A GODEMENT–JACQUET TYPE INTEGRAL AND THE METAPLECTIC
SHALIKA MODEL

JAN FRAHM AND EYAL KAPLAN

Abstract. We present a novel integral representation for a quotient of global automorphic $L$-functions, the symmetric square over the exterior square. The pole of this integral characterizes a period of a residual representation of an Eisenstein series. As such, the integral itself constitutes a period, of an arithmetic nature. The construction involves the study of local and global aspects of a new model for double covers of general linear groups, the metaplectic Shalika model. In particular, we prove uniqueness results over $p$-adic and archimedean fields, and a new Casselman–Shalika type formula.

INTRODUCTION

Let $\mathbb{A}$ be the ring of adèles of a global field and let $\pi$ be a cuspidal automorphic representation of $\text{GL}_k(\mathbb{A})$. One of the pillars of the Langlands program is the study of global automorphic $L$-functions, as mediating agents in the framework of functoriality. Integral representations are an indispensable tool in this study. Godement and Jacquet [GJ72] constructed the integral representation of the standard $L$-function of $\pi$. Jacquet, Piatetski-Shapiro and Shalika [JPSS83] developed Rankin–Selberg integrals for $\text{GL}_k \times \text{GL}_m$. Numerous authors considered integral representations for a product of a classical group and $\text{GL}_m$, e.g., [GPSR87, Gin90, Sou93, GPSR97, GRS98], to name a few.

In several settings, Rankin–Selberg integrals offer a “bonus”, namely a period integral characterizing the pole or central value of the $L$-function. For example, the pole of the exterior square $L$-function at $s = 1$ was characterized by the nonvanishing of a global Shalika period [JS90]. A similar result was obtained for the symmetric square $L$-function by Bump and Ginzburg [BG92]. Periods of automorphic forms have been studied extensively, e.g., [GRS99a, GRS99b, GJR04], and they are especially interesting when they are of an arithmetic nature [GJR04].

We present a new integral representation and use it to characterize the nonvanishing of a co-period of an Eisenstein series. Let $f$ be a global matrix coefficient of $\pi$, and let $\varphi$ and $\varphi'$ be a pair of automorphic forms in the space of the exceptional representation of a double cover of $\text{GL}_n(\mathbb{A})$, $n = 2k$. The cover and the exceptional representation were constructed by Kazhdan and Patterson [KP84]. We define a zeta integral $Z(f, \varphi, \varphi', s)$ (see § 5.2), where we integrate $f$ against the Fourier coefficients of $\varphi$ and $\varphi'$ along the Shalika unipotent subgroup and its (Shalika) character. Technically, it resembles the Godement–Jacquet zeta integral, where a matrix coefficient was integrated against a Schwartz–Bruhat function [GJ72], but they are not to be confused - our integral is not an entire function. It is absolutely convergent in some right half-plane, and for decomposable data becomes Eulerian. The computation of the resulting local integrals with unramified data lets us identify the $L$-functions represented.

2010 Mathematics Subject Classification. Primary 11F70; Secondary 11F27.

Key words and phrases. $L$-functions, co-period integral, periods, Shalika model, covering groups.
We obtain the meromorphic continuation from the known properties of these \(L\)-functions, and a detailed analysis of the local integrals.

**Theorem A** (see Theorem 5.4). The zeta integral extends to a meromorphic function on the plane, and represents the quotient of partial \(L\)-functions

\[
\frac{L^S(2s, \text{Sym}^2, \pi)}{L^S(2s + 1, \wedge^2, \pi)}.
\]

The corresponding local integrals are studied at both \(p\)-adic and archimedean places in §5.1. In light of existing Rankin–Selberg constructions, the denominator is expected to arise from the normalization of an Eisenstein series on \(SO_{2n}\). This is not evident at present (see the discussion below). On the other hand, the presence of exceptional representations and a cover of \(GL_n\) in an integral representing the symmetric square \(L\)-function is well understood, considering the role these played in the Rankin–Selberg integral of Bump and Ginzburg [BG92], and in its extension to the twisted symmetric square \(L\)-function by Takeda [Tak14].

The symmetric square \(L\)-function has been studied extensively, both via the Langlands–Shahidi method by Shahidi (e.g., [Sha81, Sha88, Sha90]), and the Rankin–Selberg method by Bump and Ginzburg [BG92]. The present zeta integral is weaker than these works, in the sense that it does not seem to yield a global functional equation. As mentioned above, the reason could be that the denominator is related to an Eisenstein series on a classical group. However, we are integrating two residues of Eisenstein series (as opposed to the series itself), namely \(\varphi\) and \(\varphi'\), against a matrix coefficient. A different approach could be to apply a Fourier transform, as in the zeta integrals of Godement and Jacquet [GJ72], but then it is not clear how to incorporate the denominator into such an equation.

At least over \(p\)-adic places, local multiplicity-one results are known (proved in [Kapa]), then any “similar” type of integral may be used to define a functional equation with a proportionality factor. See Remark 5.2 for more details. On the other hand, we can place the global integral in a different framework, that of periods.

The notion of co-period integrals was introduced by Ginzburg, Jiang and Rallis [GJR01], for the purpose of characterizing the zero of the symmetric cube \(L\)-function of a cuspidal automorphic representation of \(GL_2(\mathbb{A})\) at \(1/2\), using the spectral decomposition of a tensor of exceptional representations of the 3-fold cover of \(GL_2(\mathbb{A})\). Their co-period was an integral of an Eisenstein series against two automorphic forms in the space of an exceptional representation of the 3-fold cover of the exceptional group \(G_2(\mathbb{A})\). A co-period involving an Eisenstein series and two automorphic forms in the space of an exceptional representation of a cover of \(SO_{2n+1}(\mathbb{A})\) (or \(GSpin_{2n+1}(\mathbb{A})\)) was analyzed in [Kap15, Kapb] (following [GJS10]).

Assume that \(\pi\) is self-dual with a trivial central character. Let \(E(g; \rho, s)\) be an Eisenstein series corresponding to a vector \(\rho\) of the representation parabolically induced from \(\pi \otimes \pi\) to \(GL_n(\mathbb{A})\), such that the residue \(E_{1/2}(g; \rho)\) of the series at \(s = 1/2\) is related to the pole of \(L(s, \pi \times \pi)\) at \(s = 1\). We study a global co-period integral \(CP(E_{1/2}(\cdot; \rho), \varphi, \varphi')\) of the residue against \(\varphi\) and \(\varphi'\) (see §6.1). The integral lends itself to a close scrutiny using the truncation operator of Arthur [Art78, Art80]. Under a mild assumption regarding the archimedean places (see Assumption 5.5), we prove the following result.

**Theorem B** (see Theorem 6.2). \(CP(E_{1/2}(\cdot; \rho), \varphi, \varphi') = \int_K \text{Res}_{s=1/2} Z(f_{kp}, k\varphi, k\varphi', s) \, dk\), where \(K\) is the standard (global) compact subgroup and \(f_{kp}\) is a matrix coefficient on \(\pi\). Moreover, the co-period is not identically zero if and only if the zeta integral has a nontrivial residue at \(s = 1/2\). Equivalently, if and only if \(L^S(s, \text{Sym}^2, \pi)\) has a pole at \(s = 1\).
The results on the co-period have a local $p$-adic counterpart, proved in [Kapa]. Let $\tau$ be a self-dual irreducible supercuspidal representation of $GL_k$. The results of [Kapa] imply that if the symmetric square $L$-function of $\tau$ has a pole at $s = 0$, the Langlands quotient of the representation parabolically induced from $|\det|^{1/2} \otimes |\det|^{-1/2} \tau$ to $GL_n$ has a nontrivial bilinear $GL_n$ pairing with a tensor of exceptional representations.

We mention that in [KY17] (published since the time of writing this paper) Shunsuke Yamana and the first named author studied a similar co-period, in a more general setting, and proved a similar result, but with a different “outer period”. In the present work there is a new ingredient, the metaplectic Shalika model, which enables us to obtain a more precise outer period.

Our zeta integral involves a new model for double covers of $GL_n$, the metaplectic Shalika model, first discovered in [Kapa] during the computation of twisted Jacquet modules of the $p$-adic exceptional representation (of [KP84]). The Jacquet module of this representation along the Shalika unipotent subgroup and character is one-dimensional, and the reductive part of its stabilizer acts by a Weil factor. Therefore, this representation admits a unique embedding in a space of metaplectic Shalika functions. The key observation underlying this model for the cover, is that the restriction of the cover to the Shalika group is simple, in the sense that the cocycle is given by the quadratic Hilbert symbol. This follows immediately from the block-compatibility of the cocycle of Banks, Levy and Sepanski [BLS99]. See § 3.1 for a precise introduction to this model.

Roughly speaking, a model of a representation is an embedding, preferably unique, into a space of functions on the group, which is beneficial and convenient in terms of applications. One important model is the Whittaker model, which has had a profound impact on the study of representations, with a vast number of applications, perhaps most notably the Langlands–Shahidi theory of local coefficients. This model has a natural extension to covering groups, because the cover splits over unipotent subgroups. Indeed, the metaplectic Whittaker model has been studied, e.g., in [KP84, BFH91, McN11, CO13]. The main downside to using this model for covering groups, is that multiplicity one no longer holds. For instance over an $r$-fold cover of $GL_n$, the dimension of the space of Whittaker functionals on an unramified principal series representation is essentially $r^n$.

The (non-metaplectic) Shalika model was first introduced in a global context in the aforementioned work of Jacquet and Shalika [JS90]. Friedberg and Jacquet [FJ93] used it to characterize cuspidal automorphic representations affording linear models, in terms of the pole of the exterior square $L$-function at $s = 1$ (see also [BF90]). Ash and Ginzburg [AG94] used this model to construct $p$-adic $L$-functions. Multiplicity one results were proved by Jacquet and Rallis [JR96] over $p$-adic fields, and by Aizenbud, Gourevitch and Jacquet [AGJ09] over archimedean fields. Globally, a cuspidal automorphic representation $\pi$ affords a Shalika functional (given by an integral) if and only if $L^S(s, \wedge^2 \pi)$ has a pole at $s = 1$, and equivalently is a weak functorial lift from $SO_{2n+1}$ ([JS90, GRS99b, GRS01]). A similar local result was proved for supercuspidal representations in [JNQ08]. Further studies and applications of this model include [GRS99a, JQ07, JNQ10].

We launch a study of the metaplectic Shalika model. In contrast to the metaplectic Whittaker model, we conjecture that this model does enjoy multiplicity one. We establish the following result.

**Theorem C** (see Theorems 3.9 and 3.32). Let $I(\chi)$ be a principal series representation of a double cover of $GL_n$, over any local field, but over a $p$-adic field assume it is unramified.
Assume $\chi$ satisfies a certain regularity condition, which does not preclude reducibility. Then the space of metaplectic Shalika functionals on $I(\chi)$ is at most one-dimensional. Over a $p$-adic field and when $I(\chi)$ is irreducible, this space is one-dimensional if and only if $I(\chi)$ is a “lift” from a classical group.

It is perhaps premature to discuss lifting problems for covering groups; here we simply mean the natural condition one expects $\chi$ to satisfy, analogous to the case of $GL_n$. This result is our main tool to deduce that the (global) zeta integral is Eulerian.

Another practical starting point for the study of the metaplectic Shalika model, is the unramified setting. We prove a Casselman–Shalika formula for an unramified principal series representation of a double cover of $GL_n$. For $GL_n$ this formula was proved by Sakellaridis \cite{Sak06}, and our proof closely follows his arguments.

**Theorem D** (see Theorem 3.24 and Corollary 3.31). Let $I(\chi)$ be an unramified principal series representation of a double cover of $GL_n$, which admits a metaplectic Shalika model. There is an explicit formula for the values of the unique unramified normalized function in the model, in terms of the character $\chi$.

We apply our formula to the exceptional representation and use it to compute the local zeta integrals with unramified data.

In his proof, Sakellaridis \cite{Sak06} used a variant of the Casselman–Shalika method developed by Hironaka \cite[Proposition 1.9]{Hir99}. The original approach of Casselman and Shalika \cite{CS80} was to compute functionals on torus translates of the Casselman basis element corresponding to the longest Weyl element. A somewhat more versatile approach has been introduced by Hironaka, namely to compute the projection of the functional, say, the Shalika functional, onto the Iwahori invariant subspace. So, our first step is to extend some of the ideas from \cite{Hir99} to the cover. Hironaka established a formula for expressing a spherical function on certain spherical homogeneous spaces, which is applicable also in the absence of uniqueness results. We verify this formula for the double cover of $GL_n$, see Corollary 2.11. In fact, our proof is general and applicable to a large class of covering groups, e.g., the central extensions of unramified reductive $p$-adic groups by finite cyclic groups, studied by McNamara \cite{McN11, McN12, McN16}. See the discussion at the end of § 2.2.

Since the formula of \cite{Hir99} does not depend on a uniqueness property, it may well be more adequate for covering groups, where multiplicity one results may fail (e.g., for the Whittaker model). Our analog of \cite[Proposition 1.9]{Hir99} involves the verification of several results well known in the non-metaplectic setting. We greatly benefited from the work of Chinta and Offen \cite{CO13}, explaining how to extend the results of Casselman \cite{Cas80} to ($r$-fold) covers of $GL_n$. We must point out that the study of spherical functions on spherical varieties is vigorously expanding. The aforementioned techniques (e.g., of \cite{Hir99}) have been extended and generalized by Sakellaridis \cite{Sak13}. While his work does not include covering groups, his ideas are expected to apply to this context, albeit with some modifications.

As mentioned above, the Shalika model has global aspects and the same applies to its metaplectic analog. In a global setting, as with the Whittaker model, one requires a (perhaps) stronger notion. An automorphic representation is called globally generic if it admits a Whittaker functional given by a Fourier coefficient. We prove:

**Theorem E** (see Theorem 4.13). The global exceptional representation admits a metaplectic Shalika functional, given by a Fourier coefficient.
Theorem E plays a key role in the definition of the zeta integral. Moreover, we study a family of Fourier coefficients of exceptional representations and use their properties to prove Theorem B. We mention that Theorem E is not very surprising, in light of the relation between the exterior square $L$-function, the Shalika model and the descent construction of Ginzburg, Rallis and Soudry (e.g., [GRS99a, GRS99b]).

Theorem E demonstrates a very special phenomenon: the Fourier coefficient enjoys an additional invariance property. This can also be observed in exceptional representations of other groups (see, e.g., [Kap15, § 3.4.2]), and is intimately related to the fact that minimal, or small representations are supported on small unipotent orbits, in the sense that generic Fourier coefficients of their automorphic forms vanish on sufficiently large (with respect to the partial ordering) unipotent orbits. See [Car93, CM93, BFG03, Gin06] for more details. We mention that Shalika functionals written solely in terms of a unipotent integration already appeared in a work of Beineke and Bump [BB06] in the non-metaplectic case, for a specific representation.

As mentioned above, the proof of meromorphic continuation of the global zeta integrals involves establishing meromorphic continuation of the local integrals. Over archimedean fields, such a result is usually difficult to obtain, the analysis must be performed carefully, due to the involvement of topological vector spaces. To achieve this, we develop an asymptotic expansion of metaplectic Shalika functions over archimedean fields, using tools and techniques from Wallach [Wal83, Wal88, Wal92] and Soudry [Sou95], see § 3.2. This result may be of independent interest (it also applies to the non-metaplectic setting). Note that as a preliminary result, we write an asymptotic expansion of smooth matrix coefficients, on a whole Weyl chamber. This result applies to any real reductive group.

The rest of this work is organized as follows. In § 1 we establish notation and preliminaries. Section 2 is dedicated to $p$-adic unramified theory. Section 3 begins with a general discussion of the metaplectic Shalika model, then we compute the Casselman–Shalika formula. In § 4 we discuss exceptional representations, locally and globally. The theory of the zeta integral is contained in § 5, and the co-period in Section 6.

Acknowledgments. The first named author wishes to express his gratitude to David Ginzburg, Erez Lapid and David Soudry for many encouraging conversations. We thank Yiannis Sakellaridis for his interest in Theorem D and useful remarks. We thank Dmitry Gourevitch and Nolan Wallach for helpful correspondences. The authors are grateful to Jim Cogdell and Robert Stanton for their kind encouragement and helpful remarks. Lastly, we thank the referees for their interest in this work and helpful remarks, which helped improve the presentation.

1. Preliminaries

1.1. The groups. Let $G_n = GL_n$. Fix the Borel subgroup $B_n = T_n \times N_n$ of upper triangular matrices in $G_n$, where $T_n$ is the diagonal torus. For any $k \geq 0$, let $Q_k = M_k \ltimes U_k$ be the standard maximal parabolic subgroup of $G_n$, whose Levi part $M_k$ is isomorphic to $G_k \times G_{n-k}$. For any parabolic subgroup $Q$, $\delta_Q$ denotes its modulus character. If $U$ is a unipotent radical of a standard parabolic subgroup, let $U^-$ denote the unipotent radical opposite to $U$.

Let $W$ be the Weyl group of $G_n$. The longest Weyl element is denoted by $w_0$ and the identity element by $e$. For $w \in W$, let $\ell(w)$ be the length of $w$. Let $\Sigma_{G_n}$ be the set of roots, $\Sigma^+_G$ be the subset of positive roots and $\Delta_G$ be the set of simple roots compatible with our choice of $B_n$. For $\alpha \in \Delta_G$, denote the simple reflection along $\alpha$ by $s_\alpha$. 
All fields will be assumed to have characteristic different from 2. If $F$ is a local field, we usually denote $G_n = G_n(F)$. For a global field $F$, its ring of adèles is denoted $\mathbb{A}$. Over $\mathbb{R}$ and $\mathbb{C}$ we use small gothic letters to denote the Lie algebras of the corresponding groups, e.g., $\mathfrak{g}_n$ for $G_n$. Complex Lie algebras are regarded as real ones. Write $\mathcal{U}(\mathfrak{g}_n)$ for the universal enveloping algebra of the complexification of $\mathfrak{g}_n$ and $\mathcal{Z}(\mathfrak{g}_n)$ for its center. For any group $G$, $C_G$ denotes its center. The vector space of $k \times k$ matrices over a ring $R$ is denoted $R_{k \times k}$. If $Y < G$ and $d \in \mathbb{Z}$, $Y^d = \{y^d : y \in Y\}$. Also for $x, y \in G$, $x^y = xyx^{-1}$, $xY = \{xy : y \in Y\}$.

1.2. The local metaplectic cover. Let $F$ be a local field, $(\cdot, \cdot)_2$ be the Hilbert symbol of order 2 of $F$, and $\mu_2 = \{-1, 1\}$. Put $G_n = G_n(F)$. Kazhdan and Patterson [KP84] constructed the double cover $\tilde{G}_n$ of $G_n$, using the double cover of $\text{SL}_{n+1}$ of Matsumoto [Mat69] and the embedding $g \mapsto \text{diag}(g, \det g^{-1})$ of $G_n$ in $\text{SL}_{n+1}$.

Given a 2-cocycle $\sigma : G_n \times G_n \to \mu_2$, let $\tilde{G}_n$ be the associated central extension of $G_n$ by $\mu_2$, $p : \tilde{G}_n \to G_n$ be the natural projection and $s : G_n \to \tilde{G}_n$ be a section such that $p(s(g)) = g$ and $\sigma(g, g') = s(g)s(g')s(gg')^{-1}$ (we also require $s$ to take the identity element of $G_n$ to the identity element of $\tilde{G}_n$). For any subset $X \subset G_n$, put $\tilde{X} = p^{-1}(X)$.

We use the block-compatible cocycle $\sigma = \sigma_n$ of [BLS99]. Block-compatibility means

$$\sigma(\text{diag}(a, b), \text{diag}(a', b')) = \sigma_k(a, a')\sigma_{n-k}(b, b')(\det a, \det b')_2, \quad a, a' \in G_k, b, b' \in G_{n-k}. \quad (1.1)$$

When $n = 1$, $\sigma$ is trivial. For $n = 2$ it coincides with the cocycle of Kubota [Kub67]:

$$\sigma(g, g') = \left(\frac{x(gg')}{x(g)}, \frac{x(gg')}{x(g') \det g}ight)_2, \quad x \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{cases} c & c \neq 0, \\ d & c = 0. \end{cases} \quad (1.2)$$

The cocycle $\sigma$ and its twist $\sigma^{(1)}(g, g') = (\det g, \det g')_2\sigma(g, g')$ exhaust the nontrivial cohomology classes. Our results in this work hold for both $\sigma$ and $\sigma^{(1)}$. In fact, all of the computations will involve $g$ and $g'$ such that $(\det g, \det g')_2 = 1$.

We mention several properties of $\sigma$ (and $\sigma^{(1)}$). For any $g \in G_n$ and $v \in N_n$, $s(gv) = s(g)s(v)$ and $s(\nu g) = s(\nu)s(g)$. If $g^v \in N_n$, then $s(g^v) = s(g)s(v)s(g)^{-1}$. Consequently if $s$ is a splitting of $M < G_n$ (i.e., a homomorphism), $U < N_n$ and $MU = M \times U$, $s$ is a splitting of $MU$. Note that $C_{\tilde{G}_n} = \tilde{C}_n$, where $e = 1$ if $n$ is odd, otherwise $e = 2$.

For computations we fix a concrete subset $\mathfrak{W} \subset G_n$ as in [BLS99], which is in bijection with $W$. For $\alpha \in \Sigma_{G_n}$, let $w_\alpha$ be the image of $(1^{-1})$ in the subgroup of $G_n$ corresponding to $\alpha$. Define $\mathfrak{W}$ as the set of elements $w_{\alpha_1} \cdot \ldots \cdot w_{\alpha_{k(\omega)}}$ where $w = s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_{k(\omega)}} \in W$.

1.3. The global metaplectic cover. Let $F$ be a global field. For a finite place $v$ of $F$, let $\mathcal{O}_v$ denote the ring of integers of $F_v$, and $\mathfrak{p}_v$ be a generator of the maximal ideal. The group $G_n(\mathfrak{A})$ is the restricted direct product $\prod_v G_n(F_v)$ with respect to the compact subgroups $\{K_v\}_v$, where $K_v = G_n(\mathcal{O}_v)$ for almost all $v$, and we denote $K = \prod_v K_v$. The global double cover $\tilde{G}_n(\mathfrak{A})$ was constructed in [KP84, § 0.2] (see also [FK86, Tak14]). For all $v < \infty$ such that $q_v = |\mathfrak{p}_v \mathcal{O}_v \setminus \mathcal{O}_v|$ is odd and $q_v > 3$, there are canonical splittings $\kappa_v$ of $K_v$ ([Moo68, pp. 54-56], [KP84, Proposition 0.1.3]). We extend them to sections of $G_n(F_v)$. Also $G_n(\mathfrak{A})$ is isomorphic to a quotient of the restricted direct product $\prod_v \tilde{G}_n(F_v)$, taken with respect to $\{\kappa_v(K_v)\}_v$. The function $s = \prod_v s_v$ is a splitting of $G_n(F)$ and $N_n(\mathfrak{A})$ and is well defined on $T_n(\mathfrak{A})$. We identify $G_n(F)$ and $N_n(\mathfrak{A})$ with their image under $s$ in $\tilde{G}_n(\mathfrak{A})$. 

1.4. The Weil symbol. Let $F$ be a local field. We usually denote by $\psi$ a nontrivial additive character of $F$. Then $\gamma_\psi$ is the normalized Weil factor (see [Wei64, § 14]), $\gamma_\psi(\cdot)^4 = 1$. For $a \in F^\times$, $\psi_a$ denotes the character $\psi_a(x) = \psi(ax)$ and $\gamma_{\psi,a} = \gamma_{\psi}$.

We have the following formulas (see the appendix of [Wal83], [Wal88, Chapters 4.3, 4.4] and [Wal92, Chapter 15.2]),

$$\gamma_\psi(xy) = \gamma_\psi(x)\gamma_\psi(y)(x,y)_2, \quad \gamma_\psi(x^2) = 1, \quad \gamma_\psi^{-1} = \gamma_\psi^{-1}, \quad \gamma_{\psi,a}(x) = (a,x)\gamma_\psi(x). \quad (1.3)$$

1.5. Local representations. Let $\text{Alg} G$ be the category of complex and smooth representations of $G$, if $G$ is an $l$-group (in the sense of [BZ76]), or smooth admissible Fréchet representations of moderate growth if $G$ is a real reductive group.

For $\pi \in \text{Alg} G$ we denote by $\pi^\vee$ the contragredient representation of $\pi$. If $V$ is the space of $\pi$, $V^\vee$ will denote the space of $\pi^\vee$. For a real reductive group $G$, $\pi^\vee$ is the smooth admissible Fréchet globalization of moderate growth, of the $K$-finite vectors in the dual representation.

Then a matrix coefficient of $\pi$ is a function $f(g) = \xi^\vee(\pi(g))\xi$ where $\xi \in V$ and $\xi^\vee \in V^\vee$.

Regular induction is denoted Ind, and ind is the compact induction. Induction is not normalized.

Over an archimedean field induction denotes smooth induction. For an $l$-space or an abelian Lie group $X$, let $\mathcal{S}(X)$ be the space of Schwartz–Bruhat functions on $X$.

Let $G$ be an $l$-group, $\pi \in \text{Alg} G$ and $U < G$ be closed. The Jacquet module of $\pi$ with respect to $U$ and $\psi$ is denoted $\pi_{U,\psi}$ (or $\pi_U$ when $\psi = 1$). The action is not normalized.

Let $\pi \in \text{Alg} G$ be irreducible and $Q = M \rtimes U$ be a parabolic subgroup of $G$. Assume that the underlying field is $p$-adic. The normalized exponents of $\pi$ along $Q$ are the central characters of the irreducible constituents of $\delta_Q^{-1/2}\pi_U$.

1.6. Asymptotic expansions over archimedean fields. We derive an asymptotic expansion for matrix coefficients $f(g) = \xi^\vee(\pi(g))\xi$ of $G_k$. The key ideas are due to Wallach (see [Wal83, Theorem 7.2], [Wal88, Chapters 4.3, 4.4] and [Wal92, Chapter 15.2]), we adapt them to our setting (see also [Sou95, Theorem 1 in § 4]). The novelty is that we derive an expansion into an exponential power series, not only on a one-parameter group but on a whole Weyl chamber, depending continuously on $\xi$ and $\xi^\vee$. We carry out the details for $G_k$, but the proof extends to any real reductive group.

Let $F$ be an archimedean field. Recall the maximal parabolic subgroups $Q_\ell = M_\ell \rtimes U_\ell$ of $G_k$. The non-compact central factor of $M_\ell$ is given by $A_\ell = \exp(a_\ell)$ with $a_\ell = \mathbb{R}I_\ell + \mathbb{R}H_\ell$ and $H_\ell = \text{diag}(I_\ell, 0_{k-\ell})$. For $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ put $a_x = \exp(-\sum_{\ell=1}^k x_\ell H_\ell)$.

