ON THE GROSS-KEATING INVARIANT OF A QUADRATIC FORM OVER A NON-ARCHIMEDEAN LOCAL FIELD

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Abstract. Let $B$ be a half-integral symmetric matrix of size $n$ defined over $\mathbb{Q}_p$. The Gross-Keating invariant of $B$ was defined by Gross and Keating, and has important applications to arithmetic geometry. But the nature of the Gross-Keating invariant was not understood very well for $n \geq 4$. In this paper, we establish basic properties of the Gross-Keating invariant of a half-integral symmetric matrix of general size over an arbitrary non-archimedean local field of characteristic zero.

INTRODUCTION

Gross and Keating [3] introduced a certain invariant for a quadratic form over $\mathbb{Z}_p$. This invariant is called the Gross-Keating invariant, and has applications to arithmetic geometry (see ARGOS seminar [1], Bouw [2], Gross and Keating [3], Kudla, Rapoport, and Yang [7], Wedhorn [13]). For $p \neq 2$, the Gross-Keating invariant can be easily calculated by means of the Jordan splitting. For $p = 2$, the nature of the Gross-Keating invariant of a quadratic form of degree $n$ was understood only for $n \leq 3$. The purpose of this paper is to investigate the basic properties of the Gross-Keating invariants for a quadratic form of general degree over the ring of integers of a non-archimedean local field of characteristic zero.

Let us recall the definition of the Gross-Keating invariant. Let $F$ be a non-archimedean local field of characteristic 0, and $\mathfrak{o} = \mathfrak{o}_F$ its ring of integers. $F$ is said to be dyadic if $F$ is a finite extension of $\mathbb{Q}_2$. The order $\text{ord}(x)$ of $x \in F^\times$ is normalized so that $\text{ord}(\varpi) = 1$ for a prime element $\varpi$ of $F$. We understand $\text{ord}(0) = +\infty$.

The set of symmetric matrices $B \in M_n(F)$ of size $n$ is denoted by $\text{Sym}_n(F)$. For $B \in \text{Sym}_n(F)$ and $X \in \text{GL}_n(F)$, we set $B[X] = XBX$. We say that $B = (b_{ij}) \in \text{Sym}_n(F)$ is a half-integral symmetric matrix.
if
\[ b_{ii} \in o_F \quad (1 \leq i \leq n), \]
\[ 2b_{ij} \in o_F \quad (1 \leq i \leq j \leq n). \]
The set of all half-integral symmetric matrices of size \( n \) is denoted by \( \mathcal{H}_n(o) \). An element \( B \in \mathcal{H}_n(o) \) is non-degenerate if \( \det B \neq 0 \). The set of all non-degenerate elements of \( \mathcal{H}_n(o) \) is denoted by \( \mathcal{H}_n^{\text{nd}}(o) \).

The equivalence class of \( B \in \mathcal{H}_n(o) \) is denoted by \( \{ B \} \), i.e., \( \{ B \} = \{ B[U] \mid U \in \text{GL}_n(o) \} \). We write \( B \sim B' \) if \( B' \in \{ B \} \).

**Definition 0.1.** Let \( B = (b_{ij}) \in \mathcal{H}_n^{\text{nd}}(o) \). Let \( S(B) \) be the set of all non-decreasing sequences \( (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) such that
\[ \text{ord}(b_{ii}) \geq a_i \quad (1 \leq i \leq n), \]
\[ \text{ord}(2b_{ij}) \geq (a_i + a_j)/2 \quad (1 \leq i \leq j \leq n). \]
Put
\[ S(\{ B \}) = \bigcup_{B' \in \{ B \}} S(B') = \bigcup_{U \in \text{GL}_n(o)} S(B[U]). \]
The Gross-Keating invariant \( \text{GK}(B) \) of \( B \) is the greatest element of \( S(\{ B \}) \) with respect to the lexicographic order \( \succeq \) on \( \mathbb{Z}_{\geq 0}^n \).

Note that for \( a \in S(B) \) and \( \sigma \in \mathfrak{S}_n \), we have \( \text{ord}(2b_{i\sigma(j)}) \geq (a_i + a_{\sigma(j)})/2 \). It follows that
\[ \text{ord}(2^n b_{1\sigma(1)} \cdots b_{n\sigma(n)}) \geq a_1 + \cdots + a_n. \]
Therefore we have \( a_1 + \cdots + a_n \leq \text{ord}(\det(2B)) \). In particular, \( S(\{ B \}) \) is a finite set.

A sequence of length 0 is denoted by \( \emptyset \). When \( B \) is the empty matrix, we understand \( \text{GK}(B) = \emptyset \). By definition, the Gross-Keating invariant \( \text{GK}(B) \) is determined only by the equivalence class of \( B \). Note that \( \text{GK}(B) = (a_1, \ldots, a_n) \) is also defined by
\[ a_1 = \max_{(y_1, \ldots) \in S(\{ B \})} \{ y_1 \}, \]
\[ a_2 = \max_{(a_1, y_2, \ldots) \in S(\{ B \})} \{ y_2 \}, \]
\[ \vdots \]
\[ a_n = \max_{(a_1, a_2, \ldots, a_{n-1}, y_n) \in S(\{ B \})} \{ y_n \}. \]

**Definition 0.2.** \( B \in \mathcal{H}_n^{\text{nd}}(o) \) is optimal if \( \text{GK}(B) \in S(B) \).

By definition, a non-degenerate half-integral symmetric matrix \( B \in \mathcal{H}_n^{\text{nd}}(o) \) is equivalent to an optimal form.
Remark 0.1. If $F$ is non-dyadic, one can easily show that the Jordan splitting of $B$ is optimal (See Remark 1.1). On the other hand, if $F$ is dyadic, a Jordan splitting may not be optimal. For example, if $F$ is dyadic, then $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is not optimal (See section 2). Thus the issue is the case when $F$ is dyadic. A characterization of an optimal form will be given in section 5 (See Theorem 5.1).

For $B \in \mathcal{H}_n^\text{nd}(\mathfrak{o})$, we put $D_B = (-4)^{n/2} \det B$. $D_B$ (or its image in $F^\times/F^{\times 2}$) is often called the signed determinant ([8]) or the discriminant ([10]) of $2B$. Here, $F^\times = \{ x^2 \mid x \in F^\times \}$. If $n$ is even, we denote the discriminant ideal of $F(\sqrt{D_B})/F$ by $\mathfrak{D}_B$. We also put

$$\xi_B = \begin{cases} 1 & \text{if } D_B \in F^{\times 2}, \\ -1 & \text{if } F(\sqrt{D_B})/F \text{ is unramified and } [F(\sqrt{D_B}) : F] = 2, \\ 0 & \text{if } F(\sqrt{D_B})/F \text{ is ramified.} \end{cases}$$

We also write $\xi(B)$ for $\xi_B$, if there is no fear of confusion.

Definition 0.3. For $B \in \mathcal{H}_n^\text{nd}(\mathfrak{o})$, we put

$$\Delta(B) = \begin{cases} \ord(D_B) & \text{if } n \text{ is odd}, \\ \ord(D_B) - \ord(\mathfrak{D}_B) + 1 - \xi_B^2 & \text{if } n \text{ is even.} \end{cases}$$

Note that if $n$ is even, then

$$\Delta(B) = \begin{cases} \ord(D_B) & \text{if } \ord(\mathfrak{D}_B) = 0, \\ \ord(D_B) - \ord(\mathfrak{D}_B) + 1 & \text{if } \ord(\mathfrak{D}_B) > 0. \end{cases}$$

For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_n$, we write $|a| = a_1 + a_2 + \cdots + a_n$.

Theorem 0.1. For $B \in \mathcal{H}_n^\text{nd}(\mathfrak{o})$, we have

$$|\text{GK}(B)| = \Delta(B).$$

For a non-decreasing sequence $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_n$, we set

$$G_a = \{ g = (g_{ij}) \in \text{GL}_n(\mathfrak{o}) \mid \ord(g_{ij}) \geq (a_j - a_i)/2, \text{if } a_i < a_j \}.$$ 

Theorem 0.2. Suppose that $B \in \mathcal{H}_n^\text{nd}(\mathfrak{o})$ is optimal and $\text{GK}(B) = a$. Let $U \in \text{GL}_n(\mathfrak{o})$. Then $B[U]$ is optimal if and only if $U \in G_a$.

For $B = (b_{ij})_{1 \leq i, j \leq n} \in \mathcal{H}_n(\mathfrak{o})$ and $1 \leq m \leq n$, we denote the upper left $m \times m$ submatrix $(b_{ij})_{1 \leq i, j \leq m} \in \mathcal{H}_m(\mathfrak{o})$ by $B^{(m)}$. For $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_n$, we put $a^{(m)} = (a_1, a_2, \ldots, a_m)$ for $m \leq n$.

Theorem 0.3. Suppose that $B \in \mathcal{H}_n^\text{nd}(\mathfrak{o})$ is optimal and $\text{GK}(B) = a$. If $a_k < a_{k+1}$, then $B^{(k)}$ is also optimal and $\text{GK}(B^{(k)}) = a^{(k)}$. 
Definition 0.4. The Clifford invariant (see Scharlau [10], p. 333) of $B \in \mathcal{H}^\text{nd}_n(\mathfrak{o})$ is the Hasse invariant of the Clifford algebra (resp. the even Clifford algebra) of $B$ if $n$ is even (resp. odd).

We denote the Clifford invariant of $B$ by $\eta_B$. We also write $\eta(B)$ for $\eta_B$, if there is no fear of confusion. If $B$ is $\text{GL}_n(F)$-equivalent to a diagonal matrix $\text{diag}(b'_1, \ldots, b'_n)$, then

$$\eta_B = \langle -1, -1 \rangle^{(n+1)/4} \langle -1, \det B \rangle^{(n-1)/2} \prod_{i<j} \langle b'_i, b'_j \rangle$$

$$= \begin{cases} 
\langle -1, -1 \rangle^{m(m-1)/2} \langle -1, \det B \rangle^{m-1} \prod_{i<j} \langle b'_i, b'_j \rangle & \text{if } n = 2m, \\
\langle -1, -1 \rangle^{m(m+1)/2} \langle -1, \det B \rangle^{m} \prod_{i<j} \langle b'_i, b'_j \rangle & \text{if } n = 2m + 1.
\end{cases}$$

(See Scharlau [10] pp. 80–81.) The Clifford invariant $\eta_B$ depends only on the image of $B$ in the Witt group of $F$. In particular, if $n$ is odd, then we have

$$\eta_B = \begin{cases} 
1 & \text{if } B \text{ is split over } F, \\
-1 & \text{otherwise}.
\end{cases}$$

Theorem 0.4. Let $B, B_1 \in \mathcal{H}^\text{nd}_n(\mathfrak{o})$. Suppose that $B \sim B_1$ and both $B$ and $B_1$ are optimal. Let $a = (a_1, a_2, \ldots, a_n) = \text{GK}(B) = \text{GK}(B_1)$. Suppose that $a_k < a_{k+1}$ for $1 \leq k < n$. Then the following assertions (1) and (2) hold.

(1) If $k$ is even, then $\xi_{B^{(k)}} = \xi_{B_1^{(k)}}$.

(2) If $k$ is odd, then $\eta_{B^{(k)}} = \eta_{B_1^{(k)}}$.

Remark 0.2. It is known that $B^{(k)} \sim B_1^{(k)}$ if $F$ is non-dyadic. But it is not true if $F$ is dyadic. Suppose that $F = \mathbb{Q}_2$. Put $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B_1 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$. Then we have $B \sim B_1$ and $\text{GK}(B) = (0, 1)$. Moreover both $B$ and $B_1$ are optimal. But $B^{(1)} = (1)$ is not equivalent to $B_1^{(1)} = (3)$. Thus $B^{(k)}$ and $B_1^{(k)}$ may not be equivalent if $F$ is dyadic.

We define $a_1^* < a_2^* < \cdots < a_r^*$ by

$$\{a_1^*, \ldots, a_r^*\} = \{a_1, a_2, \ldots, a_n\}.$$
Put \( n_s = \sharp \{ i \mid a_i = a_i^* \} \) and \( n_s^* = n_1 + n_2 + \cdots + n_s \) for \( s = 1, \ldots, r \). In particular, \( n_r^* = n \).

Put
\[
\zeta_s = \begin{cases} 
\xi_{B(n^*_s)} & \text{if } n_s^* \text{ is even}, \\
\eta_{B(n^*_s)} & \text{if } n_s^* \text{ is odd}.
\end{cases}
\]

By definition, if \( n_s^* \) is even, we have
\[
\zeta_s \neq 0 \iff a_1^* n_1 + \cdots + a_s^* n_s \text{ is even}.
\]
Moreover, \( \zeta_s \neq 0 \) if \( n_s^* \) is odd. Then we can show the following theorem (see Theorem 6.1).

**Theorem 0.5.** Suppose that \( n_s^* \) is odd. Then we have

(a) Assume that \( n_s^* \) is even for any \( i < s \). Then we have
\[
\zeta_s = \zeta_1^{a_1^*} \zeta_2^{a_2^*} \cdots \zeta_{s-1}^{a_{s-1}^*}.
\]
In particular, \( \zeta_1 = 1 \) if \( n_1^* \) is odd.

(b) Assume that \( a_1^* n_1 + \cdots + a_{s-1}^* n_{s-1} + a_s^*(n_s - 1) \) is even and that \( n_s^* \) is odd for some \( i < s \). Let \( t < s \) be the largest number such that \( n_t^* \) is odd. Then we have
\[
\zeta_s = \zeta_t^{a_{t+1}^*} \cdots \zeta_{s-1}^{a_{s-1}^*}.
\]
In particular, \( \zeta_s = \zeta_t \) if \( t + 1 = s \).

The datum \( \text{EGK}(B) = (n_1, \ldots, n_r; a_1^*, \ldots, a_r^*; \zeta_1, \ldots, \zeta_r) \) is called the extended GK datum of \( B \). In general, a datum \( (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r) \) satisfying these conditions is called an EGK datum (see Definition 6.2). The EGK data have an application to the theory of Siegel series. For the theory of Siegel series, one can consult Katsurada [5] and Shimura [11]. The Siegel series arises from the local factor of the Fourier coefficients of Siegel Eisenstein series. For \( F = \mathbb{Q}_p \), an explicit formula for the Siegel series was given by Katsurada [5]. But the explicit formula in [5] was very complicated for \( p = 2 \). In our forthcoming paper [4], we will show that there exists a Laurent polynomial \( \tilde{F}(\text{EGK}(B); Y, X) \in \mathbb{Z}[X^{1/2}, X^{-1/2}, Y^{1/2}, Y^{-1/2}] \) such that the Laurent polynomial \( \tilde{F}(B, X) \) attached to the Siegel series of \( B \) is given by
\[
\tilde{F}(B, X) = \tilde{F}(\text{EGK}(B); q^{1/2}, X).
\]
In particular, the Siegel series of \( B \) is determined by \( \text{EGK}(B) \). Note that this formula holds for both the non-dyadic case and the dyadic case.

We now explain the content of this paper. In section 1, we will discuss some elementary properties of the Gross-Keating invariant. In particular, we show that if \( B_1 \in \mathcal{H}_m^\text{nd}(\mathfrak{o}) \) is represented by \( B \in \mathcal{H}_n^\text{nd}(\mathfrak{o}) \),
then \( \text{GK}(B_1) \succeq \text{GK}(B)^{(m)} \) (Lemma 1.2). This lemma is useful to calculate Gross-Keating invariants. In section 2, we calculate Gross-Keating invariants of binary forms explicitly. The results of section 3 and 4 are the technical heart of this paper. In section 3, we introduce reduced forms (see Definition 3.2) and discuss its properties. In section 4, we prove the reduction theorem (Theorem 4.1) which says that any half-integral symmetric matrix is equivalent to a reduced form. By the reduction theorem, the proofs of the theorems above are reduced to the case of reduced form. Using these results, we prove the Theorems 0.1–0.4 in section 5. In section 6, we discuss some combinatorial properties of auxiliary invariants \( \xi_B^{(k)} \) and \( \eta_B^{(k)} \). To this end, we introduce EGK data (Definition 6.2), and show that these invariants satisfy the axioms of EGK data (Theorem 6.1).

Throughout this paper, except for section 1 and section 6, we mainly discuss the dyadic case. The proof of the Theorems 0.1–0.4 for the non-dyadic case is briefly explained at the end of section 1.

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**Notation**

When \( R \) is a ring, the set of \( m \times n \) matrices with entries in \( R \) is denoted by \( M_{m,n}(R) \) or \( M_{m,n}(R) \). As usual, \( M_n(R) = M_{n,n}(R) \). The identity matrix of size \( n \) is denoted by \( 1_n \). For \( X_1 \in M_{s}(R) \) and \( X_2 \in M_{t}(R) \), the matrix \( \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in M_{s+t}(R) \) is denoted by \( X_1 \perp X_2 \).

The diagonal matrix whose diagonal entries are \( b_1, \ldots, b_n \) is denoted by \( \text{diag}(b_1, \ldots, b_n) = \underbrace{b_1 \perp \cdots \perp b_n}_{n} \).

Let \( F \) be a non-archimedean local field of characteristic 0, and \( \mathfrak{o} = \mathfrak{o}_F \) its ring of integers. The maximal ideal and the residue field of \( \mathfrak{o} \) are denoted by \( \mathfrak{p} \) and \( \mathfrak{f} \), respectively. We put \( q = [\mathfrak{o} : \mathfrak{p}] \). We fix a prime element \( \varpi \) of \( \mathfrak{o} \) once and for all. The order of \( x \in F^\times \) is given by \( \text{ord}(x) = n \) for \( x \in \varpi^n \mathfrak{o}^\times \). We understand \( \text{ord}(0) = +\infty \). Put \( F^\times^2 = \{ x^2 \mid x \in F^\times \} \) and \( \mathfrak{o}^\times^2 = \{ x^2 \mid x \in \mathfrak{o}^\times \} \).

When \( G \) is a subgroup of \( \text{GL}_n(F) \), we shall say that two elements \( B_1, B_2 \in \text{Sym}_n(F) \) are called \( G \)-equivalent, if there is an element \( X \in G \) such that \( B_1[X] = B_2 \). When \( G = \text{GL}_n(\mathfrak{o}) \), we just say they are equivalent.

The lexicographic order \( \succeq \) on \( \mathbb{Z}_{\geq 0}^n \) is, as usual, defined as follows. For distinct sequences \( (y_1, y_2, \ldots, y_n), (z_1, z_2, \ldots, z_n) \in \mathbb{Z}_{\geq 0}^n \), let \( j \) be the largest integer such that \( y_i = z_i \) for \( i < j \). Then \( (y_1, y_2, \ldots, y_n) \succeq (z_1, z_2, \ldots, z_n) \) if and only if \( y_j > z_j \) for some \( j \) with \( 1 \leq j \leq n \).
Let \((z_1, z_2, \ldots, z_n)\) if \(y_j > z_j\). We define \((y_1, y_2, \ldots, y_n) \succeq (z_1, z_2, \ldots, z_n)\) if \((y_1, y_2, \ldots, y_n) \succeq (z_1, z_2, \ldots, z_n)\) or \((y_1, y_2, \ldots, y_n) = (z_1, z_2, \ldots, z_n)\).

