ON SYMMETRIC POWER $\mathcal{L}$-INVARIANTS OF IWAHORI LEVEL HILBERT MODULAR FORMS

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Abstract. We compute the arithmetic $\mathcal{L}$-invariants (of Greenberg–Benois) of twists of symmetric powers of p-adic Galois representations attached to Iwahori level Hilbert modular forms (under some technical conditions). Our method uses the automorphy of symmetric powers and the study of analytic Galois representations on p-adic families of automorphic forms over symplectic and unitary groups. Combining these families with some explicit plethysm in the representation theory of GL(2), we construct global Galois cohomology classes with coefficients in the symmetric powers and provide formulae for the $\mathcal{L}$-invariants in terms of logarithmic derivatives of Hecke eigenvalues.

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Introduction

The p-adic interpolation of special values of L-functions has been critical to understanding their arithmetic, providing, for instance, the link between values of the Riemann zeta function and the class groups of cyclotomic fields. However, it can happen that the p-adic L-function of a motive M vanishes at a point of interpolation despite the fact that the classical L-value is non-zero. To recuperate an interpolation property,

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one predicts that a *derivative* of the p-adic L-function is related to the L-value and one defines the (analytic) $\mathcal{L}$-invariant of M as the ratio of these quantities. The name of the game then becomes to provide an *arithmetic* meaning for the $\mathcal{L}$-invariant and show that it is non-zero.

The phenomenon of $\mathcal{L}$-invariants first arose in the work of Ferrero–Greenberg [FG78] on p-adic Dirichlet L-functions. Later on, in their work [MTT86] on formulating a p-adic analogue of the Birch and Swinnerton-Dyer conjecture, Mazur–Tate–Teitelbaum encountered this behaviour in the p-adic L-function of an elliptic curve over $\mathbb{Q}$ with split multiplicative reduction at p. They conjectured a formula for the $\mathcal{L}$-invariant in terms of the p-adic Tate parameter of E (and coined the term $\mathcal{L}$-invariant). This was proved in [GS93] by Greenberg and Stevens by placing the elliptic curve in a Hida family, i.e. a p-adic family of modular forms of varying weight. Their proof proceeded in two steps: first, relate the Tate parameter to the derivative in the weight direction of the U_p-eigenvalue of the Hida family, and then use the functional equation of the two-variable p-adic L-function of the Hida family to relate this derivative to the analytic $\mathcal{L}$-invariant. We think of the Tate parameter formula as being a conjectural *arithmetic* $\mathcal{L}$-invariant and the Greenberg–Stevens method as first linking it to a derivative of a Hecke eigenvalue and then appealing to analytic properties of several-variable p-adic L-functions to connect with the analytic $\mathcal{L}$-invariant. In the case of a p-ordinary motive M, Greenberg used an in-depth study of the ordinary filtration to conjecture an arithmetic formula for the $\mathcal{L}$-invariant of M in terms of its Galois cohomology ([Gre94, Equation (23)]). In [Ben11], Benois generalized this to the non-ordinary setting by passing to the category of (φ, Γ)-modules over the Robba ring and using triangulations. This article aims to establish the first step of the Greenberg–Stevens method for symmetric powers of Hilbert modular forms with respect to the Greenberg–Benois arithmetic $\mathcal{L}$-invariant. The second step being of a different nature (and the corresponding p-adic L-functions not generally known to exist!); in the case of symmetric squares, Dasgupta ([Das14]) carried out the second step using earlier variants of Theorem A from the work of Hida and the first author.

Before presenting the main results of this article, we introduce the trivial zero conjecture as formulated by Benois in [Ben11]. Let p be an odd prime and let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_p)$ be a continuous Galois representation which is unramified at all but finitely many primes, and semistable at p. One has an L-function $L(\rho, s) = \prod_\ell L_\ell(\rho|_{G_{F_\ell}}, s)$ where $L_\ell(\rho|_{G_{\mathbb{Q}_\ell}}, s) = \det(1 - \mathrm{Frob}_\ell\, \ell^{-s} | \rho^{I_\ell})^{-1}$ for $\ell \neq p$ (here Frob_ℓ is the geometric Frobenius) while $L_p(\rho|_{G_{\mathbb{Q}_p}}, s) = \det(1 - \varphi p^{-s} | \mathrm{D}_{\mathrm{cris}}(\rho|_{G_{\mathbb{Q}_p}}))^{-1}$. It is conjectured that $L(V, s)$ has meromorphic continuation to $\mathbb{C}$ and that there exist Gamma factors $\Gamma(\rho, s)$ and $\Gamma(\rho^*(1), s)$ such that $\Gamma(\rho, s)L(\rho, s) = \varepsilon(\rho, s)\Gamma(\rho^*(1), -s)L(\rho^*(1), -s)$ for an epsilon factor $\varepsilon(\rho, s)$ of the form $A \cdot B^s$. If $D \subset \mathrm{D}_{\mathrm{st}}(\rho|_{G_{\mathbb{Q}_p}})$ is a regular submodule (see §1.2), it is expected that there exists an analytic p-adic L-function $L_p(\rho, D, s)$ such that

$$L_p(\rho, D, 0) = \mathcal{E}(\rho, D) \frac{L(\rho, s)}{\Omega_\infty(\rho)}$$

where $\mathcal{E}(\rho, D)$ is a product of Euler-like factors and $\Omega_\infty(\rho)$ is a transcendental period.

Conjecture 1 ([Ben11, Trivial Zero Conjecture]). *If ρ is critical, $L(\rho, 0) \neq 0$, $L_p(\rho, D, s)$ has order of vanishing e at $s = 0$, and D satisfies the conditions of* [Ben11, §2.2.6] *then*

$$\lim_{s \to 0} \frac{L_p(\rho, D, s)}{s^e} = (-1)^e \mathcal{L}(\rho, D) \mathcal{E}^+(\rho, D) \frac{L(\rho, 0)}{\Omega_\infty(\rho)}$$

where $\mathcal{E}^+(\rho, D)$ is defined in [Ben11, §2.3.2] *and $\mathcal{L}(\rho, D)$ is the arithmetic $\mathcal{L}$-invariant defined in* [Ben11, §2.2.2].

When F is a number field and $\rho : G_F \to \mathrm{GL}(n, \overline{\mathbb{Q}}_p)$ is a geometric Galois representation one may still formulate the above conjecture for $\mathrm{Ind}_F^{\mathbb{Q}} \rho$ and a collection of regular submodules $D_v \subset \mathrm{D}_{\mathrm{st}}(\rho|_{G_{F_v}})$ for places $v \mid p$ at least in the case when p is unramified in F. In that case, it is natural to follow Hida and define the arithmetic $\mathcal{L}$-invariant as $\mathcal{L}(\rho, \{D_v\}) = \mathcal{L}(\mathrm{Ind}_F^{\mathbb{Q}}, D)$ where $D = \bigoplus D_v$ is a regular submodule of $\mathrm{D}_{\mathrm{st}}((\mathrm{Ind}_F^{\mathbb{Q}} \rho)|_{G_{\mathbb{Q}_p}})$.

We now describe the main results of this article. Let F be a totally real field which is unramified at all places $v \mid p$ and let π be a cohomological Hilbert modular form (cf. §2.1). Let $V_{2n} = \mathrm{Sym}^{2n} \rho_{\pi,p} \otimes \det^{-n} \rho_{\pi,p}$ where $\rho_{\pi,p}$ is the p-adic representation of G_F attached to π. In the following, when computing $\mathcal{L}$-invariants of V_{2n}, we will assume that the Bloch–Kato Selmer group $H^1_f(F, V_{2n})$ vanishes (when $n = 1$ this is mostly known; see the recent article of P. Allen [All14] for up-to-date results on the subject) and technical condition (C4) from §1.2. We will also assume that π_v is Iwahori spherical at $v \mid p$. Fixing a basis e_1, e_2 for $\mathrm{D}_{\mathrm{st}}(\rho_{\pi,p}|_{G_{F_v}})$, we fix a regular submodule $D_v \subset \mathrm{D}_{\mathrm{st}}(V_{2n}|_{G_{F_v}})$ as in §2.2. We remark now that the most difficult case occurs when V_{2n} is crystalline at v and this article provides the first results in this case when $n > 3$ (in the semistable cases, the result is either quite easy or, at least in the p-ordinary case, follows from the work of Hida [Hid07]).

We begin with the case of V_2. Suppose that if π_v for $v \mid p$ is an unramified principal series then the Satake parameters are distinct. Further suppose that the Hilbert eigenvariety $\mathcal{E}$ around π and the refinement of π_v giving the ordering e_1, e_2 is étale over the weight space around the point corresponding to π (cf. §4.1). Finally, let a_v be the analytic Hecke eigenvalue corresponding to the double coset $[\mathrm{Iw}\,\mathrm{diag}(p,1)\,\mathrm{Iw}]$.

Theorem A (Theorem 28)**.** *Writing a'_v for the derivative in the direction $(1,\dots,1;-1)$ in the weight space, we have*

$$\mathcal{L}(V_2, \{D_v\}) = \prod_{v \mid p} \left(\frac{-2a'_v}{a_v} \right)$$

Such a formula for p-ordinary elliptic modular forms was obtained first by Hida [Hid04] and then by the first author [Har09] (under the assumption $H^1_f(\mathbb{Q}, V_2) = 0$ used here). In [Mok12], Mok proved this result for finite slope modular forms over $\mathbb{Q}$.

For higher symmetric powers, we must assume π is not CM, so that certain of its symmetric powers are actually cuspidal. When $F = \mathbb{Q}$, the CM case has been dealt with in [Har13] and [HL14].

For the case of V_6, we follow [Har09, Har12] and use the Ramakrishnan–Shahidi lift Π of $\mathrm{Sym}^3 \pi$ to $\mathrm{GSp}(4)$. Suppose that π is not CM and that if π_v for $v \mid p$ is an unramified principal series then the ratio of the Satake parameters is not in μ_{60} (this condition is necessary for the existence of global triangulations). Further suppose that the genus 2 Siegel–Hilbert eigenvariety $\mathcal{E}$ around Π and the p-stabilization of Π_v giving the ordering e_1, e_2 is étale over the weight space around the point corresponding to Π (cf. §4.2). Finally, let $a_{v,1}$ and $a_{v,2}$ be the analytic Hecke eigenvalues corresponding to the double cosets $[\mathrm{Iw}\,\mathrm{diag}(1, p^{-1}, p^{-2}, p^{-1})\,\mathrm{Iw}]$ and $[\mathrm{Iw}\,\mathrm{diag}(1, 1, p^{-1}, p^{-1})\,\mathrm{Iw}]$.

Theorem B (Theorem 29)**.** *If $\overrightarrow{u} = (u_1, u_2; u_0)$ is any direction in the weight space, i.e. $u_1 \geq u_2 \geq 0$, then*

$$\mathcal{L}(V_6, \{D_v\}) = \prod_{v \mid p} \left(\frac{-4\widetilde{\nabla}_{\overrightarrow{u}} a_{v,2} + 3\widetilde{\nabla}_{\overrightarrow{u}} a_{v,1}}{u_1 - 2u_2} \right)$$

where we write $\widetilde{\nabla}_{\overrightarrow{u}} f = (\nabla_{\overrightarrow{u}} f)/f$ for the logarithmic directional derivative of f evaluated at the point above Π.

This generalizes the main result of [Har09, Har12] which computes the arithmetic $\mathcal{L}$-invariant of V_6 in the case of p-ordinary elliptic modular forms.

The first computation of $\mathcal{L}$-invariants of V_{2n} for general n we present uses symplectic eigenvarieties and is, for now, conditional on the stabilization of the twisted trace formula (this is necessary for the construction of an analytic Galois representation). Suppose π is not CM, and that for $v \mid p$ such that π_v is unramified the ratio of the Satake parameters is not in μ_∞ (again, necessary for the existence of global triangulations). Suppose π satisfies the hypotheses of Theorem 8 (2) and let Π be a suitable twist of the cuspidal representation of $\mathrm{GSp}(2n, \mathbb{A}_F)$ from Theorem 11. Let $\mathcal{E}$ be Urban's eigenvariety for $\mathrm{GSp}(2n)$ and let $a_{v,i}$ be the analytic Hecke eigenvalues from the proof of Lemma 20. Suppose that the eigenvariety $\mathcal{E}$ is étale over the weight space at the p-stabilization of Π corresponding to the ordering e_1, e_2.

Theorem C (Theorem 30)**.** *If $\overrightarrow{u} = (u_1, \dots, u_n; u_0)$ is any direction in the weight space, then*

$$\mathcal{L}(V_{4n-2}, \{D_v\}) = \prod_{v \mid p} - \left(\frac{B_n \widetilde{\nabla}_{\overrightarrow{u}} a_{v,1} + B_1(\widetilde{\nabla}_{\overrightarrow{u}} a_{v,n-1} - 2\widetilde{\nabla}_{\overrightarrow{u}} a_{v,n}) + \sum_{i=2}^{n-1} B_i(\widetilde{\nabla}_{\overrightarrow{u}} a_{v,i-1} - \widetilde{\nabla}_{\overrightarrow{u}} a_{v,i})}{\sum_{i=1}^{n} u_i B_{n+1-i}} \right)$$

where we write $B_i = (-1)^i \binom{2n}{n+i} i$.

As mentioned earlier, the results of Hida [Hid07] address the case where V_{2n} is ordinary and semistable but not crystalline.

Our second computation of $\mathcal{L}$-invariants for V_{2n} uses unitary groups, is also conditional on the stabilization of the twisted trace formula, and is more restrictive. It however has the advantage that the work of Eischen–Harris–Li–Skinner [EHLS16] provides several-variable p-adic L-functions for Hida families on unitary groups and thus the second step of the Greenberg–Stevens method may be closer at hand. As above suppose π is not CM, and that for $v \mid p$ such that π_v is unramified the ratio of the Satake parameters is not in μ_∞. Assume Conjecture 12 and suppose π satisfies the hypotheses of Theorem 8 (2) and Proposition 13 (the latter requires π_v to be special at two finite places not above p). Let E/F the CM extension and Π the cuspidal representation of $U_{4n}(\mathbb{A}_F)$ to which there is a transfer of a twist of $\mathrm{Sym}^{4n-1}\pi$ as in Proposition 13. Let $\mathcal{E}$ be Chenevier's eigenvariety and let $a_{v,i}$ be the analytic Hecke eigenvalues from the proof of Corollary 25. Suppose that the eigenvariety $\mathcal{E}$ is étale over the weight space at the p-stabilization of Π corresponding to the ordering e_1, e_2.

Theorem D (Theorem 32)**.** *If $\overrightarrow{u} = (u_1, \ldots, u_n; u_0)$ is any direction in the weight space, then*

$$\mathcal{L}(V_{8n-2}, \{D_v\}) = \prod_{v|p} \left(\frac{-\sum_{i=1}^{4n} (-1)^{i-1} \binom{4n-1}{i-1} \widetilde{\nabla}_{\overrightarrow{u}} a_{v,i}}{\sum_{i=1}^{4n} (-1)^{i-1} \binom{4n-1}{i-1} u_i} \right)$$

and

$$\mathcal{L}(V_{8n-6}, \{D_v\}) = \prod_{v|p} \left(\frac{-\sum_{i=1}^{4n} B_{i-1} \widetilde{\nabla}_{\overrightarrow{u}} a_{v,i}}{\sum_{i=1}^{4n} u_i B_{i-1}} \right).$$

Here $B_i = B_{4n-1,4n-3,i}$ is the inverse Clebsch–Gordan coefficient of Proposition 34, up to a scalar independent of i given by

$$B_i = (-1)^i \binom{4n-1}{i} ((4n-1)^3 - (4i+1)(4n-1)^2 + (4i^2+2i)(4n-1) - 2i^2).$$

Remark 2. The assumption that π satisfy the hypotheses of Theorem 8 (2), i.e. that various $\mathrm{Sym}^k \pi$ be automorphic over F is necessary for our computations. Ongoing work of Clozel and Thorne provides the automorphy of such symmetric powers when k is small.

Potential automorphy results would allow us to remove this condition and we would get formulae similar to those in Theorem C and D without the assumption of modularity of symmetric powers. However, we would need to assume the fact that after a base change to a field where the relevant symmetric power is modular the eigenvariety is etale over the weight space.

Remark 3. Some of the results in this paper have been used by G. Rosso to obtain similar formulae for arithmetic $\mathcal{L}$-invariants for Siegel–Hilbert modular forms and their adjoint representations in full monodromy rank case.

The paper is organized as follows. In Section 1, we describe Benois' definition of the arithmetic $\mathcal{L}$-invariant and triangulations in p-adic families. In Section 2, we study Galois representations attached to Hilbert modular forms and functorial transfers to unitary and symplectic groups. Then, in Section 3, we describe the unitary and symplectic eigenvarieties and global triangulations of certain analytic Galois representations. In section 4, we prove Theorems A through D. Finally, the appendix discusses some plethysm for GL(2), relating the $B_{n,k,i}$ to inverse Clebsch–Gordan coefficients and proving an explicit formula for them.

Some basic notation

Throughout this article, p denotes a fixed odd prime. We will use L to denote a finite extension of $\mathbb{Q}_p$. By a p-adic representation of a group G, we mean a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is a finite-dimensional vector space over L. Let μ_{p^∞} denote the set of p-power roots of unity and let χ denote the p-adic cyclotomic character giving the action of whatever appropriate Galois group on it. Let Γ_K denote the Galois group $\mathrm{Gal}(K(\mu_{p^\infty})/K) \cong \mathbb{Z}_p^\times$ and let γ denote a fixed topological generator of Γ_K. If F is a field, G_F denotes its absolute Galois group and $H^\bullet(F, -) = H^\bullet(G_F, -)$ denotes it Galois cohomology. We fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ which gives an identification between the set of infinite places $F \hookrightarrow \mathbb{C}$ and pairs (v, τ) where $v \mid p$ and τ is an embedding $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p$.

1. (φ, Γ)-modules and $\mathcal{L}$-invariants

This section contains a review of some pertinent content from [Ben11] and [Liu15]. We refer the reader to these articles for further details.

1.1. (φ, Γ)**-modules over the Robba ring.** Let $K/\mathbb{Q}_p$ be a finite extension and denote by K' the maximal unramified subextension of $K(\zeta_{p^\infty})/K_0$ where $K_0 = K \cap \mathbb{Q}_p^{\mathrm{ur}}$. For a real number r with $0 \leq r < 1$ denote by $\mathcal{R}_K^r$ the set of power series $f(x) = \sum_{k\in\mathbb{Z}} a_k x^k$ holomorphic for $r \leq |x|_p < 1$ with $a_k \in K'$. Let $\mathcal{R}_K = \bigcup_{r<1} \mathcal{R}_K^r$ be the *Robba ring* (over K).

The Robba ring carries natural actions of a Frobenius, φ_K, and Γ_K. These actions act semilinearly on the coefficients of $f \in \mathcal{R}_K$, $\varphi(x) = (1+x)^p - 1$ and for $\tau \in \Gamma$, $\tau(x) = (1+x)^{\chi(\tau)} - 1)$.

More generally, if S is an affinoid algebra over $\mathbb{Q}_p$, define $\mathcal{R}_{K,S} = \mathcal{R}_K \widehat{\otimes}_{\mathbb{Q}_p} S$ (cf. [Liu15, Proposition 3.5]). Extending the actions of $\varphi_K \otimes 1$ and $\tau \otimes 1$ linearly, we get actions of (φ_K, Γ_K) on $\mathcal{R}_{K,S}$; i.e. this should be thought of as the Robba ring over K with coefficients in S.

For a finite extension $L/\mathbb{Q}_p$, a (φ, Γ_K)-module over $\mathcal{R}_{K,L}$ is a free $\mathcal{R}_{K,L}$-module D_L of finite rank, equipped with a φ-semilinear Frobenius map φ_{D_L} and a semilinear action of Γ_K which commute with each other, such that the induced map $\varphi_{D_L}^* D_L = D_L \otimes_\varphi \mathcal{R}_{K,L} \to D_L$ is an isomorphism. More generally, a (φ, Γ_K)-module over $\mathcal{R}_{K,S}$ is a vector bundle D_S (coherent, locally free sheaf) over $\mathcal{R}_{K,S}$ of finite rank, equipped with a semilinear Frobenius, φ_{D_S}, and a semilinear action of Γ_K, commuting with each other, such that $\varphi_{D_S}^* D_S \to D_S$ is an isomorphism.

If $\delta : K^\times \to S^\times$ is a continuous character, one can define a rank 1 (φ, Γ_K)-module $\mathcal{R}_S(\delta)$ as in [Liu15, §1.2]. For example, when $K = \mathbb{Q}_p$ the rank 1 (φ, Γ)-module is $\mathcal{R}_{\mathbb{Q}_p,S} e_\delta$ with basis e_δ with $\varphi_{\mathcal{R}_{\mathbb{Q}_p,S}(\delta)}(x e_\delta) = \varphi_{\mathcal{R}_{\mathbb{Q}_p,S}}(x)\delta(p) e_\delta$ and for $\tau \in \Gamma$, $\tau(x e_\delta) = \tau(x)\delta(\chi(\tau)) e_\delta$.

A (φ, Γ_K)-module D_S is said to be *trianguline* if there exists an increasing (separated, exhaustive) filtration $\mathrm{Fil}_\bullet D_S$ such that the graded pieces are of the form $\mathcal{R}_{K,S}(\delta) \otimes_S M$ for some continuous $\delta : K^\times \to S^\times$ and locally free one-dimensional M over S with trivial (φ, Γ_K) actions.