For a representation $\pi \in \text{Alg} G$ on a space $V$, we write $E_1(Q_\ell, \pi) \subset (a_\ell)_\ell$ for the finite set of generalized $a_\ell$-weights in $V/u_VV$. Put

$$E_1^{(\ell)}(\pi) = \{\mu(H_\ell) : \mu \in E_1(Q_\ell, \pi)\} \subset \mathbb{C}, \quad E^{(\ell)}(\pi) = E_1^{(\ell)}(\pi) + \mathbb{N},$$

$$\Lambda_{\pi,\ell} = \min\{\text{Re } z : z \in E^{(\ell)}(\pi)\} = \min\{\text{Re } z : z \in E_1^{(\ell)}(\pi)\},$$

$$\Lambda_{\pi} = (\Lambda_{\pi,1}, \ldots, \Lambda_{\pi,k}).$$

Here $\mathbb{N} = \{0, 1, 2, \ldots\}$. By [Wal92, Theorem 15.2.4] there exist continuous seminorms $q_1, q_2$ on $V$, $V^\vee$ (resp.) and $d \geq 0$ such that

$$|\xi^\vee(\pi(a_x)\xi)| \leq q_1(\xi)q_2(\xi^\vee)(1 + |x|)^d e^{-\Lambda_\pi x}, \quad \forall x \in \mathbb{R}_+^k, \xi \in V, \xi^\vee \in V^\vee. \quad (1.5)$$

For a subset $I = \{i_1, \ldots, i_r\}$ of $\{1, \ldots, k\}$ with $i_1 < \ldots < i_r$, let $x_I = (x_{i_1}, \ldots, x_{i_r})$ and $I^c = \{1, \ldots, k\} - I$. 


Theorem 1.1. Let $\pi \in \text{Alg } G_k$ be an irreducible representation on a space $V$. For any $D = (D_1, \ldots, D_k) \in \mathbb{R}^k$ there exist finite subsets $C_\ell \subset E^{(\ell)}(\pi)$ and functions $p_{I,z} : \mathbb{R}^k \times V \times V^\vee \rightarrow \mathbb{C}$ for $I \subset \{1, \ldots, k\}$, $z \in \prod_{\ell \in I} C_\ell$, such that the following holds.

(1) For all $\xi \in V$, $\xi^\vee \in V^\vee$ and $x \in \mathbb{R}^k$,
$$\xi^\vee(\pi(a_x)\xi) = \sum_{I,z} p_{I,z}(x; \xi, \xi^\vee) \cdot e^{-z \cdot x_I},$$  
(1.6)
where the (finite) summation is over all $I \subset \{1, \ldots, k\}$ and $z \in \prod_{\ell \in I} C_\ell$.

(2) Each function $p_{I,z}(x; \xi, \xi^\vee)$ is polynomial in $x_I$:
$$p_{I,z}(x; \xi, \xi^\vee) = \sum_{\alpha \in \mathbb{N}^I} c_\alpha(x_I; \xi, \xi^\vee) x_1^\alpha,$$  
(1.7)
where the coefficients $c_\alpha(x_I; \xi, \xi^\vee)$ are smooth in $x_I$, linear in $\xi$ and $\xi^\vee$, and for suitable $d \geq 0$ and continuous seminorms $q_1$, $q_2$ on $V$, $V^\vee$ (resp.),
$$|c_\alpha(x_I; \xi, \xi^\vee)| \leq q_1(\xi)q_2(\xi^\vee)(1 + |x_I|^d) e^{-D_{\Gamma,x_I}}, \quad \forall x_I \in \mathbb{R}_{\Gamma}^\times, \xi \in V, \xi^\vee \in V^\vee. \quad (1.8)$$

Proof. It is enough to show the above statement for arbitrarily large $D_\ell$, $\ell = 1, \ldots, k$. We achieve this by first treating the case $D = \Lambda_\pi$, then proving that any $D$ can be replaced by $D + \varepsilon_\ell$ for some $\ell = 1, \ldots, k$. Repeatedly applying this argument yields the claim.

For $D = \Lambda_\pi$ one may choose the sum in (1.6) to consist of only one summand, for the empty set $I$, then the statement follows from (1.5). Now let $D \in \mathbb{R}^k$ be arbitrary and assume (1.6)–(1.8). We fix $1 \leq \ell \leq k$ and prove that the same is true for $D$ replaced by $D + \varepsilon_\ell$. For brevity, set $H = H_\ell$, $t = x_\ell$ and $x' = (x_1, \ldots, x_{\ell-1}, 0, x_{\ell+1}, \ldots, x_k)$.

Let $\{X_p\}_p$ be a basis of $u_\ell$. In [Wal92, § 15.2.4] Wallach constructed $E_1 = 1, E_2, \ldots, E_r \in \mathcal{U}(g_k)$ as well as finite sets $\{Z_{i,j}\}_{i,j} \subset \mathcal{Z}(g_k)$ and $\{U_{p,i}\}_{p,i} \subset \mathcal{U}(g_k)$ such that
$$H_\ell E_i = \sum_j Z_{i,j} E_j + \sum_p X_p U_{p,i}, \quad \forall i = 1, \ldots, r. \quad (1.9)$$

Since $\pi$ is irreducible, the elements $Z_{i,j}$ act by scalars $B_{i,j} = \pi(Z_{i,j}) \in \mathbb{C}$ and we form the matrix $B = (B_{i,j})_{i,j}$. Note that the eigenvalues of $B$ are contained in the finite set $E^{(1)}(\pi) = \{\xi_1, \ldots, \xi_N\}$ (the generalized eigenvalues of $\pi(H_\ell)$ on $V/u_\ell V$). Define
$$F(t, x'; \xi, \xi^\vee) = \left( \begin{array}{c} \xi^\vee(\pi(e^{-tH})\pi(a_{x'})\pi(E_1)\xi) \\ \vdots \\ \xi^\vee(\pi(e^{-tH})\pi(a_{x'})\pi(E_r)\xi) \end{array} \right),$$
$$G(t, x'; \xi, \xi^\vee) = \sum_p \left( \begin{array}{c} \xi^\vee(\pi(e^{-tH})\pi(a_{x'})\pi(X_p)\pi(U_{p,1})\xi) \\ \vdots \\ \xi^\vee(\pi(e^{-tH})\pi(a_{x'})\pi(X_p)\pi(U_{p,r})\xi) \end{array} \right).$$

The first coordinate of $F(t, x'; \xi, \xi^\vee)$ is equal to $\xi^\vee(\pi(a_x)\xi)$ and the function $F(t, x'; \xi, \xi^\vee)$ satisfies the following differential equation:
$$\frac{d}{dt} F(t, x'; \xi, \xi^\vee) = -BF(t, x'; \xi, \xi^\vee) - G(t, x'; \xi, \xi^\vee).$$

The solution of this system is given by
$$F(t, x'; \xi, \xi^\vee) = e^{-tB} F(0, x'; \xi, \xi^\vee) - e^{-tB} \int_0^t e^{\tau B} G(\tau, x'; \xi, \xi^\vee) d\tau. \quad (1.10)$$
We claim that each coordinate on the right-hand side is a finite sum of terms of the form
\[ c(x_T; \xi, \xi') x_I^a e^{-z \cdot x} \]
with \( I \subset \{1, \ldots, k\} \), \( \alpha \in \mathbb{N}^J \) and \( z \in \prod_{i \in I} E^{(i)}(\pi) \), and a function \( c(x_T; \xi, \xi') \) which is smooth in \( x_T \), linear in \( \xi \) and \( \xi' \) and satisfies
\[ |c(x_T; \xi, \xi')| \leq q_1(\xi)q_2(\xi')(1 + |x_T|^d) e^{-(D + e)\|x_T\|}, \quad \forall x_T \in \mathbb{R}^T, \xi, \eta \in V, \xi' \in V', \] (1.12)
for some continuous seminorms \( q_1, q_2 \) and \( d \geq 0 \). This will finish the proof. Note that
\[ e^{\tau B} = \sum_{j=1}^N P_j(\tau)e^{\ell_j \tau} \] (1.13)
for some matrix-valued polynomials \( P_j(\tau) \). We treat the two summands in (1.10) separately.

(1) The coordinates of \( F(0, x'; \xi, \xi') \) are of the form \( \xi'(\pi(a_{x'})\pi(\xi)) \), and by (1.13) multiplying with \( e^{-\tau B} \) yields sums of terms of the form
\[ t^m e^{-z \cdot x} \cdot \xi'(\pi(a_{x'})\pi(\xi)), \] (1.14)
where \( m \geq 0 \). The map \( \xi \mapsto \pi(\xi) \) is continuous, hence for every continuous seminorm \( q_1 \) on \( V \) there exists a continuous seminorm \( q_1 \) such that \( q_1(\pi(\xi)) \leq q_1(\xi) \) for all \( \xi \in V \). Applying the induction assumption to the matrix coefficients \( \xi'(\pi(a_{x'})\pi(\xi)) \) then shows that the terms (1.14) are finite sums of the form
\[ t^m e^{-z \cdot x} \cdot c(x_T'; \xi, \xi')(x_I')^a e^{-z' \cdot x'}, \] (1.15)
where \( J \subset \{1, \ldots, k\} - \{\ell\} \), \( J = \{1, \ldots, k\} - (J \cup \{\ell\}) \), \( \alpha \in \mathbb{N}^J \), \( z' \in \prod_{i \in J} E^{(i)}(\pi) \) and \( c(x_T'; \xi, \xi') \) is smooth in \( x_T' \), linear in \( \xi \) and \( \xi' \) and satisfies
\[ |c(x_T'; \xi, \xi')| \leq q_1(\xi)q_2(\xi')(1 + |x_T'|)^d e^{-(D + e)\|x_T'\|}, \quad \forall x_T' \in \mathbb{R}^J, \xi, \eta \in V, \xi' \in V', \] (1.16)
for some continuous seminorms \( q_1, q_2 \) and \( d \geq 0 \). Let \( I = J \cup \{\ell\} \), \( x = x' + te_{\ell} \) and \( z = (z_j, z') \in E^{(i)}(\pi) \times \prod_{i \in J} E^{(i)}(\pi) \), then (1.15) is clearly of the form (1.11) and \( c(x_T'; \xi, \xi') \) satisfies (1.12) for \( \ell \notin J \).

(2) The coordinates of \( G(t, x'; \xi, \xi') \) are sums of the terms
\[ \xi'(\pi(e^{-\tau B})\pi(a_{x'})\pi(X_p)\pi(U_p)) \xi = e^{-t} \xi'(\pi(a_{x'})\pi(X_p)\pi(e^{-\tau B})\pi(U_p)) \xi. \]
Note that \( \pi(a_{x'})\pi(X_p) \) is a sum of terms of the form \( e^{-\beta \cdot x'} \pi(X) \pi(a_{x'}) \) with \( X \in U_k \) and \( \beta \in \mathbb{N}^{k-1} \). Hence, each coordinate of \( G(t, x'; \xi, \xi') \) is a sum of terms of the form
\[ e^{-t} e^{-\beta \cdot x'} \cdot (\xi'(\pi(X)) \xi'(\pi(e^{-\tau B}) \pi(a_{x'}) \pi(U_p)) \xi), \quad U, X \in U(U_k), \beta \in \mathbb{N}^{k-1}. \]
Since \( \xi \mapsto \pi(U) \xi \) and \( \xi' \mapsto \pi(X) \xi' \) are continuous linear operators we can apply the induction assumption to the matrix coefficients \( \pi(X) \xi'(\pi(e^{-\tau B}) \pi(a_{x'}) \pi(U_p)) \xi \) and find that the coordinates of \( G(t, x'; \xi, \xi') \) are linear combinations of terms of the form
\[ c(x_T'; \xi, \xi')(x_I')^a e^{-z' \cdot x'} t^m e^{-(s+1) \cdot t} \text{ and } c(x_T'; \xi, \xi')(x_I')^a e^{-z' \cdot x'} e^{-t} \]
with \( J \subset \{1, \ldots, k\} - \{\ell\} \), \( J = \{1, \ldots, k\} - (J \cup \{\ell\}) \), \( \alpha \in \mathbb{N}^J \), \( m \in \mathbb{N} \), \( z' \in \prod_{i \in J} E^{(i)}(\pi) \), \( z_{\ell} \in E^{(i)}(\pi) \), where the coefficients \( c(x_T'; \xi, \xi') \) and \( c(t, x_T'; \xi, \xi') \) are smooth in \( x_T' \) and also \( t \) in the second case, linear in \( \xi \) and \( \xi' \), and satisfy estimates of the form
\[ |c(x_T'; \xi, \xi')| \leq q_1(\xi)q_2(\xi')(1 + |x_T'|)^d e^{-(D + e)\|x_T'\|}, \] (1.16)
\[ |c(t, x'_T; \xi, \xi'\nu)| \leq q_1(\xi)q_2(\xi')(1 + |x'_T|)^d e^{-D_{x'_T}}(1 + t)^d e^{-D_t}t \]  

(1.17)

for all \( x'_T \in \mathbb{R}_{>0}^d, t \geq 0 \) and \( \xi \in V, \xi' \nu \in V' \). Here we have attributed the factors \( e^{-\beta_i x'_i} \) either to \( e^{-\tau_i x'_i} \) in case \( i \in J \), or to \( c(x'_T, \xi, \xi'\nu) \) and \( c(t, x'_T; \xi, \xi'\nu) \) in case \( i \notin J \). By (1.13) the components of \( e^{(\tau-t)B G(\tau, x'; \xi, \xi'\nu)} \) are linear combinations of terms of the form

\[
I_1 = t^n e^{-\zeta_j t} \cdot c(x'_T, \xi, \xi'\nu)(x'_j)^\alpha e^{-\tau_i x'_j} \cdot \tau^m e^{(\zeta_j - z_i - 1)\tau} \\
I_2 = t^n e^{-\zeta_j t} \cdot c(\tau, x'_T; \xi, \xi'\nu)(x'_j)^\alpha e^{-\tau_i x'_j} \cdot \tau^m e^{(\zeta_j - 1)\tau}.
\]

We treat the integration over \( \tau \) for these two types separately. For \( I_1 \) we know that

\[
\int_0^t \tau^m e^{(\zeta_j - z_i - 1)\tau} d\tau = p(t)e^{(\zeta_j - z_i - 1)t} + C
\]

for some polynomial \( p(t) \) and a constant \( C \). Hence

\[
\int_0^t I_1 d\tau = p(t)t^n e^{-(z_i+1)t}c(x'_T; \xi, \xi'\nu)(x'_j)^\alpha e^{-\tau_i x'_j} + C t^n e^{-\zeta_j t}c(x'_T; \xi, \xi'\nu)(x'_j)^\alpha e^{-\tau_i x'_j}.
\]

Since \( z_i + 1 \) and \( \zeta_j \) are in \( E^{(\ell)}(\pi) \), both summands are linear combinations of terms of the form (1.11) for \( I = J \cup \{\ell\} \). The estimates (1.12) follow from (1.16) since \( \ell \notin T \).

For \( I_2 \) we first assume \( D_{\ell} > \text{Re} \zeta_j - 1 \). Rewrite \( \int_0^t I_2 d\tau = \int_0^\infty I_2 d\tau - \int_t^\infty I_2 d\tau \). The first integral \( \int_0^\infty I_2 d\tau \) is of the form (1.11) for \( I = J \cup \{\ell\} \). The estimate (1.12) follows from (1.17) since \( |c(\tau, x'_T; \xi, \xi'\nu)| \tau^m e^{(\zeta_j - \ell - 1)\tau} | \) is bounded by

\[
q_1(\xi)q_2(\xi')(1 + |x'_T|)^d e^{-D_{x'_T}} \cdot \tau^m(1 + \tau)^d e^{(\text{Re} \zeta_j - D_{\ell} - 1)\tau},
\]

which is integrable over \( \mathbb{R}_{>0} \) since \( D_{\ell} > \text{Re} \zeta_j - 1 \).

The second integral \( \int_t^\infty I_2 d\tau \) is of the form (1.11) for \( I = J \). The integral is by the previous estimation smooth in \( t \) and \( x'_T \), and the estimate (1.12) follows from

\[
\left| t^n e^{-\zeta_j t} \int_t^\infty c(\tau, x'_T; \xi, \xi'\nu)\tau^m e^{(\zeta_j - 1)\tau} d\tau \right|
\leq \text{constant} \times q_1(\xi)q_2(\xi')(1 + |x'_T|)^d e^{-D_{x'_T}} \cdot (1 + t)^{m+d+n} e^{-(D_{\ell}+1)t}.
\]

It remains to consider \( I_2 \) when \( D_{\ell} \leq \text{Re} \zeta_j - 1 \). Then \( \int_0^t I_2 d\tau \) is also of the form (1.11) for \( I = J \). The estimate (1.12) now follows from (1.17). \( \square \)

2. Metaplectic Unramified Theory

2.1. Preliminaries. In this section \( F \) is a local non-archimedean field and \( G_n = G_n(F) \). Then \( \mathcal{O}, \varpi, K \) and \( q \) are defined as in § 1.3. Assume \( q > 3 \) and \( |2| = 1 \) in the field. We normalize the Haar measure \( dg \) on \( G_n \) such that \( \text{vol}(K) = 1 \). Let \( \kappa : K \to \hat{G}_n \) be the canonical splitting of \( K \). It coincides with \( \mathfrak{s} \) on \( T_n \cap K, \mathfrak{w} \) and \( N_n \cap K \) (\( \mathfrak{w} \) was defined in § 1.2, and is a subset of \( K \)). We use this freely, e.g., it implies that \( \mathfrak{s} \) is a splitting of \( \mathfrak{w} \).
2.2. Review of Casselman’s theory and a result of Hironaka. We describe a metaplectic analog of several results of Casselman [Cas80], needed to prove a result of Hironaka [Hir99]. We rely on the detailed exposition in [CO13, McN12].

Let $\mathcal{I} < K$ be the Iwahori subgroup which is compatible with $B_n$. The section $\kappa$ is in particular a splitting of $\mathcal{I}$. Put $\mathcal{I}^+ = \mathcal{I} \cap N_n$, $\mathcal{T}^0 = \mathcal{I} \cap T_n$ and $\mathcal{I}^- = \mathcal{I} \cap N_n^-$. Then

$$\mathcal{I} = \mathcal{I}^+ \mathcal{T}^0 \mathcal{I}^- = (\mathcal{I} \cap B_n) \mathcal{I}^- = (K \cap B_n) \mathcal{I}^-.$$  \hfill (2.1)

For a representation $\pi \in \text{Alg}\tilde{G}_n$ and a compact open subgroup $K_0 < K$, the space of $\kappa(K_0)$-invariants is denoted $\pi^{K_0}$ and the projection on $\pi^{K_0}$ is denoted $\mathcal{P}_{K_0}$. For any vector $\xi$ in the space of $\pi$, $\mathcal{P}_{K_0}(\xi) = \int_{K_0} \pi(\kappa(k)) \xi \, dk$, where the measure is normalized such that $\text{vol}(K_0) = 1$. Also set $T_n^\alpha = \{ t \in T_n : |\alpha(t)| \leq 1, \; \forall \alpha \in \Delta_{\tilde{G}_n} \}$. The following is a version of Jacquet’s First Lemma, which is simple to show using [BZ76, 2.33].

**Proposition 2.1.** Let $\pi \in \text{Alg}\tilde{G}_n$ be admissible. For any $\xi$ in the space of $\pi^{\mathcal{I}^0}$ and $t \in C_{T_n} \cap \tilde{T}_n^-$, $\mathcal{P}_{\mathcal{I}}(\pi(t)\xi) = \mathcal{P}_{\mathcal{I}^+}(\pi(t)\xi)$ and $\mathcal{P}_{\mathcal{I}}(\pi(t)\xi) - \pi(t)\xi$ vanishes in $\pi_{N_n}$.

The following proposition is a weak analog of [Cas80, Proposition 2.4].

**Proposition 2.2.** Let $\pi \in \text{Alg}\tilde{G}_n$ be admissible. The vector space $\pi^{\mathcal{I}}$ injects into $\pi^{\mathcal{I}^0}_{N_n}$.

**Proof.** Similar to the linear case, use Proposition 2.1 and [Sav88, Proposition 3.1.4]. \hfill \square

**Remark 2.3.** The original claim in [Cas80] also included surjectivity. In the metaplectic case this does not hold in general, e.g., when $\pi$ is a genuine unramified principal series.

We recall the parametrization of genuine unramified principal series representations of $\tilde{G}_n$. Let $\tilde{T}_{n,*}$ be the centralizer of $\tilde{T}_n \cap \kappa(K)$ in $\tilde{T}_n$, it is a maximal abelian subgroup. The group $T_{n,*}$ consists of diagonal matrices whose nonzero entries belong to $F^{*2}O^*$, multiplied by matrices $zI_n$ with $z \in F^{*e}$ (e was defined in § 1.2). Assume $s \in \mathbb{C}^n$. Let $\gamma : F^* \to \mathbb{C}$ be a function satisfying $\gamma(zz') = \gamma(z)\gamma(z')\sigma(zI_n, z'I_n)$ for $z, z' \in F^{*e}$, and $\gamma|_{F^{*2}} = \gamma|_{O^*} = 1$. The genuine unramified character $\chi = \chi_{z, s}$ of $\tilde{T}_{n,*}$ is defined by

$$\chi(\epsilon s(\text{diag}(t_1, \ldots, t_n))s(zI_n)) = \epsilon\gamma(z)|z|^{s_1 + \ldots + s_n} \prod_{i=1}^n |t_i|^{s_i}, \; \forall \epsilon \in \mu_2, t_i \in F^{*2}O^*, z \in F^{*e}.$$  

Put $B_{n,*} = T_{n,*}N_n$, it is a closed subgroup of $G_n$, open and normal in $B_n$. Then

$$I(\chi) = \text{Ind}_{B_{n,*}}^{\tilde{G}_n}(\delta_B^{3/2} \chi)$$

is a genuine unramified principal series representation, i.e., $I(\chi)^K$ is one dimensional, and any such representation takes this form, where $s_i$ is unique modulo $\frac{s_i}{\log q} \mathbb{Z}$ (as opposed to $\frac{s_i}{\log q} \mathbb{Z}$ in $G_n$). The dependency on $\gamma$ will not appear in the formulas and in fact when $n$ is even, $\gamma$ is irrelevant.

We regard the elements in the space of $I(\chi)$ as complex-valued functions. The unramified normalized element of $I(\chi)$ is the unique function $f_\chi \in I(\chi)$ such that $f_\chi(gk\kappa(k)) = f_\chi(g)$ for any $g \in \tilde{G}_n$ and $k \in K$, and $f_\chi(s(I_n)) = 1$.

Let $S^{\text{gen}}(\tilde{G}_n)$ denote the subspace of $S(\tilde{G}_n)$ consisting of genuine functions. Given an open subset $K_0 \subset K$, one can lift any $f \in S(K_0)$ to $f^{\text{gen}} \in S^{\text{gen}}(\tilde{G}_n)$ by putting

$$f^{\text{gen}}(g) = \begin{cases} \epsilon f(k_0) & g = \epsilon k_0, \epsilon \in \mu_2, k_0 \in K_0, \\ 0 & g \notin K_0. \end{cases}$$
For example, for any subset $X \subset G_n$ let $\text{ch}_X$ be the characteristic function of $X$, then $\text{ch}_X^\text{gen}$ vanishes off $\tilde{K}$ and is 1 on $\kappa(K)$.

The group $\tilde{G}_n$ acts on the right on $S^\text{gen}(\tilde{G}_n)$. We have a surjection $P_\chi : S^\text{gen}(\tilde{G}_n) \to I(\chi)$ of $\tilde{G}_n$-representations

$$P_\chi(f)(g) = \int_{B_n;} f(s(b_s)g)s_b^{1/2}(b_s)\chi^{-1}(s(b_s)) \, db_s. \quad (2.2)$$

Here $db_s$ is the left Haar measure normalized by requiring $\text{vol}(B_n) = 1$, this is consistent with the normalization in the non-metaplectic setting because $B_n \cap K = B_n \cap K$. Note that $s$ is a splitting of $B_n$. The unramified normalized function of $I(\chi)$ is then $\varphi_{K,\chi} = P_\chi(\text{ch}_K^\text{gen})$.

For $w \in W$, let $w \in \tilde{W}$ be a representative of $w$ and define $\varphi_{w,\chi} = P_\chi(\text{ch}_K^\text{gen})$, This is the function supported on $\tilde{B}_n.; \kappa(\tilde{w}) = \tilde{B}_n.; \kappa(\tilde{w}) \chi(\tilde{I})$, which is right-invariant by $\kappa(\tilde{I})$ and $\varphi_{w,\chi}(s(\tilde{w})) = 1$. Then $\{\varphi_{w,\chi} \}_{w \in W}$ is a basis of $I(\chi)^2$. Note that $\varphi_{K,\chi} = \sum_w \varphi_{w,\chi}$.

We introduce the intertwining operators. If $\alpha \in \Sigma_{G_n}$, let $N_\alpha < G_n$ denote the root subgroup of $\alpha$. For any $w \in \tilde{W}$, let $T_w = T_{w,\chi} : I(\chi) \to I(\chi)$ be the standard intertwining operator defined by

$$T_w f_\chi(g) = \int_{N(w)} f_\chi(s_\chi^{-1}v) \, dv,$$

where $w \in \tilde{W}$ is the representative of $w$ and $N_\alpha(w) = \prod_{\alpha > 0 ; w^{-1} \alpha < 0} N_\alpha < N_n$. Henceforth assume $\chi$ is regular, i.e., $\tilde{w}\chi \neq \chi$ for any $e \neq w \in W$. In this case (see [McN12, Theorem 4])

$$\dim \text{Hom}_{G_n}(I(\chi), I(\chi)) = 1, \quad \forall \chi \in \tilde{W}. \quad (2.3)$$

We recall the Gindikin-Karpelevich formula ([Cas80] Theorem 3.1), whose extension to $\tilde{G}_n$ was proved in [KP84]. For any root $\alpha$, let $R_\alpha$ be the group generated by $N_\alpha$ and $N_{-\alpha}$ ($R_\alpha \cong \text{SL}_2$), $\alpha'_\alpha$ be the embedding of $\text{diag}(\varpi^2, \varpi^{-2})$ in $R_\alpha$ and put $a_\alpha = s(a'_\alpha)$. Define for $\alpha$ and $w \in W$, $c_\alpha(\chi) = \frac{1 - \chi(a_{\alpha})}{1 - \chi(a_{\alpha})}$ and $c_w(\chi) = \prod_{\alpha > 0 ; w \alpha < 0} c_\alpha(\chi)$. Then

$$T_w \varphi_{K,\chi} = c_w(\chi) \varphi_{K,\chi}. \quad (2.4)$$

This implies the analog of [Cas80, Proposition 3.5]:

**Proposition 2.4.** $T_w (= T_{w,\chi})$ is an isomorphism if and only if $c_{w^{-1}}(\chi) c_w(\chi) \neq 0$. Furthermore, $I(\chi)$ is irreducible if and only if $T_w$ is an isomorphism.

**Proof.** For the second part one uses (2.3) and [BZ77, Theorem 2.9 and Theorem 2.4b]. \qed

In contrast with the non-metaplectic case, here $I(\chi)|_{\tilde{T}_n}$ is not linearly isomorphic with $I(\chi)|_{\tilde{T}}$ (compare to [Cas80, Proposition 2.4]). Indeed, the dimension of the former is $|W| \cdot |T_{n,*}|$.

