ON THE ALGEBRAICITY OF SOME PRODUCTS OF SPECIAL VALUES OF BARNES’ MULTIPLE GAMMA FUNCTION

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Abstract

We consider partial zeta functions \( \zeta(s, c) \) associated with ray classes \( c \)'s of a totally real field. Stark’s conjecture implies that an appropriate product of \( \exp(\zeta'(0, c)) \)'s is an algebraic number which is called a Stark unit. Shintani gave an explicit formula for \( \exp(\zeta'(0, c)) \) in terms of Barnes’ multiple gamma function. Yoshida “decomposed” Shintani’s formula: he defined the symbol \( X(c, \iota) \) satisfying that
\[
\exp(\zeta'(0, c)) = \prod \exp(X(c, \iota))
\]
where \( \iota \) runs over all real embeddings of \( F \). Hence we can decompose a Stark unit into a product of \( [F : \mathbb{Q}] \) terms. The main result is to show that \( (|F : \mathbb{Q}| - 1) \) of them are algebraic numbers. We also study a relation between Yoshida’s conjecture on CM-periods and Stark’s conjecture.

1 Introduction.

We recall the rank 1 abelian Stark conjecture. Let \( K/F \) be an abelian extension of number fields, and \( S \) a finite set of places of \( F \). In this paper, we always assume that

\( S \) contains all primes ramifying in \( K/F \) and all infinite places of \( F \), and \( |S| \geq 3 \).

We put \( G := \text{Gal}(K/F) \). The partial zeta function \( \zeta_S(s, \tau) \) associated with \( \tau \in G \) is defined by

\[
\zeta_S(s, \tau) := \sum_{a \in \mathcal{O}_F, (a, S) = 1, (\mathcal{K}/a) = \tau} Na^{-s}.
\]

Here \( a \) runs over all integral ideals of \( F \), relatively prime to any prime ideal in \( S \), whose image under the Artin map \( (\mathcal{K}/a) \) is equal to \( \tau \). This series converges for \( \text{Re}(s) > 1 \), has a meromorphic continuation to \( \mathbb{C} \), which is analytic at \( s = 0 \). The following is a version of the Stark conjectures stated in [St].

**Conjecture 1.1** (The rank 1 abelian Stark conjecture). Let \( F, K, S, G \) be as above. We denote the number of roots of unity in \( K \) by \( e_K \). Assume that there exists a place \( v \) in \( S \) splitting completely in \( K \), and fix a place \( w \) of \( K \) lying above \( v \). Then there exists an element \( \epsilon \in K^\times \), which is called a Stark unit, satisfying

(i) \( \epsilon \) is a \( v \)-unit.

(ii) \( \log |\tau(\epsilon)|_w = -e_K\zeta'_S(0, \tau) \) for all \( \tau \in G \).

(iii) \( K(\epsilon^{1/e_K})/F \) is an abelian extension.
We note that the statement of the rank 1 abelian Stark conjecture in the case $|S| = 2$ is slightly different from the above, and has been proved (for the proof, see [Da, Proposition 4.3.11]).

In the following, we only consider the case when the splitting place $v$ is a real infinite place, and focus on the property that $\epsilon$ is a unit. We note that

- If more than one place in $S$ splits completely in $K$, then $\zeta'_S(0, \tau) = 0$ for all $\tau \in G$, so Conjecture 1.1 is trivial. Therefore, when $v$ is a real infinite place, we may assume that $F$ is totally real. (Note that a complex infinite place always splits completely.)
- There exists a real infinite place of $F$ splitting completely if and only if $K$ is not totally imaginary.
- We can write a Stark unit explicitly as $\epsilon = \exp(-e_K \zeta'_S(0, \text{id}))$ when $w$ is a real infinite place.

Therefore our problem can be formulated as follows:

**Definition 1.2.** Let $\mathbb{R}_+$ be the multiplicative group of positive real numbers. For any subgroup $X$ of $\mathbb{R}_+$, we put $X^\mathbb{Q} := \{ x \in \mathbb{R}_+ \mid \text{there exists } n \in \mathbb{N} \text{ satisfying } x^n \in X \}$.

**Conjecture 1.3** (A part of Stark’s conjecture). Let $K$ be a finite abelian extension of a totally real field $F$ and $G := \text{Gal}(K/F)$. Assume that $K$ is not totally imaginary. We fix a real embedding of $K$ and regard $F, K$ as subfields of $\mathbb{R}$ (i.e., $F \subset K \subset \mathbb{R}$). Then we have

$$\exp(\zeta'_S(0, \tau)) \in (O_K^\times \cap \mathbb{R}_+)^\mathbb{Q} \quad (\tau \in G).$$

Here we denote by $O_K^\times$ the group of units in $K$.

Let $K$ be a finite abelian extension of a totally real field $F$, and $S, \tau \in G$ as above. In particular, we do not assume that $K$ is not totally imaginary. In [Yo], Yoshida decomposed $\exp(\zeta'_S(0, \tau))$ into the following form. For an integral divisor $f$ of $F$, we denote by $C_f$ the ray class group modulo $f$. Let $f_{K/F}$ be the conductor of $K/F$ and $\text{Art} : C_{f_{K/F}} \to G$ the Artin map. For an integral divisor $f$ with $f_{K/F} | f$, we denote the composite map $C_f \to C_{f_{K/F}} \to G$ by $\text{Art}_f$. Then we obtain (Theorem 2.5)

$$\exp(\zeta'_S(0, \tau)) = \prod_{\iota : F \hookrightarrow \mathbb{R}} \prod_{c \in \text{Art}_f^{-1}(\tau)} \exp(X(c, \iota)),$$

if we take

$$f_S := f_{K/F} \times \text{“the product of all places } \lambda \in S \text{ with } \lambda \not| f_{K/F}.$$

Here $\iota$ runs over all real embeddings of $F$, and $X(c, \iota)$ is Yoshida’s class invariant defined in §2, in terms of Barnes’ multiple gamma function and Shintani’s cone decomposition. The main result (Theorem 3.1) in this paper states that under the assumption of Conjecture 1.3, there exist a totally positive unit $\epsilon \in O_F^\times$ and a natural number $m$ satisfying
\[
\prod_{c \in \text{Art}^{+}_{\text{id}}(\tau)} \exp(X(c, \iota)) = \iota(\epsilon)^{\frac{1}{2 \pi}} \text{ whenever } \iota \neq \text{id}.
\]

As a direct application, we can refine the statement of Conjecture 1.3 as follows:

\[
\prod_{c \in \text{Art}^{+}_{\text{id}}(\tau)} \exp(X(c, \text{id})) \in (\mathcal{O}^{+}_{K} \cap \mathbb{R}_{+})^{Q}.
\]

Moreover, the main result in this paper has the following significance: In §5, we will discuss a relation between the algebraicity of Stark units, monomial relations among CM-periods, and Yoshida’s conjecture (Conjecture 5.2) on Shimura’s period symbol. Theorem 3.1 and its Corollary are necessary for this discussion (to be precise, for the proofs of Propositions 5.4, 5.6, 5.9).

Let us explain the outline of this paper. In §2, we introduce Yoshida’s technique for decomposing \(\zeta_{\mathbb{R}}(0, \tau)\): Associated with a ray class \(c\) and a real embedding \(\iota\) of \(F\), we define the invariant \(X(c, \iota)\) in terms of Barnes’ multiple gamma function and Shintani’s cone decomposition. Then we state a modified version of Shintani’s formula (Theorem 2.5) which gives the canonical decomposition of \(\zeta_{\mathbb{R}}(0, \tau)\). In §3, we state our main result (Theorem 3.1) and give a proof. When \([F : \mathbb{Q}] = 2\), this result is due to Yoshida, and our proof is a generalization of his proof. One of the new ideas of this paper is to define “formal multiple zeta values” in (3.3), which we need since we treat the case when usual multiple zeta function does not converges (e.g., in Definition 3.5). In §4, we give the proof of Lemma 3.4 which states that, roughly speaking, we can take a suitable cone decomposition for computing \(X(c, \iota)\)’s. This Lemma is a key step in the proof of Theorem 3.1. As an application, in §5, we study a relation between Yoshida’s conjecture on CM-periods and Stark’s conjecture: Yoshida formulated a Conjecture (Conjecture 5.2) which expresses Shimura’s period symbol in terms of \(\exp(X(c, \iota))\)’s. By using Theorem 3.1, we can reformulate this Conjecture to the form (5.1). We note that (5.1) has a natural generalization (Conjecture 5.5) which also implies a part of Stark’s conjecture. Furthermore, we give some evidence for Conjecture 5.5.

2 Yoshida’s \(X\)-invariant.

In this section, we introduce Yoshida’s \(X\)-invariant, which is defined as the sum of three invariants \(G, W, V\). Barnes’ multiple zeta function \(\zeta(s, a, z)\) for \(z \in \mathbb{R}_{+}\), \(a = (a_{1}, \ldots, a_{r}) \in \mathbb{R}_{+}^{r}\) is defined by

\[
\zeta(s, a, z) := \sum_{0 \leq m_{1}, m_{2}, \ldots, m_{r} \in \mathbb{Z}} (z + m_{1}a_{1} + \cdots + m_{r}a_{r})^{-s}.
\]

This series converges for \(\text{Re}(s) > r\), has a meromorphic continuation to \(\mathbb{C}\), which is analytic at \(s = 0\).

**Definition 2.1.** We define the multiple gamma function as

\[
\Gamma(z, a) := \exp(\zeta'(0, a, z)) \quad (z \in \mathbb{R}_{+}, \ a \in \mathbb{R}_{+}^{r}).
\]
We note that this definition is slightly different from Barnes’ definition: \( \exp(\zeta'(0, a, z)) = \Gamma_r(z, a) / \rho_r(a) \) with a correction term \( \rho_r(a) \).

Let \( F \) be a totally real field of degree \( n \), \( \mathcal{O} \) its ring of integers, and \( \infty_1, \ldots, \infty_n \) all the infinite places of \( F \). We denote the set of all totally positive elements in \( F, \mathcal{O}, \mathcal{O}^\times \) by \( F_+, \mathcal{O}_+, \mathcal{O}_+^\times \) respectively. A formal product of an integral ideal and a finite number of infinite places is called an integral divisor. Let \( \mathfrak{f} \) be an integral divisor of \( F \) of the form

\[
\mathfrak{f} = m\infty_1 \cdots \infty_n
\]

with \( m \) an integral ideal. We denote the ray class group modulo \( \mathfrak{f} \) by \( C_\mathfrak{f} \). The partial zeta function \( \zeta(s, c) \) associated with \( c \in C_\mathfrak{f} \) is defined by

\[
\zeta(s, c) := \sum_{a \in \mathcal{O}, a \equiv c} \mathfrak{N}a^{-s},
\]

where \( a \) runs over all integral ideals in the ray class \( c \). This series also has a meromorphic continuation to \( \mathbb{C} \), which is analytic at \( s = 0 \). When \( K/F \) is an abelian extension, taking \( S, \mathfrak{f}_S \) as in (1.2), we can write for \( \tau \in \text{Gal}(K/F) \)

\[
\zeta_S(s, \tau) = \sum_{c \in \text{Art}_{\mathfrak{f}^{-1}}(\tau)} \zeta(s, c).
\]

We identify \( F \otimes \mathbb{R} = \mathbb{R}^n \) by \( z \otimes \alpha \mapsto (\alpha \nu_k(z))_k \), where \( \nu_1, \ldots, \nu_n \) are all real embeddings of \( F \). We consider a “cone decomposition” of a subset of \( F \otimes \mathbb{R} \) in the following sense.

**Definition 2.2.** Assume that \( v_1, \ldots, v_r \in \mathcal{O} \) are linearly independent in \( F \otimes \mathbb{R} \) over \( \mathbb{R} \). We define an \( \langle r \rangle \)-dimensional open simplicial cone \( C(v) \) with the basis \( v := (v_1, \ldots, v_r) \) by

\[
C(v) = C(v_1, \ldots, v_r) := \left\{ \sum_{i=1}^r x_i v_i \in F \otimes \mathbb{R} \mid x_1, x_2, \ldots, x_r \in \mathbb{R}^+_r \right\}.
\]

We make the convention that whenever we consider a cone \( C(v_1, \ldots, v_r) \) in this paper, we assume that \( v_1, \ldots, v_r \) are in \( \mathcal{O} \) and linearly independent.

We put \( F \otimes \mathbb{R}_{n+} := \mathbb{R}_{n+}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n > 0\} \). Let \( D \) be a subset of \( F \otimes \mathbb{R} \). Throughout this section, we always assume that

- \( D \) has a cone decomposition of the form \( D = \coprod_{j \in J} C(v_j) \), where the symbol \( \coprod \) denotes the disjoint union, \( J \) is a finite set of indices, and \( C(v_j) \) is a cone with the basis \( v_j = (v_{j1}, \ldots, v_{jr(j)}) \in \mathcal{O}^{r(j)} \) (\( j \in J, 1 \leq r(j) \leq n \)).
- \( D \subset F \otimes \mathbb{R}_{n+} \). Namely, all of the above \( v_{ji} \)'s are totally positive.

We note that we will relax the second condition for \( D \) from the next section.