1. Elementary properties of the Gross-Keating invariant

Let \(L\) be a free module of rank \(n\) over \(\mathfrak{o}\), and \(Q\) an \(\mathfrak{o}\)-valued quadratic form on \(L\). The pair \((L, Q)\) is called a quadratic module over \(\mathfrak{o}\). The symmetric bilinear form \(\langle x, y \rangle_Q\) associated to \(Q\) is defined by

\[
\langle x, y \rangle_Q = Q(x + y) - Q(x) - Q(y), \quad x, y \in L.
\]

When there is no fear of confusion, \(\langle x, y \rangle_Q\) is simply denoted by \((x, y)\).

If \(\psi = (\psi_1, \ldots, \psi_n)\) is an ordered basis of \(L\), we call the triple \((L, Q, \psi)\) a framed quadratic \(\mathfrak{o}\)-module. Hereafter, “a basis” means an ordered basis. For a framed quadratic \(\mathfrak{o}\)-module \((L, Q, \psi)\), we define a matrix \(B = (b_{ij}) \in H_n(\mathfrak{o})\) by

\[
b_{ij} = \frac{1}{2} \langle \psi_i, \psi_j \rangle.
\]

The isomorphism class of \((L, Q, \psi)\) (as a framed quadratic \(\mathfrak{o}\)-module) is determined by \(B\). We say that \(B \in H_n(\mathfrak{o})\) is associated to the framed quadratic module \((L, Q, \psi)\). If \(B\) is non-degenerate, we also say \((L, Q)\) or \((L, Q, \psi)\) is non-degenerate. The set \(S(B)\) is also denoted by \(S(\psi)\). If \(B\) is optimal, then \(\psi\) is called an optimal basis.

We consider \(\text{Aut}(L)\) acting on \(L\) from the right. When \(U \in \text{Aut}(L)\) is given by \(\psi_j \mapsto \sum_{i=1}^n \psi_i u_{ij}\), with \((u_{ij}) \in \text{GL}_n(\mathfrak{o})\), we define an ordered basis \(\psi U = ((\psi U)_1, \ldots, (\psi U)_n)\) by \((\psi U)_j = \sum_{i=1}^n \psi_i u_{ij}\). Then the matrix associated to \((L, Q, \psi U)\) is equal to \(B[U] = B[(u_{ij})]\). In particular, the equivalence class of \(\overline{B}\) is determined by the isomorphism class of the quadratic module \((L, Q)\). The norm \(n(L)\) of \((L, Q)\) is the fractional ideal generated by \(\{Q(x) | x \in L\}\). It is known (see [12] Lemma B.1) that \(a_1 = \text{ord}(n(L))\), where \(a_1\) is the first entry of \(\text{GK}(B)\).

Let \(a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n\) be a non-decreasing sequence. We define \(n_1, n_2, \ldots, n_r\) with \(n_1 + \cdots + n_r = n\) by

\[
\begin{align*}
a_1 &= \cdots = a_{n_1} < a_{n_1+1}, \\
(a_{n_1} < a_{n_1+1} = \cdots = a_{n_1+n_2} < a_{n_1+n_2+1}, \\
\cdots \\
a_{n_1+\cdots+n_{r-1}} < a_{n_1+\cdots+n_{r-1}+1} = \cdots = a_{n_1+\cdots+n_r}.
\end{align*}
\]

For \(s = 1, 2, \ldots, r\), we set

\[
n_s = \sum_{u=1}^s n_u.
\]
We put \( n^*_s = 0 \). The \( s \)-th block \( I_s \) is defined by \( I_s = \{ n^*_s - 1 + 1, n^*_s - 1 + 2, \ldots, n^*_s \} \) for \( s = 1, 2, \ldots, r \). We put \( a^*_s = a_{n^*_s - 1 + 1} = \cdots = a_{n^*_s} \).

Let \( (L, Q, \psi) \) be the framed quadratic \( \mathfrak{o} \)-module associated to \( B = (b_{ij}) \). For \( s = 1, \ldots, r \), we denote by \( L_s \) the submodule of \( L \) generated by \( \{ \psi_k \mid n^*_s - 1 + 1 \leq k \leq n \} = \{ \psi_k \mid k \in I_s \cup \cdots \cup I_r \} \). We put \( L_{r+1} = \{ 0 \} \).

Let \( S^0(B) \) be the set of all non-decreasing sequences \( (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) such that

\[
\text{ord}(b_{ii}) > a_i \quad (1 \leq i \leq n), \\
\text{ord}(2b_{ij}) > (a_i + a_j)/2 \quad (1 \leq i \leq j \leq n).
\]

**Lemma 1.1.** Suppose that \( a = (a_1, \ldots, a_n) \in S(B) \) with \( B \in \mathcal{H}_{\mathfrak{m}}^{\text{nd}}(\mathfrak{o}) \). Let \( (L, Q, \psi) \) and \( L_1, \ldots, L_r \) be as above. If \( x \in L_s \) and \( y \in L_t \), then we have

\[
\text{ord}(Q(x)) \geq a^*_s, \quad \text{ord}((x, y)) \geq \frac{a^*_s + a^*_t}{2}.
\]

Moreover, if \( \bar{a} \in S^0(B) \), then we have

\[
\text{ord}(Q(x)) > a^*_s, \quad \text{ord}((x, y)) > \frac{a^*_s + a^*_t}{2}
\]
for \( x \in L_s \) and \( y \in L_t \).

**Proof.** The proof of this lemma is easy and is left to the reader. \( \square \)

Recall that \( B_1 \in \mathcal{H}_m(\mathfrak{o}) \) is represented by \( B \in \mathcal{H}_n(\mathfrak{o}) \), if there exists \( X \in M_{nm}(\mathfrak{o}) \) such that \( B_1 = BX \). For \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \), we put \( a^{(m)} = (a_1, a_2, \ldots, a_m) \) for \( m \leq n \).

**Lemma 1.2.** Suppose that \( a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) is a non-decreasing sequence and \( a \in S(B) \) with \( B \in \mathcal{H}_m^{\text{nd}}(\mathfrak{o}) \). If \( B_1 \in \mathcal{H}_m^{\text{nd}}(\mathfrak{o}) \) is represented by \( B \), then we have \( a^{(m)} \in S(B_1) \). In particular, \( \text{GK}(B_1) \geq a^{(m)} \).

**Proof.** Let \( (L, Q, \psi) \) be the framed quadratic module corresponding to \( B \). We define \( L_1, \ldots, L_{r+1} \) as in Lemma 1.1. Let \( (L_{B_1}, Q_1) \) be the quadratic module corresponding to \( B_1 \). We may consider \( L_{B_1} \) as a submodule of \( L \). It is enough to find an ordered basis \( \psi = (\phi_1, \ldots, \phi_m) \) of \( L_{B_1} \) such that \( a^{(m)} \in S(\psi) \). Put \( M = L_{B_1} \) and \( M_u = M \cap L_u \) for \( u = 1, \ldots, r+1 \). Then we have

\[
M = M_1 \supset M_2 \supset \cdots \supset M_{r+1} = \{ 0 \}.
\]

Note that \( M_u/M_{u+1} \subset L_u/L_{u+1} \) is a torsion free \( \mathfrak{o} \)-module for \( u = 1, \ldots, r \). Put \( m_u = \text{rank}(M_u/M_{u+1}) \) and \( m^*_u = m_1 + m_2 + \cdots + m_u \). Choose an ordered basis \( \tilde{\psi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_m) \) such that \( \{ \tilde{\phi}_{m^*_u+1}, \ldots, \tilde{\phi}_m \} \) is a basis of \( M_u \) for \( u = 1, 2, \ldots, r \). By Lemma 1.1, we have \( a^{(m)} \in S(\tilde{\phi}) \), since \( (a_1, a_2, \ldots, a_n) \) is a non-decreasing sequence. \( \square \)
Lemma 1.1 can be generalized as follows. For \( x \in \mathbb{R} \), the smallest integer \( n \) such that \( n \geq x \) is denote by \( \lceil x \rceil \).

**Lemma 1.3.** Suppose that \( \bar{a} = (a_1, \ldots, a_n) \in S(B) \) with \( B \in \mathcal{H}_n^{\text{nd}}(\mathfrak{o}) \). Let \((L, Q, \psi)\) and \( L_s (1 \leq s \leq r) \) be as in Lemma 1.1. Put

\[
\mathcal{L}_s = L_s + \sum_{u=1}^{s-1} \mathcal{O}^{\lfloor(a_u^* - a_u)\rfloor/2} L_u = \sum_{u=1}^{s} \mathcal{O}^{\lfloor(a_u^* - a_u)\rfloor/2} L_u.
\]

If \( x \in \mathcal{L}_s \) and \( y \in \mathcal{L}_t \), then we have

\[
\text{ord}(Q(x)) \geq a_s^*, \quad \text{ord}((x, y)) \geq \frac{a_s^* + a_t^*}{2}.
\]

Moreover, if \( a \in S^0(B) \), then we have

\[
\text{ord}(Q(x)) > a_s^*, \quad \text{ord}((x, y)) > \frac{a_s^* + a_t^*}{2}
\]

for \( x \in \mathcal{L}_s \) and \( y \in \mathcal{L}_t \).

**Proof.** The proof of this lemma is easy and is left to the reader. \( \square \)

Let \( \bar{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) be a non-decreasing sequence. Recall that we have defined the group \( G_\bar{a} \subset \text{GL}_n(\mathfrak{o}) \) by

\[
G_\bar{a} = \{ g = (g_{ij}) \in \text{GL}_n(\mathfrak{o}) \mid \text{ord}(g_{ij}) \geq (a_j - a_i)/2, \text{ if } a_i < a_j \}.
\]

We define subgroups \( G_\bar{a}^\Delta \) and \( G_\bar{a}^\triangledown \) of \( G_\bar{a} \) by

\[
G_\bar{a}^\Delta = \{ g = (g_{ij}) \in G_\bar{a} \mid g_{ij} = 0, \text{ if } a_i > a_j. \},
\]

\[
G_\bar{a}^\triangledown = \{ g = (g_{ij}) \in G_\bar{a} \mid g_{ij} = 0, \text{ if } a_i < a_j. \}.
\]

The symbols \( \Delta \) and \( \triangledown \) stands for upper and lower block triangular matrices, respectively. We also define

\[
N_\bar{a}^\Delta = \{ g = (g_{ij}) \in G_\bar{a}^\Delta \mid g_{ij} = \delta_{ij}, \text{ if } a_i = a_j. \},
\]

\[
N_\bar{a}^\triangledown = \{ g = (g_{ij}) \in G_\bar{a}^\triangledown \mid g_{ij} = \delta_{ij}, \text{ if } a_i = a_j. \}.
\]

Here, \( \delta_{ij} \) is the Kronecker delta. Then we have \( G_\bar{a} = N_\bar{a}^\triangledown G_\bar{a}^\Delta = G_\bar{a}^\Delta N_\bar{a}^\triangledown \).

**Definition 1.1.** For \( \bar{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \), put

\[
\mathcal{M}(\bar{a}) = \left\{ (b_{ij}) \in \mathcal{H}_n(\mathfrak{o}) \left| \begin{array}{l}
\text{ord}(b_{ii}) \geq a_i \\
\text{ord}(2b_{ij}) \geq (a_i + a_j)/2
\end{array} \right. \begin{array}{l}
1 \leq i \leq n, \\
1 \leq i < j \leq n
\end{array} \right\},
\]

\[
\mathcal{M}^0(\bar{a}) = \left\{ (b_{ij}) \in \mathcal{H}_n(\mathfrak{o}) \left| \begin{array}{l}
\text{ord}(b_{ii}) > a_i \\
\text{ord}(2b_{ij}) > (a_i + a_j)/2
\end{array} \begin{array}{l}
1 \leq i \leq n, \\
1 \leq i < j \leq n
\end{array} \right\}.
\]
In the definition of $\mathcal{M}(a)$ or $\mathcal{M}^0(a)$, we do not assume $a$ is non-decreasing. Note that when $B \in \mathcal{H}_n^{\text{nd}}(a)$, we have

\[ a \in S(B) \iff a \text{ is non-decreasing and } B \in \mathcal{M}(a), \]

\[ a \in S^0(B) \iff a \text{ is non-decreasing and } B \in \mathcal{M}^0(a). \]

**Proposition 1.1.** Suppose that $a = (a_1, a_2, \ldots, a_{n}) \in \mathbb{Z}_{\geq 0}^{n}$ is a non-decreasing sequence and that $B \in \mathcal{M}(a)$ (resp. $B \in \mathcal{M}^0(a)$). Then we have $B[U] \in \mathcal{M}(a)$ (resp. $B[U] \in \mathcal{M}^0(a)$) for any $U \in G_{a}$. In particular, if $B$ is optimal and $\text{GK}(B) = a$, then $\text{GK}(B[U]) = a$ for any $U \in G_{a}$.

**Proof.** Suppose that $U \in G_{a}$. Let $(L, Q, \psi)$ be the framed quadratic module associated to $B$. Then by definition, we have $\psi_i U \in L$ for $i \in I$. By Lemma 1.3, we have $B[U] \in \mathcal{M}(a)$ (resp. $B[U] \in \mathcal{M}^0(a)$).

The proof of the last part is clear. □

The following lemma will be frequently used in this paper.

**Lemma 1.4.** Let $a = (a_1, a_2, \ldots, a_{n}) \in \mathbb{Z}_{\geq 0}^{n}$ be a non-decreasing sequence and $1 \leq m < n$. Let $s$ be the largest integer such that $a_{m+1} = \cdots = a_{m+s}$. Put $c = a_{m+1} = \cdots = a_{m+s}$. Assume that $B \in \mathcal{M}(a)$. Write $B$ in a block form

\[
B = \begin{pmatrix}
B_{11} & B_{12} & B_{13} \\
^tB_{12} & B_{22} & B_{23} \\
^tB_{13} & ^tB_{23} & B_{33}
\end{pmatrix}
\]

Then the following two assertions (1) and (2) hold.

1. If $\text{GK}(B_{22}) \succeq (c, c, \ldots, c)$, then $\text{GK}(B) \succeq a$.
2. If $\text{GK}(B) = a$, then we have $\text{GK}(B_{22}) = (c, c, \ldots, c)$.

**Proof.** Obviously $\text{GK}(B_{22}) \succeq (c, c, \ldots, c)$. Suppose that $\text{GK}(B_{22}) \neq (c, c, \ldots, c)$. Then there exists $U \in \text{GL}_s(a)$ such that $(c, c, \ldots, c, c+1) \in S(B_{22}[U])$. Put $B' = B[1_r \perp U \perp 1_{n-m-s}]$. Then we have

\[
(a_1, a_2, \ldots, a_{m+s-1}, a_{m+s} + 1, \ldots, a_{m+s} + 1)
\]

\[
(a_1, a_2, \ldots, a_{m}, c, c, \ldots, c, c+1, c+1, \ldots, c+1) \in S(B').
\]
Hence the assertion (1) holds. The assertion (2) follows from (1) immediately.

**Remark 1.1.** We briefly explain the proofs of the Theorems 0.1–0.4 for non-dyadic case. Suppose that \( F \) is a non-dyadic field. Then it is well-known that any non-degenerate element \( B \in \mathcal{H}_n(\mathfrak{o}) \) is equivalent to a diagonal matrix of the form

\[
T = \text{diag}(t_1, t_2, \ldots, t_n), \quad \text{ord}(t_1) \leq \text{ord}(t_2) \leq \cdots \leq \text{ord}(t_n),
\]

which is called a Jordan splitting of \( B \). It is known that the Gross-Keating invariant \( \text{GK}(B) = a = (a_1, a_2, \ldots, a_n) \) is given by \( a_i = \text{ord}(t_i) \) for \( i = 1, 2, \ldots, n \). In particular, \( T \) is optimal. For a proof, see Bouw \cite{2} Proposition 2.6. The proof of \cite{2} is valid for any non-dyadic field. Using this, one can easily show Theorem 0.1. Note that \( \text{GK}(B) = (0, \ldots, 0) \) if and only if \( B \in \text{GL}_n(\mathfrak{o}) \).

The “if part” of Theorem 0.2 follows from Proposition 1.1. Let \( a = (a_1, a_2, \ldots, a_n) \) be a non-decreasing sequence. Suppose that \( B \) is optimal and \( \text{GK}(B) = a \). By Lemma 1.4, we have \( \text{GK}(B^{(n_1)}) = (a_1, \ldots, a_1) \). Then, we have \( \omega^{-a_1}B^{(n_1)} \in \text{GL}_{n_1}(\mathfrak{o}) \). It follows that there exists \( U_1 \in G_2 \) such that \( B[U_1] \) is of the form \( B^{(n_1)} \perp B' \). Repeating this argument, one can show that there exists \( U_2 \in G_2^{\Delta} \) such that \( B[U_2] \) is a Jordan splitting of \( B \). Suppose that \( B[U] \) is also optimal with \( U \in \text{GL}_n(\mathfrak{o}) \). Then by the same argument as above, there exists \( U_3 \in G_2^{\Delta} \) such that \( B[UU_3] \) is a Jordan splitting of \( B \). It is well-known that two Jordan splittings of \( B \) are \( \text{GL}_{n_1}(\mathfrak{o}) \times \cdots \times \text{GL}_{n_r}(\mathfrak{o}) \)-equivalent. Moreover, if \( T \) is a Jordan splitting of \( B \), then \( \{U \in \text{GL}_n(\mathfrak{o}) | T[U] = T\} \subset G_2 \). Thus we obtain the “only if part” of Theorem 0.2.

Suppose that \( B \in \mathcal{M}(a) \). Then as we have seen above, \( \text{GK}(B) = a \) if and only if there exists \( U \in G_2^{\Delta} \) such that \( B = T[U] \), where \( T \) is a Jordan splitting of \( B \) and \( \text{GK}(T) = a \). Equivalently, \( \text{GK}(B) = a \) if and only if \( \text{ord}(\det B^{(n_s)}) = |a^{(n_s)}| \) for \( s = 1, \ldots, r \). Theorem 0.3 follows from this. Let \( B \) and \( B_1 \) be as in Theorem 0.4. As we have seen above, there exists \( U \in G_2^{\Delta} \) such that \( B_1 = B[U] \). It follows that \( B^{(k)} \sim B_1^{(k)} \). Hence we obtain Theorem 0.4.

### 2. Binary quadratic forms

Hereafter, until the end of section 5, we assume that \( F \) is dyadic.