There is a functor $D^\dagger_{\mathrm{rig},L}$ associating to an L-linear continuous representation of G_K a (φ, Γ_K)-module over $\mathcal{R}_{K,L}$ and more generally a functor $D^\dagger_{\mathrm{rig},S}$ associating to an S-linear continuous G_K-representation a (φ, Γ_K)-module over $\mathcal{R}_{K,S}$. The functor $D^\dagger_{\mathrm{rig},L}$ induces an isomorphism of categories between the category of L-linear continuous representations of G_K and slope 0 (φ, Γ_K)-modules over $\mathcal{R}_{K,L}$.

There exist functors $\mathcal{D}_{\mathrm{cris}}$ (resp. $\mathcal{D}_{\mathrm{st}}$) attaching to a (φ, Γ_K)-module D over $\mathcal{R}_{K,L}$ a filtered φ-module (resp. (φ, N)-module) over K_0 with coefficients in L such that if V is crystalline (resp. semistable) then $\mathcal{D}_{\mathrm{cris}}(D^\dagger_{\mathrm{rig},L}(V)) \cong D_{\mathrm{cris}}(V)$ (resp. $\mathcal{D}_{\mathrm{st}}(D^\dagger_{\mathrm{rig},L}(V)) \cong D_{\mathrm{st}}(V)$).

Suppose V is a finite-dimensional L-linear continuous representation of G_K and $D = D^\dagger_{\mathrm{rig},L}(V)$ is the associated (φ, Γ_K)-module over $\mathcal{R}_{K,L}$. It is a theorem of Ruochuan Liu ([Liu08]) that the Galois cohomology $H^\bullet(K, V)$ can be computed as the cohomology $H^\bullet(D)$ of the Herr complex $0 \to D \xrightarrow{f} D \oplus D \xrightarrow{g} D \to 0$ where the transition maps are $f(x) = (\varphi_D - 1)x \oplus (\gamma - 1)x$ and $g(x, y) = (\gamma - 1)x - (\varphi_D - 1)y$. We denote by $\mathrm{cl}(x, y)$ the image of $x \oplus y \in D \oplus D$ in $H^1(D)$.

The Bloch–Kato local conditions $H^1_f(K, V) = \ker\left(H^1(K, V) \to H^1(K, V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})\right)$ and $H^1_{\mathrm{st}}(K, V)$ (analogously defined) can also be computed directly using Herr's complex. Indeed, if $\alpha = a \oplus b \in D \oplus D$ one gets an extension $D_\alpha = D \oplus \mathcal{R}_{K,L} e$, depending only on $cl(a, b)$, endowed with Frobenius and Γ_K-action defined by $(\varphi_D - 1)(0 \oplus e) = a \oplus 0$ and $(\gamma - 1)(0 \oplus e) = b \oplus 0$. Let $H^1_f(D)$ be the set of crystalline extensions D_α, i.e. those satisfying $\dim_{K_0} \mathcal{D}_{\mathrm{cris}}(D_\alpha) = \dim_{K_0} \mathcal{D}_{\mathrm{cris}}(D) + 1$ and $H^1_{\mathrm{st}}(D)$ be the set of semistable extensions (defined analogously). Then

$$H^1_f(K, V) \cong H^1_f(\mathrm{D}^\dagger_{\mathrm{rig}}(V)) \quad \text{and} \quad H^1_{\mathrm{st}}(K, V) \cong H^1_{\mathrm{st}}(\mathrm{D}^\dagger_{\mathrm{rig}}(V)).$$

We end with the following computation of Benois ([Ben11, Proposition 1.5.9]) in the special case when $K = \mathbb{Q}_p$. Let $\delta : \mathbb{Q}_p^\times \to L^\times$ be the character $\delta(x) = x^{-k}$ where $k \in \mathbb{Z}_{\geq 0}$. The rank 1 module $\mathcal{R}_L(\delta)$ is crystalline and so $\mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)) = \mathcal{R}_L(\delta)^\Gamma \subset \mathcal{R}_L(\delta)$. This allows us to define the map

$$i : \mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)) \oplus \mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)) \to H^1(\mathcal{R}_L(\delta))$$

by $i(x, y) = \mathrm{cl}(-x, y \log \chi(\gamma))$. Then i is an isomorphism and $H^1_f(\mathcal{R}_L(\delta)) \cong i(\mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)) \oplus 0)$. Moreover, defining $H^1_c(\mathcal{R}_L(\delta)) = i(0 \oplus \mathcal{D}_{\mathrm{cris}}(\mathcal{R}_L(\delta)))$, both $H^1_f(\mathcal{R}_L(\delta))$ and $H^1_c(\mathcal{R}_L(\delta))$ have rank 1.

1.2. **Regular submodules and the Greenberg–Benois $\mathcal{L}$-invariant.**

1.2.1. *$\mathcal{L}$-invariants over* $\mathbb{Q}$. (cf. [Ben11, §§2.1–2.2]) Let $\rho : G_\mathbb{Q} \to \mathrm{GL}(V)$ be a "geometric" p-adic representation of $G_\mathbb{Q}$, i.e. ρ is unramified outside a finite set of places and it is potentially semistable at p. Let S be a finite set of places containing the ramified ones as well as p and ∞. We give an overview of Benois' definition of the arithmetic $\mathcal{L}$-invariant of V. His definition requires five additional assumptions (C1–5) on V described below. We also discuss $\mathcal{L}$-invariants of representations of G_F, where F is a number field. We end by proving a lemma we later use in our computations of $\mathcal{L}$-invariants.

For $\ell \nmid p\infty$, define $H^1_f(\mathbb{Q}_\ell, V) = \ker(H^1(\mathbb{Q}_\ell, V) \to H^1(I_\ell, V))$, where I_ℓ denotes the inertia subgroup of $G_{\mathbb{Q}_\ell}$. When $\ell = p$, the Bloch–Kato local condition $H^1_f(\mathbb{Q}_p, V)$ was defined in the previous section. Finally, let $H^1_f(\mathbb{R}, V) = H^1(\mathbb{R}, V)$. Let $G_{\mathbb{Q},S} = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$, where $\mathbb{Q}_S$ is the maximal extension of $\mathbb{Q}$ unramified outside of S. Define the Bloch–Kato Selmer group of V as

$$H^1_f(V) = \ker\left(H^1(G_{\mathbb{Q},S}, V) \to \bigoplus_{v\in S} H^1(\mathbb{Q}_v, V)/H^1_f(\mathbb{Q}_v, V)\right),$$

which does not depend on the choice of S.

As in [Ben11, §2.1.2] we will assume:

(C1) $H^1_f(V) = H^1_f(V^*(1)) = 0$;
(C2) $H^0(G_{\mathbb{Q},S}, V) = H^0(G_{\mathbb{Q},S}, V^*(1)) = 0$;
(C3) $V|_{G_{\mathbb{Q}_p}}$ is semistable and the semistable Frobenius φ is semisimple at 1 and p^{-1};
(C4) $\mathrm{D}^\dagger_{\mathrm{rig}}(V|_{G_{\mathbb{Q}_p}})$ has no saturated subquotient isomorphic to some $U_{k,m}$ for $k \geq 1$ and $m \geq 0$ (cf. [Ben11, §2.1.2], where $U_{k,m}$ is the (unique) non-split crystalline (φ, Γ)-module extension of $\mathcal{R}(x^{-m})$ by $\mathcal{R}(|x|x^k)$).

A *regular submodule* D of $\mathrm{D}_{\mathrm{st}}(V)$ is a (φ, N)-submodule such that $D \cong \mathrm{D}_{\mathrm{st}}(V)/\mathrm{Fil}^0\,\mathrm{D}_{\mathrm{st}}(V)$ (as vector spaces) under the natural projection map. Given a regular submodule D, Benois constructs the filtration

$$\begin{aligned} D_{-1} &= (1 - p^{-1}\varphi^{-1})D + N(D^{\varphi=1}) \\ D_0 &= D \\ D_1 &= D + \mathrm{D}_{\mathrm{st}}(V)^{\varphi=1} \cap N^{-1}(D^{\varphi=p^{-1}}) \end{aligned}$$

The filtration $D_\bullet$ on $\mathrm{D}_{\mathrm{st}}(V)$ gives a filtration $F_\bullet\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)$ by setting

$$F_i\,\mathrm{D}^\dagger_{\mathrm{rig}}(V) = \mathrm{D}^\dagger_{\mathrm{rig}}(V) \cap (D_i \otimes_{\mathbb{Q}_p} \mathcal{R}_L[1/t])$$

(here $t = \log(1 + x) \in \mathcal{R}_L$).

Define the *exceptional subquotient* of V to be $W = F_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)/F_{-1}\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)$, which is a (φ, Γ) analogue of Greenberg's F^{00}/F^{11} (see [Gre94, p. 157]). Benois shows there are unique decompositions

$$\begin{aligned} W &\cong W_0 \oplus W_1 \oplus M \\ \mathrm{gr}_0\,\mathrm{D}^\dagger_{\mathrm{rig}}(V) &\cong W_0 \oplus M_0 \\ \mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V) &\cong W_1 \oplus M_1 \end{aligned}$$

such that W_0 has rank $\dim H^0(W^*(1))$, W_1 has rank $\dim H^0(W)$. Moreover, M_0 and M_1 have equal rank and the sequence $0 \to M_0 \xrightarrow{f} M \xrightarrow{g} M_1 \to 0$ is exact.

One has

$$\begin{aligned} H^1(W) &= \mathrm{coker}(H^1(F_{-1}\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)) \to H^1(F_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))), \\ H^1_f(W) &= \mathrm{coker}(H^1_f(F_{-1}\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)) \to H^1_f(F_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))), \end{aligned}$$

and $H^1(W)/H^1_f(W)$ has dimension $e_D = \mathrm{rk}\,M_0 + \mathrm{rk}\,W_0 + \mathrm{rk}\,W_1$.

The (dual of the) Poitou–Tate exact sequence gives an exact sequence

$$0 \to H^1_f(V) \to H^1(G_{\mathbb{Q},S}, V) \to \bigoplus_{v\in S} \frac{H^1(\mathbb{Q}_v, V)}{H^1_f(\mathbb{Q}_v, V)} \to H^1_f(V^*(1))^\vee$$

where $V^* = \mathrm{Hom}(V, \mathbb{Q}_p)$ and $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. Assumptions (C1) and (C2) above imply that

$$H^1(G_{\mathbb{Q},S}, V) \cong \bigoplus_{v\in S} \frac{H^1(\mathbb{Q}_v, V)}{H^1_f(\mathbb{Q}_v, V)}$$

Note that $\bigoplus_{v\in S}\frac{H^1(\mathbb{Q}_v,V)}{H^1_f(\mathbb{Q}_v,V)}$ contains the e_D-dimensional subspace $\frac{H^1(W)}{H^1_f(W)}\cong\frac{H^1(F_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V_p))}{H^1_f(\mathbb{Q}_p,V_p)}$. Define $H^1(D,V)\subset H^1(G_{\mathbb{Q},S},V)$ to be the set of classes whose image in $\frac{H^1(\mathbb{Q}_p,V)}{H^1_f(\mathbb{Q}_p,V)}$ lies in $\frac{H^1(W)}{H^1_f(W)}$.

From now on, assume

(C5) $W_0=0$.

Since φ acts as 1 on $\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)$, [Ben11, Proposition 1.5.9] implies, assuming that the Hodge–Tate weights are nonnegative, that $\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)\cong\oplus\mathcal{R}_L(x^{-k_i})$ where the $k_i\geq 0$ are the Hodge–Tate weights. Thus one obtains a decomposition $H^1(\mathrm{D}^\dagger_{\mathrm{rig}}(V))\cong H^1_f(\mathrm{D}^\dagger_{\mathrm{rig}}(V))\oplus H^1_c(\mathrm{D}^\dagger_{\mathrm{rig}}(V))$ by summing the decompositions for each $\mathcal{R}_L(x^{-k_i})$; as in the rank 1 case, $H^1_f(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))\cong H^1_c(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))\cong\mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))$. There exist linear maps $\rho_{D,?}:H^1(D,V)\to\mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))$ for $?\in\{f,c\}$ making the following diagram commute:

$$\begin{array}{ccc}
\mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)) & \xrightarrow[\iota_f]{\cong} & H^1_f(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))\\
\uparrow{\scriptstyle \rho_{D,f}} & & \uparrow{\scriptstyle \pi_f}\\
H^1(D,V) & \longrightarrow & H^1(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))\\
\downarrow{\scriptstyle \rho_{D,c}} & & \downarrow{\scriptstyle \pi_c}\\
\mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V)) & \xrightarrow[\iota_c]{\cong} & H^1_c(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))
\end{array}$$

Under the assumption that $W_0=0$, Benois shows that the linear map $\rho_{D,c}$ is invertible and defines the arithmetic $\mathcal{L}$-invariant as

$$\mathcal{L}(V,D):=\det\left(\rho_{D,f}\circ\rho_{D,c}^{-1}\big|\mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(V))\right).$$

1.2.2. *$\mathcal{L}$-invariants over number fields.* Let F be a number field in which p is unramified. Following Hida we may define arithmetic $\mathcal{L}$-invariants of certain p-adic representations $\rho:G_F\to\mathrm{GL}(V)$ by inducing to $\mathbb{Q}$.

Suppose the p-adic representation ρ is unramified almost everywhere and that at all $v\mid p$, $\rho|_{G_{F_v}}$ is semistable. Assume that ρ satisfies conditions (C1–2) (with appropriate modifications) and that $\mathrm{Ind}_F^{\mathbb{Q}}\rho$ satisfies conditions (C3-4). (That the representation $\mathrm{Ind}_F^{\mathbb{Q}}\rho$ is semistable at p already implies that ρ is semistable at $v\mid p$.) These assumptions imply that $\mathrm{Ind}_F^{\mathbb{Q}}\rho$ satisfies conditions (C1–4), by Shapiro's lemma.

For $v\mid p$ denote $V_v:=V|_{G_{F_v}}$. Since

$$(\mathrm{Ind}_F^{\mathbb{Q}}V)|_{G_{\mathbb{Q}_p}})\cong\oplus_{v|p}\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}V_v$$

it suffices to study each induction $\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}V_v$ separately. Note that $\mathrm{D}_{\mathrm{st}}(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}V_v)$ as a $\mathbb{Q}_p$ vector space is simply $\mathrm{D}_{\mathrm{st}}(V_v)$ (which is naturally an $F_v\cap\mathbb{Q}_p^{\mathrm{ur}}$ vector space) taken as a $\mathbb{Q}_p$ vector space, with induced φ and N.

Choose $D_v\subset\mathrm{D}_{\mathrm{st}}(V_v)$ a regular submodule, by which we mean a (φ,N)-submodule such that D_v maps isomorphically onto the de Rham tangent space $t_{V_v}(F_v)$. Since we have assume that $F_v/\mathbb{Q}_p$ is unramified D_v, as a $\mathbb{Q}_p$-vector space, is a regular submodule of $\mathrm{D}_{\mathrm{st}}(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}V_v)$. Benois' construction yields the modules $W_{0,v}$, $W_{1,v}$ and M_v for each $v\mid p$. Then $D=\oplus_{v|p}D_v\subset\oplus_{v|p}\mathrm{D}_{\mathrm{st}}(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}V_v)\cong\mathrm{D}_{\mathrm{st}}((\mathrm{Ind}_F^{\mathbb{Q}}V)|_{G_{\mathbb{Q}_p}})$ is a regular submodule and $W_0=\oplus_{v|p}W_{0,v}$, $W_1=\oplus_{v|p}W_{1,v}$ and $M=\oplus_{v|p}M_v$. Assuming $W_{0,v}=0$ for every $v\mid p$ yields $W_0=0$ and we may define

$$\mathcal{L}(\{D_v\},V)=\mathcal{L}(D,\mathrm{Ind}_F^{\mathbb{Q}}V)$$

Note that $\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}((\mathrm{Ind}_F^{\mathbb{Q}}V)|_{G_{\mathbb{Q}_p}})\cong\bigoplus_{v|p}\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}V_v)$. We remark that even if v is ramified, by [KPX14, Lemma 2.2.18], $\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}V_v)\cong\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}\mathrm{D}^\dagger_{\mathrm{rig}}(V_v)$ where for a (φ,Γ_{F_v})-module D over $\mathcal{R}_{F_v,L}$, $\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}D$ is the $\Gamma_{\mathbb{Q}_p}$-module $\mathrm{Ind}_{\Gamma_{F_v}}^{\Gamma_{\mathbb{Q}_p}}D$ with the inherited action of φ.

Lemma 4. *Suppose that* $\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p}V_v)\cong\mathcal{R}$ *for each* $v\mid p$. *Let* $H^1(\mathcal{R})\cong H^1_f(\mathcal{R})\oplus H^1_c(\mathcal{R})$ *with basis* $x=(-1,0)$ *and* $y=(0,\log_p\chi(\gamma))$. *Suppose* $c\in H^1(D,\mathrm{Ind}_F^{\mathbb{Q}}V)$ *is such that the image of* c *in*

$H^1(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}^{\mathbb{Q}_p}_{F_v} V_v))$ *is* $\xi_v = a_v x + b_v y$ *with* $b_v \neq 0$. *Then*

$$\mathcal{L}(\{D_v\}, V) = \prod_{v|p} \frac{a_v}{b_v}$$

Proof. Since

$$H^1(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}((\mathrm{Ind}^{\mathbb{Q}}_F V)|_{G_{\mathbb{Q}_p}})) \cong \bigoplus_{v|p} H^1(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}^{\mathbb{Q}_p}_{F_v} V_v))$$

and

$$\mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}((\mathrm{Ind}^{\mathbb{Q}}_F V)|_{G_{\mathbb{Q}_p}})) \cong \bigoplus_{v|p} \mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}^{\mathbb{Q}_p}_{F_v} V_v)),$$

the maps ι_f, ι_c, π_f, and π_c are direct sums of the maps

$$\iota_{f,v} : \mathcal{D}_{\mathrm{cris}}(\mathcal{R}) \cong H^1_f(\mathcal{R}), \quad \iota_{c,v} : \mathcal{D}_{\mathrm{cris}}(\mathcal{R}) \cong H^1_c(\mathcal{R}),$$

$$\pi_{f,v} : H^1(\mathcal{R}) \to H^1_f(\mathcal{R}) \quad \text{and} \quad \pi_{c,v} : H^1(\mathcal{R}) \to H^1_c(\mathcal{R}).$$

Let $\xi_v = a_v x + b_v y$ be the image of c in $H^1(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}^{\mathbb{Q}_p}_{F_v} \rho_v)) \cong H^1(\mathcal{R})$. We deduce that the maps ρ_f (resp. ρ_c) are direct sums of the $\rho_{f,v} = \iota^{-1}_{f,v} \circ \pi_{f,v}$ (resp. $\rho_{c,v} = \iota^{-1}_{c,v} \circ \pi_{c,v}$). Then $a_v \in \mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}^{\mathbb{Q}_p}_{F_v} V_v))$ is the image of ξ_v under ρ_f and $b_v \in \mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}^{\mathbb{Q}_p}_{F_v} \rho_v))$ is the image of ξ_v under ρ_c. We deduce that

$$\begin{aligned}\mathcal{L}(\{D_v\}, V) &= \det(\rho_f \circ \rho_c^{-1} | \mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}((\mathrm{Ind}^{\mathbb{Q}}_F V)|_{G_{\mathbb{Q}_p}}))) \\ &= \prod_{v|p} \det(\rho_{f,v} \circ \rho_{c,v}^{-1} | \mathcal{D}_{\mathrm{cris}}(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}^{\mathbb{Q}_p}_{F_v} V_v))) \\ &= \prod_{v|p} \frac{a_v}{b_v}\end{aligned}$$

□

1.3. **Refined families of Galois representations.** Lemma 4 provides a framework for computing arithmetic $\mathcal{L}$-invariants as long as one is able to produce cohomology classes $c \in H^1(D, \mathrm{Ind}^{\mathbb{Q}}_F \rho)$ and compute their projections to $H^1(\mathrm{gr}_1\,\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}^{\mathbb{Q}_p}_{F_v} \rho_v))$. We will produce such cohomology classes using analytic Galois representations on eigenvarieties and we will compute explicitly the projections (in effect the a_v and b_v of Lemma 4) using global triangulations of (φ, Γ)-modules.