However, we can still define an analog of the Casselman basis.

For each $w \in W$ let $\Upsilon_w$ be the functional on $I(\chi)$ given by $\Upsilon_w(f_\chi) = T_w f_\chi(s(I_n))$. Clearly

$$\Upsilon_w(t f_\chi) = t^{1/2} \, w(\chi) \Upsilon_w(f_\chi), \quad \forall f_\chi \in I(\chi), \quad t \in C_{\tilde{T}_n}. \quad (2.5)$$

For any genuine character $\omega$ of $\tilde{T}_{n,*}$, let $\rho(\omega) = \text{ind}_{\tilde{T}_{n,*}}^{\tilde{T}_n} (\omega)$. In fact $\rho(\omega)$ depends only on $\omega|_{C_{\tilde{T}_n}}$. According to the Geometric Lemma [BZ77, Theorem 5.2] and because $\chi$ is regular,

$$I(\chi)|_{\tilde{T}_n} = \bigoplus_{w \in W} t^{1/2} \rho/w(\chi). \quad (2.6)$$
Since $\Upsilon_w$ factors through $I(\chi)_{N_n}$, we can consider it as a functional on this finite direct sum, of finite-dimensional vector spaces. For any $w' \neq w$, if the image $f_\chi + I(\chi)(N_n)$ of $f_\chi$ in $I(\chi)_{N_n}$ belongs to the space of $\rho(w'\chi)$, 
\[
\Upsilon_w(tf_\chi) = \Upsilon_w(t_B^{1/2}w'\chi(t)f_\chi) = \delta^{1/2}_{B_n} w'\chi(t)\Upsilon_w(f_\chi), \quad \forall t \in C_{\tilde{T}_n}.
\] (2.7)
Combining this with (2.5) implies that $\Upsilon_w$ vanishes on $\bigoplus_{w' \neq w} t_B^{1/2} \rho(w'\chi)$.

**Proposition 2.5.** The functionals $\{\Upsilon_w\}_w$ restricted to $I(\chi)^2$ are linearly independent.

**Proof.** According to [CO13, Remark 1], if we put
\[
f_{w,\chi} = T_{w^{-1}w_0}\varphi_{w_0,(w^{-1}w_0)^{-1}\chi} \in I(\chi)^2,
\]
$\Upsilon_w(f_{w,\chi}) = \delta_{w',w}$, where $\delta_{w',w}$ is the Kronecker delta. □

Let $\{f_{w,\chi}\}_w$ be the basis of $I(\chi)^2$ dual to $\{\Upsilon_w\}_w$, given in the proof of Proposition 2.5.

**Proposition 2.6.** The vector $f_{w,\chi} + I(\chi)(N_n)$ belongs to the space of $\rho(w'\chi)$. Consequently, for any $t \in C_{\tilde{T}_n}$, the image of $tf_{w,\chi}$ in $I(\chi)_{N_n}$ is $\delta^{1/2}_{B_n} w'\chi(t)f_{w,\chi} + I(\chi)(N_n)$.

**Proof.** Looking at (2.6), we can write the image of $f_{w,\chi}$ in $I(\chi)_{N_n}$ in the form $\sum_{w \in W} \xi_w$, $\xi_w \in \rho(w'\chi)$. Fix $w' \neq w$. We prove $\xi_w = 0$, by showing that any functional on $\rho(w'\chi)$ vanishes on $\xi_{w'}$.

For a representative $b$ of $T_{n,*} \setminus T_n$, consider the linear functional on $I(\chi)$ given by
\[
\Upsilon_{w',b}(f_\chi) = T_{w'f_\chi}(s(b)).
\]
It is defined on $I(\chi)_{N_n}$ and satisfies equalities similar to (2.5) and (2.7), i.e.,
\[
\Upsilon_{w',b}(tf_\chi) = \delta^{1/2}_{B_n} w'\chi(t)\Upsilon_{w',b}(f_\chi), \quad \forall t \in C_{\tilde{T}_n},
\]
\[
\Upsilon_{w',b}(tf_\chi) = \delta^{1/2}_{B_n} w'\chi(t)\Upsilon_{w',b}(f_\chi), \quad \forall t \in C_{\tilde{T}_n},
\]
hence $\Upsilon_{w',b}$ vanishes on $\rho(w'\chi)$ for any $w'' \neq w'$.

The set $\{\Upsilon_{w',b}\}_b$ is linearly independent, since for $b_1 \in T_n$,
\[
\Upsilon_{w',b}(s(b_1)f_{w,\chi}) = T_{w_0}\varphi_{w_0,(w^{-1}w_0)^{-1}\chi}(s(b)s(b_1)) = \begin{cases} 
\delta^{1/2}_{B_n} w'\chi(s(b)s(b_1)) & b_1 \in T_{n,*}, \\
0 & \text{otherwise.}
\end{cases}
\]
This is because for $t \in T_n$, if $w_0 N_n t$ intersects the projection of the support of $\varphi_{w_0}$..., then $t \in T_{n,*}$ (use (2.1) and the fact that $w_0 t \in B_{n,*} N_n$ implies $t \in T_{n,*}$).

Thus $\{\Upsilon_{w',b}\}_b$ is a basis of the linear dual of $\rho(w'\chi)$ and moreover,
\[
\Upsilon_{w',b}(\xi_{w'}) = \Upsilon_{w',b}(f_{w,\chi}).
\]
Recall that we have to show $\xi_{w'} = 0$, we prove $\Upsilon_{w',b}(\xi_{w'}) = 0$ for all $b \in T_n$. For $b \in T_{n,*}$, $\Upsilon_{w',b}$ is a scalar multiple of $\Upsilon_{w'}$, hence vanishes on $f_{w,\chi}$ (by definition). It remains to show $\Upsilon_{w',b}(f_{w,\chi}) = 0$ for all $b \notin T_{n,*}$. Using [CO13, (3.17)],
\[
\Upsilon_{w',b}(f_{w,\chi}) = \frac{c_{w}{w_0(1)}c_{w_0(1)}^{-1}}{c_{w_0(1)}^{-1}} T_{w'u^{-1}w_0}\varphi_{w_0,(w^{-1}w_0)^{-1}\chi}(s(b)) = 0,
\]
since for any $w \in W$ and $v \in N_n$, $vwv \in B_{n,*} N_n$ implies $b \in T_{n,*}$ (use (2.1)). □

**Remark 2.7.** This result is central to the proof of the next proposition. In [Cas80], the fact that the image of $f_{w,\chi}$ in $I(\chi)_{N_n}$ belongs to $\rho(w'\chi)$ was immediate.
Proposition 2.8. ([Cas80, Proposition 3.9]) For any \( w \in W \) and \( t \in C_{T_n} \cap \widehat{T}_n^- \),

\[
\mathcal{P}_I(tf_{w, \chi}) = \delta_{B_n}^{1/2} w \chi(t) f_{w, \chi}.
\]

Proof. This follows from Propositions 2.1, 2.6 and 2.2. \( \square \)

For each \( w \in W \), if \( \mathfrak{w} \) is the representative of \( w \), put \( q(\mathfrak{w}) = [\mathfrak{T} \mathfrak{w}I : \mathfrak{I}] \). The \( \mathcal{G}_n \)-pairing on \( I(\chi) \times I(\chi^{-1}) \) is given by (see [CO13, (10.2)])

\[
\langle f_\chi, f_{\chi^{-1}} \rangle = \sum_{b \in B_n, \mathfrak{q}B_n} \delta_{B_n}^{-1}(b) \int_{\mathcal{K}} f_\chi(s(b)\kappa(k)) f_{\chi^{-1}}(s(b)\kappa(k)) \, dk. \tag{2.8}
\]

Note that for any unramified \( f_\chi, f_{\chi}(s(t)) = 0 \) on \( t \in T_n - T_n^* \) ([KP84, Lemma 1.13], see also [McN12] Lemma 2), which means that \( \langle f_\chi, f_{\chi^{-1}} \rangle = \mathcal{P}_K(f_{\chi^{-1}})s(I_n) \).

Proposition 2.9. ([Cas80, § 4]) For any \( w \in W \), \( \mathcal{P}_K(f_{w, \chi})(s(I_n)) = \frac{c_{\mathfrak{w}}(w\mathfrak{w} \chi)}{Qc_{\mathfrak{w}}(\chi)} \), where \( Q = \sum_{w \in W} q(w)^{-1} \).

Proof. As in [Cas80], this is deduced from the formula for the zonal spherical function, which in our setting was computed in [CO13, § 10]. For the argument we use (2.8), Proposition 2.8 and the fact that the characters \( \chi \) restricted to \( C_{T_n} \cap \widehat{T}_n^- \) are linearly independent. \( \square \)

Let \( I(\chi)^* \) denote the linear dual of \( I(\chi) \). For any \( w \in W \), by definition

\[
T_{w^{-1}, w\chi^{-1}}^*: I(\chi^{-1})^* \to I(w\chi^{-1})^*.
\]

Assume we have a family \( \{\Lambda_{a, \chi}\}_{a \in \mathcal{A}} \) of functionals \( \Lambda_{a, \chi} \in I(\chi^{-1})^* \) indexed by a finite (ordered) set \( \mathcal{A} \), and such that for each \( a \), \( \Lambda_{a, \chi} \) is a meromorphic function in \( \chi \). Also assume that for each \( w \in W \) there is an invertible matrix \( A(w, \chi) = (A(w, \chi)_{a,a'}) \) such that

\[
(T_{w^{-1}, w\chi^{-1}}^* \Lambda_{a, \chi})_{a \in \mathcal{A}} = A(w, \chi) (\Lambda_{a, w\chi})_{a \in \mathcal{A}}. \tag{2.9}
\]

Remark 2.10. In [Hir99] the matrix was defined after normalizing the operator \( T_{w^{-1}, w\chi^{-1}}^* \) by \( c_{\mathfrak{w}}(\chi)c_{\mathfrak{w}}(w\chi^{-1})^{-1} \), we followed the convention of [KP84, CO13].

Here is our main tool for developing a Casselman–Shalika type formula.

Corollary 2.11. ([Hir99, Proposition 1.9])

\[
(\Lambda_{a, \chi}(g\varphi_{K, \chi^{-1}}))_{a \in \mathcal{A}} = \frac{1}{Q\text{vol}(I)} \sum_{w \in W} \frac{c_{\mathfrak{w}}(w\mathfrak{w} \chi)}{c_{\mathfrak{w}}(w\chi^{-1})}A(w, \chi)(\Lambda_{a, w\chi}(g\varphi_{e, w\chi^{-1}}))_{a \in \mathcal{A}}.
\]

Proof. First extend [Hir99, Lemma 1.5, Propositions 1.6-1.7], this is easy using (2.3), (2.4) and (2.8). To complete the proof repeat the steps from [Hir99, Proposition 1.9], with the aid of Proposition 2.9. Note that in the statement of [Hir99], on the right-hand side we have \( \lambda_{a, w\chi, g}(1) \), where \( \lambda_{a, w\chi, g} \) is the image of \( \mathcal{P}_I(g^{-1} \Lambda_{a, \chi}) \) in \( I(\chi) \), instead of \( \Lambda_{a, w\chi}(g\varphi_{e, w\chi^{-1}}) \). As observed in [Sak06], we actually have \( \lambda_{a, w\chi, g}(1) = \text{vol}(I)^{-1} \Lambda_{a, \chi}(g\varphi_{e, \chi^{-1}}) \). \( \square \)

The results of this section apply to a more general setting of covering groups, for example, the \( r \)-fold covers of \( G_a \), in which case all the preliminary results we used are available (see [CO13]). McNamara [McN11, McN16] has generalized substantially several results of [CO13], and our results will become valid in that setting as well (e.g., after the zonal spherical formula is proved). To be cautious, we refrain from stating our results in this generality.
3. The metaplectic Shalika model

3.1. Definition. Let $F$ be a local field and $\psi$ be a nontrivial additive character of $F$. First we recall the notion of the Shalika model, introduced by Jacquet and Shalika [JS90].

Put $n = 2k$. Consider the unipotent radical $U_k$ of the parabolic subgroup $Q_k < G_n$, and the (Shalika) character $\psi$ of $U_k$ given by
\[
\psi(( I_k \ u \ I_k )) = \psi(\text{tr}(u)).
\]

The normalizer of $U_k$ and stabilizer of $\psi$ is
\[
G^\triangle_k = \{ c^\triangle : c \in G_k \}, \quad c^\triangle = \text{diag}(c, c).
\]

The group $G^\triangle_k \ltimes U_k$ is called the Shalika group. Let $\pi \in \text{Alg} G_n$. A $\psi$-Shalika functional on the space of $\pi$ is a functional $l$ (continuous over an archimedean field) such that for any vector $\xi$ in the space of $\pi$, $c \in G_k$ and $u \in U_k$,
\[
l(\pi(c^\triangle u)\xi) = \psi(u)l(\xi).
\]

When $\pi$ is irreducible and $l$ exists, one may define the Shalika model $\mathcal{I} (\pi, \psi)$ of $\pi$ as the space of functions $\mathcal{I}(\xi)(g) = l(\pi(g)\xi)$, as $\xi$ varies in the space of $\pi$.

We explain the metaplectic analog (following [Kapa]). According to (1.1), restriction of the cover of $G_n$ to $G^\triangle_k$ is a “simple cover” in the sense that $\sigma(c^\triangle, c'^\triangle) = (\det c, \det c')_2$. Hence
\[
\mathfrak{s}(cc'^\triangle) = \sigma(c^\triangle, c'^\triangle)\mathfrak{s}(c^\triangle)\mathfrak{s}(c'^\triangle) = (\det c, \det c')_2 \mathfrak{s}(c^\triangle)\mathfrak{s}(c'^\triangle).
\]

Then if $\psi'$ is another nontrivial additive character of $F$, any genuine representation of $G^\triangle_k$ takes the form $\tau \otimes \gamma_{\psi'}$, where $\tau$ is a representation of $G_k$ and
\[
\tau \otimes \gamma_{\psi'}(\mathfrak{s}(c^\triangle)) = \epsilon \gamma_{\psi'}(\det c)\tau(c), \quad \epsilon \in \mu_2.
\]

For a genuine $\pi \in \text{Alg} \tilde{G}_n$, we call $l$ a metaplectic $(\psi', \psi)$-Shalika functional on $\pi$ if
\[
l(\pi(\mathfrak{s}(c^\triangle u)\xi)) = \gamma_{\psi'}(\det c)\psi(u)l(\xi).
\]

The terminology is reasonable because $\gamma_{\psi'}$ corresponds to the trivial character of $G_k$. If $\pi$ is irreducible, we denote the corresponding metaplectic Shalika model by $\mathcal{I}(\pi, \psi', \psi)$. It is a subrepresentation of $I(\gamma_{\psi'} \otimes \psi) = \text{Ind}_{G^\triangle_k U_k}^{G_n}(\gamma_{\psi'} \otimes \psi)$, where $\gamma_{\psi'} \otimes \psi(\mathfrak{s}(c^\triangle u)) = \epsilon \gamma_{\psi'}(\det c)\psi(u)$.

Remark 3.1. Since $\det c^\triangle \in F^{*2}$, we also have $\sigma^{(1)}(c^\triangle, c'^\triangle) = (\det c, \det c')_2$ and (3.1) also holds for the section corresponding to $\sigma^{(1)}$. (The cocycle $\sigma^{(1)}$ was defined in § 1.2, as we mentioned there our work applies to both covers.)

We have the following bound for Shalika functions.

Lemma 3.2. Assume that $\pi \in \text{Alg} \tilde{G}_n$ is genuine irreducible and admits a metaplectic Shalika model. Then there exists $\alpha > 0$ such that for any $\xi$ in the space of $\pi$, there exists a positive $\phi \in \mathcal{S}(F_{k\times k})$ such that
\[
|\mathcal{I}(\xi)(\mathfrak{s}(\text{diag}(g, I_k)))| \leq |\det g|^{-\alpha}\phi(g), \quad \forall g \in G_k.
\]

Proof. Over $p$-adic fields, the proof is a simple adaptation of Lemma 6.1 of [JR96] for the non-metaplectic setting. The main difference is that the asymptotic expansion is written for $T^2_k$ instead of $T_k$, the group $T^2_k$ is abelian and the cover splits over it.
The archimedean case follows from [AGJ09, Lemma 3.2] (after a standard application of the Dixmier–Malliavin Lemma [DM78] to the representation \( \pi|_{U_k} \)), which also holds in the metaplectic setting since the proof merely uses the Lie algebra action.

The next lemma resembles the relation between Whittaker and Kirillov models ([GK75, Proposition 2]).

**Lemma 3.3.** Assume \( F \) is non-archimedean. Let \( \pi \) be a genuine irreducible subrepresentation of \( I(\gamma \psi \otimes \psi) \). The restriction of functions in the space of \( \pi \) to \( \tilde{Q}_n \) contains any element of \( \text{ind}_{G_k^e U_k}^{\tilde{G}_k^e U_k} (\gamma \psi \otimes \psi) \).

**Proof.** For any \( \xi \) in the space of \( \pi \) and \( \phi \in \mathcal{S}(U_k) \), the function \( \phi(\xi) \) defined by

\[
\phi(\xi)(x) = \int_{U_k} \phi(u) \xi(x s(u)) du, \quad x \in G_n,
\]

belongs to the space of \( \pi \). Given \( d_g = s(\text{diag}(g, I_k)) \in \tilde{Q}_n \), one can find \( \xi \) such that \( \xi(d_g) \neq 0 \) and we also have \( \phi(\xi)(d_g) = \hat{\phi}(g) \xi(d_g) \), where \( \hat{\phi} \) denotes the Fourier transform of \( \phi \) with respect to \( \psi \). Note that the group of characters of \( U_k \) is isomorphic to the direct product of \( k^2 \) copies of \( F \). We can select \( \phi \) such that the support of \( \hat{\phi} \) is contained in \( g \mathcal{V} \), where \( \mathcal{V} < G_k \) is a small compact open neighborhood of the identity. Then \( \phi(\xi)|_{\tilde{Q}_n} \) belongs to \( \text{ind}_{G_k^e U_k}^{\tilde{G}_k^e U_k} (\gamma \psi \otimes \psi) \). The latter space is spanned by these functions, as \( g \) and \( \mathcal{V} \) vary.

### 3.2. Asymptotic expansion of metaplectic Shalika functions.

As with Whittaker functions, one can write an asymptotic expansion for Shalika functions. Over \( p \)-adic fields, such an expansion was obtained in [JR96, § 6.2] and extends to \( \tilde{G}_n \). Over archimedean fields the problem is more difficult, because one has to deal with delicate continuity properties. Here we provide an asymptotic expansion for metaplectic Shalika functions over \( F = \mathbb{R} \) or \( \mathbb{C} \). The same result holds in the non-metaplectic setting for the Shalika functional.

**Remark 3.4.** The results of this section will be used in § 5 to prove the meromorphic continuation of the local integrals (see the proof of Theorem 5.3).

Let \( n = 2k \). We use the notation and definitions of § 1.6, e.g., \( A_\ell, a_\ell, E_1(Q_\ell, \pi) \) and \( \Lambda_{\pi, \ell} \), where now \( \pi \in \text{Alg} \tilde{G}_n \) is a genuine representation. Note that \( a_\ell \) is viewed as an element in \( G_n \) rather than \( G_k \) by embedding \( G_k \) into \( G_n \) via \( g \mapsto \text{diag}(g, I_k) \).

**Theorem 3.5.** Let \( \pi \in \text{Alg} \tilde{G}_n \) be a genuine irreducible representation on a space \( V \), that admits a metaplectic Shalika functional \( l \). Then for all \( D = (D_1, \ldots, D_k) \in \mathbb{R}^k \) there exist finite subsets \( C_\ell \subset E^{(\ell)}(\pi) \) and functions \( p_{I, z} : \mathbb{R}^k \times V \to \mathbb{C} \) for \( I \subset \{1, \ldots, k\} \), \( z \in \prod_{\ell \in I} C_\ell \), such that the following holds.

1. For all \( \xi \in V \) and \( x \in \mathbb{R}^k \),

\[
l(\pi(s(a_\ell))\xi) = \sum_{I, z} p_{I, z}(x; \xi) \cdot e^{-z \cdot x_I},
\]

where the (finite) summation is over all \( I \subset \{1, \ldots, k\} \) and \( z \in \prod_{\ell \in I} C_\ell \).

2. Each function \( p_{I, z}(x; \xi) \) is polynomial in \( x_I \):

\[
p_{I, z}(x; \xi) = \sum_{\alpha} c_\alpha(x_I; \xi) x_I^\alpha,
\]
where the coefficients \( c_\alpha(x; \xi) \) are smooth in \( x \), linear in \( \xi \), and for some \( d \geq 0 \) and a continuous seminorm \( q \) on \( V \),

\[
|c_\alpha(x; \xi)| \leq q(\xi)(1 + |x|)^d e^{-D_T x^T}, \quad \forall x \in \mathbb{R}^T, \xi \in V.
\]

**Proof.** The proof is similar to the proof of Theorem 1.1, we explain the necessary modifications. The main difference is, of course, that \( l \) is not a smooth functional. In contrast to the situation in Theorem 1.1 we cannot replace \( l \) by \( \pi^\vee(X)l \) for \( X \in \mathfrak{u}_J \), because \( \pi^\vee(X)l \) is not a metaplectic Shalika functional. However, each \( X \in \mathfrak{u}_J \) can be written as \( X = X_1 + X_2 \) with \( X_1 \in \mathfrak{u}_k \) and \( X_2 = \text{diag}(Y,0) \) for some \( Y \in \mathfrak{g}_k \) (here \( 0 \in F_{k \times k} \)). By (3.2) we obtain

\[
l(\pi(X)\pi(s(a_x))\xi) = \psi(X_1)l(\pi(s(a_x))\xi) - l(\pi(s(a_x))\pi(\text{diag}(0,Y))\xi).
\]

In the second term we can replace \( \pi(\text{diag}(0,Y))\xi \) by \( \xi \) since the map \( \xi \mapsto \pi(\text{diag}(0,Y))\xi \) is continuous and all data in the statement depends continuously on \( \xi \). This shows that the recursive argument in the proof of Theorem 1.1 extends to our setting.

Now, to show that an estimate similar to (1.5) holds for \( l \), observe that \( l \) is continuous and \( \pi \) is of moderate growth, then argue as in [Wal88, Section 4.3.5]. \( \square \)

**Remark 3.6.** One can also expand \( l(\pi(s(a_x))\xi) \) for \( x_1, \ldots, x_{k-1} \geq 0 \) and \( x_k \leq 0 \) into an exponential power series, and the remainder terms are at least bounded by \( e^{-N x_k} \) for some \( N \geq 0 \). This is because \( l(\pi(s(a_x))\xi) = l(\pi(s(a_{x'}))\pi(s(e^{-x_k H_k}))\xi) \), where \( x' = (x_1, \ldots, x_{k-1},0) \).

### 3.3. The unramified metaplectic Shalika function

We develop an explicit formula for the unramified metaplectic Shalika function. Our arguments closely follow those of Sakellaridis [Sak06], who established this formula in the non-metaplectic setting. We use the definitions and results of § 2, in particular \( |2| = 1 \) and \( \kappa \) is the canonical splitting of \( K \).

Assume \( n = 2k \). Let \( I(\chi) \) be a genuine unramified principal series representation, \( \chi = \chi_\kappa \) (since \( n \) is even, \( \gamma \) can be ignored). Let \( \psi \) and \( \psi' \) be a pair of nontrivial additive characters of \( F \), and assume they are both unramified, i.e., their conductor is \( \mathcal{O} \). We construct a metaplectic \((\psi', \psi)\)-Shalika functional on \( I(\chi) \), and provide an explicit formula for the value of such a functional on the torus translates of the normalized unramified element of \( I(\chi) \).

Identify \( \Delta_{G_n} \) with the pairs \((i, i+1), 1 \leq i < n \), and put \( \alpha_i = (i, i+1) \), then \( \Sigma_{G_n} \) consists of the pairs \((i, j) \) with \( 1 \leq i \neq j \leq n \). We will need a few simple auxiliary results for computing conjugations.

**Proposition 3.7.** Let \( \mathfrak{w} \in \mathfrak{W} \) and \( t \in T_n \). Assume that \( \mathfrak{w} \) is the representative of \( w \in W \). Then \( s^{(\mathfrak{w})}\mathfrak{s}(t) = \sigma(\mathfrak{w}, t)s^{(\mathfrak{w})}t \), where \( \sigma(\mathfrak{w}, t) = \prod_{i,j=\alpha,\omega<0}(-t_j, t_i)_2 \).

**Proof.** Apply [BLS99, § 3, Lemma 3 and Theorem 7]. \( \square \)

For any \( t, t' \in T_n \), according to the computation of \( \sigma(t, t') \) in [BLS99, § 3, Lemma 1],

\[
[s(t), s(t')] = s(t)s(t')s(t)^{-1}s(t')^{-1} = \prod_{i<j}(t_i, t'_j)_2(t'_i, t_j)_2.
\]  

**Remark 3.8.** Let \( w \) be a permutation matrix. One can find unique \( \mathfrak{w} \in \mathfrak{W} \) and \( t_0 \in T_n \), such that \( \mathfrak{w} = t_0\mathfrak{w} \), where the diagonal coordinates of \( t_0 \) are \( \pm 1 \). Then for any \( t \in T_n \),

\[
s^{(\mathfrak{w})}\mathfrak{s}(t) = s^{(t_0)}(s^{(\mathfrak{w})}\mathfrak{s}(t)), \quad \text{which can be computed using Proposition 3.7 and (3.3)}.
\]

The following theorem shows that when the inducing data is in a certain general position, which does not preclude reducibility, the space of metaplectic Shalika functionals is at most
Remark 3.11. To prove (3.4) it is enough to show (3.3) (see the discussion following Remark 2.3). Also \( s(g) (\gamma_\psi \otimes \psi^{-1}) \otimes \chi \) acts trivially on \( \mu_2 \).

By [AG94] (proof of Proposition 1.3) \( G_n = \bigcup_w G_k U_k w B_n \) where \( w \) varies over the permutation matrices. Hence we may assume \( g = wt \) for a permutation matrix \( w \) and \( t \in T_n \). Set
\[
\omega = (\delta_{i,j})_{1 \leq i,j \leq k}, \quad \omega^{-1} = \left( I_k \omega^{-1} \right).
\]

By definition \( \dim \mathcal{H}(g) \leq 1 \) for any \( g \). According to Bruhat Theory (see e.g., [Sil79, Theorems 1.9.4 and 1.9.5]), to prove (3.4) it is enough to show
\[
\mathcal{H}(g) = 0, \quad \forall g = wt \text{ such that } G_k^\Delta U_k w B_n \neq G_k^\Delta U_k \omega^{-1} w B_n,
\]
\[
\mathcal{H}(\omega^{-1} t) = 0, \quad \forall t \text{ such that } G_k^\Delta U_k \omega^{-1} w B_n \neq G_k^\Delta U_k \omega^{-1} w B_n.
\]

Claim 3.10. Equality (3.5) holds.

Remark 3.11. In [AG94] this claim already completed the proof of uniqueness, because in \( G_n \) one could a priori take \( g = w \).