Let \((L, Q)\) and \((L_1, Q_1)\) be quadratic modules of rank \( n \) over \( \mathfrak{o} \). We say that \((L, Q)\) and \((L_1, Q_1)\) are weakly equivalent if there exists an isomorphism \( \iota : L \rightarrow L_1 \) and a unit \( u \in \mathfrak{o}^\times \) such that \( uQ_1(\iota(x)) = Q(x) \) for any \( x \in L \). Similarly, we say that \( B, B_1 \in \mathcal{H}_n(\mathfrak{o}) \) are weakly
equivalent if there exists a unimodular matrix $U \in \text{GL}_n(o)$ and a unit $u \in o^\times$ such that $uB_1 = B[U]$. If $B$ and $B_1$ are weakly equivalent, then $\text{GK}(B) = \text{GK}(B_1)$.

Recall that a half-integral symmetric matrix $B \in \mathcal{H}_n(o)$ is primitive if and only if $\varpi^{-1}B \notin \mathcal{H}_n(o)$. It is well-known that $B$ is primitive if and only if $n(L) = o$, where $L$ is the quadratic module associated to $B$. Let $\text{GK}(B) = (a_1, a_2, \ldots, a_n)$. It is obvious that if $B$ is not primitive, then $a_1 > 0$. Conversely, if $B$ is primitive, then $a_1 = 0$ by Lemma 1.2. Thus $B$ is primitive if and only if $a_1 = 0$.

We define the integer $e$ by $|2|^{-1} = q^e$. Since we have assumed that $F$ is dyadic, $e$ is equal to the ramification index of $F/\mathbb{Q}_2$. It is well-known that $1 + 4p \subset o^{\times 2}$. For $\xi \in F^\times$, we denote the discriminant ideal of $F(\sqrt{\xi})/F$ by $\mathcal{D}_\xi$. Then $\text{ord}(\mathcal{D}_\xi) = 0$ if and only if $\xi \in (1 + 4o)o^{\times 2}$ for $\xi \in o^\times$. Moreover, $[(1 + 4o)o^{\times 2} : o^{\times 2}] = 2$. It is easy to see that if $\text{ord}(\mathcal{D}_\xi) = 0$, then $\text{ord}(\mathcal{D}_{\xi'}) = \text{ord}(\mathcal{D}_\xi)$ for any $\xi' \in F^\times$. (See e.g., O'Meara [9] §63A.)

Let $E/F$ be a semi-simple quadratic algebra. This means that $E$ is a quadratic extension of $F$ or $E = F \oplus F$. The non-trivial automorphism of $E/F$ is denoted by $x \mapsto \bar{x}$. Note that if $E = F \oplus F$, we have $(x_1, x_2) = (x_2, x_1)$. Let $o_E$ be the maximal order of $E$. In the case $E = F \oplus F$, $o_E = o \oplus o$. The discriminant ideal of $E/F$ is denoted by $\mathcal{D}_E$. The order $o_{E,f}$ of conductor $f$ for $E/F$ is defined by $o_{E,f} = o + p^f o_E$. Any open $o$-subring of $o_E$ is of the form $o_{E,f}$ for some non-negative integer $f$. We say that $E/F$ is unramified, if $E = F \oplus F$ or $E/F$ is an unramified quadratic extension. Then $E/F$ is unramified if and only if $\text{ord}(\mathcal{D}_E) = 0$.

**Proposition 2.1.** Let $B \in \mathcal{H}_2^{\text{nd}}(o)$ be a primitive half-integral symmetric matrix of size 2 and $(L,Q)$ its associated quadratic module. Put $E = F(\sqrt{D_B})/F$. When $D_B \in F^{\times 2}$, we understand $E = F \oplus F$. Put $f = (\text{ord}(D_B) - \text{ord}(\mathcal{D}_E))/2$. Then $f$ is an integer and $(L,Q)$ is weakly equivalent to $(o_{E,f}, N)$, where, $N$ is the norm form for $E/F$.

**Proof.** Since $B$ is primitive, there exists $x_0 \in L$ such that $u = Q(x_0)$ is a unit. By replacing $Q$ by $u^{-1}Q$, we may assume $Q(x_0) = 1$. Let $R$ be the even Clifford algebra of $(L,Q)$ over $o$ (See [6]). Then $R \otimes_o F$ is the even Clifford algebra of $(L \otimes F, Q \otimes F)$, which is isomorphic to $E$. Thus $R \simeq o_{E,f}$ for some $f \geq 0$. By Lemma 2.2.1 of Chapter 5 of [6], we have $(L,Q) \simeq (o_{E,f}, N)$. Let $\{1, \omega\}$ be an $o$-basis of $o_E$. Then $\{1, \varpi \omega, \omega^2, \varpi \omega^2\}$ is an $o$-basis of $o_{E,f}$. From this, we have $\text{ord}(D_B) = \text{ord}(\mathcal{D}_E) + 2f$, since $(D_B) = (\varpi \omega - \varpi \omega^2)$ and $\mathcal{D}_B = (\omega - \varpi \omega^2)$. \qed
We assume a basis. If \( a \) over \( 2 \)
Then \((1, \psi)\) is an ordered \( \mathfrak{o} \)-basis of \( \mathfrak{o}_E \). It follows that \( a \) also a quadratic module over \( \mathfrak{o} \). In particular, \( a \) of \( \mathfrak{o} \).

\( \text{Proposition 2.2.} \) The Gross-Keating invariant of the binary quadratic form \((L, Q) = (\mathfrak{o}_{E,f}, N)\) is given by

\[
\begin{cases}
(0, 2f) & \text{if } E/F \text{ is unramified}, \\
(0, 2f + 1) & \text{if } E/F \text{ is ramified}.
\end{cases}
\]

**Proof.** Let \((a_1, a_2)\) be the Gross-Keating invariant of \((L, Q)\). Since \((L, Q)\) is primitive, we have \(a_1 = 0\).

**Step 1.** We first consider the case \( f = 0 \). In this case, \( L = \mathfrak{o}_E \).
Assume that \( E/F \) is unramified and \((\omega_1, \omega_2)\) is any ordered \( \mathfrak{o} \)-basis of \( \mathfrak{o}_E \). Then we have \((\omega_1, \omega_2)_Q \in \mathfrak{o}^*\), since \( F \) is dyadic. It follows that \( S((\omega_1, \omega_2)) = \{(0, 0)\} \), and so \( a_1 = a_2 = 0 \) in this case.
Next, we assume \( E/F \) is ramified. Let \( \omega_E \) be a prime element of \( \mathfrak{o}_E \). Then \((1, \omega_E)\) is an ordered \( \mathfrak{o} \)-basis of \( \mathfrak{o}_E \) and \((0, 1) \in S((1, \omega_E))\). It follows that \( a_2 \geq 1 \). On the other hand, let \((\psi_1, \psi_2)\) be an optimal basis. If \( a_2 \geq 2 \), then the \( \mathfrak{o} \)-module generated by \( \{\psi_1, \omega^{-[a_2/2]}\psi_2\} \) is also a quadratic module over \( \mathfrak{o} \). This contradicts the fact that \( \mathfrak{o}_E \) is a maximal quadratic module. It follows that \( a_2 \leq 1 \), and so \( a_2 = 1 \).

**Step 2.** We assume \( f > 0 \). Let \((\psi_1, \psi_2)\) be an optimal basis of \( L \). The \( \mathfrak{o} \)-module generated by \( \{\psi_1, \omega^{-[a_2/2]}\psi_2\} \) is also a quadratic module over \( \mathfrak{o} \). It follows that \( \omega^{-[a_2/2]}\psi_2 \in \mathfrak{o}_E \). Thus we have \([a_2/2] \leq f\), i.e., \( a_2 \leq 2f + 1 \). On the other hand, let \((1, \omega)\) be an optimal basis of \( \mathfrak{o}_E \). Then \((1, \omega)\) is an ordered \( \mathfrak{o} \)-basis of \( \mathfrak{o}_{E,f} \) and

\[
\begin{cases}
(0, 2f) \in S((1, \omega^f)) & \text{if } E/F \text{ is unramified}, \\
(0, 2f + 1) \in S((1, \omega^f)) & \text{if } E/F \text{ is ramified}.
\end{cases}
\]

In particular, \( a_2 = 2f + 1 \), if \( E/F \) is ramified. Assume that \( a_2 = 2f + 1 \) and \( E/F \) is unramified. If \((\psi_1, \psi_2)\) is an optimal basis, then \( \omega^{-f}\psi_2 \in \mathfrak{o}_E \), and so \((\psi_1, \omega^{-f}\psi_2)\) is an ordered \( \mathfrak{o} \)-basis of \( \mathfrak{o}_E \). Then \((0, 1) \in S((\psi_1, \omega^{-f}\psi_2))\), this contradicts \( \text{GK}(\mathfrak{o}_E) = (0, 0) \). This proves \( a_2 = 2f \) in this case. \( \square \)

**Corollary 2.1.** If \( B \in \mathcal{H}_{2d}(\mathfrak{o}) \), then \( |\text{GK}(B)| = \Delta(B) \).

**Proof.** We may assume \( B \) is primitive. If \( B \) is primitive, then the corollary follows from Proposition 2.2. \( \square \)

Recall that an element \( B \in \mathcal{H}_{n}(\mathfrak{o}) \) is said to be decomposable if \( B \sim B_1 \perp B_2 \), for some \( B_1 \in \mathcal{H}_s(\mathfrak{o}) \), \( B_2 \in \mathcal{H}_t(\mathfrak{o}) \), \( s, t < n \). \( B \) is said to be indecomposable if it is not decomposable.
Lemma 2.1. Let $B \in \mathcal{H}^\mathrm{un}_2(\mathfrak{o})$ be a primitive binary form. Then $B$ is decomposable if and only if $D_B \in 4\mathfrak{o}$.

Proof. Suppose that $B \sim (b_1) \bot (b_2)$ is decomposable. Then we have $D_B = 4b_1b_2 \in 4\mathfrak{o}$. Conversely, suppose that $D_B \in 4\mathfrak{o}$. Then $B$ is weakly isomorphic to $\begin{pmatrix} 1 & 0 \\ 0 & -D_B/4 \end{pmatrix}$ by the remark after Proposition 2.1.

Definition 2.1. $K \in \mathcal{H}_2(\mathfrak{o})$ is a primitive unramified binary (quadratic) form if the quadratic module associated to $K$ is isomorphic to $(\mathfrak{o}_E, N)$ for an unramified quadratic algebra $E$.

Clearly, $B$ is a primitive unramified binary form if and only if $\Delta(B) = 0$. By Proposition 2.2, it is also equivalent to $\text{GK}(B) = (0, 0)$. Note also that $B \in \mathcal{H}_2(\mathfrak{o})$ is weakly equivalent to a primitive unramified binary form, then $B$ itself is a primitive unramified binary form, since $N(\mathfrak{o}_E) = \mathfrak{o}^\times$. A primitive unramified binary form is indecomposable by Lemma 2.1, If $\begin{pmatrix} a \\ b/2 \\ c \end{pmatrix} \in \mathcal{H}_2(\mathfrak{o})$ is a primitive unramified binary form, then the proof of Proposition 2.2 shows $b \in \mathfrak{o}^\times$. Conversely, If $b \in \mathfrak{o}^\times$, then $\begin{pmatrix} a \\ b/2 \\ c \end{pmatrix} \in \mathcal{H}_2(\mathfrak{o})$ is a primitive unramified binary form, since $\text{ord}(D_{b^2-4ac}) = 0$. More precisely, a primitive unramified binary form is isomorphic to either

$$H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \quad \text{or} \quad Y = \begin{pmatrix} 1 & 1/2 \\ 1/2 & c \end{pmatrix},$$

where $c \in \mathfrak{o}$ and $1 - 4c \notin \mathfrak{o}^\times$. Note that $Y \perp Y \sim H \perp H$.

We characterize optimal binary forms as follows.

Proposition 2.3. Assume that $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \in \mathcal{M}((a_1, a_2))$ and $a_1 \leq a_2$.

1. If $a_1 = a_2$, then
   $$\text{GK}(B) = (a_1, a_2) \iff \text{ord}(2b_{12}) = a_1.$$

2. If $a_2 - a_1 = 2f > 0$, with $f \in \mathbb{Z}_{>0}$, then
   $$\text{GK}(B) = (a_1, a_2) \iff \text{ord}(b_{11}) = a_1, \text{ord}(2b_{12}) = a_1 + f.$$

3. If $a_2 - a_1 = 2f + 1$, with $f \in \mathbb{Z}_{\geq 0}$, then
   $$\text{GK}(B) = (a_1, a_2) \iff \text{ord}(b_{11}) = a_1, \text{ord}(b_{22}) = a_2.$$
Proof. By replacing $B$ by $\varpi^{-a_1}B$, we may assume $a_1 = 0$. We have already seen (1). We prove (2). Since $B$ is primitive, we have $\text{ord}(b_{11}) = 0$. Put $B' = B[\text{diag}(1, \varpi^{-f})] \in \mathcal{H}_2(\mathfrak{o})$. Then $B'$ is a primitive unramified binary form if and only if $\text{ord}(2b_{12}) = f$. On the other hand, $\text{GK}(B) = 2f$ if and only if $\Delta(B) = 2f$ by Corollary 2.1. Since $\Delta(B) = \Delta(B') + 2f$, we have (2). Now we prove (3). Since $B$ is primitive, we have $\text{ord}(b_{11}) = 0$. Put $B'' = B[\text{diag}(1, \varpi^{-f})] \in \mathcal{H}_2(\mathfrak{o})$. Then $B''$ is a primitive unramified binary form if and only if $\text{ord}(2b_{12}) = f$. On the other hand, $\text{GK}(B) = 2f$ if and only if $\Delta(B) = 2f$ by Corollary 2.1. Since $\Delta(B) = \Delta(B') + 2f$, we have (2). Now we prove (3). Since $B$ is primitive, we have $\text{ord}(b_{11}) = 0$. Put $B'' = B[\text{diag}(1, \varpi^{-f})] \in \mathcal{H}_2(\mathfrak{o})$. Then $B''$ is a primitive unramified binary form if and only if $\text{ord}(2b_{12}) = f$. On the other hand, $\text{GK}(B) = 2f$ if and only if $\Delta(B) = 2f$ by Corollary 2.1. Since $\Delta(B) = \Delta(B') + 2f$, we have (2).

Lemma 2.2. Suppose that $B$ and $B'$ are primitive unramified binary forms. If $B - B' \in \varpi\mathcal{H}_2(\mathfrak{o})$, then $\xi_B = \xi_{B'}$.

Proof. One can easily show that $D_B - D_{B'} \in 4p$, and so $\xi_B = \xi_{B'}$. □

3. Reduced forms

In this section, we introduce reduced forms in a somewhat generalized way. We do not assume $\mathfrak{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n$ is non-decreasing, unless otherwise stated.

The $s$-th block $I_s \subset \{1, 2, \ldots, n\}$ is given by

- $I_1 = \{i \mid a_i = \min(a_1, a_2, \ldots, a_n)\}$,
- $I_2 = \{i \mid a_i = \min\{a_j \mid j \notin I_1\}\}$,
- $I_3 = \{i \mid a_i = \min\{a_j \mid j \notin I_1 \cup I_2\}\}$,
- $\ldots$
- $I_r = \{i \mid a_i = \max\{a_j \mid 1 \leq j \leq n\}\}$. 

Example. Suppose that $F = \mathbb{Q}_2$. Then $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not optimal. It is equivalent to an optimal form $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Note that $\text{GK}(B) = (0, 1)$. 

Lemma 2.2. Suppose that $B$ and $B'$ are primitive unramified binary forms. If $B - B' \in \varpi\mathcal{H}_2(\mathfrak{o})$, then $\xi_B = \xi_{B'}$.

Proof. One can easily show that $D_B - D_{B'} \in 4p$, and so $\xi_B = \xi_{B'}$. □
Let $\mathfrak{S}_n$ be the symmetric group of degree $n$. Recall that a permutation $\sigma \in \mathfrak{S}_n$ is an involution if $\sigma^2 = \text{id}$. For an involution $\sigma$, we set
\[
P^0 = P^0(\sigma) = \{ i \mid 1 \leq i \leq n, i = \sigma(i) \},
\]
\[
P^+ = P^+(\sigma) = \{ i \mid 1 \leq i \leq n, a_i > a_{\sigma(i)} \},
\]
\[
P^- = P^-(\sigma) = \{ i \mid 1 \leq i \leq n, a_i < a_{\sigma(i)} \}.
\]
For $s = 1, 2, \ldots, r$, put
\[
P^0_s = P^0_s(\sigma) = P^0 \cap I_s,
\]
\[
P^+_s = P^+_s(\sigma) = P^+ \cap I_s,
\]
\[
P^-_s = P^-_s(\sigma) = P^- \cap I_s.
\]

**Definition 3.1.** We shall say that an involution $\sigma \in \mathfrak{S}_n$ is $a$-admissible if the following three conditions are satisfied.

(i) $P^0$ has at most two elements. If $P^0$ has two distinct elements $i$ and $j$, then $a_i \neq a_j$ mod 2. Moreover, if $i \in P^0$, then
\[
a_i = \max\{a_j \mid j \in P^0 \cup P^+, a_j \equiv a_i \text{ mod } 2\}.
\]

(ii) For $s = 1, 2, \ldots, r$, we have
\[
\sharp P^+_s \leq 1, \quad \sharp P^-_s + \sharp P^0_s \leq 1.
\]

(iii) If $i \in P^-$, then
\[
a_{\sigma(i)} = \min\{a_j \mid j \in P^+, a_j > a_i, a_j \equiv a_i \text{ mod } 2\}.
\]

Similarly, if $i \in P^+$, then
\[
a_{\sigma(i)} = \max\{a_j \mid j \in P^-, a_j < a_i, a_j \equiv a_i \text{ mod } 2\}.
\]

It is easy to see that such an $a$-admissible involution exists for any $a \in \mathbb{Z}_{\geq 0}^n$. When $\sigma$ is an $a$-admissible involution, we call the pair $(a, \sigma)$ a GK type. Note that $\sharp P^0(\sigma) = 1$ if $n$ is odd. If $n$ is even, then $\sharp P^0(\sigma) = 0$ or $\sharp P^0(\sigma) = 2$ according as $|a|$ is even or odd.

**Definition 3.2.** Let $\sigma \in \mathfrak{S}_n$ be an $a$-admissible involution. We say that $B = (b_{ij}) \in \mathcal{M}(a)$ is a reduced form of (generalized) GK type $(a, \sigma)$ if the following conditions are satisfied.

1. If $i \notin P^0$, $j = \sigma(i)$, and $a_i \leq a_j$, then
\[
\text{GK} \left( \begin{pmatrix} b_{ii} & b_{ij} \\ b_{ij} & b_{jj} \end{pmatrix} \right) = (a_i, a_j).
\]

Note that this condition is equivalent to the following condition.

\[
\begin{align*}
\text{ord}(2b_{ij}) &= \frac{a_i + a_j}{2}, \quad i \notin P^0, \ j = \sigma(i) \\
\text{ord}(b_{ii}) &= a_i, \quad i \in P^-.
\end{align*}
\]
(2) If \( i \in \mathcal{P}^0 \), then \( \text{ord}(b_{ii}) = a_i \).

