}^{G_K} T_n \rightarrow T_n \), where the last surjection is via Lemma 4.2.1(2). It suffices to show that the composite \( h : \text{Ind}_{G_K'}^{G_K} L \rightarrow T_n \) realizes \( \mathcal{M} \). Suppose \( h \) realizes \( \mathcal{M}'(2) \), then we have \( h|_{G_K'} \) realizes \( W(k') \otimes_{W(k)} \mathcal{M}'(2) \). By Lemma 4.2.1(3), \( h|_{G_K'} \), clearly factors through \( L \rightarrow T_n|_{G_K'} \), which realizes \( \mathcal{M} \), and so \( \mathcal{M}' = W(k') \otimes_{W(k)} \mathcal{M}'(2) \). Since \( W(k') \) is faithfully flat over \( W(k) \), we must have \( \mathcal{M} = \mathcal{M}'(2) \). \( \square \)

Our next lemma shows that the maximal object \( \mathcal{M}_{(n)} \) is compatible with unramified base change. Recall that we use \( \mathcal{M}_{(n)} \) and \( \mathcal{M}'_{(n)} \) to denote the maximal liftable Kisin models of \( M_n \) and \( M'_n \) respectively.

\section{4.2.10. Lemma.} With notations in 4.2.8, we have \( \mathcal{M}'_{(n)} \simeq (W(k') \otimes_{W(k)} \mathcal{M}_{(n)}). \)
Proof. Since $\mathcal{M}_{(n)}$ can be realized by a loose crystalline lift $L \rightarrow T_n$, after restricting to $G_{K'}$, we see that $W(k') \otimes_{W(k)} \mathcal{M}_{(n)}$ is liftable. So $W(k') \otimes_{W(k)} \mathcal{M}_{(n)} \subset \mathcal{M}'_{(n)}$. Conversely, we claim that $\mathcal{M}'_{(n)}$ is stable under $\Gamma$-action. If so, then by Lemma 4.2.5, $\mathcal{M}'_{(n)}$ descent to an $\mathcal{M}$ such that $\mathcal{M}$ is a Kisin model of $M_n$. Furthermore, $\mathcal{M}$ is liftable by Lemma 4.2.9, and so $\mathcal{M} \subset \mathcal{M}_{(n)}$, concluding the proof.

Now it suffices to prove the claim. Suppose $f : L \rightarrow T_n|_{G_{K'}}$, is surjection of $G_{K'}$-representations which realizes $\mathcal{M}'_{(n)}$, then it is not hard (c.f. Lemma 4.2.7) to see that $f' : L' \rightarrow T_n'$ realizes $\gamma(\mathcal{M}'_{(n)})$. That is to say, $\gamma(\mathcal{M}'_{(n)})$ is liftable, and so $\gamma(\mathcal{M}'_{(n)}) \subset \mathcal{M}'_{(n)}$ by maximality of $\mathcal{M}'_{(n)}$ (and hence indeed, $\gamma(\mathcal{M}'_{(n)}) = \mathcal{M}'_{(n)}$). In other words, $\mathcal{M}'_{(n)}$ is $\Gamma$-stable. □

5. Torsion theory

In this section, we use the results on loose crystalline lifts to study Kisin models in the étale $\varphi$-modules corresponding to $p^m$-torsion representation of $G_K$. We use many facts on torsion Kisin modules heavily. The reader may consult [22, §2.3] for general facts on torsion Kisin modules.

5.1. Generators of torsion Kisin modules. In this subsection, we will freely use notations from 4.1.3. Recall that for $\mathcal{M}$ a $p^m$-torsion Kisin module such that $M := \mathcal{M}[[\frac{1}{p}]]$ is a finite free $\mathcal{S}_n[[\frac{1}{p}]]$-module, we have defined the modules $\mathcal{M}^{i,j}$ for $i < j$. For each $1 \leq i \leq n$, we can choose elements $\{e_j^{(i)} \in \mathcal{M}_{j=1}^d\}$, such that $\{p^{i-1}e_j^{(i)}\}_{j=1}^d$ forms a $k[[u]]$-basis of $\mathcal{M}^{i-1,i}$. The following easy lemma will be used later.

5.1.1. Lemma. (1) For $m = 1, \ldots, n$, the module $\mathcal{M}[p^m]$ is generated (over $\mathcal{S}$) by $\{e_j^{(i)}, 1 \leq i \leq m\}_{j=1}^d$. Note that when $m = n$, $\mathcal{M}[p^n] = \mathcal{M}$.

Proof. We prove by induction on $m$. The case $m = 1$ is trivial since $\mathcal{M}[p]$ is precisely $\mathcal{M}^{0,1}$. Suppose the statement is valid for $m-1$, and consider $\mathcal{M}[p^m]$. For any $x \in \mathcal{M}[p^m]$, we have $p^{m-1}x \in \mathcal{M}^{m-1,m}$. Hence there exists $y_j \in \mathcal{S}$ so that $p^{m-1}(x - \sum_j y_j e_j^{(m)}) = 0$, and so $x - \sum_j y_j e_j^{(m)} \in \mathcal{M}[p^{m-1}]$. By induction hypothesis, $x - \sum_j y_j e_j^{(m)}$ can be written as a linear combination of $e_j^{(i)}$ for $i \leq m-1$; this completes the induction. □

The following lemma is the technical key of this subsection. Its assumption ("existence of $h$") will be verified for certain Kisin modules in Proposition 5.2.1.

5.1.2. Lemma. Using notations from above. Suppose that there exists an $h \in \mathbb{Z}^{>0}$ such that $u^h \mathcal{M}^{i-1,i} \subset \mathcal{M}^{i+1,i}$ for $i = 1, \ldots, n-1$. Then the following holds.

(1) For each $i$, we have

\[
(e_1^{(i)}, \ldots, e_d^{(i)}) = (e_1^{(n)}, \ldots, e_d^{(n)})(\frac{p}{u^h})^{n-i}Y_{i,n}
\]

with $Y_{i,n} \in M_d(\mathcal{S}(\frac{p}{u^h}))$.