Before proving the claim, let us complete the proof of the theorem. We establish (3.6). Put \( g = \omega^{-1} t \). The condition on \( t \) implies that we can assume \( t = \text{diag}(t_1, \ldots, t_k, I_k) \) and for some \( 1 \leq i \leq k, t_i \notin F^{*2} \Omega^* \). Let \( i \) be minimal with this property. Note that for any \( y \in F^* - F^{*2} \Omega^* \), there is \( x \in \Omega^* \) such that \( \langle y, x \rangle_2 \neq 1 \). Then by (3.3) there is \( x \in \Omega^* \) such that for \( d = \text{diag}(I_{-1}, \ldots, I_{-i}, I_{-i}) \),
\[
\text{s(t)}^{-1} \text{s}(d) = (t_i, x) \text{s}(x) \text{s}(d) = -\text{s}(x) \text{s}(d).
\]

Then using Remark 3.8 and Proposition 3.7 we obtain \( \text{s(g)}^{-1} \text{s}(d) = -\text{s}(d) \). Clearly \( \text{s(g)}^{-1} \text{s}(d) \in (G_k^\Delta U_k)^{\text{s(g)}^{-1}} \), and we see that \( \langle \gamma_\psi \otimes \psi \rangle (\text{s}(d)) = 1 \) (because \( \gamma_\psi \) is trivial on \( \Omega^* \)) and \( \chi (\text{s(g)}^{-1} \text{s}(d)) = 1 \).
$-\chi(s(\omega d)) = -1$, because $\chi$ is trivial on $s(T_n \cap K)$. But $\delta_{B_n}$ and $\delta_g$ are modulus characters and as such, are trivial on $g^{-1} d$. Thus $\mathcal{H}(\omega^{-1}) = 0$.

Therefore the space of metaplectic Shalika functionals on $I(\chi)$ is at most one-dimensional. For the second assertion of the theorem consider $\mathcal{H}(\omega^{-1})$. In this case

$$(G_k^\Delta U_k)^{\omega} = t((T_k \cap K)T_k^2) \times \prod_{\{\alpha \in \Sigma_{G_k^\Delta}^{-1} \alpha > 0\}} t(N_\alpha),$$

where $N_\alpha$ on the right-hand side is a subgroup of $G_k$ and for any $c \in G_k$, $t(c) = \text{diag}(c, u_0 c)$. For $d = x^\Delta$ with $x \in T_k$ such that its coordinates lie in $F^{x^2}O^*$, $(\gamma_{\psi'} \otimes \psi)(s(d)) = 1$. Also $s$ is a homomorphism of $t((T_k \cap K)T_k^2)$. Hence

$$\mathcal{H}(\omega^{-1}) = \text{Hom}_{s((T_k \cap K)T_k^2)}(\chi, \delta_{B_n}^{-1/2} \delta_{\omega^{-1}}).$$

For each $1 \leq i \leq k$, $t_{i,k+\tau(i)} \in s((T_k \cap K)T_k^2))$ and $\delta_{\omega^{-1}}(t_{i,k+\tau(i)}) = \delta_{B_n}^{1/2}(t_{i,k+\tau(i)})$. Hence there is no metaplectic Shalika functional on $I(\chi)$, unless $\chi(t_{i,k+\tau(i)}) = 1$. □

**Proof of Claim 3.10.** As mentioned above, the arguments are similar to [AG94, Lemma 1.7]. One difference is that we consider a general $\tau$, whereas in [AG94] the permutation was taken (for simplicity) to be the identity. This is only a minor complication. We omit the proof. □

**Remark 3.12.** Recall that $\chi = \chi_s$, $s \in \mathbb{C}^n$ (see § 2.2). Assume $q^{2s_i} \neq q^{-2s_i}$ for all $1 \leq i < j \leq k$ and $q^{2s_i} = q^{-2s_{i+1}}$ for all $1 \leq i \leq k$, perhaps after a reordering of $(s_{n+1}, \ldots, s_n)$. By Theorem 3.9, $I(\chi)$ and $I(\chi^{-1})$ admit at most one metaplectic Shalika functional (up to scalar multiplication).

We define a functional satisfying the prescribed equivariance properties (3.2). It will be defined using an integral, which is absolutely convergent in a certain cone in $\chi$. Then we extend it to a meromorphic function, thereby obtaining a functional for all relevant $\chi$.

Following [Sak06], we realize the metaplectic Shalika functional using a similar functional on a group $H$, convenient for computational reasons. Let $J_k$ be the permutation matrix having 1 on the anti-diagonal and put $H = \omega(G_k^\Delta U_k)$, where $\omega = (j_k I_k)$. We form a section $h$ of $H$ by $h(h) = s(\omega) s(\omega^{-1}) h s(\omega^{-1})$. For $h \in H$, write $\omega^{-1} h = c_h h u_h$ with $c_h \in G_k$ and $u_h \in U_k$. The mapping $h \mapsto c_h$ is a group epimorphism, the mapping $h \mapsto u_h$ is onto $U_k$, and $h \mapsto \psi(u_h)$ is a character of $H$ (trivial on $\omega G_k^\Delta$). According to (3.1), the section $h$ satisfies

$$h(h') = (\det c_h, \det c_h') z(h) h(h').$$

A functional $l$ on a genuine $\pi \in \text{Alg} \tilde{G}_n$ is called a (metaplectic) $(H, \psi', \psi)$-functional if for any $\xi$ in the space of $\pi$ and $h \in H$,

$$l(\pi(h) \xi) = \gamma_{\psi'}(\det c_h) \psi(u_h) l(\xi).$$

It is simple to relate these functionals to the metaplectic Shalika functionals:

**Proposition 3.13.** Let $\pi \in \text{Alg} \tilde{G}_n$ be genuine. The dimension of the space of $(H, \psi', \psi)$-functionals on $\pi$ is equal to the dimension of metaplectic $(\psi', \psi)$-Shalika functionals.

**Proof.** If $l$ is an $(H, \psi', \psi)$-functional on $\pi$, $\xi \mapsto l(\pi(\omega)) \xi$ is a $(\psi', \psi)$-Shalika functional. □

We now construct $(H, \psi', \psi)$-functionals. Our main tool will be the metaplectic analog of Hironaka’s theorem, Corollary 2.11. To preserve the notation, we construct the functional on $I(\chi^{-1})$. 

---

A GODEMENT–JACQUET TYPE INTEGRAL

---

---

---

---
The group $H$ is closed and unimodular. Moreover, $B_nH$ is Zariski open in $G_n$ ([Sak06, Lemma 3.1]), thus $B_nH$ is dense and open in $G_n$, making it convenient for integration formulas. Here we are forced to work with $B_{n,*}H$, which is in particular not dense. Next we describe certain structural results involving $B_{n,*}$ and $H$, which will enable us to adapt the integration formulas from [Sak06] to the metaplectic setting.

**Proposition 3.14.** $B_nH = \coprod_{t \in T_k, s \in T_k} \text{diag}(t, I_k)B_{n,*}H$.

**Proof.** Direct verification, note that $B_{n,*}$ is normal in $B_n$. \hfill \Box

We will use the following geometric results from [Sak06, Lemma 5.1]:

$$
N_\alpha(\pi O) \subset (T_n \cap K)N_\alpha(O)N_{\alpha'}(O)(H \cap K), \quad \forall \alpha > 0, \quad (3.9)
$$

$$
T^- \subset (B_n \cap K)(H \cap K).
$$

Here for $(i, j) = 0$, $\alpha = 0$ if $i \leq k < j$ otherwise $\alpha = (n - j + 1, n - i + 1)$. The latter inclusion along with (2.1) and the fact that $B_n \cap K < B_{n,*}$ implies

$$
T \subset (B_n \cap K)(H \cap K) \subset B_{n,*}H. \quad (3.10)
$$

Consequently, $B_{n,*}H$ is open (in $G_n$). The following claims relate the sections $\mathfrak{h}$, $s$ and $\kappa$.

**Proposition 3.15.** The sections $s$ and $\kappa$ agree on $(G^\Delta U_k) \cap K$; $\mathfrak{h}$ and $\kappa$ agree on $H \cap K$.

**Proof.** First we compare $s$ and $\kappa$. We have $(G^\Delta U_k) \cap K = G_k(O)U_k(O)$, where $G_k(O)^\Delta = \{c^\Delta : c \in G_k(O)\}$. According to (3.1) and because $(O^*, O^*)_2 = 1$, $s$ is a splitting of $G_k(O)^\Delta$ and thereby of $G_k(O)U_k(O)$. The section $\kappa$ is also a splitting of the latter. Since $s|_{N_n \cap K} = \kappa|_{N_n \cap K}$, it is enough to show that $s$ and $\kappa$ agree on $G_k(O)^\Delta$. The mapping $c \mapsto s(c^\Delta)\kappa(c^\Delta)^{-1}$ is a homomorphism $G_k(O) \to \mu_2$, which is trivial on $T_k \cap G_k(O)$ because $s|_{T_k \cap K} = \kappa|_{T_k \cap K}$. Moreover, there is only one homomorphism $\text{SL}_k(O) \to \mu_2$, the trivial one (when $|2| = 1$ and $q > 3$). Thus $c \mapsto s(c^\Delta)\kappa(c^\Delta)^{-1}$ is the identity.

The section $\mathfrak{h}$ is a splitting of $H \cap K$ by definition and because $s$ is a splitting of $(G^\Delta U_k) \cap K$. The assertion for $\mathfrak{h}$ now follows from the previous result because $\kappa(\omega) = s(\omega)$. \hfill \Box

Next we consider $H \cap B_{n,*}$ (a subgroup of $T_{n,*}$), which is closed and unimodular.

**Proposition 3.16.** The sections $\mathfrak{h}$ and $s$ agree on $H \cap B_{n,*}$.

**Proof.** Let $t \in H \cap B_{n,*}$. Then $\mathfrak{h}(t) = s(\omega)s(\omega^{-1}t)$. This conjugation is explained as computed in Remark 3.8, and since $(\cdot, \cdot)_2$ is trivial on $F^{*2}O^* \times F^{*2}O^*$, it is equal to $s(t)$. \hfill \Box

Let $dh$ be a Haar measure on $H$. Consider the functional $l_H$ on $S^\text{gen}(\widetilde{H})$ given by

$$
l_H(f) = \int_H f(\mathfrak{h}(h))\gamma_{\psi'}^{-1}(\text{det } c_h)\psi^{-1}(u_h) \, dh.
$$

This integral is absolutely convergent because $|f| \in S^\text{gen}(H)$ ($|f|$ is non-genuine and $p : \widetilde{H} \to H$ is continuous). Since $f$ is genuine, (3.7) and (1.3) imply

$$
l_H(\mathfrak{h}(h)f) = \gamma_{\psi'}(\text{det } c_h)\psi(u_h)l_H(f).
$$

Hence $l_H$ satisfies (3.8) and is an $(H, \psi', \psi)$-functional on the space of $S^\text{gen}(\widetilde{H})$.

Next consider the space of $S^\text{gen}(s(H \cap B_{n,*}) \backslash \widetilde{H})$. There is a surjection

$$
S^\text{gen}(p^{-1}(B_{n,*}H)) \to S^\text{gen}(s(H \cap B_{n,*}) \backslash \widetilde{H})
$$
given by \( f \mapsto f_{B_n^* \cap H} \), where
\[
f_{B_n^* \cap H}(h) = \int_{B_n^* \cap H} f(s(b)s) h \, db, \quad h \in \tilde{H}.
\]
Note that \( h(b_s) = s(b_s) \) (Proposition 3.16). Then we have a functional \( l_{H \cap B_n^* \setminus H} \) on \( S^{\text{gen}}(s(H \cap B_n^*)) \tilde{H} \) defined by (see [Sak06, (44)])
\[
l_{H \cap B_n^* \setminus H}(f) = \int_{H \cap B_n^* \setminus H} f(h) \gamma_{\psi'}^{-1}(\det c_h) \psi^{-1}(u_h) \, dh.
\]
(3.11)
The integrand is invariant on the left by \( H \cap B_n^* \): for \( b_s \in H \cap B_n^* \) and \( h \in H \), by (3.7), Proposition 3.16, (1.3) and the fact that \( h \mapsto \gamma_{\psi'}^{-1}(\det c_h) \) is trivial on \( H \cap B_n^* \).
\[
\gamma_{\psi'}^{-1}(\det c_{b_s}h) h(b_s h) = \gamma_{\psi'}^{-1}(\det c_h) s(b_s) h(h).
\]
Hence
\[
f(h(b_s h)) \gamma_{\psi'}^{-1}(\det c_{b_s} h) = f(h) \gamma_{\psi'}^{-1}(\det c_h).
\]
Also the character \( \psi^{-1} \) satisfies \( \psi^{-1}(u_{b_s} h) = \psi^{-1}(u_h) \) because \( u_{b_s} = I_n \). Of course, \( l_{H \cap B_n^* \setminus H} \) satisfies the same equivariance properties as \( l_H \).
Henceforth assume
\[
q^{2s_i} \neq q^{2s_j}, \quad \forall 1 \leq i < j \leq k, \quad q^{2s_i} = q^{-2s_{n-i+1}}, \quad \forall 1 \leq i \leq k.
\]
(3.12)
By Theorem 3.9 and Proposition 3.13 the space of \( (H, \psi', \psi) \)-functionals on \( I(\chi^{-1}) \) is at most one-dimensional. We chose \( \chi \) in this form so that it is trivial on \( s(H \cap B_n^*) \). The modulus character \( \delta_{B_n^*} \) is also trivial on \( H \cap B_n^* \). We may then regard the elements of \( I(\chi^{-1}) \) as genuine locally constant functions on \( s(H \cap B_n^*) \tilde{H} \). Define a functional \( \Lambda_{\chi} \) on \( I(\chi^{-1}) \) by
\[
\Lambda_{\chi}(f_{\chi^{-1}}) = l_{H \cap B_n^* \setminus H}(f_{\chi^{-1}}).
\]
First we prove that this makes sense, for sufficiently many characters \( \chi \). Later we extend it to all relevant characters by meromorphic continuation. The equivariance properties of \( l_H \) imply that \( \Lambda_{\chi} \) is an \( (H, \psi', \psi) \)-functional.

**Proposition 3.17.** Assuming \( \text{Re}(s_i) < \text{Re}(s_{i+1}) \) for all \( 1 \leq i < k \) and \( \text{Re}(s_k) < 0 \), the integral \( l_{H \cap B_n^* \setminus H}(f_{\chi^{-1}}) \) given by (3.11) is absolutely convergent for all \( f_{\chi^{-1}} \).

**Proof.** The proof in [Sak06, Proposition 7.1] applies to our setting as well. \( \square \)

Recall the surjection \( P_{\chi} : S^{\text{gen}}(G_n) \to I(\chi) \) given by (2.2). We combine \( P_{\chi^{-1}} \) with (3.11) to obtain a formula convenient for computations (as in [Sak06, (45)]). For \( g \in B_n^* H \), one can write \( g = b_s h \) with \( b_s \in B_n^* \) and \( h \in H \). This writing is not unique. We still put \( b_g = b_s \) and \( h_g = h \). The arguments after (3.11) imply that the mappings \( g \mapsto \psi^{-1}(u_h) \) and \( g \mapsto s(b_g) h(h_g) \gamma_{\psi'}^{-1}(\det c_{h_g}) \) are well defined. Also \( g \mapsto \delta_{B_n^*}^{1/2}(b_g) \) is independent of the choice of \( b_g \) and the assumption \( \chi|_{s(H \cap B_n^*)} = 1 \) implies \( g \mapsto \chi(s(b_g)) \) is independent of the specific writing. Now we formally obtain
\[
\Lambda_{\chi}(P_{\chi^{-1}}(f)) = \int_{B_n^* H} \delta_{B_n^*}^{1/2}(b_g) \chi(s(b_g)) f(s(b_g) h(h_g)) \gamma_{\psi'}^{-1}(\det c_{h_g}) \psi^{-1}(u_{h_g}) \, dg.
\]
(3.13)
This formula is valid whenever the right-hand side is absolutely convergent. The measure \( dg \) is taken to be the restriction of the Haar measure on \( G_n \) to \( B_n^* H \) and on the left-hand side we normalize the measure \( dh \) accordingly (\( db_s \) was normalized after (2.2)).
For any $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$, put $\varpi^\lambda = \text{diag}(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_k})$ and $t_\lambda = \text{diag}(\varpi^\lambda, I_k)$. Let $\mathbb{Z}_+^k = \{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k : \lambda_1 \geq \cdots \geq \lambda_k \}$, $\mathbb{Z}_-^k = \{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k : \lambda_1 \geq \cdots \geq \lambda_k \geq 0 \}$. We also use $2\mathbb{Z}_+^k$, where for all $i$, $\lambda_i \in 2\mathbb{Z}$.

The following lemma describes the evaluation of $\Lambda_\chi(\varphi_{w, x^{-1}})$ for certain elements $w$, where it can be done succinctly. It is the main technical result of this section. The proof is parallel to the proof of Propositions 5.2, 8.1 and 8.2 in [Sak06], the complications involve the various sections $s$, $h$ and $k$. The results will be used below to deduce the general formula.

**Lemma 3.18.** Assume $\text{Re}(s_i) < \text{Re}(s_{i+1})$ for all $1 \leq i < k$ and $\text{Re}(s_k) < 0$. Also assume $\psi' = \psi_0$, for some $\varphi \in \mathcal{O}^*$ (i.e., $\psi'(x) = \psi(\varphi x)$). The following holds.

$$
\Lambda_\chi(s(t_{-\lambda})\varphi_{e, x^{-1}}) = \begin{cases} 
\text{vol}((I)\delta_{B^n}\chi(s(t_{\lambda}))) & \lambda \in 2\mathbb{Z}_+^k, \\
0 & \lambda \in \mathbb{Z}_-^k - 2\mathbb{Z}_+^k.
\end{cases}
$$

(3.14)

$$
\Lambda_\chi(\varphi_{s_{\alpha_k}, x^{-1}}) = \text{vol}(I \mathcal{W}I)(1 - q^{-1} + (-\varphi, \varpi)_2(\varpi, \varpi)_2^{-1}q^{-1/2}\chi^{-1/2}(a_{\alpha_k})).
$$

(3.15)

Here $\chi^{-1/2}(a_{\alpha_k}) = q^{2s_k}$. For all $1 \leq i < k$,

$$
\Lambda_\chi(\varphi_{s_{\alpha_i}, s_{\alpha_{n-i}}, x^{-1}}) = \text{vol}(I \mathcal{W}I)(1 - q^{-1} + q^{-2} + (1 - q^{-1})^2 \chi^{-1}(a_{\alpha_i}) / 1 - \chi^{-1}(a_{\alpha_i})).
$$

(3.16)

**Remark 3.19.** Note that $\chi^{-1/2}(a_{\alpha})$ is well defined because $\chi(\varphi(\text{diag}(I_{i-1}, \varpi^2, I_{n-i}))) = q^{-2s_i}$.

**Proof.** We first proceed formally using (3.13), the computation will also imply that (3.13) is absolutely convergent, providing the justification for replacing (3.11) with (3.13).

Consider (3.14) and assume $\lambda \in \mathbb{Z}_+^k$. Note that $t_{-\lambda}^{-1} = \lambda$. The integrand vanishes unless $g \in I t_{\lambda}$. For $\lambda \in \mathbb{Z}_+^k$, $t_{\lambda} \in T_{n_-}$ whence $t_{\lambda}^{-1}(I^-) t_{\lambda} < I^-$ and by (2.1) and (3.10),

$$
I t_{\lambda} \subset (B_n \cap K) t_{\lambda}(B_n \cap K)(H \cap K).
$$

In particular $I t_{\lambda} \subset t_{\lambda} B_{n, s} H$, the latter does not intersect $B_{n, s} H$ unless $\lambda \in 2\mathbb{Z}_+^k$, by Proposition 3.14. So the integrand vanishes unless $\lambda \in 2\mathbb{Z}_+^k$. In this case one can take $b_g = b_0 t_{\lambda} b_0'$ for some $b_0, b_0' \in B_n \cap K$ and $h_g \in H \cap K$. Then $\psi(u_{h_g}) = \gamma^{-1}_{\psi}(\det c_{h_g}) = 1$ because $h_g \in K$, so that both (3.13) and (3.11) equal $	ext{vol}(I \mathcal{W}I) / \delta_{B^n}^{1/2}(s(t_{\lambda})).$

Next we prove (3.15). Let $w$ be the representative of $s_{\alpha_k}$. Using (2.1) and (3.9) we obtain

$$
\mathcal{W}I = (B_n \cap K) w N_{\alpha_k}(\mathcal{O}) \prod_{\beta > 0; \beta \neq \alpha_k} N_{\beta}^{-}(\varpi \mathcal{O}) \subset (B_n \cap K) w N_{\alpha_k}(\mathcal{O})(H \cap K).
$$

The integrand vanishes unless $g \in \mathcal{W}I$. Write

$$
g = b_0 w v h_0, \quad b_0 \in B_n \cap K, \quad v \in N_{\alpha_k}(\mathcal{O}), \quad h_0 \in H \cap K.
$$

We decompose $w v = b_s h \in B_{n, s} H$, using $2 \times 2$ matrix computations, where the matrices are regarded as elements of $G_2$, embedded in $G_n$ such that its derived group is $R_{\alpha_k}$ (which was defined after (2.3)). Then $w = (\begin{smallmatrix} 1 & 1 \\
-1 & 1 \end{smallmatrix})$. If $v = (\begin{smallmatrix} 1 & \bar{z} \\
0 & 1 \end{smallmatrix})$ and $z \neq 0$, take $b_s = (\begin{smallmatrix} 1 & -\bar{z} \\
0 & 1 \end{smallmatrix})$ and $h = (\begin{smallmatrix} z^{-1} & 0 \\
0 & z \end{smallmatrix})$. Then $b_g = b_0 b_s$ and $h_g = h h_0$, hence $\psi^{-1}(u_{h_g}) = \psi^{-1}(z^{-1})$, $\gamma^{-1}_{\psi}(\det c_{h_g}) = (z, \det c_{h_0}) z^{-1} \gamma^{-1}_{\psi}(z^{-1})$, $\delta_{B^n}^{1/2}(b_g) = |z|^{-1}$ and $\chi(s(b_g)) = |z|^{2s_k + 1} = |z|^{-2s_k}$. Next we show

$$
\chi_{s_{\alpha_k}, x^{-1}}(s(b_g) h(h_g)) = (z, z)_{2}^{-k-1}(z, \det c_{h_0})_2.
$$
Because \( g \in \mathcal{I} \mathcal{W} \mathcal{I} \), this will follow immediately from

\[
\mathfrak{s}(b_g) \mathfrak{h}(h_g) = (z, z)^{k-1}(z, \det c_h) 2 \kappa(g).
\]

This is a direct (perhaps tedious) computation using (1.1) and (1.2). Then (3.13) becomes

\[
\operatorname{vol}(\mathcal{I} \mathcal{W} \mathcal{I}) \sum_{l=0}^{\infty} (q^{2s_k+1})^l q^{-l}(\varpi^{-l}, \varpi^{-l})^{k-1} \int_{\mathcal{O}^*} \gamma_{\psi}^{-1}(\varpi^{-l}x)\psi^{-1}(\varpi^{-l}x) \, dx.
\]

Note that \( dx \) is the additive measure of \( F \) (in particular \( \int_{\mathcal{O}^*} dx = 1 - q^{-1} \)).

**Remark 3.20.** Szpruch ([Szp09]) used similar \( dx \)-integrals to represent a local coefficient resembling Tate’s \( \gamma \)-factor.

For \( l > 1 \) the \( dx \)-integral vanishes, since

\[
\int_{\mathcal{O}^*} \gamma_{\psi}^{-1}(\varpi^{-l}x)\psi^{-1}(\varpi^{-l}x) \, dx = \gamma_{\psi}^{-1}(\varpi^{-l}) \sum_{\zeta \in (1+\varpi\mathcal{O}) \setminus \mathcal{O}^*} (\varpi^{-l}, \zeta) \int_{1+\varpi\mathcal{O}} \psi^{-1}(\varpi^{-l}\zeta x) \, dx = 0,
\]

because the conductor of \( \psi \) is \( \mathcal{O} \). Consider \( l = 1 \). Then (1.3) and the assumption \(|\varpi| = 1\) show \( \gamma_{\psi}^{-1}(\varpi^{-1}x) = (-\varpi, \varpi)_{\gamma_{\psi}}(\varpi^{-1}x) \), and the \( dx \)-integral equals \((-\varpi, \varpi)^{-1} \) multiplied by

\[
\int_{\mathcal{O}^*} \gamma_{\psi}(\varpi^{-1}x)\psi^{-1}(\varpi^{-1}x) \, dx.
\]

This integral was computed in [Szp09, Lemma 1.12] and equals \( q^{-1/2} \). Altogether we obtain

\[
\operatorname{vol}(\mathcal{I} \mathcal{W} \mathcal{I})(1 - q^{-1} + q^{2s_k-1/2}(-\varpi, \varpi)_{2}(\varpi, \varpi)^{k-1}).
\]

The proof of (3.16) follows along similar lines and is simpler. This is because the cocycle on \( G^\Delta_k \times G^\Delta_k \) is the Hilbert symbol, so that the arguments of [Sak06] apply in a straightforward manner. For brevity, the details are omitted. \( \square \)

Recall that we assume \( \chi \) satisfies (3.12). We say that \( f_{\chi^{-1}} \) is a polynomial (resp. rational) section if its restriction to \( \prod_{t \in T_n, \chi^{-1}} \mathfrak{s}(t) \kappa(K) \) depends polynomially (resp. rationally) on \( \chi^{-1} \), or more precisely on \( q^{\pm 2s_1}, \ldots, q^{\pm 2s_k} \). The definition is independent of the actual choice of representatives of \( T_n \setminus T_n \).

**Proposition 3.21.** The functional \( \Lambda_\chi \) has a meromorphic continuation to all \( \chi \) satisfying (3.12), in the sense that if \( f_{\chi^{-1}} \) is a rational section, \( \Lambda_\chi(f_{\chi^{-1}}) \) is a rational function, i.e., belongs to \( \mathbb{C}(q^{-2s_1}, \ldots, q^{-2s_k}) \).

**Proof.** This follows from Bernstein’s continuation principle (in [Ban98]), in light of Theorem 3.9, Propositions 3.13 and 3.17, and (3.14). See also [Sak06, § 7]. \( \square \)

**Corollary 3.22.** The results of Lemma 3.18 hold for all \( \chi \) satisfying (3.12), by means of meromorphic continuation.

Below we state the formula for the unramified function \( g \mapsto \Lambda_\chi(g\mathfrak{f}_{K,\chi^{-1}}) \), for \( g = t_\lambda \) with \( \lambda \in \mathbb{Z}^k_{\geq 2} \). The following result shows that these elements already determine \( \Lambda_\chi(g\mathfrak{f}_{K,\chi^{-1}}) \).

**Lemma 3.23.** Let \( l \) be a metaplectic \((\psi', \psi)'\)-Shalika functional (resp. \((H, \psi', \psi)'\)-functional) on a genuine unramified \( \pi \in \operatorname{Alg} \bar{G}_n \). Let \( \xi \) be an unramified vector in the space of \( \pi \) and consider the function \( l_\xi(g) = l(\pi(g)\xi) \).