(3) If \( j \neq i, \sigma(i) \), then \( \text{ord}(2b_{ij}) > \frac{a_i + a_j}{2} \).

We often say \( B \) is a reduced form of GK type \( \mathbf{a} \) without mentioning \( \sigma \). We formally think of the empty matrix as a reduced form of GK type \( \emptyset \).

**Lemma 3.1.** Suppose that \( B \in \mathcal{H}_n(\mathfrak{o}) \) is a reduced form of GK type \( \mathbf{a} = (0, \ldots, 0) \).

1. If \( n = 2m \) is even, then \( B \sim K_1 \perp \cdots \perp K_m \), where \( K_1, \ldots, K_m \) are primitive unramified binary forms.

2. If \( n = 2m + 1 \) is odd, then \( B \sim (u) \perp K_1 \perp \cdots \perp K_m \), where \( K_1, \ldots, K_m \) are primitive unramified binary forms and \( u \in \mathfrak{o}^\times \).

**Proof.** We prove this lemma by induction with respect to \( n \). For \( n \leq 2 \), the lemma is obvious. Assume that \( n > 2 \). We may assume \( \sigma(1) = 2 \) by changing the coordinates. Then \( B \) is of the form

\[
B = \begin{pmatrix} K & X \\ X & B_{22} \end{pmatrix}
\]

such that \( 2X \in \varpi M_{2n-2}(\mathfrak{o}) \) and \( B_{22} \) is a reduced form of GK type \( (0, \ldots, 0) \). Since \( K^{-1}X \in \varpi M_{2n-2}(\mathfrak{o}) \), \( B \) is equivalent to

\[
B \left[ \begin{pmatrix} 1 & -K^{-1}X \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} K & 0 \\ 0 & B_{22} - K^{-1}[X] \end{pmatrix}.
\]

Then \( B_{22} - K^{-1}[X] \) is a reduced form, since \( K^{-1}[X] \in \varpi^2 \mathcal{H}_{n-2}(\mathfrak{o}) \). Hence the lemma.

**Proposition 3.1.** Suppose that \( B \in \mathcal{H}_n(\mathfrak{o}) \) is a reduced form of GK type \( \mathbf{a} = (0, \ldots, 0, 1, 1, \ldots, 1) \). Then there exists \( U \in G_2 \) such that

\[
B[U] = K_1 \perp K_2 \perp \cdots \perp K_{(s/2)-1} \perp B' \perp \varpi(K'_1 \perp K'_2 \perp \cdots \perp K'_{(l/2)-1}),
\]

where \( K_1, K_2, \ldots, K_{(s/2)-1} \) and \( K'_1, K'_2, \ldots, K'_{(l/2)-1} \) are primitive unramified binary forms and \( B' \) is a reduced form of GK type \( \emptyset, (0), (1), \) or \( (0, 1) \).

**Proof.** First assume \( s \geq 2 \). We may assume \( \sigma(1) = 2 \). Then \( B \) is of the form

\[
B = \begin{pmatrix} K & X \\ X & B_{22} \end{pmatrix},
\]
where $K$ is a primitive unramified binary form. Moreover, $2X \in \varpi M_{2,n-2}(\mathfrak{o})$ and $B_{22}$ is a reduced form of GK type $(0,\ldots,0,1,\ldots,1)$. Put
\[
U = \begin{pmatrix} 1 & -K^{-1}X \\ 0 & 1 \end{pmatrix}.
\]
Since $K^{-1}X \in M_{2,n-2}(\mathfrak{o})$, we have $U \in G_{\mathfrak{o}}$. Then we have
\[
B[U] = \begin{pmatrix} K & 0 \\ 0 & B_{22} - K^{-1}[X] \end{pmatrix}.
\]
Since $K^{-1}[X] \in \varpi^2 H_{n-2}(\mathfrak{o})$, $B_{22} - K^{-1}[X]$ is also a reduced form of GK type $(0,\ldots,0,1,\ldots,1)$. Thus we may assume $s \leq 1$. In particular, the proposition is proved if $t = 0$. The case $s = 0$ is reduced to the case $t = 0$ by considering $\varpi^{-1}B$. Thus we may assume $s = 1$ and $t \geq 2$.

We may assume $\sigma(n) = n - 1$. Write $B$ in a block form as follows.
\[
\begin{pmatrix} 1 & t-2 & 2 \\ B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{23} \\ B_{13} & B_{23} & B_{33} \end{pmatrix} = \begin{pmatrix} B'_{11} & B'_{12} & 0 \\ tB'_{12} & tB_{22} & 0 \\ tB_{13} & tB_{23} & B_{33} \end{pmatrix}.
\]
Then we have $2B_{13} \in \varpi M_{1,2}(\mathfrak{o})$, $2B_{23} \in \varpi^2 M_{t-2,2}(\mathfrak{o})$. Moreover, $\varpi^{-1}B_{33}$ is an unramified primitive binary form. Put $X_1 = -B_{33}^{-1}\cdot B_{13}$ and $X_2 = -B_{33}^{-1}\cdot B_{23}$. Then we have
\[
X_1 \in M_{2,1}(\mathfrak{o}), \quad X_2 \in \varpi M_{2,t-2}(\mathfrak{o}).
\]
Put
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t-2 \\ X_1 & X_2 & 1 \end{pmatrix} = \begin{pmatrix} B'_{11} & B'_{12} & 0 \\ tB'_{12} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix},
\]
\[
\begin{pmatrix} B'_{11} & B'_{12} \\ tB_{12} & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ tB_{12} & B_{22} \end{pmatrix} - B_{33}[(X_1 X_2)].
\]
Then we have
\[
\begin{pmatrix} B_{11} & B_{12} \\ tB_{12} & B_{22} \end{pmatrix} - \begin{pmatrix} B'_{11} & B'_{12} \\ tB'_{12} & B'_{22} \end{pmatrix} = B_{33}[(X_1 X_2)] \in \mathcal{M}_0(0,1,\ldots,1).
\]
It follows that
\[
\begin{pmatrix} B'_{11} & B'_{12} \\ tB'_{12} & B'_{22} \end{pmatrix}
\]
is a reduced form of GK type $(0,1,\ldots,1)$. Repeating this argument, the lemma is reduced to the case $t \leq 1$. Hence the lemma is proved. \qed
Lemma 3.2. Suppose that $B \in \mathcal{H}_n(\mathfrak{o})$ is a reduced form of GK type $\underline{a} = (0, \ldots, 0, 1, \ldots, 1)$. Then we have $\Delta(B) = |\underline{a}|$.

Proof. Note that $\Delta(B \perp \varpi^cK) = \Delta(B) + 2c$ if $K$ is a primitive unramified binary form. By Proposition 3.1, it is enough to consider the case $s, t \leq 1$. The case $s = 0$ or $t = 0$ is trivial. The case $s = t = 1$ follows from Corollary 2.1 and Proposition 2.3.

Proposition 3.2. Suppose that $B \in \mathcal{H}_n(\mathfrak{o})$ is a reduced form of GK type $\underline{a} = (a_1, a_2, \ldots, a_n)$. Then we have $|\underline{a}| = \Delta(B)$.

Proof. Put

$$B' = B[\text{diag}(\varpi^{-[a_1/2]}, \varpi^{-[a_2/2]}, \ldots, \varpi^{-[a_n/2]})]$$

Then $B'$ is a reduced form of GK type $\underline{a}' = (a_1', a_2', \ldots, a_n')$, where $a_i' = a_i - 2[a_i/2]$. Since

$$|\underline{a}| = |\underline{a}'| + 2 \sum_{i=1}^{n} \left[ \frac{a_i}{2} \right],$$

$$\Delta(B) = \Delta(B') + 2 \sum_{i=1}^{n} \left[ \frac{a_i}{2} \right],$$

it is enough to consider the case $a_1, a_2, \ldots, a_n \leq 1$. By changing the coordinate, we may assume $\underline{a} = (0, \ldots, 0, 1, \ldots, 1)$. In this case, the proposition follows from Lemma 3.2.

Let $\underline{a} = (a_1, a_2, \ldots, a_n)$ be a sequence of integers whose components are allowed to be negative. For such a sequence, we put $\mathcal{M}(\underline{a}) = \varpi^{-\underline{a}}\mathcal{M}(a_0 + \underline{a})$, where $a_0$ is a sufficiently large integer and $a_0 + \underline{a} = (a_0 + a_1, a_0 + a_2, \ldots, a_0 + a_n)$. Obviously, this definition does not depend on a choice of $a_0$. Similarly, we say that $B \in \mathcal{H}_n(\mathfrak{o})$ is a reduced form of GK type $(\underline{a}, \sigma)$, if $\varpi^{a_0}B$ is a reduced form of GK type $(a_0 + \underline{a}, \sigma)$.

Lemma 3.3. Suppose that $B \in \mathcal{H}_n(\mathfrak{o})$ is a reduced form of GK type $(\underline{a}, \sigma)$. If $\mathcal{P}^0(\sigma) = \emptyset$, then $(4B)^{-1} \in \mathcal{M}(-\underline{a})$.

Proof. We first note that the lemma holds for $\underline{a} = (0, 0)$. In fact, $B$ is a primitive unramified binary form in this case. Then $(4B)^{-1}$ is also a primitive unramified binary form.

Now we consider general case. As is the proof of Proposition 3.2, we may assume that $\underline{a} = (0, \ldots, 0, 1, \ldots, 1)$. The assumption $\mathcal{P}^0(\sigma) = \emptyset$
implies that both \( s \) and \( t \) are even. By the proof of Proposition 3.1, there exist \( K_1 \in \mathcal{H}_s(\mathfrak{o}) \), \( K_2 \in \mathcal{H}_t(\mathfrak{o}) \) and \( X \in \mathcal{W}_{s,t}(\mathfrak{o}) \) such that the following conditions hold:

1. \( K_1 \) and \( K_2 \) are equivalent to direct sums of primitive unramified binary forms.
2. We have

\[
B = \begin{pmatrix} K_1 & 0 \\ 0 & \mathcal{W}K_2 \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}.
\]

Then we have

\[
\Gamma K((4K_1)^{-1}) = (0, \ldots, 0) \left| \left\{ \mathcal{W} \right\} \right. \\
\Gamma K((4K_2)^{-1}) = (0, \ldots, 0) \left| \left\{ \mathcal{W} \right\} \right.
\]

It follows that

\[
(4B)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{W}^{-1}(4K_2)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -X & 1 \end{pmatrix} \in \mathcal{M}(-\mathfrak{a}).
\]

Hence we have proved the lemma.

For \( B \in \mathcal{H}^n_n(\mathfrak{o}) \), let \( \eta_B \) be the Clifford invariant of \( B \) introduced in Definition 0.4. By [8], Chapter 5, section 3, (3.13), we have

\[
\eta_B(\mathfrak{a}) = \eta_B(\mathfrak{a}) \langle D_B, D_B \rangle \quad \text{if} \ n_1 \equiv n_2 \mod 2,
\]

\[
\eta_B(\mathfrak{a}) = \eta_B(\mathfrak{a}) \langle D_B, -D_B \rangle \quad \text{if} \ n_1 \text{ is even and } n_2 \text{ is odd}
\]

for \( B_1 \in \mathcal{H}^n_n(\mathfrak{o}) \) and \( B_2 \in \mathcal{H}^n_n(\mathfrak{o}) \).

**Lemma 3.4.** For \( B \in \mathcal{H}^n_n(\mathfrak{o}) \) such that \( B^{(n-1)} \in \mathcal{H}_{n-1}(\mathfrak{o}) \), then we have

\[
\eta_B = \eta_B^{(n-1)}(D_B, D_B^{(n-1)}).
\]

**Proof.** Note that \( B \) is \( \text{GL}_n(F) \)-equivalent to \( B^{(n-1)} \perp (-1)^{n-1} D_B D_B^{(n-1)} \). Assume that \( n \) is odd. Then we have

\[
\eta_B = \eta_B^{(n-1)}(D_B, -D_B D_B^{(n-1)}) = \eta_B^{(n-1)}(D_B, D_B^{(n-1)}).
\]

The case when \( n \) is even is similar.

**Lemma 3.5.** Let \( K \) be a primitive unramified binary form. For \( B \in \mathcal{H}^n_n(\mathfrak{o}) \), we have

\[
\eta_{B, \mathcal{W}^n_K} = \eta_B \xi_K^{a+\text{ord}(D_B)}.
\]

**Proof.** Note that \( \eta_{\mathcal{W}^n_K} = \xi_K^a \). Hence we have

\[
\eta_{B, \mathcal{W}^n_K} = \eta_B \eta_{\mathcal{W}^n_K}(D_B, D_K) = \eta_B \xi_K^{a+\text{ord}(D_B)}.
\]
Lemma 3.6. Let $B \in \mathcal{H}_n(o)$ be a half-integral symmetric matrix with $GK(B) = a = (a_1, \ldots, a_n)$. Assume that $a_1 = \cdots = a_n$. Then we have

$$
\eta_B = \begin{cases} 
1 & \text{if } n \text{ is odd}, \\
\xi_B & \text{if } n \text{ is even}.
\end{cases}
$$

Proof. By Theorem 4.1 and Lemma 3.1, we may assume

$$
B = \begin{cases} 
\varpi^{a_1}(u \perp K_1 \perp \cdots \perp K_{[n/2]}) & \text{if } n \text{ is odd}, \\
\varpi^{a_1}(K_1 \perp \cdots \perp K_{n/2}) & \text{if } n \text{ is even},
\end{cases}
$$

where $u \in o^\times$ and $K_1, \ldots, K_{[n/2]}$ are primitive unramified binary forms. Then the lemma follows from Lemma 3.5.

Proposition 3.3. Suppose that $B, T \in \mathcal{H}_n(o)$ are reduced forms of GK type $a$. If $B - T \in M^0(a)$, then the following assertions (a) and (b) hold.

(a) If $n$ is even, then $\xi_B = \xi_T$.
(b) If $n$ odd, then $\eta_B = \eta_T$.
(c) If $n$ is even and $\xi_B \neq 0$, then $\eta_B = \eta_T$.

Proof. Put

$$
B' = B[\text{diag}(\varpi^{-[a_1/2]}, \varpi^{-[a_2/2]}, \ldots, \varpi^{-[a_n/2]})], \\
T' = T[\text{diag}(\varpi^{-[a_1/2]}, \varpi^{-[a_2/2]}, \ldots, \varpi^{-[a_n/2]})].
$$

Then $B'$ and $T'$ are reduced forms of GK type $a' = (a_1', a_2', \ldots, a_n')$ and $B' - T' \in M^0(a')$, where $a_i' = a_i - 2[a_i/2]$. Thus we may assume $a = (0, \ldots, 0, 1, \ldots, 1)$. We first prove (a). If both $s$ and $t$ are odd, then $\xi_B = \xi_{B'} = 0$. We assume both $s$ and $t$ are even. Suppose that $s \geq 2$. We may assume $\sigma(1) = 2$. Write $B$ and $T$ in block forms

$$
B = \begin{pmatrix} 
B_{11} & B_{12} \\
B_{12}^t & B_{22}
\end{pmatrix}, \quad 2B_{12} \in \varpi M_{2,n-2}(o)
$$

and

$$
T = \begin{pmatrix} 
T_{11} & T_{12} \\
T_{12}^t & T_{22}
\end{pmatrix}, \quad 2T_{12} \in \varpi M_{2,n-2}(o).
$$

Then $B_{11}$ and $T_{11}$ are unramified primitive binary forms and $\xi_{B_{11}} = \xi_{T_{11}}$ by Lemma 2.2. Put

$$
B \begin{pmatrix} 
1 & -B_{11}^{-1}B_{12} \\
0 & 1
\end{pmatrix} = \begin{pmatrix} 
B_{11} & 0 \\
0 & B'
\end{pmatrix}, \quad B' = B_{22} - B_{11}^{-1}[B_{12}].$$
Then we have $\xi_B = \xi_{B_1} \xi_{B'}$. Similarly, put
\[
T \begin{pmatrix}
1 & -T_{11}^{-1}T_{12} \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
T_{11} & 0 \\
0 & T'
\end{pmatrix}, \quad T' = T_{22} - T_{11}^{-1}[T_{12}].
\]
Then we have $\xi_T = \xi_{T_{11}} \xi_{T'}$. Note that
\[
B_{11}^{-1}[B_{12}], T_{11}^{-1}[T_{12}] \in \mathcal{H}_{n-2}(o).
\]
It follows that $B'$ and $T'$ are reduced form of GK type $(0, \ldots, 0, 1, \ldots, 1)$
and $B' - T' \in M^0(0, \ldots, 0, 1, \ldots, 1)$. Thus the proof is reduced to the

case $s = 0$. The case $s = 0$ is reduced to the case $t = 0$, by replacing
$B$ and $T$ by $\mathcal{H}^{-1}B$ and $\mathcal{H}^{-1}T$, respectively. Thus we have proved (a).