(2) For each $i = 1, \ldots, n-1$, we have

\[
p(e_1^{(i+1)}, \ldots, e_d^{(i+1)}) = (e_1^{(i)}, \ldots, e_d^{(i)})\Lambda_i \left(I_d + \frac{p}{u^h}Y_{i+1,i}'\right),
\]

where $\Lambda_i \in M_d(\mathcal{S})$ such that $u^h \Lambda_i^{-1} \in M_d(\mathcal{S})$, and $Y_{i+1,i}' \in M_d(\mathcal{S}(\frac{p}{u^h}))$.\]
Proof. Since \( u^b \mathfrak{M}^{i-1,i} \subset \mathfrak{M}^{i,i+1} \subset \mathfrak{M}^{i-1,i} \), we obviously have
\[
(p^i \epsilon_1^{(i+1)}, \ldots, p^i \epsilon_d^{(i+1)}) = (p^{i-1} \epsilon_1^{(i)}, \ldots, p^{i-1} \epsilon_d^{(i)}) \tilde{A}_i, \ i = 1, \ldots, n - 1,
\]
where \( \tilde{A}_i \in M_d(\mathfrak{S}_1) \) is a matrix such that \( u^b \tilde{A}_i^{-1} \in M_d(\mathfrak{S}_1) \).

Now for each \( i \), we construct a lift \( \Lambda_i \) of \( \tilde{A}_i \) as follows. Since \( k[u] \) is PID, we can write \( \tilde{A}_i = XAY \) where \( X, Y \) are invertible matrices, and \( A \) is diagonal matrix with elements on the diagonal of the form \( u^a \) such that \( a_t \leq h \) for \( 1 \leq t \leq d \). Let \( \tilde{X}, \tilde{Y} \in M_d(\mathfrak{S}) \) be some fixed lifts of \( X, Y \) respectively. Then \( \Lambda_i := \tilde{X} \tilde{Y}^t \in M_d(\mathfrak{S}) \) is a lift of \( \Lambda_i \) and satisfies \( u^b \Lambda_i^{-1} \in M_d(\mathfrak{S}) \). And so we have
\[
(p^i \epsilon_1^{(i+1)}, \ldots, p^i \epsilon_d^{(i+1)}) = (p^{i-1} \epsilon_1^{(i)}, \ldots, p^{i-1} \epsilon_d^{(i)}) \Lambda_i, \ i = 1, \ldots, n - 1,
\]
since \( p^i \epsilon_j^{(i)} = 0, 1 \leq j \leq d \).

Now we prove by induction on \( i (\geq 2) \) that
\[
(5.1.1) \quad (\epsilon_1^{(i-k)}, \ldots, \epsilon_d^{(i-k)}) = (\epsilon_1^{(i)}, \ldots, \epsilon_d^{(i)}) \left( \frac{p}{u^h} \right)^k Y_{i-k,i}
\]
with \( Y_{i-k,i} \in M_d(\mathfrak{S}[\frac{p}{u^h}]) \) for \( 1 \leq k \leq i-1 \). When \( i = 2 \), we have \((p^2 \epsilon_1^{(2)}, \ldots, p^2 \epsilon_d^{(2)}) = (\epsilon_1^{(1)}, \ldots, \epsilon_d^{(1)}) \Lambda_2 \). The statement is valid as \( Y_{1,2} := u^b \Lambda_1^{-1} \in M_d(\mathfrak{S}) \). Now suppose the statement is valid for \( i \leq m - 1 \) \((m \geq 3)\), we consider the situation of \( i = m \). Since
\[
(p^{m-1} \epsilon_1^{(m)}, \ldots, p^{m-1} \epsilon_d^{(m)}) = (p^{m-2} \epsilon_1^{(m-1)}, \ldots, p^{m-2} \epsilon_d^{(m-1)}) \Lambda_{m-1},
\]
we have \( p(\epsilon_1^{(m)}) - (\epsilon_1^{(m-1)}) \Lambda_{m-1} \in \mathfrak{M}[p^{m-2}] \). By Lemma 5.1.1, \( \mathfrak{M}[p^{m-2}] \) is generated by \( \epsilon_j^{(i)} \) for \( 1 \leq i \leq m - 2 \) and \( j = 1, \ldots, d \). So, we have
\[
p(\epsilon_1^{(m)}, \ldots, \epsilon_d^{(m)}) - (\epsilon_1^{(m-1)}, \ldots, \epsilon_d^{(m-1)}) \Lambda_{m-1} = \sum_{j=0}^{m-2} (\epsilon_1^{(j)}, \ldots, \epsilon_d^{(j)}) X_j, \ \text{for some } X_j \in M_d(\mathfrak{S})
\]
\[
= \sum_{j=1}^{m-2} (\epsilon_1^{(m-1)}, \ldots, \epsilon_d^{(m-1)})(\frac{p}{u^h})^{m-1-j} Y_{j,m-1} X_j, \ \text{by induction hypothesis.}
\]

Hence we have
\[
(\epsilon_1^{(m)}, \ldots, \epsilon_d^{(m)}) p \Lambda_{m-1} = (\epsilon_1^{(m-1)}, \ldots, \epsilon_d^{(m-1)}) \left( I_d + \sum_{j=1}^{m-2} (\frac{p}{u^h})^{m-1-j} Y_{j,m-1} \right)
\]
Since \( u^b \Lambda_{m-1} \in M_d(\mathfrak{S}) \), we can write (using \( \frac{p}{u^h} = \frac{p}{u^2} u^h \))
\[
(5.1.2) \quad (\epsilon_1^{(m)}, \ldots, \epsilon_d^{(m)}) p \Lambda_{m-1} = (\epsilon_1^{(m-1)}, \ldots, \epsilon_d^{(m-1)}) \left( I_d + \frac{p}{u^2} Y_{m-1,m} \right),
\]
with \( Y_{m-1,m} \in M_d(\mathfrak{S}[\frac{p}{u^2}]) \). Hence
\[
(\epsilon_1^{(m-1)}, \ldots, \epsilon_d^{(m-1)}) = (\epsilon_1^{(m)}, \ldots, \epsilon_d^{(m)}) p \Lambda_{m-1} \left( I_d + \frac{p}{u^2} Y_{m-1,m} \right)^{-1}
\]
with \( Y_{m-1,m} \in M_d(\mathfrak{S}[\frac{p}{u^2}]) \) (Note that when calculating \( (I_d + \frac{p}{u^2} Y_{m-1,m})^{-1} \), we can throw away terms with high \( p \)-powers, because \( p^m \) kills \( (\epsilon_1^{(m)}, \ldots, \epsilon_d^{(m)}) \)). Now for \( k = 2, \ldots, m - 1 \), we iterate
the above to get
\[(\epsilon_1^{(m-k)}, \ldots, \epsilon_d^{(m-k)}) = (\epsilon_1^{(m-1)}, \ldots, \epsilon_d^{(m-1)})(\frac{P}{u^m})^{k-1}Y_{m-k,m-1},\]
and so \((5.1.1)\) is proved.