1. \( l_\xi(k_0)g\kappa(k) = l_\xi(g) \) for any \( k_0 \in (G_k U_k) \cap K \) (resp. \( k_0 \in H \cap K \), \( g \in \bar{G}_n \) and \( k \in K \).
(2) $l_\xi$ is uniquely determined by its values on $t_\lambda$ (resp. $t_{-\lambda}$), where $\lambda \in \mathbb{Z}_k^\pm$.
(3) $l_\xi(t_\lambda) = 0$ (resp. $t_{-\lambda}$) for $\lambda \in \mathbb{Z}_k^\pm - \mathbb{Z}_k^\pm$.

Proof. The first assertion follows from Proposition 3.15 and (3.8). The proof of parts (2) and (3) is similar to the analogous proof in the non-metaplectic setting ([JR96, §6.2]). \[\square\]

Let $\text{Sp}_k$ be the symplectic group on $n$ variables, embedded in $G_n$ as the subgroup
\[\{g \in G_n : \frac{1}{2} g \cdot (-J_k J_k^t) g = (-J_k J_k^t)\},\]
where $^t g$ denotes the transpose of $g$. We fix the Borel subgroup $\text{Sp}_k \cap B_n$ and let $\Sigma_{\text{Sp}_k}$ (resp. $\Sigma_{\text{Sp}_k}^\pm$) be the corresponding root system (resp. positive roots) of $\text{Sp}_k$. The map $\Sigma_{G_n} \to \Sigma_{\text{Sp}_k}$ is two-to-one onto the short roots, and injective onto the long roots. Recall that we assumed $q^{2s_i} = q^{-2s_{n-i+1}}$ for all $1 \leq i \leq k$ (in (3.12)). Equivalently, $\chi$ is trivial on $\mathfrak{s}(H \cap B_{n,s})$. Hence if $\beta, \beta' \in \Sigma_{G_n}$ are the two roots corresponding to a short root $\alpha \in \Sigma_{\text{Sp}_k}$, $c_\beta(\chi) = c_{\beta'}(\chi)$. Thus we may extend the notation $c_\alpha(\chi)$ to short roots $\alpha$, meaning $c_\beta(\chi)$. The same applies to $\chi^{-1}$. Similarly, we define $c_\alpha(\chi)$ for long roots (there is no ambiguity here, for any $\chi$).

Denote the Weyl group of $\text{Sp}_k$ by $W_{\text{Sp}_k}$, it is the group generated by $s_k$ and $s_i s_{n-i}$, $1 \leq i < k$. For $w \in W_{\text{Sp}_k}$, let $\ell_{\text{Sp}_k}(w)$ be the minimal number of simple reflections in $W_{\text{Sp}_k}$ whose product is $w$, i.e., the usual length with respect to $W_{\text{Sp}_k}$.

Recall $\psi' = \psi_\varepsilon$ (see Lemma 3.18), put $\varepsilon_{\theta, k} = (-\theta, \varpi)_2 (\varpi, \varpi)^{k-1}$. For any $\alpha \in \Sigma_{\text{Sp}_k}^+$ let
\[y_{\alpha}(\chi^{-1}) = \begin{cases} c_\alpha(\chi^{-1}) c_\alpha(\chi) & \alpha \text{ is short}, \\ \frac{(1 + q^{-1/2} \varepsilon_{\theta, k} \chi^{-1/2}(a_\alpha))(1 - q^{-1/2} \varepsilon_{\theta, k} \chi^{1/2}(a_\alpha))}{1 - \chi(a_\alpha)} & \alpha \text{ is long}. \end{cases}\]

As in Lemma 3.18, if $\alpha$ is the long root $(i, n-i+1)$, $\chi^{-1/2}(a_\alpha) = q^{-2s_i}$.

Theorem 3.24. Let $\lambda \in \mathbb{Z}_k^\pm$. The function $\Lambda_X(\varphi_K, \chi^{-1})$ vanishes unless $\lambda \in 2\mathbb{Z}_k^\pm$. In this case,
\[\Lambda_X(\varphi_K, \chi^{-1}) = Q^{-1} \sum_{w \in W_{\text{Sp}_k}} \prod_{\alpha > 0 : \alpha > 0} c_\alpha(\chi^{-1}) \prod_{\alpha \in \Sigma_{\text{Sp}_k}^+ : \alpha < 0} y_{\alpha}(\chi^{-1}) \epsilon_{\beta, k}^{1/2} w \chi(\varphi(t_\lambda)).\]

More compactly, put
\[\beta(\chi^{-1}) = \prod_{\alpha \in \Sigma_{\text{Sp}_k}^+} \chi^{1/2}(a_\alpha) \prod_{\alpha \in \Sigma_{\text{Sp}_k}^+ \text{ short} : \alpha \in \Sigma_{\text{Sp}_k}^+} (1 - q^{-1} \chi^{-1}(a_\alpha)) \prod_{\alpha \in \Sigma_{\text{Sp}_k}^+ \text{ long}} (1 - q^{-1/2} \varepsilon_{\theta, k} \chi^{-1/2}(a_\alpha)).\]

Then
\[\Lambda_X(\varphi_K, \chi^{-1}) = Q^{-1} c_{w_0} \chi^{-1}) \beta(\chi^{-1}) \sum_{w \in W_{\text{Sp}_k}} (-1)^{\ell_{\text{Sp}_k}(w)} \beta(w \chi^{-1}) \epsilon_{\beta, k}^{1/2} w \chi(\varphi(t_\lambda)).\]

Remark 3.25. The product $\beta(\chi^{-1})^{-1} \beta(w \chi^{-1})$ in this formula is well defined.

Remark 3.26. To obtain the formula for the (non-metaplectic) Shalika model, simply remove the product over long roots in the definition of $\beta(\chi^{-1})$. Up to a factor independent of $\lambda$ this was the formula obtained by Sakellaridis [Sak06] (see also [Sak13, §5.5.2]).

Proof. We use the notation of §2.2. The space of $(H, \psi', \psi)$-functionals on $I(\chi^{-1})$ is at most one-dimensional. Therefore (2.9) becomes
\[T_{w^{-1}, w \chi^{-1}} \Lambda_\xi = A(w, \chi) \Lambda_{w \chi},\]
where $A(w, \chi)$ is a scalar (a rational function of $\chi$). By virtue of Corollary 2.11,

$$\Lambda_\chi(s(t_\lambda)\varphi_{K,\chi^{-1}}) = \frac{1}{Q\text{vol}(I)} \sum_{w \in W} \frac{c_{\text{vol}}(w^0w^{\chi})}{c_{w^{-1}}(w^{-1})} A(w, \chi)(\Lambda_{w^0}(s(t_\lambda)\varphi_{e,w^{-1}})).$$

According to Lemma 3.18,

$$\Lambda_{w^0}(s(t_\lambda)\varphi_{e,w^{-1}}) = \begin{cases} \text{vol}(I)\delta_{B_n}^{1/2} w^{\chi}(s(t_\lambda)) & \lambda \in 2\mathbb{Z}_k^k, \\ 0 & \lambda \in \mathbb{Z}_+^k - 2\mathbb{Z}_+^k. \end{cases} \quad (3.17)$$

More precisely $\Lambda_{w^0}$ is a functional on $I^{(\chi^{-1})}$, and the domain of absolute convergence changes after the conjugation, so actually we apply Corollary 3.22 and deduce this equality in the sense of meromorphic continuation.

Therefore $\Lambda_\chi(s(t_\lambda)\varphi_{K,\chi^{-1}})$ vanishes, unless $\lambda \in 2\mathbb{Z}_k^k$. Henceforth assume this is the case. Equality (3.17) also implies $\Lambda_{w^0}(\varphi_{e,w^{-1}}) \neq 0$. Therefore $A(w, \chi)$ is given by the quotient

$$A(w, \chi) = \frac{T_{w^{-1},w^{-1}}^*\Lambda_{w^0}(\varphi_{e,w^{-1}})}{\Lambda_{w^0}(\varphi_{e,w^{-1}})} = \frac{\Lambda_\chi(T_{w^{-1},w^{-1}}^*\varphi_{e,w^{-1}})}{\text{vol}(I)}. \quad (3.18)$$

We claim that only elements of $W_{\text{Sp}_k}$ contribute to the sum.

**Claim 3.27.** If $w \in W - W_{\text{Sp}_k}$, $A(w, \chi) = 0$.

Using the fact that $c_{w^0}(w^{\chi}) = c_{w^0}(w^{-1})$, we obtain (this is [Sak06, (48)])

$$\Lambda_\chi(s(t_\lambda)\varphi_{K,\chi^{-1}}) = Q^{-1} \sum_{w \in W_{\text{Sp}_k}} \prod_{\{\alpha > 0; \omega \alpha > 0\}} c_{\alpha}(\chi^{-1}) A(w, \chi) \delta_{B_n}^{1/2} w^{\chi}(s(t_\lambda)).$$

It remains to compute $A(w, \chi)$ for $w \in W_{\text{Sp}_k}$. Let $\ell_{\text{Sp}_k}(w) = l$ and write $w = s_1' \cdots s_l'$, where $s_i'$ is either $s_{\alpha_k}$ or $s_{\alpha_i}s_{\alpha_{n-i}}$ with $1 \leq i < k$. Then

$$A(w, \chi) = A(s_1', s_2', \ldots, s_l') \cdots A(s_{l-1}', s_l') A(s_l', \chi).$$

**Claim 3.28.** We have $A(s_{\alpha_k}, \chi) = y_{\alpha_k}(\chi^{-1})$ and $A(s_{\alpha_i}s_{\alpha_{n-i}}, \chi) = y_{\alpha_i}(\chi^{-1})$.

We conclude that for any $w \in W_{\text{Sp}_k}$, $A(w, \chi) = \prod_{\{\alpha \in \Sigma_{\text{Sp}_k}; \omega \alpha > 0\}} y_{\alpha}(\chi^{-1})$. Plugging this into the last expression for $\Lambda_\chi(s(t_\lambda)\varphi_{K,\chi^{-1}})$ yields the theorem. □

**Proof of Claim 3.27.** According to (3.18) it is enough to show that $T_{w^{-1},w^{-1}}^*\Lambda_\chi$ vanishes on $\varphi_{e,w^{-1}}$. By (3.10) the support of $\varphi_{e,w^{-1}}$ is contained in $\tilde{B}_{n,*}$. Let $\Lambda$ denote the restriction of $T_{w^{-1},w^{-1}}^*\Lambda_\chi$ to the subspace of functions in $I^{(\chi^{-1})}$, whose support is contained in $\tilde{B}_{n,*}$. We prove $\Lambda = 0$. The argument was adapted from [Sak06, Proposition 5.2].

Consider the space $S_{\text{gen}}^0(p^{-1}(B_{n,*}), \delta_{B_n}^{1/2} w^{\chi^{-1}})$ of complex-valued locally constant genuine functions $f$ on $p^{-1}(B_{n,*})$ such that

$$f(s(b)) = \delta_{B_n}^{1/2} w^{-1}(s(b)) f(h), \quad \forall b \in B_{n,*}, h \in H. \quad (3.19)$$

Since $B_{n,*}$ is open in $G_{n,*}$, $\Lambda$ is the restriction of $T_{w^{-1},w^{-1}}^*\Lambda_\chi$ to this subspace.

Also consider $S_{\text{gen}}^0(s(H \cap B_{n,*}) \setminus \tilde{H}, w^{\chi^{-1}})$, where the functions satisfy (3.19) on $b \in B_{n,*} \cap H$, and note that $\delta_{B_n}$ is trivial there.
These spaces are isomorphic (use (3.7)). Thus we may regard $\Lambda$ as a functional on $S^{\text{gen}}(\mathfrak{s}(H \cap B_{n,*}) \backslash \tilde{H}, w \chi^{-1})$. Then it lifts to a functional on $S^{\text{gen}}(\tilde{H})$, by virtue of the projection

$$f^{B_{n,*} \cap H, w \chi^{-1}}(h) = \int_{B_{n,*} \cap H} w \chi(s(b_*))f(s(b_*)h) \, db_*,$$

$h \in \tilde{H}, f \in S^{\text{gen}}(\tilde{H})$.

Furthermore, since $\Lambda$ is an $(H, \psi', \psi)$-functional, it is a scalar multiple of the functional $l_H$ (defined after Proposition 3.16). Indeed, this follows from the Frobenius reciprocity: if we let $\iota$ be the identity character of $\mu_2$ and regard $\gamma_{\psi'} \otimes \psi \otimes \iota$ as the genuine character of $\widetilde{H}$ taking $h(h)$ to $\gamma_{\psi'}(c_h)\psi(u_h)$, both $\Lambda$ and $l_H$ belong to

$$\text{Hom}_{\tilde{H}}(S^{\text{gen}}(\tilde{H}), \gamma_{\psi'} \otimes \psi \otimes \iota) = \text{Hom}_{\tilde{H}}(\text{ind}_{\mu_2}(\iota), \gamma_{\psi'} \otimes \psi \otimes \iota).$$

Finally for $b_* \in B_{n,*} \cap H$ and $f \in S^{\text{gen}}(\tilde{H})$, let $L(b_*)f(h) = f(s(b_*)h)$. Since $H$ is unimodular we see that $l_H(L(b_*)f) = l_H(f)$, so assuming $\Lambda \neq 0$, the same holds for $\Lambda$.

But this contradicts the fact that $\Lambda$ factors through $S^{\text{gen}}(\mathfrak{s}(H \cap B_{n,*}) \backslash \tilde{H}, w \chi^{-1})$, because for $w \in W - W_{Sp_k}$, $w \chi^{-1}$ is not trivial on $\mathfrak{s}(H \cap B_{n,*})$. 

\textbf{Proof of Claim 3.28.} First note that by [Cas80, Theorem 3.4], for any $\alpha \in \Delta_{G_n}$,

$$T_{s_\alpha} \varphi_{e, s_\alpha \chi^{-1}} = (c_{\alpha}(s_\alpha \chi^{-1}) - 1)\varphi_{e, \chi^{-1}} + q^{-1} \varphi_{s_\alpha, \chi^{-1}}.$$ Consider $A(s_{\alpha_k}, \chi)$. Assume that $w$ represents $s_{\alpha_k}$. By Corollary 3.22,

$$\Lambda_{\chi}(T_{s_{\alpha_k}} \varphi_{e, s_{\alpha_k} \chi^{-1}}) = \text{vol}(\mathcal{I})(c_{\alpha_k}(s_{\alpha_k} \chi^{-1}) - 1) + q^{-1} \text{vol}(\mathcal{I}w\mathcal{I})(1 - q^{-1} + (-q, \omega)_2(\omega, \omega)_2^{k-1}q^{-1/2} \lambda^{-1/2}(a_{\alpha_k})).$$

Since $c_{\alpha_k}(s_{\alpha_k} \chi^{-1}) = c_{\alpha_k}(\chi)$ and $\text{vol}(\mathcal{I}w\mathcal{I}) = q \cdot \text{vol}(\mathcal{I})$, we find that $A(s_{\alpha_k}, \chi) = y_{\alpha_k}(\chi^{-1})$.

The computation of $A(s_{\alpha_k}, s_{\alpha_{n-1}}, \chi)$ can be carried out similarly, and as in the proof of Lemma 3.18, this case is a closer adaptation of [Sak06] and hence omitted. 

The following corollary strengthens Proposition 3.21.

\textbf{Corollary 3.29.} Let $\chi$ be a character satisfying (3.12). Then $\Lambda_{\chi}$ is a nonzero $(H, \psi', \psi)$-functional on $I(\chi^{-1})$. Moreover it depends polynomially on $\chi$, in the sense that for a polynomial section $f_{\chi^{-1}}$, $\Lambda_{\chi}(f_{\chi^{-1}})$ belongs to $\mathbb{C}[q^{\pm 2s_1}, \ldots, q^{\pm 2s_k}]$.

\textbf{Proof.} According to Proposition 3.21, the functional $\Lambda_{\chi}$ is defined on $I(\chi^{-1})$ by means of meromorphic continuation. It is nonzero because it does not vanish on $\varphi_{e, \chi^{-1}}$ (Corollary 3.22). It remains to show it does not have a pole. Lemma 3.23 and Theorem 3.24 show that there are no poles on the $G_n$-space spanned by $\varphi_{k, \chi^{-1}}$. Proposition 2.4 implies that for almost all $\chi$ (except for a discrete subset of $\mathbb{C}^k$), this space is all of $I(\chi^{-1})$. Since $\Lambda_{\chi}$ is determined by meromorphic continuation, the assertion holds.

\textbf{Remark 3.30.} The representation $I(\chi^{-1})$ might be reducible. In this case it may happen that the product of factors $c_{\alpha}(\chi^{-1})$ and $y_{\alpha}(\chi^{-1})$ appearing in Theorem 3.24 vanishes, for all $w \in W_{Sp_k}$, and consequently $\Lambda_{\chi}$ will vanish on the $G_n$-space spanned by $\varphi_{k, \chi^{-1}}$.

We finally relate between the $(H, \psi', \psi)$-functional and the metaplectic $(\psi', \psi)$-Shalika functional, on the unramified normalized element of $I(\chi^{-1})$.

\textbf{Corollary 3.31.} Assume $\chi$ satisfies (3.12) and let $l$ be the functional on $I(\chi^{-1})$ given by

$$l(f_{\chi^{-1}}) = \Lambda_{\chi}(s(w)f_{\chi^{-1}}).$$
(1) \( l \) is a nonzero metaplectic \((\psi', \psi)\)-Shalika functional.

(2) \( l \) depends polynomially on \( \chi \).

(3) \( l (s(t_\lambda) \varphi_{K, \chi^{-1}}) = \Lambda_\chi (s(t_{-\lambda})) \) for all \( \lambda \in \mathbb{Z}^k_+ \). In particular \( l (s(t_\lambda) \varphi_{K, \chi^{-1}}) \) vanishes on \( \lambda \in \mathbb{Z}^k_+ - 2\mathbb{Z}^k_+ \). Consequently, Theorem 3.24 gives the Casselman–Shalika formula for the metaplectic Shalika model.

**Proof.** We adapt the argument from [Sak06, § 3, p. 1104]. We have \( h^w t_\lambda = t_{-\lambda} \) with

\[
h = \text{diag}(\varpi^{-\lambda}, J_k \varpi^{-\lambda} J_k) \in H.
\]

Put \( h(h)^{s(w)} s(t_\lambda) = \epsilon s(t_{-\lambda}) \) for some \( \epsilon \in \mu_2 \). Because \( \varphi_{K, \chi^{-1}} \) is unramified,

\[
l (s(t_\lambda) \varphi_{K, \chi^{-1}}) = \Lambda_\chi (s(w) s(t_\lambda) \varphi_{K, \chi^{-1}}) = \epsilon \gamma \psi'( \det c_h)^{-1} \Lambda_\chi (s(t_{-\lambda}) \varphi_{K, \chi^{-1}}).
\]

This vanishes unless \( \lambda \in 2\mathbb{Z}^k_+ \), but then \( \epsilon = 1 \) and \( \gamma \psi'( \det \varpi^\lambda) = 1 \). The result follows. \( \square \)

3.4. **Archimedean uniqueness results.** We establish uniqueness results for the metaplectic Shalika model over \( \mathbb{R} \) or \( \mathbb{C} \), analogous to those of Theorem 3.9.

**Theorem 3.32.** Let \( I(\chi) \) be a genuine principal series representation of \( \tilde{G}_n \), satisfying the conditions of Theorem 3.9. Then the space of metaplectic Shalika functionals on \( I(\chi) \) is at most one-dimensional, and is zero dimensional unless \( \chi(t_{i,k+\tau(i)}) = 1 \) for all \( 1 \leq i \leq k \).

**Proof.** The proof is an archimedean version of the proof of Theorem 3.9, the main complication concerns the symmetric tensors appearing in the archimedean Bruhat Theory. We only present this part of the argument.

For a permutation matrix \( g \in G_n \) and \( m \geq 0 \), let \( \mathcal{H}(g, m) \) denote the space \( \mathcal{H}(g) \) in the proof of Theorem 3.9, but with \( \delta \) (the product of modulus characters) replaced by \( \delta \otimes S^m(V(g)) \), where \( V(g) = g_n / (b_n + \text{Ad}(g^{-1})(g_\Delta + u_k)) \). Then \( \mathcal{H}(g, 0) = \mathcal{H}(g) \), and we additionally need to show that \( \mathcal{H}(g, m) = 0 \) for \( m > 0 \).

Let \( i \mapsto j_i \) denote the permutation of \( \{1, \ldots, n\} \) corresponding to \( g \), i.e., \( ge_{j_i} = e_i \). The \((\tilde{G}_k^\Delta U_k)^{s(w)} \)-representation \( V(g) \) is the sum of the equivalence classes of \( \text{Ad}(g^{-1}) E_{\ell,m} = E_{j_\ell,j_m} \), with either \( \ell, m \in \{1, \ldots, k\} \) or \( \ell \in \{k+1, \ldots, n\} \), \( m \in \{1, \ldots, k\} \). We claim that if the equivalence class of some \( E_{j_\ell,j_m} \) in \( V(g) \) is nonzero, then there exists a root subgroup of \((\tilde{G}_k^\Delta U_k)^{s(w)} \) that acts nilpotently on \( E_{j_\ell,j_m} \). This root subgroup acts on the left hand side of \( \mathcal{H}(g, m) \) either trivially or by the character \( \psi \), and on the right hand side it only acts on \( S^m(V(g)) \). Hence, no homomorphism in \( \mathcal{H}(g, m) \) can have a component of \( E_{j_\ell,j_m} \) in its image. This is true for all generators of \( V(g) \), whence \( \mathcal{H}(g, m) = 0 \) if \( m > 0 \).

It remains to show that every nonzero equivalence class \( E_{j_\ell,j_m} \) in \( V(g) \) has a root subgroup in \((\tilde{G}_k^\Delta U_k)^{s(w)} \) acting nilpotently on it. First, if \( \ell \in \{k+1, \ldots, n\} \) and \( m \in \{1, \ldots, k\} \) with \( E_{j_\ell,j_m} \) nonzero in \( V(g) \), then \( E_{j_\ell,j_m} \notin b_n \) which means \( j_\ell > j_m \). But then \( E_{j_{m+\ell}} \in b_n \), and also \( E_{j_{m+\ell}} = \text{Ad}(g^{-1}) E_{m,\ell} \in \text{Ad}(g^{-1}) u_k \), so that \((\tilde{G}_k^\Delta U_k)^{s(w)} \) contains the root subgroup \( N_{j_{m+\ell}} \), which acts nilpotently on \( E_{j_\ell,j_m} \).

Next, if \( \ell, m \in \{1, \ldots, k\} \) with \( E_{j_\ell,j_m} \) non-trivial in \( V(g) \), then \( E_{j_\ell,j_m} \notin b_n \) and also \( E_{j_{\ell+k},j_{m+k}} \notin b_n \), because \( E_{j_\ell,j_m} + E_{j_{\ell+k},j_{m+k}} = \text{Ad}(g^{-1})(E_{\ell,m} + E_{\ell+k,m+k}) \in \text{Ad}(g^{-1}) g_k^\Delta \). Again, this implies \( E_{j_{m+\ell},j_{m+k+j+\ell}} \in b_n \). Since also \( E_{j_{m+\ell}} + E_{j_{m+k+j+\ell}} = \text{Ad}(g^{-1})(E_{m,\ell} + E_{m+k,\ell+k}) \in \text{Ad}(g^{-1}) g_k^\Delta \), the one-parameter subgroup of \( E_{j_{m+\ell}} + E_{j_{m+k+j+\ell}} \) is contained in \((\tilde{G}_k^\Delta U_k)^{s(w)} \) and acts nilpotently on \( E_{j_\ell,j_m} \). This completes the proof. \( \square \)
4. Metaplectic Shalika model for exceptional representations

4.1. Exceptional representations. The exceptional representations were introduced and studied in [KP84], locally and globally. Locally, the exceptional representation $\theta_n$ of $G_n$ is the unique irreducible quotient of a principal series representation corresponding to a genuine character, which is a lift of $\delta^{1/4}_{B_n}|r_2^n$ to $C_{T_n}$. When $n$ is odd, there are $[F^* : F^{*2}]$ possible lifts of $\delta^{1/4}_{B_n}|r_2^n$ to $C_{T_n}$, resulting in non-isomorphic representations, but $\theta_{2k}$ is unique.

Any exceptional representation of $\tilde{G}_n$ can be obtained by twisting $\theta_n$ with a character of $F^*$ (pulled back to $G_n$ via det). Here for simplicity we take this character to be trivial.

Over a global field we denote this representation by $\Theta_n$, it has an automorphic realization as the residual representation of an Eisenstein series on $B_n(F) \backslash G_n(F)$. Abstractly, it is isomorphic to the restricted tensor product of local exceptional representations. For a detailed description of the properties of exceptional representations see [KP84, BG92, Kab01].

4.2. Local metaplectic Shalika model. Let $F$ be a $p$-adic field, $n = 2k$ and $\theta = \theta_n$. In [Kapa, Theorem 3.1] we proved that $\theta_{u_k,\psi}$ is the one-dimensional representation of $\tilde{G}_k^{\Delta}$ given by $\epsilon_\tau(c^\Delta) \to \epsilon_\tau\psi_\tau(-1)^k(\det c)$ ($\epsilon \in \mu_2$). In particular $\theta$ admits a unique metaplectic Shalika model $\mathcal{J}(\theta, \psi_\tau(-1)^k, \psi)$.

Assume all data are unramified as in § 2.1. Then $\theta$ is unramified. Let $\mathcal{J} \in \mathcal{J}(\theta, \psi_\tau(-1)^k, \psi)$ be the unramified normalized function.

Lemma 4.1. For any $\lambda \in \mathbb{Z}_\geq^k$, if $\lambda \in 2\mathbb{Z}_+^k$, $\mathcal{J}(s(t_\lambda)) = \delta_{B_n}^{1/4}(t_\lambda)$, otherwise $\mathcal{J}(s(t_\lambda)) = 0$.

Proof. In the notation of § 2.2, $\theta$ is a quotient of $I(\chi)$ where $\chi = \delta_{B_n}^{1/4}$. This character satisfies (3.12). Put $\psi' = \psi_\varphi$ with $\varphi = (-1)^k$. Let $l_\varphi$ be the metaplectic $(\psi', \psi)$-Shalika functional on $\theta$ such that $\mathcal{J}(g) = l_\varphi(\theta(g)\varphi_{K,\chi})$. It extends to a similar nonzero functional on $I(\chi)$.

Now let $l$ be the nonzero metaplectic $(\psi', \psi)$-Shalika functional on $I(\chi)$, whose existence was guaranteed by Corollary 3.31. By Theorem 3.9 the functionals $l_\varphi$ and $l$ are proportional, in particular $l$ is nonzero on $\varphi_{K,\chi}$, so that $l_\varphi = l(\varphi_{K,\chi})^{-1}l$ and

$$\mathcal{J}(g) = l(\varphi_{K,\chi})^{-1}l(g\varphi_{K,\chi}), \quad \forall g \in \tilde{G}_n.$$ 

Looking at the formula of Theorem 3.24 for $I(\chi)$ and $\varphi_{K,\chi}$, we see that none of the terms has a pole, and by definition for any $\alpha \in \Delta_{G_n}$ (a simple root), $\chi(a_\alpha) = q^{-1}$ so that $c_\alpha(\chi^{-1}) = 0$. Now consider the product of factors $y_\alpha(\chi)$. If $w_\alpha < 0$ for some short root $\alpha \in \Sigma_{Sp_k}$, $y_\alpha(\chi) = c_\alpha(\chi)c_\alpha(\chi^{-1}) = 0$. Otherwise if $w \neq e$, we must have $w_\alpha_k < 0$. But then since $\epsilon_{e,k} = (-(-1)^k, w)^{-1} = 1$, $y_k(\chi) = 0$.