Next, we show (b). By the same argument as above, the proof is
reduced to the case $s \leq 1$ by using Lemma 3.5. If $s = 0$, then $\eta_B = \eta_T = 1$ by Lemma 3.6. Assume now $s = 1$. In this case $t$ is even.
Since the case $t = 0$ is trivial, we may assume $t \geq 2$. We may assume
$\sigma(n) = n - 1$. Write $B$ and $T$ in block forms as follows.

\[
B = \begin{pmatrix}
\underbrace{1}_{1} & \underbrace{t-2}_{2} & \underbrace{2}_{t} \\
\underbrace{B_{11}}_{1} & \underbrace{B_{12}}_{t-2} & \underbrace{B_{13}}_{t} \\
\underbrace{tB_{12}}_{1} & \underbrace{B_{22}}_{t-2} & \underbrace{B_{23}}_{t} \\
\underbrace{tB_{13}}_{1} & \underbrace{tB_{23}}_{t-2} & \underbrace{B_{33}}_{t}
\end{pmatrix}
\]

\[
T = \begin{pmatrix}
\underbrace{1}_{1} & \underbrace{t-2}_{2} & \underbrace{2}_{t} \\
\underbrace{T_{11}}_{1} & \underbrace{T_{12}}_{t-2} & \underbrace{T_{13}}_{t} \\
\underbrace{tT_{12}}_{1} & \underbrace{T_{22}}_{t-2} & \underbrace{T_{23}}_{t} \\
\underbrace{tT_{13}}_{1} & \underbrace{tT_{23}}_{t-2} & \underbrace{T_{33}}_{t}
\end{pmatrix}
\]

Put
\[
X_1 = -B_{33}^{-1} \cdot tB_{13}, \quad X_2 = -B_{33}^{-1} \cdot tB_{23},
\]
\[
Y_1 = -T_{33}^{-1} \cdot tT_{13}, \quad Y_2 = -T_{33}^{-1} \cdot tT_{23}.
\]

\[
B \begin{pmatrix}
1 & 0 & 0 \\
0 & 1_{t-2} & 0 \\
X_1 & X_2 & 1_2
\end{pmatrix} = \begin{pmatrix}
B'_{11} & B'_{12} & 0 \\
\underbrace{tB'_{12}}_{1} & \underbrace{B'_{22}}_{t-2} & \underbrace{B'_{23}}_{t} \\
0 & 0 & B_{33}
\end{pmatrix},
\]
\[
T \begin{pmatrix}
1 & 0 & 0 \\
0 & 1_{t-2} & 0 \\
Y_1 & Y_2 & 1_2
\end{pmatrix} = \begin{pmatrix}
T'_{11} & T'_{12} & 0 \\
\underbrace{tT'_{12}}_{1} & \underbrace{T'_{22}}_{t-2} & \underbrace{T'_{23}}_{t} \\
0 & 0 & T_{33}
\end{pmatrix}.
\]
As in the proof of Proposition 3.1, we have
\[
\begin{pmatrix}
B_{11} & B_{12} \\
B'_{12} & B_{22}
\end{pmatrix} - \begin{pmatrix}
B'_{11} & B'_{12} \\
B'_{12} & B'_{22}
\end{pmatrix} \in \mathcal{M}_n^0(0, 1, \ldots, 1),
\]
\[
\begin{pmatrix}
T_{11} & T_{12} \\
T'_{12} & T_{22}
\end{pmatrix} - \begin{pmatrix}
T'_{11} & T'_{12} \\
T'_{12} & T'_{22}
\end{pmatrix} \in \mathcal{M}_n^0(0, 1, \ldots, 1).
\]

By Lemma 2.2, we have \(\xi_{B_{33}} = \xi_{T_{33}}\). On the other hand, by induction hypothesis, we have \(\eta_B = \eta_T\), where
\[
B' = \begin{pmatrix}
B'_{11} & B'_{12} \\
B'_{12} & B_{22}
\end{pmatrix}, \quad T' = \begin{pmatrix}
T'_{11} & T'_{12} \\
T'_{12} & T_{22}
\end{pmatrix}.
\]

By Lemma 3.5, we have \(\eta_B = \xi_{B_{33}}\eta_B = \xi_{T_{33}}\eta_T = \eta_T\).

Now we prove (c). As in the previous cases, we may assume \(a = (0, \ldots, 0, 1, \ldots, 1)\). Since \(\xi_B \neq 0\), both \(s\) and \(t\) are even. We proceed by induction with respect to \(s\). The case \(s = 0\) follows from (a) and Lemma 3.6. Suppose that \(s \geq 2\). As in the proof of (a), we can show
\[
B \sim B_{11} \perp B', \quad \text{GK}(B_{11}) = (0, 0), \quad \text{GK}(B') = \begin{pmatrix}
0, & \ldots, & 0, & 1, & \ldots, & 1
\end{pmatrix},
\]
\[
T \sim T_{11} \perp T', \quad \text{GK}(T_{11}) = (0, 0), \quad \text{GK}(T') = \begin{pmatrix}
0, & \ldots, & 0, & 1, & \ldots, & 1
\end{pmatrix},
\]
where
\[
B_{11} - T_{11} \in \mathcal{M}_n^0((0, 0)), \quad B' - T' \in \mathcal{M}_n^0((0, \ldots, 0, 1, \ldots, 1)).
\]

By (a) and Lemma 3.5, we have \(\eta_B = \eta_{B'} = \eta_{T'} = \eta_T\), as desired.

\[\Box\]

**Remark 3.1.** If \(n\) is even and \(\xi_B \neq 0\), then (a) and (c) imply that \(B\) and \(T\) are \(\text{GL}_n(F)\)-equivalent, but not \(\text{GL}_n(\mathfrak{o})\)-equivalent in general. In the case \(\xi_B = 0\), \(B\) and \(T\) may not be \(\text{GL}_n(F)\)-equivalent. For example, put \(B = \begin{pmatrix}
-1 & 0 \\
0 & 2
\end{pmatrix}\) and \(T = \begin{pmatrix}
-1 & 1 \\
1 & -2
\end{pmatrix}\). Then \(\text{GK}(B) = \text{GK}(T) = (0, 1)\) and \(B - T \in \mathcal{M}_n^0((0, 1))\). By easy calculation, \(\eta_B = 1\) and \(\eta_T = -1\). Note that \(D_B = 2\) and \(D_T = -1\), and so the discriminant fields are different.

**Proposition 3.4.** Suppose that \(B \in \mathcal{H}_n(\mathfrak{o})\) and \(\text{GK}(B) = (0, 0, \ldots, 0)\).

1. If \(n = 2m\) is even, then \(B \sim K_1 \perp \cdots \perp K_m\), where \(K_1, \ldots, K_m\) are primitive unramified binary forms.
2. If \(n = 2m + 1\) is odd, then \(B \sim (u) \perp K_1 \perp \cdots \perp K_m\), where \(K_1, \ldots, K_m\) are primitive unramified binary forms and \(u \in \mathfrak{o}^\times\).
Proof. By Proposition 1.1, any half-integral symmetric matrix equivalent to $B$ is optimal, since $G_a = \text{GL}_n(\mathfrak{o})$ for $a = (0, \ldots, 0)$. It is well-known that $B$ is isomorphic to a direct sum of matrices of size 1 or 2. By Lemma 1.4, the Gross-Keating invariant of any direct summand is of the form $(0, 0, \ldots, 0)$. Thus $B$ is isomorphic to

$$(u_1) \perp (u_2) \perp \cdots \perp (u_r) \perp K_1 \perp K_2 \perp \cdots \perp K_s,$$

where $u_1, u_2, \ldots, u_r$ are units and $K_1, K_2, \ldots, K_s$ are primitive unramified binary forms. Note that $\text{GK}((u_1) \perp (u_2)) \neq (0, 0)$, since it is not a primitive unramified binary form. Thus $B$ cannot contain a direct summand of the form $(u_1) \perp (u_2)$. This shows that $r \leq 1$. \hfill \Box

Lemma 3.7. If $B \in \mathcal{M}_n(\mathfrak{o})$, then we have

$$|a| \leq \Delta(B).$$

Proof. As in the proof of Proposition 3.2, we may assume $a = (\underbrace{0, \ldots, 0}_{s}, 1, \ldots, 1)$ by replacing $B$ by $B[\text{diag}(\varpi^{-a_1/2}, \varpi^{-a_2/2}, \ldots, \varpi^{-a_n/2})]$. Write $B$ in a block form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^t & B_{22} \end{pmatrix}.\begin{cases} s \\ t \end{cases}$$

If $\text{GK}(B_{11}) \cong (0, \ldots, 0)$, then $B[U \perp 1_t] \in \mathcal{M}(0, \ldots, 0, 1, \ldots, 1)$ for some $U \in \text{GL}_s(\mathfrak{o})$ by the proof of Lemma 1.4. Replacing $B$ and $t$ by $B[U \perp 1_t]$ and $t+1$, respectively, we may assume $\text{GK}(B_{11}) = (0, \ldots, 0)$.

Moreover, if $\text{GK}(B_{22}) \cong (1, \ldots, 1)$, then we can find a non-decreasing sequence $a'$ and $U' \in \text{GL}_t(\mathfrak{o})$ such that $|a'| > |a|$ and $B[1_s \perp U'] \in \mathcal{M}(a')$ by Lemma 1.4, (1). In this case, we go back to the case $a_n > 1$.

Repeating this argument, we may assume $a = (0, \ldots, 0, 1, \ldots, 1)$, $\text{GK}(B_{11}) = (0, \ldots, 0)$, and $\text{GK}(B_{22}) = (1, \ldots, 1)$. In this case, $B$ is equivalent to a reduced form of $\text{GK}$ type $a$ by Proposition 3.4. Thus the lemma follows from Proposition 3.2. \hfill \Box

Lemma 3.8. Suppose that $B \in \mathcal{H}^\text{nd}_n(\mathfrak{o})$. Assume that $a = (a_1, \ldots, a_n) \in S(B)$. If $B_1 \in \mathcal{H}^\text{nd}_m(\mathfrak{o})$ is represented by $B$, then we have

$$|a^{(m)}| \leq \Delta(B_1).$$
Proof. We may assume $a \in S(B)$. We can find $U \in \text{GL}_m(\mathfrak{o})$ such that $B_1[U] \in \mathcal{M}(a^{(m)})$ by Lemma 1.2. Then we have

$$|a^{(m)}| \leq \Delta(B_1[U]) = \Delta(B_1)$$

by Lemma 3.7.

\begin{lemma}
Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}$ be a sequence such that $a^{(k)} = (\underbrace{0, \ldots, 0}_{s}, 1, \ldots, 1)$ and $a_{k+1}, \ldots, a_n \geq 1$, $s + t = k$. Suppose that $B = (b_{ij}) \in \mathcal{M}(a)$. Assume that $B^{(k)}$ is a reduced form of GK type $a^{(k)}$.

Let $(L, Q, \psi)$ be a framed quadratic module associated to $B$. Assume that $x = \sum_{i=1}^n x_i\psi_i \in L \otimes F$ satisfies the following conditions (a), (b) and (c).

(a) $x_1, \ldots, x_k \in F$ and $x_{k+1}, \ldots, x_n \in \mathfrak{o}$.
(b) $(x, y)_Q \in \mathfrak{p}$ for any $y = \sum_{i=1}^k y_i\psi_i$, $y_1, \ldots, y_k \in \mathfrak{o}$.
(c) $Q(x) \in \mathfrak{p}$.

Then we have $x_1, \ldots, x_s \in \mathfrak{p}$ and $x_{s+1}, \ldots, x_k \in \mathfrak{o}$.

Proof. Note that the group $G_{y^{(k)}}$ preserves both $\sum_{i=1}^k \mathfrak{o}\psi_i$ and $\sum_{i=1}^s \mathfrak{p}\psi_i + \sum_{i=s+1}^k \mathfrak{o}\psi_i$. Write $B$ in a block form

$$B = \begin{pmatrix} B^{(k)}_{12} & B_{12} \\ tB_{12} & B_{22} \end{pmatrix}. $$

Here, $2B_{12} \in \varpi\mathcal{M}_{k,n-k}(\mathfrak{o})$ and $B_{22} \in \varpi\mathcal{H}_{n-k}(\mathfrak{o})$ by assumption. By Proposition 3.1, we may assume $B^{(k)}$ is of the form

$$K_1 \perp K_2 \perp \cdots \perp K_{[s/2]} \perp B' \perp \varpi(K'_1 \perp K'_2 \perp \cdots \perp K'_{[t/2]}),$$

where $K_1, K_2, \ldots K_{[s/2]}$ and $K'_1, K'_2, \ldots K'_{[t/2]}$ are primitive unramified binary forms and $B'$ is a reduced form of GK type $\emptyset$, $(0)$, $(1)$, or $(0, 1)$. We consider only the case $\text{GK}(B') = (0, 1)$, since the other cases are similar. In this case, the condition (b) is equivalent to

$$\text{ord}(\sum_{i=1}^n 2b_{ij}x_i) \geq 1 \quad \text{for } j = 1, 2, \ldots, k.$$

It follows that $x_1, x_2, \ldots, x_{s-1} \in \mathfrak{p}$, $x_{s+2}, x_{s+3}, \ldots, x_k \in \mathfrak{o}$.

We fix $x_1, x_2, \ldots, x_{s-1} \in \mathfrak{p}$ and $x_{s+2}, x_{s+3}, \ldots, x_n \in \mathfrak{o}$. We need to show $x_s \in \mathfrak{p}$ and $x_{s+1} \in \mathfrak{o}$. Put $E = F(\sqrt{D_{B'}})$. Then $E$ is a ramified quadratic extension of $F$. Moreover, there exists a prime element $\varpi_E$ of $E$ such that the framed quadratic module associated to $B'$ is weakly isomorphic to $(\mathfrak{o}_E, N, (1, \varpi))$. By multiplying $B$ by some unit, we may
assume \( B' \left[ \begin{pmatrix} x_s \\ x_{s+1} \end{pmatrix} \right] = N(x_s + \varpi_E x_{s+1}) \). Put \( X = x_s + \varpi_E x_{s+1} \in E \).

Then the condition (c) implies

\[
N(X) + \beta_1 x_s + \beta_2 x_{s+1} \in \mathfrak{p},
\]

where, \( \beta_1 = \sum_{i=k+1}^{n} b_{si} x_i \in \mathfrak{p} \) and \( \beta_2 = \sum_{i=k+1}^{n} b_{s+1,i} x_i \in \mathfrak{p} \). Note that

\[
\text{ord}(N(X)) = 2 \text{ord}_E(X), \quad \text{ord}(\beta_1 x_s + \beta_2 x_{s+1}) \geq \left[ \frac{\text{ord}_E(X)}{2} \right] + 1,
\]

where \( \text{ord}_E \) is the order for \( E \). It follows that \( X \in p_E \), and so \( x_s \in \mathfrak{p} \) and \( x_{s+1} \in \mathfrak{o} \).

**Lemma 3.10.** Let \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) be a sequence. Put \( A = \max(a_1, a_2, \ldots, a_k) \). We assume that \( a_{k+1}, \ldots, a_n \geq A \). Suppose that \( B = (b_{ij}) \in M(a) \) and that \( B^{(k)} \) is a reduced form of \( \text{GK type } a^{(k)} \). Let \( (L, Q, \psi) \) be the framed quadratic module associated to \( B \). Assume that \( x = \sum_{i=1}^{n} x_i \psi_i \in L \otimes F \) satisfies the following conditions (a), (b) and (c).

(a) \( x_1, \ldots, x_k \in F \) and \( x_{k+1}, \ldots, x_n \in \mathfrak{o} \).

(b) \( \text{ord}((\psi_j, x)_Q) \geq (a_j + A)/2 \) for \( j = 1, \ldots, k \).

(c) \( \text{ord}(Q(x)) \geq A \).

Then we have

\[
\text{ord}(x_i) \geq \frac{A - a_i}{2} \quad (i = 1, 2, \ldots, k).
\]

**Proof.** By multiplying \( B \) by \( \varpi \) if necessary, we may assume \( A \) is odd. The condition (b) is equivalent to

\[
\text{ord} \left( \sum_{i=1}^{k} 2b_{ij} x_i \right) \geq \frac{a_j + A}{2}
\]

for \( j = 1, 2, \ldots, k \).

Put

\[
B' = B[\text{diag}(\varpi^{-[a_1/2]}, \varpi^{-[a_2/2]}, \ldots, \varpi^{-[a_k/2]}, \varpi^{-[A/2]}, \ldots, \varpi^{-[A/2]})]
\]

and

\[
x' = t(x_1', \ldots, x_n'), \\
x_i' = \begin{cases} 
\varpi^{[a_i/2]-[A/2]} x_i & \text{if } i \leq k, \\
x_i & \text{if } k < i \leq n.
\end{cases}
\]

Then the conditions (b) and (c) are equivalent to the following conditions (b’) and (c’), respectively.

(b’) \( \text{ord}(\sum_{i=1}^{k} 2b_{ij} x_i') \geq (a_j + 1)/2 - [a_j/2] \).

(c’) \( \text{ord}(B'[x']) \geq 1 \).
Changing the coordinate, we may assume \( \alpha^{(k)} \) is of the form \((0, \ldots, 0, 1, \ldots, 1)\).

In this case, the lemma follows from Lemma 3.9. \( \square \)

Suppose that \( \alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) is a non-decreasing sequence. Let \( B \in \mathcal{H}_n(\mathfrak{o}) \) be a reduced form of GK type \( \alpha \) and \((L, Q, \psi)\) the framed quadratic module associated to \( B \). We define \( L_s \) and \( \mathcal{L}_s \) as in section 1, i.e.,

\[
L_s = \sum_{i=n_{s-1}+1}^{n} a_i \psi_i, \\
\mathcal{L}_s = L_s + \sum_{u=1}^{s-1} \mathfrak{m}^{[a_u^* - a_s^*]/2} L_u.
\]

**Lemma 3.11.** Let \( B \) and \((L, Q, \psi)\) be as above. Suppose that \( x \in L \). Then \( x \in \mathcal{L}_s \) if and only if the following conditions (1) and (2) are satisfied.

1. For any \( y \in \mathcal{L}_t \), we have
   \[ \text{ord}((x, y)Q) \geq a_t^* + a_s^*/2 \text{ for } t = 1, 2, \ldots, s - 1. \]
2. \( \text{ord}(Q(x)) \geq a_s^* \).

**Proof.** We denote by \( M \) the set of all \( x \in L \) which satisfies (1) and (2). By Lemma 1.3, we have \( \mathcal{L}_s \subset M \). Conversely, \( M \subset \mathcal{L}_s \) by Lemma 3.10, since \( \mathcal{L}_t \subset \mathcal{L}_t \). \( \square \)

The following lemma will be used in our forthcoming paper [4]. Recall that \( e = \text{ord}(2) \).

**Lemma 3.12.** Let \( \alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) be a sequence and \( \sigma \in \mathfrak{S}_n \) an \( \alpha \)-admissible involution. Let \( B = (b_{ij}) \in \mathcal{H}_n(\mathfrak{o}) \) be a reduced form of GK type \((\alpha, \sigma)\). We assume \( n \) is even and \( a_1 + \cdots + a_n \) is odd. Put \( B^{-1} = (b_{ij}') \) and \( \text{ord}(\mathfrak{D}_B) = d \). Then we have the following.

(a) \( \text{ord}(b_{ii}') = 2e + 1 - d - a_i \) \( (i \in \mathcal{P}^0(\sigma)). \)
(b) \( \text{ord}(b_{ij}') \geq \frac{(2e + 1 - d - a_i - a_j)}{2} \) \( (i, j \in \mathcal{P}^0(\sigma)). \)
(c) \( \text{ord}(b_{ij}') > \frac{(2e + 1 - d - a_i - a_j)}{2} \) \( (i \in \mathcal{P}^0(\sigma), j \notin \mathcal{P}^0(\sigma)). \)
(d) \( \text{ord}(b_{ii}') > 2e + 1 - d - a_i \) \( (i \notin \mathcal{P}^0(\sigma)). \)
(e) \( \text{ord}(b_{ij}') > \frac{(2e + 1 - d - a_i - a_j)}{2} \) \( (i, j \notin \mathcal{P}^0(\sigma)). \).
Theorem. Note that $1 < d \leq 2e + 1$ by our assumption. We first consider the case $n = 2$. In this case, $\text{ord}(b_{ii}) = a_i$ for $i = 1, 2$. By Corollary 2.1, we have

$$\text{ord}(b_{12}^2 - b_{11}b_{22}) = a_1 + a_2 + d - 2e - 1.$$  

Since $d \leq 2e + 1$, we have $\text{ord}(b_{11}b_{22}) = a_1 + a_2 \geq \text{ord}(b_{12}^2 - b_{11}b_{22})$. Hence we have $\text{ord}(b_{12}) \geq (a_1 + a_2 + d - 2e - 1)/2$. This proves the lemma for $n = 2$. Note that $\text{ord}(b_{ii}) \geq (2a_i + d - 2e - 1)/2$ for $i = 1, 2$ also holds, since $d \leq 2e + 1$.

