ON THE ANALYTICITY OF CR-DIFFEOMORPHISMS

I. KOSSOVSKIY AND B. LAMEL

Abstract. In any positive CR-dimension and CR-codimension we provide a construction of real-analytic embedded CR-structures, which are C^∞ CR-equivalent, but are inequivalent holomorphically.

Contents

1. Introduction

1.1. **Overview.** Let M be a smooth manifold. A *CR-structure* on M is a couple $(\mathcal{B}, J)$, where $\mathcal{B}$ is an even-dimensional subbundle of TM (called *the complex tangent bundle* and denoted by $T^{\mathbb{C}}M$), and $J_p\colon \mathcal{B}_p \longrightarrow \mathcal{B}_p$ is a linear operator with $J_p^2 = -\mathrm{Id}$, smoothly depending on $p \in M$ and called the *complex structure operator*. A manifold M endowed with a CR-structure is called an *(abstract) CR-manifold*. If the CR-structure is *integrable* [39] (e.g. if it is real-analytic), then there exists a (local) generic embedding $\mathfrak{i} : M \hookrightarrow \mathbb{C}^N$ such that the complex tangent bundle of M is given by the $T^{\mathbb{C}}M = TM \cap iTM$, and the complex structure J is induced by multiplication with i in $\mathbb{C}^N$. Natural morphisms of CR-structures are *CR-maps*, that is, smooth maps $F\colon M \to M$ such that $F^*\mathcal{B}' = \mathcal{B}$ and $dF \circ J = J' \circ dF$. For CR-manifolds embedded into $\mathbb{C}^N$ (*CR-submanifolds*) immediate examples of CR-maps are given by restrictions of holomorphic maps between the real submanifolds (or, more generally, by boundary values of holomorphic maps).

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The study of CR-mappings between real submanifolds in complex space goes back to the classical work of Poincare [56] and Cartan [13]. In particular, in [13], Cartan showed that every smooth CR-map F between smooth Levi-nondegenerate real hypersurfaces $M_1, M_2 \subset \mathbb{C}^2$ lifts to a smooth map $\hat{F}$ between certain associated vector bundles $\pi_j : B_j \longrightarrow M_j$, $j = 1, 2$, and the lifted map $\hat{F}$ transforms certain distinguished frames on B_1, B_2 respectively into each other (so-called *reduction to* $\{e\}$*-structure*). This gives, among other things, a complete first order PDE system for the map. In the real-analytic case, both frames are real-analytic, thus the map appears to be analytic. In this way, Cartan shed light on the nature of CR-mappings by establishing the following intriguing phenomenon:

smooth CR-diffeomorphisms between Levi-nondegenerate real-analytic hypersurfaces in $\mathbb{C}^2$ *are automatically analytic (that is, such diffeomorphism are always restrictions of biholomorphisms of domains in* $\mathbb{C}^2$*).*

Accordingly, since the inception of CR-geometry, the following general question is of fundamental importance for the field:

Do CR-maps between sufficiently nondegenerate real CR-submanifolds coincide with the ones induced by holomorphic maps?

Celebrated results of Chern-Moser, Lewy and Pinchuk in the 1970's [15, 55], and Trepreau, Baouendi, Jacobowitz, Treves and Tumanov in the 1980's [62, 8, 5, 65] (as well as a big number of subsequent publications in the field) address question A in a sufficiently large generality. They give a characterization of CR-maps, under certain nondegeneracy assumptions on the CR-manifolds, as restrictions or boundary values of holomorphic maps of domains (we refer to the next section for details).

In the case of equidimensional CR-manifolds, one comes to the consideration of *CR-equivalences*. Here, in addition to the analyticity issue, a much more subtle question occurs:

Problem A. *Are CR-equivalent real-analytic CR-structures also equivalent holomorphically?*

In other words, Problem A asks whether a (possibly non-analytic) smooth CR-equivalence between real-analytic CR-manifolds can be always replaced by a holomorphic one.

Problem A, formulated for the class of C^∞ CR-diffeomorphisms, is a long-standing problem in CR-geometry. We note here that a positive solution for Problem A in the case of *minimal* CR-manifolds follows from the work of Baouendi, Mir and Rothschild [6].

The main goal of this paper is to give the *negative* resolution to Problem A, in any positive CR-dimension and CR-codimension. We do so by developing and extending the very recent Dynamical technique in CR-geometry, due to Shafikov and the first author [44, 45]. We reduce the above regularity problem for CR-maps to a classical setting in Dynamics, where its negative resolution follows from *Stockes phenomenon* (see [38]). The negative solution for Problem A implies, in particular, the negative resolution to a long-standing conjecture on the analyticity of CR-equivalences between real-analytic Levi nonflat 3-dimensional CR-manifolds, apparent in the work of Huang [34, 24] (Conjecture 1 below).

We give in the next two sections a short background, outline the history with more details, and give a detailed formulation of our results.

1.2. **Background information and historic outline.** Let M be a real-analytic submanifold in $\mathbb{C}^N$. The *complex tangent bundle* of M is given by $T^cM = TM \cap iTM$, and we say that M is a CR-manifold if the fiber dimension of this bundle is constant. The complex fiber dimension of $T^{\mathbb{C}}M$ is called the *CR-dimension*, and the (real) codimension of $T^{\mathbb{C}}M$ in TM is called the

CR-codimension. A germ of a map $F\colon (M,p) \to (M',p')$ is CR if $TF(T^cM) \subset T^cM'$ and TF is complex linear on T^cM. (Here TF denotes the differential of a map). Equivalently, F is CR if its components are germs of CR-functions, where a CR-function is defined as a CR-map $(M,0) \to \mathbb{C}$. It turns out that a function is CR if and only if it is annihilated by every section of $\mathcal{V}(M)$, the *CR-bundle* of M, which is defined by

$$\mathcal{V}(M) = T^{(0,1)}\mathbb{C}^N \cap \mathbb{C}TM.$$

Thus CR-maps satisfy a certain system of PDEs, also known as the tangential Cauchy-Riemann equations. Boundary values of holomorphic maps are the primary examples of such maps. Note that a real-analytic CR-map is always a restriction of a map, holomorphic in an open neighborhood of the source manifold [63].

The naturally arising problem of regularity of CR-mappings is of fundamental importance for the study of boundary regularity of holomorphic mappings (see, e.g., the discussions in the survey [26] of Forstnerič on the subject). On the other hand, the problem of analyticity of CR-mappings is equally important for Linear PDEs, where the latter property is addressed as *analytic hypoellipticity* and can be of substantial help for studying regularity of solutions for a wide range of PDE systems (see [10]).

It turns out that the systems of PDEs, determining the space of CR-mappings between real submanifolds in complex space, are rather hard to satisfy. Actually, a heuristic going back to Poincare tells us that there are no CR-maps between two randomly chosen smooth CR-manifolds. This lack of richness is made up for by a number of beautiful properties CR-maps possess: in particular, they have an uncanny tendency to be very regular. In the case of hypersurfaces in $\mathbb{C}^2$ this regularity is already apparent in E. Cartan's work on Levi-nondegenerate germs [13], as was discussed before. Actually, every formal map between such hypersurfaces is convergent, and every smooth CR-diffeomorphism is the restriction of a germ of a holomorphic map. The work of Pinchuk [55] and Chern-Moser [15] extends this result to $\mathbb{C}^N$, $N \geq 3$. In fact, regularity results of this sort hold under less stringent conditions. For hypersurfaces in $\mathbb{C}^2$, it has been known for some time that if M is minimal at p, then every germ of a smooth CR diffeomorphism (it is enough to assume just continuity) is actually the restriction of a germ of a holomorphic map (see Huang [34]). Here minimality (or finite type, which in the case of real-analytic hypersurfaces is the same) refers to the fact that the tangential CR-equations satisfy Hormander's bracket condition, or, equivalently, that there does not exist a germ of a complex curve $X \subset M$ through p.

This regularity property relies on two crucial ingredients. One uses the minimality to obtain a one-sided extension of the map, which relies on the one-sided extension of the component CR-functions, possible by the result of Trepreau [62]. One then obtains the extension across the hypersurface by reflection methods (regularity results of this form are therefore also known as *reflection principles*). The nondegeneracy properties of real-analytic submanifolds governing reflection are by now well understood. One of the most useful results in that regard is the Baouendi-Jacobowitz-Treves theorem [5] which states that every smooth diffeomorphic boundary value of a holomorphic map in a wedge actually extends to a germ of a holomorphic map, if the target real submanifold is *essentially finite*. The reflection principle for merely continuous CR-maps between real-analytic hypersurfaces which are of D'Angelo finite type (meaning they do not contain any complex varieties) in $\mathbb{C}^N$, $N \geq 3$, is contained in the work of Diederich and Pinchuk [19]. For notable results on the reflection principle for CR-mappings between CR-submanifolds of different dimension see Forstnerič [27, 28], Huang [36], Coupet, Pinchuk and Sukhov [17], Meylan, Mir and Zaitsev [49], Mir [50], and Berhanu and Xiao [11] (compare also with the results in [21]).

However, these positive results are not applicable in the *nonminimal* situation, or when one sided extension of the mapping is not a priori assumed. To discuss the nonminimal case in detail, let us consider the class of 3-dimensional CR-manifolds (i.e., hypersurfaces in $\mathbb{C}^2$), leading to Conjecture 1 below. Within this class, the concepts of essential finiteness and minimality actually agree, so that violation of either of these conditions leads to the consideration of nonminimal hypersurfaces. As CR-mappings between Levi flat hypersurfaces can trivially be non-analytic, we restrict the considerations to Levi nonflat hypersurfaces (in $\mathbb{C}^2$ the latter property is equivalent to *holomorphic nondegeneracy,* see [3]). Since in the nonminimal case CR-functions do not admit a one-sided extension in general, one should use a different strategy for proving the analyticity in this setting. The most natural approach is to derive a complete system for the components of a CR-map $F : M \longrightarrow M'$ from the *basic identity*

$$\rho'(F(z), \overline{F(z)})|_{z \in M} = 0$$

(here $M' = \{\rho'(z', \bar{z}') = 0\}$). In order to do so, one should be able to derive differential corollaries of sufficiently high order from the basic identity. In addition, one should presume the (generic) invertibility of the resulting system with respect to high-order derivatives, which amounts to nondegeneracy conditions for the map. In this way, we come to the consideration of *(generically) invertible C^∞ CR-maps.* Elementary examples (see, e.g., [20]) show that these two conditions can not be dropped. That is, one cannot hope for diffeomorphism of class C^k for finite k to be analytic in general (in particular, the C^∞ condition helps to avoid branching phenomena). Similarly, CR-mappings of low rank can not enjoy the analyticity property in the degenerate setting.

We shall also outline that, upon completion of the minimal case in [34], Huang (with his coauthors) attacked the case of CR-mappings between nonminimal hypersurfaces, admitting one-sided holomorphic extension, in [34, 35, 37, 24]. The most general result in this direction was obtained by in [24], were it was shown that merely continuous boundary values have the analyticity property, as long as M, M' are Levi nonflat. However, the general question whether a smooth CR-diffeomorphism between Levi nonflat hypersurfaces is necessarily the restriction of a holomorphic map remained open, even in dimension 2. The result of Ebenfelt [20] on the analyticity of C^∞ CR-maps in the 1-*nonminimal* case, as well as evidence in the algebraic case due to Baouendi, Huang and Rothschild [4] provided some basis for hopes in that direction, and the following was conjectured in [34, 24].

Conjecture 1 (see [34, 24])**.** *Let $M, M' \subset \mathbb{C}^2$ be real-analytic Levi nonflat hypersurfaces. Then any C^∞-smooth CR (local) diffeomorphism $F : M \to M'$ extends holomorphically to an open neighborhood of M in $\mathbb{C}^2$.*

We note that Conjecture 1 is closely related to Problem A, formulated in the C^∞ category for the $\mathbb{C}^2$-case.

Our main result provides the negative resolution for Problem A, formulated in the C^∞ category for any positive CR-dimension and CR-codimension. In particular, this implies the negative answer to Conjecture 1 (that is, we prove the existence of Levi nonflat hypersurfaces in $\mathbb{C}^2$, possessing a smooth CR-diffeomorphism between them which is not the restriction of a holomorphic map).

1.3. **Main results.** In order to discuss our results in more detail, let us introduce a number of natural spaces of maps between real-analytic CR-manifolds. We will write $\mathrm{Diff}^{\mathrm{k}}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),\mathrm{M}')$ for the space of germs of CR-diffeomorphisms of class C^k, where $k \in \mathbb{N} \cup \{\infty, \omega\}$, and

$\mathrm{Diff}^{\mathrm{k}}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),(\mathrm{M}',\mathrm{p}'))$ for those diffeomorphisms H which in addition satisfy $H(p)=p'$. We will also need the space of *formal* CR-diffeomorphisms for which we will write $\mathrm{Diff}^{\mathrm{f}}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),\mathrm{M}')$ and $\mathrm{Diff}^{\mathrm{f}}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),(\mathrm{M}',\mathrm{p}'))$, respectively. In the case $M'=M$ we use the notation $\mathrm{Hol}^{\mathrm{k}}(\mathrm{M},\mathrm{p})=\mathrm{Diff}^{\mathrm{k}}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),\mathrm{M})$ and $\mathrm{Aut}^{\mathrm{k}}(\mathrm{M},\mathrm{p})=\mathrm{Diff}^{\mathrm{k}}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),(\mathrm{M},\mathrm{p}))$, $k\in\mathbb{N}\cup\{\infty,\omega,f\}$. Then our first main result is as follows.

Theorem 2. *For any positive integers $n,k>0$ there exist germs of real-analytic holomorphically nondegenerate CR-submanifolds (M,p), (M',p') in $\mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k such that*

$$\mathrm{Diff}^{\infty}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),(\mathrm{M}',\mathrm{p}'))\neq\emptyset,\ \ \textit{but}\ \ \mathrm{Diff}^{\omega}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),(\mathrm{M}',\mathrm{p}'))=\emptyset.$$

We shall emphasize that we provide a proof of existence here, but not explicit counterexamples. Theorem 2 should be compared with other divergence phenomena in CR-geometry, e.g. due to Gong [30] and due to Shafikov and the first author [44].

An immediate crucial corollary from Theorem 2 is that, in any positive CR-dimension and CR-codimension, the holomorphic and the C^∞ CR equivalence problems are *distinct.* To formulate this corollary in detail, we fix two integers $n,k\geq 0$ and introduce the C^∞ *CR moduli space* $\mathfrak{M}^{n,k}_{\infty}$ and the *holomorphic moduli space* $\mathfrak{M}^{n,k}_{\omega}$ as the space of C^∞ CR-equivalence classes and the space of biholomorphic equivalence classes for germs of real-analytic CR-submanifolds in $\mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k at the origin, respectively. We have the natural surjective map $\mathfrak{i}_{n,k}:\mathfrak{M}^{n,k}_{\omega}\to\mathfrak{M}^{n,k}_{\infty}$.

Corollary 3. *For any integers $n,k>0$ the map $\mathfrak{i}_{n,k}:\mathfrak{M}^{n,k}_{\omega}\to\mathfrak{M}^{n,k}_{\infty}$ is not injective.*

Thus, in any positive CR-dimension and CR-codimension, the holomorphic moduli space of germs at the origin of real-analytic CR-submanifolds is *bigger* than the corresponding C^∞ CR moduli space.

Setting in Theorem 2 $n=k=1$, we immediately obtain the negative answer to Conjecture 1.

We recall that the so-called *nonminimality order* of a hypersurface (see Section 3 for details of the concept) is an integer $m\geq 1$ measuring the "flatness" of a hypersurface at a nonminimal point. As discussed above, C^∞ CR-diffeomorphisms of 1-nonminimal hypersurfaces are always analytic due to Ebenfelt [20]. As we will see from the proof of Theorem 2, the 1-nonminimality restriction in the work [20] can not be dropped in general in the sense that for $m\geq 2$ there are counterexamples to the analyticity statement.

We note that Theorem 2 also implies that, in the nonminimal case, the approximation property for CR-equivalences between real-analytic submanifolds $M,M'\subset\mathbb{C}^N$ akin to the Baouendi-Treve's property [8] of CR-functions or CR Artin's Approximation Property for CR-mappings (see Mir [51] and Sunye [61]) fails.

Corollary 4. *For any integers $n,k>0$ there exist real-analytic CR-submanifolds $M,M'\subset\mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k, $M\ni p$, $M'\ni p'$, and a C^∞ CR-diffeomorphism $F:(M,p)\longrightarrow(M',p')$ which, for any fixed open neighbourhood U of p in $\mathbb{C}^{n+k}$, can not be approximated by holomorphic mappings $M\cap U\longrightarrow M'$; the formal Taylor series $\hat F$ of F at p also cannot be approximated by holomorphic series taking M into M'.*

This result shows that $\mathrm{Diff}^{\infty}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),(\mathrm{M}',\mathrm{p}'))$ is in general not an appropriate "closure" of $\mathrm{Diff}^{\omega}_{\mathrm{CR}}((\mathrm{M},\mathrm{p}),(\mathrm{M}',\mathrm{p}'))$.