We recall here the main result of [Liu15, §5.3] on triangularization in refined families. Let L be a finite extension of $\mathbb{Q}_p$. Suppose X is a separated and reduced rigid analytic space over L and V_X is a locally free, coherent $\mathcal{O}_X$-module of rank d with a continuous $\mathcal{O}_X$-linear action of the Galois group G_K where $K/\mathbb{Q}_p$ is a finite unramified extension with uniformizer ϖ_K. The family V_X of Galois representations is said to be *refined* if there exist $\kappa_1, \ldots, \kappa_d \in \mathcal{O}(X) \otimes_{\mathbb{Q}_p} K$, $F_1, \ldots, F_d \in \mathcal{O}(X)$, and a Zariski dense set of points $Z \subset X$ such that:

1. for $x \in X$ and any embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$ the Hodge–Tate weights of $\mathrm{D}_{\mathrm{dR}}(V_x)_\tau$ are $\kappa_1(x)_\tau, \ldots, \kappa_d(x)_\tau$,
2. for $z \in Z$, the representation V_z is crystalline,
3. for $z \in Z$ and any embedding $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$, $\kappa_1(z)_\tau < \ldots < \kappa_d(z)_\tau$,
4. the eigenvalues of φ acting on $\mathrm{D}_{\mathrm{cris}}(V_z)$ are distinct and equal to $\{\prod_\tau \tau(\varpi_K)^{\kappa_i(z)_\tau} F_i(z)\}$,
5. for $C \in \mathbb{Z}_{\geq 0}$ the set Z_C, consisting of $z \in Z$ such that for all distinct subsets $I, J \subset \{1, \ldots, d\}$ of equal cardinality one has $|\sum_{i\in I} \kappa_i(z)_\tau - \sum_{j \in J} \kappa_j(z)_\tau| > C$, accumulates at every $z \in Z$ for every $\tau : K \hookrightarrow \overline{\mathbb{Q}}_p$,
6. for each $1 \leq i \leq d$, there exists a continuous character $\chi_i : \mathcal{O}_K^\times \to \mathcal{O}(X)^\times$ such that the derivative of χ_i at $1 \in \mathcal{O}_K^\times$ is $\kappa_i \in \mathcal{O}(X)^\times \otimes_{\mathbb{Q}_p} K$ and $z(\chi_i)(u) = \prod_\tau \tau(u)^{\kappa_i(z)_\tau}$ for all $z \in Z$.

Given a refined family V_X, we define $\Delta_i : K^\times \to \mathcal{O}(X)^\times$ by $\Delta_i(\varpi_K) = \prod_{j=1}^i F_j$ and for $u \in \mathcal{O}_K^\times$, $\Delta_i(u) = \prod_{j=1}^i \chi_j(u)$. Let $\delta_i = \Delta_i / \Delta_{i-1}$.

Let V_X be a refined family. For all $z \in Z$, there is an induced *refinement* of V_z, i.e. a filtration $0 = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \ldots \subsetneq \mathcal{F}_d = \mathrm{D}_{\mathrm{cris}}(V_z)$ of φ-submodules. It is determined by the condition that the eigenvalue of φ_{cris} on

$\mathcal{F}_i/\mathcal{F}_{i-1}$ is $\prod_\tau \tau(\varpi_K)^{\kappa_i(z)_\tau} F_i(z)$. We say that z is *noncritical* if $\mathrm{D}_{\mathrm{dR}}(V_z)_\tau = \mathcal{F}_{i,\tau} + \mathrm{Fil}^{\kappa_{i+1}(z)} \mathrm{D}_{\mathrm{dR}}(V_z)_\tau$. We say that z is *regular* if $\det\varphi$ on $\mathcal{F}_i$ has multiplicity one in $\mathrm{D}_{\mathrm{cris}}(\wedge^i V_z)$ for all i.

Then, [Liu15, Theorem 5.42] gives:

Theorem 5. *If $z \in Z$ is regular and noncritical, then in an affinoid neighborhood U of z, V_U is trianguline with graded pieces isomorphic to $\mathcal{R}_U(\delta_1), \dots, \mathcal{R}_U(\delta_d)$.*

Remark 6. In fact, [Liu15, Theorem 5.45] shows that V_x is trianguline at all $x \in X$, but the triangulation is only made explicit over a proper birational transformation of X.

2. Symmetric powers of Hilbert modular forms

2.1. **Hilbert modular forms and their Galois representations.** Let $F/\mathbb{Q}$ be a totally real field of degree d and let $I \subseteq \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$ be a parametrization of the infinite places. We fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ thus identifying I with a subset of $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$. We also fix $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ which identifies $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p)$ with $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}})$. This determines a partition $I = \bigcup_{v|p} I_v$. Let ϖ_v be a uniformizer for F_v and e_v be the ramification index of $F_v/\mathbb{Q}_p$.

Before defining Hilbert modular forms, we need some notation on representations of $\mathrm{GL}(2,\mathbb{R})$. Recall that the Weil group of $\mathbb{R}$ is $W_{\mathbb{R}} = \mathbb{C}^\times \rtimes \{1, j\}$ where $j^2 = -1$ and $jz = \overline{z}j$ for $z \in \mathbb{C}$. For an integer $n \geq 2$, let $\mathcal{D}_n$ be the essentially discrete series representation of $\mathrm{GL}(2,\mathbb{R})$ whose Langlands parameter is $\varphi_n : W_{\mathbb{R}} \to \mathrm{GL}(2,\mathbb{C})$ given by $\varphi_n(z) = \begin{pmatrix} (z/\overline{z})^{(n-1)/2} & \\ & (\overline{z}/z)^{(n-1)/2} \end{pmatrix}$ and $\varphi_n(j) = \begin{pmatrix} & 1 \\ (-1)^{n-1} & \end{pmatrix}$. The representation $\mathcal{D}_n$ is unitary and has central character sign^n and Blattner parameter n. More generally, if $t \in \mathbb{C}$, the representation $\mathcal{D}_n \otimes |\det|^t$ has associated Langlands parameter $\varphi_n \otimes |\cdot|^{2t}$. If $w \in \mathbb{Z}$, then

$$H^1(\mathfrak{gl}_2, \mathrm{SO}(2), \mathcal{D}_n(-w/2) \otimes V^\vee_{(w+n-2)/2,(w-n+2)/2}) \neq 0$$

where $V_{(a,b)} = \mathrm{Sym}^{a-b}\,\mathbb{C}^2 \otimes \det^b$ is the representation of highest weight (a,b).

By a cohomological Hilbert modular form of infinity type $(k_1, \dots, k_d, w)$, we mean a cuspidal automorphic representation π of $\mathrm{GL}(2,\mathbb{A}_F)$ such that

1. $k_i \equiv w \pmod 2$ with $k_i \geq 2$ and
2. for every $i \in I$, $\pi_i \cong \mathcal{D}_{k_i} \otimes |\det|^{-w/2}$.

This is equivalent to the fact that π_i has Langlands parameter $z \mapsto |z|^{-w} \begin{pmatrix} (z/\overline{z})^{(k_i-1)/2} & \\ & (z/\overline{z})^{-(k_i-1)/2} \end{pmatrix}$, $j \mapsto \begin{pmatrix} & 1 \\ (-1)^{k_i-1} & \end{pmatrix}$. Since $H^1(\mathfrak{gl}_2, \mathrm{SO}(2), \pi_i \otimes V^\vee_{(-w+k_i-2)/2,(-w-k_i+2)/2}) \neq 0$, the representation π can realized in the cohomology of the local system $\left(\bigotimes_i \left(\mathrm{Sym}^{k_i-2} \otimes \det^{(-w-k_i+2)/2}\right)\right)^\vee$ over a suitable Hilbert modular variety (cf. [RT11, §3.1.9]). When $F = \mathbb{Q}$ and $w = 2-k$ we recover the usual notion of an elliptic modular form of weight k.

If p is a prime number, then (by Eichler, Shimura, Deligne, Wiles, Taylor, Blasius–Rogawski) there exists a continuous p-adic Galois representation $\rho_{\pi,p} : G_F \to \mathrm{GL}(2, \overline{\mathbb{Q}}_p)$ such that $L^S(\pi, s-1/2) = L^S(\rho_{\pi,p}, s)$ for a finite set S of places of F. Moreover, one has local-global compatibility: if $v \in S$, then $\mathrm{WD}(\rho_{\pi,p}|_{G_{F_v}})^{\text{Fr-ss}} \cong \mathrm{rec}(\pi_v \otimes |\cdot|^{-1/2})$. When $v \nmid p$ this follows from the work of Carayol ([Car86]) and when $v \mid p$ from the work of Saito ([Sai09]) and Skinner ([Ski09]). Finally, fixing a place $v \mid p$ and an embedding $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p$ we denote by $i \mid \infty$ the corresponding infinite place. Then the Hodge–Tate weights of $\mathrm{D}_{\mathrm{dR}}(\rho_{\pi,p}|_{G_{F_v}})_\tau$ are $(w-k_i)/2$ and $(w+k_i-2)/2$. (For weight k elliptic modular forms with $w = 2-k$ this amounts to Hodge–Tate weights $1-k$ and 0.)

We end this discussion with the following result on the irreducibility of symmetric powers.

Lemma 7. *Suppose π is not CM. Then $\rho_{\pi,p}$ and $\mathrm{Sym}^n \rho_{\pi,p}$ are Lie irreducible.*

Proof. First, the remark at the end of [Ski09] shows that $\rho_{\pi,p}$ is irreducible. We will apply [Pat12, Proposition 1.0.14] to $\rho_{\pi,p}$ which states that an irreducible Galois representation of a compatible system is of the form $\mathrm{Ind}(\tau \otimes \sigma)$ where τ is Lie irreducible and σ is Artin. Since π is not CM, the irreducibility of $\rho_{\pi,p}$ implies that $\rho_{\pi,p}$ is either Lie irreducible or Artin. But π is cohomological and $\rho_{\pi,p}|_{G_{F_v}}$ has Hodge–Tate weights 0 and $1-k_v < 0$ and so it cannot be Artin. Finally, the restriction of $\rho_{\pi,p}$ to any open subgroup will contain

$\mathrm{SL}(2,\overline{\mathbb{Q}}_p)$. Since the symmetric power representation of SL(2) is irreducible, it follows that the restriction of $\mathrm{Sym}^n \rho_{\pi,p}$ to any open subgroup will be irreducible. □

2.2. **p-adic Hodge theory of Hilbert modular forms.** As previously discussed, p-adic L-functions are expected to be attached to Galois representations and a choice of regular submodule. In this section, we will describe the possible regular submodules in the case of twists of symmetric powers of Galois representations.

Suppose F is a totally real field and π is a cohomological Hilbert modular form of infinity type $(k_1,\dots,k_d,w)$ over F. Let $V_{2n}=\mathrm{Sym}^{2n}\rho_{\pi,p}\otimes\det^{-n}\rho_{\pi,p}$ and $V_{2n,v}=V_{2n}|_{G_{F_v}}$. We will classify the regular submodules of $\mathrm{D}_{\mathrm{st}}(V_{2n,v})$ in the case when p is unramified in F and π is Iwahori spherical at places $v\mid p$.

Local-global compatibility describes the representation $\rho_{\pi,p}|_{G_{F_v}}$ completely whenever $v\nmid p$, but not so when $v\mid p$. We now make explicit the possibilities for the p-adic Galois representation at places $v\mid p$ and in the process we choose a suitable regular submodule of $\mathrm{D}_{\mathrm{st}}(V_{2n,v})$.

For an unramified extension $K/\mathbb{Q}_p$ and a semistable G_K-representation V on a finite dimensional $\overline{\mathbb{Q}}_p$-vector space recall that $\mathrm{D}_{\mathrm{st}}(V)$ is a $\dim V$-rank module over $K\otimes_{\mathbb{Q}_p}\overline{\mathbb{Q}}_p\cong\oplus_{\tau:K\hookrightarrow\mathbb{C}_p}\overline{\mathbb{Q}}_p$ with Frobenius φ and monodromy N. For each $\tau:K\hookrightarrow\mathbb{C}_p$ get a filtration $\mathrm{Fil}^\bullet\mathrm{D}_{\mathrm{dR}}(V)_\tau$, where $\mathrm{D}_{\mathrm{dR}}(V)_\tau$ is the $\overline{\mathbb{Q}}_p$-module given by projection to the τ component.

Back to the Hilbert modular form π over the totally real field F. For every $v\mid p$ and $\tau:F_v\hookrightarrow\mathbb{C}_p$ corresponds an infinite place $i=i(v,\tau)$ and we denote by k_i the infinite weight. Then $V_v=\rho_{\pi,p}|_{G_{F_v}}$ is de Rham and $\mathrm{D}_{\mathrm{dR}}(V)_\tau$ has Hodge–Tate weights $(w-k_i)/2$ and $(w+k_i-2)/2$. Since π is Iwahori spherical there are two possibilities: either $\pi_v=\mathrm{St}\otimes\mu$, where μ is an unramified character, or π_v is the unramified principal series with characters μ_1 and μ_2.

If $\pi_v=\mathrm{St}\otimes\mu$ then V is semistable but not crystalline, $\mathrm{D}^*_{\mathrm{st}}(V)$ is two-dimensional with basis e_1,e_2 over $\overline{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}F_v$, $\varphi=\begin{pmatrix}\lambda&\\&p\lambda\end{pmatrix}$ for $\lambda=\mu(\mathrm{Frob}_v)$ and $N=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. The Hodge-Tate filtration on $\mathrm{D}_{\mathrm{dR}}(V)_\tau$ jumps in degrees $(w-k_i)/2$ and $(w+k_i-2)/2$ and the proper filtered piece is given by $\overline{\mathbb{Q}}_p(e_2-\mathcal{L}_\tau e_1)$ for some $\mathcal{L}_\tau\in\overline{\mathbb{Q}}_p$. Writing $f_i=e_1^{n+i}e_2^{n-i}$, with $n\ge i\ge -n$, for the basis of $\mathrm{D}_{\mathrm{st}}(V_{2n,v})$, the Frobeniuns map $\varphi(f_i)=p^{-i}f_i$ is represented by a diagonal matrix while the monodromy is upper triangular with off-diagonal entries $2n,2n-1,\dots,1$. The (φ,N)-stable submodules of $\mathrm{D}_{\mathrm{st}}(V_{2n,v})$ are the spans $\langle f_n,f_{n-1},\dots,f_i\rangle$. Note that $\mathrm{Fil}^0\mathrm{D}^*_{\mathrm{dR}}(V_{2n,v})_\tau$ is

$$\left\langle(e_2-\mathcal{L}_\tau e_1)^{2n},(e_2-\mathcal{L}_\tau e_1)^{2n-1}e_1,(e_2-\mathcal{L}_\tau e_1)^{2n-1}e_2,\dots,(e_2-\mathcal{L}_\tau e_1)^n e_1e_2^{n-1},(e_2-\mathcal{L}_\tau e_1)^n e_2^n\right\rangle$$

and hence is $(n+1)$-dimensional. Thus, a regular submodule must be n-dimensional. The only n-dimensional (φ,N)-stable submodule is $D=\langle f_n,\dots,f_1\rangle$ and if $\mathcal{L}\neq 0$ (as is expected), then D is regular. In this case, $\mathrm{D}_{\mathrm{st}}(V_{2n,v})^{\varphi=1}=\langle f_0\rangle$ and $D^{\varphi=p^{-1}}=\langle f_1\rangle$. Thus, $D_1=\langle f_n,\dots,f_0\rangle$ (since $N^{-1}(\langle f_1\rangle)=\langle f_0\rangle$) and $D_{-1}=\langle f_n,\dots,f_2\rangle$.

If π_v is unramified then V is crystalline. Order the roots $\alpha=\mu_1(\varpi_v)$ and $\beta=\mu_2(\varpi_v)$ of $x^2-a_vx+p^{k_v-1}$ such that $v_p(\alpha)\le v_p(\beta)$. Then $\mathrm{D}_{\mathrm{cris}}(V)^*=(F_v\otimes\overline{\mathbb{Q}}_p)e_1\oplus(F_v\otimes\overline{\mathbb{Q}}_p)e_2$. There are now two possibilities. Either the local representation V splits as $\mu\oplus\mu^{-1}\chi_{\mathrm{cycl}}^{k-1}$, where μ is unramified, or V is indecomposable. The former case is expected to occur only when π is CM.

If V splits then π_v is ordinary and $\mathrm{D}^*_{\mathrm{cris}}(V_{2n,v})=\bigoplus_{i=-n}^n(F_v\otimes\overline{\mathbb{Q}}_p)t^{i(k_v-1)}$ (basis $f_i=e_1^{n+i}e_2^{n-i}$), φ has eigenvalues $\alpha^{2n}p^{n(k_v-1)},\dots,\alpha^{-2n}p^{-n(k_v-1)}$ where $\mu(\mathrm{Frob}_p)=\alpha$. The τ de Rham tangent space is $\mathrm{D}^*_{\mathrm{dR}}(V_{2n})_\tau/\mathrm{Fil}^0\mathrm{D}^*_{\mathrm{dR}}(V_{2n})_\tau=\overline{\mathbb{Q}}_pf_1\oplus\cdots\oplus\overline{\mathbb{Q}}_pf_n$ and so the only regular subspace is $D=(F_v\otimes\overline{\mathbb{Q}}_p)f_1\oplus\cdots\oplus(F_v\otimes\overline{\mathbb{Q}}_p)f_n$. In this case the filtration on D is given by $D_{-1}=D_0=D$ and $D_1=D\oplus(F_v\otimes\overline{\mathbb{Q}}_p)f_0$.

If V is not split, we will assume that $\alpha/\beta\notin\mu_\infty$. Then, we choose e_1 and e_2 to be eigenvectors of φ so that $\mathrm{D}^*_{\mathrm{cris}}(V)=(F_v\otimes\overline{\mathbb{Q}}_p)e_1\oplus(F_v\otimes\overline{\mathbb{Q}}_p)e_2$ with $\varphi=\begin{pmatrix}\alpha&\\&\beta\end{pmatrix}$. Moreover, we can scale e_1 and e_2 so that the one-dimensional filtered pieces are $\langle e_1+e_2\rangle$. We remark that V is reducible if and only if it is ordinary. Again taking $f_i=e_1^{n+i}e_2^{n-i}$ as a basis, Frobenius on $\mathrm{D}^*_{\mathrm{cris}}(V_{2n,v})$ is diagonal with $\varphi(f_i)=(\alpha/\beta)^i$. The de Rham tangent space is generated by homogeneous polynomials in e_1 and e_2 which are not divisible by $(e_1+e_2)^n$. Thus, any choice of n basis vectors in $f_n,\dots,f_{-n}$ will generate a regular submodule. The assumption that $\alpha/\beta\notin\mu_\infty$ implies that $\varphi(f_i)=f_i$ only for $i=0$. Since α and β are Weil numbers of the same complex absolute value, the eigenvalue p^{-1} does not show up. Therefore, no matter what choice of D we take, we have $D_{-1} = D_0 = D$. We choose the regular submodule $D = \langle f_n, f_{n-1}, \ldots, f_1 \rangle$. Since f_0 is not among the chosen basis vectors, $D_1 = D \oplus \langle f_0 \rangle = \langle f_n, f_{n-1}, \ldots, f_0 \rangle$.

In all cases, we have $D_0 = D = \langle f_n, f_{n-1}, \ldots, f_1 \rangle$ and $D \oplus \langle f_0 \rangle = \langle f_n, f_{n-1}, \ldots, f_0 \rangle$. Therefore,

$$\mathrm{gr}_1 \, \mathrm{D}^{\dagger}_{\mathrm{rig}}(V_{2n,v}) \cong \mathcal{R}_L.$$

2.3. **Automorphy of symmetric powers.** Let π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_F)$ over a totally real field F as in §2.1. We say that $\mathrm{Sym}^n \pi$ is (cuspidal) automorphic on $\mathrm{GL}(n+1, \mathbb{A}_{F'})$ for a number field F'/F if there exists a (cuspidal) automorphic representation Π_n of $\mathrm{GL}(n+1, \mathbb{A}_{F'})$ such that $L(\Pi_n, \mathrm{std}, s) = L(\mathrm{BC}_{F'/F}(\pi), \mathrm{Sym}^n, s)$. A cuspidal representation σ of $\mathrm{GL}(2, \mathbb{A}_F)$ is said to be dihedral if it is isomorphic to its twist by a quadratic character, in which case there exists a CM extension E/F and a character $\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times$ such that $\sigma \cong \mathrm{AI}_{E/F} \, \psi$; we say that σ is tetrahedral (resp. octahedral) if $\mathrm{Sym}^2 \sigma$ (resp. $\mathrm{Sym}^3 \sigma$) is cuspidal and is isomorphic to its twist by a cubic (resp. quadratic) character; we say that σ is solvable polyhedral if σ is dihedral, tetrahedral or octahedral; we say that σ is icosahedral if $\mathrm{Sym}^5 \sigma \cong \mathrm{Ad}(\tau) \boxtimes \sigma \otimes \omega_\sigma^2$ for some cuspidal representation τ of $\mathrm{GL}(2, \mathbb{A}_F)$ and if σ is not solvable polyhedral. The cuspidal representation π, being associated with a regular Hilbert modular form, is either dihedral (if the Hilbert modular form is CM) or not polyhedral.

Theorem 8. *Suppose π is as in §2.1. Then*

(1) $\mathrm{Sym}^m \pi$ *is automorphic for $m = 2$ ([GJ78, Theorem 9.3]), $m = 3$ ([KS02b, Corollary 1.6] and [KS02a, Theorem 2.2.2]) and $m = 4$ ([Kim03, Theorem B] and [KS02a, 3.3.7]); it is cuspidal unless π is CM.*
(2) *Suppose π is not CM. If $\mathrm{Sym}^5 \pi$ is automorphic then it is cuspidal. If $m \geq 6$ suppose either that $\mathrm{Sym}^k \pi$ is automorphic for $k \leq 2m$ or that $\pi \boxtimes \tau$ is automorphic for any cuspidal representation τ of $\mathrm{GL}(r, \mathbb{A}_F)$ where $r \leq \lfloor m/2 + 1 \rfloor$. Then $\mathrm{Sym}^m \pi$ is cuspidal ([Ram09, Theorem A']).*

Remark 9. Results on the automorphy of $\mathrm{Sym}^n \pi$ for small values of n have been obtained by Clozel and Thorne assuming conjectures on level raising and automorphy of tensor products. Potential automorphy results follow from automorphy lifting theorems.