It is clear \((5.1.1)\) implies Item (1). And Item (2) is already proved in Equation \((5.1.2)\). \(\square\)

5.2. Existence of \(h\). The following proposition will play a key role to prove the later overconvergence result. As we mentioned earlier, in this subsection, we will use the notation system in Convention 3.3.3.

5.2.1. Proposition. Use notations in 4.1.4. There exists a constant \(h\) only depending on \(p, f, e\) and \(d\) such that \(u^h\mathcal{M}_{(n-1,1)} \subset \mathcal{M}_{(n,1)}\) for all \(n \geq 1\). Consequently \(u^h\mathcal{M}_n^{i-1,i} \subset \mathcal{M}_n^{i,i+1}\) for all \(i = 1, \ldots, n - 1\) by Corollary 4.1.6.

Proof. The proof is quite involved, so we break the proof into several steps. We first present the main strategy of the proof, assuming two difficult lemmas which will be proved later.

We first fix an \(n\), so we can apply Theorem 3.3.2 to our \(T_n\), and we will freely use notations there. By Lemma 4.2.10, for any \(n\), if we let \(M'_n := W(k') \otimes_{W(k)} M_n\), the maximal liftable Kisin model of \(M'_n\) is \(\mathcal{M}_n' = W(k') \otimes_{W(k)} \mathcal{M}_n\). Thus, it is easy to see that \(\mathcal{M}_n' = W(k') \otimes_{W(k)} \mathcal{M}_n\) for all \(n\). To prove our proposition, it suffices to show that \(u^h\mathcal{M}_{(n-1,1)} \subset \mathcal{M}_{(n,1)}\).

We divide the following argument into two steps. In Step 1, we will construct another Kisin model (denoted as \(\mathcal{M}'_{n,1}\)) of \(M'_1\) such that \(\mathcal{M}'_{n,1} \subset \mathcal{M}_{(n,1)}\). Then in Step 2, we show that \(u^h\mathcal{M}_{(n-1,1)} \subset \mathcal{M}_{(n,1)}\).

Step 1. The loose crystalline lift \(\tilde{L}'_n \to \tilde{W}_n|G_K' \to T_n|G_K' \to T_1|G_K'\). realizes a liftable Kisin model \(\mathcal{M}'_n\) of \(M'_n\), and so \(\mathcal{M}'_{n,1} \subset \mathcal{M}_{(n,1)}\). The following composite of \(G_K'-\)representations
\[
(5.2.1) \quad \tilde{L}'_n \longrightarrow \tilde{W}_n|G_K' \longrightarrow T_n|G_K' \longrightarrow T_1|G_K',
\]
realizes a Kisin model \(\mathcal{M}'_{n,1}\) in \(M'_1\), and we certainly have \(\mathcal{M}'_{n,1} \subset \mathcal{M}_{(n,1)}\).

By diagram (3.3.1), The composite \((5.2.1)\) is the same as
\[
\tilde{L}'_n \longrightarrow \tilde{W}_n \longrightarrow \tilde{L}_{n-1} \longrightarrow T_{n-1} \longrightarrow T_1,
\]
which is the same as
\[
\tilde{L}'_n \longrightarrow \tilde{W}_n \longrightarrow \tilde{W}_n/p\tilde{W}_n = Z_n \longrightarrow \tilde{L}_{n-1}/p\tilde{L}_{n-1} \longrightarrow T_1,
\]
which, by (3.3.6), is the same as
\[
\tilde{L}'_n \longrightarrow \tilde{Z}_n' \longrightarrow Z_n \longrightarrow \tilde{L}_{n-1}/p\tilde{L}_{n-1} \longrightarrow T_1,
\]
which, by diagram (3.3.4), is the same as
\[
(5.2.2) \quad \tilde{L}'_n \longrightarrow \tilde{Z}_n' \longrightarrow \tilde{L}_{n-1}(-s) \longrightarrow \tilde{L}_{n-1}/p\tilde{L}_{n-1} \longrightarrow T_1.
\]
And so (5.2.2) also realizes $\mathcal{M}_{n,1}$.

**Step 2.** The last surjection of (5.2.2) is induced by the following composite of maps:

\[
\begin{array}{c}
\tilde{L}_{n-1} \\
\rightarrow T_{n-1} \\
\rightarrow T_1
\end{array}
\]

Since we may assume that $\tilde{L}_{n-1} \twoheadrightarrow T_{n-1}$ realize $\mathcal{M}_{(n-1)}$, and the composite (5.2.3) restricted to $G_{K'}$ realizes $\mathcal{M}'_{(n-1,1)}$.