It is then natural to ask whether analyticity results hold for CR-automorphisms of holomorphically nondegenerate hypersurfaces, i.e., whether the isotropy groups $\mathrm{Aut}^{\infty}(\mathrm{M},\mathrm{p})$ and $\mathrm{Aut}^{\omega}(\mathrm{M},\mathrm{p})$

coincide for a germ of a real-analytic hypersurface (M,p). Our next result shows that the answer is also negative, even for the infinitesimal automorphism algebras. Recall that the *infinitesimal automorphism algebra* for a real submanifold $M\subset\mathbb{C}^N$ at a point $p\in M$ is the algebra $\mathfrak{hol}^k(M,p)$ of holomorphic ($k=\omega$) or smooth ($k=\infty$) vector fields

$$X=f_1\frac{\partial}{\partial z_1}+...+f_n\frac{\partial}{\partial z_N},$$

defined near p such that each f_j is a real-analytic ($k=\omega$) or smooth ($k=\infty$) CR-function on M and $X+\bar X$ is tangent to M near p. Vector fields $X\in\mathfrak{hol}^\omega(M,p)$ (resp. $X\in\mathfrak{hol}^\infty(M,p)$) are exactly the vector fields generating flows of holomorphic (resp. smooth CR) transformations, preserving M locally. The *stability subalgebras* $\mathfrak{aut}^k(M,p)\subset\mathfrak{hol}^k(M,p)$ are determined by the condition $X|_p=0$.

Theorem 5. *For any integer $N\geq 2$ there exist real-analytic holomorphically nondegenerate hypersurfaces $M\subset\mathbb{C}^N$, $M\ni p$, with $\mathfrak{hol}^\omega(M,p)\subsetneq\mathfrak{hol}^\infty(M,p)$ and $\mathfrak{aut}^\omega(M,p)\subsetneq\mathfrak{aut}^\infty(M,p)$.*

Theorem 5, read together with the results in [44], poses an interesting problem of finding the relations between the, respectively, holomorphic, CR and formal stability algebras $\mathfrak{aut}^\omega(M,p),\mathfrak{aut}^\infty(M,p)$ and $\mathfrak{aut}^f(M,p)$ for a real-analytic nonminimal Levi nonflat hypersurface $M\subset\mathbb{C}^2$. Note that the results in [20] and [40] show that the three algebras coincide in the case of 1-nonminimal hypersurfaces. We also point out that a recent result of Shafikov and the first author in [45] provides the sharp upper bound $\dim\mathfrak{aut}^\omega(M,p)\leq 5$ for an arbitrary Levi nonflat real-analytic hypersurface $M\subset\mathbb{C}^2$. However, no known results imply the same bound for the algebras $\mathfrak{aut}^\infty(M,p)$ and $\mathfrak{aut}^f(M,p)$. This motivates the following two problems.

Problem 6. Establish optimal nondegeneracy conditions for a real-analytic nonminimal Levi nonflat hypersurface $M\subset\mathbb{C}^2$, generalizing the 1-nonminimality and guaranteeing the coincidence of the algebras $\mathfrak{aut}^\omega(M,p),\mathfrak{aut}^\infty(M,p)$ and $\mathfrak{aut}^f(M,p)$.

Problem 7. Find the sharp upper bound for the dimension of the algebras $\mathfrak{aut}^\infty(M,0)$ and $\mathfrak{aut}^f(M,p)$ for a real-analytic Levi nonflat hypersurface $M\subset\mathbb{C}^2$.

Remark 8. Shortly before this work had been accepted for publication, Problem 6 was solved by the authors in [42].

1.4. **Principal method.** The main tool of the paper is a development of a recent CR $\longrightarrow$DS (Cauchy-Riemann manifolds $\longrightarrow$ Dynamical Systems) technique introduced by Shafikov and the first author [44, 45]. The technique suggests to replace a given real hypersurface M with a CR-degeneracy (such as nonminimality) by an appropriate holomorphic dynamical system $\mathcal{E}(M)$, and then study mappings of CR-submanifolds accordingly. (An explicit realization of this general strategy is performed so far for nonminimal hypersurfaces in $\mathbb{C}^2$). This method previously enabled to show [44] that, in any positive CR-dimension and CR-codimension, there are more holomorphic moduli for real-analytic CR-submanifolds than formal ones (compare with the results in [6],[7]). The possibility to replace a real-analytic CR-manifold by a complex dynamical system is based on the fundamental connection between CR-geometry and the geometry of completely integrable PDE systems, first observed by E. Cartan and Segre [13, 57], and recently revisited in the work of Sukhov [59, 60] (see also [29, 52] for some further properties of the connection). The "mediator" between a CR-manifold and the associated PDE system is the Segre family of the CR-manifold. By choosing real hypersurfaces $M,M'\subset\mathbb{C}^2$ in such a way that mappings between the associated dynamical systems $\mathcal{E}(M), \mathcal{E}(M')$ have certain "wedge"-type regularity, but are not regular in an open neighborhood of the singular point, we obtained the desired counterexamples.

It is notable that singular ODEs in the context of regularity of CR-maps are apparent already in the above mentioned work of Ebenfelt [20] on the 1-nonminimal case. The CR $\longrightarrow$ DS makes the connection of CR-manifolds with singular ODEs *canonical*, as well as more explicit and systematic.

We shall also note that the paper contains an important intermediate result which is a complete characterization of all real-analytic hypersurfaces in $\mathbb{C}^2$, which are nonminimal at the origin and spherical outside the complex locus $X \ni 0$ (see Theorem 21 and Corollary 23 below). The latter class of hypersurfaces was previously studied in a long sequence of publications [46, 23, 9, 41, 43, 44, 45] and appears to be highly nontrivial. The results of Section 3 below completes the study of hypersurfaces of this class.

We briefly describe the structure of the paper. In Section 2 we provide necessary background information. In Section 3 we establish a class of singular meromorphic complex differential equations that are associated with a class of nonminimal hypersurfaces in $\mathbb{C}^2$ (namely, the class of nonminimal hypersurfaces, spherical outside the complex locus). We call them *ODEs with a real structure* (compare with the work [25] of Faran, where Segre families with a real structure were studied). This gives us a freedom in the choice of nonminimal hypersurfaces, for which the associated ODEs have the prescribed properties. We also obtain in the same section the above mentioned characterization theorem for nonminimal spherical hypersurfaces. In Section 4 we provide a one-parameter family $\mathcal{E}_\gamma$ of ODEs with a real structure, any two of which are equivalent by means of a *sectorial* transformation (see Section 4 for details), while each ODE $\mathcal{E}_\gamma$ is inequivalent to $\mathcal{E}_0$ holomorphically for $\gamma \neq 0$. Remarkably, all ODEs $\mathcal{E}_\gamma$ have trivial monodromy of solutions. It follows immediately that a real hypersurface M_γ corresponding to an ODE $\mathcal{E}_\gamma$ with $\gamma \neq 0$ is holomorphically inequivalent to M_0, and the rest of the section is dedicated to the proof of the fact that all M_γ are sectorially equivalent. For that we introduce and use the class of so-called *sectorial coupled gauge transformation*. It is not difficult then to deduce the proof of Theorem 2. In Section 5 we apply the non-analytic sectorial mapping of M_γ into M_0 to describe the Lie algebras $\mathfrak{hol}^\omega(M_\gamma,0), \mathfrak{hol}^\infty(M_\gamma,0), \mathfrak{aut}^\omega(M_\gamma,0), \mathfrak{aut}^\infty(M_\gamma 0)$ $(\gamma \neq 0)$ and to deduce from there the proof of Theorem 5.

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2. Preliminaries

2.1. **Segre varieties.** Let M be a smooth real-analytic submanifold in $\mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k, $n,k>0$, $0\in M$, and U a neighbourhood of the origin where $M\cap U$ admits a real-analytic defining function $\phi(Z,\overline{Z})$ with the property that $\phi(Z,\zeta)$ is a holomorphic function for for $(Z,\zeta)\in U\times\bar{U}$. For every point $\zeta\in U$ we associate its Segre variety in U by

$$Q_\zeta=\{Z\in U : \phi(Z,\overline{\zeta})=0\}.$$

Segre varieties depend holomorphically on the variable $\overline{\zeta}$, and for small enough neighbourhoods U of 0, they are actually holomorphic submanifolds of U of codimension k.

One can choose coordinates $Z=(z,w)\in\mathbb{C}^n\times\mathbb{C}^k$ and a neighbourhood $U=\ U^z\times U^w\subset\mathbb{C}^n\times\mathbb{C}^k$ such that, for any $\zeta\in U$,

$$Q_\zeta=\left\{(z,w)\in U^z\times U^w : w=h(z,\bar\zeta)\right\}$$

is a closed complex analytic graph. h is a holomorphic function on $U^z\times\bar U$. The antiholomorphic $(n+k)$-parameter family of complex submanifolds $\{Q_\zeta\}_{\zeta\in U_1}$ is called *the Segre family* of M at the origin. The following basic properties of Segre varieties follow from the definition and the reality condition on the defining function:

$$\begin{aligned} Z\in Q_\zeta &\Leftrightarrow \zeta\in Q_Z,\\ Z\in Q_Z &\Leftrightarrow Z\in M,\\ \zeta\in M &\Leftrightarrow \{Z\in U\colon Q_\zeta=Q_Z\}\subset M. \end{aligned}\tag{2.1}$$

The fundamental role of Segre varieties for holomorphic maps is due to their *invariance property*: If $f:U\to U'$ is a holomorphic map which sends a smooth real-analytic submanifold $M\subset U$ into another such submanifold $M'\subset U'$, and U is chosen as above (with the analogous choices and notations for M'), then

$$f(Q_Z)\subset Q'_{f(Z)}.$$

For more details and other properties of Segre varieties we refer the reader to e.g. [67], [18],[19], or [3].

A particularly important case arises when M is a *real hyperquadric*, i.e., when

$$M=\left\{[\zeta_0,\dots,\zeta_N]\in\mathbb{CP}^N : H(\zeta,\bar\zeta)=0\right\},$$

where $H(\zeta,\bar\zeta)$ is a nondegenerate Hermitian form on $\mathbb{C}^{N+1}$ with $k+1$ positive and $l+1$ negative eigenvalues, $k+l=N-1$, $0\le l\le k\le N-1$. In that case, the Segre variety of a point $\zeta\in\mathbb{CP}^N$ is the globally defined projective hyperplane $Q_\zeta=\{\xi\in\mathbb{CP}^N : H(\xi,\bar\zeta)=0\}$, and the Segre family $\{Q_\zeta,\,\zeta\in\mathbb{CP}^N\}$ coincides in this case with the space $(\mathbb{CP}^N)^*$ of all projective hyperplanes in $\mathbb{CP}^N$.

The space of Segre varieties $\{Q_Z : Z\in U\}$, for appropriately chosen U, can be identified with a subset of $\mathbb{C}^K$ for some $K>0$ in such a way that the so-called *Segre map* $\lambda : Z\to Q_Z$ is antiholomorphic. This can be seen from the fact that if we write

$$h(z,\bar\zeta)=\sum_{\alpha\in\mathbb{N}^n}h_\alpha(\bar\zeta)z^\alpha,$$

then $\lambda(Z)$ can be identified with $\left(h_\alpha(\bar Z)\right)_{\alpha\in\mathbb{N}^n}$. After that the desired fact follows from the Noetherian property.

If M is a hypersurface, then its Segre map is one-to-one in a neighbourhood of every point p where M is Levi nondegenerate. When such a real hypersurface M contains a complex hypersurface X, for any point $p\in X$ we have $Q_p=X$ and $Q_p\cap X\neq\emptyset\Leftrightarrow p\in X$, so that the Segre map λ sends the entire X to a unique point in $\mathbb{C}^N$ and, accordingly, λ is not even finite-to-one near each $p\in X$ (i.e., M is *not essentially finite* at points $p\in X$).

2.2. **Real hypersurfaces and second order differential equations.** To every Levi nondegenerate real hypersurface $M\subset\mathbb{C}^N$ we can associate a system of second order holomorphic PDEs with 1 dependent and $N-1$ independent variables, using the Segre family of the hypersurface. This remarkable construction goes back to E. Cartan [14],[13] and Segre [57], and was recently revisited in [59],[60],[52],[29] (see also references therein).

Let us describe this procedure in the case $N=2$ relevant for our purposes. We denote the coordinates in $\mathbb{C}^2$ by (z,w), and put $z=x+iy$, $w=u+iv$. Let $M\subset\mathbb{C}^2$ be a smooth real-analytic hypersurface, passing through the origin, and choose $U = U_z \times U_w$ as described above. In this case we associate a second order holomorphic ODE to M, which is uniquely determined by the condition that the equation is satisfied by all the graphing functions $h(z,\zeta) = w(z)$ of the Segre family $\{Q_\zeta\}_{\zeta \in U}$ of M in a neighbourhood of the origin.

More precisely, since M is Levi-nondegenerate near the origin, the Segre map $\zeta \longrightarrow Q_\zeta$ is injective and the Segre family has the so-called transversality property: if two distinct Segre varieties intersect at a point $q \in U$, then their intersection at q is transverse. Thus, $\{Q_\zeta\}_{\zeta \in U}$ is a 2-parameter family of holomorphic curves in U with the transversality property, depending holomorphically on $\bar\zeta$. It follows from the holomorphic version of the fundamental ODE theorem (see, e.g., [38]) that there exists a unique second order holomorphic ODE $w'' = \Phi(z, w, w')$, satisfied by all the graphing functions of $\{Q_\zeta\}_{\zeta \in U}$.

To be more explicit we consider the so-called *complex defining equation* (see, e.g., [3]) $w = \rho(z, \bar z, \bar w)$ of M near the origin, which one obtains by substituting $u = \frac{1}{2}(w + \bar w)$, $v = \frac{1}{2i}(w - \bar w)$ into the real defining equation and applying the holomorphic implicit function theorem. The complex defining function ρ of a real hypersurface satisfies the *reality condition*

$$w \equiv \rho(z, \bar z, \bar\rho(\bar z, z, w)). \tag{2.2}$$

We shall again assume that U is a neighbourhood of the origin chosen as above. The Segre variety Q_p of a point $p = (a, b) \in U$ is now given as the graph

$$w(z) = \rho(z, \bar a, \bar b). \tag{2.3}$$

Differentiating (2.3) once, we obtain

$$w' = \rho_z(z, \bar a, \bar b). \tag{2.4}$$

Considering (2.3) and (2.4) as a holomorphic system of equations with the unknowns $\bar a, \bar b$, an application of the implicit function theorem yields holomorphic functions A, B such that

$$\bar a = A(z, w, w'),\ \bar b = B(z, w, w').$$

The implicit function theorem applies here because the Jacobian of the system coincides with the Levi determinant of M for $(z, w) \in M$ ([3]). Differentiating (2.3) twice and substituting for $\bar a, \bar b$ finally yields

$$w'' = \rho_{zz}(z, A(z, w, w'), B(z, w, w')) =: \Phi(z, w, w'). \tag{2.5}$$

Now (2.5) is the desired holomorphic second order ODE $\mathcal{E} = \mathcal{E}(M)$.

More generally, the association of a completely integrable PDE with a CR-manifold is possible for a wide range of CR-submanifolds (see [59, 60, 29]). The correspondence $M \longrightarrow \mathcal{E}(M)$ has the following fundamental properties:

(1) Every local holomorphic equivalence $F : (M, 0) \longrightarrow (M', 0)$ between CR-submanifolds is an equivalence between the corresponding PDE systems $\mathcal{E}(M), \mathcal{E}(M')$ (see subsection 2.3);
(2) The complexification of the infinitesimal automorphism algebra $\mathfrak{hol}^\omega(M, 0)$ of M at the origin coincides with the Lie symmetry algebra of the associated PDE system $\mathcal{E}(M)$ (see, e.g., [53] for the details of the concept).

We emphasize here that if $M \subset \mathbb{C}^2$ is a real hypersurface which is nonminimal at the origin, there is a priori *no* way to associate to M a second order ODE or even a more general PDE system near the origin. However, in [45] the authors discovered an injective correspondence between real hypersurfaces which are nonminimal at the origin and spherical outside the complex locus hypersurfaces $M \subset \mathbb{C}^2$ and certain *singular* complex ODEs $\mathcal{E}(M)$ with an isolated meromorphic singularity at the origin. In Section 3 we complete the study initiated in [45] by finding a precise description of the image for the above injective correspondence.

2.3. **Equivalence problem for second order ODEs.** We start with a description of the jet prolongation approach to the equivalence problem (which is a simple interpretation of a more general approach in the context of *jet bundles*). In what follows all variables are assumed to be complex, all mappings biholomorphic, and all ODEs to be defined near their zero solution $y(x) = 0$.

Consider two ODEs, $\mathcal{E}$ given by $y'' = \Phi(x, y, y')$ and $\tilde{\mathcal{E}}$ given by $y'' = \tilde{\Phi}(x, y, y')$, where the functions Φ and $\tilde{\Phi}$ are holomorphic in some neighbourhood of the origin in $\mathbb{C}^3$. We say that a germ of a biholomorphism $F : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ transforms $\mathcal{E}$ into $\tilde{\mathcal{E}}$, if it sends (locally) graphs of solutions of $\mathcal{E}$ into graphs of solutions of $\tilde{\mathcal{E}}$. We define the 2-*jet space* $J^{(2)}$ to be a 4-dimensional linear space with coordinates x, y, y_1, y_2, which correspond to the independent variable x, the dependent variable y and its derivatives up to order 2, so that we can naturally consider $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as complex submanifolds of $J^{(2)}$.