For each \( c \in C_\mathfrak{f} \), we fix an integral ideal \( \mathfrak{a}_c \) satisfying that \( \mathfrak{a}_c \mathfrak{f} \) and \( c \) belong to the same narrow ideal class (in \( C_{(1)\infty_1 \cdots \infty_n} \)). For \( j \in J \) we put

\[
R(c, \mathfrak{a}_c, v_j) := \left\{ x = (x_1, \ldots, x_{r(j)}) \in (\mathbb{Q} \cap (0, 1])^{r(j)} \mid z := \sum_{i=1}^{r(j)} x_i v_{ji} \in (\mathfrak{a}_c \mathfrak{f})^{-1}, \ z \mathfrak{a}_c \mathfrak{f} \in c \right\}.
\]
We see that $R(c, a_c, v_j)$ is a finite set (or the empty set) since the condition $\sum_{i=1}^{r(j)} x_i v_j \in (a_f)^{-1}$ implies that the denominators of $x_i$ are bounded. Moreover, we can write [Yo, Chapter II, Lemma 3.2]

\[
\left\{ z \in (a_f)^{-1} \cap C(v_j), \; z a_c \in c \right\}
\]

(2.1) $= \prod_{x \in R(c, a_c, v_j)} \left\{ (x_1 + m_1) v_{j_1} + \cdots + (x_r + m_r) v_{j_r} \mid 0 \leq m_1, \ldots, m_r \in \mathbb{Z} \right\}.$

Hence for any real embedding $\iota$ of $F$, we obtain

\[
\sum_{z \in (a_f)^{-1} \cap D(\epsilon) a_c \in c} \iota(z)^{-s} = \sum_{j \in J} \sum_{x \in R(c, a_c, v_j)} \zeta(s, \iota(v_j), \iota(x^t v_j)).
\]

In particular, this series has a meromorphic continuation to $\mathbb{C}$, which is analytic at $s = 0$. Then the $G$-invariant [Yo, Chapter III, (3.6), (3.28)] is defined by

\[
G(c, \iota; D, a_c) := \left[ \frac{d}{ds} \sum_{z \in (a_f)^{-1} \cap D(\epsilon) a_c \in c} \iota(z)^{-s} \right] = \sum_{j \in J} \sum_{x \in R(c, a_c, v_j)} \log \Gamma(\iota(x^t v_j), \iota(v_j)).
\]

In [Yo, Chapter III, (3.8), (3.31)], the $W$-invariant was defined by

\[
W(c, \iota; D, a_c) := \frac{1}{n} \log N a_c f \cdot \left[ \sum_{z \in (a_f)^{-1} \cap D(\epsilon) a_c \in c} \iota(z)^{-s} \right]_{s=0}.
\]

In the present paper, we slightly modify this definition (we employ the definition in [KY1]). Let $\text{FI}_F$ be the group of all fractional ideals of $F$. We define a group homomorphism $\log_c : \text{FI}_F \to \mathbb{R}$ for each real embedding $\iota$ of $F$ in the following manner. For each prime ideal $p$, we choose $\pi_p \in \mathcal{O}_+$ satisfying $p^{h_F^+} = (\pi_p)$, where $h_F^+$ is the narrow class number. Then we put $\log_c p := \frac{1}{h_F} \log \iota(\pi_p)$ for every $\iota$, and extend this linearly to $\log_c : \text{FI}_F \to \mathbb{R}$. We easily see the following (2.3), (2.4). For $a \in \text{FI}_F$, we have

\[
(2.3) \quad \log N a = \sum_{\iota : F \to \mathbb{R}} \log_c a.
\]

For a principal ideal $(\alpha)$ with $\alpha \in F^\times$, there exist $\epsilon \in \mathcal{O}_+^\times$, $m \in \mathbb{N}$ satisfying

\[
(2.4) \quad \log_c(\alpha) = \log |\iota(\alpha)| + \frac{1}{m} \log \iota(\epsilon)
\]

for all real embeddings $\iota$ of $F$. Then we define the $W$-invariant by

\[
W(c, \iota; D, a_c) := -\log_c a_c f \cdot \left[ \sum_{z \in (a_f)^{-1} \cap D(\epsilon) a_c \in c} \iota(z)^{-s} \right]_{s=0} = -\log_c a_c f \cdot \sum_{j \in J} \sum_{x \in R(c, a_c, v_j)} \zeta(0, \iota(v_j), \iota(x^t v_j)).
\]
By using the equation (2.5) below, we see that $W(c; id; D, a_c)$ is equal to $W(c)$ in [KY1, (4.3)] when $\iota = id$ and $D$ is a Shintani domain (Definition 2.3). Finally, the $V$-invariant [Yo, Chapter III, (3.7), (3.29)] is defined by

$$V(c, \iota; D, a_c) := \frac{2}{n} \sum_{k=2}^{n} v_{1,k} - \frac{2}{n^2} \sum_{1 \leq i < k \leq n} v_{i,k},$$

where we put $f_1 := \iota, \iota_2, \ldots, \iota_n$ to be all real embeddings of $F$. We define

$$X(c, \iota; D, a_c) := G(c, \iota; D, a_c) + W(c, \iota; D, a_c) + V(c, \iota; D, a_c).$$

We consider fundamental domains of the following form, for the natural action $\epsilon(z \otimes \alpha) := (\epsilon z) \otimes \alpha$ of $\epsilon \in O_F^\times$ on $z \otimes \alpha \in F \otimes \mathbb{R}_{n+}$. We have

**Definition 2.3.** We say that a subset $D \subset F \otimes \mathbb{R}_{n+}$ is a Shintani domain if and only if we can write

$$D = \bigoplus_{j \in J} C(\mathbf{v}_j), \quad F \otimes \mathbb{R}_{n+} = \bigoplus_{\epsilon \in O_F^\times} \epsilon D$$

with a finite number of cones $C(\mathbf{v}_j)$ ($j \in J$, $|J| < \infty$, $\mathbf{v}_j \in O_F^+(j)$, $1 \leq r(j) \leq n$).

Shintani showed that there exists a Shintani domain for any $F$ ([Shin1, Proposition 4]). If $D$ is a Shintani domain, and if $D, a_c$ are fixed, then $X(c, \iota; D, a_c)$ is also written as $X(c, \iota)$. For later use, we introduce the following. When $D$ is a Shintani domain, by [Yo, Chapter IV, Corollary 6.3-2], we have

$$\left( \sum_{z \in (a_c f \cdot D) \cap D, (z) a_c \in c} \left( \iota(z)^{-s} - \iota(z)^{-s} \right) \right)_{s=0} = \zeta(0, c) \in \mathbb{Q}. \quad (2.5)$$

For a proof of the last part ($\zeta(0, c) \in \mathbb{Q}$), which seems to be well-known to experts, see e.g. [Yo, Chapter II, Theorem 3.3].

The following Lemma, which is a part of Lemma 3.11, explains the reason why we modified the $W$-invariant in this paper: If we replace $\iota(O_F^+)^Q$ in the statement by $\iota(F_+)^Q$, then it was proved by Yoshida [Yo, Chapter III, §3.6, §3.7] for the original $W$-invariant.

**Lemma 2.4.** $\exp(X(c, \iota)) \bmod \iota(O_F^+)^Q$ does not depend on the choices of a Shintani domain $D$ and an integral ideal $a_c$.

Now we state a modified version of Shintani’s formula: Shintani [Shin2] expressed $\zeta'(0, c)$ in terms of log of Barnes’ multiple gamma function, with certain correction terms. Yoshida found a nice decomposition of the correction terms, which can be written as follows.
Theorem 2.5 ([Yo, Chapter III, (3.11)]). Let \(c, a_c\) be as above. Assume that \(D\) is a Shintani domain. Then we have

\[
\zeta'(0, c) = \sum_{\iota : F \to \mathbb{R}} \chi(c, \iota; D, a_c).
\]

Here \(\iota\) runs over all real embeddings of \(F\).

Proof. In [Yo, Chapter III, (3.11)], the equation (2.6) was proved for the original \(W\)-invariant (2.2). Therefore we need to show that \(\sum \chi(c, \iota; D, a_c)\) does not change when we modify the \(W\)-invariant. This follows from (2.3) since

\[
\left[ \sum_{x \in (a, \bar{f})^{-1} \cap D, (z) a_c(x)} \iota(z)^{-s} \right]_{s=0}^{1}
\]

does not depend on \(\iota\) by (2.5).

3 Monomial relations between \(\exp(X(c, \iota))\)’s.

Let \(F\) be a totally real field of degree \(n\), and \(\mathfrak{f} = m \infty \cdots \infty\) an integral divisor of \(F\), as in the previous section. We denote the maximal ray class field modulo \(\mathfrak{f}\) by \(H_{\mathfrak{f}}\). Then the Artin map gives rise to a canonical isomorphism \(C_{\mathfrak{f}} \cong \operatorname{Gal}(H_{\mathfrak{f}}/F)\). Let \(\iota_1, \ldots, \iota_n\) be all real embeddings of \(F\). We denote the complex conjugation in \(\operatorname{Gal}(H_{\mathfrak{f}}/F)\) at \(\iota_i\) by \(c_i\). That is, taking a lift \(\tilde{\iota}_i : H_{\mathfrak{f}} \to \mathbb{C}\) of \(\iota_i : F \to \mathbb{R}\), we put \(\tilde{c}_i := i \circ \tilde{\iota}_i \circ \iota_i^{-1}\) where \(\iota_i\) is the complex conjugation on \(\mathbb{C}\). The following is the main result in this paper. When \(n = 2\), this Theorem is due to Yoshida [Yo, Chapter III, Theorems 5.8, 5.12].

Theorem 3.1. Assume that \(n \geq 2\). Then there exist \(\epsilon \in \mathcal{O}_F^\times\) and \(m \in \mathbb{N}\) satisfying

\[
\exp(X(c, \iota_i)) \cdot \exp(X(c_j c, \iota_i)) = \iota_i(\epsilon)^{\frac{1}{m}} \quad \text{whenever} \quad i \neq j \quad (1 \leq i, j \leq n).
\]

Remark 3.2. When \(n = 2\), Yoshida gave explicit formulas [Yo, Chapter III, Theorems 5.8, 5.12] for \(\iota_i(\epsilon)^{\frac{1}{m}}\) in Theorem 3.1, in terms of the fundamental unit, the Bernoulli polynomials, and a 2-dimensional cone. Although we can compute \(\iota_i(\epsilon)^{\frac{1}{m}}\) by the arguments in our proof, the expression is complicated and far from an “explicit formula”.

We prepare some Lemmas for the proof of Theorem 3.1. The statement of Lemma 3.3 seems to be well-known to experts. For a proof, see [Yo, Chapter III, the first paragraph of §5.1].

Lemma 3.3. For \(1 \leq i \leq n\), take \(\nu_i \in \mathcal{O}\) so that \(\nu_i \equiv 1 \mod m\), \(\iota_i(\nu_i) < 0\), \(\iota_j(\nu_i) > 0\) \((1 \leq j \leq n, j \neq i)\). Then we have \((\nu_i) \in c_i\).

We fix a numbering of real embeddings \(\iota_1, \ldots, \iota_n\) of \(F\), and consider the following domain in \(F \otimes \mathbb{R} = \mathbb{R}^n\):

\[
F \otimes \mathbb{R}_{(n-1)+} := \mathbb{R}^{n-1}_+ \times \mathbb{R} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_{n-1} > 0\}.
\]

In the following, we consider (disjoint unions of) cones contained in \(F \otimes \mathbb{R}_{(n-1)+}\) (not necessarily in \(F \otimes \mathbb{R}_{n+}\)).
Lemma 3.4. There exist a Shintani domain $D$, an element $\nu \in F$, subsets $X_t \subset F \otimes \mathbb{R}_{(n-1)+}$, and elements $\epsilon_t \in \mathcal{O}_T^+$ ($t \in T$, $T$ is a finite set of indices) satisfying that

(i) Each $X_t$ has a cone decomposition (i.e., can be expressed as a disjoint union of a finite number of cones).

(ii) $\nu \in F \otimes \mathbb{R}_{(n-1)+}$ (i.e., $\nu_1(\nu), \ldots, \nu_{n-1}(\nu) > 0$, $\nu_n(\nu) < 0$).

(iii) We have the following equality of multisets:

$$\left(D \coprod \nu D\right) \cup \left(\bigcup_{t \in T} \epsilon_t X_t\right) = \bigcup_{t \in T} X_t.$$

Here we denote by the symbol $\bigcup$ the multiset sum.

We postpone the proof of Lemma 3.4 to §4 since it is rather technical. Here we only give an example in the case $[F : \mathbb{Q}] = 2$. We take the fundamental unit $\epsilon$ of a real quadratic field $F$, normalized so that $\nu_1(\epsilon) > 1$. If $\epsilon$ is not totally positive, replace $\epsilon$ by $\epsilon^2$. Then $D := C(1) \coprod C(1, \epsilon)$ is a Shintani domain. Take any element $\nu \in \mathcal{O}$ satisfying $\nu_1(\nu) > 0$, $\nu_2(\nu) < 0$. Then $X_1 := C(1, \nu)$, $\epsilon_1 := \epsilon (T := \{1\})$ satisfy the required conditions.

In order to prove Theorem 3.1, we generalize the definition of Yoshida’s invariants to the case $D \subset F \otimes \mathbb{R}_{(n-1)+}$. If $D \not\subset F \otimes \mathbb{R}_{n+}$, then (2.1) does not always hold, so we need some modifications: Consider a cone $C(\nu)$ with $\nu = (v_1, \ldots, v_r) \in \mathcal{O}^r$, $C(\nu) \subset F \otimes \mathbb{R}_{(n-1)+}$. We assume that $a, f$ and $c$ belong to the same narrow ideal class, as in the previous section. We take $z \in (a, f)^{-1} \cap C(\nu)$ satisfying $za, f \in c$. In particular, $(za, f, f) = 1$. Then for $1 \leq i \leq r$ we have

$$(z \pm v_i)(z)^{-1} = (1 \pm v_i/z) \in (1) \text{ or } (\nu_n),$$

where we denote by $(\alpha)$ the ray class in $C_1$ of a principal ideal $(\alpha)$ ($\alpha = 1, \nu_n$), $\nu_n$ is defined in Lemma 3.3, and we additionally assume that $z - v_i \in C(\nu)$ in the case $(z - v_i)(z)^{-1}$. Therefore, instead of (2.1), we obtain

$$\{z \in (a, f)^{-1} \cap C(\nu), za, f \in c \text{ or } c_n c\}$$

$$= \coprod_{x \in R(\{c, c_n c\}, a, v)} \{x_1 + m_1 v_1 + \cdots + (x_r + m_r) v_r \mid 0 \leq m_1, \ldots, m_r \in \mathbb{Z}\}$$

where

$$R(\{c, c_n c\}, a, v) := \left\{x \in (\mathbb{Q} \cap (0, 1])^r \mid \sum_{i=1}^r x_i v_i \in (a, f)^{-1}, za, f \in c \text{ or } c_n c\right\}.$$

Hence we can write for $\nu \neq \nu_n$

$$\sum_{z \in (a, f)^{-1} \cap C(\nu)} C(z)^{-8} = \sum_{x \in R(\{c, c_n c\}, a, v)} \zeta(s, \nu(x, n(\nu)))\zeta(s, \nu(1, v)).$$

Therefore we can generalize the $G, W$-invariants as follows: Let $c, a, f$ be as above. We assume that $D \subset F \otimes \mathbb{R}$ satisfies that
• \( D \) has a cone decomposition of the form \( D = \bigsqcup_{j \in J} C(v_j) \) (\(|J| < \infty, v_j \in \mathcal{O}^{r(j)}, 1 \leq r(j) \leq n\)).