So in order to prove $u^h\mathcal{M}'_{(n-1,1)} \subset \mathcal{M}'_{(n,1)}$, it suffices to show $u^h\mathcal{M}'_{(n-1,1)} \subset \mathcal{M}'_{n,1}$. And so it suffices to show that the cokernel of the following composite (which are maps of Kisin modules corresponding to (5.2.2)) is killed by $u^h$

\[
\begin{array}{c}
\tilde{\mathcal{L}}_n' \\
\rightarrow \tilde{\mathcal{Y}}_n' \\
\rightarrow \tilde{\mathcal{L}}'_{n-1}(s) \\
\rightarrow \mathcal{L}'_{n-1}/p\mathcal{L}'_{n-1}
\end{array}
\]

By Lemma 5.2.2 below, the map $\tilde{\mathcal{L}}_n' \rightarrow \tilde{\mathcal{Y}}_n'$ is in fact surjective, so we only need to consider the cokernel of the following composite of maps:

\[
\begin{array}{c}
\tilde{\mathcal{Y}}_n' \\
\rightarrow \tilde{\mathcal{L}}'_{n-1}(s) \\
\rightarrow \mathcal{L}'_{n-1}/p\mathcal{L}'_{n-1}
\end{array}
\]

Denote the image of the composite (5.2.5) as $\mathcal{L}'_{n-1}$, which is contained in $\mathcal{L}'_{n-1}(s)/p\mathcal{L}'_{n-1}(s)$. So we can choose basis $e_1, \ldots, e_m$ of $\mathcal{L}'_{n-1}$, such that $\mathcal{L}'_{n-1}$ has a $k'[u]$-basis formed by $u^{a_1}e_1, \ldots, u^{a_m}e_m$, where $a_i = \frac{e_i}{p^b_i}$ with $b_i \geq 0$ (note that the rank of $\mathcal{L}'_{n-1}$ is also $m$, because of the surjective maps in (5.2.2)).

Finally, we will show in Lemma 5.2.6 that $a_i$’s are bounded by a constant $h$, and this will conclude the proof. \qed

Now we will prove Lemma 5.2.2 and Lemma 5.2.6 to complete the proof of Proposition 5.2.1. Before we prove Lemma 5.2.2, let us first recall the following subtle fact: suppose that we have an exact sequence of lattices inside semi-stable representations:

\[0 \rightarrow L^{(1)} \rightarrow L^{(2)} \rightarrow L^{(3)} \rightarrow 0\]

Then the corresponding sequence of Kisin modules

\[0 \rightarrow \mathcal{L}^{(1)} \rightarrow \mathcal{L}^{(2)} \rightarrow \mathcal{L}^{(3)} \rightarrow 0\]

is only left exact [24, Lem. 2.19.], i.e., the map $\mathcal{L}^{(2)} \rightarrow \mathcal{L}^{(3)}$ is not necessarily surjective.

5.2.2. **Lemma.** The sequence of finite free Kisin modules corresponding to the first row of Diagram (3.3.6)

\[0 \rightarrow p\mathcal{L}'_{n-1} \rightarrow \mathcal{L}'_n \xrightarrow{\theta} \mathcal{Y}'_n \rightarrow 0\]

is exact, i.e., $\theta$ is surjective.

5.2.3. **Convention.** In the proof of this lemma, all the representations that we consider are $G_{K'}$-representations, and all Kisin modules are over $\mathcal{G}' = W(k') \otimes_{W(k)} \mathcal{G}$. To be completely rigorous, we will need to restrict many representations from $G_K$ to $G_{K'}$, and use prime notation over Kisin modules (i.e., notations like $\mathcal{M}'$). For notational simplicity, from now on (i.e., in the proof of Lemma 5.2.2 and Lemma 5.2.6), we will drop these prime notations.
Proof. **Step 0: strategy.** We first sketch the strategy of the proof as follows. We will not directly show that \( \theta \) is surjective. Instead, we will construct a Kisin model \( \mathfrak{M}_n \) inside \( \mathcal{W}_n \), which sits in a short exact sequence

\[
0 \to p\mathfrak{L}_{n-1} \to \mathfrak{M}_n \to \mathfrak{Z}_n \to 0,
\]

(5.2.6)

corresponding to the exact sequence (3.3.2) (restricted to \( G_K \)), where \( \mathfrak{Z}_n \) is the Kisin model realized by \( \tilde{Z}_n \to \mathcal{Z}_n \).

We then define the product \( \mathfrak{L}_n^{(2)} := \mathfrak{M}_n \times_{\mathfrak{Z}_n} \mathfrak{Z}_n \), which is a Kisin model of \( \mathfrak{L}_n \), and naturally sits inside the short exact sequence:

\[
0 \to p\mathfrak{L}_{n-1} \to \mathfrak{L}_n^{(2)} \to \mathfrak{Z}_n \to 0.
\]

This forces \( \mathfrak{L}_n^{(2)} \) to be finite free, and so it has to be equal to \( \mathfrak{L}_n \) (because the finite free Kisin model of \( \mathcal{L}_n \) is unique), concluding the proof.

**Step 1: a basis for \( \mathfrak{Z}_n \).** As the strategy suggests, we first construct a basis for \( \mathfrak{Z}_n \).

From diagram (3.3.4), we have the following

From (5.2.7), we have the following

Note that in this diagram, only the map \( \tilde{L}_{n-1}(-s) \to \tilde{L}_{n-1}/p\tilde{L}_{n-1} \) is a strict crystalline \( \mathbb{Z}_p \)-lift (i.e., it is a modp map); the other two projections \( \tilde{N}_n \to p^{n-1}T_n \) and \( \tilde{Z}_n \to \mathcal{Z}_n \) are just loose crystalline lifts.