Therefore all coefficients except the one for $w = e$ vanish and for $\lambda \in 2\mathbb{Z}_+^k$,

$$l(s(t_\lambda)\varphi_{K,\chi}) = Q^{-1}c_{w_0}(\chi)\delta_{B_n}^{1/2}\chi^{-1}(s(t_\lambda)) = Q^{-1}c_{w_0}(\chi)\delta_{B_n}^{1/4}(t_\lambda).$$

We conclude $l_\varphi = Qc_{w_0}(\chi)^{-1}l$ and the result immediately follows.

Remark 4.2. One could attempt to consider $\theta$ as a subrepresentation of $I(\chi^{-1})$. As such, it contains $\varphi_{K,\chi^{-1}}$ and the formula of Theorem 3.24 for $I(\chi^{-1})$, $\varphi_{K,\chi^{-1}}$ may be used. However, the functional $\Lambda_\chi$ vanishes on $\theta$ (see Remark 3.30). This also means that the metaplectic Shalika functional on $\theta$ does not extend to $I(\chi^{-1})$.

Over archimedean fields, when $F = \mathbb{R}$ the space of metaplectic Shalika functionals on $\theta$ is at most one-dimensional by Theorem 3.32. This remains true over $\mathbb{C}$, because then the cover
is split over $G_n$ and since $\theta$ is an irreducible admissible smooth Fréchet representation, the bound follows from [AGJ09]. We will prove that the space of metaplectic Shalika functionals on $\theta$ is precisely one-dimensional over both $\mathbb{R}$ and $\mathbb{C}$: existence of the functional will follow from a local–global argument in § 4.3 (Corollary 4.15).

### 4.3. Fourier coefficients of exceptional representations

Let $F$ be a global field and $\Theta = \Theta_n$. For an automorphic form $\varphi$ in the space of $\Theta$, a unipotent subgroup $U$, and a character $\psi$ of $U(\mathbb{A})$ which is trivial on $U(F)$, let $\varphi^{U,\psi}$ be the corresponding Fourier coefficient, given by

$$
\varphi^{U,\psi}(g) = \int_{U(F)\setminus U(\mathbb{A})} \varphi(ug)\psi^{-1}(u)\,du, \quad g \in \tilde{G}_n(\mathbb{A}).
$$

These coefficients are related to local twisted Jacquet modules at the finite places. More explicitly, if a Fourier coefficient does not vanish identically on $\Theta$, then for each place $v < \infty$ of $F$, $(\Theta_v)_{U,\psi} \neq 0$. Conversely if $(\Theta_v)_{U,\psi} = 0$ for some $v < \infty$, the coefficient (4.1) is zero for any $\varphi$ and $g$ (see e.g., [JR92, Proposition 1]).

Let $\theta = \Theta_v$ for some $v < \infty$. In [Kapa] we described the twisted Jacquet modules of $\theta$ along the unipotent subgroups $U_k$. We use these results to deduce several properties of the Fourier coefficients.

For $0 < k < n$, $M_k < G_n$ acts on the set of characters of $U_k$, with $\min(k, n-k) + 1$ orbits. We choose representatives

$$
\psi_j(u) = \psi(\sum_{i=1}^j u_{k-i+1,k+j-i+1}), \quad 0 \leq j \leq \min(k, n-k).
$$

(Here $u$ is regarded as an $n \times n$ matrix.) In particular $\psi_0$ is trivial and when $n = 2k$, $\psi_k$ is the Shalika character. The stabilizer of $\psi_j$ in $M_k$ is

$$
\text{St}_{n,k}(\psi_j) = \{\text{diag}(\begin{pmatrix} b & v \\ c & d \end{pmatrix}) : b \in G_{k-j}, c \in G_j, d \in G_{n-k-j}\}.
$$

Let $V_j$ and $Y_j$ be the unipotent subgroups of $\text{St}_{n,k}(\psi_j)$ defined by the coordinates of $v$ and $y$. Note that $V_0$ and $Y_0$ are trivial.

For any $0 \leq j \leq \min(k, n-k)$, the Jacquet module $\theta_{U_k,\psi_j}$ is a representation of $p^{-1}(\text{St}_{n,k}(\psi_j))$, which was computed in [Kapa, Theorem B]. In particular it was proved that $V_j(F_v)$ and $Y_j(F_v)$ act trivially on $\theta_{U_k,\psi_j}$. Therefore globally,

**Proposition 4.3.** For any $\varphi$ in the space of $\Theta$, $0 < k < n$ and $0 \leq j \leq \min(k, n-k)$, $\varphi^{U_k,\psi_j}$ is trivial on $V_j(\mathbb{A})$ and $Y_j(\mathbb{A})$.

The next theorem describes the constant term of automorphic forms in the space of $\Theta$, along any unipotent radical of a standard parabolic subgroup. The constant term defines an automorphic representation of the Levi part, and as such, it is often convenient to identify it with a tensor representation. Unfortunately, this is not simple to do for covering groups of $G_n$, because the preimages in the cover, of the direct factors of the Levi part, do not commute (see e.g., [BG92, Kab01, Tak14]). To avoid this problem, at least to some extent, we define exceptional representations of Levi subgroups. This approach was suggested by Bump and Ginzburg [BG92, § 1], and was perhaps implicit in [Kab01, § 5]. Let $\Delta \subset \Delta_{G_n}$ and $Q = Q_\Delta$ be the corresponding standard parabolic subgroup, $Q = M \ltimes U$. Let $B_M < M$ be the Borel subgroup such that $B_n = B_M \ltimes U$, $B_M = T_n \ltimes N_M$. Denote by $\delta_{B_M}$ the modulus
character of $B_M$, as a subgroup of $M$. Also let $W_M$ be the Weyl group of $M$ and $w_M$ its longest element.

Consider a local context first. A genuine character $\chi$ of $C_T$ is called $M$-exceptional if $\chi(\alpha) = |q|$ for all $\alpha \in \Delta$ ($\alpha$ was defined after (2.3)). The representation $\text{Ind}_{BM}^M(\delta_{BM}^{1/2} \chi')$, where $\chi'$ is an irreducible genuine representation of $T$ corresponding to $\chi$, has a unique irreducible quotient called an exceptional representation of $\tilde{M}$. This follows from the Langlands Quotient Theorem because $\chi$ belongs to the positive Weyl chamber in $W_M$. Note that this theorem was proved for covering groups over $p$-adic fields by Ban and Jantzen [BJ13], and over archimedean fields the proof of Borel and Wallach [BW80] is applicable to covering groups. The exceptional representation of $\tilde{M}$ is isomorphic to the image of the intertwining operator with respect to $w_M$ (over archimedean fields this follows from [BW80], over $p$-adic fields one can argue as in [KP84, Theorem I.2.9]).

Globally, we construct exceptional representations as residual representations of Eisenstein series. The discussion in [KP84, § II] (see also [BFG03, § 3]) can be extended to Levi subgroups. For $s = (s_1, \ldots, s_n) \in \mathbb{C}^n$, put

$$\chi_s(\text{diag}(t_1, \ldots, t_n)) = \prod_{i=1}^n |t_i|^{s_i}.$$ 

We extend $\chi_s$ to a right $K$-invariant function on $M(\mathbb{A})$ using the Iwasawa decomposition, and lift it to a function on the cover. Let $f$ be an element of $\text{Ind}_{BM, M(A)}^M(\delta_{BM}^{1/2} \chi')$, where $\chi'$ corresponds to a global $M$-exceptional character $\chi$ (defined similarly to the local one). Denote by $f_s$ the standard section given by $f_s(m) = \chi_s(m)f(m)$, $m \in \tilde{M}(\mathbb{A})$. Form the Eisenstein series

$$E_{BM}(m; f, s) = \sum_{m_0 \in B_M(F) \setminus M(F)} f_s(m_0m), \quad m \in \tilde{M}(\mathbb{A}).$$

Identify $\Sigma_{G_n}$ with the pairs $(i, j)$, $1 \leq i \neq j \leq n$, and $\Delta_{G_n}$ with the pairs $(i, i+1)$, $1 \leq i < n$. For $\alpha = (i, j) \in \Sigma_{G_n}$, put $s_\alpha = s_i - s_j$. The global exceptional representation is spanned by the functions

$$m \mapsto \text{Res}_{s = 0} E_{BM}(m; f, s) = \lim_{s \to 0} \prod_{\alpha \in \Delta} s_\alpha E_{BM}(m; f, s).$$

It is isomorphic to the restricted direct product of local exceptional representations. The proof is analogous to [KP84, Proposition II.1.2 and Theorems II.1.4 and II.2.1] and [BFG03, Proposition 3.1 and Theorem 3.2].

Let $\Theta_M$ (resp., $\theta_{M,v}$) be the global (resp., local over $F_v$) exceptional representation corresponding to a genuine character, which is a lift of $\delta_{BM}^{1/4}$ to $C_T$. Then $\Theta_M = \otimes_v \Theta_{M,v}$. In particular $\Theta_{G_n} = \Theta$. We state the result regarding the constant term of elements of $\Theta$.

**Theorem 4.4.** Let $\varphi$ belong to the space of $\Theta$. The function $m \mapsto \varphi^L(m)$ on $\tilde{M}(\mathbb{A})$ belongs to the space of $\delta_{Q}^{1/4} \Theta_M$.

**Remark 4.5.** The case $Q = B_n$ was proved in [KP84]; if $F$ is a function field, the theorem follows immediately from the computation of the Jacquet modules of $\theta$ over $p$-adic fields in [Kab01, Theorem 5.1]; for similar results concerning the small representation of $\text{SO}_{2n+1}$ and $\text{GSpin}_{2n+1}$ see [BFG03, Kap15, Kapb].
Remark 4.6. A priori, the constant term does not vanish identically, because it does not vanish for \(N_n\) (see [KP84, § II]).

Proof. As explained above and keeping the same notation (with \(M = G_n\)), we can assume
\[
\varphi(g) = \text{Res}_{s=0} E_{B_n}(g; \hat{f}, s),
\]
for some \(f\) in the space of \(\text{Ind}_{G_n}^{\hat{G}_n(\mathbb{A})} \delta_{B_n}^{1/2} \chi'\), where \(\chi'\) corresponds to a genuine lift of \(\delta_{B_n}^{1/4}|_{T_n}\).

Let \(W \subset W\) be with \(G_n = \prod_{w \in W} B_n w^{-1} Q\). For \(X < Q\) and \(w \in W\), set \(X^w = w B_n \cap X\). Define
\[
f_{2,w}(m) = M(w, \chi_{2}) f_{2}(m) = \int_{U^w(F) \setminus U(\mathbb{A})} f_{2}(w^{-1} um) du, \quad (4.2)
\]
\[
E_{M^w}(f_{2,w}(m)) = \sum_{m_0 \in M^w(F) \setminus M(F)} f_{2,w}(m_0 m). \quad (4.3)
\]
Here \(m \in \tilde{M}(\mathbb{A})\). According to Mœglin and Waldspurger [MW95, § II.1.7],
\[
\varphi^U(m) = \sum_{w \in W} \text{Res}_{s=0} E_{M^w}(f_{2,w}(m)). \quad (4.4)
\]
We can take the representatives \(W\) such that for all \(w \in W\),
\[
N_M < w N_n. \quad (4.5)
\]
Indeed this follows from [BZ77, 2.11] (applied to \(W_{T_n,M}^w\) in their notation, \(w^{-1}(M \cap B_n) < B_n\)).

Claim 4.7. \(f_{2,w}\) is holomorphic at \(s \to 0\), except for simple poles in the variables \(s_\alpha\) with \(\alpha \in \Delta_{G_n}\) such that \(w \alpha < 0\).

Claim 4.8. \(E_{M^w}(f_{2,w}(m))\) is holomorphic at \(s \to 0\), except for at most \(|\Delta_{G_n}|\) simple poles. There are less than \(\frac{1}{|\Delta_{G_n}|}\) poles unless \(w = w_M w_0\), in which case \(m \mapsto E_{M^w}(f_{2,w}(m))\) belongs to the space of \(\Theta_M\).

Granted the claims, the result follows: by Claim 4.8, any summand with \(w \neq w_M w_0\) vanishes when we take the residue, and we are left with \(E_{M^w M^{-w_0}}(f_{2,w_M w_0}(m))\). Since at most one summand is nonzero, this summand is nonzero for some data (see Remark 4.6). \(\square\)

Proof of Claim 4.7. Since \(f\) is left-invariant under \(N_n(\mathbb{A})\) and \(U^w < N_n\), we may rewrite the integration in (4.2) over \(U^w(\mathbb{A}) \setminus U(\mathbb{A})\). Moreover (4.5) implies \(N_M U^w = N_{w}^w\), hence \(U^w \setminus U = N_n \setminus N_n\) so that we can write the integral in (4.2) over \(N_{w}^w(\mathbb{A}) \setminus N_n(\mathbb{A})\). Now we proceed as in [Kapb, Claim 3.5]. We can assume that \(f\) is a pure tensor. At any finite place, the local intertwining operator is holomorphic for \(s\) in a small neighborhood of 0, because then the local component of \(\chi' \chi_2\) belongs to the positive Weyl chamber. Thus the poles are global, in the sense that they appear in the product of local intertwining operators applied to the unramified normalized components of \(f\). Now we use the Gindikin-Karpelevich formula (2.4) to identify the possible poles.

Let \(\alpha \in \Sigma_{G_n}^+\) and assume \(w \alpha < 0\). Recall the function \(c_\alpha(\cdots)\) (defined before (2.4)). The poles occur in the products
\[
\prod_v c_\alpha((\chi' \chi_2)_v) = \frac{\zeta(C_\alpha + 2s_\alpha)}{\zeta(C_\alpha + 2s_\alpha + 1)},
\]
where \( \zeta \) is the partial Dedekind zeta function of \( F \) and \( 0 < C_\alpha \in \mathbb{Z} \) is an integer depending on \( \chi' \). As a function of \( s_\alpha \), there is a pole when \( s_\alpha \to 0 \) if and only if \( C_\alpha = 1 \), this occurs precisely when \( \alpha \in \Delta_{G_n} \). We also deduce that the poles are simple. \( \square \)

**Proof of Claim 4.8.** The poles of \( E_{M^w}(f_{s,M}(m)) \) are either poles of \( f_{s,M}(m) \), which were identified in Claim 4.7, or poles incurred by the summation over \( M^w \setminus M \). Because \( U^w \setminus U = N_n \setminus N_n \), \( f_{s,M} \) is an element in the space of

\[
\text{Ind}_{B_n(A)}^G \delta^{1/2}_{B_n}(\chi' \chi_2).
\]

Then the restriction of \( f_{s,M} \) to \( \widetilde{M}(A) \) belongs to

\[
\delta^{1/2}_Q \text{Ind}_{B_M(A)}^M \delta^{1/2}_{B_M}(\chi' \chi_2).
\]

Since \( N_M < \setminus B_n \cap M \) (by (4.5)), \( B_M = \setminus B_n \cap M \) \((w \) cannot conjugate more positive roots from \( N_n \) into \( M \), because \( N_M \) is a maximal unipotent subgroup in \( M \). Hence (4.3) is a series over \( B_M \setminus M \). To locate its poles we consider the constant term along \( N_M \). Arguing as in [KP84, Proposition II.1.2], the constant term is a sum of intertwining operators \( \text{Ind}(\omega, \chi_2) f_{s,M} \), where \( \omega \) varies over \( W_M \).

Fix \( \omega \) and let \( \Sigma_M \) (resp., \( \Sigma_M^+ \)) be the set of roots (resp., positive roots) spanned by the roots in \( \Delta \). Assumption (4.5) implies that \( \setminus \chi_2 \) belongs to the positive Weyl chamber of \( W_M \) (when \( s \to 0 \)). Therefore, as in the proof of Claim 4.7 any pole must be located in a quotient of \( \zeta \) functions

\[
\prod_v C_{w,\alpha}(\chi' \chi_v) = \frac{\zeta(C_{w,\alpha} + 2s_{w^{-1} \alpha})(\chi' \chi_v)}{\zeta(C_{w,\alpha} + 2s_{w^{-1} \alpha})}.
\]

Here \( \alpha \in \Sigma_M^+ \) and \( \omega \alpha < 0 \); \( 0 < C_{w,\alpha} \in \mathbb{Z} \) depends on \( \omega \chi' \), specifically \( \omega \chi_v = q^{-C_{w,\alpha}} \). The quotient has a pole if and only if \( C_{w,\alpha} = 1 \). Since \( C_\alpha = 1 \) if and only if \( \alpha \in \Delta_{G_n} \),

\[
\{ \alpha \in \Sigma_M^+ : C_{w,\alpha} = 1 \} = \{ \alpha \in \Delta_{G_n} : w \alpha \in \Sigma_M^+ \}.
\]

Thus the number of poles of the series is bounded by the size of (4.7). Since the conditions \( w \alpha \in \Sigma_M^+ \) and \( w \alpha < 0 \) are disjoint, the number of poles of \( E_{M^w}(f_{s,M}(m)) \) is at most \( |\Delta_{G_n}| \). The poles of \( f_{s,M} \) are simple, they occur in the variables \( s_\beta \) where \( \beta > 0 \) and \( w \beta < 0 \). A pole of (4.6) is also simple, it occurs in \( s_{w^{-1} \alpha} \) with \( \alpha > 0 \), and because \( w(w^{-1} \alpha) = \alpha > 0 \), this pole does not appear in the set of poles of \( f_{s,M} \). Thus the poles of \( E_{M^w}(f_{s,M}(m)) \) are simple.

Assuming \( |\Delta_{G_n}| \) poles are obtained, we prove \( w = w_M w_0 \). In this case we must have

\[
\Delta_{G_n} = \{ \alpha \in \Delta_{G_n} : w \alpha < 0 \} \cap \{ \alpha \in \Delta_{G_n} : w \alpha \in \Sigma_M^+ \},
\]

and then (4.5) immediately implies \( w = w_M w_0 \). Since in this case \( w \delta_{B_n} = \delta_{B_M}^{-1} \), the restriction of \( f_{s,M} \) to \( \widetilde{M}(A) \) belongs to \( \delta^{1/4}_Q \text{Ind}_{B_M(A)}^M \delta^{1/2}_{B_M}(\chi' \chi_2) \), where \( \chi'' \) corresponds to a lift of \( \delta^{1/4}_{B_M} T_n^{2} \), and the definition of \( \Theta_M \) implies that \( m \mapsto \text{Res}_{s=0} E_{M^w}(f_{s,M}(m)) \) belongs to the space of \( \delta^{1/4}_Q \Theta_M \). \( \square \)

**Remark 4.9.** Takeda [Tak14, Proposition 2.45] computed the constant term along the unipotent radical corresponding to the partition \((2, \ldots, 2)\), for the twisted exceptional representation (his proof does not apply to the non-twisted version, i.e., to our setting).
Now we describe the Fourier coefficient along the Shalika unipotent subgroup and character. Assume \( n = 2k \). The stabilizer of \( \psi_k \) in \( M_k \) is now \( G_k^\Delta \).

**Proposition 4.10.** The section \( s \) is well defined on \( G_k^\Delta(\mathbb{A}) \) and satisfies a global relation similar to (3.1), with the Hilbert symbol replaced by the product of local Hilbert symbols at all places.

**Proof.** If \( g \in G_k^\Delta(\mathbb{A}) \), for almost all \( v \), \( g_v \in G_k^\Delta(\mathcal{O}_v) \) and then by Proposition 3.15, \( s_v(g_v) = \kappa_v(g_v) \). Hence \( s \) is well-defined. The global relation follows from the local one, since \( s = \prod_v s_v \).

Henceforth we identify \( G_k^\Delta(\mathbb{A}) \) with its image in \( \tilde{G}_n(\mathbb{A}) \) under \( s \).

Let \( \pi \) be any automorphic representation of \( \tilde{G}_n(\mathbb{A}) \). We say that \( l \) is a global metaplectic Shalika functional on \( \pi \), if \( l \) satisfies the global analog of (3.2), where \( c \in G_k(\mathbb{A}) \), \( u \in U_k(\mathbb{A}) \) and \( \gamma_\psi \) is now the global Weil factor. As in the case of the Whittaker model, we are interested in concrete realizations of such functionals, e.g., given as integrals. A global integral for the Shalika model of \( G_n \) was studied in [JS90, FJ93], involving an integration over \( G_k^\Delta(\mathbb{A}) \times U_k(\mathbb{A}) \). For exceptional representations, we will show that \( \varphi^{U_k,\psi_k} \) already satisfies the required properties, i.e., the reductive part of the integration is not needed.

To study the properties of \( \varphi^{U_k,\psi_k} \), we first relate it to the Fourier coefficient of \( \Theta \) along \( N_n \) and its degenerate character \( \psi^\circ \) given by \( \psi^\circ(v) = \sum_{i=1}^{k} \psi((-1)^{k-i}v_{2i-1,2i}) \) (called a semi-Whittaker coefficient in [BG92]).

Define \( w \in G_n(F) \) as follows. For all \( 1 \leq i \leq k \), it has 1 on the \((2i-1,i)\)-th coordinate and \((-1)^{k-i}\) on the \((2i,k+i)\)-th coordinate. Its other entries are zero. Let \( V < U_k \) be the subgroup obtained from \( U_k \) by zeroing all upper diagonal entries except those on rows \( i+1, \ldots, k \) of column \( k+i \), for all \( 1 \leq i \leq k-1 \).

**Example 4.11.** When \( k = 2 \),

\[
w = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad V = \left\{ \begin{pmatrix} 1 & v_1 \\ 1 & 1 \end{pmatrix} \right\}.
\]

**Lemma 4.12.** For any \( \varphi \) in the space of \( \Theta \),

\[
\varphi^{U_k,\psi_k}(g) = \int_{V(\mathbb{A})} \varphi^{N_n,\psi^\circ}(wvg) \, dv.
\]

**Proof.** The local analog of this result was proved in [Kapa, Theorem 3.1], using a local “exchange of roots” and the fact that \( \theta_n \) does not have a Whittaker model for \( n \geq 3 \) (see [KP84, BG92, Kab01, Kapa]). The proof of the global case is analogous, and follows by a repeated application of global exchange of roots [GRS11, Lemma 7.1] (see also [Gin90, GRS01, Sou05]).

**Theorem 4.13.** The mapping \( \varphi \mapsto \varphi^{U_k,\psi_k} \) on the space of \( \Theta(= \Theta_{2k}) \) is nonzero and satisfies

\[
\varphi^{U_k,\psi_k}(c^\Delta) = \gamma_{\psi,(-1)^{k}(\det c)} \varphi^{U_k,\psi_k}(1), \quad \forall c \in G_k(\mathbb{A}).
\]

Consequently, \( \Theta \) admits a global metaplectic Shalika model given by a Fourier coefficient. Moreover, the functional \( \varphi \mapsto \varphi^{U_k,\psi_k}(1) \) is factorizable.
Proof. We begin with the proof of the formula, the non triviality will be deduced during the computation. First we show that for \( c \in \text{SL}_k(\mathbb{A}) \), \( \varphi^{U_k,\psi_k}(c^\Delta) = \varphi^{U_k,\psi_k}(1) \). Since \( \varphi \) is invariant on the left by \( G_n(F) \), it is enough to consider the Fourier expansion along the abelian subgroup of matrices \( c^\Delta \), where \( c \) is a unipotent matrix in \( N_k \) with only one nonzero entry above the diagonal - the top right coordinate, which varies over \( \mathbb{A} \). But the nontrivial coefficients vanish by [Kapa, Theorem 3.1].

It remains to consider \( c = t = \text{diag}(t_1, \ldots, t_k) \in T_k \). We compute using Lemma 4.12. The conjugation of \( V \) by \( t \) multiplies the measure by \( \delta_{B_k}^{-1}(\text{diag}(t_1, \ldots, t_k)) \). To compute \( u_t \) we use [BLS99, § 2 Lemma 2, § 3 Lemmas 3 and 1],

\[
\phi(t_1, t_1, \ldots, t_k) = \prod_{i=1}^{k-1} (t_i, t_i)_{2i} \prod_{i=1}^{k} (t_i, t_i)_{2i} \phi(1).
\]

Then we claim:


\[
\varphi^{N_n,\psi}(\text{diag}(t_1, t_1, \ldots, t_k, t_k)) = \prod_{i=1}^{k} \gamma_{\psi,(-1)^{k+1-i}}(t_i) \delta_{B_k}(\text{diag}(t_1, \ldots, t_k)) \varphi(1).
\]

This result, equality (4.8) and a quick computation using (1.3), yield the formula.

As a corollary, \( \varphi \mapsto \varphi^{U_k,\psi_k} \) is nontrivial on \( \Theta \). Indeed, the arguments in Lemma 4.12 can be repeated in “the opposite direction” to relate \( \varphi^{N_n,\psi} \) to an integration of \( \varphi^{U_k,\psi_k} \). The former is nonzero by Claim 4.14, hence \( \varphi^{U_k,\psi_k} \) cannot be zero for all \( \varphi \). The assertion regarding factorizability follows from the local uniqueness (dimension at most one) of the metaplectic Shalika model at all the local components of \( \Theta \) (see § 4.2). \( \square \)

Proof of Claim 4.14. Let \( Q = M \ltimes U < G_n \) be the standard parabolic subgroup, whose Levi part \( M \) is isomorphic to \( k \) copies of \( G_2 \). According to Theorem 4.4, the mapping \( m \mapsto \varphi^U(m) \) belongs to the space of \( \delta_{Q}^{1/4} \Theta_M \). The representation \( \Theta_M \) is isomorphic to the global Weil representation \( II \otimes \tilde{\varphi}^{-1} \) constructed by Takeda [Tak14, § 2.3 and p. 204] (this is \( II \otimes \tilde{\varphi}_p^{-1} \) in his notation with the unitary character \( \chi = 1 \), see [Tak14, 2.26]). The mapping \( \tilde{\varphi} \) was defined in [Tak14, § A.1, p. 261] to correct a global block-compatibility issue. Therefore \( \varphi^{N_n,\psi} \) is the application of a Fourier coefficient corresponding to \( (N_M, \psi^\otimes|_{N_M}) \), to an automorphic form in the space of \( \delta_{Q}^{1/4} (II \otimes \tilde{\varphi}^{-1}) \). The representation \( II \) is a metaplectic tensor of \( k \) copies of the global Weil representation of \( \tilde{G}_2(\mathbb{A}) \), and by definition, the Fourier coefficient is a Whittaker functional applied to each block. The action of \( s(\text{diag}(t_i, t_i)) \) on the \( i \)-th copy is given by \( \gamma_{\psi,(-1)^{k+1-i}}(t_i) \). \( \square \)

Corollary 4.15. Let \( \theta(= \theta_{2k}) \) be an exceptional representation over an archimedean field (in fact, over any local field). Then \( \theta \) admits a metaplectic \( (\psi_{(-1)^k}, \psi) \)-Shalika functional.