For any biholomorphism F as above one may consider its 2-*jet prolongation* $F^{(2)}$, which is defined on a neighbourhood of the origin in $\mathbb{C}^4$ as follows. The first two components of the mapping $F^{(2)}$ coincide with those of F. To obtain the remaining components we denote the coordinates in the preimage by (x, y) and in the target domain by (X, Y). Then the derivative $\frac{dY}{dX}$ can be symbolically recalculated, using the chain rule, in terms of x, y, y', so that the third coordinate Y_1 in the target jet space becomes a function of x, y, y_1. In the same manner one obtains the fourth component of the prolongation of the mapping F. Thus the mapping F transforms the ODE $\mathcal{E}$ into $\tilde{\mathcal{E}}$ if and only if the prolonged mapping $F^{(2)}$ transforms $(\mathcal{E}, 0)$ into $(\tilde{\mathcal{E}}, 0)$ as submanifolds in the jet space $J^{(2)}$. A similar statement can be formulated for certain singular differential equations, for example, for linear ODEs (see, e.g., [38]).

The local equivalence problem for (nonsingular!) second order ODEs was solved in the celebrated papers of E. Cartan [14] and A. Tresse [64]. We briefly describe below Tresse's approach, as it is of particular importance for us. A *semi-invariant of order* k for the action of the group $\mathrm{Diff}(\mathbb{C}^2, 0)$ of biholomorphisms of $(\mathbb{C}^2, 0)$ on the space of germs at the origin of right-hand sides $\Phi(x, y, y_1)$ of second order holomorphic ODEs, is a polynomial L on the space $j^k(\mathbb{C}^3, \mathbb{C})$ of k-jets of functions $\Psi : (\mathbb{C}^3, 0) \mapsto \mathbb{C}$ which has the following property. Given two arbitrary ODEs $\mathcal{E} = \{y'' = \Phi(x, y, y')\}$, $\tilde{\mathcal{E}} = \{y'' = \tilde{\Phi}(x, y, y')\}$, and a transformation $F : (\mathbb{C}^2, 0) \mapsto (\mathbb{C}^2, 0)$ which takes $\mathcal{E}$ into $\tilde{\mathcal{E}}$, the value $L(j^k\tilde{\Phi}(X, Y, Y_1))$ differs from the value $L(j^k\Phi(x, y, y_1))$ by a factor $\lambda(x, y, y_1)$ non-vanishing near the origin (here $(x, y, y_1) \in (\mathbb{C}^3, 0)$ is arbitrary, and $(X, Y, Y_1) \in (\mathbb{C}^3, 0)$ is its image under the 1-jet prolongation of F). For simplicity, for the expression $L(j^k\Phi(x, y, y_1))$ we write simply $L(\Phi)$ in what follows.

In [64] Tresse found the complete system of semi-invariants for the equivalence problem for 2nd order ODEs. In particular, he found the two basic (lowest order) semi-invariants

$$\begin{aligned} L_1(\Phi) &= \Phi_{y_1y_1y_1y_1} \\ L_2(\Phi) &= D^2\Phi_{y_1y_1} - 4D\Phi_{yy_1} - \Phi_{y_1} \cdot D\Phi_{y_1y_1} + 4\Phi_{y_1}\Phi_{yy_1} - 3\Phi_y\Phi_{y_1y_1} + 6\Phi_{yy}, \end{aligned} \tag{2.6}$$

where the differential operator D is defined by

$$D := \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + \Phi\frac{\partial}{\partial y_1}.$$

Then he proved the following.

Tresse's Theorem. *A second order ODE is locally equivalent to the flat (or simplest) ODE* $Y''=0$ *if and only if the two basic invariants vanish:*

$$L_1(\Phi)=L_2(\Phi)=0.$$

The concept of the *dual second order ODE* connects the two basic invariants. For the family of solutions $\mathcal{S}=\{y=\Phi(x,\xi,\eta)\}_{\xi,\eta\in(\mathbb{C}^2,0)}$ of a second order ODE $\mathcal{E}:\,y''=\Phi(x,y,y')$, considered near the zero solution $y=0$, the two-parameter family $\mathcal{S}^*$, given by the implicit equation $\eta=\Phi(\xi,x,y)$, is called *dual* for $\mathcal{S}$. The unique second order ODE $\mathcal{E}^*$, satisfied by the family $\mathcal{S}^*$ (see subsection 2.2), is called *dual* for $\mathcal{E}$. A dual ODE is not unique, as it depends on the parametrization of the family $\mathcal{S}$, but its equivalence class with respect to the action of $\mathrm{Diff}(\mathbb{C}^2,0)$ is unique and well defined. Remarkably, for any choice of the dual ODE $\mathcal{E}^*=\{y''=\Phi^*(x,y,y_1)\}$ there exist two non-vanishing near the origin factors $\lambda(x,y,y_1),\mu(x,y,y_1)$ such that

$$L_1(\Phi)=\lambda\cdot L_2(\Phi^*),\quad L_2(\Phi)=\mu\cdot L_1(\Phi^*).$$

In particular, $\mathcal{E}$ is locally equivalent to the simplest ODE if and only if both $\mathcal{E}$ and $\mathcal{E}^*$ are cubic with respect to y_1.

For a modern treatment of the problem and some further developments we refer to the book of V. Arnold [1], and also to the work of B. Kruglikov [47] and P. Nurowski and G. Sparling [52].

2.4. **Complex linear differential equations with an isolated singularity.** Complex linear ODEs are important classical objects, whose geometric interpretations are plentiful. We refer to the excellent sources [38], [2], [12], [66],[16] on complex linear differential equations, gathering here the facts that we will need in the sequel.

A *first order linear system* of n complex ODEs in a domain $G\subset\mathbb{C}$ (or simply a *linear system* in a domain G) is a holomorphic ODE system $\mathcal{L}$ of the form $y'(w)=A(w)y(w)$, where $A(w)$ is a holomorphic function in G, taking values in the space of $n\times n$ matrices, and $y(w)=(y_1(w),...,y_n(w))$ is an n-tuple of unknown functions. Solutions of $\mathcal{L}$ near a point $p\in G$ form a linear space of dimension n. Moreover, any germ of a solution near a point $p\in G$ of $\mathcal{L}$ extends analytically along any path $\gamma\subset G$, starting at p, so that any solution $y(w)$ of $\mathcal{L}$ is defined globally in G as a (possibly multiple-valued) analytic function. A *fundamental system of solutions for* $\mathcal{L}$ is a matrix whose columns form some collection of n linearly independent solutions of $\mathcal{L}$.

If G is a punctured disc, centered at 0, we say that $\mathcal{L}$ is a system *with an isolated singularity at* $w=0$. An important (and sometimes even a complete) characterization of an isolated singularity is its *monodromy operator*, which is defined as follows. If $Y(w)$ is some fundamental system of solutions of $\mathcal{L}$ in G, and γ is a simple loop about the origin, then it is not difficult to see that the monodromy of $Y(w)$ with respect to γ is given by the right multiplication by a constant nondegenerate matrix M, called the *monodromy matrix*. The matrix M is defined up to a similarity, so that it defines a linear operator $\mathbb{C}^n\longrightarrow\mathbb{C}^n$, which is called the monodromy operator of the singularity.

If $A(w)$ has a pole at the isolated singularity $w=0$, we say that the system has a *meromorphic singularity*. As the solutions of $\mathcal{L}$ are holomorphic in any proper sector $S\subset G$ of a sufficiently small radius with vertex at $w=0$, it is important to study the behaviour of the solutions as $w\to 0$. If all solutions of $\mathcal{L}$ admit a bound $||y(w)||\leq C|w|^a$ in any such sector (with some constants $C>0,\ a\in\mathbb{R}$, depending possibly on the sector), then $w=0$ is a *regular singularity*, otherwise it is an *irregular singularity*. In particular, if the monodromy is trivial, then the singularity is regular if and only if all the solutions of $\mathcal{L}$ are meromorphic in G.

L. Fuchs introduced the following condition: the singular point $w = 0$ is *Fuchsian*, if $A(w)$ is meromorphic at $w = 0$ and has a pole of order ≤ 1 there. The Fuchsian condition turns out to be sufficient for the regularity of a singular point. Another remarkable property of Fuchsian singularities can be described as follows. We say that two complex linear systems with an isolated singularity $\mathcal{L}_1, \mathcal{L}_2$ are *(formally) equivalent*, if there exists a (formal) transformation $F : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$ of the form $F(w, y) = (w, H(w)y)$ for some (formal) invertible matrix-valued function $H(w)$, which transforms (formally) $\mathcal{L}_1$ into $\mathcal{L}_2$. It turns out that two Fuchsian systems are formally equivalent if and only if they are holomorphically equivalent (in fact, any formal equivalence between them as above must be convergent). Any Fuchsian system can be brought to a special polynomial form (in the sense that the matrix $wA(w)$ is polynomial) called the *Poincare-Dulac normal form for Fuchsian systems*, and moreover, the normalizing transformation is always convergent.

However, in the *non*-Fuchsian case the behavior of solutions and mappings between linear systems is totally different. Generically, solutions of a non-Fuchsian system

$$y' = \frac{1}{w^m} B(w)y, \quad m \geq 2$$

do *not* have polynomial growth in sectors, and formal equivalences between non-Fuchsian systems are divergent, as a rule. Also the transformation bringing a *non-resonant* non-Fuchsian system (see next paragraph for the definition) to a special polynomial form called *Poincare-Dulac normal form for non-Fuchsian systems* is usually also divergent. As a compensation for this divergence phenomenon, we formulate below a remarkable result, *Sibuya's sectorial normalization theorem*, which is of fundamental importance for our constructions.

Consider a system $y' = \frac{1}{w^m} B(w)y$, $m \geq 2$ which is non-resonant (i.e., the leading matrix $B_0 = B(0)$ has pairwise distinct eigenvalues $\{\lambda_1, ..., \lambda_n\}$). For all possible pairs (λ_i, λ_j), $i, j = 1, ..., n$, $i \neq j$, of eigenvalues, we call each of the $2(m-1)$ rays $R_{ij} = \left\{\text{Re}\,\left((\lambda_i - \lambda_j)w^{1-m}\right) = 0\right\}$ a *separating ray for the system*. Next, we recall that for a function $f(w)$, holomorphic in a sector S with the vertex at 0, a formal series

$$\hat{f}(w) = \sum_{j \geq 0} c_j w^j$$

represents $f(w)$ *in* S *asymptotically* (one uses the notation $f(w) \sim \hat{f}(w)$), if for every $k \geq 0$

$$\frac{1}{w^k} \left(f(w) - \sum_{j=0}^{k} c_j w^j \right) \longrightarrow 0, \quad w \to 0, w \in S.$$

We refer to [66] for further details and properties. Finally, in what follows a *conjugacy* between systems means a mapping, transforming the first system into the second.

Theorem 9 (Y. Sibuya, 1962, see [58],[38]). *Assume that a non-Fuchsian linear system* $\mathcal{E}$

$$y' = \frac{1}{w^m} B(w)y, \quad m \geq 2$$

is non-resonant and $S \subset (\mathbb{C}, 0)$ *is an arbitrary sector with vertex at* 0 *not containing two separating rays for any pair of the eigenvalues. Then for any formal conjugacy* $w \mapsto w$, $y \mapsto \hat{H}(w)y$, *conjugating the system with its Poincare-Dulac polynomial normal form, there exists a holomorphic function* $H_S(w)$ *defined in* S *and taking values in* $GL(n, \mathbb{C})$ *such that* $H_S(w)$ *asymptotically represents* $\hat{H}(w)$ *in* S *and* $w \mapsto w$, $y \mapsto H_S(w)y$ *conjugates* $\mathcal{E}$ *with its Poincare-Dulac normal* *form in S. If a sector S has opening bigger than $\frac{\pi}{m-1}$, then the sectorial normalization $H_S(w)$ is unique.*

Alternatively, one can require for the uniqueness in Sibuya's theorem that the sector S contains a separating ray for each pair of eigenvalues of the leading matrix.

We note that the holomorphic sectorial normalization in Theorem 9 does usually *not* extend to one holomorphic near the origin. The reason is that, somewhat surprisingly, the sectorial normalization $H_S(w)$ might change from sector to sector by means of multiplication by a constant matrix $C \in \mathrm{GL}(n,\mathbb{C})$ called a *Stokes matrix.* This phenomenon is known as the *Stokes phenomenon*, and the entire collection $\{C_{ij}\}$ of Stokes matrices, corresponding to all separating rays, is called the *Stokes collection.* Generically this collection is non-trivial (i.e., contains non-identical matrices). Actually, the Stokes phenomenon is the conceptual reason for the irregularity phenomena demonstrated in this paper.

A scalar linear complex ODE of order n in a domain $G \subset \mathbb{C}$ is an ODE $\mathcal{E}$ of the form

$$z^{(n)} = a_n(w)z + a_{n-1}(w)z' + ... + a_1(w)z^{(n-1)},$$

where $\{a_1(w), \dots a_n(w)\}$ is a given collection of holomorphic functions in G and $z(w)$ is the unknown function. By a reduction of $\mathcal{E}$ to a first order linear system (see the above references and also [31] for various approaches of doing that) one can naturally transfer most of the definitions and facts, relevant to linear systems, to scalar equations of order n. The main difference here is contained in the appropriate definition of Fuchsian: a singular point $w = 0$ for an ODE $\mathcal{E}$ is said to be *Fuchsian*, if the orders of poles p_j of the functions $a_j(w)$ satisfy the inequalities $p_j \leq j$, $j = 1, 2, \dots, n$. It turns out that the condition of Fuchs becomes also necessary for the regularity of a singular point in the case of n-th order scalar ODEs.

Further information on the classification of isolated singularities (including details of Poincare-Dulac normalizations in the Fuchsian and non-Fuchsian cases respectively) can be found in [38], [66] or [16].

3. Characterization of nonminimal spherical hypersurfaces

In this section we establish a class of (in general nonlinear) second order complex ODEs with a meromorphic singularity, which correspond to real hypersurfaces in $\mathbb{C}^2$ which are nonminimal at the origin and spherical in the complement of their complex locus. Using the connection between hypersurfaces and ODEs, this finally gives a complete description of nonminimal hypersurfaces, spherical in the complement to the complex locus. We start with necessary definitions and denote by Δ_ε a disc in $\mathbb{C}$, centered at $w = 0$ of radius ε, and by Δ^*_ε the corresponding punctured disc.

Definition 10. A second order complex ODE

$$z'' = (p_0 + p_1 z)z' + (q_3 z^3 + q_2 z^2 + q_1 z + q_0), \tag{3.1}$$

where the functions $p_i(w), q_j(w)$ are meromorphic in a domain $\Omega \subset \mathbb{C}$, is called a $\mathcal{P}_0$*-ODE*, if the meromorphic coefficients satisfy

$$q_3(w) = -\frac{1}{9}p_1^2(w), \quad q_2(w) = \frac{1}{3}(p_1' - p_0 p_1). \tag{3.2}$$

In the special case when Ω is a disc Δ_ε and the coefficients $p_i(w), q_j(w)$ have a unique meromorphic singularity at the point $w = 0$, we call (3.1) a $\mathcal{P}_0$ *- ODE with an isolated meromorphic singularity.* A $\mathcal{P}_0$ - ODE with an isolated meromorphic singularity can be always represented as

$$z'' = \frac{1}{w^m}(Az + B)z' + \frac{1}{w^{2m}}(Cz^3 + Dz^2 + Ez + F), \tag{3.3}$$

where $m \geq 1$ is an integer and $A(w), B(w), C(w), D(w), E(w), F(w)$ are holomorphic near the origin coefficients, satisfying the special relations

$$C(w) = -\frac{1}{9}A^2(w),\ D(w) = \frac{1}{3}w^{2m}\left(\frac{A(w)}{w^m}\right)' - \frac{1}{3}A(w)B(w). \tag{3.4}$$

Note that it is possible, by scaling the holomorphic coefficients and the denominators w^m and w^{2m} simultaneously, to change the integer m without changing an ODE (3.3). To avoid the uncertainty, we call the smallest possible integer $m \geq 1$ for a fixed ODE (3.3) its *singularity order*. It is straightforward to check that the special relations (3.2), applied to a $\mathcal{P}_0$ - ODE $\mathcal{E}$, are equivalent to the fact that the two Tresse semi-invariants (2.6) vanish identically for $w \in \Omega$. Thus, (3.2) is equivalent to the fact that $\mathcal{E}$ is locally equivalent to $z'' = 0$ near each *regular* point (z_0, w_0), $w_0 \in \Omega$.

The $\mathcal{P}_0$-notation is caused by the fact that the map, transforming (3.1) into the simplest ODE $z'' = 0$, is in fact linear fractional in z (see, e.g., the proof of Theorem 3.3 in [45]). In his celebrated work [54] Painlevé classified all second order complex ODEs, rational in the dependent variable z and its derivative, meromorphic in some domain Ω in the independent variable w, and having no movable critical points besides poles. (A movable critical point is a singular point of a particular solution which changes or disappears after a small perturbation of the Cauchy data). ODEs of the latter kind are called ODEs of class $\mathcal{P}$. The mapping which brings an ODE of class $\mathcal{P}$ to its standard form in this classification, is locally biholomorphic in $\mathbb{CP}^1 \times \Omega$ and is linear-fractional in the dependent variable (see, e.g., [2] for details). In our case the standard form is flat ($z'' = 0$), which motivates the $\mathcal{P}_0$ notation.