Recall that as in Step 2 of the proof of Proposition 5.2.1, \( \mathfrak{L}_{n-1} \) is the torsion Kisin module induced by the surjection \( \tilde{Z}_n \to \tilde{L}_{n-1}(-s) \to \tilde{L}_{n-1}/p\tilde{L}_{n-1} \). Let \( \mathfrak{Z}_n \) be the torsion Kisin module induced by the map \( \tilde{Z}_n \to \mathcal{Z}_n \). Let \( \mathfrak{M}_n \) be the kernel of the surjection \( \mathfrak{Z}_n \to \mathfrak{L}_{n-1} \). The following short exact sequence

\[
0 \to \mathfrak{M}_n \to \mathfrak{Z}_n \to \mathfrak{L}_{n-1} \to 0.
\]

(5.2.8)

corresponds to the bottom row of (5.2.7), and so \( \mathfrak{M}_n \) is a Kisin model of \( p^{n-1}M_n \), and hence finite free over \( k\llbracket u \rrbracket \). We already have the basis \( u^a_1 \varepsilon_1, \ldots, u^a_m \varepsilon_m \) for \( \mathfrak{L}_{n-1} \). Pick any \( k\llbracket u \rrbracket \)-basis \( \alpha_1, \ldots, \alpha_t \) for \( \mathfrak{M}_n \), and then \( \{ \alpha_1, \ldots, \alpha_t, u^a_1 \varepsilon_1, \ldots, u^a_m \varepsilon_m \} \) form a basis of \( \mathfrak{Z}_n \).

**Step 2: a Kisin model inside \( \mathcal{W}_n \).** Now we construct a Kisin model \( \mathfrak{M}_n \) inside \( \mathcal{W}_n \). Consider \( e_1, \ldots, e_m \) the \( \mathcal{G} \)-basis of \( \mathfrak{L}_{n-1} \), since \( \mathfrak{W}_n \to \mathfrak{L}_{n-1} \), we can pick lifts \( \check{\varepsilon}_i \in \mathcal{W}_n \) of \( e_i \). Consider the \( \mathcal{G} \)-module \( \mathfrak{M}_n \) generated by \( p\check{\varepsilon}_i, u^a_i \check{\varepsilon}_i \) and \( \alpha_j \); we claim this is a Kisin model of \( \mathcal{W}_n \).

We see that \( \check{\varepsilon}, \alpha \) generate \( \mathcal{W}_n \) as étale \( \varphi \)-modules, so it suffices to check that \( \mathfrak{M}_n \) is \( \varphi \)-stable. To see this, we first observe that

\[
\varphi(\check{\varepsilon}_1, \ldots, \check{\varepsilon}_m) = (\check{\varepsilon}_1, \ldots, \check{\varepsilon}_m)A + (\alpha_1, \ldots, \alpha_t)B,
\]

where \( A \in M_{m \times m}(\mathcal{G}), B \in M_{t \times m}(O_\mathcal{G}) \). Now we check step by step.

- Because \( p\alpha_i = 0 \), we have \( \varphi(p\check{\varepsilon}_1, \ldots, p\check{\varepsilon}_m) = (p\check{\varepsilon}_1, \ldots, p\check{\varepsilon}_m)A \) with \( A \in M_{m \times m}(\mathcal{G}) \).
\[ \varphi(u^{\alpha_1}e_1, \ldots, u^{\alpha_m}e_m) = (e_1, \ldots, e_m)A\varphi(\Lambda) + (\alpha_1, \ldots, \alpha_m)B\varphi(\Lambda) \] 
where \( \Lambda \) is the diagonal matrix \([u^{\alpha_1}, \ldots, u^{\alpha_m}] \). Since \( \{\alpha_i, u^{\alpha_i}e_j\} \) forms a \( k[u] \)-basis of \( \mathcal{Z}_n \), we must have 
\[ (e_1, \ldots, e_m)A\varphi(\Lambda) \equiv (e_1, \ldots, e_m)AC \pmod{p} \]
with \( C \) having coefficients in \( \mathcal{S} \) and \( B\varphi(\Lambda) \) have entries in \( k[u] \). Hence \( \{\varphi(u^{\alpha_i}e_i)\} \subset \mathfrak{M}_n \).

Finally it is obvious that \( \varphi(\alpha_1, \ldots, \alpha_m) \subset \mathfrak{M}_n \).

So we conclude that \( \mathfrak{M}_n \) is indeed a Kisin model of \( W_{(n)} \), and the sequence (5.2.6) is short exact; as we noted in Step 0, this concludes the proof. \( \square \)

We now proceed to Lemma 5.2.6. First, let us introduce a useful definition.

5.2.4. Definition. Let \( \mathfrak{M} \) be a \( p \)-torsion Kisin module (over \( k'[u] \)). Then \( v_R(\det(\varphi|_{\mathfrak{M}})) \) makes sense and does not depend on the choice of \( k'[u] \)-basis of \( \mathfrak{M} \) (here we normalize \( v_R \) on \( k'[u] \) by setting \( v_R(u) = 1 \)). Denote \( \alpha(\mathfrak{M}) = v_R(\det(\varphi|_{\mathfrak{M}})) \).

5.2.5. Lemma. (1) If we have an exact sequence of \( p \)-torsion Kisin modules \( 0 \to \mathfrak{M}^{(1)} \to \mathfrak{M}^{(2)} \to \mathfrak{M}^{(3)} \to 0 \), then \( \alpha(\mathfrak{M}^{(2)}) = \alpha(\mathfrak{M}^{(1)}) + \alpha(\mathfrak{M}^{(3)}) \).

(2) Let \( L \) be a \( \mathcal{G}_K \)-stable \( \mathbb{Z}_p \)-lattice in a semi-stable representation \( V \) with non-positive Hodge-Tate weights \( \text{HT}(V) \), and \( \mathcal{L} \) the corresponding Kisin module. Then \( \alpha(\mathcal{L}/p\mathcal{L}) = e(\sum_{i \in \text{HT}(V) - i}) \) where \( e = e(K') \) the ramification index.

5.2.6. Lemma. As we claimed at the end of proof of Proposition 5.2.1, there exists a constant \( h \) only depending on \( d = \dim_{\mathbb{F}_p} T \), \( p \), \( f \) and \( e \) such that \( a_i \leq h \).