2.4. **Functorial transfers to unitary and symplectic groups.** To compute $\mathcal{L}$-invariants, we use p-adic families of Galois representations. However, since $\mathrm{GL}(n)$ for $n > 2$ has no associated Shimura variety, to construct p-adic families of automorphic representations, we transfer to unitary or symplectic groups.

We begin with $\mathrm{GSp}(2n)$. Let ω_0 be the $n \times n$ antidiagonal matrix with 1-s on the antidiagonal and let $\mathrm{GSp}(2n)$ be the reductive group of matrices $X \in \mathrm{GL}(2n)$ such that $X^T J X = \nu(X) J$ where $J = \begin{pmatrix} & \omega_0 \\ -\omega_0 & \end{pmatrix}$ and $\nu(X) \in \mathbb{G}_m$ is the multiplier character. The diagonal maximal torus T consists of matrices $t(x_1, \ldots, x_g, z)$ with $(x_1, \ldots, x_g, z x_g^{-1}, \ldots, z x_1^{-1})$ on the diagonal. The Weyl group W of $\mathrm{GSp}(2n)$ is $S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$ and thus any $w \in W$ can be written as a pair $w = (\nu, \varepsilon)$ where $\nu \in S_n$ is a permutation and $\varepsilon : \{1, \ldots, n\} \to \{-1, 1\}$ is a function. The Weyl group acts by conjugation on T and (ν, ε) takes $t(x_1, \ldots, x_n, z)$ to $t(x'_{\nu(1)}, \ldots, x'_{\nu(n)}, z)$ where $x'_i = x_i$ if $\varepsilon(i) = 1$ and $x'_i = z x_i^{-1}$ if $\varepsilon(i) = -1$. We choose as a basis for $X^\bullet(T)$ the characters $e_i(t(x_i; z)) = x_i z^{-1/2}$ and $e_0((x_i; z)) = z^{1/2}$. Let B be the Borel subgroup of upper triangular matrices, which corresponds to the choice of simple roots $e_i - e_{i+1}$ for $i < n$ and $2e_n$. Half the sum of positive roots is then $\rho = \sum_{i=1}^n (n + 1 - i) e_i$. The compact roots are $\pm(e_i - e_j)$ and so half the sum of compact roots is $\rho_c = \sum_{i=1}^n (n - 2i + 1)/2 e_i$. This gives the relationship between Harish-Chandra and Blattner parameters for $\mathrm{GSp}(4)$ as $\lambda_{\mathrm{HC}} + \sum_{i=1}^n i e_i = \lambda_{\mathrm{Blattner}}$.

Theorem 10 (Ramakrishnan–Shahidi)**.** *Let π be as in §2.1 of infinity type $(k_1, \ldots, k_d, w)$. If π is not CM there exists a cuspidal automorphic representation Π of $\mathrm{GSp}(4, \mathbb{A}_F)$ which is a strong lift of $\mathrm{Sym}^3 \pi$ from $\mathrm{GL}(4, \mathbb{A}_F)$ such that for every infinite place τ, Π_τ is the holomorphic discrete series with Harish-Chandra parameters $(2(k_\tau - 1), k_\tau - 1; -3w/2)$.*

Proof. When $F = \mathbb{Q}$ and the weight is even this is [RS07, Theorem A$'$]. Ramakrishnan remarks that the proof of this result should also work for totally real fields. We give a proof using Arthur's results on the discrete spectrum of symplectic groups.

Let σ be the cuspidal representation of $\mathrm{GL}(4, \mathbb{A}_F)$ whose L-function coincides with that of $\mathrm{Sym}^3 \pi$. Then $L(\wedge^2 \sigma \otimes \det^3 \rho_{\pi,p}, s) = L(\wedge^2 \mathrm{Sym}^3 \rho_{\pi,p} \otimes \det^3 \rho_{\pi,p}, s) = \zeta(s) L(\mathrm{Sym}^4 \rho_{\pi,p} \otimes \det^2 \rho_{\pi,p}, s)$ has a pole at $s = 1$ (for example because $\mathrm{Sym}^4 \pi$ is cuspidal automorphic) so [GT11, Theorem 12.1] shows that there exists a globally generic (i.e., having a global Whittaker model) cuspidal representation Π^g of $\mathrm{GSp}(4, \mathbb{A}_F)$ strongly equivalent to σ, and thus to $\mathrm{Sym}^3 \pi$. For every infinite place τ the representation Π^g_τ will be a generic (nonholomorphic) discrete series representation. Let ψ be the global A-parameter attached to Π^g; since Π^g is globally generic with cuspidal lift to $\mathrm{GL}(4)$, ψ will be simple generic and therefore the A-parameter ψ is in fact an L-parameter and the component group of ψ is trivial. Let $\Pi = (\otimes_{\tau|\infty} \Pi^h_\tau) \otimes (\otimes_{v\nmid\infty} \Pi^g_v)$ be the representation of $\mathrm{GSp}(4, \mathbb{A}_F)$ obtained using the holomorphic discrete series Π^h_τ in the same archimedean local L-packet as Π^g_τ. Then Arthur's description of the discrete automorphic spectrum for $\mathrm{GSp}(4)$ implies that Π is an automorphic representation (for convenience, see [Mok14, Theorem 2.2]). Since the representations Π_τ at infinite places τ are discrete series, they are also tempered and so [Wal84, Theorem 4.3] implies that Π will also be cuspidal. By construction, Π will be strongly equivalent to $\mathrm{Sym}^3 \pi$.

It remains to compute the Harish-Chandra parameter of Π_τ. Let φ_{π_τ} and φ_{Π_τ} be the Langlands parameters of π_τ respectively Π_τ. Then $\varphi_{\Pi_\tau} = \mathrm{Sym}^3 \varphi_{\pi_\tau}$ and so

$$\varphi_{\Pi_\tau}(z) = |z|^{-3w} \begin{pmatrix} (z/\overline{z})^{(3(k_\tau-1))/2} & & & \\ & (z/\overline{z})^{(k_\tau-1)/2} & & \\ & & (z/\overline{z})^{-(k_\tau-1)/2} & \\ & & & (z/\overline{z})^{-(3(k_\tau-1))/2} \end{pmatrix}$$

The recipe from [Sor10, §2.1.2] shows that the L-packet defined by φ_{Π_τ} consists of the holomorphic and generic discrete series with Harish-Chandra parameters $(2(k_\tau - 1), k_\tau - 1; -3w/2)$.

□

For higher n one does not yet have transfers from $\mathrm{GL}(n)$ to similitude symplectic groups, although one may first transfer from $\mathrm{GL}(2n + 1)$ to $\mathrm{Sp}(2n)$ and then lift to $\mathrm{GSp}(2n)$.

Theorem 11. *Let F be a totally real field and π be a regular algebraic cuspidal automorphic self-dual representation of $\mathrm{GL}(2n + 1, \mathbb{A}_F)$ with trivial central character. Then there exists a cusidal automorphic representation $\overline{\sigma}$ of $\mathrm{Sp}(2n, \mathbb{A}_F)$ which is a weak functorial transfer of π such that $\overline{\sigma}$ is a holomorphic discrete series at infinite places. If, moreover, π is the symmetric $2n$-th power of a cohomological Hilbert modular form then there exists a cuspidal representation σ of $\mathrm{GSp}(2n, \mathbb{A}_F)$ which is a holomorphic discrete series at infinity and such that any irreducible component of the restriction $\sigma|_{\mathrm{Sp}(2n,\mathbb{A}_F)}$ is in the same global L-packet at $\overline{\sigma}$.*

Proof. We will use [GRS11, Theorem 3.1] to produce an irreducible cuspidal globally generic functorial transfer τ of a cuspidal self-dual representation π of $\mathrm{GL}(2n + 1, \mathbb{A}_F)$ to $\mathrm{Sp}(2n, \mathbb{A}_F)$, and all citations in this paragraph are from [GRS11]. In the notation of §2.3 the group H is taken to be the metaplectic double cover $\widetilde{\mathrm{Sp}}(4n+2)$ of $\mathrm{Sp}(4n+2)$ (this is case 10 from (3.43)) in which case, in the notation of §2.4, $L(\pi, \alpha^{(1)}, s) = 1$ and $L(\pi, \alpha^{(2)}, s) = L(\pi, \mathrm{Sym}^2, s)$ which has a pole at $s = 1$ since $\pi \cong \pi^\vee$. Let E_π be the irreducible representation of $H(\mathbb{A}_F)$ of Theorem 2.1 generated by the residues of certain Eisenstein series attached to the representation π thought of as a representation of the maximal parabolic of H. One finds τ as an irreducible summand of the automorphic representation generated by the Fourier–Jacobi coefficients of E_π where, in the notation of §3.6, one takes $\gamma = 1$. The fact that τ is cuspidal and globally generic and a strong transfer of π then follows from Theorem 3.1.

Next, Arthur's global classification of the discrete spectrum of $\mathrm{Sp}(2n)$ ([Art13, Theorem 1.5.2]) implies that there exists a cuspidal automorphic representation τ' which is isomorphic to τ at all finite places and the representation τ'_∞ is a holomorphic discrete series in the same L-packet at τ_∞.

Finally, Proposition 12.2.2, Corollary 12.2.4 and Proposition 12.3.3 (the fact that π is the symmetric $2n$-th power of a *cohomological* Hilbert modular form implies that hypothesis (2) of this proposition is satisfied) of [Pat12] imply the existence of a regular algebraic cuspidal automorphic representation σ of $\mathrm{GSp}(2n, \mathbb{A}_F)$ such that if v is either an infinite place or a finite place such that σ_v is unramified then $\sigma_v|_{\mathrm{Sp}(2n,F_v)}$ contains τ'_v. Moreover, the discrete series σ_∞ is holomorphic or else its restriction to $\mathrm{Sp}(2n, \mathbb{R})$ would not contain the holomorphic discrete series τ'_∞. □

One reason to seek a formula for symmetric power $\mathcal{L}$-invariants in terms of p-adic families on a certain reductive group is that it might yield a proof of a trivial zero conjecture for symmetric powers of Hilbert modular forms following the template of the Greenberg–Stevens proof of the Mazur–Tate–Teitelbaum conjecture. This method for proving such conjectures requires the existence of p-adic L-functions for these p-adic families. Whereas there has been little progress towards such p-adic families of p-adic L-functions on symplectic groups in general, the work of Eischen–Harris–Li–Skinner is expected to yield such p-adic L-functions in the case of unitary groups. We will therefore present a computation (however, under some restrictions) of the symmetric power $\mathcal{L}$-invariants using unitary groups.

For a CM extension E of a totally real field F, let U_n be the unitary group defined in [BC09, Definition 6.2.2] which is definite at every finite place and if $n \neq 2 \pmod 4$ is quasi-split at every nonsplit place of F. We denote by BC the local base change (see, for example, [Shi09, §2.3]). Assume the following conjecture on strong base change that would follow from stabilization of the trace formula.

Conjecture 12. *Let π be a regular algebraic conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}(n, \mathbb{A}_E)$ such that π_v is the base change from U of square integrable representations at ramified places of E and is either unramified or the base change from U of square integrable represenations at places of F which are inert in E. Suppose, moreover, that for at least one inert prime the local representation is not unramified. Then there exists a (necessarily cuspidal) automorphic representation Π of $U_n(\mathbb{A}_F)$ such that $\Pi_w = \mathrm{BC}(\pi_v)$ for all places v.*

We will use this conjecture to transfer symmetric powers of Hilbert modular forms to unitary groups.

Proposition 13. *Assume Conjecture 12. Let π be an Iwahori spherical cohomological non-CM Hilbert modular form over a number field F in which p is unramified. Suppose that there exist places w_1 and w_2 not above p with the property that π_w is special for $w \in \{w_1, w_2\}$ and suppose that $\mathrm{Sym}^n \pi$ is cuspidal automorphic over F. Then there exists a CM extension E/F in which every $v \mid p$ splits completely and a Hecke character ψ of E such that $\psi \otimes \mathrm{BC}_{E/F} \mathrm{Sym}^n \pi$ is the base change of a cuspidal automorphic representation of $U_n(\mathbb{A}_F)$.*

Proof. There exists $\alpha \in \{\varpi_{v_1}, \varpi_{v_2}, \varpi_{v_1}\varpi_{v_2}\}$ having a square root in $\mathbb{F}_{p^{[F:\mathbb{Q}]}}$. Then $E = F(\sqrt{\alpha})$ is ramified over F only at w_1 or w_2 (or both) and $v \mid p$ splits completely in E.

Suppose that $\Pi = \mathrm{BC}_{E/F} \mathrm{Sym}^n \pi$ is cuspidal automorphic. Then $\Pi^{c\vee} \cong \Pi \otimes \mathrm{BC}_{E/F} \omega_\pi^{-n}$. Choose a Hecke character $\psi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ such that $\psi|_{\mathbb{A}_F^\times} = \omega_\pi^{-n}$. Then $(\psi \otimes \Pi)^{c\vee} \cong \psi^{c\vee}\omega_\pi^{-n} \otimes \Pi \cong \psi \otimes \Pi$. Since $\mathrm{Sym}^n \pi$ is Iwahori spherical it follows that Π is Iwahori spherical. Moreover, if v is a ramified place of E/F, π_v is special by assumption and thus Π_v is the base change from U of a square integrable representation. Finally, at every finite place Π_v is either unramified or special. Therefore, the hypotheses of Conjecture 12 are satisfied and the conclusion follows.

It remains to show that $\Pi \cong \mathrm{Sym}^n \mathrm{BC}_{E/F} \pi$ is cuspidal automorphic. Writing $\pi_E = \mathrm{BC}_{E/F} \pi$ we note that π_E is cuspidal since π is not CM. Therefore one may apply [Ram09, Theorem A'] to study the cuspidality of $\mathrm{Sym}^n \pi_E$. If π_E is dihedral then $\pi_E \cong \mathrm{Ind}_L^E \eta$ for some quadratic extension L/E contradicting the fact that $\rho_{\pi,p}$ is Lie irreducible.

If π_E is tetrahedral then $\mathrm{Sym}^3 \pi$ is cuspidal but $\mathrm{BC}_E \mathrm{Sym}^3 \pi$ is not which implies that $\mathrm{Sym}^3 \pi \cong \mathrm{Ind}_E^F \tau$ for an automorphic representation τ of $\mathrm{GL}(2, \mathbb{A}_F)$. It suffices to show that there exists a Galois representation attached to τ since then $\mathrm{Sym}^3 \pi$ is not Lie irreducible contradicting Lemma 7. Since a real place v of F splits completely in E, the L-parameter of $(\mathrm{Ind}_E^F \tau)_v$ is the direct sum of the L-parameters of the local components of τ at the complex places over v. Therefore τ is regular algebraic, necessarily cuspidal, representation of $\mathrm{GL}(2, \mathbb{A}_E)$. Galois representations have been attached to such cuspidal representations by Harris–Lan–Taylor–Thorne, although one does not need such a general result; indeed, by twisting one may guarantee that the central character of τ is trivial in which case the Galois representation has been constructed by Mok ([Mok14]).

If π_E is icosahedral, i.e., $\mathrm{Sym}^6 \pi_E$ is not cuspidal but all lower symmetric powers are cuspidal then $\mathrm{Sym}^6 \pi_E = \eta \boxplus \eta'$ where η is a cuspidal representation of $\mathrm{GL}(3, \mathbb{A}_F)$ and η' is a cuspidal representation of $\mathrm{GL}(4, \mathbb{A}_F)$ (cf. [Ram09, §4]). Lemmas 4.10 and 4.18 of [Ram09] carry over to this setting and show that η and η' are regular, algebraic conjugate self-dual representations and thus that $\mathrm{Sym}^6 \rho_{\pi,p}|_{G_E}$ is decomposable, which contradics Lemma 7.

Finally, if π_E is octahedral then $\mathrm{Sym}^4 \pi_E$ is not cuspidal. As in the case of Sym^6 we may write $\mathrm{Sym}^4 \pi_E = \eta \boxplus \eta'$ where η is a regular algebraic cuspidal conjugate self-dual representation of $\mathrm{GL}(2, \mathbb{A}_E)$ and η' is a regular algebraic cuspidal conjugate self-dual representation of $\mathrm{GL}(3, \mathbb{A}_E)$ yielding a contradiction as above. ∎

3. p-adic families and Galois representations

As previously mentioned, we will compute explicitly the arithmetic $\mathcal{L}$-invariants attached to V_{2n} using triangulations of (φ,Γ)-modules attached to analytic Galois representations on eigenvarieties. Which eigenvariety we choose will be dictated by the requirement that the formula from Lemma 4 needs to make sense. This translates into a lower bound for the rank of the reductive group whose eigenvariety we will use. In this section we make explicit the eigenvarieties under consideration and the analytic Hecke eigenvalues whose derivatives will control the arithmetic $\mathcal{L}$-invariant.

3.1. **Urban's eigenvarieties.** We recall Urban's construction of eigenvarieties from [Urb11, Theorem 5.4.4], to which we refer for details. Let G be a split reductive group over $\mathbb{Q}$ such that $G(\mathbb{R})$ has discrete series representations. (This condition is satisfied by the restriction to $\mathbb{Q}$ of $\mathrm{GL}(2)$ and $\mathrm{GSp}(2n)$ over totally real fields and compact unitary groups attached to CM extensions.) We denote by T a maximal (split) torus, by B a Borel subgroup containing T and by B^- the opposite Borel, obtained as $B^-=w_BBw_B^{-1}$ for some $w_B\in W_{G,T}=N_G(T)/T$. For a character $\lambda\in X^\bullet(T)$ let $V_\lambda=\mathrm{Ind}_{B^-}^G\lambda$ induced from the opposite Borel, the irreducible algebraic representation of G of highest weight λ with respect to the set of positive roots defined by the Borel B. Its dual is $V_\lambda^\vee\cong\mathrm{Ind}_B^G((-w_B)\lambda)$.

Let π be a cuspidal automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ and assume that it has regular cohomological weight $\lambda\in X^\bullet(T)$, i.e. that π can be realized in the cohomology of the local system $V_\lambda^\vee$; equivalently π_∞ has central character equal to the central character of λ and has a twist which is the discrete series representation of $G(\mathbb{R})$ of Harish-Chandra parameter λ (cf. [Urb11, §1.3.4]). For example, if f is a classical modular form of weight $k\geq 2$ (as before $w=2-k$) then the cohomological weight of the associated automorphic representation is $(k-2,0)\in X^\bullet(\mathbb{G}_m^2)$ as the Eichler–Shimura isomorphism implies that f appears in the cohomology of $(\mathrm{Sym}^{k-2})^\vee$.

Let $\mathbb{A}_{\mathbb{Q},f}^{(p)}$ be the finite adeles away from p and let $K^p\subset G(\mathbb{A}_{\mathbb{Q},f}^{(p)})$ be a compact open subgroup. Let I'_m be a pro-p-Iwahori subgroup of $G(\mathbb{Q}_p)$ such that $\pi_f^{K^pI'_m}\neq 0$. We denote by $\mathcal{H}^p$ the Hecke algebra of K^p, $\mathcal{U}_p$ the Hecke algebra of the Iwahori subgroup $I_m=\{g\in G(\mathbb{Z}_p): g \pmod p\in B(\mathbb{Z}_p)\}$ and $\mathcal{H}=\mathcal{H}^p\otimes\mathcal{U}_p$. A *$p$-stabilization* ν of π is an irreducible constituent of $\pi_f^{K^pI'_m}\otimes\varepsilon^{-1}$ as a $\mathcal{H}$-representation, where ε is a character of $I_m/I'_m\cong T(\mathbb{Z}/p^m\mathbb{Z})$ acting on $\pi_f^{K^pI'_m}$ (cf. [Urb11, p. 1689]). For ν finite slope of weight λ, Urban rescales by $|\lambda(t)|_p^{-1}$ the eigenvalue of the Hecke operator $U_t=I_mtI_m\in\mathcal{U}_p$, where $t\in T(\mathbb{Q}_p)$ is such that $|\alpha(t)|_p\leq 1$ for all positive roots α (*finite slope* means the U_t act invertibly; cf. [Urb11, p. 1689–1690]). This rescaled eigenvalue is denoted $\theta(U_t)$. The p-stabilization ν of π is said to have *non-critical slope* if there is a Hecke operator U_t, with $|\alpha(t)|_p<1$ for all positive roots α, such that for every simple root α

$$v_p(\theta(U_t))<(\lambda(\alpha^\vee)+1)v_p(\alpha(t)).$$

Theorem 14 ([Urb11, Theorem 5.4.4])**.** *Suppose ν has non-critical slope. Then there exists a rigid analytic "weight space" $\mathcal{W}$ of the weight λ of π, a rigid analytic variety $\mathcal{E}$, a generically finite flat morphism $w:\mathcal{E}\to\mathcal{W}$ and a homomorphism $\theta:\mathcal{H}\to\mathcal{O}_\mathcal{E}$ such that:*

1. *there exists a dense set of points $\Sigma\subset\mathcal{E}$ and*
2. *for every point $z\in\Sigma$ there exists an irreducible cuspidal representation π_z of $G(\mathbb{A}_\mathbb{Q})$ of weight $w(z)$, with Iwahori spherical local representations at places over p, and a p-stabilization ν_z such that $\theta(T)$ is the normalized eigenvalue of $T\in\mathcal{H}$ on ν_z and*
3. *there exists $z_0\in\Sigma$ such that $\pi_{z_0}=\pi$ and $\nu_{z_0}=\nu$.*

Suppose F is a totally real field in which p splits completely and G/F is a split reductive group over F. Let $G'=\mathrm{Res}_{F/\mathbb{Q}}G$ be the Weil restriction of scalars. Then a cuspidal automorphic representation π of $G(\mathbb{A}_F)$ can be thought of as a cuspidal representation of $G'(\mathbb{A}_\mathbb{Q})$ and we may apply Urban's construction above. Let T be a maximal torus of G and $T'=\mathrm{Res}_{F/\mathbb{Q}}T$. Then $X^\bullet(T')\cong X^\bullet(T)\cong\oplus_{G_{F/\mathbb{Q}}}X^\bullet(T_{/\mathbb{Q}_p})$ gives a one-to-one correspondence between the weights of automorphic representations of $G(\mathbb{A}_F)$ and tuples $(w_1,\ldots)$ where $w_i\in X^\bullet(T_{/\mathbb{Q}_p})$. For each place $v\mid p$, given $t\in T(\mathbb{Q}_p)$ one gets a Hecke operator U_t acting on π_v and the refinement ν_v with eigenvalue α_v. Then the eigenvalue of U_t on the refinement ν of π thought of as a representation of $G'(\mathbb{A}_\mathbb{Q})$ is $\prod_v\alpha_v$ and the slope of the renormalized eigenvalue is

$$v_p(\theta(U_t))=\sum_v v_p(\lambda_v(t))+\sum_v v_p(\alpha_v)=\sum_v v_p(\theta_v(U_t))$$

where $\theta_v(U_t)$ is the eigenvalue of U_t on π_v renormalized by the weight λ_v. If Φ, Φ^+ and Ψ are the set of roots, positive roots, and simple roots of $G_{/\mathbb{Q}_p}$ with respect to a Borel, then the set of roots, positive roots, and simple roots of G are $\prod_i \Phi$, $\prod_i \Phi^+$, and $\{0 \oplus \cdots \oplus \alpha \oplus \cdots \oplus 0 | \alpha \in \Psi\}$.