We also note that for $A(w) = C(w) = D(w) = F(w) \equiv 0$ an ODE (3.3) is linear (the latter case was considered in [44]), and its Fuchsianity is equivalent to the fact that its singularity order equals 1.

A direct calculation shows that if a germ $z(w)$ of a solution of (3.3) is invertible in some domain, then the inverse function $w(z)$ satisfies in the image domain the ODE

$$w'' = -\frac{1}{w^m}(Az + B)(w')^2 - \frac{1}{w^{2m}}(Cz^3 + Dz^2 + Ez + F)(w')^3. \tag{3.5}$$

We call (3.5) *the inverse ODE* for (3.3) (i.e., we interchange the dependent and the independent variables).

We next introduce a class of anti-holomorphic 2-parameter families of planar complex curves that potentially can be the family of solutions for a $\mathcal{P}_0$-ODE with an isolated meromorphic singularity and, at the same time, the family of Segre varieties of a real hypersurface in $\mathbb{C}^2$.

Definition 11. An *m-admissible Segre family* is a 2-parameter antiholomorphic family of planar holomorphic curves in a polydisc $\Delta_\delta \times \Delta_\varepsilon$ which can be parameterized in the form

$$w = \bar\eta e^{\pm i\bar\eta^{m-1}\varphi(z,\bar\xi,\bar\eta)}, \tag{3.6}$$

where $m \geq 1$ is an integer, $\xi \in \Delta_\delta, \eta \in \Delta_\varepsilon$ are holomorphic parameters, and the function $\varphi(x, y, u)$ is holomorphic in the polydisc $\Delta_\delta \times \Delta_\delta \times \Delta_\varepsilon$ and has there an expansion

$$\varphi(x, y, u) = xy + \sum_{k,l\geq 2} \varphi_{kl}(u)x^k y^l, \quad \varphi_{kl}(u) \in \mathcal{O}(\Delta_\varepsilon).$$

To avoid confusion in terminology we will call m-admissible families of the form

$$\mathcal{S} = \left\{ w = \bar\eta e^{\pm i\bar\eta^{m-1}\left(z\bar\xi + \sum_{k\geq 2}\psi_k(\bar\eta)z^k\bar\xi^k\right)} \right\},$$

which were considered in [44], *m-admissible with rotations.* Thus an m-admissible family has the form

$$\mathcal{S}=\left\{w=\bar\eta e^{\pm i\bar\eta^{m-1}\left(z\bar\xi+\sum_{k,l\geq 2}\varphi_{kl}(\bar\eta)z^k\bar\xi^l\right)},\ (\xi,\eta)\in\Delta_\delta\times\Delta_\varepsilon\right\}. \tag{3.7}$$

m-admissibility of an anti-holomorphic 2-parameter family of planar complex curves can be checked easily: a family defined by $w=\rho(z,\bar\xi,\bar\eta)$, where ρ is holomorphic in some polydisc $U\subset\mathbb{C}^3$, centered at the origin, is m-admissible if and only if the defining function ρ has the expansion $\rho(z,\bar\xi,\bar\eta)=\bar\eta\pm i\bar\eta^m z\bar\xi+O(\bar\eta^m z^2\bar\xi^2)$.

Recall that, according to Meylan [48], a nonminimal real-analytic hypersurface $M\subset\mathbb{C}^N$ at a point p is called *m-nonminimal at p*, if in some local coordinates, vanishing at p, M can be represented as

$$\operatorname{Im}w=(\operatorname{Re})^m H(z,\bar z,\operatorname{Re}w),\ H(z,\bar z,0)\not\equiv 0.$$

Here $(z,w)\in\mathbb{C}^{N-1}\times\mathbb{C}$ denote the coordinates in $\mathbb{C}^N$ and $m\in[1,\infty)$ is an integer, known to be a biholomorphic invariant of (M,p). For a real-analytic hypersurface $M\subset\mathbb{C}^2$ which is nonminimal at the origin with nonminimality order m and is defined by an equation of the form

$$v=u^m\left(\pm|z|^2+\sum_{k,l\geq 2}h_{kl}(u)z^k\bar z^l\right), \tag{3.8}$$

it is not difficult to check that its Segre family is an m-admissible Segre family. We call a real hypersurface of the form (3.8) an *m-admissible nonminimal hypersurface.* Note that in the case of m-admissible Segre families (respectively, nonminimal hypersurfaces) the integer m is uniquely determined by the Segre family (respectively, by the hypersurface). Depending on the sign in the exponent $e^{\pm i\bar\eta^{m-1}\varphi(z,\bar\xi,\bar\eta)}$ we say that an m-admissible Segre family is *positive* or *negative*, respectively, and apply these notions for real hypersurfaces. In analogy with the case of real hypersurfaces, we call the holomorphic curve in the family (3.6), corresponding to the values $\xi=a,\eta=b$ of parameters, *the Segre variety of a point* $p=(a,b)\in\Delta_\delta\times\Delta_\varepsilon$ and denote it by Q_p. We call the hypersurface

$$X=\{w=0\}\subset\Delta_\delta\times\Delta_\varepsilon$$

the *singular locus* of an m-admissible Segre family. As a consequence of (3.6), we have the equivalences

$$Q_p\cap X\neq\emptyset\iff p\in X\iff Q_p=X.$$

Also note that the fact that $w(0)=\bar\eta,\ w'(0)=\pm i\bar\xi\bar\eta^m$ shows that the *Segre mapping* $\lambda:\ p\longrightarrow Q_p$ is injective in $(\Delta_\delta\times\Delta_\varepsilon)\setminus X$.

We next describe a way to connect admissible Segre families with $\mathcal{P}_0$ - ODEs.

Definition 12. We say that an m-admissible Segre family $\mathcal{S}$ *is associated with a* $\mathcal{P}_0$ *- ODE* $\mathcal{E}$ *of singularity order* $\leq m$, if after an appropriate shrinking of the basic neighbourhood $\Delta_\delta\times\Delta_\varepsilon$ of the origin all the elements $Q_p\in\mathcal{S}$ with $p\notin X$, considered as graphs $w=w(z)$, satisfy the inverse ODE for $\mathcal{E}$ (given by (3.5)).

Note that we may always substitute the Segre varieties into (3.5). Given an ODE $\mathcal{E}$, we denote an associated m-admissible Segre family by $\mathcal{S}^{\pm}_m(\mathcal{E})$, depending on the sign of the Segre family.

Proposition 13. *For any integer* $m\geq 1$ *and any* $\mathcal{P}_0$ *- ODE* $\mathcal{E}$ *of singularity order* $\leq m$*, as in* (3.3)*, there is a unique positive and a unique negative* m*-admissible Segre family* $\mathcal{S}$*, associated* *with $\mathcal{E}$. The ODE $\mathcal{E}$ and the associated Segre families $\mathcal{S}^{\pm}_m(\mathcal{E})$ given by (3.7), satisfy the following relations:*

$$F(w) = 2\varphi_{23}(w),\ A(w) = \pm 6i\varphi_{32}(w),\ B(w) = \pm 2i\varphi_{22}(w) - w^{m-1},$$
$$E(w) = 6\varphi_{33} \pm 2i(m-1)\varphi_{22}w^{m-1} - 8(\varphi_{22})^2 \mp 2i\varphi'_{22}w^m. \tag{3.9}$$

In particular, for any fixed m the correspondences $\mathcal{E} \longrightarrow \mathcal{S}^+_m(\mathcal{E})$ and $\mathcal{E} \longrightarrow \mathcal{S}^-_m(\mathcal{E})$ are injective.

Proof. Consider a positive m-admissible Segre family $\mathcal{S}$, as in (3.6), and a $\mathcal{P}_0$-ODE with an isolated meromorphic singularity $\mathcal{E}$. We first express the condition that $\mathcal{S}$ is associated with $\mathcal{E}$ in the form of a differential equation. Fix $p = (\xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon$ and consider the Segre variety Q_p, given by (3.6), as a graph $w = w(z)$. For the function $\varphi(x, y, u)$ we denote by $\dot\varphi$ and $\ddot\varphi$ its first and second derivatives respectively with respect to the first argument. Then one computes

$$\begin{aligned} w' &= i\bar\eta^m e^{i\bar\eta^{m-1}\varphi(z,\bar\xi,\bar\eta)}\dot\varphi(z,\bar\xi,\bar\eta),\\ w'' &= i\bar\eta^m e^{i\bar\eta^{m-1}\varphi(z,\bar\xi,\bar\eta)}\ddot\varphi(z,\bar\xi,\bar\eta) - \bar\eta^{2m-1}e^{i\bar\eta^{m-1}\varphi(z,\bar\xi,\bar\eta)}(\dot\varphi(z,\bar\xi,\bar\eta))^2. \end{aligned}$$

Plugging these expressions into (3.5) yields after simplifications

$$\begin{aligned} \ddot\varphi = -i(\dot\varphi)^2 \left(\bar\eta^{m-1} + (A(\bar\eta e^{i\bar\eta^{m-1}\varphi})z + B(\bar\eta e^{i\bar\eta^{m-1}\varphi}))e^{i(1-m)\bar\eta^{m-1}\varphi}\right) + \qquad (3.10)\\ +(\dot\varphi)^3\left(C(\bar\eta e^{i\bar\eta^{m-1}\varphi})z^3 + D(\bar\eta e^{i\bar\eta^{m-1}\varphi})z^2 + E(\bar\eta e^{i\bar\eta^{m-1}\varphi})z + F(\bar\eta e^{i\bar\eta^{m-1}\varphi})\right)e^{i(2-2m)\bar\eta^{m-1}\varphi}, \end{aligned}$$

where $\varphi = \varphi(z, \bar\xi, \bar\eta)$. The differential equation (3.10) is a second order holomorphic ODE, depending holomorphically on the parameters $\bar\xi, \bar\eta$. Considering now the Cauchy problem for the ODE (3.10) with the initial data $\varphi(0) = 0$, $\dot\varphi(0) = \bar\xi$, we get from the theorem on the analytic dependence of solutions of a holomorphic ODE on holomorphic parameters (see, e.g., [38]) that its solution $\varphi = \varphi(z, \bar\xi, \bar\eta)$ is unique and holomorphic in $z, \bar\xi, \bar\eta$ in some polydisc $U \subset \mathbb{C}^3$, centered at the origin. Observe that the above arguments are reversible.

For the proof of the proposition, given a $\mathcal{P}_0$-ODE $\mathcal{E}$ of singularity order $\leq m$, we solve the corresponding equation (3.10) with the initial data $\varphi(0) = 0$, $\dot\varphi(0) = \bar\xi$, and obtain a solution $\varphi = \varphi(z, \bar\xi, \bar\eta)$. Since $\varphi(0, \bar\xi, \bar\eta) \equiv 0$, $\varphi_z(0, \bar\xi, \bar\eta) \equiv \bar\xi$, we conclude that

$$\varphi(z, \bar\xi, \bar\eta) = z\bar\xi + \sum_{k\geq 2, l\geq 0} \varphi_{kl}(\bar\eta)z^k\bar\xi^l. \tag{3.11}$$

However, substituting (3.11) into (3.10) and gathering terms of the form $z^{k-2}\bar\xi^0$ with $k \geq 2$ yields first $\varphi_{20} \equiv 0$ and then by induction $\varphi_{k0} \equiv 0$ for all $k \geq 2$. Using the latter fact and gathering in (3.10) terms of the form $z^{k-2}\bar\xi^1$ with $k \geq 2$, we get (since, after the substitution of (3.10), the right hand side in (3.10) becomes divisible by $\bar\xi^2$) that $\varphi_{k1} \equiv 0$ for all $k \geq 2$. Thus φ has the form required for the m-admissibility and

$$w = \bar\eta e^{i\bar\eta^{m-1}\varphi(z,\bar\xi,\bar\eta)}$$

is the desired positive m-admissible Segre family $\mathcal{S} = \mathcal{S}^+_m(\mathcal{E})$ associated with $\mathcal{E}$. The uniqueness of $\mathcal{S}^+_m(\mathcal{E})$ also follows from the uniqueness of the solution to the Cauchy problem.

To prove the relations (3.9), we substitute (3.6) into (3.5). We rewrite both sides of this identity as power series in z and $\bar\xi$ with coefficients depending on $\bar\eta$. If we equate the coefficients of $\bar\xi^3$, we obtain $2\varphi_{23}(\bar\eta) = F(\bar\eta)$. Equating terms of the form $z\bar\xi^2$ we obtain $6i\varphi_{32}(\bar\eta) = A(\bar\eta)$. Similar computations for $\bar\xi^2$ and $z^3\bar\xi$ give the formulas for B and E. Finally, to prove the injectivity one needs to use, in addition to (3.9), the special relations (3.4), and this enables to express all the coefficients of $\mathcal{E}$ in terms of $\mathcal{S}$. This proves the proposition in the positive case. The proof in the negative case is analogous. $\square$

Proposition 13 gives an effective algorithm for computing the m-admissible Segre family for a given $\mathcal{P}_0$-ODE with an isolated meromorphic singularity. Our goal is, however, to identify those ODEs that produce Segre families with a reality condition, that is, Segre families of nonminimal real hypersurfaces.

Definition 14. We say that an m-admissible Segre family has a *real structure* if it is the Segre family of an m-admissible real hypersurface $M \subset \mathbb{C}^2$. We also say that a $\mathcal{P}_0$-ODE $\mathcal{E}$ of nonsingularity order at most m has *m-positive (respectively, m-negative) real structure*, if the associated positive (respectively, negative) m-admissible Segre family $\mathcal{S}^{\pm}_m(\mathcal{E})$ has a real structure. We say that the corresponding real hypersurface M is *associated* with $\mathcal{E}$.

We then need a development, in singular settings, of the concepts of the dual family and dual ODE, described in Section 2.3. For that, it is convenient to allow re-parametrizations of parametrized families under consideration. Let $\rho(z, y, u)$ be a holomorphic function near the origin in $\mathbb{C}^3$ with $\rho(0,0,0) = 0$, and $d\rho(0,0,0) = du$. For $z, \xi \in \Delta_\delta, w, \eta \in \Delta_\varepsilon$, let

$$\mathcal{S} = \{w = \rho(z, \bar{\xi}, \bar{\eta})\}$$

be a 2-parameter antiholomorphic family of holomorphic curves near the origin, parametrized by (ξ, η). An *admissible* (re)-parametrization of $\mathcal{S}$ is given by a function $\tilde{\rho}(z, \bar{\xi'}, \bar{\eta'})$ such that

$$\mathcal{S} = \{w = \tilde{\rho}(z, \bar{\xi'}, \bar{\eta'})\}$$

and there exists a germ of a biholomorphism $(\xi, \eta) \mapsto (\xi', \eta')$ such that $\rho(z, \bar{\xi}, \bar{\eta}) = \tilde{\rho}\left(z, \overline{\xi'(\xi,\eta)}, \overline{\eta'(\xi,\eta)}\right)$. Fixing a parametrization and considering all admissible (re)-parametrizations gives rise to the notion of a *general Segre family.*

For each point $p = (\xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon$ we call the corresponding holomorphic curve $Q^\rho_p = \{w = \rho(z, \bar{\xi}, \bar{\eta})\} \in \mathcal{S}$ its *Segre variety.* Clearly, an m-admissible Segre family is a particular example of a general Segre family. Note that the Segre varieties of a general Segre family do depend on the parametrization, but admissible parametrizations give rise to a (holomorphic) relabeling of the Segre varieties.

We say that two general Segre families $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are equivalent if there exists a germ of a biholomorphism $H = (f, g)$ of $(\mathbb{C}^2, 0)$ such that $\tilde{\mathcal{S}} = H^{-1}(\mathcal{S})$, and such that the solution of the implicit function problem $g(z, w) = \tilde{\rho}(f(z, w)), \xi, \eta)$ for w is an admissible parametrization of $\mathcal{S}$.

Further, given a (general) Segre family $\mathcal{S}$, from the implicit function theorem one concludes that the antiholomorphic family of planar holomorphic curves

$$\mathcal{S}^{*,\rho} = \{\bar{\eta} = \rho(\bar{\xi}, z, w)\}$$

is also a general Segre family for some, possibly, smaller polydisc $\Delta_{\tilde{\delta}} \times \Delta_{\tilde{\varepsilon}}$, which depends on the chosen parametrization ρ. We note that for every admissible parametrization of $\mathcal{S}$, we obtain an equivalent Segre family.

Definition 15. The Segre family $\mathcal{S}^{*,\rho}$ is called the *dual Segre family* for $\mathcal{S}$ with the parametrization ρ.

The dual Segre family has a simple interpretation: in the defining equation of the family S one should consider the parameters $\bar{\xi}, \bar{\eta}$ as new coordinates, and the variables z, w as new parameters. If we denote the Segre variety of a point p with respect to the family $\mathcal{S}^{*,\rho}$ by $Q^{*,\rho}_p$, this just means that $Q^{*,\rho}_p = \{(z, w) \colon \bar{p} \in Q^\rho_{(\bar{z}, \bar{w})}\}$. In the following, we will suppress the dependence on ρ from the notation whenever we make claims which hold for all admissible parametrizations of a given Segre family.