• \( D \subset F \otimes \mathbb{R}^{(n-1)_+}\). Then for \( t \neq t_n \), we put

\[
G(\{c, c_n c\}, t; D, a_c) := \left[ \frac{d}{ds} \sum_{z \in (a_c)^{-1} \cap D, (z)a_c \in \mathbb{R}} t(z)^{-s} \right]_{s=0} = \sum_{j \in J} \sum_{x \in R(\{c, c_n c\}, a_c, v_j)} \log \Gamma(t(x^t v_j), t(v_j)),
\]

\[
W(\{c, c_n c\}, t; D, a_c) := -\log \sum_{j \in J} \sum_{x \in R(\{c, c_n c\}, a_c, v_j)} \zeta(0, t(v_j), t(x^t v_j)).
\]

Concerning the definition of the \( V \)-invariant in the case \( D \subset F \otimes \mathbb{R}^{(n-1)_+} \), the series

\[
\sum_{z \in (a_c)^{-1} \cap D, (z)a_c \in \mathbb{R}} \left( (t_1(z)t_k(z))^{-s} - t_i(z)^{-s} - t_k(z)^{-s} \right)
\]

may not be defined when \( i = n \) or \( k = n \), since \( t_n(z) < 0 \). In order to avoid this problem, we introduce the following “formal multiple zeta values”: Let \( a := (a_1, \ldots, a_r), x := (x_1, \ldots, x_r), m \in \mathbb{N} \) with \( a_i > 0, x_i \geq 0, x \neq 0 \). By abuse of notation, we put

\[
(3.1) \quad \zeta(s, a, x) := \zeta(s, a, x^t a) = \sum_{0 \leq m_1, m_2, \ldots, m_r \in \mathbb{Z}} \left( (x_1 + m_1)a_1 + \cdots + (x_r + m_r)a_r \right)^{-s}.
\]

Then by [Shin1, Corollary to Proposition 1], we have

\[
(3.2) \quad \zeta(1 - m, a, x) = (-1)^r(m - 1)! \sum_{|l|=m+r-1} \prod_{i=1}^{r} \frac{B_l(x_i)a_i^{l-1}}{l!}.
\]

Here we denote by \( B_l(x) \) the \( l \)th Bernoulli polynomial, and \( l = (l_1, \ldots, l_r) \) runs over all \( r \)-tuples of non-negative integers satisfying \(|l| := l_1 + \cdots + l_r = m + r - 1\). We define “formal multiple zeta values” as the same rational functions of \( a_i, x_i:\)

\[
(3.3) \quad \zeta_{\text{fml}}(1 - m, a, x) := (-1)^r(m - 1)! \sum_{|l|=m+r-1} \prod_{i=1}^{r} \frac{B_l(x_i)a_i^{l-1}}{l!} \quad (a \in (\mathbb{R}^r)^r, x \in \mathbb{R}^r).
\]

For later use, we note the following relations. We obtain

\[
(3.4) \quad \zeta_{\text{fml}}(1 - m, a, x) = -\zeta_{\text{fml}}(1 - m, (-a_1, a_2, \ldots, a_r), (1 - x_1, x_2, \ldots, x_r)),
\]
(3.5) \( \zeta_{\text{fml}}(1 - m; a_i, (0, x_2, \ldots, x_r)) = \zeta_{\text{fml}}(1 - m; a_i, (1, x_2, \ldots, x_r)) \)
+ \( \zeta_{\text{fml}}(1 - m, (a_2, \ldots, a_r), (x_2, \ldots, x_r)) \),

(3.6) \( \zeta_{\text{fml}}(1 - m; a, x) = \sum_{h=0}^{n-1} \zeta_{\text{fml}}(1 - m, (na_1, a_2, \ldots, a_r), (x_1 + h, x_2, \ldots, x_r)) \)

by well-known formulas \( B_l(1 - x) = (-1)^l B_l(x) \), \( B_l(0) = B_l(1) - 1 \), \( B_l(nx) = n^{l-1} \sum_{h=0}^{n-1} B_l(x + h/n) (n \in \mathbb{N}) \). Furthermore we have

(3.7) \( \zeta_{\text{fml}}(1 - m; a, x) = \zeta_{\text{fml}}(1 - m; (a_1, a_1 + a_2, a_3, \ldots, a_r), (x_1 - x_2 + 1, x_3, \ldots, x_r)) \)
+ \( \zeta_{\text{fml}}(1 - m; (a_1 + a_2, a_3, \ldots, a_r), (x_2, x_1 - x_1, x_3, \ldots, x_r)) \).

We can show (3.7) as follows: When we can replace each \( \zeta_{\text{fml}}(\cdots) \) by \( \zeta(\cdots) \) (that is, when \( a_i > 0, x_i \geq 0, 1 \geq x_2 - x_1 \geq 0, \) and \( x, (x_1 - x_2 + 1, x_3, \ldots, x_r), (x_1 - x_2 - x_1, x_3, \ldots, x_r) \neq 0) \), this follows by decomposing the sum \( \sum_{0 \leq m_1, m_2, \ldots, m_r \in \mathbb{Z}} \) in (3.1) as

\[
\sum_{0 \leq m_1, m_2, \ldots, m_r \in \mathbb{Z}} = \sum_{0 \leq m_1, m_2, \ldots, m_r \in \mathbb{Z}, m_1 > m_2} + \sum_{0 \leq m_1, m_2, \ldots, m_r \in \mathbb{Z}, m_1 \leq m_2}.
\]

Since \( \zeta_{\text{fml}}(1 - m; a, x) \) is a rational function, the same equality holds for any \( a_i, x_i \). Similarly we can show that

(3.8) \( \zeta_{\text{fml}}(1 - m; a, x) \)
= \( \zeta_{\text{fml}}(1 - m; (a_1, a_1 + a_2, a_3, \ldots, a_r), (x_1 - x_2, x_3, \ldots, x_r)) \)
+ \( \zeta_{\text{fml}}(1 - m; (a_1 + a_2, a_3, \ldots, a_r), (x_1, x_2 - x_1 + 1, x_3, \ldots, x_r)) \).

We can derive the following formula from (3.5), (3.7) by substituting \( x := x_1 = x_2 \).

(3.9) \( \zeta_{\text{fml}}(1 - m; a, (x, x, x_3, \ldots, x_r)) \)
= \( \zeta_{\text{fml}}(1 - m; (a_1, a_1 + a_2, a_3, \ldots, a_r), (1, x, x_3, \ldots, x_r)) \)
+ \( \zeta_{\text{fml}}(1 - m; (a_1 + a_2, a_3, \ldots, a_r), (x_1, x_3, \ldots, x_r)) \)
+ \( \zeta_{\text{fml}}(1 - m, (a_1 + a_2, \ldots, a_r), (x, x_3, \ldots, x_r)) \).

By combining (3.4) with (3.5), substituting \( x_1 = 1 \), we obtain

(3.10) \( \zeta_{\text{fml}}(1 - m; a, (1, x_2, \ldots, x_r)) + \zeta_{\text{fml}}(1 - m, (-a_1, a_2, \ldots, a_r), (1, x_2, \ldots, x_r)) \)
+ \( \zeta_{\text{fml}}(1 - m, (a_2, \ldots, a_r), (x_2, \ldots, x_r)) = 0. \)

Next we introduce some notations and results in order to rearrange the expression of the \( V \)-invariant in [Yo, Chapter III, (3.7)]. Let \( A = (a_{ij}) \) be an \( n \times r \) matrix with \( a_{ij} > 0 \), and \( x = (x_j) \) an \( r \)-dimensional vector with \( x_j \geq 0, x \neq 0 \). For \( l = (l_1, \ldots, l_r), j, k \) with \( 0 \leq l_1, \ldots, l_r, 1 \leq j, k \leq n, j \neq k, \) we put

\[
C_{l,j,k}(A) := \int_0^1 \left\{ \prod_{m=1}^r (a_{jm} + a_{km} u)^{l_m-1} - \prod_{m=1}^r a_{jm}^{l_m-1} \right\} \frac{du}{u}. 
\]
If $a_{jp}a_{kp} - a_{jp}a_{kq} \neq 0$ for all $p, q$ with $p \neq q$, then we have [Yo, Chapter I, Lemma 2.3]

$$C_{1,j,k}(A) + C_{1,k,j}(A) = \sum_{p \text{ with } l_p = 0} \prod_{q \neq p} (a_{jq}/a_{jp} - a_{kq}/a_{kp})^{l_q-1} \log \frac{a_{jp}}{a_{kp}}.$$ 

Therefore we can write

$$(-1)^r \sum_{|l|=r} (C_{1,j,k}(A) + C_{1,k,j}(A)) \prod_{q=1}^r B(l_q(x_q)) l_q^{|l_q|} \log \frac{a_{jp}}{a_{kp}}$$

$$= (-1)^r \sum_{|l|=r} \left( \sum_{p \text{ with } l_p = 0} \prod_{|l|=r} (a_{jq}/a_{jp} - a_{kq}/a_{kp})^{l_q-1} \log \frac{a_{jp}}{a_{kp}} \right) \prod_{q=1}^r B(l_q(x_q)) l_q^{|l_q|} \log \frac{a_{jp}}{a_{kp}}$$

(3.11)

$$= - \sum_{p=1}^r (-1)^{r-1} \sum_{|l|=r, l_p = 0} B(l_q(x_q)) (a_{jq}/a_{jp} - a_{kq}/a_{kp})^{l_q-1} \log \frac{a_{jp}}{a_{kp}}$$

$$= - \sum_{p=1}^r \zeta_{ml}(1, (a_{jq}/a_{jp} - a_{kq}/a_{kp})_{q \neq p}, (x_q)_{q \neq p}) \log \frac{a_{jp}}{a_{kp}}.$$

Here for an $r$-dimensional vector $(x_q) = (x_1, \ldots, x_r)$, we denote by $(x_q)_{q \neq p}$ the $(r-1)$-dimensional vector $(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_r)$. We define the “formal $V$-invariant” as follows.

**Definition 3.5.** Let $c, a_c, D = \prod_{j \in J} C(j_v) \subset F \otimes \mathbb{R}_{(n-1)+}$ be as above. Then for any real embedding $\iota$ of $F$, we define

$$V_{\text{ml}}(\{c, c_n c\}, \iota; D, a_c) := \sum_{j \in J} \sum_{x \in \mathcal{R}(\{c, c_n c\}, a_c, j_v)} V(j_v, x, \iota).$$

(3.12)

Here, for $\mathbf{v} = (v_1, v_2, \ldots, v_r) \in \mathcal{O}^r$, $\mathbf{x} = (x_1, x_2, \ldots, x_r) \in (\mathbb{Q} \cap (0, 1))^r$, we put

$$V(\mathbf{v}, \mathbf{x}, \iota) := \frac{1}{n} \sum_{k=2}^n \sum_{p=1}^r \zeta_{ml}(1, \frac{i'(v_k)}{i'(v_p)} - \frac{i'(v_k)}{i'(v_p)})_{|q \neq p|} (x_q)_{|q \neq p|} \log \frac{i'(v_p)}{i'(v_k)}$$

$$+ \frac{1}{n^2} \sum_{1 \leq i < k \leq n} \sum_{p=1}^r \zeta_{ml}(1, \frac{i'(v_k)}{i'(v_p)} - \frac{i'(v_k)}{i'(v_p)})_{|q \neq p|} (x_q)_{|q \neq p|} \log \frac{i'(v_p)}{i'(v_k)}$$

with $\{\iota_1 = \iota, \iota_2, \ldots, \iota_n\}$ all real embeddings of $F$.

We note that, by (3.11) and [Yo, Chapter III, (3.7)], the equation (3.12) holds if we replace $\{c, c_n c\}, V_{\text{ml}}(\cdots), \zeta_{ml}(\cdots), D \subset F \otimes \mathbb{R}_{(n-1)+}$ by $c, V(\cdots), \zeta(\cdots), D \subset F \otimes \mathbb{R}_{n+}$ respectively. In particular, if $\{z \in (a_c)^{-1} \cap C(\mathbf{v})\}, z a_c a_c \in c$ or $c_n c = \{z \in (a_c)^{-1} \cap C(\mathbf{v})\}, z a_c a_c \in c$, then we have

$$V_{\text{ml}}(\{c, c_n c\}, \iota; D, a_c) = V(c, \iota; D, a_c).$$

The following Lemmas are modifications of [Yo, Chapter III, §3.5, §3.6].
Lemma 3.6. The definition of $V_{\text{inf}}(\{c, c_n c\}, \nu; D, a_c)$ does not depend on the choice of the cone decomposition of $D$.

Proof. Since the intersection of two cones is a disjoint union of a finite number of cones (or the empty set), it suffices to show that $V_{\text{inf}}(\{c, c_n c\}, \nu; D, a_c)$ does not change when we replace the cone decomposition $D = \bigsqcup_{j \in J} C(\nu_j)$ by its refinement. We have to consider the following cases for each $j \in J$. We write $v := \nu_j$, $r := r(j)$ for simplicity.

(I) Change the order of the basis $v = (v_1, \ldots, v_r)$.

(II) Replace $v_i$ by $nv_i$ with $n \in \mathbb{N}$.

(III) Decompose $C(v)$ into $C(v^*) \bigsqcup C(v^*) \bigsqcup C(v^*)$ with

$$v^* = (v^*_{1_1}, v^*_{2_1}, v^*_{3_1}, \ldots, v^*_{r_1}) := (v_1, v_1 + v_2, v_3, \ldots, v_r),$$

$$v^* = (v^*_{1_2}, v^*_{2_2}, v^*_{3_2}, \ldots, v^*_{r_2}) := (v_1 + v_2, v_2, v_3, \ldots, v_r),$$

$$v^* = (v^*_{1_3}, v^*_{2_3}, v^*_{3_3}, \ldots, v^*_{r_3}) := (v_1 + v_2, v_1, v_3, \ldots, v_r).$$

In general, the cone $C(v)$ is separated by an $(r - 1)$-dimensional cone $C(v')$ which is not necessarily equal to $C(v^*)$. When a 2-dimensional cone $C(v_i, v_j)$ ($1 \leq i < j \leq r$) and $C(v')$ intersect, the intersection line is a 1-dimensional cone in the form of $C(mv_i + nv_j)$ ($m, n \in \mathbb{N}$). Therefore, by considering a further refinement if necessary, we can reduce the problem to the case $v' = (mv_i + nv_j, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_r)$. We may assume that $i = 1$, $j = 2$ by the operation (I), and that $m = n = 1$ by the operation (II). Hence it suffices to consider the operations (I), (II) and the case when $v' = v^*$ as in (III).