Proof. Apply Theorem 4.13 and a standard globalization argument. \( \square \)

5. A Godement–Jacquet Type Integral

5.1. Local theory. Let \( F \) be a local field and \( \pi \in \text{Alg} G_k \) be irreducible. Set \( n = 2k \) and \( \theta = \theta_n \) (\( \theta_n \) was defined in § 4.1). Regard \( G_k \) as a subgroup of \( G_n \) via \( g \mapsto \text{diag}(g, I_k) \).
Lemma 3.23 (1), for any $k$ the genuine. Henceforth we omit it from the notation. Theorem 5.1. Assume $F$ is a $p$-adic field.

(1) The integral (5.1) is absolutely convergent for $\text{Re}(s) \gg 0$, the domain of convergence depends only on the representations.

(2) One can choose data $(f, \mathcal{I}, \mathcal{I}')$ such that $Z(f, \mathcal{I}, \mathcal{I}', s)$ is absolutely convergent and equals 1, for all $s$.

(3) The integral has a meromorphic continuation to a rational function in $\mathbb{C}(q^{-s})$.

(4) When all data are unramified,

$$Z(f, \mathcal{I}, \mathcal{I}', s) = \frac{L(2s, \text{Sym}^2, \pi)}{L(2s + 1, \Lambda^2, \pi)}.$$  

(5) Assume that $\pi$ satisfies the following condition: if $\varepsilon$ is a normalized exponent of $\pi$ along a standard parabolic subgroup $Q$ (see § 1.5 for the definition), then $\delta^{1/2}_{Q_k} |\varepsilon|$ lies in the open cone spanned by the positive roots in the Levi part of $Q$. In particular, this condition holds when $\pi$ is tempered. Then $Z(f, \mathcal{I}, \mathcal{I}', s)$ is holomorphic at $\text{Re}(s) = 1/2$.

Proof. The first three assertions have already been proved in [Kapa], in a slightly different form. By Lemma 3.2 the integral is bounded by the zeta integral of Godement and Jacquet $\int_{G} f(g) \phi(g) |\det(g)|^r \, dg$, where $0 \leq \phi \in \mathcal{S}(F_{kxk})$ and $r \in \mathbb{R}$ depends only on $\theta$, which is absolutely convergent for $r > r_0$, where $r_0$ is independent of $f$ ([GJ72, p. 30]).

Applying Lemma 3.3 to $\theta$, one can choose elements $\mathcal{I}$ and $\mathcal{I}'$ such that the mapping $g \mapsto \mathcal{I}(g) \mathcal{I}'(g)$ is an arbitrary function in $\text{ind}_{G^\kappa_k U_k}^G \text{Sym}^2 \mathcal{O}_k \cong C'_c(G_k)$. This implies (2), when we take some $f$ which does not vanish at the identity.

Meromorphic continuation: in its domain of absolute convergence (5.1) satisfies

$$Z(mf, mu \mathcal{I}, mu \mathcal{I}', s) = \delta^{1/2-s/k}_{Q_k}(m) Z(f, \mathcal{I}, \mathcal{I}', s).$$

Indeed, for any $g \in G_k, m = \text{diag}(a, b) \in M_k$ and $u \in U_k$,\n
$$(\mathcal{I} \mathcal{I}')(gm) = \gamma_{\psi}(\det b) \gamma_{\psi}^{-1}(\det b)(\mathcal{I} \mathcal{I}')(b^{-1}ga) = (\mathcal{I} \mathcal{I}')(b^{-1}ga),$$
\n$$(\mathcal{I} \mathcal{I}')(gu) = \psi(gu) \gamma_{\psi}^{-1}(gu)(\mathcal{I} \mathcal{I}')(g) = (\mathcal{I} \mathcal{I}')(g).$$

Therefore (5.1) is a nonzero element of

$$\text{Hom}_{M_k}((\theta \otimes \theta')_{U_k}, \delta^{1/2}_{Q_k} |\det|^{-s} \pi^\kappa \otimes |\det|^{s} \pi).$$

In [Kapa, Claim 4.7] we proved that this space is at most one-dimensional, outside of a finite set of values of $q^{-s}$. Combined with the first two assertions, this implies (3), by virtue of Bernstein’s continuation principle (in [Ban98]).

Assume all data are unramified. We prove (4). The function $f$ is $\text{bi-}G_k(\mathcal{O})$-invariant and by Lemma 3.23 (1), for any $k_1, k_2 \in G_k(\mathcal{O})$, $(\mathcal{I} \mathcal{I}')(k_1 g k_2) = (\mathcal{I} \mathcal{I}')(g)$. Note that on its own, $\mathcal{I}$ is not $\text{bi-}\kappa(G_k(\mathcal{O}))$-invariant because elements of $\kappa(\text{diag}(I_k, G_k(\mathcal{O})))$ and $\mathfrak{s}(\text{diag}(G_k, I_k))$ do not commute in $\tilde{G}_n$. Nonetheless, the integrand is $\text{bi-}G_k(\mathcal{O})$-invariant. Hence we can
write (5.1) using the Cartan decomposition $\prod_{\lambda \in \mathbb{Z}_+^k} G_k(\mathcal{O}) \varpi^\lambda G_k(\mathcal{O})$ and Lemma 4.1 implies that the integrand vanishes unless $\lambda \in 2\mathbb{Z}_+^k$ (see § 3.3 for the notation).

Let $t_\pi \in G_k(\mathbb{C})$ be the Satake parameters of $\pi$. For $\lambda \in \mathbb{Z}_+^k$, the formulas of Macdonald [Mac95, Chapter V § 2 (2.9) and § 3 (3.4)] $(f(\varpi^\lambda) \text{ is } \omega_s(\pi^{-\lambda}) \text{ in his notation})$ show

$$f(\varpi^\lambda) = \text{vol}(G_k(\mathcal{O}) \varpi^\lambda G_k(\mathcal{O}))^{-1} \delta_{B_k}^{-1/2}(\varpi^\lambda) P_\lambda(t_\pi; q^{-1}).$$

Here $P_\lambda$ is the Hall–Littlewood polynomial in the Satake parameters, see [Mac95, Chapter III § 2] for the definition. Using Lemma 4.1 and $\delta_{B_k}^{-1/2}(t_\lambda) | \det \varpi^\lambda|^{-k/2} = \delta_{B_k}^{-1/2}(\varpi^\lambda)$, the integral becomes

$$\sum_{\lambda \in 2\mathbb{Z}_+^k} P_\lambda(t_\pi; q^{-1}) q^{-|\lambda|s}, \quad |\lambda| = \lambda_1 + \ldots + \lambda_k.$$

Since $P_\lambda(t_\pi; q^{-1}) q^{-|\lambda|s} = P_\lambda(t_{|\det|}; q^{-1})$, the required formula follows immediately from the formal identity of [Mac95, Chapter III § 5, example 2]:

$$\sum_{\lambda \in 2\mathbb{Z}_+^k} P_\lambda(x_1, \ldots, x_k; r) = \prod_{i<j} (1 - r x_i x_j),$$

Finally consider (5). Assume $\pi$ satisfies the assumption on the exponents. We prove that the integral is absolutely convergent, uniformly on compact sets, if $r = \text{Re}(s) \geq 1/2 - \epsilon_0$, for some $\epsilon_0 > 0$ (depending on the exponents of $\pi$). Let $\Delta \subset \Delta_{G_k}$ and $P = P_\Delta$ be the corresponding standard parabolic subgroup, $P = M \ltimes U$. Let $\zeta > 0$. Let $C_M^+$ be the set of $\varpi^\lambda \in C_M$ such that $|\alpha(t)| < \zeta$ for all $\alpha \in \Delta_{G_k} - \Delta$. Let $K$ be a principal congruence subgroup on which the integrand is bi-$\mathcal{K}$-invariant. For the proof of convergence, we may replace $K$ in the Cartan decomposition with $\mathcal{K}$, and we see that it is enough to prove the convergence of the following sum, for each $\Delta$,

$$\sum_{\varpi^\lambda \in C_M^+} |f(\varpi^\lambda) \varpi^\lambda \delta_{B_k}^{-1}|(\varpi^\lambda) q^{-|\lambda|(r-k/2)}.$$

Here the modulus character $\delta_{B_k}^{-1}$ is bounding $\text{vol}(\mathcal{K} \varpi^\lambda \mathcal{K})$.

This sum can be bounded using the exponents of the representations. Let $P^\lambda$ be the standard parabolic subgroup of $G_n$, whose Levi part is isomorphic to $M \times G_k$. According to [JR96, Lemma 6.2], if $\zeta$ is sufficiently small with respect to $\varphi$, $\mathcal{S}$ is bounded on $C_M^+$ by the exponents of $\theta$ along $P^\lambda$. Kable proved that the normalized Jacquet module of $\theta$ along the unipotent radical of $P^\lambda$ is an irreducible representation with a central character $\omega$ such that $|\omega| = \delta_{P^\lambda}^{-1/4}$ [Kab01, Theorem 5.1]. The normalized exponent of $\theta$ along $P^\lambda$ is thus $\delta_{P^\lambda}^{-1/4}$. Since on $C_M$, $|\det|^{-k/4} \delta_{B_k}^{-1/4} = \delta_{P^\lambda}^{-1/4}$ (on the right-hand side $C_M$ is embedded in $M$), we obtain (up to a constant) $\sum_{\varpi^\lambda \in C_M^+} |\varpi^\lambda| \delta_{B_k}^{-1/2} f(\varpi^\lambda) q^{-|\lambda|r}$. This is bounded in the proper right half-plane, given our assumption on $\pi$. \hfill \Box

Remark 5.2. The one-dimensionality result of [Kapa, Claim 4.7] for (5.2) is the usual property underlying the definition of local $\gamma$-factors for Rankin–Selberg integrals.

Theorem 5.3. Assume $F$ is an archimedean field.

1. The integral (5.1) is absolutely convergent for $\text{Re}(s) \gg 0$, the domain of convergence depends only on the representations.
For each $r > 0$ one can choose data $(f, \mathcal{I}, \mathcal{I}')$ such that $Z(f, \mathcal{I}, \mathcal{I}', s)$ is absolutely convergent and nonzero for $s$ in the strip $\{ s \in \mathbb{C} : |\text{Im}(s)| \leq r \}$.

(3) For fixed $(f, \mathcal{I}, \mathcal{I}')$ the function $Z(f, \mathcal{I}, \mathcal{I}', s)$ has a meromorphic continuation.

Proof. As in the $p$-adic case, by Lemma 3.2 the convergence is reduced to the convergence of a Godement–Jacquet type integral, which was proved in [GJ72, Theorem 8.7].

Consider (2). Let $f$ be a matrix coefficient with $f(I_k) = 1$. We choose a compact neighborhood $\mathcal{V}$ of the identity in $G_k$, such that both $f$ and $\det |^{s-k/2}$ take values in $\{ z \in \mathbb{C} : \arg(z) < \pi/8 \}$, on $\mathcal{V}$ and whenever $|\text{Im}(s)| \leq r$. Imitating the argument used in the proof of Lemma 3.3, one can choose $\mathcal{I}$ and $\mathcal{I}'$ such that $\mathcal{I}'(g)$ is supported in $\mathcal{V}$, equals 1 at the identity, and takes values in $\{ \arg(z) < \pi/4 \}$. With these choices the integral converges and is nonzero when $|\text{Im}(s)| \leq r$.

Now we prove (3). By virtue of the Dixmier–Malliavin Lemma [DM78] (cf. the proof of Lemma 3.2) it is enough to prove a meromorphic continuation for an integral of the form

$$\int_{G_k} f(g) \mathcal{I}'(g) \phi(g) |\det(g)|^{s-k/2} dg, \quad \phi \in \mathcal{S}(F_{k \times k}).$$

We use the integration formula

$$\int_{G_k} \phi(g) dg = \int_K \int_K \int_{\mathbb{R}^{k-1}} \phi(k_1 a_x k_2) J(x) d(x_1, \ldots, x_{k-1}) dx_k dk_1 dk_2,$$

where $K$ is the standard maximal compact subgroup of $G_k$ and for $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$,

$$a_x = \text{diag}(e^{-(x_1 + \cdots + x_k)}, e^{-(x_2 + \cdots + x_k)}, \ldots, e^{-x_k}), \quad J(x) = \prod_{1 \leq i < j \leq k} \sinh (x_i + \cdots + x_j)$$

(if $F = \mathbb{C}$, $J(x)$ is replaced by $J(x)^2$). Since $J(x)$ is a finite sum of terms of the form $e^{p \cdot x'}$ with $p \in \mathbb{Z}^{k-1}$ and $x' = (x_1, \ldots, x_{k-1})$, it suffices to show that every integral of the form

$$\int_K \int_K \int_{\mathbb{R}^{k-1}} f(\mathcal{I}'(k_1 a_x k_2) e^{-(s_1 + s_2 + \cdots s_k)} e^{-(2s_1 + 2s_2 + \cdots 2s_k)} d(x))$$

with $s_1, \ldots, s_k \in \mathbb{R}$ (coming from the exponent $k/2$ in $|\det(g)|^{s-k/2}$ and from $e^{p \cdot x'}$) converges absolutely for $\text{Re}(s) \gg 0$ and extends to a meromorphic function in $s \in \mathbb{C}$.

Note that $f(g) = \xi^\vee(\pi(g) \xi)$ for some $\xi \in V$ and $\xi^\vee \in V^\vee$, where $V$ is the space of $\pi$, hence

$$f(k_1 a_x k_2) = \pi^\vee(k_1^{-1}) \xi^\vee(\pi(a_x) \pi(k_2) \xi).$$

This is a smooth function in $k_1$ and $k_2$. Hence Theorem 1.1 can be applied to $\pi^\vee(k_1^{-1}) \xi^\vee$ and $\pi(k_2) \xi$. Furthermore $\mathcal{I}'$ is of the form $\mathcal{I}'(g) = l(\theta(g) \varphi)$ with $\varphi$ in the space of $\theta$ and a metaplectic Shalika functional $l$ (similarly, $\mathcal{I}'(g) = l'(\theta(g) \varphi')$). Then (3.2) implies

$$\mathcal{I}'(k_1 a_x k_2) = \mathcal{I}'(a_x \text{diag}(k_2, k_1^{-1})),$$

and we can apply Theorem 3.5 with $\theta(g(\text{diag}(k_2, k_1^{-1}))) \varphi$ instead of $\varphi$ (similarly for $\varphi'$).

We split the $dx_k$-integration in (5.3) into $x_k > 0$ and $x_k \leq 0$. First assume $x_k > 0$. By the above considerations and Theorems 1.1 and 3.5, $f(\mathcal{I}'(k_1 a_x k_2)$ can be written as a finite sum of terms of the form

$$c(x_7; k_1, k_2)x_7^\alpha e^{-x_7}$$
where \( I \subset \{1, \ldots, k\}, \alpha \in \mathbb{N}^I, z \in \mathbb{C}^I \) with \( \text{Re} \; z_j \geq \Lambda_j = \Lambda_{\pi,j} + \Lambda_{\theta,j} + \Lambda_{\vartheta,j} \) for all \( j \in I \) (see (1.4) for the definition of \( \Lambda_{\pi,j} \)), and \( c(x_T; k_1, k_2) \) is smooth in \( x_T, k_1 \) and \( k_2 \), and satisfies
\[
|c(x_T; k_1, k_2)| \leq Ce^{-D_T x_T}, \quad \forall x_T \in \mathbb{T}^7_{>0}, k_1, k_2 \in K.
\]
Here the numbers \( D_j \in \mathbb{R} \) can be chosen arbitrarily large. The resulting integral is
\[
\int_K \int_K \int_{\mathbb{R}^k_{>0}} c(x_T; k_1, k_2)x_1^\alpha e^{-z_T x_T} \phi(k_1 a_x k_2) \left( \prod_{j=1}^k e^{-(j s_j + s_j) x_j} \right) d(\cdots).
\]
Substitute \( y_j = e^{-x_j} \) and obtain
\[
\int_K \int_K \int_0^1 \cdots \int_0^1 c'(y_T; k_1, k_2) \left( \prod_{j \in I} y_j^{s_j + s_j + z_j} \log^{a_j} y_j \right) \left( \prod_{j \not\in I} y_j^{s_j} \right) \phi(k_1 b_y k_2) d(\cdots),
\]
where \( b_y = \text{diag}(y_1 \cdots y_k, y_2 \cdots y_k, \ldots, y_k) \) and \( c'(e^{-x_T}; k_1, k_2) = c(x_T; k_1, k_2) \) satisfies
\[
|c'(y_T; k_1, k_2)| \leq Cy_T^{D_T}, \quad \forall y_T \in (0, 1)^7, k_1, k_2 \in K.
\]
The function \( \phi(k_1 b_y k_2) \) together with all its \( y \)-derivatives is bounded for all \( k_1, k_2 \in K \) and \( y_1, \ldots, y_k \in (0, 1) \). Hence, the integral
\[
B_c(y) = \int_K \int_K c'(y_T; k_1, k_2) \phi(k_1 b_y k_2) dk_1 dk_2
\]
converges absolutely and defines a smooth function in \( y_1, \ldots, y_k \in (0, 1) \) such that for all differential operators \( D \) in \( y \) with constant coefficients we have \( |DB_c(y)| \leq C_D y_T^{M_T}, \) for all \( y \in (0, 1)^k \). This implies that the integral
\[
\int_0^1 \cdots \int_0^1 B_c(y) \left( \prod_{j \in I} y_j^{s_j + s_j + z_j} \log^{a_j} y_j \right) \left( \prod_{j \not\in I} y_j^{s_j} \right) dy_1 \cdots dy_k
\]
converges absolutely whenever \( \text{Re} \; s > -\frac{\text{Re} \Lambda_j + s_j + 1}{j} \geq -\frac{\text{Re} \; z_j + s_j + 1}{j} \) for \( j \in I \) and \( \text{Re} \; s > -\frac{D_j + s_j + 1}{j} \) for \( j \not\in I \). Writing \( (s+1)y^s = d \frac{dy}{y^{s+1}} \) and integrating by parts, we see that the integral extends to a meromorphic function when \( \text{Re} \; s > -\frac{D_j + s_j + 1}{j} \) for \( j \not\in I \) (in this range the integrals over \( y_T \) always converge absolutely). Since \( D_j \) can be chosen arbitrarily large, this shows the meromorphic extension.

Now assume \( x_k \leq 0 \). Let \( x' = (x_1, \ldots, x_{k-1}, 0) \) with \( x_1, \ldots, x_{k-1} \geq 0 \) and \( x_k \leq 0 \). By Theorems 1.1 and 3.5 and Remark 3.6 the product \( f \mathcal{F} \mathcal{F}'(k_1 a_x k_2) \) can be written as a finite sum of terms of the form \( c(x_T', x_k; k_1, k_2)(x_j')^{\alpha} e^{-z' x_j} \), where \( J \subset \{1, \ldots, k-1\}, \mathcal{T} = \{1, \ldots, k-1\} - J, \alpha \in \mathbb{N}^J, z' \in \mathbb{C}^J \) with \( \text{Re} \; z'_j \geq \Lambda_j = \Lambda_{\pi,j} + \Lambda_{\theta,j} + \Lambda_{\vartheta,j} \) for all \( j \in J \), and \( c(x_T', x_k; k_1, k_2) \) is smooth in \( x_T', x_k, x_1 \) and \( k_2 \), and satisfies
\[
|c(x_T', x_k; k_1, k_2)| \leq Ce^{-D_T x_T' x_k} e^{-N x_k}, \quad \forall x_T' \in \mathbb{T}^7_{>0}, x_k \leq 0, k_1, k_2 \in K.
\]
We take \( D_j' \gg 0 \), and \( N \in \mathbb{R} \) depends on \( D_j' \). The resulting integral is
\[
\int_K \int_K \int_{-\infty}^0 \int_{\mathbb{R}^k_{>0}} c(x_T', x_k; k_1, k_2)(x_j')^{\alpha} e^{-z' x_j} \phi(k_1 a_x k_2) \left( \prod_{j=1}^k e^{-(j s_j + s_j) x_j} \right) d(\cdots).
\]
As above, the integrals over \( K \) converge absolutely and define a smooth function in \( y_1, \ldots, y_{k-1} \in (0, 1), y_k \in (1, \infty) \). Since \( \phi \in \mathcal{S}(F_k \times k) \), \( \phi(k_1 a_x k_2) \) together with all its \( x' \)-derivatives is
bounded above by a constant times $e^{Lx_k}$ with $L \in \mathbb{R}$ arbitrarily large. Then the integral converges absolutely whenever $\Re s \geq \frac{-\Re(\pi_j + \pi_j)}{j}$ for $j \in J$, $\Re s > \frac{-D_j + s_j}{j}$ for $j \in J'$ and $\Re s < \frac{L - N - s_k - 1}{k}$. Since $L$ is arbitrarily large the last condition is superfluous. \hfill \Box

5.2. Global theory. Let $F$ be a global field. Let $\pi$ be a cuspidal (irreducible) automorphic representation of $G_k(\mathbb{A})$. We construct a global integral representation.

Put $n = 2k$ and let $\varphi$ and $\varphi'$ be a pair of automorphic forms belonging to the space of the global exceptional representation $\Theta$ of $\widetilde{G}_n(\mathbb{A})$ (defined in § 4.1). The Fourier coefficient $\varphi^U_k,\psi_k(g)$ is a global Shalika functional on $\Theta$. By restriction it is a function on $\widetilde{G}_k(\mathbb{A})$, where $G_k$ is embedded in $G_n$ in the top left block of $M_k$. Then $g \mapsto \varphi^U_k,\psi_k(g)\varphi'^U_k,\psi_k^{-1}(g)$ is well defined on $G_k(\mathbb{A})$ and trivial on $G_k(F)$.

Let $f$ be a matrix coefficient of $\pi$, i.e.,

$$f(g) = \int_{G_k(\mathbb{A})/G_k(F) \backslash G_k(\mathbb{A})} v(hg)v^\vee(h) \, dh,$$

where $v$ (resp. $v^\vee$) is a cusp form in the space of $\pi$ (resp. $\pi^\vee$). Also let $s \in \mathbb{C}$. Consider the integral

$$Z(f, \varphi^U_k,\psi_k, \varphi'^U_k,\psi_k^{-1}, s) = \int_{G_k(\mathbb{A})} f(g)\varphi^U_k,\psi_k(g)\varphi'^U_k,\psi_k^{-1}(g) | \det g |^{r-k/2} \, dg. \quad (5.4)$$

**Theorem 5.4.** The integral $Z(f, \varphi^U_k,\psi_k, \varphi'^U_k,\psi_k^{-1}, s)$ satisfies the following properties.

1. It is absolutely convergent for $\Re(s) \gg 0$.
2. For factorizable data $f = \prod_v f_v$, $\varphi$ and $\varphi'$, write $\varphi^U_k,\psi_k = \prod_v \mathcal{I}_v$ and $\varphi'^U_k,\psi_k^{-1} = \prod_v \mathcal{I}'_v$.

   In the domain of absolute convergence, the integral is equal to a product

   $$\prod_v Z(f_v, \mathcal{I}_v, \mathcal{I}'_v, s).$$

   Moreover, for any sufficiently large finite set of places $S$ of $F$,

   $$Z(f, \varphi^U_k,\psi_k, \varphi'^U_k,\psi_k^{-1}, s) = \prod_{v \in S} Z(f_v, \mathcal{I}_v, \mathcal{I}'_v, s) \frac{L^S(2s, \text{Sym}^2, \pi)}{L^S(2s + 1, \wedge^2, \pi)}.$$

3. For a suitable choice of data, the integral is not identically zero (as a function of $s$).
4. It admits a meromorphic continuation to the complex plane.

**Proof.** We merely have to establish convergence: the integral is Eulerian by virtue of Theorem 4.13; the other properties follow immediately from the corresponding local results of Theorems 5.1 and 5.3, and the properties of $L^S(s, \text{Sym}^2, \pi)$ and $L^S(s, \wedge^2, \pi)$.

As in [GJ72, § 12], first we show that for $r \gg 0$,

$$\int_{G_k(\mathbb{A})} |\varphi^U_k,\psi_k(g)\varphi'^U_k,\psi_k^{-1}(g)| | \det g |^{r-k/2} \, dg < \infty. \quad (5.5)$$

This implies the absolute convergence of $Z(f, \varphi^U_k,\psi_k, \varphi'^U_k,\psi_k^{-1}, s)$, because $|f(g)| \leq C_f | \det g |^{r_0}$ for some $C_f > 0$ independent of $g$, and a fixed $r_0$ depending only on $\pi$ (when $\pi$ is unitary, $r_0 = 0$). It is enough to consider factorizable data $\varphi$ and $\varphi'$. Then we need to bound

$$\prod_v \int_{G_k(F_v)} |\mathcal{I}_v(g)\mathcal{I}'_v(g)| | \det g |_{v}^{r-k/2} \, dv. g. \quad (5.6)$$
At any place \( v \), the integral is finite by Theorem 5.1 (1) and Theorem 5.3 (1). Let \( S \) be a finite set of places such that for \( v \notin S \), all data are unramified. Arguing as in the proof of Theorem 5.1 (4), for \( v \notin S \),

\[
\int_{G_k(F_v)} |\mathcal{L}_v(g) \mathcal{L}_v'(g)| \cdot |\det g|_v^{-k/2} \, dg = \sum_{\lambda \in \mathbb{Z}^k_+} \text{vol}(G_k(\mathcal{O}_v) \mathcal{W}_v^\lambda G_k(\mathcal{O}_v)) \delta_{B_k}^{1/2}(\mathcal{W}_v^\lambda) q^{-|\lambda|r}.
\]

Since \( \text{vol}(G_k(\mathcal{O}_v) \mathcal{W}_v^\lambda G_k(\mathcal{O}_v)) \leq \delta_{B_k}^{-1}(\mathcal{W}_v^\lambda) \), this sum is bounded by \( \prod_{i=1}^k (1 - q_v^{-2r-2i+k+1})^{-1} \). According to Weil [Wei95, Chapter VII § 1, Proposition 1] the product of these factors over all the finite places of \( F \) is finite for \( r \geq (k+1)/2 \), and then

\[
\prod_{v \notin S} \int_{G_k(F_v)} |\mathcal{L}_v(g) \mathcal{L}_v'(g)| \cdot |\det g|_v^{-k/2} \, dg < \infty.
\]

Therefore (5.6) is finite, when we take \( r \gg (k+1)/2 \) to guarantee the convergence of the integrals also at \( v \in S \). \( \square \)

The next theorem will be used in § 6. It relies on the following assumption on \( \pi \) and \( \Theta \).

**Assumption 5.5.** At each archimedean place \( v \) of \( F \), there is a neighborhood of 1/2 such that the local zeta integral is holomorphic for all data, i.e., matrix coefficients of \( \pi_v \) and local metaplectic Shalika functions.