It is not difficult to see that if $\mathcal{S}$ is a positive (respectively, negative) m-admissible Segre family, then $\mathcal{S}^*$ is a negative (respectively, positive) m-admissible Segre family. Indeed, to obtain the defining function $\rho^*(z,\bar\xi,\bar\eta)$ of the general Segre family $\mathcal{S}^*$ we need to solve for w in the equation

$$\bar\eta = we^{\pm iw^{m-1}\left(z\bar\xi+\sum_{k,l\geq 2}\varphi_{kl}(w)z^k\bar\xi^l\right)}. \tag{3.12}$$

Note that (3.12) implies

$$w = \bar\eta e^{\mp iw^{m-1}(z\bar\xi+O(z^2\bar\xi^2))}. \tag{3.13}$$

We then obtain from (3.13) $w=\rho^*(z,\bar\xi,\bar\eta)=\bar\eta(1+O(z\bar\xi))$. Substituting the latter representation into (3.13) gives $w=\rho^*(z,\bar\xi,\bar\eta)=\bar\eta e^{\mp i\bar\eta^{m-1}(z\bar\xi+O(z^2\bar\xi^2))}$, as required.

We also need the following Segre family, connected with S:

$$\bar{\mathcal{S}} = \{w=\bar\rho(z,\bar\xi,\bar\eta)\},$$

where for a power series of the form

$$f(x)=\sum\nolimits_{\alpha\in\mathbb{Z}^d} c_\alpha x^\alpha$$

we denote by $\bar f(x)$ the series $\sum_{\alpha\in\mathbb{Z}^d}\bar c_\alpha x^\alpha$. Note that $\bar{\mathcal{S}}$ does not depend on the particular admissible parametrization, in contrast to the dual family.

Definition 16. The Segre family $\bar{\mathcal{S}}$ is called the *conjugated family* of $\mathcal{S}$.

If $\sigma:\mathbb{C}^2\longrightarrow\mathbb{C}^2$ is the antiholomorphic involution $(z,w)\longrightarrow(\bar z,\bar w)$, then one simply has $\sigma(Q_p)=\overline{Q_{\sigma(p)}}$. We will denote the Segre variety of a point p with respect to the family $\bar{\mathcal{S}}$ by $\bar Q^\rho_p$. It follows from the definition that if $\mathcal{S}$ is a positive (respectively, negative) m-admissible Segre family, then $\bar{\mathcal{S}}$ is a negative (respectively, positive) m-admissible Segre family.

In the same manner as for the case of an m-admissible Segre family, we say that a (general) Segre family $\mathcal{S}=\{w=\rho(z,\bar\xi,\bar\eta)\}$ has a *real structure,* if there exists a smooth real-analytic hypersurface $M\subset\mathbb{C}^2$, passing through the origin, such that $\mathcal{S}$ is the Segre family of M.

The use of the dual and the conjugated Segre families is illuminated by the fact that

A (general) Segre family $\mathcal{S}$ has a real structure if and only if the conjugated Segre family $\bar{\mathcal{S}}$ is also a dual family, i.e. if there exists an admissible parametrization ρ such that $\mathcal{S}^{*,\rho}=\bar{\mathcal{S}}$

This fact proved, for example, in [44] (see Proposition 3.10 there) is a corollary of the reality condition (2.2) for a real-analytic hypersurface. Our goal is to transfer the above real structure criterion from m-admissible families to the associated ODEs. In this case, we can somewhat simplify matters with regard to different parametrizations: when working with admissible families, we will always use the (unique) parametrization ρ which satisfies the conditions in (3.7) for our constructions.

Definition 17. Let $\mathcal{E}$ be a $\mathcal{P}_0$-ODE with an isolated meromorphic singularity of order $\leq m$. We say that a $\mathcal{P}_0$-ODE $\mathcal{E}^*$ with an isolated meromorphic singularity of order $\leq m$ is *m-dual* to $\mathcal{E}$, if the negative m-admissible Segre family which is dual to the family $\mathcal{S}^+_{\mathcal{E}}$ is in fact associated with $\mathcal{E}^*$, i.e.,

$$\mathcal{E}^* \text{ is } m\text{-dual to } \mathcal{E} \iff (\mathcal{S}^+_m(\mathcal{E}))^* = S^-_m(\mathcal{E}^*).$$

In the same manner, we say that a $\mathcal{P}_0$-ODE $\overline{\mathcal{E}}$ with an isolated meromorphic singularity of order $\leq m$ is *m-conjugated to $\mathcal{E}$,* if the negative m-admissible Segre family which is conjugated to the family $\mathcal{S}^+_m(\mathcal{E})$ is in fact associated with $\overline{\mathcal{E}}$, i.e.,

$$\bar{\mathcal{E}} \text{ is } m\text{-conjugated to } \mathcal{E} \iff \overline{\mathcal{S}}^+_m(\mathcal{E}) = S^-_m(\bar{\mathcal{E}}).$$

From Proposition 13 we conclude that for each fixed integer m not preceding the order of a given ODE $\mathcal{E}$, both the conjugated and the dual ODEs are unique (if they exist). The existence of the conjugated ODE for any m as above is obvious: if $\mathcal{E}$ is given by $z'' = \frac{1}{w^m}(Az+B)z' + \frac{1}{w^{2m}}(Cz^3+Dz^2+Ez+F)$, then, clearly, the desired ODE $\overline{\mathcal{E}}$ is given explicitly by

$$z'' = \frac{1}{w^m}(\bar{A}z+\bar{B})z' + \frac{1}{w^{2m}}(\bar{C}z^3+\bar{D}z^2+\bar{E}z+\bar{F}), \tag{3.14}$$

where $\bar{A} = \bar{A}(w)$ and similarly for the other coefficients of $\mathcal{E}$. In particular, the conjugated ODE does not depend on m and we skip this parameter for the conjugated ODE in what follows. The existence of the dual ODE is a much more delicate issue, which uses the triviality of Tresse semi-invariants of $\mathcal{P}_0$ - ODEs in a significant way.

Proposition 18. *For any $\mathcal{P}_0$-ODE $\mathcal{E}$ with an isolated meromorphic singularity of order $\leq m$ the m-dual ODE always exists.*

Proof. Suppose first that $\mathcal{S}$ is positive. Consider the family $\mathcal{T} = (S^+_m(\mathcal{E}))^*$ and denote by $\Delta_\delta \times \Delta_\varepsilon$ the polydisc where $\mathcal{T}$ is defined. Take then an arbitrary $p = (\xi,\eta) \in \Delta_\delta \times \Delta^*_\varepsilon$ and consider Segre varieties Q^*_p of $\mathcal{T}$ as graphs $w = w(z) = \bar{\eta}e^{-i\bar{\eta}^{m-1}(z\bar{\xi}+O(z^2\bar{\xi}^2))}$. Then we have

$$w = \bar{\eta} + O(z\bar{\xi}\bar{\eta}^m), \quad \frac{w'}{w^m} = -i\bar{\xi} + O(z\bar{\xi}), \quad w'' = O(\bar{\xi}^2\bar{\eta}^m). \tag{3.15}$$

and use the relations (3.15) in order to obtain a second order ODE satisfied by all $Q^*_p, p \in \Delta_\delta \times \Delta^*_\varepsilon$. An application of the implicit function theorem to the first two equations in (3.15) yields functions $\Lambda(z,w,\zeta) = i\zeta + O(z\zeta)$ and $\Omega(z,w,\zeta) = w + O(zw\zeta)$, such that

$$\bar{\xi} = \Lambda\left(z,w,\frac{w'}{w^m}\right), \bar{\eta} = \Omega\left(z,w,\frac{w'}{w^m}\right).$$

Substituting $\bar{\xi} = \Lambda(z,w,\frac{w'}{w^m})$, $\bar{\eta} = \Omega(z,w,\frac{w'}{w^m})$ into the equation for w'' in (3.15) gives us a second order ODE

$$w'' = \Phi\left(z,w,\frac{w'}{w^m}\right) \tag{3.16}$$

for some function $\Phi(z,w,\zeta)$, holomorphic in a polydisc $\tilde{V} \subset \mathbb{C}^3$, centered at the origin (compare this with the elimination procedure in Section 2.2). The ODE (3.16) is satisfied by all Q^*_p with $p \in \Delta_\delta \times \Delta^*_\varepsilon$. The function $\Phi(z,w,\zeta)$ also satisfies $\Phi(z,w,\zeta) = O(\zeta^2 w^m)$.

On the other hand, the holomorphic 2-parameter family $\mathcal{S}$ can be locally biholomorphically mapped into the family of affine straight lines in $\mathbb{C}^2$ near each regular point of it, i.e., near each point with $w \neq 0$ (see the discussion in the beginning of the section). According to Section 2, the same property holds for the dual family $\mathcal{T}$. In particular, Tresse's semi-invariants (2.6) vanish identically for the ODE (3.16), and hence $\frac{\partial^4}{(\partial w')^4}\left[\Phi\left(z,w,\frac{w'}{w^m}\right)\right] \equiv 0$. The latter means that the function $\Phi(z,w,\zeta)$ is at most cubic in its third argument. Since, in addition, $\Phi(z,w,\zeta) = O(\zeta^2 w^m)$, we conclude that we can write

$$\Phi(z,w,\zeta) = w^m(\Phi_2(z,w)\zeta^2 + \Phi_3(z,w)\zeta^3)$$

for some functions $\Phi_2(z,w)$ and $\Phi_3(z,w)$ holomorphic in a polydisc $\Delta_r \times \Delta_R$. Then the substitution $\zeta = \frac{w'}{w^m}$ turns (3.16) into an ODE

$$w'' = \frac{\Phi_2(z,w)}{w^m}(w')^2 + \frac{\Phi_3(z,w)}{w^{2m}}(w')^3. \tag{3.17}$$

We claim that the functions $\Phi_2(z,w)$ and $\Phi_3(z,w)$ in (3.17) are actually polynomials in z of degree 1 and 3 respectively. Let $(z_0,w_0) \in \Delta_\delta \times \Delta^*_\varepsilon$ and choose a small enough polydisk U centered at (z_0,w_0) such that there exists a locally biholomorphic mapping $\mathcal{F}:\ Z = f(z,w),\ W = g(z,w)$ of the polydisc U into $\mathbb{C}^2$, transforming (3.17) into the ODE $W'' = 0$. Performing a recalculation of first and second order derivatives in the coordinates (z,w) (see Section 2.3) we get that (3.17) is given in U by

$$w'' = I_0(z,w) + I_1(z,w)w' + I_2(z,w)(w')^2 + I_3(z,w)(w')^3, \tag{3.18}$$

where

$$\begin{aligned}
I_0 &= \frac{1}{f_w g_z - g_w f_z}\left(f_z g_{zz} - g_z f_{zz}\right),\\
I_1 &= \frac{1}{f_w g_z - g_w f_z}\left(f_w g_{zz} - g_w f_{zz} + 2 f_z g_{zw} - 2 g_z f_{zw}\right),\\
I_2 &= \frac{1}{f_w g_z - g_w f_z}\left(f_z g_{ww} - g_z f_{ww} + 2 f_w g_{zw} - 2 g_w f_{zw}\right),\\
I_3 &= \frac{1}{f_w g_z - g_w f_z}\left(f_w g_{ww} - g_w f_{ww}\right).
\end{aligned} \tag{3.19}$$

(since $\mathcal{F}$ is biholomorphic in U, the Jacobian $J = f_w g_z - f_z g_w$ is nonzero in U). Comparing (3.17) and (3.18) we conclude that the two functions $I_0(z,w), I_1(z,w)$ vanish identically in U and that $\frac{\Phi_2(z,w)}{w^m} = I_2(z,w)$, $\frac{\Phi_3(z,w)}{w^{2m}} = I_3(z,w)$. In particular, we have that (f,g) satisfies the PDE system

$$\begin{aligned}
f_z g_{zz} - g_z f_{zz} &= 0\\
f_w g_{zz} - g_w f_{zz} + 2 f_z g_{zw} - 2 g_z f_{zw} &= 0.
\end{aligned} \tag{3.20}$$

As was shown in [45] (see the proof of Theorem 3.3 there), any solution (f,g) of the system (3.20) with $J(z,w) \neq 0$ is linear-fractional in z in the polydisc U, i.e., there exists six holomorphic in U functions $\alpha_j(w), \beta_j(w)$, $j = 0,1,2$ such that

$$f = \frac{\alpha_1(w)z + \beta_1(w)}{\alpha_0(w)z + \beta_0(w)}, \quad g = \frac{\alpha_2(w)z + \beta_2(w)}{\alpha_0(w)z + \beta_0(w)}.$$

After composing $\mathcal{F}$ with an appropriate element $\sigma \in \mathrm{Aut}(\mathbb{CP}^2)$ (this group preserves the target ODE $W'' = 0$) if needed, we can assume without loss of generality that $\alpha_0(w) \not\equiv 0$, and rewrite f and g as

$$f(z,w) = \frac{\alpha}{z+\delta} + \beta, \quad g(z,w) = \frac{a}{z+\delta} + b, \tag{3.21}$$

for appropriate $\alpha(w), \beta(w), \delta(w), a(w), b(w)$, meromorphic near w_0; one checks that these need to satisfy $\alpha b' = \beta a'$ if (3.20) is satisfied.

If we now substitute the expressions (3.21) of f and g into $I_2(z,w)$ and $I_3(z,w)$, we obtain affine-linear and cubic expressions in z, respectively, more precisely, we have

$$I_2(z,w)=\left[\frac{a\alpha''-\alpha a''}{a'\alpha-\alpha' a}\right]+3\left[\frac{b'\alpha'-\beta' a'}{a'\alpha-\alpha' a}\right](z+\delta),$$

$$I_3(z,w)=\left[\delta''+\delta'\frac{a\alpha''-\alpha a''}{a'\alpha-\alpha' a}\right]+\left[\frac{a''\alpha'-\alpha'' a'}{a'\alpha-\alpha' a}+3\delta'\frac{b'\alpha'-\beta' a'}{a'\alpha-\alpha' a}\right](z+\delta)+$$
$$+\left[\frac{\beta' a''-b'\alpha''+\alpha' b''-a'\beta''}{a'\alpha-\alpha' a}\right](z+\delta)^2+\left[\frac{\beta' b''-b'\beta''}{a'\alpha-\alpha' a}\right](z+\delta)^3.$$

This implies the desired polynomial dependence of $\Phi_2(z,w)$ and $\Phi_3(z,w)$ on z.

Clearly, the obtained property of Φ_2,Φ_3 is equivalent to the fact that (3.17) has the form (3.3). Since (3.17) is mappable into the simplest ODE $w''=0$ near its regular points (see the arguments above), its Tresse semi-invariants vanish identically, which yields the special relations (3.4). Thus (3.17) is a $\mathcal{P}_0$-ODE with an isolated meromorphic singularity of order $\leq m$, which proves the proposition. □

We immediately get the following criterion for identifying ODEs with a real structure.

Corollary 19. *A $\mathcal{P}_0$-ODE with an isolated meromorphic singularity of order $\leq m$ has an m-positive real structure if and only if its m-dual ODE coincides with the conjugated one: $\mathcal{E}^*_m=\bar{\mathcal{E}}$.*

Before providing the real structure criterion for $\mathcal{P}_0$-ODEs we need a computational

Lemma 20. *Let $\mathcal{S}$ be a positive m-admissible Segre family, and*

$$\mathcal{S}=\left\{w=\bar\eta e^{i\bar\eta^{m-1}\varphi}\right\},\quad \bar{\mathcal{S}}=\left\{w=\bar\eta e^{-i\bar\eta^{m-1}\tilde\varphi}\right\},\quad \mathcal{S}^*=\left\{w=\bar\eta e^{-i\bar\eta^{m-1}\varphi^*}\right\}.$$

Then

$$\tilde\varphi_{kl}(w)=\bar\varphi_{kl}(w),\ k,l\geq 2 \tag{3.22}$$

$$\varphi^*_{22}(w)=\varphi_{22}(w)-i(m-1)w^{m-1},\quad \varphi^*_{32}(w)=\varphi_{23}(w),\quad \varphi^*_{23}(w)=\varphi_{32}(w), \tag{3.23}$$

$$\varphi^*_{33}=\varphi_{33}(w)+\frac{3}{2}(m-1)^2w^{2m-2}-2i(m-1)w^{m-1}\varphi_{22}(w)-iw^m\varphi'_{22}(w). \tag{3.24}$$

Proof. The relations (3.22) follow directly from the definition of $\bar{\mathcal{S}}$. To prove (3.23),(3.24) we write $\mathcal{S}^*_m$ up first by definition as

$$\bar\eta=w\exp\left[iw^{m-1}\left(z\bar\xi+\sum_{k,l\geq 2}\varphi_{kl}(w)z^l\bar\xi^k\right)\right],$$

and then as

$$w=\bar\eta\exp\left[-i\bar\eta^{m-1}\left(z\bar\xi+\sum_{k,l\geq 2}\varphi^*_{kl}(\bar\eta)z^k\bar\xi^l\right)\right].$$

Substituting the first representation into the second and simplifying, we get

$$\exp\left[i(m-1)w^{m-1}\left(z\bar\xi+\sum\nolimits_{k,l\geq 2}\varphi_{kl}(w)z^l\bar\xi^k\right)\right]\times$$
$$\times\left(z\bar\xi+\sum\nolimits_{k,l\geq 2}\varphi^*_{kl}(\bar\eta)z^k\bar\xi^l\right)\Big|_{\bar\eta=we^{iw^{m-1}\varphi(\bar\xi,z,w)}}=z\bar\xi+\sum\nolimits_{k,l\geq 2}\varphi_{kl}(w)z^l\bar\xi^k. \tag{3.25}$$

Gathering the terms with $z^2\bar\xi^2, z^3\bar\xi^2, z^2\bar\xi^3$ respectively in (3.25), we get the first, the second and the third identities in (3.23). Gathering then terms with $z^3\bar\xi^3$ and using (3.23), we obtain (3.24), which proves the lemma. □

We are in the position now to prove the main result of this section.