We easily see that $V_{\text{inf}}(\{c, c_n c\}, \nu; D, a_c)$ does not change under (I), (II) by definition and (3.6). Let us consider the case (III). We take $(x_1, \ldots, x_r) \in R(\{c, c_n c\}, a_c, v)$ and put

$$x^* := \begin{cases} (x_1 - x_2 + 1, x_2, x_3, \ldots, x_r) & \text{if } x_1 < x_2, \\ (x_1 - x_2, x_2, x_3, \ldots, x_r) & \text{if } x_1 > x_2, \\ (1, x, x_3, \ldots, x_r) & \text{if } x_1 = x_2 =: x, \end{cases}$$

$$x^* := \begin{cases} (x_1 - x_2 - x_1, x_3, \ldots, x_r) & \text{if } x_1 < x_2, \\ (x_1 - x_2 - x_1 + 1, x_3, \ldots, x_r) & \text{if } x_1 > x_2, \\ (x_1, x_3, \ldots, x_r) & \text{if } x_1 = x_2 =: x, \end{cases}$$

$$x^* := (x, x_3, x_4, \ldots, x_r) \quad \text{if } x_1 = x_2 =: x.$$
Therefore, when \( x_1 = x_2 \), it suffices to show that for all \( i, k \) with \( 1 \leq i < k \leq n \)
\[
\sum_{p=1}^{r} \zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p, \ell'_k(v_p)} \log \left| \frac{\ell'_i(v_q)}{\ell'_k(v_p)} \right|)
= \sum_{p=1}^{r} \zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p, \ell'_k(v_p)} \log \left| \frac{\ell'_i(v_q)}{\ell'_k(v_p)} \right|)
+ \sum_{p=1}^{r} \zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p, \ell'_k(v_p)} \log \left| \frac{\ell'_i(v_q)}{\ell'_k(v_p)} \right|)
+ \sum_{p=1}^{r-1} \zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p, \ell'_k(v_p)} \log \left| \frac{\ell'_i(v_q)}{\ell'_k(v_p)} \right|).
\]

When \( x_1 \neq x_2 \), we drop the terms with the symbol "\( \emptyset \)" on the right-hand side. By comparing the coefficients of \( \log \cdot \cdot \cdot \) on both sides, we can reduce the problem to the following relations among formal multiple zeta values:
\[
\zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p})
= \begin{cases} 
\zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p}) \\
+ \zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p}) \\
\zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p, \ell'_k(v_p)} \log \left| \frac{\ell'_i(v_q)}{\ell'_k(v_p)} \right|) \\
+ \zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p, \ell'_k(v_p)} \log \left| \frac{\ell'_i(v_q)}{\ell'_k(v_p)} \right|) \\
\zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p, \ell'_k(v_p)} \log \left| \frac{\ell'_i(v_q)}{\ell'_k(v_p)} \right|) \end{cases} (x_1 = x_2, p \neq 1, 2),
\]

\[
\zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p}) + \zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p}) = 0 \quad (x_1 \neq x_2),
\]

\[
\zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p}) + \zeta_{\text{fmI}}(-1, \frac{\ell'_i(v_q)}{\ell'_k(v_p)} - \frac{\ell'_k(v_q)}{\ell'_k(v_p)} (x_q)_{q\neq p}) = 0 \quad (x_1 = x_2).
\]

Indeed, the first relation follows from (3.7), (3.8), (3.9), and the remaining relations follow from (3.4), (3.10) respectively. Then the assertion is clear. \( \square \)
Lemma 3.7. Let $c, a_c$ be as above, \{v_1', \ldots, v_n'\} all real embeddings of $F$, and $C(v)$ an $r + 1$-dimensional cone with the basis $v = (v_1, \ldots, v_r) \in O^r$. We assume that $C(v) \subset F \otimes \mathbb{R}_{(r-1)+}$.

(i) Let $\alpha \in F^*_+ \cap a_c^{-1}$. We assume that $\alpha^{-1}v \in O^r$ (multiplying $v_i$ by an integer, if necessary). Then we have

\[
V_{fml}(\{c, c_n c\}, \iota; C(\alpha^{-1}v), (\alpha) a_c) - V_{fml}(\{c, c_n c\}, \iota; C(v), a_c) = \frac{1}{n} \log N(\alpha) - \log \iota(\alpha) \times \left( \iota(\zeta_{fml}(0, v, x)) - \frac{1}{n} \sum_{k=1}^n \iota_k' \left( \zeta_{fml}(0, v, x) \right) \right).
\]

(ii) Let $v_n$ be as in Lemma 3.3. We assume that $v_n^{-1}v \in O^r$. Then we have

\[
V_{fml}(\{c, c_n c\}, \iota; C(v_n^{-1}v), (v_n) a_c) - V_{fml}(\{c, c_n c\}, \iota; C(v), a_c) = \frac{1}{n} \log |N(v_n)| - \log \iota(v_n) \times \left( \iota(\zeta_{fml}(0, v, x)) - \frac{1}{n} \sum_{k=1}^n \iota_k' \left( \zeta_{fml}(0, v, x) \right) \right).
\]

Here we note that the roles of $c, c_n c$ in the symbol $V_{fml}(\{c, c_n c\}, \iota; C(v_n^{-1}v), (v_n) a_c)$ are exchanged: $c_n c, (v_n) a_c$ belong to the same narrow ideal class, and we have $c_n c_n c = c$.

Proof. First we prove (i). We have $R(\{c, c_n c\}, a_c, v) = R(\{c, c_n c\}, (\alpha) a_c, \alpha^{-1}v)$ by definition, so it suffices to show that

\begin{equation}
V(\alpha^{-1}v, x, \iota_1) - V(v, x, \iota_1') = \left( \frac{1}{n} \log N(\alpha) - \log \iota_1(\alpha) \right) \left( \iota_1' \left( \zeta_{fml}(0, v, x) \right) - \frac{1}{n} \sum_{k=1}^n \iota_k' \left( \zeta_{fml}(0, v, x) \right) \right).
\end{equation}

Since $\zeta_{fml}(\alpha, \frac{\iota_1'(v_q)}{\iota_k'(v_p)})_{q \neq p}, (x_q)_{q \neq p}$ does not change when we multiply $v$ by $\alpha^{-1}$, we see that

\begin{equation}
V(\alpha^{-1}v, x, \iota_1') - V(v, x, \iota_1) = \frac{1}{n} \sum_{k=2}^n \left( \sum_{p=1}^r \zeta_{fml}(\alpha, \frac{\iota_1'(v_q)}{\iota_1'(v_p)} - \frac{\iota_k'(v_q)}{\iota_k'(v_p)} \right) \log \frac{\iota_1'(\alpha)}{\iota_k'(\alpha)} - \frac{1}{n^2} \sum_{1 \leq i < k \leq n} \left( \sum_{p=1}^r \zeta_{fml}(\alpha, \frac{\iota_1'(v_q)}{\iota_1'(v_p)} - \frac{\iota_k'(v_q)}{\iota_k'(v_p)} \right) \log \frac{\iota_1'(\alpha)}{\iota_k'(\alpha)}.
\end{equation}
By definition, we have

$$
(3.15) \sum_{p=1}^{r} \zeta_{\text{fin}}(-1, \frac{t_i'(v_q)}{t_i'(v_p)}) = \left( \frac{t_k'(v_q)}{t_k'(v_p)} \right)_{q \neq p}, (x_q)_{q \neq p}
$$

$$
= -(-1)^{r} \sum_{|l|=r} \left( \prod_{p \text{ with } l_p=0 \ q \neq p} \left( \frac{t_i'(v_q)}{t_i'(v_p)} - \frac{t_k'(v_q)}{t_k'(v_p)} \right) \right) \prod_{q=1}^{r} B_{l_q}(x_q) l_q!.
$$

We fix \( l = (l_1, \ldots, l_r), i, k \) with \( l_1 + \cdots + l_r = r, \ 1 \leq i < k \leq n \) and substitute the following values for the variables in Proposition 3.8 below: Let \( t := |\{p \mid 1 \leq p \leq r, l_p = 0\}| \). Then there exist \( 1 \leq f_1, \ldots, f_t, g_1, \ldots, g_{r-t} \leq r \) satisfying \( l_{f_1} = \cdots = l_{f_t} = 0, l_{g_1}, \ldots, l_{g_{r-t}} > 0 \), \( \{f_1, \ldots, f_t, g_1, \ldots, g_{r-t}\} = \{1, 2, \ldots, r\} \). We put \( a_j := t_i'(v_{f_j})^{-1}, a_j' := t_k'(v_{f_j})^{-1} \ (1 \leq j \leq t) \). Concerning \( b_1, \ldots, b_t \), we define

$$
b_1 = b_2 = \cdots = b_{l_{g_1}} := t_i'(v_{g_1}), b_{l_{g_1}+1} = b_{l_{g_1}+1} = \cdots = b_{l_{g_1}+l_{g_2}} = t_i'(v_{g_2}),
$$

$$
\vdots \hspace{1cm}

\vdots \hspace{1cm}
$$

$$
\vdots \hspace{1cm}
$$

$$
= b_{l_{g_1}+l_{g_2}+(r_{g_{r-t}-1})+1} = b_{l_{g_1}+l_{g_2}+(r_{g_{r-t}-1}+1)} = \cdots = b_t := t_i'(v_{g_{r-t}}).
$$

For \( b_1', \ldots, b_t' \), we replace \( t_i' \) by \( t_k' \). Then Proposition 3.8 states that

$$
(3.16) \prod_{q=1}^{r} t_i'(v_q)^{q-1} - \prod_{q=1}^{r} t_k'(v_q)^{q-1} = \sum_{p \text{ with } l_p=0 \ q \neq p} \left( \frac{t_i'(v_q)}{t_i'(v_p)} - \frac{t_k'(v_q)}{t_k'(v_p)} \right) l_q^{-1}.
$$

By (3.14), (3.15), (3.16) and (3.3), we obtain

$$
V(\alpha^{-1}v,x,t_i') - V(v,x,t_i') = -\frac{1}{n} \sum_{k=2}^{n} \left( \frac{t_i'(\zeta_{\text{fin}}(0,v,x))}{t_k'(\zeta_{\text{fin}}(0,v,x)))} \right) \log \frac{t_i'(\alpha)}{t_k'(\alpha)}
$$

$$
+ \frac{1}{n^2} \sum_{1 \leq i < k \leq n} \left( \frac{t_i'(\zeta_{\text{fin}}(0,v,x))}{t_k'(\zeta_{\text{fin}}(0,v,x)))} \right) \log \frac{t_i'(\alpha)}{t_k'(\alpha)}.
$$

It is easy to see that the right-hand side of the above formula is equal to that of (3.13). Then the assertion (i) is clear. The same proof works for (ii) by using \( R(\{c,c_n c\}, a_c, v) = R(\{c,c_n c\}, (\nu_n) a_c, \nu_n^{-1} v). \)

**Proposition 3.8.** Let \( a_1, \ldots, a_t, a_1', \ldots, a_t', b_1, \ldots, b_t, b_1', \ldots, b_t' \) be variables. Then we have

$$
(3.17) \prod_{i=1}^{t} a_i b_i - \prod_{i=1}^{t} a_i' b_i' = \sum_{i=1}^{t} \left( \prod_{j=1}^{t} (a_i b_j - a_i' b_j') \right) \left( \prod_{1 \leq j \leq t, j \neq i} \left( \frac{a_i}{a_j} - \frac{a_i'}{a_j'} \right)^{-1} \right).
$$

**Proof.** We put \( d(i,i) := a_i b_i - a_i' b_i', \ d(i,j) := (a_i b_j - a_i' b_j')/(a_i/a_j - a_i'/a_j') \ (i \neq j) \). Then the right-hand side of (3.17) can be written as \( \sum_{i=1}^{t} \prod_{j=1}^{t} d(i,j) \). We use induction. The case \( t = 1 \) is trivial. Assume that (3.17) holds for \( t \). We easily see that for \( i \leq t \)

$$
a_{t+1} b_{t+1} + \frac{a_i'}{a_{t+1}} \frac{d(t+1, t+1)}{a_i/a_{t+1} - a_i'/a_{t+1}} = d(i,t+1).
$$
Hence we can write
\[
\prod_{i=1}^{t+1} a_i b_i - \prod_{i=1}^{t+1} a'_i b'_i = \left( \prod_{i=1}^{t} a_i b_i - \prod_{i=1}^{t} a'_i b'_i \right) a_{t+1} b_{t+1} + d(t+1, t+1) \left( \prod_{i=1}^{t} a'_i b'_i \right)
\]
\[
= \sum_{i=1}^{t} \left( d(i, t+1) - \frac{a'_i}{a_{t+1} a_i - a'_t a_i - a'_t a'_t} \prod_{j=1}^{t} d(i, j) \right) + d(t+1, t+1) \prod_{i=1}^{t} a'_i b'_i.
\]
Therefore it suffices to show that
\[ (3.18) \quad \sum_{i=1}^{t} \left( \frac{a'_i}{a_{t+1} a_i - a'_t a_i - a'_t a'_t} \prod_{j=1}^{t} d(i, j) \right) = \prod_{j=1}^{t} d(t+1, j) - \prod_{i=1}^{t} a'_i b'_i. \]
Substituting \( a_i := \frac{1}{a_{t+1} a_i - a'_t a_i - a'_t a'_t} \), \( b_i := a_{t+1} b_i - a'_t b'_i \) into (3.17), we obtain
\[
\prod_{j=1}^{t} d(t+1, j) - \prod_{i=1}^{t} a'_i b'_i = \sum_{i=1}^{t} \frac{\prod_{j=1}^{t} \left( \frac{a_{t+1} b_j - a'_t b'_j}{a_{t+1} a_i - a'_t a_i - a'_t a'_t} - a'_i b'_i \right)}{\prod_{1 \leq j \leq t, j \neq i} \left( \frac{a_{t+1} a_j - a'_t a_j - a'_t a'_t}{a_{t+1} a_i - a'_t a_i - a'_t a'_t} \right)}.
\]
Then (3.18) follows since we have
\[
\frac{a_{t+1} b_j - a'_t b'_j}{a_{t+1} a_i - a'_t a_i - a'_t a'_t} - a'_i b'_i = d(i, j) \quad (i \neq j),
\]
\[
\frac{a_{t+1} a_j - a'_t a_j - a'_t a'_t}{a_{t+1} a_i - a'_t a_i - a'_t a'_t} - a'_i a'_j = \frac{a'_i}{a_{t+1} a_i - a'_t a_i - a'_t a'_t} - d(i, i).
\]
Hence the assertion is clear.

**Definition 3.9.** Let \( c, a_c, \ D = \prod_{j \in J} C(v_j) \subset F \otimes R_{(n-1)+} \) be as above, and \( \tau \) a real embedding of \( F \) other than \( \iota_n \). Then we define
\[
X_{\text{fml}}(\{c, c_n c\}, \tau; D, a_c) := G(\{c, c_n c\}, \tau; D, a_c) + W(\{c, c_n c\}, \tau; D, a_c) + V_{\text{fml}}(\{c, c_n c\}, \tau; D, a_c).
\]

By the definitions of the \( G, W \)-invariants and Lemma 3.6, we see that the definition of \( X_{\text{fml}}(\{c, c_n \}, \tau; D, a_c) \) does not depend on the choice of a cone decomposition of \( D \). We also see that if \( D, D' \) satisfy \( D \cap D' = \emptyset \) then we have
\[ (3.19) \quad X_{\text{fml}}(\{c, c_n c\}, \tau; D \prod D', a_c) = X_{\text{fml}}(\{c, c_n c\}, \tau; D, a_c) + X_{\text{fml}}(\{c, c_n c\}, \tau; D', a_c).
\]
The following Lemmas 3.10, 3.11 are modifications of [Yo, Chapter III, (3.35), (3.46)].
Lemma 3.10. Let $c, a_c, \epsilon \neq \epsilon_n, D \subset F \otimes \mathbb{R}_{(n-1)+}$ be as above. We put

$$Z := \left[ \sum_{z \in (a_f)^{-1} \cap D; \{z\} \in c \text{ or } c_n c} \nu(z)^{-s} \right]_{s=0}.$$ 

Then $Z \in \nu(F)$. Moreover the following assertions hold.