In the next sections, we will make explicit first what non-critical slope means in the case of the groups under consideration, and second, the relationship between the renormalized action of Hecke operators on p-stabilizations and their action on smooth representations of p-adic groups. The latter will allow us to study global triangulations of Galois representations in terms of analytic Hecke eigenvalues.

3.2. **Eigenvarieties for Hilbert modular forms.** Suppose π is a cohomological Hilbert modular form of infinity type $(k_1, \dots, k_d, w)$ over a totally real field F of degree d in which p is unramified. The representation π has cohomological weight $\bigoplus_i((-w + k_i - 2)/2, (-w - k_i + 2)/2)$. For each place $v \mid p$, let α_v be an eigenvalue of U_v (corresponding to $t = \begin{pmatrix} 1 & \\ & \varpi_v^{-1} \end{pmatrix}$) acting on a refinement ν of π_v in which case $\theta(U_p) = \prod_\tau \tau(\varpi_v)^{(w+k_{v,\tau}-2)/2}\alpha_v$. Then $v_p(\theta(U_v)) = \sum_\tau (w + k_{v,\tau} - 2)/2 + v_p(\alpha_v)$. The simple roots of $\mathrm{GL}(2)_{/F}$ are of the form $(1, -1)_i$ where i is an infinite place. As $((-w + k - 2)/2, (-w - k + 2)/2) \cdot (1, -1) = k - 2$, the refinement ν has noncritical slope if and only if

$$\sum_{v,\tau}(w + k_{v,\tau} - 2)/2 + \sum_v v_p(\alpha_v) < k_i - 1$$

for every i, which is equivalent to

$$\sum_{v,\tau}(w + k_{v,\tau} - 2)/2 + \sum_v v_p(\alpha_v) < \min(k_i) - 1$$

We remark that for modular forms of weight k (with $w = 2 - k$) this is the usual definition of noncritical slope.

Lemma 15. *Let π be a cohomological Hilbert modular form, ν a refinement of π of noncritical slope as above, and $\mathcal{E} \to \mathcal{W}$ Urban's eigenvariety around (π, ν). Shrinking $\mathcal{W}$ if necessary, there exists an analytic Galois representation $\rho_{\mathcal{E}} : G_F \to \mathrm{GL}(2, \mathcal{O}_{\mathcal{E}})$ such that for $z \in \Sigma$, $z \circ \rho_{\mathcal{E}} = \rho_{\pi_z}$. If π has Iwahori level at $v \mid p$ then $\rho_{\mathcal{E}}$ admits a refinement in the sense of §1.3.*

Proof. The existence of $\rho_{\mathcal{E}}$ is done by a standard argument, but we reproduce it here for convenience. If $v \nmid p$ is a place such that π_w is unramified and $z \in \Sigma$ then $z(\theta(T_v))$ is equal to the eigenvalue of T_v on π_z. Local-global compatibility implies that $\mathrm{Tr}\,\rho_{\pi_z}(\mathrm{Frob}_v) = z \circ \theta(T_v)$ and so $\theta(T_v)$ is an analytic function $T(\mathrm{Frob}_v)$ specializing at points $z \in \Sigma$ to $\mathrm{Tr}\,\rho_{\pi_z}(\mathrm{Frob}_v)$. As $\{\mathrm{Frob}_v | v \nmid Np\}$ is dense in G_F we may define by continuity $T(g) = \lim T(\mathrm{Frob}_v)$ where $\mathrm{Frob}_v \to g$. The function $T : G_F \to \mathcal{O}_{\mathcal{E}}$ is an analytic two-dimensional pseudorepresentation. Since π is cuspidal, ρ_π is irreducible (cf. [Ski09, p. 256]). This implies that shrinking $\mathcal{W}$, T is the trace of an analytic Galois representation $\rho_{\mathcal{E}}$ (cf. [Jor10, 4.2.6]).

If π has Iwahori level at $v \mid p$ then π_z will also have Iwahori level at $v \mid p$ for $z \in \Sigma$. Thus $\rho_{z,v}$ for $v \mid p$ is either unramified or Steinberg. If Steinberg then the unique refinement of $\pi_{z,v}$ has slope $\sum_\tau (k_{v,\tau} - 2 + w/2)$. Shrinking the neighborhood $\mathcal{W}$ we may guarantee that the slope is constant. The classical points on $\mathcal{E}$ are dense and the ones satisfying $k_{v,\tau} - 2 + w/2$ constant lie in a subvariety and so there is a dense set Σ of points on $\mathcal{E}$ corresponding to cohomological regular Hilbert modular forms.

Suppose $z \in \Sigma$ in which case $\rho_{\pi_z,p}|_{G_{F_v}}$ is crystalline for $v \mid p$. By local-global compatibility the eigenvalues of φ_{cris} acting on $\mathrm{D}_{\mathrm{cris}}(\rho_{\pi_z,p}|_{G_{F_v}})$ are $\alpha_v \prod \tau(\varpi_v)^{-1/2}$ and $\beta_v \prod \tau(\varpi_v)^{-1/2}$ and by choice of central character for π we deduce that $\alpha_v \beta_v = \prod \tau(\varpi_v)^{-w}$. Writing $a_v(z) = z(\theta(U_v))$ we get an analytic function on $\mathcal{E}$ such that for $z \in \Sigma$, the eigenvalues of φ_{cris} are $\alpha_v \prod \tau(\varpi_v)^{-1/2} = a_v(z) \prod \tau(\varpi_v)^{(-w-k_{v,\tau}+1)/2}$ and $\beta_v \prod \tau(\varpi_v)^{-1/2} = \prod \tau(\varpi_v)^{-w-1/2}\alpha_v^{-1} = a_v(z)^{-1} \prod \tau(\varpi_v)^{(-w+k_{v,\tau}-3)/2}$. Let $\kappa_{v,\tau,1}(z) = (w - k_{v,\tau})/2$ and $\kappa_{v,\tau,2}(z) = (w + k_{v,\tau} - 2)/2$; let $F_1(z) = a_v(z)^{-1} \prod \tau(\varpi_v)^{3/2}$ and $F_2(z) = a_v(z) \prod \tau(\varpi_v)^{1/2}$. Then for $z \in \Sigma$, $\kappa_{1,v,\tau}(z) < \kappa_{2,v,\tau}(z)$ are the Hodge–Tate weights of the crystalline representation $\rho_{\pi_z,p}|_{G_{F_v}}$ and the eigenvalues of φ_{cris} are $F_1(z) \prod \tau(\varpi_v)^{\kappa_{1,v,\tau}(z)}$ and $F_2(z) \prod \tau(\varpi_v)^{\kappa_{2,v,\tau}(z)}$ which implies that $\rho_{\mathcal{E}}|_{G_{F_v}}$ admits a refinement as desired. □

Corollary 16. *Let π be as in Lemma 15. For $v \mid p$, assume that π_v is Iwahori spherical; if π_v is an unramified principal series assume that $\alpha_v \neq \beta_v$. Then there exists a global triangulation*

$$\mathscr{D}^\dagger_{\mathrm{rig}}(\rho_{\mathcal{E},p}|_{G_{F_v}}) \sim \begin{pmatrix} \delta_1 & * \\ & \delta_2 \end{pmatrix}$$

with $\delta_1(\varpi_v) = a_v(z)^{-1}\prod \tau(\varpi_v)^{3/2}$, $\delta_1(u) = \prod \tau(u)^{(w-k_{v,\tau})/2}$, $\delta_2(\varpi_v) = a_v(z)\prod \tau(\varpi_v)^{-1/2}$ and $\delta_2(u) = \prod \tau(u)^{(w+k_{v,\tau}-2)/2}$.

Proof. We only need to check that the refinement attached to the triangulation of $\rho_{\pi,p}|_{G_{F_v}}$ is noncritical and regular by Theorem 5. Noncriticality is immediate from the requirement that the Hodge–Tate weights be ordered increasingly (cf. [Liu15, §5.3]). Regularity is equivalent to the fact that φ_{cris} has distinct eigenvalues, i.e., that $\alpha_v \neq \beta_v$. □

3.3. **Eigenvarieties for** $\mathrm{GSp}(2n)$**.** Suppose $F/\mathbb{Q}$ is a number field in which p is unramified, $n \geq 2$ an integer, and π a cuspidal automorphic representation of $\mathrm{GSp}(2n, \mathbb{A}_F)$ such that for $v \mid \infty$ (writing $v \mid p$ for the associated place above p) the representation π_v is the holomorphic discrete series representation with Langlands parameter, under the 2^n-dimensional spin representation $\mathrm{GSpin}(2n+1, \mathbb{C}) \hookrightarrow \mathrm{GL}(2^n, \mathbb{C})$, uniquely identified by

$$z \mapsto |z|^{\mu_0} \operatorname{diag}((z/\overline{z})^{1/2\sum \varepsilon(i)(\mu_{v,i}+n+1-i)})$$

where $\varepsilon : \{1, 2, \dots, n\} \to \{-1, 1\}$. If V_μ is the algebraic representation of $\mathrm{GSp}(2n)$ with highest weight $\mu_v = (\mu_{v,1}, \dots, \mu_{v,n}; \mu_0)$ then $H^\bullet(\mathfrak{gsp}_{2n}, \mathrm{SU}(n), \pi_v \otimes V_\mu^\vee) \neq 0$ and so π_v has cohomological weight μ in the sense of Urban.

For simplicity of exposition we will take $\varpi_v = p$ as a uniformizer in the unramified $F_v/\mathbb{Q}_p$. We denote by Iw the Iwahori subgroup of $\mathrm{GSp}(2n)$ of matrices which are upper triangular mod p. Let $\beta_0 = t(1, \dots, 1; p^{-1})$ and $\beta_j = t(1, \dots, 1, p^{-1}, \dots, p^{-1}; p^{-2})$ where p^{-1} appears j times. The double coset $[\mathrm{Iw}\,\beta_i\,\mathrm{Iw}]$ acts on the finite dimensional vector space π_v^{Iw} for $v \mid p$.

Lemma 17. *Let $\chi = \chi_1 \otimes \dots \otimes \chi_n \otimes \sigma$ be an unramified character of the diagonal torus of $\mathrm{GSp}(2n, F_v)$ and assume that the unitary induction $\mu = \mathrm{Ind}_B^{\mathrm{GSp}(2n,F_v)} \chi$ is irreducible. Then μ^{Iw} is $2^n n!$ dimensional and has a basis in which the operator $U_t = [\mathrm{Iw}\,t\,\mathrm{Iw}]$ for any $t = (p^{a_1}, \dots, p^{a_n}; p^{a_0})$ is upper triangular. The basis is $\{e_w\}$ indexed by the Weyl group elements $w = (\nu, \varepsilon) \in W$ in the Bruhat ordering. The diagonal element of U_t corresponding to e_w is*

$$p^{\frac{1}{4a_0}n(n+1)-\sum(n+1-j)a_j}\sigma(p)^{a_0}\prod_{j=1}^n \chi_j(p)^{a_{\nu(j)} \text{ or } a_0-a_{\nu(j)}}$$

where the exponent depends on whether $\varepsilon(\nu(j)) = 1$ or -1.

In particular, the Hecke operators $U_{p,i} = [\mathrm{Iw}\,\beta_{n-i}\,\mathrm{Iw}]$ are upper triangular whose diagonal elements are

$$\left\{ p^{c_{i,\nu,\varepsilon}}\sigma^{-2}(p) \prod_{\nu(j)>i} \chi_j^{-1}(p) \prod_{\nu(j)\leq i, \varepsilon(j)=-1} \chi_j^{-2}(p) \right\} \text{ for } 1 \leq i \leq n-1$$

and

$$\left\{ p^{c_{n,\nu,\varepsilon}}\sigma^{-1}(p) \prod_{\varepsilon(j)=-1} \chi_j^{-1}(p) \right\} \text{ for } i = n.$$

Here

$$c_{i,\nu,\varepsilon} = \sum_{\nu(j)>i}(n+1-j) + 2\sum_{\nu(j)\leq i, \varepsilon(j)=-1}(n+1-j) - n(n+1)/2$$

and

$$c_{n,\nu,\varepsilon} = \sum_{\varepsilon(j)=-1}(n+1-j) - n(n+1)/4.$$

Proof. The proof is inspired by [GT05, Proposition2n 3.2.1]. Indeed, the Iwahori decomposition gives $\mathrm{GSp}(2n, F_v) = \cup_{w \in W} Bw\,\mathrm{Iw}$. Writing e_w for $w \in W$ the function defined on the cell $Bw\,\mathrm{Iw}$ gives a basis for μ^{Iw} and the fact that the Hecke action is upper triangular with respect to the basis $\{e_w\}$ in the Bruhat ordering is the first part of the proof of [GT05, Proposition 3.2.1]. The computation of the diagonal elements only requires replacing the parahoric subgroup by the Iwahori subgroup and therefore the quotient of the Weyl group by the Weyl group of the parabolic with the whole Weyl group.

Let $\widetilde{\chi} = \chi \otimes \delta^{1/2}$ where δ is the modulus character of the upper triangular Borel subgroup, given by $\delta(t) = |\rho(t)|_p^2$. Write ${}^w\widetilde{\chi}(t) = \delta^{1/2}(t)\delta^{1/2}(w^{-1}tw)\widetilde{\chi}(w^{-1}tw)$ and ${}^w\chi(t) = \chi(w^{-1}tw)$. Let S be the Satake isomorphism from the Iwahori–Hecke algebra to the spherical Hecke algebra of the maximal torus T. Analogously to the second part of [GT05, Proposition 3.2.1], the eigenvalue of U_t on the one-dimensional subspace of μ^{Iw} generated by e_w is equal to ${}^w\widetilde{\chi}(S(t)) = \delta^{1/2}(t)\chi(w^{-1}tw)$. We now make explicit these eigenvalues. Recall that $w^{-1}tw = (p^{a'_{\nu(1)}}, \ldots, p^{a'_{\nu(n)}}; p^{a_0})$ where $a'_i = a_i$ if $\varepsilon(i) = 1$ and $a'_i = a_0 - a_i$ if $\varepsilon(i) = -1$. Moreover, $\delta^{1/2}(t) = p^{n(n+1)/4a_0 - \sum(n+1-j)a_j}$. Thus the e_w eigenvalue of U_t is

$$p^{n(n+1)/4a_0 - \sum(n+1-j)a_j}\sigma(p)^{a_0}\prod_{j=1}^{n}\chi_j(p)^{a_{\nu(j)} \text{ or } a_0 - a_{\nu(j)}}$$

where the exponent depends on whether $\varepsilon(\nu(j)) = 1$ or -1.

We remark that in the final computation of $\mathcal{L}$-invariants the constants $c_{i,\nu,\varepsilon}$ do not matter. □

Lemma 18. *Let π be a cuspidal automorphic representation of $\mathrm{GSp}(2n, \mathbb{A}_F)$ of cohomological weight $\lambda = \bigoplus(\mu_{v,1}, \ldots, \mu_{v,n}; \mu_0)$ as above. Suppose ν is a p-stabilization of π and $t \in \mathrm{Res}_{F/\mathbb{Q}} T(\mathbb{Q}_p)$ with associated Hecke operator $U_{v,t} = [\mathrm{Iw}\, t\, \mathrm{Iw}]$ acting on π_v. There exists an integer m such that the refinement $\nu \otimes |\cdot|_{\mathbb{A}_F}^m$ of $\pi \otimes |\cdot|_{\mathbb{A}_F}^m$ has noncritical slope with respect to U_t.*

Proof. Recall Urban's convention that $\theta(U_{v,t}) = |\lambda_v(t)|_p^{-1}a_{v,t}$ where $a_{v,t}$ is the $U_{v,t}$ eigenvalue on the refinement ν. The condition that ν have noncritical slope is that

$$\sum_{v|p} v_p(\theta(U_{v,t})) < (\lambda(\alpha^\vee) + 1)v_p(\alpha(t))$$

for every simple root α of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSp}(2n)$. This is equivalent to

$$\sum_{v|p}(v_p(\lambda_v(t)) + v_p(\alpha_{v,t})) < \min\left((\mu_{v,i} - \mu_{v,i+1} + 1)(a_i - a_{i+1}), 2(2\mu_{v,n} + 1)a_n\right)$$

Suppose one replaces π by $\pi_m = \pi \otimes |\cdot|^m$ for an integer m. Then the cohomological weight of π_m is $\lambda'_m = \oplus_{v|p}(\mu_{v,i}; \mu_0 + m)$ which implies that $v_p(\lambda'_v(t)) = v_p(\lambda_v(t)) + ma_0/2$. $v_p(\alpha'_{v,t}) = v_p(\alpha_{v,t}) + m$. Lemma 17 implies that $v_p(\alpha'_{v,t}) = v_p(\alpha_{v,t}) - ma_0$ because $(\chi_1 \times \cdots \times \chi_n \rtimes \sigma) \otimes \eta \cong \chi_1 \times \cdots \times \chi_n \rtimes \sigma\eta$. The result is now immediate as the right-hand side of the inequality does not change with m. □

Before discussing analytic Galois representations over Siegel eigenvarieties we explain how to attach Galois representations to cohomological cuspidal automorphic representations of $\mathrm{GSp}(2n, \mathbb{A}_F)$ for a totally real F using the endoscopic classification of cuspidal representations of symplectic groups due to Arthur, which is conditional on the stabilization of the twisted trace formula.

Theorem 19. *Let π be a cuspidal representation of $\mathrm{GSp}(2n, \mathbb{A}_F)$ of cohomological weight $\oplus(\mu_{v,1}, \ldots, \mu_{v,n}; \mu_0)$.*

1. *If $n = 2$ there exists a spin Galois representation $\rho_{\pi,\mathrm{spin},p} : G_F \to \mathrm{GSp}(4, \overline{\mathbb{Q}}_p)$ such that if $v \nmid \infty p$ with π_v unramified then $\mathrm{WD}(\rho_{\pi,\mathrm{spin},p}|_{G_{F_v}})^{\mathrm{Fr\text{-}ss}} \cong \mathrm{rec}_{\mathrm{GSp}(4)}(\pi_v \otimes |\cdot|^{-3/2})$. If $v \mid p$ and π_v is unramified then the crystalline representation $\rho_{\pi,\mathrm{spin},p}|_{G_{F_v}}$ has Hodge–Tate weights $(\mu_0 - \mu_{v,\tau,1} - \mu_{v,\tau,2})/2 + \{0, \mu_{v,\tau,2} + 1, \mu_{v,\tau,1} + 2, \mu_{v,\tau,1} + \mu_{v,\tau,2} + 3\}$ as $\tau : F_v \hookrightarrow \mathbb{C}_p$.*
2. *If $n \geq 2$ there exists a standard Galois representation $\rho_{\pi,\mathrm{std},p} : G_F \to \mathrm{GL}(2n+1, \mathbb{C})$ such that if $v \nmid \infty p$ with π_v the unramified principal series $\chi_1 \times \cdots \times \chi_n \rtimes \sigma$ then $\rho_{\pi,\mathrm{std},p}|_{G_{F_v}}$ is unramified and local-global compatibility is satisfied. If $v \mid p$ and π_v is the unramified principal series $\chi_1 \times \cdots \times \chi_n \rtimes \sigma$ then $\rho_{\pi,\mathrm{std},p}|_{G_{F_v}}$ is crystalline with Hodge–Tate weights $0, \pm(\mu_{v,\tau,i} + n + 1 - i)$ and the eigenvalues of φ_{cris} are $\chi_1(p), \ldots, \chi_n(p), 1, \chi_1^{-1}(p), \ldots, \chi_n^{-1}(p)$.*

Proof. The first part follows from [Mok14, Theorem 3.5].