Theorem 21. *Let*

$$\mathcal{E}:\, z''=\frac{1}{w^m}(Az+B)z'+\frac{1}{w^{2m}}(Cz^3+Dz^2+Ez+F)$$

be a $\mathcal{P}_0$-ODE with an isolated meromorphic singularity of order $\leq m$, $w\in\Delta_r, r>0$, $m\in\mathbb{N}$. Then $\mathcal{E}$ has an m-positive real structure if and only if the functions $A(w),B(w),C(w),D(w),E(w),F(w)$ are given by

$$\begin{aligned}A(w)&=3c(w),\\ B(w)&=2ia(w)-mw^{m-1},\\ C(w)&=\bar c(w)^2,\\ D(w)&=w^mc'(w)-2ia(w)c(w)\\ E(w)&=b(w)+iw^ma'(w),\\ F(w)&=i\bar c(w)\end{aligned}\tag{3.26}$$

for some power series

$$a(w)=\sum_{j=0}^{\infty}a_jw^j,\ b(w)=\sum_{j=0}^{\infty}b_jw^j\in\mathbb{R}\{w\}\,,\ \textit{and}\ c(w)=\sum_{j=0}^{\infty}c_jw^j\in\mathbb{C}\{w\}$$

which converge in Δ_r. Moreover, if $\mathcal{E}$ has an m-positive real structure, then the associated real hypersurface $M\subset\mathbb{C}^2$ is Levi nondegenerate and spherical outside the complex locus $X=\{w=0\}$.

Proof. As previously observed, the conjugated ODE $\overline{\mathcal{E}}$ has the form

$$z''=\frac{1}{w^m}(\bar Az+\bar B)z'+\frac{1}{w^{2m}}(\bar Cz^3+\bar Dz^2+\bar Ez+\bar F).$$

We write the dual ODE $\mathcal{E}^*_m$ as

$$\mathcal{E}^*:\, z''=\frac{1}{w^m}(A^*z+B^*)z'+\frac{1}{w^{2m}}(C^*z^3+D^*z^2+E^*z+F^*)$$

and assume that the families $\mathcal{S}=\mathcal{S}^+_m(\mathcal{E})$, $\mathcal{S}^*$, and $\bar{\mathcal{S}}$ are given in a polydisc $U=\Delta_\delta\times\Delta_\varepsilon$ by

$$\mathcal{S}=\left\{w=\bar\eta e^{i\bar\eta^{m-1}\varphi(z,\bar\xi,\bar\eta)}\right\},\quad \mathcal{S}^*=\left\{w=\bar\eta e^{-i\bar\eta^{m-1}\varphi^*(z,\bar\xi,\bar\eta)}\right\},\quad \bar{\mathcal{S}}=\left\{w=\bar\eta=e^{-i\bar\eta^{m-1}\bar\varphi(z,\bar\xi,\bar\eta)}\right\},$$

with φ, φ^* as in (3.7). According to Corollary 19, $\mathcal{E}$ has an m-positive real structure if an only if

$$\bar A(w)=A^*(w),\ \bar B(w)=B^*(w),\ \bar C(w)=C^*(w),\ \bar D(w)=D^*(w),\ \bar E(w)=E^*(w),\ \bar F(w)=F^*(w).$$

It follows directly from (3.9) and (3.4) that the latter conditions are equivalent to

$$\bar\varphi_{kl}=\varphi^*_{kl},\ k,l\in\{2,3\}.\tag{3.27}$$

Lemma 20 now implies that (3.27) is equivalent to the existence of power series $a(w), \tilde{b}(w) \in \mathbb{R}\{w\}$ and $c(w) \in \mathbb{C}\{w\}$, convergent in some disc Δ_r, such that

$$\begin{aligned} \varphi_{22}(w) &= a(w) + i\frac{m-1}{2}w^{m-1}, \\ \varphi_{23}(w) &= \frac{i}{2}\bar{c}(w), \\ \varphi_{32}(w) &= -\frac{i}{2}c(w) \\ \varphi_{33}(w) &= \tilde{b}(w) + \frac{i}{2}w^m a'(w) + i(m-1)w^{m-1}a(w). \end{aligned} \tag{3.28}$$

Applying (3.9) and (3.4) again, we conclude that with

$$b(w) := 6\tilde{b}(w) - 8a^2(w) + 2(m-1)^2 w^{2m-2} \tag{3.29}$$

(3.28) is equivalent to (3.26).

It remains to prove that if $\mathcal{E}$ has an m-positive real structure, then the associated nonminimal real hypersurface $M \subset \mathbb{C}^2$ is Levi nondegenerate and spherical in the complement to the singular set $X = \{w = 0\}$. Recall that the Segre family of M near the origin coincides with $\mathcal{S}$. To prove the Levi nondegeneracy of M in $M \setminus X$ we first note that the Segre map $\lambda\colon p \mapsto Q_p$ is one-to-one in $U \setminus X$ (see the arguments in the beginning of the section). Consider now any two distinct points $p, q \notin X$ and their Segre varieties Q_p, Q_q, $Q_p \cap Q_q \ni r$. The fact that Q_p, Q_q are two distinct solutions of the *nonsingular* ODE $\mathcal{E}$ in $U \setminus X$ implies that their intersection at $r \in U \setminus X$ is transverse. Accordingly, any Segre variety of M near an arbitrary point $s \in M \setminus X$ is uniquely determined by its 1-jet at a given point, and hence M is Levi nondegenerate at s (see, e.g., [19],[3]).

Finally, to prove that M is spherical at any $s \in M \setminus X$, we note that the Segre family $\mathcal{S}$ of M satisfies the $\mathcal{P}_0$-ODE $\mathcal{E}$ and hence is locally biholomorphically mappable in a neighborhood V of s onto the family of straight affine lines in $\mathbb{C}^2$. It is not difficult to verify from here that the image of $M \cap V$ under such a mapping is contained in a quadric $\mathcal{Q} \subset \mathbb{CP}^2$ (see, for example, the proof of Theorem 6.1 in [43]), which implies sphericity of M at s. □

A completely analogous argument as in the case of positive Segre families gives a complete characterization of ODEs with a negative real structure: these are obtained by conjugating ODEs with a positive real structure. Thus we can formulate

Corollary 22. *Let*

$$\mathcal{E}\colon z'' = \frac{1}{w^m}(Az + B)z' + \frac{1}{w^{2m}}(Cz^3 + Dz^2 + Ez + F)$$

be a $\mathcal{P}_0$-ODE with an isolated meromorphic singularity of order $\leq m$, $w \in \Delta_r, r > 0, m \in \mathbb{N}$. Then $\mathcal{E}$ has an m-negative real structure if and only if the the conjugated ODE

$$\bar{\mathcal{E}}\colon z'' = \frac{1}{w^m}(\bar{A}z + \bar{B})z' + \frac{1}{w^{2m}}(\bar{C}z^3 + \bar{D}z^2 + \bar{E}z + \bar{F})$$

satisfies the relations (3.26) *for some power series $a(w), b(w) \in \mathbb{R}\{w\}$, $c(w) \in \mathbb{C}\{w\}$, which converge in some Δ_r. Moreover, if $\mathcal{E}$ has an m-negative real structure, then the associated real hypersurface $M \subset \mathbb{C}^2$ is Levi nondegenerate and spherical outside the complex locus $X = \{w = 0\}$.*

Theorem 21 and the proof of Proposition 13 enable us to complete the study of the class of real-analytic hypersurfaces $M \subset \mathbb{C}^2$ which are nonminimal at the origin and spherical outside their complex locus $X \ni 0$. More precisely, we present an effective algorithm for obtaining real-analytic hypersurfaces $M \subset \mathbb{C}^2$, nonminimal at the origin, with prescribed nonminimality order $m \geq 1$, which are Levi nondegenerate and spherical outside the nonminimal locus $X \subset M$. Moreover, one can prescribe to M an arbitrary 6-jet, satisfying the reality condition (2.2). In fact, the result of [45] shows that for any hypersurface M as above there exists appropriate local holomorphic coordinates near the origin and a $\mathcal{P}_0$-ODE $\mathcal{E}$ such that the Segre family of M in these coordinates is associated with $\mathcal{E}$, and thus this algorithm describes *all* possible hypersurfaces of our class. We summarize this algorithm below.

Algorithm 1: Algorithm for obtaining nonminimal spherical real hypersurfaces

1. Take three power series $a(w), b(w), c(w)$, where $a(w), b(w) \in \mathbb{R}\{w\}$, $c(w) \in \mathbb{C}\{w\}$, which converge in some disk centered at the origin, an integer $m \geq 1$, and compute six functions $A(w), B(w), C(w), D(w), E(w), F(w)$ by the formulas (3.26). This gives a $\mathcal{P}_0$-ODE (3.3) with an isolated meromorphic singularity of order $\leq m$.
2. Solve the holomorphic ODE (3.10) with holomorphic parameters $\bar{\xi}, \bar{\eta}$ and the initial data $\dot{\varphi}(0) = 0$, $\dot{\varphi}(0) = \bar{\xi}$ to obtain a function $\varphi(z, \bar{\xi}, \bar{\eta})$, holomorphic near the origin in $\mathbb{C}^3$.
3. Either of the two equation $w = \bar{w}e^{i\bar{w}^{m-1}\varphi(z,\bar{z},\bar{w})}$ and $w = \bar{w}e^{-i\bar{w}^{m-1}\bar{\varphi}(z,\bar{z},\bar{w})}$ determines a real-analytic hypersurface $M^{\pm} \subset \mathbb{C}^2$, nonminimal at the origin of nonminimality order m, Levi nondegenerate and spherical outside the nonminimal locus $X = \{w = 0\}$. The 6-jets of $M^{\pm}$ in z are determined by finding $\tilde{b}(w)$ from (3.29) and then $\varphi_{22}, \varphi_{23}, \varphi_{32}, \varphi_{33}$ by formulas (3.28).

Corollary 23. *Algorithm 1 gives a complete description, up to a local biholomorphic equivalence, of all possible real-analytic hypersurfaces $M \subset \mathbb{C}^2$ which are nonminimal at the origin and spherical outside their complex locus $X \ni 0$.*

4. Smoothly but not analytically equivalent analytic CR-structures

In this section we provide a construction of real-analytic holomorphically nondegenerate real hypersurfaces, which are C^∞ CR-equivalent, but are not equivalent holomorphically.

The desired real hypersurfaces are associated with singular ODEs with a real structure as studied in section 3. We make the particular choice

$$a(w) \equiv 1, \quad b(w) = \gamma w^4, \quad c(w) \equiv 0,$$

where $\gamma \in \mathbb{R}$ is a real constant, and apply Algorithm 1. In the first step, applying formulas (3.26) with $m = 4$, we obtain a one-parameter family $\mathcal{E}_\gamma$ of $\mathcal{P}_0$-ODEs (in fact, *linear* ODE) with an isolated meromorphic singularity of order 4, which have a 4-positive real structure:

$$\mathcal{E}_\gamma\colon z'' = \left(\frac{2i}{w^4} - \frac{4}{w}\right) z' + \frac{\gamma}{w^4} z. \tag{4.1}$$

Each ODE $\mathcal{E}_\gamma$ has a non-Fuchsian singularity at the origin. We denote by M_γ the 4-nonminimal at the origin real hypersurfaces, associated with $\mathcal{E}_\gamma$. Each M_γ is Levi nondegenerate and spherical outside the complex locus $X = \{w = 0\}$. Note that the ODE $\mathcal{E}_0$ coincides with the ODE $\mathcal{E}_0^4$, studied in [44], while the ODEs $\mathcal{E}_\gamma$ with $\gamma \neq 0$ are different from that in [44].

After introducing $u := z'w^3$ as a new dependent variable we rewrite (4.1) as the first order system

$$\begin{pmatrix} z \\ u \end{pmatrix}' = \frac{1}{w^4}\left(A_0 + A_1 w + A_3 w^3\right)\begin{pmatrix} z \\ u \end{pmatrix}, \tag{4.2}$$

which possesses a non-Fuchsian singularity at the origin, where

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & 0 \\ \gamma & -1 \end{pmatrix}.$$

We need to consider three different kinds of local transformations in the following: holomorphic, formal and sectorial. To introduce the latter ones, we denote by $S^\pm_\alpha$ the unbounded sectors

$$S^+_\alpha = \{-\alpha < \mathrm{Arg} w < \alpha\}, \quad S^-_\alpha = \{\pi - \alpha < \mathrm{Arg} w < \pi + \alpha\},$$

where $0 < \alpha < \frac{\pi}{2}$, and by $S^\pm_{\alpha,r}$ the bounded sectors $S^\pm_\alpha \cap \Delta_r$, where $r > 0$.

Definition 24. We say that $F(z,w) : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ is a *(formal) gauge transformation* if there exist (formal) power series $f(w)$, $g(w)$ satisfying $f(0) \neq 0$, $g(0) = 0$, $g'(0) \neq 0$ such that

$$F(z,w) = (zf(w), g(w)).$$

A *sectorial gauge transformation* $F(z,w)$ is a holomorphic map $F : \Delta_R \times S^\pm_{\alpha,r} \to \Delta_{\tilde R} \times S^\pm_{\tilde\alpha,\tilde r}$ which is of the form $F(z,w) = (zf(w), g(w))$ where $f(w)$ and $g(w)$ are holomorphic on $S^\pm_{\alpha,r}$, and whose asymptotic expansion $\hat F$ is a formal gauge transformation.

We will denote the groups of holomorphic or formal gauge transformations by $\mathcal{G}$ and $\mathcal{FG}$, respectively, and for any integer $m \geq 2$, $\mathcal{G}_m \subset \mathcal{G}$ and $\mathcal{FG}_m \subset \mathcal{FG}$ will denote the subgroups whose elements $(zf(w), g(w))$ satisfy the normalization conditions $f(0) = 1$, $g(w) = w + O(w^{m+1})$.

One can define, in the natural way, equivalence of $\mathcal{P}_0$-ODEs by means of homomorphic, formal or sectorial gauge transformations.

Proposition 25. *For any $\gamma \in \mathbb{R}$ the ODE $\mathcal{E}_\gamma$ is formally equivalent to the ODE $\mathcal{E}_0$ by means of a transformation $F \in \mathcal{FG}_4$.*

Proof. The main tool of the proof is the Poincare-Dulac normalization procedure for nonresonant non-Fuchsian systems (see, e.g., [38],[66]). Such a normal form enables one to find the fundamental system of formal solutions of a non-Fuchsian system.

It is straightforward to verify that the function $\exp\left(-\frac{2i}{3}w^{-3}\right)$ is a solution of the ODE $\mathcal{E}_0$, so that the fundamental system of solutions for $\mathcal{E}_0$ is $\left\{1, \exp\left(-\frac{2i}{3}w^{-3}\right)\right\}$. For the ODE $\mathcal{E}_\gamma$ with $\gamma \neq 0$ we consider the corresponding system (4.2) and note that the principal matrix

$$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}$$

is diagonal with distinct eigenvalues; hence the system is nonresonant. When we perform a transformation of the form

$$\begin{pmatrix} z \\ u \end{pmatrix} \longrightarrow (I + wH)\begin{pmatrix} z \\ u \end{pmatrix},$$

where I is the identity matrix and H is a constant 2×2 matrix, we obtain the transformed system

$$\begin{pmatrix} z \\ u \end{pmatrix}' = \frac{1}{w^4}\tilde A(w)\begin{pmatrix} z \\ u \end{pmatrix} = \frac{1}{w^4}\left(\tilde A_0 + \tilde A_1 w \dots\right),$$

where $\tilde{A}(w) = (I + wH)^{-1}\big(A(w)(I + wH) - Hw^4\big)$. By comparing coefficients of $w^k, k \geq -4$, one computes that

$$\tilde{A}_0 = A_0,$$
$$\tilde{A}_1 = [A_0, H] + A_1.$$

One can choose H so that $\tilde{A}_1 = 0$ by solving the equation $[A_0, H] = -A_1$, which can be done explicitly:

$$H = \begin{pmatrix} 0 & \frac{i}{2} \\ 0 & 0 \end{pmatrix}.$$

Note that $H^2 = HA_1 = A_1H = 0$. We then get

$$\tilde{A}_2 = A_1H - HA_0H - HA_1 + H^2A_0 = H[H, A_0] = HA_1 = 0$$

and

$$\tilde{A}_3 = A_3 - HA_1H + H^2A_0H + H^2A_1 - H^3A_0 = A_3.$$

Thus $\tilde{A}(w) = A_0 + A_3w^3 + O(w^4)$.