(i) Assume that $D$ is a Shintani domain. Then $Z \in \mathbb{Q}$. More precisely, if $c = c_n c$, or if $\mathcal{O}^\times \cap F \otimes \mathbb{R}_{(n-1)+} = \emptyset$, then we have

$$Z = \zeta(0, c).$$

Otherwise we have

$$Z = \zeta(0, c) + \zeta(0, c_n c).$$

(ii) For $\alpha \in F_+ \cap a_c^{-1}$, we have

$$X_{\text{frl}}(\{c, c_n c\}, \nu; \alpha^{-1} D, (\alpha) a_c) - X_{\text{frl}}(\{c, c_n c\}, \nu; D, a_c)$$

$$= (\log \nu(\alpha) - \log \epsilon_n(\alpha)) Z + \left(\frac{1}{n} \log N(\alpha) - \log \nu(\alpha)\right) \left(1 - \frac{1}{n} \nu(F)/\mathbb{Q} Z\right).$$

In addition, assume that $D$ is a Shintani domain. Then we have

$$X_{\text{frl}}(\{c, c_n c\}, \nu; \alpha^{-1} D, (\alpha) a_c) - X_{\text{frl}}(\{c, c_n c\}, \nu; D, a_c) = (\log \nu(\alpha) - \log \epsilon_n(\alpha)) Z.$$

(iii) For $\epsilon \in \mathcal{O}^\times_+$, we have

$$X_{\text{frl}}(\{c, c_n c\}, \nu; \epsilon D, a_c) - X_{\text{frl}}(\{c, c_n c\}, \nu; D, a_c) = \frac{\nu(F)/\mathbb{Q}}{n} \log \nu(\epsilon).$$

In particular, if $D, D'$ are Shintani domains and $a_c, a'_c$ are integral ideals satisfying that $a_c, a'_c, c$ belong to the narrow ideal class, then there exist $\epsilon \in \mathcal{O}^\times_+, m \in \mathbb{N}$ satisfying

$$(3.20) \quad X_{\text{frl}}(\{c, c_n c\}, \nu; D', a'_c) - X_{\text{frl}}(\{c, c_n c\}, \nu; D, a_c) = \frac{1}{m} \log \nu(\epsilon)$$

for all real embeddings $\nu$ of $F$ such that $\nu \neq \nu_n$.

Proof. The fact that $Z \in \nu(F)$ follows from (3.2). For (i), assume that $D$ is a Shintani domain. First we claim that

$$(\ast) \quad \text{if } \{z \in (a_f)^{-1} \cap D \mid (z) a_f \in c_n c\} \neq \emptyset, \text{ then } \mathcal{O}^\times \cap F \otimes \mathbb{R}_{(n-1)+} \neq \emptyset.$$ 

Indeed, take $z_1 \in (a_f)^{-1} \cap D$ satisfying $(z_1) a_f \in c_n c$. Then $z_1 \in F_+$, by $D \subset F \otimes \mathbb{R}_{n+}$, so both $c$ and $c_n c$ belong to the narrow ideal class of $a_f$. Hence the narrow ideal class of $(\nu_n)$ is equal to that of (1), i.e., there exist $z_2 \in F_+, \epsilon_n \in \mathcal{O}^\times$ satisfying $1 = z_2 \nu_n \epsilon_n$. Hence the claim $(\ast)$ holds since $\epsilon_n \in \mathcal{O}^\times \cap F \otimes \mathbb{R}_{(n-1)+}$. Therefore, if $\mathcal{O}^\times \cap F \otimes \mathbb{R}_{(n-1)+} = \emptyset$, then $\{z \in (a_f)^{-1} \cap D \mid (z) a_f \in c \text{ or } c_n c\} = \{z \in (a_f)^{-1} \cap D \mid (z) a_f \in c\}$, so we have

$Z = \zeta(0, c) \in \mathbb{Q}$ by (2.5). The same holds when $c = c_n c$. Next assume that $c \neq c_n c$ and
that there exists an element \( \epsilon_n \in \mathcal{O}^\times \cap F \otimes \mathbb{R}_{(n-1)+} \). Let \( \nu_n \) be as in Lemma 3.3. Then we have the following bijection:
\[
\{ z \in ((\nu_n) a, f)^{-1} \cap \nu_n^{-1} e_n D \mid (z)(\nu_n) a, f \in c_n c \} \rightarrow \{ z \in (a, f)^{-1} \cap D \mid (z)a, f \in c_n c \}
\]
Namely, we can write
\[
(3.21)
\{ z \in (a, f)^{-1} \cap D \mid (z)a, f \in c \}
= \{ z \in (a, f)^{-1} \cap D \mid (z)a, f \in c_n c \}
\]
Hence we obtain
\[
\prod \nu_n \epsilon_n \{ z \in ((\nu_n) a, f)^{-1} \cap \nu_n^{-1} e_n D \mid (z)(\nu_n) a, f \in c_n c \}
\]
which is equal to \( \zeta(0, c) + \zeta(0, c_n c) \in \mathbb{Q} \) by (2.5). This complete the proof of (i). For (ii), a similar statement was proved by Yoshida [Yo, Chapter III, §3.7]. The same proof works, as follows. Since \( \{ z \in ((\alpha)a, f)^{-1} \cap \alpha^{-1} D \mid (z)(\alpha)a, f \in c \} \rightarrow \{ z \in (a, f)^{-1} \cap D \mid (z)a, f \in c \} \), \( z \mapsto \alpha z \) is a bijection, we have
\[
\sum_{z \in ((\alpha)a, f)^{-1} \cap \alpha^{-1} D, (z)(\alpha)a, f \in c} \tau(z)^{-s} = \tau(\alpha)^s \sum_{z \in (a, f)^{-1} \cap D, (z)a, f \in c} \tau(z)^{-s}.
\]
Hence, by definition, we obtain
\[
G(\{ c, c_n c \}, \tau; \alpha^{-1} D, (\alpha)a_c) = G(\{ c, c_n c \}, \tau; D, a_c) + \log \tau(\alpha) \cdot Z,
\]
\[
W(\{ c, c_n c \}, \tau; \alpha^{-1} D, (\alpha)a_c) = W(\{ c, c_n c \}, \tau; D, a_c) - \log \tau(\alpha) \cdot Z.
\]
Lemma 3.7-(i) implies that
\[
V_{\text{fin}}(\{ c, c_n c \}, \tau; \alpha^{-1} D, (\alpha)a_c)
= V_{\text{fin}}(\{ c, c_n c \}, \tau; D, a_c) + \left( \frac{1}{n} \log N(\alpha) - \log \tau(\alpha) \right) \left( \tau(Z) - \frac{1}{n} \text{Tr}_{(F)/Q} Z \right),
\]
so the first assertion of (ii) follows. If \( D \) is a Shintani domain, then \( Z \in \mathbb{Q} \) by (i), so we have \( \tau(Z) - \frac{1}{n} \text{Tr}_{(F)/Q} Z = 0 \). This completes the proof of (ii). The assertion (iii) follows from (ii) by putting \( \alpha := c \), and by noting that \( \log N(\epsilon) = \log_c(\epsilon) = 0 \). Finally, we prove (3.20). By the assumption on \( a_c, a'_c \), there exists \( \alpha \in F^+ \cap a_c^{-1} \) satisfying \( a'_c = (\alpha)a_c \). Hence, by (ii) and (2.4), we can write \( X_{\text{fin}}(\{ c, c_n c \}, \tau; \alpha^{-1} D, a'_c) = X_{\text{fin}}(\{ c, c_n c \}, \tau; D, a_c) \) in the form \( \frac{1}{m} \log \epsilon(\epsilon) \) with \( \epsilon \in \mathcal{O}_F^\times, m \in \mathbb{N} \). Since \( D', \alpha^{-1} D \) are Shintani domains, by [Yo, Chapter III, Lemma 3.13], we can write these in the form
\[
D' = \prod_{j \in J} C(v_j), \quad \alpha^{-1} D = \prod_{j \in J} \epsilon_j C(v_j)
\]
with \( \epsilon_j \in \mathcal{O}_F^\times \). Therefore, by (iii), \( X_{\text{fin}}(\{ c, c_n c \}, \tau; D', a'_c) - X_{\text{fin}}(\{ c, c_n c \}, \tau; \alpha^{-1} D, a'_c) \) also can be written in the form \( \frac{1}{m} \log \epsilon(\epsilon) \). Then the assertion (3.20) is clear. □
Lemma 3.11. Let \( c \in C_f \). If \( D, D' \) are Shintani domains and \( a_c, a'_c \) are integral ideals satisfying that \( a_c, a'_c, c \) belong the same narrow ideal class, then there exist \( \epsilon \in \mathcal{O}_+^\times \), \( m \in \mathbb{N} \) satisfying

\[
X(c, \nu; D', a'_c) - X(c, \nu; D, a_c) = \frac{1}{m} \log \nu(\epsilon)
\]

for all real embeddings \( \nu \) of \( F \).

**Proof.** We proceed similarly to the above proof. Let \( Z := \left[ \sum_{z \in (a, f)^{-1} \cap D, (z)_{a_c} \in \epsilon(z)^{-1}} \right] \) and \( \alpha \in F_+ \cap a_c^{-1} \). We easily see that

\[
G(c, \nu; \alpha^{-1}, (\alpha)a_c) = G(c, \nu; D, a_c) + \log \nu(\alpha) \cdot Z,
\]

\[
W(c, \nu; \alpha^{-1}, (\alpha)a_c) = W(c, \nu; D, a_c) - \log \nu(\alpha) \cdot Z.
\]

In addition, by [Yo, Chapter III, (3.44)], we have

\[
V(c, \nu; \alpha^{-1}, (\alpha)a_c) = V(c, \nu; D, a_c) + \left( \frac{1}{n} \log N(\alpha) - \log \nu(\alpha) \right) \left( Z - \frac{1}{n} \text{Tr}_{v(F)/Q} Z \right).
\]

Therefore we obtain

\[
X(c, \nu; \alpha^{-1}, (\alpha)a_c) - X(c, \nu; D, a_c)
= (\log \nu(\alpha) - \log_0(\alpha)) Z + \left( \frac{1}{n} \log N(\alpha) - \log \nu(\alpha) \right) \left( Z - \frac{1}{n} \text{Tr}_{v(F)/Q} Z \right),
\]

which corresponds to Lemma 3.10-(ii). The remaining arguments are exactly the same as in the proof of Lemma 3.10.

\[\square\]

Lemma 3.12. Let \( c, a_c, \nu \neq \nu_n \) be as above, \( \nu_n \) as in Lemma 3.3, and \( D \) a Shintani domain. Then the following assertions hold.

(i) We have

\[
X_{\text{finm}}\{c, c_n c\}, \nu; \nu_n^{-1}D, (\nu_n) a_c - X_{\text{finm}}\{c, c_n c\}, \nu; D, a_c = (\log \nu(\nu_n) - \log_0(\nu_n)) \zeta(0, c).
\]

Here we note that the roles of \( c, c_n c \) in the symbol \( X_{\text{finm}}\{c, c_n c\}, \nu; \nu_n^{-1}D, (\nu_n) a_c \) are exchanged.

(ii) If \( c = c_n c \) or if \( \mathcal{O}_+^\times \cap F \otimes \mathbb{R}_{n-1} = \emptyset \), then we have

\[
X_{\text{finm}}\{c, c_n c\}, \nu; D, a_c = X(c, \nu; D, a_c).
\]

Otherwise, we have

\[
X_{\text{finm}}\{c, c_n c\}, \nu; D, a_c = X(c, \nu; D, a_c) + X(c_n c, \nu; \nu_n^{-1}E_n D, (\nu_n) a_c) + (\log \nu(\nu_n^{-1}) - \log_0(\nu_n^{-1})) \zeta(0, c_n c).
\]
Proof. We again proceed similarly to the above proofs. In particular, for (i), the same proof as in Lemma 3.10-(ii) works, using Lemma 3.7-(ii) and a bijection \( \{ z \in ((v_n)_{a, f})^{-1} \cap \nu_n^{-1} D \mid (z)_{(v_n)_{a, f}} \in \mathbb{C} \text{ or } c_n c, z \rightarrow \nu_n z \} \). We prove (ii). In the former case, we have \( \{ z \in (a, f)^{-1} \cap D \mid (z)_{a, f} \in \mathbb{C} \text{ or } c_n c, z \rightarrow \nu_n z \} \) by (*). We prove as in Lemma 3.10-(ii) works, using Lemma 3.7-(ii) and a bijection \( \| \) in the proof of Lemma 3.10. Then we have \( G_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) = G(c, \nu D, \mathbf{a}_c), W_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) = W(c, \nu D, \mathbf{a}_c) \) by definition. Furthermore we obtain \( V_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) = V(c, \nu D, \mathbf{a}_c) \), as we noted after Definition 3.5. Thus the assertion follows in this case. Next, we assume that \( c \neq c_n c \) and that there exists an element \( \epsilon_n \in O^\times \cap F \otimes \mathbb{R}_{(n-1)+} \). We put \( Z := \sum_{z \in ((v_n)_{a, f})^{-1} \cap \nu_n^{-1} D \mid (z)_{(v_n)_{a, f}} \in \mathbb{C} \cap c_n c, z \rightarrow \nu_n z} \), which is equal to \( \zeta(0, c_n c) \in \mathbb{Q} \) by (2.5). Then by (3.21), we have
\[
G(\{ c, c_n c \}, \nu \alpha^{-1} D, (\alpha)_{\mathbf{a}_c}) = G(c, \nu D, \mathbf{a}_c) + G(c_n c, \nu \nu_n^{-1} \epsilon_n D, (\nu_n)_{\mathbf{a}_c}) + \log \nu_n \epsilon_n \cdot Z, \\
W(\{ c, c_n c \}, \nu \alpha^{-1} D, (\alpha)_{\mathbf{a}_c}) = W(c, \nu D, \mathbf{a}_c) + W(c_n c, \nu \nu_n^{-1} \epsilon_n D, (\nu_n)_{\mathbf{a}_c}) - \log \nu_n \epsilon_n \cdot Z.
\]
By further using (3.13) and \( \nu(Z) - \frac{1}{n} \text{Tr}_F(\frac{\mathbf{g}}{\mathbf{a}_c}) = 0 \), we obtain
\[
V_{fml}(\{ c, c_n c \}, \nu \alpha^{-1} D, (\alpha)_{\mathbf{a}_c}) = V(c, \nu D, \mathbf{a}_c) + V(c_n c, \nu \nu_n^{-1} \epsilon_n D, (\nu_n)_{\mathbf{a}_c}).
\]
Summing up the above, we obtain the desired formula.