Now we consider the case $n > 2$. Without loss of generality, we may assume $\mathcal{P}^0(\sigma) = \{n - 1, n\}$. Write $B$ in a block form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix} \in M_{n \times n}.$$  

Then $B_{11}$ is a reduced form of GK type $(a', \sigma')$, where $a'_i = a_i$ and $\sigma'(i) = \sigma(i)$ for $i = 1, \ldots, n - 2$. In particular, $\mathcal{P}^0(\sigma') = \emptyset$ and $\text{ord}(\mathcal{D}B_{11}) = 0$. Put $X = -B_{11}^{-1}B_{12}$ and $U = (u_{ij}) = \begin{pmatrix} 1 & X \\ O & 1 \end{pmatrix}$. Then we have

$$B[U] = \begin{pmatrix} B_{11} & O \\ O & T \end{pmatrix},$$

where $T = B_{22} + B_{11}[X]$.

By Lemma 3.3, we have $\text{ord}(u_{ij}) > (a_j - a_i)/2$ for $i \in \{1, \ldots, n - 2\}$ and $j \in \{n - 1, n\}$. Hence we have $B_{11}[X] \in \mathcal{M}^0((a_{n-1}, a_n))$. It follows that $T$ is a reduced form of GK type $(a_{n-1}, a_n)$. Note that $\text{ord}(\mathcal{D}T) = \text{ord}(\mathcal{D}B) = d$, since $\text{ord}(\mathcal{D}B_{11}) = 0$. Hence we have

$$\text{ord}(t'_{ii}) \geq 2e + 1 - d - a_{i+n-2} \quad (i = 1, 2),$$

$$\text{ord}(t'_{12}) \geq (2e + 1 - d - a_{n-1} - a_n)/2,$$

where $(t'_{ij}) = T^{-1}$. This proves (a) and (b), since $t'_{ij} = b'_{i+n-2,j+n-2}$.

As we have observed as above, these two inequalities imply

$$\text{ord}(t'_{ij}) \geq (2e + 1 - d - a_{i+n-2} - a_{j+n-2})/2 \quad (i, j = 1, 2).$$

Next, we prove (c). Since

$$B^{-1} = \begin{pmatrix} 1 & -X \\ O & 1 \end{pmatrix} \begin{pmatrix} B_{11}^{-1} & O \\ O & T^{-1} \end{pmatrix} \begin{pmatrix} 1 & O \\ -X & 1 \end{pmatrix},$$

we have

$$\text{ord}(b'_{ij}) \geq \min\{\text{ord}(u_{i,n-1}t'_{1,j+n-2,n}), \text{ord}(u_{i,n}t'_{2,j+n-2})\} \geq (2e + 1 - d - a_i - a_j)/2.$$
for \(i \in \{1, \ldots, n-2\}\) and \(j \in \{n-1, n\}\). Hence we have (c).

By Lemma 3.3, we have

\[
\text{ord}((B_{11})_{ii}) \geq 2e - a_i, \quad (i = 1, \ldots, n-2),
\]

\[
\text{ord}((B_{11})_{ij}) \geq (2e - a_i - a_j)/2, \quad (i, j = 1, \ldots, n-2).
\]

Here, \((B_{11})_{ij}\) is the \(ij\)-th entry of \(B_{11}\). One can easily show

\[
\text{ord}((X^tX)_{ii}) \geq 2e + 1 - d - a_i, \quad (i = 1, \ldots, n-2),
\]

\[
\text{ord}((X^tX)_{ij}) \geq (2e + 1 - d - a_i - a_j)/2, \quad (i, j = 1, \ldots, n-2).
\]

Hence we have (d) and (e).

\(\square\)

4. Reduction theorem

Suppose that \(a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}\) is a non-decreasing sequence. In this case, the integers \(n_1, n_2, \ldots, n_r\) are given by

\[
a_1 = \cdots = a_{n_1} < a_{n_1+1},
\]

\[
a_{n_1} < a_{n_1+1} = \cdots = a_{n_1+n_2} < a_{n_1+n_2+1},
\]

\[
\vdots
\]

\[
a_{n_1+\cdots+n_{r-1}} < a_{n_1+\cdots+n_{r-1}+1} = \cdots = a_{n_1+\cdots+n_r},
\]

with \(n = n_1 + n_2 + \cdots + n_r\). For \(s = 1, 2, \ldots, r\), we set

\[
n_s^* = \sum_{v=1}^{s} n_v, \quad a_s^* = a_{n_s^*+1} = \cdots = a_{n_r^*}.
\]

We set \(n_0^* = 0\). The \(s\)-th block \(I_s\) is defined by \(I_s = \{n_{s-1}^* + 1, n_s^* + 2, \ldots, a_s^*\}\).

We say that two \(a\)-admissible involutions \(\sigma, \sigma' \in \mathfrak{S}_n\) are equivalent if they are conjugate by an element of \(\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}\). In each equivalence class of \(a\)-admissible involutions, there exists a unique \(a\)-admissible involution \(\sigma\) satisfying the following properties (i), (ii) and (iii).

(i) If \(i \in \mathcal{P}^0_s \cup \mathcal{P}^-_s\), then \(i\) is the maximal element of \(I_s\).
(ii) If \(i \in \mathcal{P}^+_s\), then \(i\) is the minimal element of \(I_s\).
(iii) If \(a_i = a_{\sigma(i)}\), then \(|\sigma(i) - i| \leq 1\).

We say that an \(a\)-admissible involution \(\sigma\) is standard if \(\sigma\) satisfies these conditions. Thus the set of standard \(a\)-admissible involutions is a complete set of representatives for the equivalence classes of \(a\)-admissible involutions. We shall say that a GK type \((a, \sigma)\) is a standard GK type if \(\sigma\) is standard.

Thus an involution \(\sigma \in \mathfrak{S}_n\) is a standard \(a\)-admissible involution if the following conditions are satisfied:
$(i)$ $\mathcal{P}^0$ has at most two elements. If $\mathcal{P}^0$ has two distinct elements $i$ and $j$, then $a_i \neq a_j \mod 2$. Moreover, if $i \in I_s \cap \mathcal{P}^0$, then $i$ is the maximal element of $I_s$, and
\[
i = \max\{j \mid j \in \mathcal{P}^0 \cup \mathcal{P}^+, a_j \equiv a_i \mod 2\}.
\]
$(ii)$ For $s = 1, \ldots, r$, there is at most one element in $I_s \cap \mathcal{P}^-$. If $i \in I_s \cap \mathcal{P}^-$, then $i$ is the maximal element of $I_s$ and
\[
\sigma(i) = \min\{j \in \mathcal{P}^+ \mid j > i, a_j \equiv a_i \mod 2\}.
\]
$(iii)$ For $s = 1, \ldots, r$, there is at most one element in $I_s \cap \mathcal{P}^+$. If $i \in I_s \cap \mathcal{P}^+$, then $i$ is the minimal element of $I_s$ and
\[
\sigma(i) = \max\{j \in \mathcal{P}^- \mid j < i, a_j \equiv a_i \mod 2\}.
\]
$(iv)$ If $a_i = a_{\sigma(i)}$, then $|i - \sigma(i)| \leq 1$.

We draw a picture of an example of a standard GK type. Let us consider a standard GK type given by
\[
a = (0, 0, 0, 1, 2, 2, 2, 3, 3, 5, 5, 6, 6, 6, 7, 7, 7),
\]
\[
\sigma = (12)(35)(4, 17)(67)(8, 13)(9, 10)(11, 12)(14, 15)(18, 19).
\]

Then this GK type can be picturized as follows.

Here, the upper line shows blocks $I_s$ with $a_s^*$ even, and the lower line shows blocks $I_s$ with $a_s^*$ odd.

Let $(a, \sigma)$ be a standard GK type. For $1 \leq k \leq n$, we define $\sigma^{(k)} \in \mathfrak{S}_k$ by
\[
\sigma^{(k)}(i) = \begin{cases} i & \text{if } \sigma(i) > k, \\ \sigma(i) & \text{otherwise.} \end{cases}
\]
If $\sigma^{(k)}$ is $a^{(k)}$-admissible, then $\sigma^{(k)}$ is also standard. In this case, we say that the standard GK type $(a^{(k)}, \sigma^{(k)})$ is a restriction of the standard GK type $(a, \sigma)$. We also say $(a, \sigma)$ is an extension of $(a^{(k)}, \sigma^{(k)})$. The proof of the following lemma is easy and omitted.

**Lemma 4.1.** Let $(a, \sigma)$ be a standard GK type, and $B = (b_{ij}) \in \mathcal{H}_n(a)$ a reduced form of standard GK type $(a, \sigma)$.

$(1)$ If $a_k < a_{k+1}$, then $\sigma^{(k)}$ is $a^{(k)}$-admissible and $B^{(k)}$ is a reduced form of GK type $(a^{(k)}, \sigma^{(k)})$. 
(2) If $n \in \mathcal{P}_0 \cup \mathcal{P}^+$, then $\sigma^{(n-1)}$ is $\underline{a}^{(n-1)}$-admissible and $B^{(n-1)}$ is a reduced form of GK type $(\underline{a}^{(n-1)}, \sigma^{(n-1)})$.

(3) If $a_{n-1} = a_n$ and if $\sigma(n) = \sigma(n-1)$, then $\sigma^{(n-2)}$ is $\underline{a}^{(n-2)}$-admissible and $B^{(n-2)}$ is a reduced form of GK type $(\underline{a}^{(n-2)}, \sigma^{(n-2)})$.

**Lemma 4.2.** Suppose that $B = (b_{ij}) \in \mathcal{M}(\underline{a})$, where $\underline{a} = (a_1, \ldots, a_n)$. Write $B$ in a block form

$$
B = \begin{pmatrix}
B^{(m)} & C' \\
 i' & D'
\end{pmatrix} = (b_{ij}).
$$

We assume that $B^{(m)}$ is a reduced form of GK type $(\underline{a}^{(m)}, \sigma_m)$ for some $\underline{a}^{(m)}$-admissible involution $\sigma_m \in \mathfrak{S}_m$. Then there exists $U \in G_{\underline{a}}^{\Delta}$ satisfying the following conditions.

1. $U$ is of the form
$$
U = \begin{pmatrix} 1_m & X \\
0 & 1_{n-m} \end{pmatrix}.
$$

2. Put
$$
B[U] = \begin{pmatrix} B^{(m)} & C' \\
 i' & D' \end{pmatrix}.
$$

Then $i$-th row of $C'$ is 0 unless $i \in \mathcal{P}_0(\sigma_m)$.

**Proof.** We first consider the case when $\mathcal{P}_0(\sigma_m) = \emptyset$. Put

$$
X = -(B^{(m)})^{-1}C, \quad U = \begin{pmatrix} 1_m & X \\
0 & 1_{n-m} \end{pmatrix}.
$$

Put $(B^{(m)})^{-1} = (y_{ij})$. By Lemma 3.3, $\text{ord}(2^{-1}y_{ij}) \geq -(a_i + a_j)/2$ for $1 \leq i \leq m$ and $1 \leq j \leq m$. On the other hand, $\text{ord}(2r_{ij}) \geq (a_i + a_m + j)/2$ for $1 \leq i \leq m$ and $1 \leq j \leq n - m$, where $c_{ij}$ is the $ij$-th entry of $C$. Hence we have

$$
\text{ord}(x_{ij}) \geq \min_{1 \leq k \leq m} \{\text{ord}(y_{ik}c_{kj})\} \geq (a_{m+j} - a_i)/2,
$$

where $x_{ij}$ is the $ij$-th entry of $X$. It follows that $U \in G_{\underline{a}}^{\Delta}$. Thus in this case, the conditions are satisfied.

Next, we consider the case $\mathcal{P}_0(\sigma_m) \neq \emptyset$. For simplicity, we assume $\sharp \mathcal{P}_0(\sigma) = 1$. Put $\mathcal{P}_0(\sigma_m) = \{i_0\}$. Write

$$
B^{(m)} = \begin{pmatrix} B_{11} & * & B_{12} \\
* & b_{i_0i_0} & * \\
i'B_{22} & * & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\
* \\
C_2 \end{pmatrix}.
$$
Then \( \left( \begin{array}{cc} B_{11} & B_{12} \\ tB_{12} & B_{22} \end{array} \right) \) is a reduced form with GK type \((a', \sigma')\), where \((a', \sigma')\) is the GK type obtained by removing \(i_0\)-th component from \((a^{(m)}, \sigma_m)\). In particular, \(P^0(\sigma') = \emptyset\). Put

\[ \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right) = \left( \begin{array}{cc} B_{11} & B_{12} \\ tB_{12} & B_{22} \end{array} \right)^{-1} \left( \begin{array}{c} C_1 \\ C_2 \end{array} \right) \]

and

\[ X = \left( \begin{array}{c} X_1 \\ 0 \\ X_2 \end{array} \right) \cdots i_0. \]

Then \( U = \left( \begin{array}{cc} 1_m & X \\ 0 & 1_{n-m} \end{array} \right) \) satisfies the required conditions. The case \( \#P^0(\sigma_m) = 2 \) can be treated in a similar way.

The following lemma will be used in our forthcoming paper

**Lemma 4.3.** Let the notation be as in Lemma 4.2. Put \( B[U] = (b'_{ij}) \). If \( \text{ord}(b_{m+1,m+1}) > a_{m+1} \) and \( \text{ord}(2b_{i,m+1}) > (a_i + a_{m+1})/2 \) for any \( 1 \leq i \leq m \), then we have \( \text{ord}(b'_{m+1,m+1}) > a_{m+1} \).

**Proof.** If \( \text{ord}(2b_{i,m+1}) > (a_i + a_{m+1})/2 \) for any \( 1 \leq i \leq m \), then \( \text{ord}(x_{i1}) > (a_{m+1} - a_i)/2 \) for any \( 1 \leq i \leq m \), and so \( \text{ord}(b'_{m+1,m+1}) > a_{m+1} \). \( \square \)

**Lemma 4.4.** Suppose that \( a_1, a_2 \in \mathbb{Z}_{\geq 0} \) and \( a_2 - a_1 \) is an even integer. We assume

\[ \text{ord}(b_{11}) = a_1, \quad \text{ord}(2b_{12}) > \frac{a_1 + a_2}{2}, \quad \text{ord}(b_{22}) = a_2. \]

Then there exists \( x \in F \) such that

\[ \text{ord}(x) \geq \frac{a_2 - a_1}{2}, \quad \text{ord}(b_{22} + 2b_{12}x + b_{11}x^2) > a_2. \]

**Proof.** It is enough to consider the case \( a_1 = a_2 = 0 \). We denote the image of \( b_{11} \) and \( b_{22} \) in \( \mathfrak{f} = \mathfrak{o}/\mathfrak{p} \) by \( \bar{b}_{11} \) and \( \bar{b}_{22} \), respectively. Since \( \mathfrak{f} \) is a finite field of characteristic 2, there exists \( t \in \mathfrak{f} \) such that \( \bar{b}_{22} = \bar{b}_{11} t^2 \). Then one can choose \( x \in \mathfrak{o} \) such that \( \bar{x} = t \). \( \square \)

**Lemma 4.5.** Let \( \underline{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n \) be a non-decreasing sequence. Suppose that \( B = (b_{ij}) \in H^\text{nd}(\underline{a}) \) is optimal and \( \text{GK}(B) = \underline{a} \). We assume that \( m < n \) and \( B^{(m)} \) is a reduced form of a standard GK type \((\underline{a}^{(m)}, \sigma_m)\) for some \( \underline{a}^{(m)} \)-admissible involution \( \sigma_m \in \mathfrak{S}_m \). Then there exists a standard GK type \((\underline{a}^{(k)}, \sigma_k)\) and \( U \in C^\Delta_{\underline{a}} \) satisfying the following conditions.
(1) \( k > m \) and \( (a^{(k)}, \sigma_k) \) is an extension of \( (a^{(m)}, \sigma_m) \).

(2) \( B[U]^{(k)} \) is a reduced form of GK type \( (a^{(k)}, \sigma_k) \).

Proof. Put \( c = a_{m+1} \). Let \( s \) be the maximal integer such that \( c = a_{m+1} = \cdots = a_{m+s} \). Write \( B \) in a block form as follows.

\[
B = \begin{pmatrix}
B_{11} & B_{12} & B_{13} \\
\bar{t}B_{12} & B_{22} & B_{23} \\
\bar{t}B_{13} & \bar{t}B_{23} & B_{33}
\end{pmatrix}
\begin{array}{l}
\{ m \} \\
\{ s \} \\
\{ n-m-s \}
\end{array}
\]

By Lemma 4.2, we may assume \( (B_{12})_{ij} = 0 \) for \( 1 \leq i \leq m, \ i \not\in \mathcal{P}^0(\sigma_m) \).

Suppose that there exists \( h \in \mathcal{P}^0(\sigma_m) \) such that

\[
\min_{1 \leq j \leq s} (\text{ord}(2(B_{12})_{hj})) = \frac{a_h + c}{2}.
\]

We claim that

\[
\min_{1 \leq j \leq s} (\text{ord}(2(B_{12})_{ij})) > \frac{a_i + c}{2} \quad \text{for } i \neq h.
\]

In fact, if \( h' \in \mathcal{P}^0(\sigma_m) \) and \( h' \neq h \), then \( a_h \neq a_{h'} \mod 2 \). It follows that \( (a_{h'} + c)/2 \notin \mathbb{Z} \), and so \( \text{ord}(2(B_{12})_{hj}) > (a_{h'} + c)/2 \) for \( j = 1, 2, \ldots, s \).

By changing the coordinates, we may assume

\[
\text{ord}(2(B_{12})_{h1}) = \text{ord}(2b_{h,m+1}) = \frac{a_h + c}{2}.
\]

In this case, put \( k = m + 1 \) and

\[
\sigma_k(i) = \begin{cases} 
  i & \text{for } 1 \leq i \leq m, \ i \neq h, \\
  m + 1 & \text{for } i = h \\
  h & \text{for } i = m + 1
\end{cases}
\]

Then \( (a^{(k)}, \sigma_k) \) is a standard GK type, which is an extension of \( (a^{(m)}, \sigma_m) \).

Moreover, \( B^{(k)} \) is a reduced form of GK type \( (a^{(k)}, \sigma_k) \).