Proof. Recall that \( \mathcal{L}_{n-1} \) has a \( k'[u] \)-basis formed by \( u^{\alpha_1}e_1, \ldots, u^{\alpha_m}e_m \), where \( e_1, \ldots, e_m \) is a \( k'[u] \)-basis of \( \mathcal{L}_{n-1}/p\mathcal{L}_{n-1} \). We clearly have
\[ \alpha(\mathcal{L}_{n-1}) = (p-1) \sum_{1 \leq i \leq m} a_i + \alpha(\mathcal{L}_{n-1}/p\mathcal{L}_{n-1}) = (p-1) \sum_{1 \leq i \leq m} b_i + \alpha \left( \mathcal{L}_{n-1}(-s)/p\mathcal{L}_{n-1}(-s) \right) \]
Since \( b_i \leq \sum_{1 \leq i \leq m} b_i \), so we only need to bound \( \alpha(\mathcal{L}_{n-1}) - \alpha(\mathcal{L}_{n-1}/p\mathcal{L}_{n-1}) \).

By Lemma 5.2.5(1) and Equation (5.2.8), \( \alpha(\mathcal{L}_{n-1}) \leq \alpha(\mathfrak{M}_n) \). Recall from diagram (3.3.4), \( \mathfrak{M}_n \) is realized via \( \mathbb{Z}_n \to \mathbb{Z}_n \otimes_{\mathbb{Z}_p} k_E \to \mathbb{Z}_n \). That is, we have
\[ \mathcal{L}_n \to \mathbb{Z}_n/p\mathcal{L}_n = \oplus \mathfrak{M}_n \to \mathfrak{M}_n \]
where \( \mathfrak{M}_n \otimes_{\mathbb{Z}_E} \mathfrak{M}_n \) is a direct sum of \( ([k_E : \mathbb{F}_p]) \)-copies of \( \mathfrak{M}_n \). So we have
\[ \alpha(\mathcal{L}_{n-1}) \leq \alpha(\mathfrak{M}_n) \leq \alpha(\mathcal{L}_n/p\mathcal{L}_n) \leq \alpha(\mathfrak{M}_n/p\mathfrak{M}_n) \]
So we have
\[ \alpha(\mathcal{L}_{n-1}) - \alpha \left( \mathcal{L}_{n-1}(-s)/p\mathcal{L}_{n-1}(-s) \right) \leq \alpha(\mathfrak{M}_n/p\mathfrak{M}_n) - \alpha \left( \mathcal{L}_{n-1}(-s)/p\mathcal{L}_{n-1}(-s) \right) = e(\sum_{i \in \text{HT}(\mathfrak{N}_n)} -i) \]
by Lemma 5.2.5(2).

Here, the representation \( \mathcal{N}_n \) comes from the first row of (3.3.4). Recall that by Lemma 3.1.3, we could have chosen our \( E \) to be \( K_d \). By Corollary 3.2.2, we have \( \sum_{i \in \text{HT}(\mathfrak{N}_n)} -i \leq ef^d(p^{jd} - 2) \)
This finishes the inductive definition. (since generators \( \{ e \} \) overconvergent basis by taking inverse limit. Use them to define a basis for \( M \).

**Proof.** For the proof, we first inductively construct a specific set of generators for \( M \).

By Equation (3.3.5), we can let \( s = p^f d + p - 2 \). So for example, we can let

\[
(5.2.9)\quad h := 3f e^2 d^2 p^f d.
\]

6. **Overconvergent basis and main theorem**

In this section, we prove our main theorem, namely, the overconvergence of \((\varphi, \tau)\)-modules associated to finite free \( \mathbb{Z}_p \)-representations. To do so, we first show the existence of an “overconvergent basis”, with respect to which all entries of the matrices for \( \varphi \) and \( \hat{G} \) are overconvergent elements.

6.1. **Existence of overconvergent basis.**

6.1.1. **Theorem.** Let \( \rho : G_K \to \text{GL}_d(\mathbb{Z}_p) \) be a continuous representation and \( \hat{M} = (M, \varphi_M, \hat{\varphi}, \hat{\tau}) \) the associated \((\varphi, \tau)\)-module. Then there exists an \( \mathcal{O}_\mathbb{Z}\)-basis of \( M \), and a constant \( \alpha = \alpha(p, e, f, d) \) that only depends on \( p, e, f \) and \( d \), such that with respect to this basis,

- the matrix of \( \varphi_M \) is in \( M_d(\mathbb{G}\[\frac{p}{\mathbb{Z}}\]) \),
- the matrix of \( \tau \) is in \( M_d(W(\mathbb{R}\[\frac{p}{\mathbb{Z}}\]) \) for any \( \tau \in \hat{G} \).

Before we proceed to the proof, we will adopt the following convention on notations.

6.1.2. **Convention.** To ease the notation, we will use notations like \( (e_j)_{j=1}^d \) to mean a row vector \( (e_j)_{j=1}^d = (e_1, \ldots, e_d) \).

**Proof.** For the proof, we first inductively construct a specific set of generators for \( \mathfrak{m}_{(n)} \). Then we use them to define a basis for \( M_n \). We show that the bases are compatible, and gives us the desired overconvergent basis by taking inverse limit.

**Step 1.** **Generators of \( \mathfrak{m}_{(n)} \).** First of all, by induction on \( n \), we construct a specific set of generators \( \{ \epsilon^{(i)}_{(n,j)} \}_{i=1}^d \) of \( \mathfrak{m}_{(n)} \) such that \( \{ p^{-1} \epsilon^{(i)}_{(n,j)} \}_{j=1}^d \) forms a \( k[u] \)-basis of \( \mathfrak{m}_{(n)}^{-1,i} \).

We choose \( \{ \epsilon^{(i)}_{(1,j)} \}_{j=1}^d \) any \( k[u] \)-basis of \( \mathfrak{m}_{(1)} \). Suppose we have defined \( \{ \epsilon^{(i)}_{(n-1,j)} \}_{1 \leq i \leq n-1} \) \( j=1 \) the generators for \( \mathfrak{m}_{(n-1)} \). Recall that in Corollary 4.1.6, we have used the map \( \iota_{n-1,n} : M_{n-1} \to M_n \) to identify \( \mathfrak{m}_{(n-1)}^{-1,i} \) with \( \mathfrak{m}_{(n)}^{-1,i} \) when \( 1 \leq i \leq n-1 \). Now define

- \( \epsilon^{(i)}_{(n,\j)} := \iota_{n-1,n}(\epsilon^{(i)}_{(n-1,j)}) \) for \( 1 \leq i \leq n-1, 1 \leq \j \leq d \),
- choose any \( \{ \epsilon^{(i)}_{(n,\j)} \}_{\j=1}^d \) in \( \mathfrak{m}_{(n)} \), so that \( p^{-1} \epsilon^{(n)}_{(n,\j)} \) is a \( k[u] \)-basis of \( \mathfrak{m}_{(n)}^{-1,n} = p^n \mathfrak{m}_{(n)} \).