Proof of Theorem 3.1. By reordering the embeddings \( \nu_1, \ldots, \nu_n \) of \( F \) if necessary, we may assume that \( j = n, i < n \). We take \( \nu_n \in O^\times \) as in Lemma 3.3, \( D, \nu, X \epsilon, \epsilon (t \in T) \) as in Lemma 3.4, and integral ideals \( \mathbf{a}_d \) satisfying that \( \mathbf{a}_d \) and \( d \) belong to the same narrow ideal class for \( d = c_n c, c \). For real numbers \( \alpha(a) \) associated with real embeddings \( \nu \) of \( F \), we write \( \alpha(a) \sim b(\nu) \) if there exist \( \alpha \in O^\times F \) and \( m \in \mathbb{N} \) satisfying \( \alpha(a) - b(\nu) = \frac{1}{m} \log \nu \) for all \( \nu \neq \nu_n \). This is an equivalence relation. Then Theorem 3.1 states that
\[
X(c, \nu D, \mathbf{a}_c) + X(c_n c, \nu D, \mathbf{a}_c) \sim 0.
\]
On the other hand, by Lemma 3.4-(iii), (3.19), Lemma 3.10-(iii), we can write
\[
X_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) = \sum_{t \in T} X_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) - \sum_{t \in T} X_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) \sim 0.
\]
In the following of the proof, we show that (3.22) follows from (3.23). We consider two cases:

- Case 1. When \( c = c_n c \) or when \( O^\times \cap F \otimes \mathbb{R}_{(n-1)+} = \emptyset \).

- Case 2. Cases other than Case 1.

In Case 1, we can write
\[
X_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) = X_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) + X_{fml}(\{ c, c_n c \}, \nu D, \mathbf{a}_c) = X(c, \nu D, \mathbf{a}_c) + X(c_n c, \nu D, \mathbf{a}_c) \sim X(c, \nu D, \mathbf{a}_c) + X(c_n c, \nu D, \mathbf{a}_c).
\]
by using (3.19), Lemma 3.12-(ii), (3.20), Lemma 3.12-(i) and (2.4), Lemma 3.12-(ii), Lemma 3.11, respectively. In Case 2, we take an element $\epsilon_n \in \mathcal{O}^\times \cap F \otimes \mathbb{R}_{(n-1)+}$. Then we obtain

\begin{equation}
X_{\text{fin}}(c, c_n c), \nu D, a_c) = X_{\text{fin}}(c, c_n c), \nu D, a_c) + X_{\text{fin}}(c, c_n c), \nu D, a_c)
\end{equation}

by (3.19),

\begin{equation}
X_{\text{fin}}(c, c_n c), \nu D, a_c) \sim X(c, \nu D, a_c) + X(c, \nu_n^{-1} \epsilon_n D, (\nu_n)a_c)
\end{equation}

by Lemma 3.12-(ii) and (2.4), and

\begin{equation}
X_{\text{fin}}(c, c_n c), \nu D, a_c) \sim X_{\text{fin}}(c, c_n c), \nu D, a_c) + X(c, \nu_n^{-1} \epsilon_n D, (\nu_n)a_c)
\end{equation}

by (3.20), Lemma 3.12-(i) and (2.4), Lemma 3.12-(ii) and (2.4), respectively. Hence, by (3.24), (3.25), (3.26) and Lemma 3.11, we obtain

\begin{equation}
X_{\text{fin}}(c, c_n c), \nu D, a_c) \sim 2X(c, \nu D, a_c) + 2X(c, \nu_n^{-1} \epsilon_n D, (\nu_n,a_c)).
\end{equation}

Therefore we see that (3.23) implies (3.22) in both cases. This completes the proof. \hfill \Box

**Corollary 3.13.** Let $K$ be a CM-field which is an abelian extension of a totally real field $F$, $\rho$ the unique complex conjugation in $G := \text{Gal}(K/F)$, $f$ an integral divisor with $f_{K/F}|\mathcal{f}$, and $\text{Art}_f$ the composite map $C_f \to C_{(K/F)\to G}$. Then we have

\begin{equation}
\prod_{c \in \text{Art}_f^{-1}(\tau)} \exp(X(c, \nu D, a_c)) \cdot \prod_{c \in \text{Art}_f^{-1}(\tau \rho)} \exp(X(c, \nu D, a_c)) \in \begin{cases} \mathcal{O}_+(Q) \quad (F \neq Q), \\ (K \times \mathbb{R}_+)^Q \quad (F = Q), \end{cases}
\end{equation}

for any real embedding $\nu$ of $F$ and $\tau \in G$. Here $K^\times \cap \mathbb{R}_+$ in the case $F = Q$ is well-defined since $K$ is normal over $\mathbb{Q}$.

**Proof.** Let $c_j \in C_f$ be as in Theorem 3.1. Since $K$ is a CM-field, $\text{Art}_f(c_j)$ are equal to $\rho$ for all $j$. Therefore we have $\{c \in \text{Art}_f^{-1}(\tau \rho) \} = \{c_j \mid c \in \text{Art}_f^{-1}(\tau)\}$. If $[F : \mathbb{Q}] \geq 2$, then there exists $j$ satisfying $\nu \neq \nu_j$, so the assertion follows from Theorem 3.1. If $F = \mathbb{Q}$, then $X(c, \text{id}) = \zeta'(0, c)$, so the assertion is a part of Stark’s conjecture for $(K \cap \mathbb{R})/\mathbb{Q}$, which has been proved. \hfill \Box

### 4 Proof of Lemma 3.4.

We prove Lemma 3.4 in this section. We take an arbitrary Shintani domain $D = \prod_{j \in J} C(v_j)$ with $v_j = (v_{j_1}, \ldots, v_{j_r}) \in \mathcal{O}_+(Q)$, and put $J_n := \{j \in J \mid r(j) = n\}$. We denote the set of all faces of $n$-dimensional cones $C(v_j)$ $(j \in J_n)$ by $F(D)$. Namely,

\begin{equation}
F(D) := \{C(v_{j_1}, v_{j_2}, \ldots, v_{j_r}) \mid j \in J_n, 1 \leq r \leq n-1, 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}.
\end{equation}
Then we see that

\[(4.1) \quad F \otimes \mathbb{R}_{n+} = \bigcup_{\epsilon \in \mathcal{O}_{n+}^\epsilon, j \in J_n} \epsilon \overline{C(v_j)} = \left( \bigcap_{\epsilon \in \mathcal{O}_{n+}^\epsilon, j \in J_n} \epsilon \overline{C(v_j)} \right) \bigcup \left( \bigcap_{\epsilon \in \mathcal{O}_{n+}^\epsilon, C \in F(D)} \epsilon C \right), \]

where \( \overline{C(v_j)} \) denotes the topological closure of \( C(v_j) \) in \( F \otimes \mathbb{R}_{n+} \). We may assume the following (replacing the cone decomposition \( F = \bigcup_{j \in J} C(v_j) \) by its refinement, if necessary).

\((\star)\) If \( C, C' \in F(D) \) and \( \epsilon \in \mathcal{O}_{n+}^\epsilon \) satisfy \( (\epsilon C) \cap C' \neq \emptyset \), then \( \epsilon C = C' \).

\((\star\star)\) If \( C \in F(D), j \in J - J_n \) and \( \epsilon \in \mathcal{O}_{n+}^\epsilon \) satisfy \( C(v_j) \cap \epsilon C \neq \emptyset \), then \( C(v_j) \subset \epsilon C \).

In the following, we shall replace lower-dimensional cones \( C(v_j) \) \( (j \in J - J_n) \) so that \( D \) can be expressed as the disjoint union of \( n \)-dimensional cones \( C(v_j) \) \( (j \in J_n) \) and their faces on “the upper side” with respect to the \( x_n \)-axis. First we define which face is on “the upper side”: Consider the hyper plane \( P := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0 \} \). We denote the line passing through two points \( z, z' \in \mathbb{R}^n \) by \( \mathcal{L}(z, z') := \{ sz + (1 - s)z' \mid s \in \mathbb{R} \}. \)

Take \( \nu \in F \cap F \otimes \mathbb{R}_{(n-1)+} \) such that \( |t_1(\nu) - 1|, |t_2(\nu) - 1|, \ldots, |t_{n-1}(\nu) - 1|, |t_n(\nu) + 1| \) are sufficiently small. Then we may assume that for any \( C = C(v_1, \ldots, v_r) \in F(D) \), the angles between \( L(v_i, \nu v_i) \) and \( P \) \( (i = 1, \ldots, r) \) are closer to a right angle than that between \( C \) and \( P \). (Note that \( C \in F(D) \) and \( P \) do not meet at a right angle since the basis of \( C \) is in \( F \).) In particular, we see that for any \( C \in F(D), z \in C \), the line \( L(z, \nu z) \) is not contained in \( C \). Namely, the assumption on \( \nu \) implies that

\[(4.2) \quad L(z, \nu z) \cap C = \{ z \} \quad (C \in F(D), \ z \in C). \]

Actually, the following argument works whenever \( \nu \in F \cap F \otimes \mathbb{R}_{(n-1)+} \) satisfies \( (4.2) \).

**Proposition 4.1.** Let \( \nu, D = \bigcup_{j \in J} C(v_j) \) be as above. Then \( L(z, \nu z) \cap \epsilon C(v_j) \) is a one-point set or the empty set for any \( z \in F_+, j \in J - J_n, \epsilon \in \mathcal{O}_{n+}^\epsilon \).

**Proof.** Assume that there exists an element \( z_1 \in L(z, \nu z) \cap \epsilon C(v_j) \). By \((4.1)\) and \((\star\star)\), there exist \( \epsilon_1 \in \mathcal{O}_{n+}^\epsilon \) and \( C \in F(D) \) satisfying \( \epsilon C(v_j) \subset \epsilon_1 C \). Put \( z_2 := \epsilon_1^{-1} z_1 \). Then \( z_2 \in C \), so \( L(z_2, \nu z_2) \cap C \) is a one-point set by \((4.2)\). We easily see that \( L(z_2, \nu z_2) = \epsilon_1^{-1} L(z_1, \nu z_1) = \epsilon_1^{-1} L(z, \nu z) \), \( L(z, \nu z) \cap \epsilon C(v_j) \subset L(z, \nu z) \cap \epsilon_1 C = \epsilon_1 \epsilon_1^{-1} L(z, \nu z) \cap C \). Hence we have \( L(z, \nu z) \cap \epsilon C(v_j) \subset \epsilon_1(L(z_2, \nu z_2) \cap C) \). Then the assertion is clear. \( \square \)

**Definition 4.2.** Let \( C(v) \) be an \( n \)-dimensional cone with the basis \( v = (v_1, \ldots, v_n) \in \mathcal{O}^n \).

We say that a face \( C = C(v_1, v_2, \ldots, v_r) \) \( (1 \leq r \leq n - 1, 1 \leq i_1 < i_2 < \cdots < i_r \leq n) \) of \( C(v) \) is on the upper side (resp. on the lower side) of \( C(v) \) if and only if for each \( z \in C \), there exists \( \delta > 0 \) satisfying \( \{ svz + (1 - s)z \mid 0 < s < \delta \} \subset C(v) \) (resp. \( \{ svz + (1 - s)z \mid -\delta < s < 0 \} \subset C(v) \)).

We put \( \{ C(v_t) \mid t \in T \} \) to be the set of all faces in \( F(D) \) which are on the upper side of some \( C(v_j) \) with \( j \in J_n \). We replace \( D \) by

\[ D_0 := \left( \bigcap_{j \in J_n} C(v_j) \right) \bigcup \left( \bigcap_{t \in T} C(v_t) \right). \]
Proposition 4.3. The above $D_0$ is again a Shintani domain.

Proof. Let $z \in F \otimes \mathbb{R}_+ - \bigsqcup_{j \in J_n} \epsilon C(v_j)$. It suffices to show that there exist unique $t \in T$, $z' \in C(v_{j_1})$, $\epsilon \in \mathcal{O}_+^x$, and $s \in [0,1]$, satisfying $z = \epsilon z' + s$. Take $\delta > 0$ so that $I := \{svz + (1 - s)z' \mid 0 \leq s \leq \delta\} \subset F \otimes \mathbb{R}_+$. Then we can write $I \subset \bigsqcup_{k=1}^m \epsilon_k C(v_{j_k})$ with $j_k \in J$, $\epsilon_k \in \mathcal{O}_+^x$, $m \in \mathbb{N}$. Replacing $\delta$ by a smaller positive number if necessary, we may assume that $m = 2$. Then we can write $z \in \epsilon_1 C(v_{j_1})$, $\{svz + (1 - s)z' \mid 0 < s < \delta\} \subset \epsilon_2 C(v_{j_2})$ with $j_1 \in J - J_n$, $j_2 \in J_n$, $\epsilon_1, \epsilon_2 \in \mathcal{O}_+^x$ by Proposition 4.1. Therefore $z' := \epsilon_2^{-1} z$ is contained in a face on the upper side of $C(v_{j_2})$, as desired. In order to prove the uniqueness, assume that $t_k \in T$, $z_k \in C(v_{j_k})$ $(k = 1, 2)$, $\epsilon \in \mathcal{O}_+^x$ satisfy $z_1 = \epsilon z_2$. Then there exist $\delta_k > 0$ and $j_k \in J_n$ satisfying $\{svz_k + (1 - s)z_k \mid 0 < s < \delta_k\} \subset C(v_{j_k})$ $(k = 1, 2)$. In particular we have $C(v_{j_1}) \cap \epsilon C(v_{j_2}) \neq \emptyset$. Since $\bigsqcup_{j \in J_n, \epsilon \in \mathcal{O}_+^x} \epsilon C(v_j)$ is a disjoint union, we have $C(v_{j_1}) = C(v_{j_2})$, $\epsilon = 1$. Hence we obtain $z_1 = z_2$. This completes the proof.