Let π_0 be any irreducible constituent of $\pi|_{\mathrm{Sp}(2n,\mathbb{A}_F)}$. If $v \mid \infty$ then $\pi_{0,v}$ will then be a discrete series representation with L-parameter uniquely defined by

$$z \mapsto \mathrm{diag}((z/\overline{z})^{\mu_{v,i}+n+1-i}, 1, (z/\overline{z})^{-\mu_{v,i}-(n+1-i)})$$

cohomological weight $\oplus(\mu_{v,i})$.

Arthur's endoscopic classification implies the existence of a transfer of π_0 from $\mathrm{Sp}(2n)$ to $\mathrm{GL}(2n+1)$ as follows (cf. [Sch15, Corollary 5.1.7]). Let $\eta : {}^L\mathrm{Sp}(2n) \to {}^L\mathrm{GL}(2n+1)$ be the standard inclusion of $\mathrm{SO}(2n+1,\mathbb{C}) \hookrightarrow \mathrm{GL}(2n+1,\mathbb{C})$. There exists a partition $2n+1 = \sum_{i=0}^r \ell_i k_i$ and cuspidal automorphic representations Π_i of $\mathrm{GL}(k_i,\mathbb{A}_F)$ such that:

(1) $\Pi_i^\vee \cong \Pi_i$ for all i;
(2) writing $\varphi_{\pi_{0,v}}$ for the L-parameter of $\pi_{0,v}$ and $\varphi_{\Pi_{i,v}}$ for the L-parameter of $\Pi_{i,v}$, if v is archimedean or $\pi_{0,v}$ is unramified then

$$\eta \circ \varphi_{\pi_{0,v}} = \oplus_{i=1}^r \oplus_{j=1}^{\ell_i} \varphi_{\Pi_{i,v}} |\cdot|_v^{j-(\ell_i+1)/2}$$

Next, if τ is a cuspidal automorphic representation of $\mathrm{GL}(k,\mathbb{A}_F)$ with cohomological weight $\oplus(a_{v,1},\dots,a_{v,k})$ and $\tau^\vee \cong \tau \otimes \chi$ for some Hecke character χ such that $\chi_v(-1)$ is independent of v then by [BLGHT11, Theorem 1.1] there exists a continuous Galois representation $\rho_\tau : G_F \to \mathrm{GL}(k,\overline{\mathbb{Q}}_p)$ such that:

(1) if τ_v is unramified and $v \nmid p$ then $\mathrm{WD}(\rho_\tau|_{G_{F_v}})^{\text{Fr-ss}} \cong \mathrm{rec}(\pi_v \otimes |\cdot|^{(1-k)/2})$;
(2) if τ_v is unramified and $v \mid p$ then $\rho_\tau|_{G_{F_v}}$ is crystalline with Hodge–Tate weights $-a_{v,\tau,k+1-i}+k-i$ and $\mathrm{WD}(\rho_\tau|_{G_{F_v}})^{\text{Fr-ss}} \cong \mathrm{rec}(\pi_v \otimes |\cdot|^{(1-k)/2})$. (The Hodge–Tate weights are $(k-1)/2+$ the Harish-Chandra parameter.)

The discrepancy between the description above and [BLGHT11, Theorem 1.1] arises because we defined cohomological weight as the highest weight of the algebraic representation with the same central and infinitesimal characters as the discrete series, whereas in [BLGHT11, Theorem 1.1] one uses the dual.

We will apply this to (a suitable twist of) Π_i and denote ρ_{Π_i} the resulting Galois representation. We define

$$\rho_{\pi,\mathrm{std},p} = \rho_{\pi_0} = \oplus_{i=1}^r \oplus_{j=1}^{\ell_i} \rho_{\Pi_i} \otimes |\cdot|^{(k_i-1)/2+j-(\ell_i+1)/2}$$

First, from the description of Hodge–Tate weights above in the case of $\mathrm{GL}(k)$ it is immediate that the Hodge–Tate weights of the crystalline representation $\rho_{\pi_0,p}|_{G_{F_v}}$ for $v \mid p$ are the entries of the Harish-Chandra parameter of $\pi_{0,v}$ for the corresponding $v \mid \infty$. Each Galois representation ρ_{Π_i} was twisted by $|\cdot|^{(k_i-1)/2}$ so that local-global compatibility holds without twisting. As a result, the eigenvalues of φ_{cris} on $\mathrm{D}_{\mathrm{cris}}(\rho_{\pi_0,p}|_{G_{F_v}})$ are $\chi_1(p),\dots,\chi_n(p),1,\chi_1^{-1}(p),\dots,\chi_n^{-1}(p)$, as desired. The statement at $v \nmid p$ with π_v unramified is analogous. □

Lemma 20. *Let π be a cohomological cuspidal automorphic representation of $\mathrm{GSp}(2n,\mathbb{A}_F)$, ν a p-stabilization of π of noncritical slope and $\mathcal{E} \to \mathcal{W}$ Urban's eigenvariety around (π,ν). If $\rho_{\pi,\mathrm{std},p}$ is irreducible, shrinking $\mathcal{W}$, there exists an analytic Galois representation $\rho_{\mathcal{E},\mathrm{std}} : G_F \to \mathrm{GL}(2n+1,\mathcal{O}_{\mathcal{E}})$ such that for $z \in \Sigma$, $z \circ \rho_{\mathcal{E},\mathrm{std}} = \rho_{\pi_z,\mathrm{std},p}$. If π has Iwahori level at $v \mid p$ then $\rho_{\mathcal{E},\mathrm{std}}$ admits a refinement in the sense of §1.3.*

In the case $n = 2$, if $\rho_{\pi,\mathrm{spin},p}$ is irreducible one obtains an analogous analytic spin Galois representation $\rho_{\mathcal{E},\mathrm{spin}} : G_F \to \mathrm{GSp}(4,\mathcal{O}_{\mathcal{E}})$ which admits a refinement if π_v is Iwahori spherical at $v \mid p$.

Proof. We will follow the proof of Lemma 15. First, the existence of $\rho_{\mathcal{E},\mathrm{spin}}$ and $\rho_{\mathcal{E},\mathrm{std}}$ follows analogously.

Next, we need to show that if π_v is Iwahori spherical at $v \mid p$ then there is a dense set of points $\Sigma' \subset \Sigma \subset \mathcal{E}$ such that if $z \in \Sigma'$ then $\pi_{z,v}$ is unramified at $v \mid p$.

Let $z \in \Sigma$ and let π_z be the associated cuspidal representation. Suppose $\pi_{z,v}$ is not unramified. Since it is Iwahori spherical it follows from [Tad94, Theorem 7.9] that for every $v \mid p$, $\pi_{z,v} = \mathrm{Ind}\,\chi$ where $\chi = \chi_1 \times \cdots \times \chi_n \rtimes \sigma$ such that one of the following is satisfied:

(1) $\chi_i^2 = 1$ but $\chi_i \neq 1$ for at least 3 indices i,
(2) $\chi_i = |\ \ |^{\pm 1}$ for at least one index i, or
(3) $\chi_i \chi_j^{\pm 1} = |\ \ |$ for at least one pair (i,j) and choice of exponent.

Denote by $\alpha_{v,i}$ the eigenvalue of $[\mathrm{Iw}\,\beta_{n-i}\,\mathrm{Iw}]$ acting on $\pi_{z,v}$. There exists a permutation ν and a function $\varepsilon : \{1,\dots,n\} \to \{-1,1\}$ such that $\alpha_{v,i}$ are given by Lemma 17. Solving, one obtains

$$\chi_{\nu^{-1}(i)}(p)^{\varepsilon(i)} = p^{c_{i,\nu,\varepsilon}-c_{i-1,\nu,\varepsilon}} \frac{\alpha_{v,i-1}}{\alpha_{v,i}}$$

for $1 < i < g$, $\chi_{\nu^{-1}(n)}(p) = p^{2c_{n,\nu,\varepsilon}-c_{n-1,\nu,\varepsilon}} \alpha_{v,n-1}/\alpha_{v,n}^2$ and $\chi_{\nu^{-1}(1)}(p)^{\varepsilon(1)} = p^{\mu_0 - c_{1,\nu,\varepsilon}} \alpha_{v,1}$ where the last equality comes from the fact that $\det \pi_{z,v}(p) = \chi_1 \cdots \chi_n \sigma^2(p) = p^{\mu_0}$.

But $\alpha_{v,i} = |\lambda_v(\beta_{n-i})|_p \cdot \theta(U_{v,i})$ where $\lambda_v(\beta_{n-i}) = p^{\sum_\tau \mu_{v,\tau,1}+\cdots+\mu_{v,\tau,i}-\mu_0}$ for $1 \le i < n$ and $\lambda_v(\beta_0) = p^{\sum_\tau(\mu_{v,\tau,1}+\cdots+\mu_{v,\tau,n}-\mu_0)/2}$. We deduce that

$$\chi_{\nu^{-1}(i)}(p)^{\varepsilon(i)} = p^{c_{i,\nu,\varepsilon}-c_{i-1,\nu,\varepsilon}+\sum_\tau \mu_{v,\tau,i}}\frac{\theta(U_{v,i-1})}{\theta(U_{v,i})}$$

for $1 < i < n$,

$$\chi_{\nu^{-1}(n)}(p) = p^{2c_{n,\nu,\varepsilon}-c_{n-1,\nu,\varepsilon}+\sum_\tau \mu_{v,\tau,n}}\theta(U_{v,n-1})/\theta(U_{v,n})^2$$

and

$$\chi_{\nu^{-1}(1)}(p)^{\varepsilon(1)} = p^{\mu_0-c_{1,\nu,\varepsilon}+\sum_\tau \mu_{v,\tau,1}-\mu_0}\theta(U_{v,1}) = p^{-c_{1,\nu,\varepsilon}+\sum_\tau \mu_{v,\tau,1}}\theta(U_{v,1}).$$

The functions $\theta(U_{v,i})$ are analytic on $\mathcal{E}$ and so $v_p(\theta(U_{v,i}))$ is locally constant. By shrinking $\mathcal{E}$ we may even assume they are constant. Since $\pi_{z,v}$ is Iwahori spherical but ramified it follows from the conditions listed above that $v_p(\chi_i(p)) \in \{0,1\}$ or $v_p(\chi_i(p)\chi_j(p)^{\pm1}) = 1$. This, however, implies certain linear combinations of the weights $\mu_{v,\tau,i}$ and μ_0 are constant, which is a contradiction as $\mathcal{E}$ maps to a full-dimensional open set in the weight space.

Suppose $z \in \Sigma'$ in which case $\rho_{\pi_z,\mathrm{std},p}|_{G_{F_v}}$ is crystalline for $v \mid p$. Consider the analytic functions $a_{v,i}(z) = z(\theta(U_{v,i}))$ on $\mathcal{E}$. Let $\kappa_{n+1,\tau}(z) = 0$, $\kappa_{n+1\pm i,\tau}(z) = \pm(\mu_{v,\tau,n+1-i}(z)+i)$ for $1 \le i \le n$; Theorem 19 these are the Hodge–Tate weights, arranged increasingly, of $z \circ \rho_{\mathcal{E}}|_{G_{F_v}}$. By local-global compatibility the eigenvalues of φ_{cris} acting on $\mathrm{D}_{\mathrm{cris}}(\rho_{\pi_z,p}|_{G_{F_v}})$ are $\chi_i(p)^{\pm1}$ and 1. We will use the formulae above to construct the analytic functions F_k.

Let $F_{n+1}(z) = 1$. For $1 < i < n$ let

$$F_{n+1\pm(n+1-i)}(z) = \left(p^{c_{i,\nu,\varepsilon}-c_{i-1,\nu,\varepsilon}-i}\frac{a_{v,i-1}(z)}{a_{v,i}(z)}\right)^{\pm1}.$$

Let $F_{n+1\pm(n+1-n)}(z) = \left(p^{2c_{n,\nu,\varepsilon}-c_{n-1,\nu,\varepsilon}-n}a_{v,n-1}/a_{v,n}^2\right)^{\pm1}$ and $F_{n+1\pm(n+1-1)}(z) = \left(p^{-c_{1,\nu,\varepsilon}-1}a_{v,1}\right)^{\pm1}$. Thus $p^{\sum_\tau \kappa_{n+1\pm i,\tau}}F_{n+1\pm i} = \chi_{\nu^{-1}(i)}(p)^{\pm\varepsilon(i)}$ for $1 < i \le n$ and $p^{\sum_\tau \kappa_{n+1\pm1,\tau}}F_{n+1\pm1} = \chi_{\nu^{-1}(1)}(p)^{\pm1}$. By Theorem 19 these are the eigenvalues of φ_{cris} and so $\rho_{\mathcal{E},\mathrm{std},p}$ admits a refinement.

Finally, we need to construct a refinement for $\rho_{\mathcal{E},\mathrm{spin},p}$ in the genus $n = 2$ case. The eigenvalues of φ_{cris} in this case acting on $\chi_1 \times \chi_2 \rtimes \sigma$ are $p^{-3/2} \times \{\sigma(p), \sigma(p)\chi_1(p), \sigma(p)\chi_2(p), \sigma(p)\chi_1(p)\chi_2(p)\}$. Let $\kappa_{1,\tau} = (\mu_0 - \mu_{v,\tau,1} - \mu_{v,\tau,2})/2$, $\kappa_{2,\tau} = (\mu_0 - \mu_{v,\tau,1} + \mu_{v,\tau,2})/2 + 1$, $\kappa_{3,\tau} = (\mu_0 + \mu_{v,\tau,1} - \mu_{v,\tau,2})/2 + 2$ and $\kappa_{4,\tau} = (\mu_0 + \mu_{v,\tau,1} + \mu_{v,\tau,2})/2 + 3$. For simplicity of notation we will assume that $\nu = 1$ and $\varepsilon = 1$, the other cases being analogous. (Later we will choose this refinement anyway.) Then $\sigma(p) = p^{c_2+\sum_\tau(\mu_0-\mu_{v,\tau,1}-\mu_{v,\tau,2})/2}a_{v,2}^{-1}$, $\chi_2(p) = p^{\sum_\tau \mu_{v,\tau,1}-c_1}a_{v,1}$ and $\chi_1(p) = p^{c_1-2c_2+\sum_\tau \mu_{v,\tau,2}}a_{v,2}^2a_{v,1}^{-1}$. Write $F_1 = p^{c_2-3/2}a_{v,2}^{-1}$, $F_2 = p^{c_1-c_2-1-3/2}(a_{v,2}/a_{v,1})$, $F_3 = p^{c_2-c_1-2-3/2}(a_{v,1}/a_{v,2})$ and $F_4 = p^{-c_2-3-3/2}a_{v,2}$ which are analytic and satisfy $p^{\sum\kappa_{1,\tau}}F_1 = p^{-3/2}\sigma(p)$, $p^{\sum\kappa_{2,\tau}}F_2 = p^{-3/2}\sigma(p)\chi_1(p)$, $p^{\sum\kappa_{3,\tau}}F_3 = p^{-3/2}\sigma(p)\chi_2(p)$ and $p^{\sum\kappa_{4,\tau}}F_4 = p^{-3/2}\sigma(p)\chi_1(p)\chi_2(p)$, which are the eigenvalues of φ_{cris}. Thus $\rho_{\mathcal{E},\mathrm{spin},p}$ has a refinement. □

Corollary 21. *Let π be a Hilbert modular form of infinity type $(k_1, \ldots, k_d; w)$. Suppose π is not CM and let Π be the cuspidal representation of $\mathrm{GSp}(4, \mathbb{A}_F)$ from Theorem 10. Let ν be a p-stabilization of Π and let $m \in \mathbb{Z}$ such that $\Pi \otimes |\cdot|^m$ has noncritical slope. For $v \mid p$, assume that π_v is Iwahori spherical; if π_v is an unramified principal series assume that $\alpha_v/\beta_v \notin \mu_{60}$. Then $\rho_{\mathcal{E},\mathrm{spin},p}$ has a global triangulation whose graded pieces $\mathcal{R}(\delta_i)$ are such that $z(\delta_i(u)) = \prod \tau(u)^{\kappa_{i,\tau}(z)}$ for $z \in \Sigma$ and $\delta_i(p) = F_i$ from the proof of Lemma 20.*

Proof. Since the Hodge–Tate weights in the triangulation are ordered increasingly the only thing left to check is that the associated refinement is regular, i.e., that $\det\varphi$ on the filtered piece $\mathcal{F}_i$ has multiplicity one in $\mathrm{D}_{\mathrm{cris}}(\wedge^i\rho_\Pi)$. This is equivalent to showing that for each $i \in \{1,2,3,4\}$, each product of i terms in $\{\alpha_v^3, \alpha_v^2\beta_v, \alpha_v\beta_v^2, \beta_v^3\}$ occurs once, which can be checked if $\alpha_v/\beta_v \notin \mu_{60}$. □

Corollary 22. *Let π be a Hilbert modular form of infinity type $(k_1, \ldots, k_d; w)$. Suppose π is not CM and let Π be the cuspidal representation of $\mathrm{GSp}(2n, \mathbb{A}_F)$ from Theorem 11. Let ν be a p-stabilization of Π and let $m \in \mathbb{Z}$ such that $\Pi \otimes |\cdot|^m$ has noncritical slope. For $v \mid p$, assume that π_v is Iwahori spherical; if π_v is an unramified principal series assume that $\alpha_v/\beta_v \notin \mu_\infty$. Then $\rho_{\mathcal{E},\mathrm{std},p}$ has a global triangulation whose graded pieces $\mathcal{R}(\delta_i)$ are such that $z(\delta_i(u)) = \prod \tau(u)^{\kappa_{i,\tau}(z)}$ for $z \in \Sigma$ and $\delta_i(p) = F_i$ from the proof of Lemma 20.*

Proof. As in the previous corollary we only need to check that for each $1 \le i \le 2n+1$ each product of i terms in $\{(\alpha_v/\beta_v)^k \mid -n \le k \le n\}$ occurs only once. Again, this can be checked when $\alpha_v/\beta_v \notin \mu_\infty$. □

3.4. **Eigenvarieties for unitary groups.** One could reproduce the results of §3.3 in the context of unitary groups. Indeed, the endoscopic classification for unitary groups was completed by Mok and compact unitary groups of course have discrete series so all the results translate into this context, again under the assumption of stabilization of the twisted trace formula. The main reason for redoing the computations using unitary groups is work in progress of Eischen–Harris–Li–Skinner and Eischen–Wan which will produce p-adic L-functions for unitary groups.

Let $F/\mathbb{Q}$ be a totally real field in which p is unramified and E/F a CM extension such that each place $v \mid p$ of F splits in E. Suppose U is a definite unitary group over F, in n variables, attached to E/F. Suppose π is an irreducible (necessarily cuspidal) automorphic representation π of $U(\mathbb{A}_F)$ of cohomological weight $\bigoplus_{v|\infty}(\mu_{v,1},\dots,\mu_{v,n})$. Then the restriction to $W_\mathbb{C}$ of the L-parameter of π_v is given by

$$z \mapsto \operatorname{diag}((z/\overline{z})^{\mu_{v,i}+(n+1)/2-i})$$

(cf. [BC09, §6.7]).

If for some $v \mid p$ the representation π_v has Iwahori level then the Hecke operators $U_{v,i} = [\mathrm{Iw}\, e_i^\vee(p)\,\mathrm{Iw}]$ act on π_v where $e_i^\vee$ is dual to the character e_i isolating the i-th entry on T. For consistency of notation with the previous section we remark that Urban's $\theta(U_{v,i})$ is denoted by $\delta^{1/2}\psi_{\pi,\mathcal{R}}$ in [BC09, §7.2.2]. If $\pi_v = \chi_1\times\cdots\times\chi_n$ is an unramified principal series then π_v^{Iw} is n dimensional and the Hecke operators $U_{v,i}$ can be simultaneously written in upper triangular form with diagonal entries $\chi_i(p)p^{-(n-1)/2}$.

Theorem 23. *Suppose U and π are as above, with π of cohomological weight $\bigoplus_{v|\infty}(\mu_{v,1},\dots,\mu_{v,n})$. Then there exists a continuous Galois representation $\rho_{\pi,p}: G_E \to \mathrm{GL}(n,\overline{\mathbb{Q}}_p)$ such that:*

1. *If $v \nmid p\infty$ and π_v is unramified then $\rho_{\pi,p}|_{G_{E_w}}$ is unramified for $w \mid p$ and $\mathrm{WD}(\rho_{\pi,p}|_{G_{E_w}})^{\text{Fr-ss}} \cong \mathrm{rec}(\mathrm{BC}_{E_w/F_v}(\pi_v)\otimes|\cdot|^{-(n-1)/2})$.*
2. *If $v \mid p$, since it splits in E we may write $v = w\overline{w}$ where w is the finite place of E corresponding to $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. If $\pi_v = \chi_1\times\cdots\times\chi_n$ is unramified then $\rho_{\pi,p}|_{G_{E_w}}$ is crystalline with Hodge–Tate weights $-\mu_{v,\tau,i}+i$ and φ_{cris} has eigenvalues $\chi_i(p)p^{-(n-1)/2}$.*

Proof. The proof is analogous to that of Theorem 19 as the transfer from U to GL is the content of [Mok15] (cf. [Sch15, Corollary 5.1.7]). The statement about Hodge-Tate weights follows by appealing to [BLGHT11, Theorem 1.2] rather than [BLGHT11, Theorem 1.1]. □

Remark 24. The literature contains base change results for both isometry unitary and similitude unitary groups to various degrees of generality. We remark that one may deduce base change for isometry unitary groups from the analogous results for similitude results using algebraic liftings of automorphic representations ([Pat12, Proposition 12.3.3]).