A computation, similar to the above one, shows that the offdiagonal element $-\gamma$ of the matrix $\tilde{A}_3$ can be removed by a transformation

$$\begin{pmatrix} z \\ u \end{pmatrix} \longrightarrow (I + w^3\tilde{H})\begin{pmatrix} z \\ u \end{pmatrix}$$

for an appropriate 2×2 constant matrix $\tilde{H}$ without changing the 2-jet of $\tilde{A}(w)$ and the diagonal of $\tilde{A}_3$. The matrix $\tilde{H}$ can be found from the equation $\tilde{A}_3 + [A_0, \tilde{H}] = 0$, and one can choose, for example,

$$\tilde{H} = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ -\gamma & 0 \end{pmatrix}.$$

Finally, the matrices $\tilde{A}_k$ with $k \geq 4$ correspond to holomorphic terms in the expansion of $\frac{1}{w^4}\tilde{A}(w)$ and hence can be removed by the Poincare-Dulac formal normalization procedure for nonresonant non-Fuchsian systems, without changing the 3-jet of the matrix $\tilde{A}(w)$ (see, e.g., [38], Theorem 20.7). Thus the formal normal form of the system (4.2) becomes

$$\begin{pmatrix} z \\ u \end{pmatrix}' = \left[\frac{1}{w^4}\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} + \frac{1}{w}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\right]\begin{pmatrix} z \\ u \end{pmatrix}. \tag{4.3}$$

This implies that systems (4.2) for different γ are formally gauge equivalent.

We need now to deduce the same fact for the initial ODEs (4.1) with respect to formal gauge equivalences $(\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$, which is a different issue. In order to do so we use a strategy similar to the one used in the proof of Proposition 4.2 in [44], and first consider the fundamental system of solutions for the normal form (4.3), which is given by

$$e^{-\frac{1}{3}w^{-3}\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}} \cdot w^{\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}},$$

implying that the fundamental system of formal solutions for (4.2) is given by

$$\hat{\Phi}_\gamma(w) \cdot e^{-\frac{1}{3}w^{-3}\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}} \cdot w^{\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}}, \tag{4.4}$$

where

$$\hat{\Phi}_\gamma(w) = \begin{pmatrix} \hat{f}_\gamma(w) & \hat{g}_\gamma(w) \\ \hat{h}_\gamma(w) & \hat{s}_\gamma(w) \end{pmatrix} = I + \sum_{k\geq 2} \Phi_k w^k$$

is a matrix-valued formal power series. The columns of (4.4) are linearly independent (over the quotient field of $\mathbb{C}[[w]]$). From (4.4) we conclude that the ODE (4.1) possesses a fundamental system of formal solutions $\left\{\hat{f}_\gamma(w), \hat{g}_\gamma(w)\cdot w^{-1}\cdot \exp\left(-\frac{2i}{3}w^{-3}\right)\right\}$ for two formal power series

$$\hat{f}_\gamma(w) = 1 + O(w), \quad \hat{g}_\gamma(w) = w + O(w^2). \tag{4.5}$$

The expansion of $\hat{g}_\gamma$ can be deduced from

$$w^3\left(\hat{g}_\gamma(w)w^{-1}\exp\left(-\frac{2i}{3}w^{-3}\right)\right)' = w^{-1}\hat{s}_\gamma(w)\exp\left(-\frac{2i}{3}\right),$$

which holds by the initial substitution $u = z'w^3$, and since $\hat{s}_\gamma(w) = 1 + O(w)$, we get $\operatorname{ord}_0 \hat{g}_\gamma = 1$. Hence we can scale $\hat{g}_\gamma(w)$ to obtain $\hat{g}_\gamma(w) = w + O(w^2)$.

We set

$$\hat{\chi}(w) := \frac{1}{\hat{f}_\gamma(w)}, \quad \hat{\tau}(w) := w\left(1 - \frac{3}{2i}w^3 \ln\frac{\hat{g}_\gamma(w)}{w\hat{f}_\gamma(w)}\right)^{-\frac{1}{3}}. \tag{4.6}$$

In view of (4.5), $\hat{\tau}(w)$ is a well defined formal power series of the form $w + O(w^5)$, and $\hat{\chi}(w)$ is a well defined formal power series of the form $1 + O(w)$. We claim now that

$$(z, w) \longrightarrow (\hat{\chi}(w)z, \hat{\tau}(w)) \tag{4.7}$$

is the desired formal gauge transformation of class $\mathcal{FG}_4$, sending $\mathcal{E}_\gamma$ into $\mathcal{E}_0$.

As it is shown in [1], if two functions $z_1(w), z_2(w)$ are some linearly independent holomorphic solutions of a second order linear ODE $z'' = p(w)z' + q(w)z$, then the transformation $z \longrightarrow \frac{1}{z_1(w)}z$, $w \longrightarrow \frac{z_2(w)}{z_1(w)}$ transfers the initial ODE into the simplest ODE $z'' = 0$. The same fact can be verified, by a simple computation, for certain spaces of formal series (as soon as all above operations are well defined). In particular, the transformation

$$z \longrightarrow \frac{1}{\hat{f}_\gamma(w)}z,\ w \longrightarrow \frac{\hat{g}_\gamma(w)}{w\hat{f}_\gamma(w)}\exp\left(-\frac{2i}{3}w^{-3}\right) \tag{4.8}$$

transforms formally $\mathcal{E}_\gamma$ into $z'' = 0$, and

$$z \longrightarrow z,\ w \longrightarrow \exp\left(-\frac{2i}{3}w^{-3}\right) \tag{4.9}$$

transforms $\mathcal{E}_0$ into $z'' = 0$. It follows then that the formal substitution of (4.7) into (4.9) gives (4.8). Since the chain rule agrees with the above formal substitutions, this shows that (4.7) transfers $\mathcal{E}_\gamma$ into $\mathcal{E}_0$. This proves the proposition. □

On the other hand, the ODEs $\mathcal{E}_0$ and $\mathcal{E}_\gamma$ with $\gamma \neq 0$ are different from the analytic point of view, even though *all* $\mathcal{E}_\gamma$, $\gamma \in \mathbb{R}$ have trivial monodromy.

Proposition 26. *i) For any $\gamma \in \mathbb{R}$ the ODE $\mathcal{E}_\gamma$ has a trivial monodromy; ii) For any $\gamma \in \mathbb{R}\setminus\{0\}$ the ODE $\mathcal{E}_\gamma$ has no non-zero holomorphic solutions, while $\mathcal{E}_0$ has the holomorphic solution $z \equiv 1$.*

Proof. We first obtain the monodromy matrix for an arbitrary system (4.2). In order to do that we consider ∞ as an isolated singular point for (4.2) and perform the change of variables $t := \frac{1}{w}$. We obtain the system

$$\begin{pmatrix} z \\ u \end{pmatrix}' = \left[t^2 \begin{pmatrix} 0 & 0 \\ 0 & -2i \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} 0 & 0 \\ -\gamma & 1 \end{pmatrix} \right] \begin{pmatrix} z \\ u \end{pmatrix} \tag{4.10}$$

with an isolated *Fuchsian* singularity at $t = 0$. The singular points of (4.2) in $\overline{\mathbb{C}}$ are $w = 0$ and $w = \infty$, hence it is sufficient to prove that the monodromy matrix at $t = 0$ for each system (4.10) is the identity. To compute the monodromy we apply the Poincare-Dulac normalization procedure for Fuchsian systems (see e.g. [38], Theorem 26.15). Note that the residue matrix

$$R_\gamma = \begin{pmatrix} 0 & 0 \\ -\gamma & 1 \end{pmatrix}$$

for (4.10) at $t = 0$ has eigenvalues 0 and 1 and hence is resonant. However, the only possible resonances of this system correspond to the resonant monomials with *zero* degree in t, which are already removed from (4.10). All higher degree monomials can be removed from (4.10) by the Poincare-Dulac procedure after diagonalizing the residue matrix R_γ. Hence the normal form of the system (4.10) at $t = 0$ is the diagonal Euler system

$$\begin{pmatrix} z \\ u \end{pmatrix}' = \frac{1}{t} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ u \end{pmatrix},$$

which has trivial monodromy (as its solutions are given by $z = c_1$, $u = c_2 t$ for arbitrary constants $c_1, c_2 \in \mathbb{C}$). Convergence of the Poincare-Dulac normalizing transformation in the Fuchsian case now implies i).

To prove ii) we substitute the formal power series $h(w) = \sum_{j \geq 0} a_j w^j$ into the ODE (4.1) with $\gamma \neq 0$ and obtain

$$\begin{aligned} a_1 &= -\frac{\gamma a_0}{2i}, \\ a_2 &= -\frac{\gamma a_1}{4i}, \\ a_{k+3} &= \frac{1}{2i} k a_k - \frac{\gamma}{2i(k+3)} a_{k+2}, \quad k \geq 0. \end{aligned} \tag{4.11}$$

Clearly, $a_0 = 0$ implies $h \equiv 0$ so that we assume $a_0 = 1$ and get $a_1 = -\frac{\gamma}{2i}$, $a_2 = -\frac{\gamma^2}{8}$. Note that there exists no $s > 0$ such that $a_s \neq 0$ and $a_k = 0$ for all $k > s$, as follows from the relation (4.11) with $k = s$. Let

$$p := \sup_{a_k \neq 0} \left| \frac{a_{k+2}}{a_k} \right|, \quad q := \sup_{a_k \neq 0} \left| \frac{a_{k+3}}{a_k} \right|.$$

If either $p = +\infty$ or $q = +\infty$, then the power series $h(w)$ is divergent for all $w \neq 0$; however, if we had $p, q < +\infty$, (4.11) would imply that $k \leq \frac{p|\gamma|}{k+3} + 2q$ for large k, which is impossible. This proves the proposition. ☐

In fact, Proposition 26 proves that all the ODEs $\mathcal{E}_\gamma$ are pairwise holomorphically gauge equivalent near the singular point $w = \infty$. We can also formulate

Corollary 27. *The nonminimal real hypersurfaces M_γ associated with the ODEs $\mathcal{E}_\gamma$ have a trivial monodromy in the sense of* [43] *for all $\gamma \in \mathbb{R}$.*

Proof. Let $h_1(w), h_2(w)$ be two linearly independent solutions of an ODE $\mathcal{E}_\gamma$, defined in $\mathbb{C} \setminus \{0\}$. Proposition 26 implies that h_1 and h_2 are single-valued. We now use the fact (see [1]) that one of the possible mappings of the linear ODE $\mathcal{E}_\gamma$ onto the ODE $z'' = 0$ is given by the single-valued gauge transformation

$$z \mapsto \frac{1}{h_2(w)} z, \quad w \mapsto \frac{h_1(w)}{h_2(w)}.$$

Hence this mapping takes the associated hypersurface M_γ at a Levi nondegenerate point (z_0, w_0) onto a quadric $\mathcal{Q} \subset \mathbb{CP}^2$, and we conclude that the monodromy of M_γ is trivial. □

Remark 28. Corollary 27, compared with Theorem 32 below, shows that the monodromy does not help to decide whether irregularity phenomena for CR-mappings between given nonminimal hypersurfaces appear (for both the divergence phenomenon discovered in [44] and the one in the present paper).

We now fix some $\gamma \in \mathbb{R} \setminus \{0\}$ and apply Sibuya's sectorial normalization theorem (see section 2) to connect formal and sectorial equivalences of $\mathcal{E}_\gamma$ with $\mathcal{E}_0$. The separating rays for each of the systems (4.2) are determined by Re $\left(\frac{2i}{w^3}\right) = 0$, so that we get the six rays

$$\left\{w = \pm\mathbb{R}^+,\, w = \pm\mathbb{R}^+ e^{\frac{\pi i}{3}},\, w = \pm\mathbb{R}^+ e^{-\frac{\pi i}{3}}\right\}.$$

It follows from Sibuya's theorem that the formal matrix function

$$\hat{\Phi}_\gamma(w) = \begin{pmatrix} \hat{f}_\gamma(w) & \hat{g}_\gamma(w) \\ \hat{h}_\gamma(w) & \hat{s}_\gamma(w) \end{pmatrix}$$

introduced in (4.4) admits (for some $r > 0$) unique sectorial asymptotic representations $\Phi^\pm_\gamma(w) \sim \hat{\Phi}_\gamma$ in sectors $S^\pm_{\pi/3,r}$, respectively, for functions $\Phi^\pm_\gamma(w)$ which are holomorphic in $S^\pm_{\pi/3,r}$. Accordingly, we obtain by formulas, identical to (4.6), two functions $\chi(w), \tau(w)$, asymptotically represented in both sectors $S^\pm_{\pi/3,r}$ by the functions $\hat{\chi}(w), \hat{\tau}(w)$, respectively.

In what follows we use the notation $\mathcal{SG}^\pm_m$ for the class of gauge transformations $z \to zf(w)$, $w \to g(w)$ such that the functions f and g are holomorphic in a sector $S^\pm_{\alpha,r}$ respectively for some $r > 0, 0 < \alpha < \frac{\pi}{2}$ and, in addition, having in the sector $S^\pm_{\alpha,r}$ asymptotic power series representations with the properties $f(w) = 1 + O(w), g(w) = w + O(w^{m+1})$. Considering then the formal gauge equivalence between $\mathcal{E}_\gamma$ and $\mathcal{E}_0$, given by (4.7), we see from the proof of Proposition 25 that the map

$$(z, w) \longrightarrow (z\chi^\pm(w), \tau^\pm(w)) \tag{4.12}$$

is of class $\mathcal{SG}^\pm_4$ and, moreover, transfers the ODE $\mathcal{E}_\gamma$ into $\mathcal{E}_0$. The latter statement follows from the fact that, according to Sibuya's theorem, the map

$$\begin{pmatrix} z \\ u \end{pmatrix} \mapsto \Phi^\pm_\gamma(w) \cdot \begin{pmatrix} z \\ u \end{pmatrix}, \quad w \mapsto w$$

transforms the system (4.2) into its normal form (4.3). Arguing then identically to the proof of Proposition 25, we see that (4.12) transfers $\mathcal{E}_\gamma$ into $\mathcal{E}_0$.

We also need a uniqueness statement for normalized gauge equivalences between $\mathcal{E}_\gamma$ and $\mathcal{E}_0$. We note that a statement similar to Proposition 29 below for the systems (4.2), associated with the ODEs $\mathcal{E}_\gamma$ and $\mathcal{E}_0$ respectively, follows directly from the uniqueness in Sibuya's theorem. However, gauge equivalences of ODEs is a *different* issue, which needs a separate treatment.

Proposition 29. *The only transformation $F \in \mathcal{SG}^\pm_4$, transferring $\mathcal{E}_\gamma$ into $\mathcal{E}_0$, is given by* (4.12).

Proof. It is sufficient to prove that the unique transformation $F \in \mathcal{SG}_4$, transferring $\mathcal{E}_0$ into itself, is the identity. If a gauge transformation $F = (zf(w), g(w)) \in \mathcal{SG}_4^\pm$ preserves $\mathcal{E}_0$, we know that $\{z = \frac{1}{f(w)}\}$ is (locally) the graph of a solution of $\mathcal{E}_0$ (as it is the preimage of the graph $\{z = 1\}$). Since each solution of $\mathcal{E}_0$ is a linear combination of 1 and $\exp\left(-\frac{2i}{3w^3}\right)$, and $\frac{1}{f}$ has an asymptotic expansion of the form $1 + O(w)$ in a sector $S^\pm_{\alpha,r}$, we conclude that $f \equiv 1$. Thus $F = (z, g(w))$. Substituting F into $\mathcal{E}_0$, we get in the preimage

$$\frac{1}{(g')^2}z'' - \frac{g''}{(g')^3}z' = \frac{1}{g'}\left(\frac{2i}{g^4} - \frac{4}{g}\right)z'.$$

Since $\mathcal{E}_0$ is preserved, we obtain

$$g'\left(\frac{2i}{g^4} - \frac{4}{g}\right) + \frac{g''}{g'} = \frac{2i}{w^4} - \frac{4}{w}. \tag{4.13}$$

We now argue similarly to the proof of Proposition 4.4 in [44] and study the ODE (4.13). Assuming that $g(w) \not\equiv w$, (4.13) can be rewritten as a differential relation

$$2i\left(-\frac{1}{3g^3}\right)' + 2i\left(\frac{1}{3w^3}\right)' + (\ln g')' - 4\left(\ln\frac{g}{w}\right)' = 0,$$

which gives $-\frac{2i}{3}\left(\frac{1}{g^3} - \frac{1}{w^3}\right) + \ln g' - 4\ln\frac{g}{w} = C_1$ for some constant $C_1 \in \mathbb{C}$. It follows then that the function $-\frac{1}{3}\left(\frac{1}{g^3} - \frac{1}{w^3}\right)$ has an asymptotic representation by a formal power series in a sector $S^\pm_{\alpha,r}$, and a straightforward computation shows that the substitution $-\frac{1}{3}\left(\frac{1}{g^3} - \frac{1}{w^3}\right) := u$ transforms the latter equation for g into $2iu + \ln(w^4u' + 1) = C_1$. Shifting u, we get the equation $2iu + \ln(w^4u' + 1) = 0$, where $u(w)$ is represented in $S^\pm_{\alpha,r}$ by a formal power series with zero free term. Hence we finally obtain the following meromorphic first order ODE for the shifted function $u(w)$:

$$u' = \frac{1}{w^4}(e^{-2iu} - 1). \tag{4.14}$$

However, if $u \not\equiv 0$, (4.14) can be represented as $-\frac{1}{2i}u'\left(\frac{1}{u} + H(u)\right) = \frac{1}{w^4}$, where $H(t)$ is a holomorphic at the origin function. Hence we get that the logarithmic derivative $\frac{u'}{u}$ is asymptotically represented in $S^\pm_{\alpha,r}$ by a formal Laurent series $-\frac{2i}{w^4} + \ldots$, where the dots denote a formal power series in w. But this clearly contradicts the existence of an asymptotic representation of $u(w)$ by a power series in $S^\pm_{\alpha,r}$. Hence $u \equiv 0$, and, returning to the unknown function g, we get $\frac{1}{g^3} - \frac{1}{w^3} = C$ for some constant $C \in \mathbb{C}$, so that $g(w) = \frac{w}{(1+Cw^3)^{\frac{1}{3}}}$. Taking into account the asymptotic representation $g(w) = w + O(w^5)$, we conclude that $C = 0$ and $g(w) = w$. This proves the proposition. ☐

Let $\mathcal{S} = \{w = \rho(z, \bar{\xi}, \bar{\eta})\}$ be a (general) Segre family in a polydisc $\Delta_\delta \times \Delta_\varepsilon$. According to [44], we call the complex submanifold

$$\mathcal{M}_\rho = \{(z, w, \xi, \eta) \in \Delta_\delta \times \Delta_\varepsilon \times \Delta_\delta \times \Delta_\varepsilon : w = \rho(z, \xi, \eta)\} \subset \mathbb{C}^4, \tag{4.15}$$

the foliated submanifold associated with ρ. If $\mathcal{S}$ is associated with a $\mathcal{P}_0$-ODE with an isolated meromorphic singularity $\mathcal{E}$, then $\mathcal{M}_\rho$ is called *the associated foliated submanifold of* $\mathcal{E}$, and if $\mathcal{S}$ is m-admissible, then $\mathcal{M}_\mathcal{S}$ is said to be m-admissible (as before, we use the unique ρ satisfying the conditions of m-admissiblity). If $\mathcal{S}$ is the Segre family of a real hypersurface $M \subset \mathbb{C}^2$, then the associated foliated submanifold is simply the complexification of M.