This finishes the inductive definition.

**Step 2.** **Basis of \( M_n \).** With above, now we define basis for \( M_n \). By Lemma 5.1.2(2), for any \( x \geq 1 \), we can write

\[
(6.1.1)\quad p \epsilon_{(x)}^{(x)} = (\iota_{x-1,x}(\epsilon_{(x-1)}^{(x-1)})) \Lambda_{x-1}(I_d + \frac{p}{y^2h} z_{x-1})
\]
with $Z_{x-1} \in M_d(\mathcal{G}_{[\frac{p}{u^{2h}}]})$. Now define

\begin{equation}
(6.1.2) \quad (e_j^{(n)}) := (\epsilon_{(n),j}) \left( \prod_{x=1}^{n-1} (A_{x-1}(I_d + \frac{p}{u^{2h}} Z_{x-1})) \right)^{-1}.
\end{equation}

We denote $Y_n := \prod_{x=1}^{n-1} (A_{x-1}(I_d + \frac{p}{u^{2h}} Z_{x-1}))$, and so $(e_j^{(n)}) = (\epsilon_{(n),j}) Y_n^{-1}$. Since $(\epsilon_{(n),j})$ are killed by $p^n$, we can take $Y_n$ and $Y_n^{-1}$ to be in $M_d(\mathcal{O}_{\mathcal{E},n})$, and we do so. More precisely, we can arrange that $Y_n \in M_d(\frac{1}{u^{(n-1)} - \mathcal{E}})$, and $Y_n^{-1} \in M_d(\frac{1}{u^{(n-1)} - \mathcal{E}})$; these are only rough estimates, but they are sufficient for our use.

Since $\epsilon_{(n),j}, 1 \leq j \leq d$ lifts an $\mathcal{O}_{\mathcal{E},n}$-basis of $M_1$, by Nakayama Lemma, $(\epsilon_1^{(n)}, \ldots, \epsilon_d^{(n)})$ is an $\mathcal{O}_{\mathcal{E},n}$-basis of $M_n$.

**Step 3. Compatibility of basis.** We check that $(e_j^{(n-1)}) = (e_j^{(n)} \mod p^{n-1})$. Note that the composite of the map $M_n \rightarrow M_{n-1} \rightarrow M_n$ is precisely the multiplication by $p$ map, where the first map is modulo $p^{n-1}$ and the second map is $\tau_{n-1,n}$. So it suffices to check that $\tau_{n-1,n}(e_j^{(n-1)}) = p(e_j^{(n)})$. But this is easy consequence of (6.1.1) and (6.1.2).

So now we can define $(e_j) := \lim_{n \rightarrow \infty}(e_j^{(n)})$, which is a basis of $M$.

**Step 4. Matrices for $\varphi$ and $\tau$.** Let $\tau$ be any element in $\hat{G}$, we claim that we have

- $\varphi(e_j) = (e_j)A$ with $A \in M_d(\mathcal{G}_{[\frac{p}{u^{n}}]})$.
- $\tau(e_j) = (e_j)B$ with $\varphi(B) \in M_d(\hat{R}_{[\frac{p}{u^{n}}]})$.

By Lemma 6.1.3, it suffices to prove that

- $\varphi(e_j^{(n)}) = (e_j^{(n)})A_n$ with $A_n \in M_d(\frac{1}{u^{(n-1)n}} \mathcal{G})$.
- $\tau(e_j^{(n)}) = (e_j^{(n)})B_n$ with $\varphi(B_n) \in M_d(\frac{1}{u^{(n-1)n}} \hat{R})$.

Since $\mathcal{M}(n)$ comes from a loose crystalline lift, we can write

- $\varphi(\epsilon_{(n),j}^{(i)}) = \sum_{i=1}^{n} (\epsilon_{(n),j}^{(i)}) P_i^{(1)}$, with $P_i^{(1)} \in M_d(\mathcal{G})$,
- $\tau(1 \otimes \varphi \epsilon_{(n),j}^{(i)}) = \sum_{i=1}^{n} (1 \otimes \varphi \epsilon_{(n),j}^{(i)}) Q_i^{(1)}$, with $Q_i^{(1)} \in M_d(\hat{R})$.

By Lemma 5.1.2, for all $i < n$, we can write

\begin{equation}
(\epsilon_{(n),j}^{(i)}) = (\epsilon_{(n),j}^{(n)}) Y_{i,n} \text{ with } Y_{i,n} \in M_d(\mathcal{G}_{[\frac{p}{u^{2h}}]})
\end{equation}

So we can write

- $\varphi(\epsilon_{(n),j}^{(n)}) = (\epsilon_{(n),j}^{(n)}) P_n$, with $P_n \in M_d(\mathcal{G}_{[\frac{p}{u^{2h}}]})$,
- $\tau(1 \otimes \varphi \epsilon_{(n),j}^{(n)}) = (1 \otimes \varphi \epsilon_{(n),j}^{(n)}) Q_n$, with $Q_n \in M_d(\hat{R}_{[\frac{p}{u^{2h}}]})$.

So

- $\varphi(e_j^{(n)}) = (e_j^{(n)}) Y_n P_n \varphi(Y_n^{-1})$,
- $\tau(1 \otimes \varphi e_j^{(n)}) = (1 \otimes \varphi e_j^{(n)}) Y_n Q_n \tau(Y_n^{-1})$.

and so (using the fact $e_j^{(n)}$ is killed by $p^n$ and then $(\frac{p}{u^{2h}})^m$ will vanishes for $m \geq n$)

- $A_n = Y_n P_n \varphi(Y_n^{-1}) \in M_d(\frac{1}{u^{2(n-1)n+2(n-1)n+3(n-1)n}} \mathcal{G})$,
- $\varphi(B_n) = \varphi(Y_n) Q_n \tau(\varphi(Y_n^{-1})) \in M_d(\frac{1}{u^{2(n-1)n+2(n-1)n+3(n-1)n}} \hat{R})$. 