Next, we consider the case

\[
\min_{1 \leq j \leq s} (\text{ord}(2(B_{12})_{ij})) > \frac{a_i + c}{2} \quad \text{for any } 1 \leq i \leq m.
\]

In this case, we have

\[
\begin{pmatrix}
0 & B_{12} & 0 \\
\bar{t}B_{12} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \in \mathcal{M}^0(\mathcal{G})
\]
and GK\((B_{22}) = (c, \ldots, c)\) by Lemma 1.4. If \(s \geq 2\), then we may assume \(B_{22} = \varpi^c K \perp B'\), for some primitive unramified binary form \(K\) and \(B' \in \mathcal{H}_{s-2}(\mathfrak{o})\) by Proposition 3.4. In this case, put \(k = m + 2\) and
\[
\sigma_k(i) = \begin{cases} 
  i & 1 \leq i \leq m, \\
  m + 2 & i = m + 1 \\
  m + 1 & i = m + 2.
\end{cases}
\]
Then \((a^{(k)}, \sigma_k)\) is a standard GK type, which is an extension of \((a^{(m)}, \sigma_m)\). Moreover, \(B^{(k)}\) is a reduced form of GK type \((a^{(k)}, \sigma_k)\).

Finally, we consider the case when
\[
\begin{pmatrix} 
  0 & B_{12} & 0 \\
  B_{12} & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix} \in \mathcal{M}^0(a)
\]
and \(s = 1\). Note that \(B_{22} = (b_{m+1,m+1})\) and \(\text{ord}(b_{m+1,m+1}) = c\). In this case, we claim that \(\{h \in \mathcal{P}^0(\sigma_k) \mid a_h \equiv c \mod 2\} = \emptyset\). Suppose \(h \in \mathcal{P}^0(\sigma_k)\) and \(a_h \equiv c \mod 2\). Then there exists \(x \in \mathfrak{o}\) such that
\[
\text{ord}(x) \geq \frac{c - a_h}{2}, \quad \text{ord}(b_{m+1,m+1} + 2b_{h,m+1}x + b_{hh}x^2) > c
\]
by Lemma 4.4. Put \(B' = B[U]\), where \(U\) is the upper triangular unipotent matrix whose \(U_{h,m+1} = x\) and \(U_{ij} = 0\) for \(i < j, (i, j) \neq (h, m+1)\). Then \(U \in G_\Sigma\) and GK\((B') \supseteq a\) by Lemma 1.4. This contradicts the assumption GK\((B) = a\). Thus we have \(\{h \in \mathcal{P}^0(\sigma_k) \mid a_h \equiv c \mod 2\} = \emptyset\). In this case, put \(k = m + 1\) and
\[
\sigma_k(i) = \begin{cases} 
  i & 1 \leq i \leq m, \\
  m + 1 & i = m + 1.
\end{cases}
\]
Then \((a^{(k)}, \sigma_k)\) is a standard GK type, which is an extension of \((a^{(m)}, \sigma_m)\). Moreover, \(B^{(k)}\) is a reduced form of GK type \((a^{(k)}, \sigma_k)\). \qed

By using Lemma 4.5 repeatedly, we obtain the following theorem.

**Theorem 4.1 (Reduction theorem).** Let \(a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n\) be a non-decreasing sequence. Suppose that \(B \in \mathcal{H}_{n}^\text{ad}(\mathfrak{o})\) is optimal and GK\((B) = a\). Then there exists \(U \in G_\Sigma\), and a standard \(a\)-admissible involution \(\sigma\) such that \(B|U\) is a reduced form of GK type \((a, \sigma)\). In particular, a non-degenerate half-integral symmetric matrix is equivalent to a reduced form.

**Remark 4.1.** We shall say a reduced form \(B = (b_{ij})\) of GK type \((a, \sigma)\) is a strongly reduced form if the following condition hold:
(1) If \( i \notin \mathcal{P}_0(\sigma) \), then \( b_{ij} = 0 \) for \( j > \max\{i, \sigma(i)\} \).

The proof of Theorem 4.1 shows that a non-degenerate half-integral symmetric matrix is equivalent to a strongly reduced form.

Recall that two \( a \)-admissible involutions \( \sigma, \sigma' \in S_n \) are equivalent if they are conjugate by an element of \( S_{n_1} \times \cdots \times S_{n_r} \). The equivalence class of \( \sigma \) is determined by
\[
\#\mathcal{P}_1^+, \ldots, \#\mathcal{P}_r^+, \#\mathcal{P}_1^-, \ldots, \#\mathcal{P}_r^-, \#\mathcal{P}_1^0, \ldots, \#\mathcal{P}_r^0,
\]
since
\[
\sigma(i) = \min\{j \in \mathcal{P}^+ \mid j > i, a_j \equiv a_i \mod 2\} \quad \text{for} \ i \in \mathcal{P}^-,
\]
\[
\sigma(i) = \max\{j \in \mathcal{P}^- \mid j < i, a_j \equiv a_i \mod 2\} \quad \text{for} \ i \in \mathcal{P}^+.
\]

Note that for each block \( I_s \), exactly one of the following possibilities occur:

1. \( n_s \) is even and \( \#\mathcal{P}_s^+ = \#\mathcal{P}_s^- = \#\mathcal{P}_s^0 = 0 \).
2. \( n_s \) is even and \( \#\mathcal{P}_s^+ = \#\mathcal{P}_s^- + \#\mathcal{P}_s^0 = 1 \).
3. \( n_s \) is odd and \( \#\mathcal{P}_s^+ = 1, \#\mathcal{P}_s^- = \#\mathcal{P}_s^0 = 0 \).
4. \( n_s \) is odd and \( \#\mathcal{P}_s^+ = 0, \#\mathcal{P}_s^- + \#\mathcal{P}_s^0 = 1 \).

Moreover, if \( i \in \mathcal{P}_0 \), then
\[
i = \max\{j \in \mathcal{P}_0 \cup \mathcal{P}^+ \mid a_i \equiv a_j \mod 2\}.
\]

It follows that the equivalence class of \( \sigma \) is determined by
\[
\#\mathcal{P}_1^+, \ldots, \#\mathcal{P}_r^+.
\]

We determine the number of equivalence classes of GK types for given \( a \). For a block \( I_s \), let \( k_s \) be the number of blocks \( I_u \) such that
\[
1 \leq u < s, \quad a_u^* \equiv a_s^* \mod 2, \quad n_u \equiv 0 \mod 2.
\]

If \( n_s \) is odd, then the possibility (3) (resp. the possibility (4)) occurs if and only if \( k_s \) is odd (resp. even). Suppose that \( n_s \) is even. If \( k_s \) is even, only the possibility (1) occurs. If \( k_s \) is odd, both (1) and (2) are possible. Note also that \( \#\mathcal{P}_s^0 = 1 \) if and only if \( k_s \) is even and
\[
i = \max\{j \in \mathcal{P}_0 \cup \mathcal{P}^+ \mid a_i \equiv a_j \mod 2\}.
\]

Thus the number of equivalence classes of GK types is equal to \( 2^K \), where \( K \) is the number of blocks \( I_s \) such that \( n_s \equiv 0 \mod 2 \) and \( k_s \equiv 0 \mod 2 \).

The proof of the following lemma is easy and will be omitted.

**Lemma 4.6.** Suppose that \( B \in \mathcal{H}_n(\sigma) \) is a reduced form of GK type \((a, \sigma)\). If \( U \in N_\omega^\vee \), then \( B[U] \) is also a reduced form of GK type \((a, \sigma)\).
Theorem 4.2. Let \( B = (b_{ij}) \in \mathcal{H}_n(\mathfrak{o}) \) and \( T = (t_{ij}) \in \mathcal{H}_n(\mathfrak{o}) \) be reduced forms with standard GK types \((\underline{a}, \sigma_1)\) and \((\underline{a}, \sigma_2)\), respectively. If \( B \sim T \), then we have \( \sigma_1 = \sigma_2 \).

Proof. Assume that both \( B[U] = T \) for some \( U \in \text{GL}_n(\mathfrak{o}) \). Since both \( B \) and \( T \) are optimal, we have \( U \in G_{\underline{a}} \). Since \( G_{\underline{a}} = N^\Delta G_{\underline{a}}^\Delta \), there exist \( U_1 \in N_{\underline{a}}^\Delta \) and \( U_2 \in G_{\underline{a}}^\Delta \) such that \( U = U_1 U_2 \). Note that \( B[U_1] \) is a reduced form of GK type \((\underline{a}, \sigma_1)\) by Lemma 4.6. Replacing \( B \) by \( B[U_1] \), we may assume \( U \in G_{\underline{a}}^\Delta \).

Suppose that \( \sigma_1 \not\sim \sigma_2 \). Let \( I_1, \ldots, I_r \) be the blocks for \( \underline{a} \). Then we have
\[
\sharp \mathcal{P}_s^+(\sigma_1) \neq \sharp \mathcal{P}_s^+(\sigma_2)
\]
for some \( s \). Let \( s \) be the smallest integer with this property. We may assume \( \mathcal{P}_s^+(\sigma_1) \neq \emptyset \) and \( \mathcal{P}_s^+(\sigma_2) = \emptyset \). By replacing \( B \) and \( T \) by \( B(n_s^*) \) and \( T(n_s^*) \), we may assume \( n = n_s \). Put \( m = n_{s-1} = n - n_s \).

Write \( B \) and \( T \) in block forms as follows.
\[
B = \begin{pmatrix}
B_{11} & B_{12} \\
\tilde{B}_{12} & B_{22}
\end{pmatrix}_{m \times n_s} \quad T = \begin{pmatrix}
T_{11} & T_{12} \\
\tilde{T}_{12} & T_{22}
\end{pmatrix}_{m \times n_s}
\]
Then we have \( \begin{pmatrix} 0 & T_{12} \\ \tilde{T}_{12} & 0 \end{pmatrix} \in \mathcal{M}^0(\underline{a}) \), since \( \mathcal{P}_s^+(\sigma_2) = \emptyset \). Decompose \( U \in G_{\underline{a}}^\Delta \) into
\[
U = \begin{pmatrix}
1_m & X \\
0 & 1_{n_s}
\end{pmatrix}
\begin{pmatrix}
U_{11} & 0 \\
0 & U_{22}
\end{pmatrix},
\]
where
\[
U_{11} \in G_{\underline{a}^{(m)}}, \quad U_{22} \in \text{GL}_{n_s}(\mathfrak{o}), \quad \begin{pmatrix} 1_m & X \\ 0 & 1_{n_s} \end{pmatrix} \in N_{\underline{a}}^\Delta.
\]
Put
\[
B' = \begin{pmatrix}
B_{11}' & B_{12}' \\
\tilde{B}_{12}' & B_{22}'
\end{pmatrix} = B \begin{pmatrix} 1_m & X \\ 0 & 1_{n_s} \end{pmatrix}.
\]
Then we have
\[
\begin{pmatrix} 0 & B_{12}' \\ \tilde{B}_{12}' & 0 \end{pmatrix} = \begin{pmatrix} 0 & T_{12} \\ \tilde{T}_{12} & 0 \end{pmatrix} \begin{pmatrix} U_{11}^{-1} & 0 \\ 0 & U_{22}^{-1} \end{pmatrix} \in \mathcal{M}^0(\underline{a}).
\]
Since \( \mathcal{P}_s^+(\sigma_1) \neq \emptyset \), there exists \( h \in \mathcal{P}^-(\sigma_1) \) such that \( \sigma_1(h) \in I_s \). Now look at the \( h \)-th row of
\[
B_{12}' = B_{12} + B_{11}X.
\]
Put \( Y = (y_{ij}) = B_{11}X \). We claim that
\[
\text{ord}(2y_{hj}) > \frac{a_h + a_i^*}{2} \quad \text{for } j = 1, 2, \ldots, n_s.
\]
In fact, 
\[ y_{hj} = \sum_{i=1}^{m} b_{hi} x_{ij}, \]
where \( x_{ij} \) is the \((i, j)\)-th entry of \( X \). Since \( B \) is a reduced form of GK type \((a, \sigma_1)\), we have
\[ \text{ord}(2b_{hj}) > \frac{a_{h}+a_{i}}{2}, \quad \text{for } 1 \leq i \leq m, i \neq h. \]
Note that \( \text{ord}(2b_{hh}) > a_{h} \), since \( F \) is dyadic. Note also that
\[ \text{ord}(x_{ij}) \geq \frac{a_{s}^{*} - a_{i}}{2}, \quad \text{for } i = 1, 2, \ldots, m, \]
since \( \begin{pmatrix} 1_{m} & X \\ 0 & 1_{n_{s}} \end{pmatrix} \in N_{2}^{\Delta} \). This proves the claim.

Put \( \sigma_1(h) = m + k \in \mathcal{P}_{s}^{+}(\sigma_1) \). Then we have
\[ \text{ord}(2(B_{12})_{hk}) = \text{ord}(2b_{h,m+k}) = \frac{a_{h} + a_{s}^{*}}{2}, \]
where \((B_{12})_{hk}\) is the \((h, k)\)-th entry of \( B_{12} \). This is a contradiction. \( \square \)

**Example.** Suppose that \( F = \mathbb{Q}_2 \). Put
\[ B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}. \]

Then \( B_1 \) is a reduced form of GK type \(((0, 2, 2), \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix})\) and \( B_2 \) is a reduced form of GK type \(((0, 2, 2), \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix})\). Note that \( B_1^{(2)} \) is reduced, but \( B_2^{(2)} \) is not. For more examples, see Bouw [2] and Yang [12].

5. **Optimal Forms**

**Theorem 5.1.** Suppose that \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^{n} \) is a non-decreasing sequence and that \( B \in \mathcal{M}(a) \). Then \( \text{GK}(B) = a \) if and only if \( \Delta(B^{(s_{n})}) = |a^{(s_{n})}| \) for \( s = 1, \ldots, r \). Moreover, \( B \) is optimal in this case.

**Proof.** Assume that \( \Delta(B^{(s_{n})}) = |a^{(s_{n})}| \) for \( s = 1, \ldots, r \). Put \( \text{GK}(B) = \overline{b} = (b_1, \ldots, b_n) \). Since \( a \in S(B) \), we have \( b \succeq a \). We shall show that \( (a_1, \ldots, a_{n_{s}}) = (b_1, \ldots, b_{n_{s}}) \) for \( s = 0, 1, \ldots, r \) by induction. The case \( s = 0 \) is trivial.
Assume that \((a_1, \ldots, a_n^*) = (b_1, \ldots, b_n^*)\). Then, we have \(b_{n^*_1 + 1} \geq a_{n^*_2 + 1}\), since \(b \succeq a\). Note that
\[
\begin{align*}
& a_{n^*_1 + 1} = a_{n^*_2 + 2} = \cdots = a_{n^*_r + 1}, \\
& b_{n^*_1 + 1} \leq b_{n^*_2 + 2} \leq \cdots \leq b_{n^*_r + 1}.
\end{align*}
\]
It follows that
\[
\sum_{i=1}^{n^*_1 + 1} a_i \leq \sum_{i=1}^{n^*_1} b_i.
\]
By applying Lemma 3.8 for \(B_1 = B^{(n^*_1 + 1)}\), we have
\[
\sum_{i=1}^{n^*_1 + 1} b_i \leq \Delta(B^{(n^*_1 + 1)}) = \sum_{i=1}^{n^*_1 + 1} a_i.
\]
It follows that \((a_1, \ldots, a_{n^*_r + 1}) = (b_1, \ldots, b_{n^*_r + 1})\), as desired.

Conversely, assume that \((a_1, \ldots, a_{n^*_r + 1}) = (b_1, \ldots, b_{n^*_r + 1})\), as desired.

By Theorem 4.1, there exist a reduced form \(B_1\) and \(U \in G_{\underline{\triangle}}\) such that \(B = B_1[U]\). Then, we have \(B^{(n^*_1)} = B_1^{(n^*_1)}[U(n^*_1)]\). In particular, we have \(\Delta(B^{(n^*_1)}) = \Delta(B_1^{(n^*_1)})\). Since \(B_1^{(n^*_1)}\) is a reduced form of \(GK\) type \(a^{(n^*_1)}\), we see \(\Delta(B_1^{(n^*_1)}) = |a^{(n^*_1)}|\). The last part of the theorem is clear.

**Corollary 5.1.** Suppose that \(a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n\) is a non-decreasing sequence. If \(B \in \mathcal{H}_n(a)\) is a reduced form of \(GK\) type \(a\), then we have \(GK(B) = a\). In particular, \(B\) is optimal.

**Proof.** Note that \(B^{(n^*_1)}\) is reduced with \(GK\) type \(a^{(n^*_1)}\) for \(s = 1, \ldots, r\). By Proposition 3.2, we have \(\Delta(B^{(n^*_1)}) = |a^{(n^*_1)}|\). By Theorem 5.1, \(B\) is optimal and \(GK(B) = a\).

**Proof of Theorem 0.1.** By Theorem 4.1, there exists \(U \in G_{\underline{\triangle}}\) such that \(B[U]\) is a reduced form of \(GK\) type \(a\). By Proposition 3.2, we have \(|a| = \Delta(B[U]) = \Delta(B)|\).

**Proof of Theorem 0.2.** The “if” part is Proposition 1.1. We prove the “only if” part. Suppose that both \(B\) and \(B[U]\) are optimal with \(U \in GL_n(a)\). By Theorem 4.1 and Corollary 5.1, we may assume \(B\) is a reduced form of \(GK\) type \(a\). By Lemma 3.11, \(\psi_i U \in L_s\) for any \(i \in I_s\) \((s = 1, 2, \ldots, r)\). It follows that \(U \in G_a\), as desired.

Theorem 0.3 follows from Theorem 5.1 immediately.

**Proof of Theorem 0.4.** We may assume that \(B\) is a reduced form of \(GK\) type \(a\). By Theorem 0.2, there exists an element \(U \in G_{\underline{\triangle}}\) such that \(B_1 = B[U]\). Since \(G_a = N_\triangle G_{\underline{\triangle}}\), there exist \(U_1 \in N_{\triangle}^\perp\) and \(U_2 \in G_{\underline{\triangle}}\) such
that \( U = U_1U_2 \). Then \( B_1^{(k)} \) is equivalent to \( B_1[U_2^{-1}]^{(k)} = B[U_1]^{(k)} \), since \( U_2 \in G_a^\Delta \). Replacing \( B_1 \) by \( B_1[U_2^{-1}] \), we may assume that \( B_1 = B[U_1] \) with \( U_1 \in N_a^\nabla \). In this case, \( B_1 \) is a reduced form of GK type \( a \) and \( B^{(k)} - B_1^{(k)} \in \mathcal{M}(a^{(k)}) \). Then the theorem follows from Proposition 3.3.

Corollary 5.2. Suppose that \( n \) is even. Let \( B \in \mathcal{H}_n^\text{ad}(a) \) be a half-integral symmetric matrix such that \( \text{GK}(B) = (a, \sigma) \). Then the following four conditions are equivalent.

1. \(|a|\) is odd.
2. \( \#P_0(\sigma) = 2 \).
3. \( \text{ord}(\mathfrak{D}_B) > 0 \).
4. \( \xi_B = 0 \).

Proof. The equivalence of (1) and (2) follows from the definition of admissible involutions. The equivalence of (1) and (4) follows from Theorem 0.1. The equivalence of (3) and (4) follows from the definition of \( \xi_B \).

6. Extended GK Data

In this section, we discuss combinatorial properties of the invariants \( \text{GK}(B), \xi_B^{(k)}, \eta_B^{(k)} \). We do not assume \( F \) is dyadic in this section. The results of this section will be used in our forthcoming paper [4].