The main theorem of [Che09] implies that if $4 \mid n$, which we will asume, then the conclusion of Theorem 14 holds for π and a p-stabilization ν. Moreover, if $\rho_{\pi,p}$ is irreducible then Theorem 23 implies the existence of an analytic Galois representation $\rho_{\mathcal{E},p}: G_E \to \mathrm{GL}(n,\mathcal{O}_\mathcal{E})$ interpolating, as before, the Galois representations attached to the classical regular points on $\mathcal{E}$.

Corollary 25. *Let F be a totally real field in which p is unramified. Let π be a Hilbert modular form over F, of infinity type $(k_1,\dots,k_d;w)$, suppose there exist finite places w_1,w_2 not above p with the property that π_w is special for $w\in\{w_1,w_2\}$, and suppose that π_v is Iwahori spherical for $v \mid p$ and that if π_v is unramified with Satake parameters α_v and β_v then $\alpha_v/\beta_v \notin \mu_\infty$. Suppose π is not CM. Let E be a CM extension of F, ψ a Hecke character of E, and Π a cuspidal automorphic representation of $U(\mathbb{A}_F)$ such that $\Pi = \psi\otimes\mathrm{BC}_{E/F}\operatorname{Sym}^n\pi$ as in Proposition 13. Let $\mathcal{E}$ and $\rho_{\mathcal{E},p}$ as above. Then $\rho_{\mathcal{E},p}|_{G_{E_w}}$ for $w \mid v \mid p$ a finite place of E admits a triangulation with graded pieces $\mathcal{R}(\delta_i)$ such that $\delta_i(u) = \prod\tau(u)^{\kappa_{i,\tau}}$ for $u\in\mathcal{O}_{F_v}^\times$ and $\delta_i(p) = F_i$ where $\kappa_{i,\tau} = -\mu_{v,\tau,i}+i$ and $F_i = p^{(n-1)/2-i}a_{v,i}$ where $a_{v,i} = \theta(U_{v,i})$ is analytic over $\mathcal{E}$.*

Proof. Theorem 23 implies that at regular classical points which are unramified at $v \mid p$ the analytic functions κ_i give the Hodge–Tate weights of $\rho_{\mathcal{E},p}|_{G_{E_w}}$. Thus it suffices to check that $p^{\sum\kappa_{i,\tau}}F_i$ gives the eigenvalues of φ_{cris} at such regular unramified crystalline points. This follows from the fact that the eigenvalue of $U_{v,i}$ on π_v equals $a_{v,i}$ times $|\lambda(e_i^\vee(p))|_p^{-1}\delta^{-1/2}(e_i^\vee(p))$. Finally, the condition $\alpha_v/\beta_v \notin \mu_\infty$ implies the existence of the global triangulation as in the proof of Corollary 22. □

4. Computing the $\mathcal{L}$-invariants

Let F be a totally real field in which the prime p is unramified and let π be a non-CM cohomological Hilbert modular form of infinity type $(k_1, \ldots, k_d; w)$. Let $V_{2n} = \rho_{\pi,p} \otimes \det^{-n} \rho_{\pi,p}$. Suppose that for $v \mid p$, π_v is Iwahori spherical, which is equivalent to the requirement that $\rho_{\pi,p}|_{G_{F_v}}$ be semistable. For each such v let $D_v \subset \mathrm{D}_{\mathrm{st}}(V_{2n,v})$ be the regular submodule chosen in §2.2.

Under the hypotheses (C1–4) for $\mathrm{Ind}_F^{\mathbb{Q}} V_{2n}$, we will compute $\mathcal{L}(V_{2n}, \{D_v\})$ in terms of logarithmic derivatives of analytic Hecke eigenvalues over eigenvarieties. We will assume the existence of a rigid analytic space $\mathcal{E} \to \mathcal{W}$ which is étale at a weight w_0 over which one has a point $z_0 \in \mathcal{E}$ such that $z_0 \circ \rho_{\mathcal{E},p} \cong \psi \otimes \mathrm{Sym}^m \rho_{\pi,p}$ for some Hecke character ψ and some $m \geq n$. We will moreover assume that z_0 corresponds to the p-stabilization of Π_v coming from the p-stabilization of π_v that gave rise to the regular submodule D_v. Assume there exists a global analytic triangulation of $\mathscr{D}^\dagger_{\mathrm{rig}}(\rho_{\mathcal{E},p}|_{G_{F_v}})$ with graded pieces $\mathcal{R}(\delta_i)$. We remark that the weight space $\mathcal{W}$ contains analytic functions $\kappa_{i,v,\tau}$ corresponding to the infinite places (v, τ) as $v \mid p$ and $\tau : F_v \hookrightarrow \mathbb{C}_p$.

Before we state the lemma about cohomology classes arising from analytic Galois representations we introduce a "trace" map acting on cohomology. Suppose $F/\mathbb{Q}$ is a finite extension and $c \in H^1(F, V)$ for a G_F-representation V on a finite dimensional $\overline{\mathbb{Q}}_p$ vector space. The trace $\mathrm{Tr}_{F/\mathbb{Q}}(c)$ is the image of c under the composite map $H^1(F, V) \cong H^1(\mathbb{Q}, \mathrm{Ind}_F^{\mathbb{Q}} V) \xrightarrow{\mathrm{res}} H^1(F, (\mathrm{Ind}_F^{\mathbb{Q}} V)|_{G_F}) \to H^1(F, V) \cong H^1(\mathbb{Q}, \mathrm{Ind}_F^{\mathbb{Q}} V)$ where the last middle map is the natural projection. We would like to describe $\mathrm{res}_p \mathrm{Tr}_{F/\mathbb{Q}}(c) \in \oplus_{v|p} H^1(\mathbb{Q}_p, \mathrm{Ind}_{F_v}^{\mathbb{Q}_p} V_v)$ directly in terms of (φ, Γ)-modules in the form $\mathrm{res}_p \mathrm{Tr}_{F/\mathbb{Q}} c = \oplus_{v|p} \mathrm{Tr}_{F_v/\mathbb{Q}_p} \mathrm{res}_v c$. For $v \mid p$ unramified $\mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p} V_v) \cong \mathrm{Ind}_{F_v}^{\mathbb{Q}_p} \mathrm{D}^\dagger_{\mathrm{rig}}(V_v)$ which is defined as follows: as a $(\varphi, \Gamma_{\mathbb{Q}_p})$-module over $\mathcal{R}_{\mathbb{Q}_p} \otimes \overline{\mathbb{Q}}_p$ it is just the (φ, Γ_{F_v})-module $\mathrm{D}^\dagger_{\mathrm{rig}}(V_v)$ over $\mathcal{R}_{F_v} \otimes \overline{\mathbb{Q}}_p \cong \mathcal{R}_{\mathbb{Q}_p} \otimes_{\mathbb{Q}_p} F_v \otimes \overline{\mathbb{Q}}_p$ interpreted as a module over $\mathcal{R}_{\mathbb{Q}_p} \otimes \overline{\mathbb{Q}}_p$ on which $\gamma \in \Gamma_{\mathbb{Q}_p} \cong \Gamma_{F_v}$ acts as on D and $\varphi_{\mathrm{Ind}_{F_v}^{\mathbb{Q}_p} \mathrm{D}^\dagger_{\mathrm{rig}}(V)}$ acts as $\varphi_{\mathrm{D}^\dagger_{\mathrm{rig}}(V)}$.

Suppose $c \in H^1(\mathrm{D}^\dagger_{\mathrm{rig}}(V))$ is given by a pair $(a, b) \in \mathrm{D}^\dagger_{\mathrm{rig}}(V) \oplus \mathrm{D}^\dagger_{\mathrm{rig}}(V)$ such that $(\gamma - 1)a = (\varphi - 1)b$, the pair (a, b) being determined up to the image of $\mathrm{D}^\dagger_{\mathrm{rig}}(V)$ under $(\varphi - 1, \gamma - 1)$. To the class c corresponds the extension $0 \to \mathrm{D}^\dagger_{\mathrm{rig}}(V_v) \to D_c \to \mathcal{R}_{F_v} \otimes \overline{\mathbb{Q}}_p e \to 0$ with $\varphi_{D_c} v = v + a$ and $\gamma_{D_c} v = v + b$. Under $H^1(\mathrm{D}^\dagger_{\mathrm{rig}}(V_v)) \cong H^1(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p} \mathrm{D}^\dagger_{\mathrm{rig}}(V_v))$ corresponds the extension obtained by taking the preimage of $\mathcal{R}_{\mathbb{Q}_p} \otimes \overline{\mathbb{Q}}_p e$ under the projection map, again characterized by (a, b). Restriction to G_{F_v} on the Galois representation side is $\otimes_{\mathbb{Q}_p} F_v$ on the (φ, Γ)-module side and, after taking traces from F_v to $\mathbb{Q}_p$ on the level of scalars we get the extension $0 \to \mathrm{D}^\dagger_{\mathrm{rig}}(V_v) \to D_{\mathrm{Tr}\, c} \to \mathcal{R}_{F_v} \to 0$ given by $(\mathrm{Tr}\, a, \mathrm{Tr}\, b)$ where $\mathrm{Tr}\, a, \mathrm{Tr}\, b$ are now vectors over $\mathcal{R}_{\mathbb{Q}_p} \otimes \overline{\mathbb{Q}}_p$. This extension gives the class $\mathrm{Tr}\, c \in H^1(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p} \mathrm{D}^\dagger_{\mathrm{rig}}(V_v))$ and is compatible with the global construction. Indeed, for $c \in H^1(F, V)$, $\mathrm{res}_p \mathrm{Tr}_{F/\mathbb{Q}} c = \oplus_{v|p} \mathrm{Tr}_{F_v/\mathbb{Q}_p} \mathrm{res}_v c$.

Lemma 26. *Let $\overrightarrow{u} = (u_{i,v,\tau})$ be a direction in $\mathcal{W}$ and let $\nabla_{\overrightarrow{u}} \rho_{\mathcal{E},p}$ be the tangent space to $\rho_{\mathcal{E},p}$ in the $\overrightarrow{u}$-direction, which makes sense under the assumption that $\mathcal{E} \to \mathcal{W}$ is étale at z_0. Then $c_{\overrightarrow{u}} = (z_0 \circ \rho_{\mathcal{E},p})^{-1} \nabla_{\overrightarrow{u}} \rho_{\mathcal{E},p}$ is a cohomology class in $H^1(F, \mathrm{End}(z_0 \circ \rho_{\mathcal{E},p}))$. Its natural projection $c_{\overrightarrow{u},n} \in H^1(F, V_{2n}) \cong H^1(\mathbb{Q}, \mathrm{Ind}_F^{\mathbb{Q}} V_{2n})$ has the property that $\mathrm{Tr}_{F/\mathbb{Q}}(c_{\overrightarrow{u},n})$ lies in fact in the Selmer group $H^1(\oplus D_v, \mathrm{Ind}_F^{\mathbb{Q}} V_{2n})$.*

Proof. Note that $\mathrm{End}(z_0 \circ \rho_{\mathcal{E},p}) \cong \mathrm{End}(\psi \otimes \mathrm{Sym}^m \rho_{\pi,p}) \cong \oplus_{i=0}^m V_{2i}$ and the natural projection on cohomology arises from the natural projection of this representation to V_{2n}.

One needs to check two things. The first, that the cohomology classes are unramified at $v \notin S \cup \{w \mid p\}$ follows along the same lines as [Hid07, Lemma 1.3] as S contains the ramified places of F. The second is that the image of the cohomology class $\mathrm{Tr}_{F/\mathbb{Q}}(c_{\overrightarrow{u},n})$ in $H^1(\mathbb{Q}_p, \mathrm{Ind}_{F_v}^{\mathbb{Q}_p} V_{2n,v})/H^1_f(\mathbb{Q}_p, \mathrm{Ind}_{F_v}^{\mathbb{Q}_p} V_{2n,v})$ lands in $H^1(F_1 \mathrm{D}^\dagger_{\mathrm{rig}}(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p} V_{2n,v}))/H^1_f(\mathbb{Q}_p, \mathrm{Ind}_{F_v}^{\mathbb{Q}_p} V_{2n})$. But Proposition 34 implies that the natural projection $c_{\overrightarrow{u},n}$ (in the notation of this lemma) lies entirely in the span of $e_1^i e_2^{n-i}$ for $2i \leq n$ in $H^1(\mathrm{D}^\dagger_{\mathrm{rig}}(V_{2n,v}))$. By the choice of regular submodular D_v this implies that $\oplus_v \mathrm{res}_v \mathrm{Tr}_{F/\mathbb{Q}} c_{\overrightarrow{u},n} \in H^1(F_1 \mathrm{Ind}_F^{\mathbb{Q}} \mathrm{D}^\dagger_{\mathrm{rig}}(V_{2n})_p)$, as desired. □

Proposition 27. *Suppose that F is as before, unramified at p. In the notation of the previous lemma, suppose $\mathscr{D}^\dagger_{\mathrm{rig}}(\rho_{\mathcal{E},p}|_{G_{F_v}})$ has an analytic triangulation with graded pieces $\delta_1, \ldots, \delta_{m+1}$ where $\delta_i(\varpi_v) = \prod_{\tau: F_v \hookrightarrow \overline{\mathbb{Q}}_p} \tau(\varpi_v)^{\kappa_{i,\tau}} F_i$ for analytic functions $F_i \in \mathcal{O}(\mathcal{E})^\times$ and $z(\delta_i(u)) = \prod_{\tau: F_v \hookrightarrow \mathbb{C}_p} \tau(u)^{\kappa_{v,\tau}(z)}$ for a dense set of $z \in \mathcal{E}$ and any*

$u \in \mathcal{O}_{F_v}^\times$. *Then* $\mathrm{gr}_1\, \mathrm{D}_{\mathrm{rig}}^\dagger(\mathrm{Ind}_{F_v}^{\mathbb{Q}_p} V_{2n,v}) \cong \mathcal{R}_{\mathbb{Q}_p}$ *and*

$$\mathcal{L}(V_{2n}, \{D_v\}) = \prod_{v|p} \left(- \frac{\sum_i [F_v : \mathbb{Q}_p] B_{m,n,i-1}(\nabla_{\overrightarrow{u}} F_i)/F_i}{\sum_{i,\tau} B_{m,n,i-1} \nabla_{\overrightarrow{u}} \kappa_{i,\tau}} \right)$$

as long as this formula makes sense. Here, the coefficients $B_{m,n,i}$ *are given in Remark 35 and are basically alternating binomial coefficients multiplied by inverse Clebsch–Gordan coefficients.*

Proof. Lemma 4 implies that $\mathcal{L}(V_{2n}, \{D_v\}) = \prod_{v|p}(a_v/b_v)$ where the projection $\mathrm{Tr}_{F_v/\mathbb{Q}_p} c_{\overrightarrow{u},v}$ to $H^1(\mathrm{gr}_1\, \mathrm{Ind}_{F_v}^{\mathbb{Q}_p} \mathrm{D}_{\mathrm{rig}}^\dagger(V_{2n,v})) \cong H^1(\mathcal{R})$ is written as $a_v(-1,0) + b_v(0, \log_p \chi(\gamma))$. Proposition 34 implies that

$$c_{\overrightarrow{u},v} = \sum B_{m,n,i-1}(\nabla_{\overrightarrow{u}} \delta_i)/\delta_i : F_v^\times \to \overline{\mathbb{Q}}_p^\times$$

and this corresponds to the extension given by the pair $(c_{\overrightarrow{u},v}(\varpi_v), c_{\overrightarrow{u},v}(u))$ for any $u \in \mathcal{O}_v^\times$. We compute $a = c_{\overrightarrow{u},v}(\varpi_v) = \sum B_{m,n,i-1}(\nabla_{\overrightarrow{u}} F_i)/F_i$. Since for z in a dense subset of $\mathcal{E}$ we have $z(\delta_i(u)) = \prod_\tau \tau(u)^{\kappa_{i,\tau}(z)}$ we compute $\nabla_{\overrightarrow{u}} \delta_i(u)/\delta_i(u) = \sum_\tau \nabla_{\overrightarrow{u}} \tau(u)^{\kappa_{i,\tau}}/\tau(u)^{\kappa_{i,\tau}} = \sum_\tau \nabla_{\overrightarrow{u}} \kappa_{i,\tau} \log_p \tau(u)$. Then we get the explicit formula

$$\frac{a_v}{b_v} = \frac{\mathrm{Tr}_{F_v/\mathbb{Q}_p}(a)}{\mathrm{Tr}_{F_v/\mathbb{Q}_p}(b)/-\log_p N_{F_v/\mathbb{Q}_p} u} = - \frac{\sum [F_v : \mathbb{Q}_p] B_{m,n,i-1}(\nabla_{\overrightarrow{u}} F_i)/F_i}{\sum_{i,\tau} B_{m,n,i-1} \nabla_{\overrightarrow{u}} \kappa_{i,\tau}}$$

as $\mathrm{Tr}_{F_v/\mathbb{Q}_p} \log_p \tau(u) = \log_p N_{F_v/\mathbb{Q}_p}(u)$ and the norm map identifies $\mathcal{O}_v^\times$ with $\mathbb{Z}_p^\times$ compatibly with the isomorphism $\Gamma_{F_v} \cong \Gamma_{\mathbb{Q}_p}$.

□

In the remaining sections we apply Proposition 27 to obtain explicit formulae for $\mathcal{L}$-invariants for relevant symmetric powers in terms of logarithmic derivatives of analytic Hecke eigenvalues.

We will assume that F is a totally real field in which p is unramified and π is a cohomological Hilbert modular form with infinity type $(k_1, \ldots, k_d; w)$. At $v \mid p$ we assume that π_v is Iwahori spherical. In the computation of the $\mathcal{L}$-invariant of V_{2n} we will assume that $H_f^1(F, V_{2n}) = 0$. Throughout we will consider the refinement of π corresponding to the ordering e_1, e_2 of the basis of $\mathrm{D}_{\mathrm{st}}(\rho_{\pi,p}|_{G_{F_v}})$, which gives a suitable refinement of any automorphic form equivalent to $\mathrm{Sym}^m \pi$ using the ordering $e_1^m, e_1^{m-1} e_2, \ldots, e_2^m$. We will assume that if π_v is special then $\mathrm{Ind}_{F_v}^{\mathbb{Q}_p} \mathrm{D}_{\mathrm{rig}}^\dagger(V_{2n,v})$ satisfies condition (C4). (By the second remark of [Ben11, p. 1610] this is automatically true when π_v is unramified.)

4.1. **Symmetric squares.** Suppose for $v \mid p$ such that π_v is unramified that the two Satake parameters are distinct. Let $\mathcal{E}$ be the eigenvariety from Lemma 15. Suppose that $\mathcal{E}$ is étale over the weight space at the chosen refinement of π_v.

Theorem 28. *Writing* a_v' *for the derivative in the direction* $(1, \ldots, 1; -1)$ *in the weight space we compute*

$$\mathcal{L}(V_2, \{D_v\}) = \prod_{v|p} \left(\frac{-2a_v'}{a_v} \right)$$

Proof. Recall from Corollary 16 that $\kappa_{1,\tau} = (w - k_{v,\tau})/2$, $\kappa_{2,\tau} = (w + k_{v,\tau} - 2)/2$, $F_1 = a_v^{-1} p^{3/2}$ and $F_2 = a_v p^{1/2}$. The result now follows directly from Proposition 27 and the fact that $B_{1,1,0} = 1$ and $B_{1,1,1} = -1$. □

4.2. **Symmetric sixth powers.** Assume that π is not CM. Suppose for $v \mid p$ such that π_v is unramified that $\alpha_v/\beta_v \notin \mu_{60}$. Let Π be a suitably twisted Ramakrishnan–Shahidi lift of $\mathrm{Sym}^3 \pi$ such that the chosen refinement has noncritical slope (cf. Theorem 10 and Lemma 18). Let $\mathcal{E}$ be Urban's eigenvariety for $\mathrm{GSp}(4)$ and let $a_{v,1}$ and $a_{v,2}$ be the analytic Hecke eigenvalues from the proof of Lemma 20. Suppose that the eigenvariety $\mathcal{E}$ is étale over the weight space at the chosen refinement of Π.

Theorem 29. *If* $\overrightarrow{u} = (u_1, u_2; u_0)$ *is any direction in the weight space, i.e.* $u_1 \geq u_2 \geq 0$, *such that the denominator below is non-zero, then*

$$\mathcal{L}(V_6, \{D_v\}) = \prod_{v|p} \left(\frac{-4\widetilde{\nabla}_{\overrightarrow{u}} a_{v,2} + 3\widetilde{\nabla}_{\overrightarrow{u}} a_{v,1}}{u_1 - 2u_2} \right)$$

where we write $\widetilde{\nabla}_{\overrightarrow{u}} f = (\nabla_{\overrightarrow{u}} f)/f$.