Finally, we can simply let
\[ \alpha = 7ph \]
to conclude. Note that since \( \varphi(B) \in M_d(\hat{\mathbb{R}}[\frac{p}{u^\infty}]) \), we have \( B \in M_d(W(R)[\frac{p}{u^\infty}]) \).

6.1.3. Lemma. Let \( B \) be the ring \( \mathfrak{S} \), or \( \hat{\mathbb{R}} \), or \( W(R) \). Suppose that \( y_n = \frac{x_{n+1}}{u^{m+1}} \in W_n(FR) \) such that \( y_{n+1} \equiv y_n \mod p^n \) in \( W_{n+1}(FR) \). If \( x_n \in B \) for all \( n \), then \( y_n \) converges to a \( y \in B[\frac{p}{u^\infty}] \).

**Proof.** It suffices to show that \( y_n \in B_n[\frac{p}{u^n}] \) for all \( n \). This is true when \( n = 1 \). Suppose it is valid for \( n = m \). Let us consider the case that \( n = m + 1 \). Since \( y_m \in B_m[\frac{p}{u^m}] \), we can lift it to an element in \( B_{m+1}[\frac{p}{u^{m+1}}] \) and still denote it by \( y_m \), then \( x_m = u^{(m-1)\alpha}y_m \in B_{m+1} \). Now
\[ y_{m+1} = y_m + (y_{m+1} - y_m) = y_m + \frac{x_{m+1} - u^\alpha x_m}{u^\alpha m}. \]
Since \( y_{m+1} \equiv y_m \mod p^m \), we have \( x_{m+1} - u^\alpha x_m \in p^m W(R)_{m+1} \cap B_{m+1} = p^m B_{m+1} \), and so \( x_{m+1} - u^\alpha x_m = p^m x_{m+1}^{(m+1)} \) with \( x_{m+1} \in B_{m+1} \). To prove the induction and proves the lemma.

6.2. Overconvergence. Finally, we can prove our main theorem.

6.2.1. Theorem. Let \( T \) be a continuous finite free \( \mathbb{Z}_p \)-representation of \( G_K \), and let \( \hat{M} = (M, \varphi, \tilde{G}) \) the associated \((\varphi, \tau)\)-module. Then \( \hat{M} \) is overconvergent.

**Proof.** Let \((e_1, \ldots, e_d)\) be the basis of \( M \) as in Theorem 6.1.1. Let \((t_1, \ldots, t_d)\) be any basis of \( T \), and let \((e_1, \ldots, e_d) = (t_1, \ldots, t_d)X \) where \( X \in M_d(O_{\hat{\mathfrak{S}}}^\infty) \). Then we have \( \varphi(X) = XA \). In order to prove the theorem, it suffices to show that \( X \in M_d(W(FR)^{1/r}) \) for some \( r > 0 \). We claim that \( X \in M_d(W(R)[\frac{p}{u^\infty}]) \).

To prove the claim, by Lemma 6.1.3, it suffices to show that \( u^{(n-1)\alpha}X_n \in M_d(W_n(R)) \) where \( X_n := X(\mod p^n) \). We prove this by induction on \( n \). When \( n = 1 \), we have \( \varphi(X_1) = X_1A_1 \). Since \( A_1 \in M_d(\mathfrak{S}) \) by Theorem 6.1.1, we have \( X_1 \in M_d(R) \) by Lemma 6.2.2. Suppose the claim is true when \( n \leq m \), and let us consider the case \( n = m + 1 \). We can always write \( X = \sum_{\ell=0}^m p^\ell X_\ell \) with \( X_\ell \in M_d([FR]) \), where we use \([FR]\) to mean the set of Teichmüller lifts. We have \( X_{m+1} = \sum_{\ell=0}^m p^\ell X_\ell^\prime \). To finish the induction process, it suffices to show \( u^{ma}X_m^\prime \in M_d([R]) \). From \( \varphi(X_{m+1}) = X_{m+1}A_{m+1} \), we can deduce
\[ p^m \varphi(u^{ma}X_m^\prime) = p^m u^{ma}X_m^\prime u^{(p-1)ma}A_{m+1} + u^{ma}X_m A_{m+1} - \varphi(u^{ma}X_m) \]
By induction hypothesis, \( u^{(m-1)\alpha}X_m \in M_d(W(R)) \), and by Theorem 6.1.1, \( u^{ma}A_{m+1} \in M_d(W(R)) \), so the above equation is reduced to the form
\[ p^m \varphi(u^{ma}X_m^\prime) - p^m u^{ma}X_m^\prime u^{(p-1)ma}A_{m+1} = C, \]
for some \( C \in M_d(W(R)) \). Since \( p^m W(FR) \cap W(R) = p^m W(R) \), we have \( C \in M_d(p^m W(R)) \). Divide both sides of the equation by \( p^m \) and modulo \( p \) on both sides, then we can apply Lemma 6.2.2 to conclude that \( u^{ma}X_m^\prime \in M_d([R]) \). This completes the proof of our theorem.
As a final note, it is easy to see that the overconvergence radius \( r \) can be any number \(< \frac{1}{\alpha} \). Combined with Equation (6.1.3) and Equation (5.2.9), it suffices that

\[
(6.2.1) \quad r < \frac{1}{21pfe^2d^2pfd}. \]

6.2.2. Lemma. Suppose \( Y \in M_d(F_R) \) and \( B, C \in M_d(R) \) such that \( \varphi(Y) = YB + C \), then we must have \( Y \in M_d(R) \).

Proof. Suppose otherwise, consider an entry \( y_{i,j} \) in the matrix \( Y \) with minimal \( v_R \) valuation, then one can easily deduce a contradiction. □

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