First we introduce some definitions. Put \( \mathbb{Z}_3 = \{0, 1, -1\} \).

Definition 6.1. An element \( H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n) \) of \( \mathbb{Z}_{\geq 0}^n \times \mathbb{Z}_3^n \) is said to be a naive EGK datum of length \( n \) if the following conditions hold:

(N1) \( a_1 \leq \cdots \leq a_n \).

(N2) Assume that \( i \) is even. Then \( \varepsilon_i \neq 0 \) if and only if \( a_1 + \cdots + a_i \) is even.

(N3) If \( i \) is odd, then \( \varepsilon_i \neq 0 \).

(N4) \( \varepsilon_1 = 1 \).

(N5) If \( i \geq 3 \) is odd and \( a_1 + \cdots + a_{i-1} \) is even, then \( \varepsilon_i = \varepsilon_{i-2}^{a_i + a_{i-1}} \).

We denote the set of naive EGK data of length \( n \) by \( \mathcal{NEGK}_n \).

Proposition 6.1. Suppose that \( F \) is non-dyadic field. Let \( T = (t_1) \perp \cdots \perp (t_n) \) be a diagonal matrix such that \( \text{ord}(t_1) \leq \text{ord}(t_2) \leq \cdots \leq \text{ord}(t_n) \). Put \( a_i = \text{ord}(t_i) \) and

\[
\varepsilon_i = \begin{cases} 
\xi_T^{(i)} & \text{if } i \text{ is even}, \\
\eta_T^{(i)} & \text{if } i \text{ is odd}.
\end{cases}
\]

Then \( (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n) \) is a naive EGK datum.
Proof. Only (N5) needs a proof. Suppose \( i \geq 3 \) is odd and \( a_1 + \cdots + a_{i-1} \) is even. Then \( F(\sqrt{D_{T(i-1)}})/F \) is unramified, since \( F \) is non-dyadic. It follows that \( \langle D_{B(i-1)}, t \rangle = \xi_{B(i-1)}^{\text{ord}(t)} = \xi_{i-1}^{\text{ord}(t)} \) for \( t \in F^\times \). By Lemma 3.4, we have

\[
\eta_{T(i)} = \eta_{T(i-2)} \langle D_{B(i-1)}, D_B D_{B(i-2)} \rangle = \xi_{i-2} a_i + a_{i-1}.
\]

Hence the lemma. \( \square \)

We set \( \text{NEGK}(T) = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n) \) and call it the naive EGK datum associated to \( T \).

Remark 6.1. Conversely, for a given \( H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{NEGK}_n \), there exists a diagonal matrix

\[
T = \text{diag}(t_1, \ldots, t_n), \quad \text{ord}(t_1) \leq \cdots \leq \text{ord}(t_n),
\]
such that \( \text{NEGK}(T) = H \). The proof is easy and left to the reader.

Remark 6.2. If \( F \) is a dyadic field, then \( (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n) \) may not be a naive EGK datum for a diagonal matrix \( T = \text{diag}(t_1, \ldots, t_n) \) such that \( \text{ord}(t_1) \leq \cdots \leq \text{ord}(t_n) \). For example, put \( T = (1) \times (1) \). Then \( a_1 = a_2 = 0, \varepsilon_1 = 1, \varepsilon_2 = 0 \), and so (N2) does not hold.

Definition 6.2. Let \( G = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r) \) be an element of \( \mathbb{Z}_{\geq 0}^r \times \mathbb{Z}_{\geq 0}^r \times \mathbb{Z}_3^r \). Put \( n^*_s = \sum_{i=1}^s n_i \) for \( s \leq r \). We say that \( G \) is an EGK datum of length \( n \) if the following conditions hold:

(E1) \( n^*_r = n \) and \( m_1 < \cdots < m_r \).

(E2) Assume that \( n^*_s \) is even. Then \( \zeta_s \neq 0 \) if and only if \( m_1 n_1 + \cdots + m_s n_s \) is even.

(E3) Assume that \( n^*_s \) is odd. Then \( \zeta_s \neq 0 \). Moreover, we have

(a) Assume that \( n^*_s \) is even for any \( i < s \). Then we have

\[
\zeta_s = \zeta_1^{m_1 + m_2 + m_3 + \cdots + m_s}. 
\]

In particular, \( \zeta_1 = 1 \) if \( n_1 \) is odd.

(b) Assume that \( m_1 n_1 + \cdots + m_{s-1} n_{s-1} + m_s (n_s - 1) \) is even and that \( n^*_s \) is odd for some \( i < s \). Let \( t < s \) be the largest number such that \( n^*_t \) is odd. Then we have

\[
\zeta_s = \zeta_t \zeta_{t+1}^{m_{t+1} + m_{t+2} + m_{t+3} + \cdots + m_s}. 
\]

In particular, \( \zeta_s = \zeta_t \) if \( t + 1 = s \).

We denote the set of EGK data of length \( n \) by \( \mathcal{EGK}_n \). Thus \( \mathcal{EGK}_n \subset \prod_{r=1}^n (\mathbb{Z}_{\geq 0}^r \times \mathbb{Z}_{\geq 0}^r \times \mathbb{Z}_3^r) \).

Let \( H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n) \) be a naive EGK datum. Let \( n_1, n_2, \ldots, n_r \) and \( n_1^*, n_2^*, \ldots, n_r^* \) be as in section 1. For \( s = 1, 2, \ldots, r \), we
set $m_s = a_n^s$ and $\zeta_s = \varepsilon_n^s$. The following proposition can be easily verified.

**Proposition 6.2.** Let $H = (a_1, \ldots, a_n; \varepsilon_1, \ldots, \varepsilon_n)$ be a naive EGK datum. Then $G = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r)$ is an EGK datum.

We define a map $\Upsilon = \Upsilon_n : \mathcal{NEGK}_n \to \mathcal{EGK}_n$ by $\Upsilon(H) = G$. We call $G = \Upsilon(H)$ the EGK datum associated to a naive EGK datum $H$. We also write $\Upsilon(a) = (n_1, \ldots, n_r; m_1, \ldots, m_r)$, if there is no fear of confusion.

**Proposition 6.3.** The map $\Upsilon : \mathcal{NEGK}_n \to \mathcal{EGK}_n$ is surjective. Thus for any EGK datum $G = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r)$ of length $n$, there exists a naive EGK datum $H$ such that $\Upsilon(H) = G$.

**Proof.** Note that $(a_1, \ldots, a_n)$ is determined by $(n_1, \ldots, n_r; m_1, \ldots, m_r)$. We proceed by induction with respect to $n$. When $n = 1$, the proposition is trivial. First consider the case $n_r = 1$. In this case, put $G' = (n_1, n_2, \ldots, n_{r-1}; m_1, \ldots, m_{r-1}; \zeta_1, \ldots, \zeta_{r-1})$. Then $G'$ is an EGK datum. By the induction hypothesis, there exists a naive EGK datum $H' = (a_1, \ldots, a_{n-1}; \varepsilon_1, \ldots, \varepsilon_{n-1})$ such that $\Upsilon(H') = G'$. Then $H = (a_1, \ldots, a_{n-1}, a_n; \varepsilon_1, \ldots, \varepsilon_{n-1}, \zeta_r)$ satisfies the condition.

Now, we assume $n_r \geq 2$. We define an EGK datum $G' = (n_1, \ldots, n_r-1; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_{r-1}, \zeta'_r)$ of length $n - 1$ as follows. If $n$ and $m_1n_1 + \cdots + m_r(n_r - 1)$ are odd, then put $\zeta'_r = 0$.

Assume that $n$ and $m_1n_1 + \cdots + m_rn_r$ are even. If $n_i^s$ is even for any $i < r$, then we put

$$\zeta'_r = \zeta_m^{m_1 + m_2} \cdots \zeta_{r-1}^{m_{r-1} + m_r}.$$ 

Let $t < r$ be the largest number such that $n_t^s$ is odd. Then we put

$$\zeta'_r = \zeta_t^{m_{t+1} + m_{t+2}} \cdots \zeta_{r-1}^{m_{r-1} + m_r}.$$ 

We put $\zeta'_r = \pm 1$ arbitrarily, in other cases. Then one can easily see $G'$ is an EGK datum. By the induction hypothesis, there exists a naive EGK datum $H' = (a_1, \ldots, a_{n-1}; \varepsilon_1, \ldots, \varepsilon_{n-1})$. Then

$$H = (a_1, \ldots, a_{n-1}, a_n; \varepsilon_1, \ldots, \varepsilon_{n-1}, \zeta_r)$$

is a naive EGK datum such that $\Upsilon(H) = G$. \qed
Definition 6.3. Let $B \in \mathcal{H}_n(o)$ be an optimal form such that $\text{GK}(B) = \varrho$. Put $Y(o) = (n_1, \ldots, n_r; m_1, \ldots, m_r)$. We define $\zeta_s = \zeta_s(B)$ by

$$
\zeta_s = \zeta_s(B) = \begin{cases} 
\xi_B(n_1) & \text{if } n_s^* \text{ is even,} \\
\eta_B(n_s) & \text{if } n_s^* \text{ is odd.}
\end{cases}
$$

Then put $\text{EGK}(B) = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r)$. For $B \in \mathcal{H}_n^\text{nd}(o)$, we define $\text{EGK}(B) = \text{EGK}(B')$, where $B'$ is an optimal form equivalent to $B$.

By Theorem 0.4, this definition does not depend on the choice of $B'$. Thus $\text{EGK}(B)$ depends only on the isomorphism class of $B$.

We will show that $\text{EGK}(B)$ is in fact an EGK datum. If $F$ is non-dyadic, this follows from Proposition 6.1 and Proposition 6.2, since $B$ has a Jordan splitting. To treat the dyadic case, we need some lemmas.

We assume that $F$ is a dyadic field in Lemma 6.1–6.4.

Lemma 6.1. Let $B = (b_{ij}) \in \mathcal{H}_n(o)$ be a reduced form such that $\text{GK}(B) = \varrho = (a_1, \ldots, a_n)$. Assume that $B^{(n-1)}$ is a reduced form with $\text{GK}(B^{(n-1)}) = \varrho^{(n-1)}$.

1. Assume that both $n$ and $a_1 + \cdots + a_n$ are even. Then we have
   $$
   \eta_B = \eta_{B^{(n-1)}} \xi_B^{a_n}.
   $$

2. Assume that $n$ is odd and $a_1 + \cdots + a_n - 1$ is even. Then we have
   $$
   \eta_B = \eta_{B^{(n-1)}} \xi_B^{a_n}.
   $$

Proof. We prove (1). Let $n$, $\varrho$, and $B$ be as in (1). Then we have

$$
\eta_B = \eta_{B^{(n-1)}} \langle D_B, D_{B^{(n-1)}} \rangle = \eta_{B^{(n-1)}} \xi_B^{a_1 + \cdots + a_n - 1} = \eta_{B^{(n-1)}} \xi_B^{a_n}
$$

by Lemma 3.4. Hence we have proved (1). Similarly, if $n$, $\varrho$, and $B$ are as in (2), then we have

$$
\eta_B = \eta_{B^{(n-1)}} \langle D_B, D_{B^{(n-1)}} \rangle = \eta_{B^{(n-1)}} \xi_B^{a_1 + \cdots + a_n - 1} = \eta_{B^{(n-1)}} \xi_B^{a_n}
$$

Hence we have proved (2). \hfill \Box

Let $B = (b_{ij}) \in \mathcal{H}_n(o)$ be a reduced form of standard GK type $(\varrho, \sigma)$, where $\varrho = (a_1, \ldots, a_n)$. If $n \in \mathcal{P}^+ \cup \mathcal{P}^0$, then $B^{(n-1)}$ is a reduced form with $\text{GK}(B) = \varrho^{(n-1)}$. Note that if $n \in \mathcal{P}^+$, then $a_{n-1} < a_n$.

Assume that $a_{n-1} = a_n$ and $a_{\sigma(n)} = a_{n-1}$. Since $\sigma$ is standard, we have $\sigma(n) = n - 1$. In this case, $B^{(n-2)}$ is a reduced form of GK type $(\varrho^{(n-2)}, \sigma^{(n-2)})$.

Lemma 6.2. Let $B = (b_{ij}) \in \mathcal{H}_n(o)$ be a reduced form of standard GK type $(\varrho, \sigma)$. Assume that $a_{n-1} = a_n$ and $\sigma(n) = n - 1$.

1. Assume that both $n$ and $a_1 + \cdots + a_n$ are even. Then we have
   $$
   \eta_B = \eta_{B^{(n-2)}} \xi_B^{a_n} \xi_B^{a_n}.
   $$


Lemma 6.3. For a non-decreasing sequence \( \eta = (\eta_1, \ldots, \eta_{n-1}) \) of integers, let \( \sigma \) be an \( \eta \)-admissible involution. Let \( \mathcal{B} \subset \mathcal{H}_n(\mathfrak{a}) \) be a reduced form of standard GK type \((\mathfrak{a}, \sigma)\). Assume that \( r \geq 2 \). Put \( k = n_{r-1}^* \).

1. Assume that both \( n \) and \( k \) are odd and that \( \eta_1 + \cdots + \eta_{n-1} \) is even. Then \( \eta_\mathcal{B} = \eta_\mathcal{B}(k) \).
2. Assume that \( n \) is odd, \( k \) is even, and that \( \eta_1 + \cdots + \eta_k \) is even. Then \( \eta_\mathcal{B} = \eta_\mathcal{B}(k) \xi_{\mathcal{B}(k)}^{\mathfrak{a}_k} \).
3. Assume that both \( n \) and \( k \) are even and that \( \eta_1 + \cdots + \eta_n \) is even. Then \( \eta_\mathcal{B} = \eta_\mathcal{B}(k) \xi_{\mathcal{B}(k)}^{\mathfrak{a}_k} \).
4. Assume that \( n \) is even, \( k \) is odd, and that \( \eta_1 + \cdots + \eta_n \) is even. Then \( \eta_\mathcal{B} = \eta_\mathcal{B}(k) \xi_{\mathcal{B}}^{\mathfrak{a}_k} \).

Proof. We proceed by induction with respect to \( n_r \). If \( n_r = 1 \), then (2) and (4) follow from Lemma 6.1 (2) and Lemma 6.1 (1), respectively. If \( n_r = 2 \), then (1) and (3) follow from Lemma 6.2 (2) and Lemma 6.2 (1), respectively. If \( n_r \geq 3 \), the lemma follows by using Lemma 6.2 repeatedly.

Lemma 6.4. Let \( \mathcal{B} \subset \mathcal{H}_n(\mathfrak{a}) \) be a reduced form of GK type \((\mathfrak{a}, \sigma)\). Put

\[
\text{EGK}(\mathcal{B}) = (n_1, \ldots, n_r; m_1, \ldots, m_r; \xi_1, \ldots, \xi_r).
\]
(1) Assume that $n_i^*$ is even for $i = 1, 2, \ldots, r$. Then we have
\[
\eta_B = \xi_1^{m_1+m_2} \cdots \xi_r^{m_r-1+m_r} \cdot \xi_r^{m_r}.
\]

(2) Assume that $n$ is even and that $t < r$ is the largest number such that $n_t^*$ is odd. Assume also that $m_1n_1 + \cdots + m_rn_r$ is even. Then we have
\[
\eta_B = \zeta_t \xi_t^{m_1+1+m_2} \cdots \xi_r^{m_r-1+m_r} \cdot \xi_r^{m_r-1}.
\]

Proof. Let $n$, $B$, and $(a, \sigma)$ be as in (1). If $r = 1$, then (1) is a special case of Lemma 3.6. For $r > 1$, (1) can be proved by applying Lemma 6.3 (3), repeatedly.

Next, we shall prove (2). Let $n$, $B$, and $(a, \sigma)$ be as in (2). Note that $m_1n_1 + \cdots + m_t+1n_{t+1}$ is even. Then we have $\eta_B(n_{t+1}) = \eta_B(n_t)$ by applying Lemma 6.3 (4). By using Lemma 6.3 (3) repeatedly, we have (2).

**Theorem 6.1.** Let $F$ be a non-archimedean local field. Suppose that $B \in \mathcal{H}_n^\text{nd}(\mathcal{O})$. Then $\text{EGK}(B)$ is an EGK datum of length $n$.

Proof. It is enough to consider the case when $F$ is dyadic. We may assume $B$ is a reduced form of standard GK type $(a, \sigma)$. Put
\[
\text{EGK}(B) = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r).
\]

The condition (E1) is obvious, and the condition (E2) follows from Theorem 0.1.

We will prove the condition (E3) holds. Suppose that $n_t^*$ is odd. Replacing $B$ by $B(s)$, we may assume $s = r$. It is obvious that $\zeta_r = \eta_B \neq 0$.

Assume that $n_t^*$ is even for any $i < r$. By Lemma 6.4 (1), we have
\[
\eta_B(n_{t-1}) = \xi_1^{m_1+m_2} \cdots \xi_r^{m_r-2+m_r} \cdot \xi_r^{m_r}.
\]

Then by Lemma 6.3 (2), we have
\[
\zeta_r = \eta_B = \xi_1^{m_1+m_2} \cdots \xi_r^{m_r-1+m_r}.
\]

Hence (a) of the condition (E3) holds.

Next, assume $m_1n_1 + \cdots + m_{r-1}n_{r-1} + m_r(n_r - 1)$ is even and that $n_i^*$ is odd for some $i < r$. Let $t < r$ be the largest number such that $n_t^*$ is odd. If $r = t + 1$, then we have $\eta_B = \eta_B(n_t^*)$ by Lemma 6.3 (1). Hence (b) of the condition (E3) holds in this case. Now, assume that $r > t + 1$. By Lemma 6.4 (2), we have
\[
\eta_B(n_{t-1}) = \xi_t \xi_{t+1}^{m_1+1+m_2} \cdots \xi_r^{m_r-2+m_r} \cdot \xi_r^{m_r-1}.
\]
By Lemma 6.3 (2), we have

$$\zeta_r = \eta_B = \zeta_i^{m_{i+1}+m_{i+2}} \cdots \zeta_{r-1}^{m_r}.$$ 

Hence, EGK($B$) satisfies (b) of the condition (E3). \qed

We call EGK($B$) the extended GK datum associate to $B$. One can prove the following proposition, but as we do not use it later, we omit a proof.

**Proposition 6.4.** Suppose that $F$ is a dyadic local field. Let $G = (n_1, \ldots, n_r; m_1, \ldots, m_r; \zeta_1, \ldots, \zeta_r)$ be an EGK datum and $(\underline{a}, \sigma)$ a standard GK type such that $(n_1, \ldots, n_r; m_1, \ldots, m_r) = \Upsilon(\underline{a})$. Then there exists a reduced form $B \in \mathcal{H}_n^{\text{red}}(\sigma)$ of GK type $(\underline{a}, \sigma)$ such that EGK($B$) = $G$.

**References**


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