As we show in the proof below, if \( \pi \) is unitary with a trivial central character, the analogous assumption at \( p \)-adic places is valid. This gap will probably be closed once the exponents of \( \theta_n \) are determined over archimedean fields; for \( n = 2 \) this can be done directly using the realization of \( \theta_2 \) as a Weil representation and the assumption holds.

**Theorem 5.6.** If \( \pi \) is unitary with a trivial central character, under Assumption 5.5,

1. The global integral has a pole at \( s = 1 \) if and only if \( L^S(s, \text{Sym}^2, \pi) \) has a pole at \( s = 1 \).
2. For any data \((f, \varphi_{U_k, \psi}, \varphi_{U_k, \psi}^{-1})\), there is a punctured neighborhood \( \mathcal{U} \) of 1/2 such that the integral \( Z(kf, k\varphi_{U_k, \psi}, k\varphi_{U_k, \psi}^{-1}, s) \) is holomorphic for all \( k \in K \) and \( s \in \mathcal{U} \).

**Proof.** Observe that \( L^S(s, \Lambda^2, \pi) \) is holomorphic and nonzero for \( \text{Re}(s) > 1 \) ([Sha81, JS90]). In fact, nonvanishing also follows from the equality \( L^S(s, \Lambda^2, \pi) L^S(s, \text{Sym}^2, \pi) = L^S(s, \pi \times \pi) \), since \( L^S(s, \pi \times \pi) \) is nonvanishing in this region ([JS81, Theorem 5.3]) and the only possible poles of \( L^S(s, \text{Sym}^2, \pi) \) are at \( s = 0 \) or 1 ([BG92]). Therefore, the pole at \( s = 1 \) of the quotient \( L^S(s, \text{Sym}^2, \pi)/L^S(s + 1, \Lambda^2, \pi) \) is determined by the numerator.

The first part follows from Theorem 5.4 (2), assuming we can control the integrals at the places in \( S \). One can choose data, for which these do not contribute poles or zeros in a neighborhood of 1/2, by Theorem 5.1 (2) and Theorem 5.3 (2). Hence if \( L^S(s, \text{Sym}^2, \pi) \) has a pole at \( s = 1 \), the integral will have a pole at \( s = 1/2 \), for some choice of data. In the other direction, we need to show that the integrals at the places of \( S \) cannot contribute any pole. At archimedean places this is guaranteed by Assumption 5.5. At any \( p \)-adic place \( v \in S \) we can use Theorem 5.1 (5): since \( \pi_v \) is a local component of a unitary cuspidal automorphic representation of \( G_v(\mathbb{A}) \), it satisfies the prescribed condition on the exponents, by virtue of the results of Luo, Rudnick and Sarnak [LRS99] towards Ramanujan type bounds (see [CKPSS04, p. 221], over a function field \( \pi_v \) is already tempered by [Laf02a, Laf02b]).

For the second part, first note that if \( f \) is a pure tensor, \( k \)-translations leave most of its components unchanged. The global partial \( L \)-functions are meromorphic, so one can find a
punctured neighborhood of 1/2 where their quotient is holomorphic, and we only have to deal with a finite number of local integrals. At archimedean places we resort to Assumption 5.5. At p-adic places we use Theorem 5.1 (5) and its proof, to choose a neighborhood of 1/2 depending only on the exponents of $\pi_v$, at which the local integrals are holomorphic. □

6. The co-period

6.1. The co-period integral. Let $F$ be a global field and $\pi$ be a cuspidal self-dual automorphic representation of $G_k(\mathbb{A})$ with a trivial central character. Put $n = 2k$ and denote by $A^+$ the subgroup of idèles of $F$ whose finite components are trivial and archimedean components are equal and positive.

Let $\rho$ be a smooth complex-valued function on $G_n(\mathbb{A}) \times M_k(\mathbb{A})$ satisfying the following properties: for $g \in G_n(\mathbb{A})$, the function $g \mapsto \rho(g, I_n)$ is right $K$-finite; for any $m, m_1 \in M_k(\mathbb{A})$ and $u \in U_k(\mathbb{A})$, $\rho(mug, m_1) = \delta_{Q_k}^{1/2}(m)\rho(g, m_1m)$; and the function $m \mapsto \rho(g, m)$ is a cusp form in the space of $\pi \otimes \pi$. The standard section corresponding to $\rho$ is defined by $\rho_s(muk, m_1) = \delta_{Q_k}^{s/k}(m)\rho(muk, m_1)$, for any $k \in K$ and $s \in \mathbb{C}$. The function $\rho_s$ belongs to the space of

$$\text{Ind}_{Q_k(\mathbb{A})}^{G_n(\mathbb{A})}(\delta_{Q_k}^{1/2+s/k} \pi \otimes \pi).$$

For simplicity, denote $\rho_s(g) = \rho_s(g, I_n)$. Form the Eisenstein series

$$E(g; \rho, s) = \sum_{g_0 \in Q_k(F) \backslash G_n(F)} \rho_s(g_0g), \quad g \in G_n(\mathbb{A}).$$

Let $E_{1/2}(g; \rho) = \text{Res}_{s=1/2} E(g; \rho, s)$ denote the residue of the series at $s = 1/2$. The residue here is in the sense $\lim_{s \to 1/2} (s - 1/2) E(g; \rho, s)$. If we let $\rho$ vary, then $E_{1/2}(g; \rho)$ is nontrivial if and only if $L(s, \pi \times \pi)$ has a pole at $s = 1$; equivalently, $L^S(s, \pi \times \pi)$ has a pole at $s = 1$ for any sufficiently large finite set of places $S$ ([JS81, 1.5 and 3.16]). Since $\pi$ is cuspidal and unitary, this is indeed the case, i.e., the residues are nontrivial. In fact, they span the Spel representation $\text{Speh}(\pi, 2)$.

Also let $\varphi$ and $\varphi'$ be a pair of automorphic forms belonging to the space of the global exceptional representation $\Theta$ of $\tilde{G}_n(\mathbb{A})$. The central character of $\Theta$ is trivial on $\mathfrak{g}(C_{G_n}(\mathbb{A}))$. The function $g \mapsto \varphi(g)\varphi'(g)$ is well defined and moreover, it is left-invariant on a subgroup $C' < C_{G_n}(\mathbb{A})$ of finite index, which contains $C_{G_n}(\mathbb{A})$ and is also contained in $G_{C_n}(\mathbb{A})$ (see [BG92, pp. 159-160]). Specifically, let $S'$ be a finite set of places of $F$ such that for all $v \notin S'$, $g \mapsto \varphi(g)\varphi'(g)$ is right $K_v$-invariant, then $C' = \{zI_n : z \in F^* \mathbb{A}^2 \prod_{v \in S'} \mathcal{O}_v^2\}$.

The global co-period integral is

$$\text{CP}(E_{1/2}(\cdot; \rho); \varphi, \varphi') = \int_{C'\text{Ind}_{G_n(\mathbb{A})}^{G_n(\mathbb{A})}} E_{1/2}(g; \rho)\varphi(g)\varphi'(g) \, dg. \quad (6.1)$$

This integral was defined in [KY17] (albeit using a slightly different quotient for the integration domain). One may replace $E_{1/2}(\cdot; \rho)$ with any complex-valued function on the quotient, such that the integral is absolutely convergent. As long as $\varphi$ and $\varphi'$ are fixed, this does not change the integration domain (in particular, e.g., the co-period is a linear function in the first variable).

Exactly as in [Kap15, Claim 3.1], since the cuspidal support of $E_{1/2}(\cdot; \rho)$ is $\delta_{B_n}^{-1/2k} \pi \otimes \pi$ and the only cuspidal exponent of $\Theta$ is $\delta_{B_n}^{1/4}$ [KP84, Proposition II.1.2, Theorems II.1.4 and II.2.1], and using [MW95, Lemma I.4.1], one proves the following proposition:
Proposition 6.1. Integral (6.1) is absolutely convergent.

For the function \( \rho \), and with \( C' \) selected as described above (depending on \( \varphi \) and \( \varphi' \)) define

\[
f_\rho(g) = \int_{C'G^\Delta_k(F)\backslash G^\Delta_k(F)} \rho(1_n, c \text{ diag}(g, I_k)) \, dc, \quad g \in G_k(\mathbb{A}).
\]

As noted above, \( C' < G^\Delta_k(\mathbb{A}) \). Then \( f_\rho \) is a matrix coefficient of \( \pi \). Since the integrand is left-invariant on \( C_{G_n}(\mathbb{A}) \), taking a different \( C' \) only multiplies \( f_\rho \) by some nonzero constant.

Theorem 6.2. Assume \( F \) is a function field or Assumption 5.5 holds. The co-period (6.1) satisfies the following properties.

1. \( CP(E_{1/2}(\cdot ; \rho), \varphi, \varphi') = \int_K \text{Res}_{s=1/2} Z(f_{kp}, k\varphi; U_k, \psi_k, k\varphi; U_k, \psi_k^{-1}, s) \, dk \).
2. It is nonzero for some choice of data \( (\rho, \varphi, \varphi') \) if and only if

\[
\text{Res}_{s=1/2} Z(f_{\rho,1}; \varphi_1; U_k, \psi_k, \varphi'_1; U_k, \psi_k^{-1}, s)
\]

does not vanish for some matrix coefficient \( f_{\rho,1} \) of \( \pi \) and \( \varphi_1, \varphi'_1 \) in the space of \( \Theta \).
3. The co-period is nonzero if and only if \( L^2(s; \text{Sym}^2, \pi) \) has a pole at \( s = 1 \).

Proof. We apply the truncation operator of Arthur [Art78, Art80] to \( E(g; \rho, s) \) as in, e.g., [JR92, Jia98, GRS99b, GJR01, Kap15]. For a real number \( d > 1 \), let \( ch_{c>d} : \mathbb{R}_{>0} \to \{0,1\} \) be the characteristic function of \( \mathbb{R}_{>d} \) and set \( ch_{c>d} = 1 - ch_{>d} \). Also extend \( \delta_Q \) to \( G_n(\mathbb{A}) \) via the Iwasawa decomposition. The truncation operator is given by

\[
\Lambda_d E(g; \rho, s) = E(g; \rho, s) - \sum_{g_0 \in Q_k(F)\backslash G_n(F)} E^U_k(g_0 g; \rho, s) ch_{>d}(\delta_Q(g_0 g)),
\]

where \( E^U_k \) is the constant term of the series along \( U_k \). Since \( \pi \) is cuspidal, \( E^U_k(g; \rho, s) = \rho_s(g) + M(\omega, s)\rho_s(g) \). Here \( M(\omega, s) \) is the global intertwining operator corresponding to a Weyl element \( w \) taking \( U_k \) to \( U_k \). Write \( \Lambda_d E(g; \rho, s) = \mathcal{E}_1^d(g; s) - \mathcal{E}_2^d(g; s) \), an equality between meromorphic functions in \( s \), where

\[
\mathcal{E}_1^d(g; s) = \sum_{g_0 \in Q_k(F)\backslash G_n(F)} \rho_s(g_0 g) ch_{\leq d}(\delta_Q(g_0 g))
\]

and

\[
\mathcal{E}_2^d(g; s) = \sum_{g_0 \in Q_k(F)\backslash G_n(F)} M(\omega, s)\rho_s(g_0 g) ch_{>d}(\delta_Q(g_0 g)).
\]

Since \( \varphi \) and \( \varphi' \) have moderate growth and \( \rho \) is a cusp form on \( M_k(\mathbb{A}) \), it is simple to prove:

Claim 6.3. Fix \( d > 1 \). The integral obtained from (6.1) by replacing \( E_{1/2}(\cdot ; \rho) \) with \( \mathcal{E}_j^d(\cdot ; s) \), and \( \rho_s(M(\omega, s)\rho_s \) for \( j = 2 \). \( \varphi \) and \( \varphi' \) with their absolute values, is finite for \( \text{Re}(s) \gg 0 \). \( \square \)

It follows that for fixed \( d > 1 \) and \( \text{Re}(s) \gg 0 \), \( CP(\Lambda_d E(\cdot ; \rho, s), \varphi, \varphi') \) is absolutely convergent and

\[
CP(\Lambda_d E(\cdot ; \rho, s), \varphi, \varphi') = CP(\mathcal{E}_1^d(\cdot ; s), \varphi, \varphi') - CP(\mathcal{E}_2^d(\cdot ; s), \varphi, \varphi').
\]

The following result is the heart of the argument.

Proposition 6.4. Assume \( d \) is sufficiently large. The integrals \( CP(\mathcal{E}_j^d(\cdot ; s), \varphi, \varphi') \) admit meromorphic continuation to the plane. There exist complex-valued functions \( \alpha_0(s) \), \( \alpha_1(s, d) \), \( \alpha_2(s) \) and \( \alpha_3(d, s) \) such that

\[
CP(\mathcal{E}_1^d(\cdot ; s), \varphi, \varphi') = \frac{d^{s/k}}{sk} \alpha_0(s) + \alpha_1(d, s),
\]

and
The function $\alpha_0(s)$ is entire; for $j > 0$ the functions $\alpha_j(\cdot \cdot \cdot)$ are meromorphic in $s$ (for fixed $d$ if $j = 1, 3$);

$$
\text{Res}_{s=1/2} \alpha_1(d, s) = \int_K \text{Res}_{s=1/2} Z(f_{kp}, k\varphi^{U_k, \psi_k}, k\varphi^{U_k, \psi_k^{-1}}, s) \, dk,
$$

$\alpha_2(s)$ has at most a simple pole at $s = 1/2$, and

$$
\lim_{d \to \infty} \text{Res}_{s=1/2} \alpha_3(d, s) = 0.
$$

Furthermore, when $F$ is a number field, $\alpha_0(s) = \alpha_0$ is a constant, and if $F$ is a function field, we in fact have $CP(\mathcal{E}_2^d(\cdot; s), \varphi, \varphi') = 0$.

Exactly as argued in [Kap15, pp. 2181-2182], the properties of the truncated series imply that when we take the residue in (6.2),

$$
CP(A_dE_{1/2}(\cdot; \rho), \varphi, \varphi') = \text{Res}_{s=1/2} CP(\mathcal{E}_2^d(\cdot; s), \varphi, \varphi') - \text{Res}_{s=1/2} CP(\mathcal{E}_2^d(\cdot; s), \varphi, \varphi').
$$

According to Labesse and Waldspurger [LW13, Proposition 4.3.3], $A_dE_{1/2}(g; \rho)$ is bounded by a sum $E_{1/2}(g; \rho) + \vartheta(g, d; \rho)$, where $\vartheta(g, c; \rho)$ is uniformly rapidly decreasing in $d$ and rapidly decreasing in $g$. Since $|A_dE(\cdot; \rho, s)| \leq |E(\cdot; \rho, s)|$, Proposition 6.1 and the Dominated Convergence Theorem imply that we can take the limit $d \to \infty$ under the integral sign on the left-hand side of the last equality. Then according to the proposition,

$$
CP(E_{1/2}(\cdot; \rho), \varphi, \varphi') = \int_K \text{Res}_{s=1/2} Z(f_{kp}, k\varphi^{U_k, \psi_k}, k\varphi^{U_k, \psi_k^{-1}}, s) \, dk.
$$

This completes part (1) of the theorem.

The $dk$-integral has a pole at $s = 1/2$ for some choice of data $(\rho, \varphi, \varphi')$ if and only if $\text{Res}_{s=1/2} Z(\cdot \cdot \cdot) \neq 0$, for some choice of data $(f_{\rho_1}, \varphi_1, \varphi'_1)$. This follows exactly as in [GJR01, Theorem 3.2] (see also [JR92, Proposition 2] and [Jia98]), their arguments are general and apply to our setting. The proof of the second part of the theorem is complete. The third part follows immediately from the second and Theorem 5.6 (1). □

**Proof of Proposition 6.4.** We provide the proof over a number field, the proof over function fields follows along the same lines and is simpler. First consider $CP(\mathcal{E}_2^d(\cdot; s), \varphi, \varphi')$. We take $s$ in the domain of absolute convergence, where our integral manipulations are justified. Collapsing the summation and using the Iwasawa decomposition,

$$
CP(\mathcal{E}_1^d(\cdot; s), \varphi, \varphi') = \int_K \int_{C' \backslash M_k(F) \backslash M_k(\mathbb{A})} \rho_s(mk) \chi_{\leq d}(\delta_{Q_k}(m)) \varphi(umk) \varphi'(umk) \, du \, \delta_{Q_k}^{-1}(m) \, dm \, dk.
$$

Next we appeal to the results of § 4.3 and replace $\varphi$ with its Fourier expansion along $U_k$. In the notation of § 4.3, the integral becomes a sum of integrals $\sum_{j=0}^k I_j$, where

$$
I_j = \int_K \int_{C' \backslash \text{St}_{m(k)}(\mathbb{A})} \rho_s(mk) \chi_{\leq d}(\delta_{Q_k}(m)) \varphi_{U_k, \psi_j}(mk) \varphi^{U_k, \psi_j^{-1}}(mk) \delta_{Q_k}^{-1}(m) \, dm \, dk.
$$

This formal manipulation is justified using the fact that $\varphi$ has uniform moderate growth. See [Kap15, p. 2184] for the complete argument leading to the sum of integrals $I_j$, the
identities here are very similar (and simpler). By virtue of Proposition 4.3, the integrals $I_j$ with $0 < j < k$ vanish, because we may introduce an inner integration $\int_{V_j(F)}V_j(\mathbb{A}) \rho_s(\nu g) \, dv$ along a unipotent subgroup $V_j$ of $M_k$, which vanishes because $m \mapsto \rho(m)$ is a cusp form.

Consider $I_0$. Then $\psi_0 = 1$ and $\text{St}_{n,k}(\psi_0) = M_k$. For a subgroup $H < M_k(\mathbb{A})$, let $H^1 = \{ h \in H : |\delta_{Q_k}(h)| = 1 \}$. Since $C'M_k(F) < M_k(\mathbb{A})^1$ and $M_k(\mathbb{A})$ is the direct product of $M_k(\mathbb{A})^1$ and $\{ \text{diag}(tI_k, I_k) : t \in A^+ \}$, we obtain

$$I_0 = \int_K \int_{C'M_k(F) \setminus M_k(\mathbb{A})^1} \rho_s \varphi'^U_k \varphi'^U_k(\text{diag}(tI_k, I_k)mk)$$

$$\times ch_{\leq d}(t^{k^2}) \delta_{Q_k}^{-1}(\text{diag}(tI_k, I_k)) \, t^{-1} \, dt \, dm \, dk.$$  

Here we identified $A^+$ with $\mathbb{R}_{>0}$, $dt$ is the standard Lebesgue measure on $\mathbb{R}$ and the constant normalizing the measure $dm$ on the right-hand side is omitted. For $m \in M_k(\mathbb{A})^1$,

$$\rho_s(\text{diag}(tI_k, I_k)mk) = \delta_{Q_k}^{1/2+s/k}(\text{diag}(tI_k, I_k)) \rho(m).$$  

By virtue of Theorem 4.4, the function $\varphi'^U_k$ on $\widetilde{M_k}(\mathbb{A})$ belongs to the space of $\delta_{Q_k}^{1/4} \Theta_{M_k}$, where $\Theta_{M_k}$ corresponds to a character which is a lift of $(\theta_{B_k} \otimes \theta_{B_k})_{|T_2}$ (the chosen Borel subgroup of $M_k$ is $B_k \times B_k$). Therefore

$$\varphi'^U_k \varphi'^U_k(\text{diag}(tI_k, I_k)mk) = \delta_{Q_k}^{1/2}(\text{diag}(tI_k, I_k)) \varphi'^U_k(\varphi'^U_k(mk)).$$  

Combining (6.3) and (6.4) we obtain an inner integral $\int_{0 < \nu^2 \leq \delta} t^{sk-1} \, dt = \frac{dv_k}{sk}$, multiplied by a $dmdk$-integral, which is independent of $s$ and $d$. Therefore $I_0 = \frac{dv_k}{sk} \alpha_0$.

Regarding $I_k$, $\psi_k$ is the Shalika character and $\text{St}_{n,k}(\psi_k) = G_\Delta$. According to Theorem 4.13,

$$\varphi'^U_k \varphi'^U_k(\text{diag}(g, I_k)mk) = \varphi'^U_k \varphi'^U_k\varphi'^U_k(\psi_k^{-1}(mk)), \quad \forall c \in G_k(\mathbb{A}).$$

Therefore we may introduce the inner integration given by the formula for $f_\rho$ and

$$I_k = \int_K \int_{G_k(\mathbb{A})} f_{k\rho}(g) \varphi'^U_k \psi_k \varphi'^U_k\psi_k^{-1}(\text{diag}(g, I_k)k) \, | \det g |^{s-k/2} \, ch_{\leq d}(| \det g |^k) \, dg \, dk.$$  

Consider the inner integral, for a fixed $k \in K$. Without the function $ch_{\leq d}$, it is the integral $Z = Z(f_{k\rho}, k\varphi'^U_k\psi_k, k\varphi'^U_k\psi_k^{-1}, s)$ given by (5.4). Denote by $Z_{\leq d}$ (resp. $Z_{> d}$) the integral obtained when the integrand is multiplied by $ch_{\leq d}$ (resp. $ch_{> d}$). Then for $\text{Re}(s) \gg 0$, $Z = Z_{\leq d} + Z_{> d}$. The following claim is the main observation needed to complete the proof.

### Claim 6.5

**Claim 6.5.** Fix $d$, sufficiently large. For any $s$, $Z_{> d}$ is absolutely convergent, uniformly for $s$ in compact subsets of $\mathbb{C}$. Hence $Z_{> d}$ has a holomorphic continuation to the plane. Moreover for any fixed $s$, $Z_{> d} = 0$ as $d \to \infty$.

The proof will be given below. Assume that $d$ is large enough. By Theorem 5.4 (4), $Z$ admits a meromorphic continuation to $\mathbb{C}$, and therefore the claim implies that $Z_{\leq d}$ has a meromorphic continuation to the plane. Furthermore, because $Z_{> d}$ is holomorphic, we deduce that for any $s_0$, $\text{Res}_{s=s_0} Z = \text{Res}_{s=s_0} Z_{\leq d}$, and since the left-hand side is independent of $d$, the residues of $Z_{\leq d}$ are also independent of $d$.

Because $K$ is compact, $I_k$ also has a meromorphic continuation. It follows that

$$\text{Res}_{s=1/2} I_k = \text{Res}_{s=1/2} \int_K Z_{\leq d}(f_{k\rho}, k\varphi'^U_k\psi_k, k\varphi'^U_k\psi_k^{-1}, s) \, dk.$$
According to Theorem 5.6 (2) and its proof, at \( s = 1/2 \) the integral \( Z \) has at most a simple pole, and there is a punctured neighborhood of \( 1/2 \) where the integral is holomorphic for all \( k \in K \). The same applies to \( Z_{\leq d} \), hence by virtue of Cauchy’s Residue Theorem and Fubini’s Theorem we can take the residue under the integral sign and obtain

\[
\int_K \text{Res}_{s=1/2} Z_{\leq d}(f_{k\rho}, k\varphi U_k, \psi_k, k\varphi U_k, \psi_k^{-1}, s) \, dk = \int_K \text{Res}_{s=1/2} Z(f_{k\rho}, k\varphi U_k, \psi_k, k\varphi U_k, \psi_k^{-1}, s) \, dk.
\]

The right-hand side is independent of \( d \). Hence \( I_k \) qualifies as the function \( \alpha_1(s, d) \). In addition, now that we proved that both \( I_0 \) and \( I_k \) admit meromorphic continuations, the same holds for \( CP(C_0^d(\cdot; s), \varphi, \varphi') \).

Next we handle \( CP(C_0^d(\cdot; s), \varphi, \varphi') \). We may proceed as above and obtain a sum \( I_0 + I_k \), where \( \rho_s \) and \( ch_{\leq d} \) are replaced by \( M(w, s) \rho_s \) and \( ch_{> d} \). Now \( I_0 \) takes the form

\[
I_0 = \int_K \int_{C'/M_k(F) \setminus M_k(k)^1} \int_{R_{> 0}} M(w, s) \rho_s \varphi U_k \varphi U_k^{-1}(\text{diag}(tI_k, I_k) mk) \times ch_{> d}(t^{k^2}) \delta_{Q_k}(\text{diag}(tI_k, I_k)) \quad t \, dt \, dm \, dk.
\]

Since

\[
M(w, s) \rho_s (\text{diag}(tI_k, I_k) mk) = \delta_{Q_k}(t^{k^2}) \delta_{Q_k}(\text{diag}(tI_k, I_k)) M(w, s) \rho_s (mk),
\]

the inner integral in this case becomes \( \int_{d< t < \infty} t^{-s} \, dt \). Thus

\[
I_0 = \frac{d^{-s/k}}{sk} \int_K \int_{C'/M_k(F) \setminus M_k(k)^1} M(w, s) \rho_s \varphi U_k \varphi U_k^{-1}(mk) \, dm \, dk.
\]

The \( dm \, dk \)-integral still depends on \( s \), because of the intertwining operator, but it is a meromorphic function in \( s \) and if it has a pole at \( s = 1/2 \), this pole is simple. Hence the constant \( \alpha_0 \) is replaced by a function \( \alpha_2(s) \) with the stated properties.

Regarding \( I_k \), with the above notation

\[
I_k = \int_K Z_{> d}(f_{kM(w, s) \rho_s}, k\varphi U_k, \psi_k, k\varphi U_k, \psi_k^{-1}, -s) \, dk.
\]

Note that since \( m \mapsto \rho(g, m) \) is a cusp form, the matrix coefficient \( f_{M(w, s) \rho_s} \) is a meromorphic function in \( s \) and \( \text{Res}_{s=1/2} f_{M(w, s) \rho_s} = \text{Res}_{s=1/2} f_{M(w, s) \rho_s} \). The convergence properties of \( Z_{> d} \) stated above, imply that \( I_k \) admits a meromorphic continuation whose poles are contained in the set of poles of \( M(w, s) \), and also

\[
\text{Res}_{s=1/2} I_k = \int_K Z_{> d}(f_{\text{Res}_{s=1/2} kM(w, s) \rho_s}, k\varphi U_k, \psi_k, k\varphi U_k, \psi_k^{-1}, -s) \, dk.
\]

Hence when we let \( d \to \infty \), \( \text{Res}_{s=1/2} I_k = 0 \).

**Proof of Claim 6.5.** Consider the function \( \phi(g) = |\varphi U_k, \psi_k| \varphi U_k, \psi_k^{-1}(\text{diag}(g, I_k))| \) on \( G_k(\mathbb{A}) \). The global matrix coefficient \( f_\rho \) is bounded, therefore it is enough to establish the stated convergence properties for an integral of the form

\[
\int_{G_k(k)} \phi(g) ch_{> d}(|\det g|) |\det g|^r \, dg = \int_{G_k(k)^1} \int_{R_{> d^{1/k}}} \phi(g) t^{kr-1} \, dt \, dg, \quad r \in \mathbb{R}.
\]

But according to (5.5), for \( d > 1 \), this integral is finite even for \( r > 0 \). 

\( \square \)
References


Department Mathematik, FAU Erlangen–Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany

E-mail address: frahm@math.fau.de

Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA

current address: Department of Mathematics, Bar Ilan University, Ramat Gan 5290002, Israel

E-mail address: kaplaney@gmail.com