By definition, for $t \in T$, there exists a unique $j(t) \in J_n$ satisfying that $C(v_t)$ is on the upper side of $C(v_{j(t)})$. On the other hand, for $t \in T$, there exist a unique $j(t)' \in J_n$ and a unique $\epsilon(t) \in \mathcal{O}_+^x$ satisfying that $\epsilon(t) C(v_t)$ is on the lower side of $C(v_{j(t)'})$. This follows by a similar argument to the above proof, by using the line $\{svz + (1 - s)z' \mid -\delta < s < 0\}$ and the assumption (§). Put $X_t := \{svz + (1 - s)z \mid z \in C(v_t), 0 \leq s \leq 1\}$, $Y_t := \epsilon(t) X_t$ for $t \in T$. We write $v_t := (v_{t_1}, \ldots, v_{t_r})$, and take $n \in \mathbb{N}$ so that $\nu v \in \mathcal{O}$. Then $X_t$ can be expressed as a finite disjoint union of cones whose bases are subsets of $\{v_{t_1}, \ldots, v_{t_r}, n \nu v_{t_1}, \ldots, n \nu v_{t_r}\}$. Now we shall show that

$$
(D_0 \bigsqcup \nu D_0) \bigsqcup \bigoplus_{t \in T} Y_t = \bigoplus_{t \in T} X_t.
$$

For $j \in J_n$, we put $T_j := \{t \in T \mid j(t) = j\}$, $T_j' := \{t \in T \mid j(t)' = j\}$, and $D_{0,j} := C(v_j) \bigsqcup (\bigsqcup_{t \in T_j} C(v_t))$. Then it suffices to show that

$$(4.3)\quad D_{0,j} \bigsqcup \nu D_{0,j} \bigsqcup \bigoplus_{t \in T_j} Y_t = \bigoplus_{t \in T_j} X_t \quad (j \in J_n).$$

Here we note that

- $\{C(v_t) \mid t \in T_j\}$ is the set of all faces on the upper side of $C(v_j)$.
- $\{\epsilon(t) C(v_t) \mid t \in T_j'\}$ is the set of all faces on the lower side of $C(v_j)$.
- $\{\nu C(v_t) \mid t \in T_j\}$ is the set of all faces on the lower side of $\nu C(v_j)$.
- $\{\epsilon(t) \nu C(v_t) \mid t \in T_j'\}$ is the set of all faces on the upper side of $\nu C(v_j)$.

We also note that $\nu C(v_j)$ is almost equal to the “mirror image” of $C(v_j)$ with respect to the hyperplane $P$. Let $z_0 \in C(v_{l_0})$ with $l_0 \in T$. Then the line segment $\{svz_0 + (1 - s)z_0 \mid 0 \leq s \leq 1\}$ contains the point $z_0$ in the face $C(v_{l_0})$ on the upper side of $C(v_j)$, a point $z_1$ in a face on the lower side of $C(v_j)$, a point $z_2$ in a face on the upper side of
\[ \nu C(v_j), \text{ and the point } \nu z_0 \text{ in the face } \nu C(v_j) \text{ on the lower side of } \nu C(v_j). \] We write 
\[ z_1 = s_1 \nu z + (1 - s_1) z, z_2 = s_2 \nu z + (1 - s_2) z \] with \( 0 < s_1 < s_2 < 1 \). Then we can decompose 
\[ \{ s \nu z_0 + (1 - s) z_0 \mid 0 \leq s \leq 1 \} \subset X_0 \] into 
\[ \{ s \nu z_0 + (1 - s) z_0 \mid 0 < s < s_1 \} \prod \{ s \nu z_0 + (1 - s) z_0 \mid s_1 \leq s \leq s_2 \} \prod \{ s \nu z_0 + (1 - s) z_0 \mid s_2 < s < 1 \} \prod \{ \nu z_0 \}, \] which are subsets of \( \prod_{i \in T_2} C(v_i), C(v_j), \prod_{i \in T_2} Y_i, \nu C(v_j), \nu \prod_{i \in T_2} C(v_i) \), respectively. Hence we obtain the decomposition (4.3). This completes the proof of Lemma 3.4.

5 A relation between CM-periods and Stark units.

In this section, we discuss a relation between monomial relations among \( \exp(X(c, \ell)) \)'s, those among CM-periods, and Yoshida’s conjecture on Shimura’s period symbol \( p_K \). We note that some results (Propositions 5.4, 5.6, 5.9) are applications of Theorem 3.1. First we recall some properties of \( p_K \).

**Theorem 5.1** ([Shim, Theorem 32.5]). Let \( K \) be a CM-field, \( J_K \) the set of all embeddings of \( K \) into \( \mathbb{C} \), and \( I_K := \bigoplus_{\sigma \in J_K} \mathbb{Z} \sigma \) the free abelian group generated by elements in \( J_K \). We denote the complex conjugation by \( \rho \). Then there exists a bilinear map \( p_K : I_K \times I_K \to \mathbb{C}^\times / \mathbb{Q}^\times \) satisfying the following properties.

1. Let \( \Phi = \sum_{i=1}^{[K: \mathbb{Q}]/2} \sigma_i \) be a CM-type of \( K \) (i.e., \( \sigma_i \in J_K, \Phi + \Phi \rho = \sum_{\sigma \in J_K} \sigma \)). Then \( p_K(\sigma_i, \Phi) \) (for \( i = 1, \ldots, [K: \mathbb{Q}]/2 \)) are given as follows: Let \( A \) be an abelian variety with CM of type \( (K, \Phi) \). Namely, for \( i = 1, \ldots, [K: \mathbb{Q}]/2 \), there exists a non-zero holomorphic differential 1-form \( \omega_i \) on \( A \) where each \( a \in K \cong \text{End}(A) \otimes \mathbb{Z} \mathbb{Q} \) acts as scalar multiplication by \( \sigma_i(a) \). We assume that \( A, \omega_i \) are defined over \( \overline{\mathbb{Q}} \). Then we have
   \[ \int_c \omega_i \mod \overline{\mathbb{Q}}^\times = \pi p_K(\sigma_i, \Phi) \]
   for every \( c \in H_1(A(\mathbb{C}), \mathbb{Z}) \) with \( \int_c \omega_i \neq 0 \).

2. \( p_K(\xi \rho, \eta) = p_K(\xi, \eta \rho) = p_K(\xi, \eta)^{-1} \) for \( \xi, \eta \in I_K \).

3. Let \( L \) be a CM-field which is an extension of \( K \). Then \( p_K(\xi, \text{Res}(\zeta)) = p_L(\text{Res}(\xi), \zeta), p_K(\text{Res}(\zeta), \xi) = p_L(\zeta, \text{Res}(\xi)) \) for \( \xi \in I_K, \zeta \in I_L \). Here linear maps \( \text{Res} : I_L \to I_K \), \( \text{Inf} : I_K \to I_L \) are defined by \( \text{Res}(\sigma) := \sigma|_K \) (\( \sigma \in J_L \)), \( \text{Inf}(\sigma) := \sum_{\tau \in J_L, \tau|_K = \sigma} \tau \) respectively.

4. Let \( K' \) be a CM-field with an isomorphism \( \gamma : K' \cong K \). Then \( p_K(\gamma \xi, \gamma \eta) = p_K(\xi, \eta) \) for \( \xi, \eta \in I_K \).

Let \( K \) be a CM-field which is an abelian extension of a totally real field \( F \). We put \( f_{K/F} \) to be the conductor and \( G := \text{Gal}(K/F) \). Hereinafter we take \( \ell, D, a_\ell \) for Yoshida’s invariant \( X(c, \ell; D, a_\ell) \) in the following manner: We regard a number field as a subfield
of $\mathbb{C}$. Then $\iota := \text{id} \in J_F$ has a meaning. Fix a Shintani domain $D$ of $F$ and a complete set of representatives $\{a_\mu\}_\mu$ of the narrow ideal class group of $F$. We assume that all of $a_\mu$ are integral ideals. We choose $a_c$ (satisfying that $a, f$ and $c$ belong to the same narrow ideal class) from among $\{a_\mu\}_\mu$, and put $X(c) := X(c, \text{id}; D, a_c)$. Then Yoshida’s absolute period symbol [Yo, Chapter III, (3.10)] is defined by

$$g_K(\text{id}, \tau) := \pi^{-\mu(\tau)/2} \exp\left(\frac{1}{|G|} \sum_{\chi \in \hat{G} \cap \hat{f}} \sum_{\chi \in (\hat{G}-\hat{f})} \frac{\chi(\tau)}{L(0, \chi)} \sum_{c \in C_f} \chi(c) X(c)\right) \quad (\tau \in G).$$

Here we put $\mu(\tau) := 1, -1, 0$ if $\tau = \text{id}, \rho$, otherwise, respectively. The sum $\sum_{\chi \in \hat{G} \cap \hat{f}}$ runs over all integral divisors $f$ dividing $\hat{f}$, and put $(\hat{G}-\hat{f}) := \{\chi \in \hat{G} \mid \text{the conductor of } \chi = f\}$. Then we may regard each $\chi \in (\hat{G}-\hat{f})$ as a character of $C_f$.

**Conjecture 5.2** ([Yo, Chapter III, Conjecture 3.9]). Let $K, F, G$ be as above. Then we have

$$g_K(\text{id}, \tau) \mod \overline{Q}^x = p_K(\text{id}, \tau) \quad (\tau \in G).$$

**Remark 5.3.** Strictly speaking, Yoshida defined the symbol $g_K(\text{id}, \tau)$ by using the original $W$-invariant and formulated the above Conjecture, but this does not matter: We see that $g_K(\text{id}, \tau) \mod \overline{Q}^x$ does not change due to the modification of the $W$-invariant, since the $W$-invariants in [Yo] and in this paper are of the form $\alpha \log \alpha$ with $\alpha \in \overline{Q}$. We also note that $g_K(\text{id}, \tau) \mod \overline{Q}^x$ does not depend on the choices of a Shintani domain $D$ and integral ideals $\{a_\mu\}_\mu$ by [Yo, Chapter III, §3.6, 3.7].

Under the above Conjecture, we can express any values of Shimura’s period symbol, not only of the form $p_K(\text{id}, \tau)$, in terms of $\exp(X(c))$ (for details, see [Yo, Chapter III, the paragraph following Conjecture 3.9]). Let us consider the opposite direction: a (conjectural) formula for $\exp(X(c))$ in terms of $p_K$. For simplicity, we extend Shimura’s period symbol $p_K$ to a bilinear map $I_K \otimes_{\mathbb{Z}} \mathbb{Q} \times I_K \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{C}^x/\overline{\mathbb{Q}}^x$, which is also denoted by $p_K$.

**Proposition 5.4.** Conjecture 5.2 is equivalent to the following: Let $F, K, G, \hat{f}$ of $\hat{f}$ be as above, $\hat{f}_0$ an integral divisor of $F$ satisfying $\hat{f}_0 \mid \hat{f}$. Then we have

$$\prod_{c \in \text{Art}^{-1}(\tau)} \exp(X(c)) \mod \overline{Q}^x = \pi^{\zeta_{\hat{f}_0}(0, \tau)} p_K(\tau, \sum_{\sigma \in G} \zeta_{\hat{f}_0}(0, \sigma)\sigma).$$

Here $\text{Art}_{\hat{f}_0}$ is the composite map $C_{\hat{f}_0} \rightarrow C_{\hat{f}} \xrightarrow{\text{Art}} G$, and we put $\zeta_{\hat{f}_0}(s, \sigma) := \zeta_s(s, \sigma)$ with $S := \{\text{all places of } F \text{ dividing } \hat{f}_0\}$.

**Proof.** First we note that $\zeta_{\hat{f}_0}(0, \sigma) \in \overline{Q}$ by (2.5). We prove that Conjecture 5.2 implies (5.1). The converse follows by a similar but simpler argument. The right-hand side
of (5.1) is equal to \( \pi^{G_0(0, \tau)} \rho_K(\text{id}, \sum_{\sigma \in G} \zeta_0(0, \sigma \tau) \sigma) \) by Theorem 5.1-(iv). Conjecture 5.2 states that this is equal to

\[
\exp\left( \frac{1}{|G|} \sum_{\chi \in \hat{G}} \sum_{c \in C_{f_0}} \chi(\tau)^{-1} \prod_{p \mid f_0} (1 - \chi(p)) \sum_{c \in C_{\ell}} \chi(c) X(c) \right) \mod \mathbb{Q}^x.
\]

Here we used the following relations:

\[
\sum_{\sigma \in G} \frac{-\mu(\sigma)}{2} \zeta_0(0, \sigma \tau) = -\frac{1}{2} \zeta_0(0, \tau) + \frac{1}{2} \zeta_0(0, \tau \rho)^{-1},
\]

\[
\sum_{\sigma \in G} \chi(\sigma) \zeta_0(0, \sigma \tau) = \chi(\tau)^{-1} L(0, \chi_0) = \chi(\tau)^{-1} \prod_{p \mid f_0} (1 - \chi(p)) L(0, \chi),
\]

where the last equality in the first line follows from the fact that \( L(0, \chi) = 0 \) for all even characters \( \chi \), and the symbol \( \chi_0 \) in the second line denotes the composite map \( C_{f_0} \to C_{f} \to C^x \). Therefore it suffices to show that

\[
\prod_{p \mid f_0} (1 - \chi(p)) \sum_{c \in C_{f_0}} \chi(c) X(c) \text{ in (5.2) can be replaced by } \sum_{c \in C_{f_0}} \chi(c) X(c).
\]

Indeed by using (5.3), we see that (5.2) is equal to

\[
\exp\left( \frac{1}{|G|} \sum_{\chi \in \hat{G}} \sum_{c \in C_{f_0}} \chi(c) X(c) \right) \mod \mathbb{Q}^x
\]

\[
= \exp\left( \frac{1}{2} \sum_{c \in \text{Art}_{f_0}^1(\tau)} X(c) \right) - \frac{1}{2} \sum_{c \in \text{Art}_{f_0}^1(\tau \rho)} X(c) \mod \mathbb{Q}^x.
\]

Here we used the relation: \( \sum_{\chi \in \hat{G}} \chi(\tau) = \frac{|G|}{2}, -\frac{|G|}{2}, 0 \) if \( \tau = \text{id}, \rho, \) otherwise, respectively. Then the assertion follows from Corollary 3.13.