Proof. Recall from Corollary 21 that $\kappa_1 = (\mu_0 - \mu_{v,1} - \mu_{v,2})/2$, $\kappa_2 = (\mu_0 - \mu_{v,1} + \mu_{v,2})/2 + 1$, $\kappa_3 = (\mu_0 + \mu_{v,1} - \mu_{v,2})/2 + 2$, $\kappa_4 = (\mu_0 + \mu_{v,1} + \mu_{v,2})/2 + 3$ giving $\nabla_{\vec{u}}\kappa_1 = (u_0 - u_1 - u_2)/2$, $\nabla_{\vec{u}}\kappa_2 = (u_0 - u_1 + u_2)/2$, $\nabla_{\vec{u}}\kappa_3 = (u_0 + u_1 - u_2)/2$, $\nabla_{\vec{u}}\kappa_4 = (u_0 + u_1 + u_2)/2$. Similarly, $\widetilde{\nabla}_{\vec{u}} F_1 = -\widetilde{\nabla}_{\vec{u}} a_{v,2}$, $\widetilde{\nabla}_{\vec{u}} F_2 = \widetilde{\nabla}_{\vec{u}} a_{v,2} - \widetilde{\nabla}_{\vec{u}} a_{v,1}$, $\widetilde{\nabla}_{\vec{u}} F_3 = \widetilde{\nabla}_{\vec{u}} a_{v,1} - \widetilde{\nabla}_{\vec{u}} a_{v,2}$, $\widetilde{\nabla}_{\vec{u}} F_4 = \widetilde{\nabla}_{\vec{u}} a_{v,2}$. Using that $(B_{3,3,i})_i \sim (1, -3, 3, -1)$ we deduce the formula. □

4.3. **Symmetric powers via symplectic groups.** We remark that the results of this paragraph are conditional on the stabilization of the twisted trace formula (cf. Theorem 19).

Assume that π is not CM. Suppose for $v \mid p$ such that π_v is unramified that $\alpha_v/\beta_v \notin \mu_\infty$. Suppose π satisfies the hypotheses of Theorem 8 (2) and let Π be a suitable (as before, from Lemma 18) twist of the cuspidal representation of $\mathrm{GSp}(2n, \mathbb{A}_F)$ from Theorem 11. Let $\mathcal{E}$ be Urban's eigenvariety for $\mathrm{GSp}(2n)$ and let $a_{v,i}$ be the analytic Hecke eigenvalues from the proof of Lemma 20. Suppose that the eigenvariety $\mathcal{E}$ is étale over the weight space at the chosen refinement of Π.

Theorem 30. *If $\vec{u} = (u_1, \ldots, u_n; u_0)$ is any direction in the weight space, such that the denominator below is non-zero, then*

$$\mathcal{L}(V_{4n-2}, \{D_v\}) = \prod_{v|p} -\left(\frac{B_n \widetilde{\nabla}_{\vec{u}} a_{v,1} + B_1(\widetilde{\nabla}_{\vec{u}} a_{v,n-1} - 2\widetilde{\nabla}_{\vec{u}} a_{v,n}) + \sum_{i=2}^{n-1} B_i(\widetilde{\nabla}_{\vec{u}} a_{v,i-1} - \widetilde{\nabla}_{\vec{u}} a_{v,i})}{\sum_{i=1}^{n} u_i B_{n+1-i}}\right)$$

where we write $B_i = (-1)^i \binom{2n}{n+i} i$.

Remark 31. Given that Remark 35 gives explicit values for the $B_{n,k,i}$, we know there will be directions where the denominator is non-zero.

Proof. Combine the formulae for the triangulation of the standard Galois representation from the proof of Lemma 20 with Proposition 27. Finally, compute

$$\begin{aligned} B_i &= B_{2n,2n-1,n+i} - B_{2n,2n-1,n-i} \\ &= (-1)^{n+i}\binom{2n}{n+i}(2n - 2(n+i)) - (-1)^{n-i}\binom{2n}{n-i}(2n - 2(n-i)) \\ &= (-1)^{n+1} 2 \cdot (-1)^i \binom{2n}{n+i} i \end{aligned}$$

and the result follows because all the B_i can be scaled by the same factor. □

4.4. **Symmetric powers via unitary groups.** We remark that the results of this paragraph as well are conditional on the stabilization of the twisted trace formula.

Assume that π is not CM. Suppose for $v \mid p$ such that π_v is unramified that $\alpha_v/\beta_v \notin \mu_\infty$. Suppose π satisfies the hypotheses of Theorem 8 (2) and Proposition 13. Let E/F the CM extension and Π the cuspidal representation of $U_{4n}(\mathbb{A}_F)$ which a transfer of a twist of $\mathrm{Sym}^{4n-1}\pi$ as in Proposition 13. Let $\mathcal{E}$ be Chenevier's eigenvariety and let $a_{v,i}$ be the analytic Hecke eigenvalues from the proof of Corollary 25. Suppose that the eigenvariety $\mathcal{E}$ is étale over the weight space at the chosen refinement of Π.

Theorem 32. *If $\vec{u} = (u_1, \ldots, u_n; u_0)$ is any direction in the weight space, then*

$$\mathcal{L}(V_{8n-2}, \{D_v\}) = \prod_{v|p}\left(\frac{-\sum_{i=1}^{4n}(-1)^{i-1}\binom{4n-1}{i-1}\widetilde{\nabla}_{\vec{u}} a_{v,i}}{\sum_{i=1}^{4n}(-1)^{i-1}\binom{4n-1}{i-1}u_i}\right)$$

and

$$\mathcal{L}(V_{8n-6}, \{D_v\}) = \prod_{v|p}\left(\frac{-\sum_{i=1}^{4n} B_{i-1}\widetilde{\nabla}_{\vec{u}} a_{v,i}}{\sum_{i=1}^{4n} u_i B_{i-1}}\right)$$

Here $B_i = B_{4n-1,4n-3,i}$ is the inverse Clebsch–Gordan coefficient of Proposition 34, up to a scalar independent of i given by

$$B_i = (-1)^i \binom{4n-1}{i}((4n-1)^3 - (4i+1)(4n-1)^2 + (4i^2+2i)(4n-1) - 2i^2).$$

Proof. Note that we cannot simply appeal to Proposition 27 as the analytic Galois representation on the unitary eigenvariety is a representation of G_E and not G_F. However, note that since E/F is finite and V_{2m} is a characteristic 0 vector space, the inflation-restriction sequence gives $H^1(G_{F,S}, V_{2m}) \cong H^1(G_{E,S_E}, V_{2m})^{G_{E/F}}$. We have constructed a cohomology class $c_{\vec{u}} \in H^1(G_{E,S_E}, V_m)$ and by construction $c_{\vec{u}}$ is invariant under the nontrivial element of $\mathrm{Gal}(E/F)$ (complex conjugation acts trivially on V_{2m}). Thus it descends to a cohomology class in $H^1(G_{F,S}, V_{2m})$. Since p splits completely in both E and F, this local component of the descended class is the same as that of the original class. This implies that the conclusion of Proposition 27 stays the same and the formulae follow from Corollary 25 as before. □

5. Appendix: Plethysm for GL(2)

Let V denote the standard two-dimensional representation of GL(2) (or SL(2)) (via matrix multiplication). In this section, we study the decomposition

$$\operatorname{End}\operatorname{Sym}^n V \cong \operatorname{Sym}^n V \otimes (\operatorname{Sym}^n V)^\vee \cong \bigoplus_{k=0}^{n} \operatorname{Sym}^{2k} V \otimes \det{}^{-k}$$

as representations of GL(2), or alternatively, of $\operatorname{End}\operatorname{Sym}^n V \cong \bigoplus_{k=0}^n \operatorname{Sym}^{2k} V$ as representations of SL(2).

In general, if $\mathfrak{g}$ is a Lie algebra acting on a finite dimensional vector space W, then $\mathfrak{g}$ acts on $W^\vee$ by $(Xf)(w) = f(-Xw)$. Moreover, $\mathfrak{g}$ acts on endomorphisms $f : W \to W$ by $(Xf)(w) = X(f(w)) + f(-Xw)$ and on $W \otimes W^\vee$ by $X(u \otimes w^\vee) = (Xu) \otimes w^\vee + u \otimes (Xw^\vee)$. We deduce that there exists a $\mathfrak{g}$-equivariant isomorphism $W \otimes W^\vee \cong \operatorname{End}(W)$ sending $u \otimes w^\vee$ to the endomorphism $x \mapsto w^\vee(x)u$.

Let $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ be lowering and raising matrices in $\mathfrak{sl}(2)$. We choose the basis (e_1, e_2) of V such that $Le_1 = e_2, Le_2 = 0$ and $Re_1 = 0, Re_2 = e_1$. For $m \geq 0$, we denote $V_m = \operatorname{Sym}^m V$ the representation of $\mathfrak{sl}(2)$ of highest weight m. Define $g_{m,0}, \ldots, g_{m,m}$ as the basis of V_m, thought of as a subset of $V^{\otimes m}$, by

$$g_{m,i} = \binom{m}{i}^{-1} \sum_{\underline{j}} e_{j_1} \otimes \cdots \otimes e_{j_m} := e_1^{m-1} e_2^i$$

where the sum is over m-tuples $\underline{j} = (j_1, \ldots, j_m) \in \{1, 2\}^m$ with $j_k = 1$ for exactly $m - i$ values of k. (When $i \notin \{0, \ldots, m\}$ we simply define $g_i^{(m)}$ to be 0.) The operators L and R act on $\operatorname{Sym}^m V$ via

$$Lg_{m,i} = (m - i)g_{m,i+1}$$
$$Rg_{m,i} = ig_{m,i-1}.$$

Let $m, n, p \in \mathbb{Z}$ such that V_p appears as a subrepresentation of $V_m \otimes V_n$, i.e. $p \in \{m+n, m+n-2, \ldots, |m-n|\}$. Denote by $\Xi_{m,n,p} : V_m \otimes V_n \to V_p$ the nontrivial $\mathfrak{sl}(2)$-equivariant projection from [CFS95, Lemma 2.7.4] and let

$$\psi_{m,n,p} = \left((\sqrt{-1})^{(m+n-p)/2} \frac{m!n!}{((m + n - p)/2)!} \right) \Xi_{m,n,p}$$

which will again be $\mathfrak{sl}(2)$-equivariant. Denote by $C^{u,v,w}_{m,n,p}$ be the "inverse Clebsch–Gordan" coefficients such that

$$\psi_{m,n,p}(g_{m,u} \otimes g_{n,v}) = \sum_{w=0}^{p} C^{u,v,w}_{m,n,p} g_{p,w}$$

Lemma 33. *The coefficients $C^{u,v,w}_{m,n,p}$ satisfy the recurrence relations*

$$(p - w)C^{u,v,w}_{m,n,p} = (m - u)C^{u+1,v,w+1}_{m,n,p} + (n - v)C^{u,v+1,w+1}_{m,n,p}$$
$$wC^{u,v,w}_{m,n,p} = uC^{u-1,v,w-1}_{m,n,p} + vC^{u,v-1,w-1}_{m,n,p}$$

and the initial values in the case when $u + v = (m + n - p)/2$ are given by

$$C^{u,v,0}_{m,n,p} = (-1)^u (m - u)!(n - v)!$$

and $C^{u,v,w}_{m,n,p} = 0$ for $w > 0$.

Proof. Since the map $\psi_{m,n,p}$ is $\mathfrak{sl}(2)$-equivariant we get

$$\begin{aligned} L\psi_{m,n,p}(g_{m,u}\otimes g_{n,v}) &= \psi_{m,n,p}((Lg_{m,u})\otimes g_{n,v}+g_{m,u}\otimes(Lg_{n,v})) \\ &= (m-u)\psi_{m,n,p}(g_{m,u+1}\otimes g_{n,v})+(n-v)\psi_{m,n,p}(g_{m,u}\otimes g_{n,v+1}) \\ &= (m-u)\sum_w C^{u+1,v,w}_{m,n,p}g_{p,w}+(n-v)\sum_w C^{u,v+1,w}_{m,n,p}g_{p,w}. \end{aligned}$$

At the same time this is

$$\begin{aligned} L\psi_{m,n,p}(g_{m,u}\otimes g_{n,v}) &= \sum_w C^{u,v,w}_{m,n,p}Lg_{p,w} \\ &= \sum_w C^{u,v,w}_{m,n,p}(p-w)g_{p,w+1}, \end{aligned}$$

which gives, after identifying the coefficients of $g_{p,w+1}$, the recurrence

$$(p-w)C^{u,v,w}_{m,n,p}=(m-u)C^{u+1,v,w+1}_{m,n,p}+(n-v)C^{u,v+1,w+1}_{m,n,p}$$

The second recurrence formula is obtained analogously applying the operator R to the definition of the coefficients $C^{u,v,w}_{m,n,p}$.

In [CFS95, Lemma 2.7.4] $g_{m,u}$ is denoted $e_{m/2,m/2-u}$ and the content of the lemma is that

$$\Xi_{m,n,p}(g_{m,u}\otimes g_{n,v})=\left((\sqrt{-1})^{(m+n-p)/2}(-1)^u\frac{((m+n-p)/2)!(m-u)!(n-v)!}{m!n!}\right)g_{p,0}$$

when $u+v=(m+n-p)/2$. The result follows from the definition of $\psi_{m,n,p}$. □

Lemma 33 gives an explicit map $V_n\otimes V_n\to V_{2k}$ for $k\le n$. To make explicit the map $\mathrm{End}(V_n)\to V_{2k}$, we start with $V_n\cong V_n^\vee$, which is noncanonical as an isomorphism of vector spaces, but can be chosen uniquely (up to scalars) as follows to make the isomorphism $\mathfrak{g}$-equivariant.

Let $(e_1^\vee,e_2^\vee)$ be the basis of V dual to (e_1,e_2), in which case the dual basis to $g_{n,i}$ is

$$g^\vee_{n,i}=\sum_{\underline{j}}e^\vee_{j_1}\otimes\cdots\otimes e^\vee_{j_n},$$

where again the sum is over $\underline{j}$ with $j_k=1$ for exactly $n-i$ values of k. The map φ sending $e_1\mapsto -e_2^\vee$ and $e_2\mapsto e_1^\vee$ is an $\mathfrak{sl}(2)$-equivariant isomorphism $V\cong V^\vee$ and this leads to the $\mathfrak{sl}(2)$-equivariant isomorphism $\varphi_n:V_n\to V_n^\vee$ sending $g_{n,i}\mapsto(-1)^{n-i}\binom{n}{i}^{-1}g^\vee_{n,n-i}$. This implies

$$\begin{aligned} Lg_i^\vee &= -(n+1-i)g^\vee_{i-1} \\ Rg_i^\vee &= -(i+1)g^\vee_{i+1}. \end{aligned}$$

We remark that, as an $\mathfrak{sl}(2)$-representation, V_m has weights $\{m,m-2,\dots,-m\}$ and L maps the weight w eigenspace to the weight $w-2$ eigenspace, while R goes in the other direction. Moreover, the vector $g_{n,i}\otimes g^\vee_{n,j}\in V_n\otimes V_n^\vee$ has weight $2(j-i)$ and this implies that

$$v_{2k}=\sum_{i=0}^{n-k}\binom{k+i}{i}g_{n,i}\otimes g^\vee_{n,k+i}$$

has (highest) weight $2k$. Computationally, this vector suffices to make explicit the projection $\mathrm{End}(V_n)\to V_{2k}$. Indeed, V_{2k} has basis $\{(2k-i)!L^iv_{2k}|i=0,\dots,2k\}$ and this basis is proportional to the $(g_{2k,0},\dots,g_{2k,2k})$. Thus the projection $\mathrm{End}(V_n)\cong V_n\otimes V_n^\vee\to V_{2k}$ can be computed by finding the projection of $g_{n,i}\otimes g^\vee_{n,j}$ to V_{2k} in terms of the basis $\{(2k-i)!L^iv_{2k}|i=0,\dots,2k\}$, which amounts to a matrix inversion.

However, we will obtain a closed expression for the projection map using the inverse Clebsch–Gordan coefficients from Lemma 33. The endomorphism $g_{n,i}\otimes g^\vee_{n,j}\in V_n\otimes V_n^\vee\cong\mathrm{End}(V_n)$ projects to V_{2k} and the composite map is

$$\begin{aligned} \psi_{n,n,2k}\circ(1\otimes\varphi_n^{-1})(g_{n,i}\otimes g^\vee_{n,j}) &= (-1)^j\binom{n}{j}\psi_{n,n,2k}(g_{n,i}\otimes g_{n,n-j}) \\ &= (-1)^j\binom{n}{j}\sum_{w=0}^{2k}C^{i,n-j,w}_{n,n,2k}g_{2k,w} \end{aligned}$$

We arrive at the main result of this section:

Proposition 34. *Suppose the representation V has basis (e_1, e_2) and $V_n = \mathrm{Sym}^n V$ has basis $(g_{n,0}, \dots, g_{n,n})$. Suppose $T \in \mathrm{End}(\mathrm{Sym}^n V)$ has an upper triangular matrix with $(a_0, \dots, a_n)$ on the diagonal with respect to this basis. Then the projection of T to V_{2k} is*

$$\begin{pmatrix} * & \cdots & * & \sum_{i=0}^{n} B_{n,k,i} a_i & 0 & \cdots & 0 \end{pmatrix}$$

with respect to the basis $(g_{2k,i})$ of V_{2k} where

$$B_{n,k,i} = \sum_{a+b=k} (-1)^a \binom{n}{i} \binom{i}{a} \binom{n-i}{b} (n-i+a)!(i+b)!$$

Here, the explicit coordinate is the middle one, i.e. the coefficient of $g_{2k,k}$, and we use the usual convention that $\binom{x}{y} = 0$ if $y < 0$ or $y > x$.

Proof. That T is upper triangular implies that it is a linear combination of the form $\sum_{i \le j} \alpha_{i,j} g_{n,i} \otimes g_{n,j}^\vee$ (here $a_i = \alpha_{i,i}$). But $g_{n,i} \otimes g_{n,j}^\vee$ has weight $2(j-i) \ge 0$ and so $T \subset \bigoplus_{w \ge 0} (\mathrm{End}\, V_n)_w$. Therefore its projection to V_{2k} belongs to $\bigoplus_{w \ge 0} (V_{2k})_w$, which is spanned by $g_{2k,u}$ for $u \le k$. This implies that the coefficients of $g_{2k,u}$ for $u > k$ are 0. Note that the projection $\mathrm{End}\, V_n \to (V_{2k})_0$ to the weight 0 eigenspace factors through $(\mathrm{End}\, V_n)_0 \to (V_{2k})_0$ and so the projection of T to $(V_{2k})_0$ only depends on the image $\sum a_i g_i \otimes g_i^\vee$ of T in $(\mathrm{End}\,\mathrm{Sym}^n V)_0$.

The coefficient $B_{n,k,i}$ is then the coefficient of $g_{2k,k}$ in the projection to V_{2k} of $g_{n,i} \otimes g_{n,i}^\vee$. By the discussion above this is

$$B_{n,k,i} = (-1)^i \binom{n}{i} C_{n,n,2k}^{i,n-i,k}$$

Lemma 33 implies inductively that for $m \le k$:

$$C_{n,n,2k}^{i,n-i,k} = \sum_{a+b=m} \binom{i}{a} \binom{n-i}{b} \binom{k}{m}^{-1} C_{n,n,2k}^{i-a,n-i-b,k-m}$$

and so

$$C_{n,n,2k}^{i,n-i,k} = \sum_{a+b=k} (-1)^{i-a} \binom{i}{a} \binom{n-i}{b} (n-i+a)!(i+b)!$$

and the formula follows. □

Remark 35. We end with a computation of the special values of $B_{n,k,i}$ which are involved in our formulae for $\mathcal{L}$-invariants. In the main formula for $B_{n,k,i}$ in Proposition 34, the indices a and b satisfy $a+b=k$, $a \le i$ and $b \le n-i$. When $k = n$ the only posibility is $(a,b) = (i, n-i)$, when $k = n-1$ the two possibilities are $a \in \{i-1, i\}$, and when $k = n-2$ the three possibilities are $a \in \{i-2, i-1, i\}$. Thus

$$\begin{aligned} B_{n,n,i} &= (-1)^i (n!)^2 \binom{n}{i}, \\ B_{n,n-1,i} &= \binom{n}{i} \left((-1)^{i-1} \binom{i}{i-1} \binom{n-i}{n-i} (n-1)!n! + (-1)^i \binom{i}{i} \binom{n-i}{n-1-i} n!(n-1)! \right) \\ &= (-1)^i n!(n-1)! \binom{n}{i} (n-2i), \\ B_{n,n-2,i} &= (-1)^i \binom{n}{i} \left(\binom{i}{2} (n-2)!n! - \binom{i}{1} \binom{n-i}{1} ((n-1)!)^2 + \binom{n-i}{2} n!(n-2)! \right) \\ &= (-1)^i \binom{n}{i} (n-2)!(n-1)!(n^3 - (4i+1)n^2 + (4i^2+2i)n - 2i^2) \end{aligned}$$

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