In the rest of the proof, we will show (5.3), which can be reduced to the following: Let \( \mathfrak{f}, \mathfrak{f}' \) be integral divisors satisfying \( \mathfrak{f} \mid K/F, \mathfrak{f} \mid \mathfrak{f}' \parallel f_0 \). Then for any prime ideal \( \mathfrak{q} \mid f_0^{\mathfrak{f}} \), we have

\[
\exp\left( \sum_{\chi \in \hat{G}} \chi(\tau)^{-1} \prod_{p \mid f_0} (1 - \chi_{\mathfrak{f} q}(p)) \sum_{c \in C_{\mathfrak{f} q}} \chi_{\mathfrak{f} q}(c) X(c) \right)
\]

\[
= \exp\left( \sum_{\chi \in \hat{G}} \chi(\tau)^{-1} \prod_{p \mid f_0} (1 - \chi_{\mathfrak{f} q}(p)) \sum_{c \in C_{\mathfrak{f} q}} \chi_{\mathfrak{f} q}(c) X(c) \right) \mod \mathbb{Q}^x.
\]

In order to prove (5.4), we recall a result from [KY1]. We note that \( \{ a_{\mu} \}_\mu \) is a complete system of representatives of the narrow ideal class group, so is \( \{ a_{\mu} q \}_\mu \). We write \( X(c) \) as \( X(c, \{ a_{\mu} \}_\mu) \), \( X(c, \{ a_{\mu} q \}_\mu) \) when we choose \( a_c \) from among \( \{ a_{\mu} \}_\mu, \{ a_{\mu} q \}_\mu \) respectively. Then [KY1, Lemma 5.3] states that

\[
\sum_{c \in C_{\mathfrak{f} q}} \chi_{\mathfrak{f} q}(c) X(c, \{ a_{\mu} \}_\mu)
\]

\[
= \sum_{c \in C_{\mathfrak{f} q}} \chi_{\mathfrak{f} q}(c) X(c, \{ a_{\mu} q \}_\mu) - \chi_{\mathfrak{f} q}(q) \sum_{c \in C_{\mathfrak{f} q}} \chi_{\mathfrak{f} q}(c) X(c, \{ a_{\mu} \}_\mu) + \chi_{\mathfrak{f} q}(q) L(0, \chi_{\mathfrak{f} q}) \log \text{id} q.
\]
On the other hand, by Lemma 3.11, there exists an element $\alpha_{c,\{a_\mu\},q} \in (F_+)^\mathbb{Q}$ satisfying

$$X(c, \{a_\mu\}_\mu) = X(c, \{a_\mu\}_\mu) + \log \alpha_{c,\{a_\mu\},q}.$$

Hence the ratio of both sides of (5.4) can be written in the form

$$\exp\left(\sum_{i=1}^k \sum_{\chi \in (\hat{G}_-)_I} \chi(a_i) \log(\alpha_i)\right).$$

(5.5)

Here $k$ is a natural number, $a_i$ are integral ideals, and $\alpha_i \in (F_+)^\mathbb{Q}$. More precisely, each $\log \alpha_i$ is equal to one of $\pm \log \alpha_{c,\{a_\mu\},q}, \pm \zeta_0(0,c) \log q$, and each $a_i$ is a certain product of prime ideals dividing $f_0$, an integral ideal whose image under the Artin map is $\tau$, and representatives of classes in $C_p$. In particular, $a_i$'s and $\alpha_i$'s do not depend on $\chi \in (\hat{G}_-)_I$. In addition, we have $\sum_{\chi \in (\hat{G}_-)_I} \chi(a_i) \in \mathbb{Q}$ since $(\hat{G}_-)_I$ is stable under the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence we see that (5.5) is an algebraic number, so (5.4) follows. This completes the proof. $\square$

The expression (5.1) and Theorem 5.1-(iii) suggest that it is natural to generalize Conjecture 5.2 to the following form.

**Conjecture 5.5.** Let $K$ be a finite abelian extension of a totally real field $F$. We assume that there exists a CM-subfield of $K$ and put $K_{CM}$ to be the maximal CM-subfield of $K$. Let $G := \text{Gal}(K/F), f_{K/F}|f_0, \text{Art}_{f_0}$ be as in Proposition 5.4. Then we have

$$\prod_{c \in \text{Art}_{f_0}^{-1}(\tau)} \exp(X(c)) \mod \overline{\mathbb{Q}}^\times = \pi^{\zeta_0(0,\tau)} p_{K_{CM}}(\tau|K_{CM}, \text{Inf}^{-1}(\sum_{\sigma \in G} \zeta_{f_0}(0,\sigma)\sigma)) \quad (\tau \in G).$$

For the well-definedness of the right-hand side, we need to show that $\sum_{\sigma \in G} \zeta_{f_0}(0,\sigma)\sigma$ is in the image of $\text{Inf}: I_{K_{CM}} \otimes \mathbb{Z} \mathbb{Q} \to I_K \otimes \mathbb{Z} \mathbb{Q}$. Although this fact may be well-known to experts, we give a brief proof here: An algebraic number field $L \subset \mathbb{C}$ is a CM-field if and only if the complex conjugation $\rho$ on $\mathbb{C}$ induces a non-trivial automorphism of $L$ such that $\rho \tau = \tau \rho$ for every $\tau \in J_L$ ([Shim, Proposition 5.11]). Therefore

(5.6) $K_{CM}$ is the fixed subfield of $K$ under $\langle \rho_i, \rho_j, \iota, \iota' \in J_F \rangle$,

where $\rho_i \in G$ is the complex conjugation at $\iota$. Hence it suffices to show that $\zeta_{f_0}(0,\sigma) = \zeta_{f_0}(0,\sigma \rho_i, \rho_j)$ for all $\sigma \in G, \iota, \iota' \in J_F$, which follows from the fact that $L(0,\chi) = 0$ if $\chi$ is not totally odd. More precisely, we can write

$$\zeta_{f_0}(0,\sigma|_{K_{CM}}) = [K:K_{CM}] \zeta_{f_0}(0,\sigma) \quad (\sigma \in G),$$

(5.7) $\sum_{\sigma \in \text{Gal}(K_{CM}/F)} \zeta_{f_0}(0,\sigma)\sigma = [K:K_{CM}] \text{Inf}^{-1}(\sum_{\sigma \in G} \zeta_{f_0}(0,\sigma)\sigma)$.

Conjecture 5.5 also implies a weaker version of Stark’s conjecture in the following sense.

**Proposition 5.6.** Let $F, K, S$ be as in Conjecture 1.3. Assume that there exists an integral divisor $f_0$ satisfying that
• The conductor $f_{K/F}$ of $K/F$ divides $f_0$.
• Any place dividing $f_0$ is in $S$.
• The maximal ray class field $H_0$ modulo $f_0$ contains a CM-subfield.

Then Conjecture 5.5 implies that we have

$$\exp(\zeta_S'(0, \tau)) \in \mathbb{Q}^\times \quad (\tau \in \text{Gal}(K/F)).$$

**Proof.** Let $f_0$ satisfy the above assumptions. By Theorem 3.1, for $\iota \neq \text{id}$ we have

$$\prod_{c \in \text{Art}^{-1}(\tau)} \exp(X(c)) \cdot \prod_{c \in \text{Art}^{-1}(\tau \rho_{ad})} \exp(X(c)) \in \mathbb{Q}^\times \quad (\tau \in \text{Gal}(H_0/F)),$$

where $\rho_{ad}$ is the complex conjugation on $H_0$ at id $\in J_F$. Conjecture 5.5 implies that the same holds for $\iota = \text{id}$, by Theorem 5.1-(ii). Hence, for $S' := \{\text{all places of } F \text{ dividing } f_0\}$, $\tau \in \text{Gal}(H_0/F)$, we obtain $\exp(\zeta_S'(0, \tau)) \exp(\zeta_S'(0, \tau \rho_{ad})) \in \mathbb{Q}^\times$. Then the assertion follows, since $S' \subset S$ and $K$ is fixed under $\rho_{ad}$. □

**Example 5.7.** We introduce an example which is discussed in [Yo, Chapter III, Example 6.3]. Let $F = \mathbb{Q}(\sqrt{5})$, $K = F(\sqrt{\epsilon})$ with $\epsilon = \frac{1+\sqrt{5}}{2}$. We denote by $\infty_1, \infty_2$ the infinite places of $F$ corresponding to the real embeddings $\iota_1 = \text{id}, \iota_2 : \sqrt{5} \leftrightarrow -\sqrt{5}$, respectively. Then the conductor of $K/F$ is $(4)\infty_2$. We put $S := \{(2), \infty_1, \infty_2\}$, $f_0 := (4)\infty_1\infty_2$. Then the maximal ray class field $H_0$ is $F(\sqrt{\epsilon}, \sqrt{-1})$, and its maximal CM-subfield is $F(\sqrt{-1})$. We see that $C_{f_0} = \{c_1, c_2, c_3, c_4\}$ where $c_1, c_2, c_3, c_4$ are the classes of $(1), (3), (4+\sqrt{5}), (6+\sqrt{5})$ respectively. The element in $\text{Gal}(H_0/F)$ corresponding to $c_1$ under the Artin map $C_{f_0} \cong \text{Gal}(H_0/F)$ is denoted by $\tau_1$. We put $G := \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})} \prod_{n=1}^{19} \Gamma(\frac{a}{20})^{\psi(a)/4}$ with $\psi$ the Dirichlet character corresponding to the quadratic extension $\mathbb{Q}(\sqrt{-5})/\mathbb{Q}$. The following was shown in [Yo, Chapter III, Example 6.3]:

$$\exp(X(c_1)) \equiv \exp(X(c_2)) \equiv \exp(4^4, \exp(X(c_3)) \equiv \exp(X(c_4)) \equiv G^{-1} \text{ mod } \mathbb{Q}^\times,$$

$$\zeta(0, c_1) = \zeta(0, c_2) = \frac{1}{4}, \quad \zeta(0, c_3) = \zeta(0, c_4) = -\frac{1}{4},$$

$$G \text{ mod } \mathbb{Q}^\times = \pi p_{F(\sqrt{-1})}(\text{id}, \text{id})^2.$$

Therefore we see that Conjecture 5.5 holds true for $H_0/F$: For example, let $\tau := \tau_1 = \text{id} \in \text{Gal}(H_0/F)$. Then Conjecture 5.5 states that

$$\exp(X(c_1)) \text{ mod } \mathbb{Q}^\times = \pi^4 p_{F(\sqrt{-1})}(\text{id}, \text{Inf}^{-1}(\frac{1}{4}\tau_1 + \frac{1}{4}\tau_2 - \frac{1}{4}\tau_3 - \frac{1}{4}\tau_4)).$$

Since $F(\sqrt{-1})$ is the fixed subfield of $H_0$ under $\{\tau_1, \tau_2\}$, we see that the right-hand side of (5.10) is equal to $\pi^4 p_{F(\sqrt{-1})}(\text{id}, \frac{1}{4}\text{id} - \frac{1}{4}\rho)$, which is again equal to $G^4 \text{ mod } \mathbb{Q}^\times$ by Theorem 5.1-(ii). Therefore (5.10) holds by (5.9). The cases $\tau = \tau_2, \tau_3, \tau_4$ can be proved similarly. Moreover (5.9), which is a special case of Conjecture 5.5, implies the algebraicity of Stark units $\exp(-2\zeta_S'(0, \tau))$ with $\tau \in \text{Gal}(K/F)$, as we show in Proposition 5.6. For example, let $\tau := \text{id} \in \text{Gal}(K/F)$. Since $K$ is the fixed subfield of $H_0$ under $\{\tau_1, \tau_3\}$, we have $\exp(\zeta_S'(0, \text{id})) = \exp(\zeta'(0, c_1)) \exp(\zeta'(0, c_3)) \equiv \exp(X(c_1)) \exp(X(c_3)) \equiv 1 \text{ mod } \mathbb{Q}^\times$ by Theorem 3.1 and (5.9).
Remark 5.8. Proposition 5.6 and Example 5.7 suggest that by studying the relation between CM-periods and the multiple gamma function, we will obtain partial results for Stark’s conjecture. When \( F = \mathbb{Q} \), we obtained a more precise result in [Ka]. We showed that Rohrlich’s formula in [Gr], which is a special case of Conjecture 5.2, and Coleman’s fixed subfield of \( \mathbb{Q} \) Stark’s conjecture. When between CM-periods and the multiple gamma function, we will obtain partial results for Remark 5.8. Proposition 5.6 and Example 5.7 suggest that by studying the relation between CM-periods and the multiple gamma function, we will obtain partial results for Stark’s conjecture in the case \( F = \mathbb{Q} \).

The opposite direction of Proposition 5.6 also holds, in the following sense.

Proposition 5.9. Let \( F, K, K_{CM}, f_{K/F} | f_0 \) be as in Conjecture 5.5. We put \( K_{St} \) to be the fixed subfield of \( K \) by the complex conjugation \( \rho_{id} \) at \( id \in J_F \), and \( S := \{ \text{all places of } F \text{ dividing } f_0 \} \). Then Conjecture 5.2 for \( K_{CM}/F \) and (5.8) for \( K_{St}/F \) imply Conjecture 5.5.

We note that, in the above Proposition, \( K_{St} \) is the maximal subfield of \( K \) in which the real place corresponding to \( id \) splits completely, and that (5.8) is a weaker version of Conjecture 1.3.

Proof. Let Art\(_{f_0} : C_{f_0} \to \text{Gal}(K/F) \) be as in Proposition 5.6. Art\(_{f_0,CM} : C_{f_0} \to \text{Gal}(K_{CM}/F) \) the associated map. First we note that (5.8) for \( K_{St}/F \) states that

\[
\prod_{c \in \text{Art}_{f_0}^{-1}(\tau)} \exp(X(c)) \cdot \prod_{c \in \text{Art}_0^{-1}(\tau \rho_{id})} \exp(X(c)) \in \mathbb{Q}^\times \quad (\tau \in \text{Gal}(K/F)).
\]

This equivalence follows from the same argument used in the proof of Proposition 5.6. Conjecture 5.2 states that for \( \tau \in \text{Gal}(K/F) \) we have

\[
\prod_{c \in \text{Art}_{f_0,CM}^{-1}(\tau|_{K_{CM}})} \exp(X(c)) \mod \mathbb{Q}^\times \quad = \pi^{[K:K_{CM}]}|_{\mathbb{Q}^\times(K_{CM} / K)}(\tau|_{K_{CM}}, \text{Inf}^{-1}(\sum_{\sigma \in \mathbb{G}} \zeta_{f_0}(0, \sigma))|_{K:K_{CM}}),
\]

by Proposition 5.4 and (5.7). On the other hand, we can write for \( \tau_0 \in \text{Gal}(K_{CM}/F) \)

\[
\prod_{c \in \text{Art}_{f_0,CM}^{-1}(\tau_0)} \exp(X(c)) = \prod_{\tau \in \text{Gal}(K/F), \tau|_{K_{CM}} = \tau_0} \prod_{c \in \text{Art}_{f_0}^{-1}(\tau)} \exp(X(c)).
\]

Hence in order to derive Conjecture 5.5 from (5.13), we need the following statement: If \( \tau, \tau' \in \text{Gal}(K/F) \) satisfy \( \tau|_{K_{CM}} = \tau'|_{K_{CM}} \), then we have

\[
\prod_{c \in \text{Art}_{f_0}^{-1}(\tau)} \exp(X(c)) \equiv \prod_{c \in \text{Art}_{f_0}^{-1}(\tau')} \exp(X(c)) \mod \mathbb{Q}^\times.
\]
Indeed, for $\tau \in \text{Gal}(K/F)$, $\iota, \iota' \in J_F$, we can write

\begin{equation}
(5.15) \prod_{c \in \text{Art}^{-1}_{l_0}(\iota)} \exp(X(c)) \equiv \prod_{c \in \text{Art}^{-1}_{l_0}(\iota \rho)} \exp(X(c))^{-1} \equiv \prod_{c \in \text{Art}^{-1}_{l_0}(\iota \rho, \rho', \iota')} \exp(X(c)) \mod \mathbb{Q}^x
\end{equation}

by Theorem 3.1 (when $\iota, \iota' \neq \text{id}$) and by (5.12) (when $\iota, \iota' = \text{id}$). By (5.6), we see that (5.15) implies (5.14). Hence the assertion is clear. 

\section*{References}