A foliated submanifold $\mathcal{M}_{\mathcal{S}}$ possesses two natural foliations, induced by the projections on the first two and the last two coordinates, respectively; i.e. the first one is the initial foliation $\mathcal{S}$ with leaves $\{(z,w,\xi,\eta)\in\mathcal{M}_{\mathcal{S}}:\,\xi=\xi_0,\,\eta=\eta_0\}$. The second one is the family of dual Segre varieties with leaves $\{(z,w,\xi,\eta)\in\mathcal{M}_{\mathcal{S}}:\,z=z_0,\,w=w_0\}$. It is natural to consider the so-called *coupled equivalences* between foliated submanifolds. The latter have the form

$$(z,w,\xi,\eta)\longrightarrow(F(z,w),G(\xi,\eta)),\tag{4.16}$$

where $F(z,w),G(\xi,\eta)$ are biholomorphisms $(\mathbb{C}^2,0)\longrightarrow(\mathbb{C}^2,0)$, and thus preserve both the foliated submanifolds and the above two foliations. We shall mention in this regard the work [25] of Faran who was probably the first to use this sort of transformations.

Our next goal is to show that for any $F\in\mathcal{SG}^{\pm}_m$, conjugating two linear $\mathcal{P}_0$-ODEs with an isolated meromorphic singularity, a transformation $G\in\mathcal{SG}^{\pm}_m$ can be chosen in such a way that the direct product (F,G) conjugates the associated foliated submanifolds. We first note that for any m-admissible foliated submanifold $\mathcal{M}$ each of the intersections $\mathcal{M}\cap\{\pm\mathrm{Re}\,\eta>0\}$ lies in a domain $G^{\pm}_{R,r\alpha}=\Delta_R\times S^{\pm}_{\alpha,r}\times\Delta_R\times S^{\pm}_{\alpha,r}$ for sufficiently small R,r,α. Also note that for any $(F,G)\in\mathcal{SG}^{\pm}_m\times\mathcal{SG}^{\pm}_m$ and sufficiently small R,r,α the image of a domain $G^{\pm}_{R,r,\alpha}$ satisfies $G^{\pm}_{R_1,r_1,\alpha_1}\subset(F,G)(G^{\pm}_{R,r,\alpha})\subset G^{\pm}_{R_2,r_2,\alpha_2}$ for appropriate $0<R_1<R_2$. Moreover, the asymptotic expansion of F,G implies that, by choosing R,r,α small enough, one can make (F,G) arbitrarily close to the identity in the sense that the mapping $(F,G)-\mathrm{Id}$ is Lipschitz with an arbitrarily small constant. Hence we can assume that (F,G) is biholomorphic in $G^{\pm}_{R,r,\alpha}$. For the inverse mapping (F^{-1},G^{-1}) one has $(F^{-1},G^{-1})\in\mathcal{SG}^{\pm}_m\times\mathcal{SG}^{\pm}_m$. Thus the image $(F,G)(\mathcal{M}\cap\{\pm\mathrm{Re}\,\eta>0\})$ is a holomorphic graph of kind $w=\varphi(z\xi,\eta)$ for some $\varphi\in\mathcal{O}(\Delta_{R^2}\times S^{\pm}_{\alpha,r})$. Indeed, $(F,G)(\mathcal{M}\cap\{\pm\mathrm{Re}\,\eta>0\})$ is obtained by substituting (F^{-1},G^{-1}) into the defining equation of $\mathcal{M}$ and applying the implicit function theorem, so that (in a sufficiently small domain $G^{\pm}_{R_1,r_1,\alpha_1}$) $(F,G)(\mathcal{M}\cap\{\pm\mathrm{Re}\,\eta>0\})$ can be represented as a graph $w=\varphi(z\xi,\eta)$ locally near each point of it. This implies the required global representation. Moreover, $\varphi(z\xi,\eta)$ has the form $\varphi(z\xi,\eta)=\eta e^{i\eta^{m-1}\varphi^*(z\xi,\eta)}$ with $\varphi^*(z\xi,\eta)=\sum\limits_{k\geq 0}\varphi^*_k(\eta)z^k\xi^k$ for some $\varphi^*_k\in\mathcal{O}(S^{\pm}_{\alpha_1,r_1})$. The series converges uniformly on compact subsets in $S^{\pm}_{\alpha_1,r_1}$ and each φ^*_j has some asymptotic expansion in a, possibly, smaller sector $S^{\pm}_{\alpha_1,r^*}$ (the latter fact follows from the implicit function theorem for asymptotic series, applied for fixed z,ξ).

We now need

Proposition 30. *For any m-admissible foliated submanifold $\mathcal{M}$ and any map $F\in\mathcal{SG}^{\pm}_m$ there exists a unique $G\in\mathcal{SG}^{\pm}_m$ such that for the function $\varphi^*(z,\xi,\eta)$ as above one has $\varphi^*_{0,k}(\eta)=\varphi^*_{k,0}=0$, $\varphi^*_{1,1}(\eta)=1$, and $\varphi^*_{1,k}=\varphi^*_{k,1}=0$ for $k>1$.*

Proof. Denote the components of the inverse sectorial mapping (F^{-1},G^{-1}) by $(zf(w),g(w),\xi\lambda(\eta),\mu(\eta))$. Then (after choosing sufficiently small domains $G^{\pm}_{R,r,\alpha}$ as above) $(F,G)(\mathcal{M}\cap\{\pm\mathrm{Re}\,\eta>0\})$ is described as

$$g(w)=\mu(\eta)e^{i\mu(\eta)^{m-1}\psi(zf(w),\xi\lambda(\eta),\mu(\eta))},\tag{4.17}$$

where $w=\eta e^{i\eta^{m-1}\psi}$ is the defining equation of $\mathcal{M}$. The fact that φ^*_0,φ^*_1, determined by (4.17), have the desired form, reads (for each fixed η) as

$$g(\eta)+i\eta^m g'(\eta)z\xi=\mu(\eta)+i\mu(\eta)^m z\xi f(\eta)\lambda(\eta)+O(z^2\xi^2).$$

The latter is equivalent to

$$g(\eta)=\mu(\eta),\ \eta^m g'(\eta)=\mu(\eta)^m f(\eta)\lambda(\eta). \tag{4.18}$$

Equations (4.18) determine $\lambda(\eta),\mu(\eta)$ with the desired properties uniquely, and this proves the proposition. $\square$

Recall now that, by assumption, the mapping $F\in\mathcal{SG}^{\pm}_m$ transfers a linear $\mathcal{P}_0$-ODE $\mathcal{E}$ with an isolated meromorphic singularity onto another linear $\mathcal{P}_0$-ODE $\tilde{\mathcal{E}}$ with an isolated meromorphic singularity. This means that F transfers germs of leaves of the foliation $\mathcal{M}\cap\{\xi=const,\eta=const\}$ with $\pm\mathrm{Re}\,\eta>0$ into germs of holomorphic graphs, satisfying the ODE $\tilde{\mathcal{E}}$. Thus the substitution of $w=\eta e^{\eta^{m-1}\varphi^*(z,\xi,\eta)}$ into $\tilde{\mathcal{E}}$, where φ^* is the unique defining function, obtained in Proposition 30, gives an identity. Fixing ξ and η and performing the substitution, we obtain a second order ODE for the function $\varphi(\cdot,\xi,\eta)$, identical to (3.10). The uniqueness of a solution for this ODE with the initial data $\varphi(0,\xi,\eta)=0,\dot\varphi(0,\xi,\eta)=\xi$ implies that the defining functions of $\mathcal{M}_{\tilde{\mathcal{E}}}$ and $(F,G)(\mathcal{M}\cap\{\pm\mathrm{Re}\,\eta>0\})$ coincide for all $\eta\in S^{\pm}_{r,\alpha}$. This proves that the sectorial mapping (F,G), obtained by Proposition 30, transfers $\mathcal{M}\cap\{\pm\mathrm{Re}\,\eta>0\}=\mathcal{M}_{\mathcal{E}}\cap\{\pm\mathrm{Re}\,\eta>0\}$ into $\mathcal{M}_{\tilde{\mathcal{E}}}\cap\{\pm\mathrm{Re}\,\eta>0\}$ (after intersecting $M_{\mathcal{E}},M_{\tilde{\mathcal{E}}}$ with sufficiently small polydiscs). Proposition 30 also implies that (F,G) with such property is unique.

On the other hand, if a mapping $(F,G)\in\mathcal{SG}^{\pm}_m\times\mathcal{SG}^{\pm}_m\times$ transfers $\mathcal{M}_{\mathcal{E}}$ into $\mathcal{M}_{\tilde{\mathcal{E}}}$, where $\mathcal{E},\tilde{\mathcal{E}}$ are two linear $\mathcal{P}_0$-ODEs with an isolated meromorphic singularity, then F transfers germs of leaves of the foliation $\mathcal{M}_{\mathcal{E}}\cap\{\xi=const,\eta=const\}$ with $\pm\mathrm{Re}\,\eta>0$ into germs of leaves of the foliation $\mathcal{M}_{\tilde{\mathcal{E}}}\cap\{\xi=const,\eta=const\}$ with $\pm\mathrm{Re}\,\eta>0$. This implies that $F\in\mathcal{SG}^{\pm}_m$ is an equivalence of the ODEs $\mathcal{E}$ and $\tilde{\mathcal{E}}$.

All above arguments prove the following

Proposition 31. *Let $\mathcal{E},\tilde{\mathcal{E}}$ be two linear $\mathcal{P}_0$-ODEs with an isolated meromorphic singularity, and $\mathcal{M}_{\mathcal{E}}$, $M_{\tilde{\mathcal{E}}}\subset\mathbb{C}^4$ the associated foliated submanifolds. Then there is a one-to-one correspondence $F(z,w)\longrightarrow(F(z,w),G(\xi,\eta))$ between sectorial equivalences $F(z,w)\in\mathcal{SG}^{\pm}_m$, transferring $\mathcal{E}$ into $\tilde{\mathcal{E}}$, and coupled sectorial transformations $(F(z,w),G(\xi,\eta))\in\mathcal{SG}^{\pm}_m\times\mathcal{SG}^{\pm}_m$, sending $\mathcal{M}_{\mathcal{E}}\cap\{\pm\mathrm{Re}\,\eta>0\}$ into $\mathcal{M}_{\tilde{\mathcal{E}}}\cap\{\pm\mathrm{Re}\,\eta>0\}$ (where G is given in fact by Proposition 29).*

We are now in the position to prove our principal result.

Theorem 32. *For any $\gamma\neq0$ the germs at 0 of the real-analytic hypersurfaces M_γ and M_0, associated with the ODEs $\mathcal{E}_\gamma$ and $\mathcal{E}_0$ respectively, are C^∞ CR-equivalent, but are holomorphically inequivalent.*

Proof. In what follows we assume that the real hypersurfaces M_γ and M_0, as well as their complexifications $\mathcal{M}_{\mathcal{E}_\gamma}$ and $\mathcal{M}_{\mathcal{E}_0}$, are intersected with a sufficiently small neighborhood of the origin, if necessary. As was discussed above, the sectorial map $F^+\in\mathcal{SG}^+_4$, as in (4.12), transfers $\mathcal{E}_\gamma$ into $\mathcal{E}_0$. According to Proposition 29, there exists a unique $G^+\in\mathcal{SG}^+_4$, such that (F^+,G^+) transfers $\mathcal{M}_{\mathcal{E}_\gamma}\cap\{\mathrm{Re}\,\eta>0\}$ into $\mathcal{M}_{\mathcal{E}_0}\cap\{\mathrm{Re}\,\eta>0\}$. Considering now the reality condition (2.2) for the real hypersurfaces M_γ and complexifying it, we conclude that every $\mathcal{M}_{\mathcal{E}_\gamma}=(M_\gamma)^{\mathbb{C}}$ is invariant under the anti-holomorphic linear mapping $\sigma:\mathbb{C}^4\longrightarrow\mathbb{C}^4$ given by

$$(z,w,\xi,\eta)\longrightarrow(\bar\xi,\bar\eta,\bar z,\bar w). \tag{4.19}$$

Thus the sectorial mapping $\sigma\circ(F^+(z,w),G^+(\xi,\eta))\circ\sigma=(\overline{G^+}(z,w),\overline{F^+}(\xi,\eta))\in\mathcal{SG}^+_4\times\mathcal{SG}^+_4$ also transfers $\mathcal{M}_{\mathcal{E}_\gamma}\cap\{\mathrm{Re}\,\eta>0\}$ into $\mathcal{M}_{\mathcal{E}_0}\cap\{\mathrm{Re}\,\eta>0\}$ (here $\overline{F^+}(z,w):=\overline{F^+(\bar z,\bar w)}$ and similarly for $\overline{G^+}$). Now the uniqueness, given by Proposition 29, implies $F^+(z,w)=\overline{G^+}(z,w)$. In particular, this means that $F^+(z,w)$ transfers $M_\gamma^+ = M_\gamma \cap \{\mathrm{Re}\, w > 0\}$ into $M_0^+ = M_0 \cap \{\mathrm{Re}\, w > 0\}$. Similarly, we get that the sectorial mapping $F^-(z,w)$, as in (4.12), transfers $M_\gamma^- = M_\gamma \cap \{\mathrm{Re}\, w < 0\}$ into $M_0^- = M_0 \cap \{\mathrm{Re}\, w < 0\}$.

We now define the desired C^∞ CR-equivalence as follows:

$$F(z,w) = \begin{cases} F^-(z,w),\ (z,w) \in M_\gamma^- \\ (z,0),\ (z,w) \in X \\ F^+(z,w),\ (z,w) \in M_\gamma^+ \end{cases} \tag{4.20}$$

The arguments above imply that $F(M_\gamma) \subset M_0$, and the asymptotic expansion for the mappings (4.12) shows that F is a local C^∞ diffeomorphism on M. Now F is CR because it is actually analytic on each of the CR-manifolds M^+ and and M^-, and thus it satisfies the tangential CR-equations on all of M. This proves the CR-equivalence of the germs $(M_\gamma, 0)$ and $(M_0, 0)$.

To prove the holomorphic nonequivalence of M_γ and M_0 we finally note that each local holomorphic equivalence $\varphi : (M_\gamma, 0) \longrightarrow (M_0, 0)$ extends to a holomorphic equivalence of the associated ODEs $\mathcal{E}_\gamma$ and $\mathcal{E}_0$ near the singular point $(z,w) = (0,0)$, as follows from the invariance property of Segre varieties. However , $\mathcal{E}_0$ has a non-zero holomorphic solution, while $\mathcal{E}_\gamma$ does not have one, which shows that such a local holomorphic equivalence does not exist. This completely proves the theorem. $\square$

Theorem 32 enables us to give the negative answer to Conjecture 1.

Corollary 33. *The mapping* (4.20) *is a C^∞ CR-equivalence between the germs of the real-analytic holomorphically nondegenerate hypersurfaces $M_\gamma, M_0 \subset \mathbb{C}^2$, which is not analytic at* 0.

(Note that, in fact, the map (4.20) is not analytic at any point of the exceptional hypersurface X, as a consequence of the Hanges-Treves propagation theorem [33]).

It is not difficult now to deduce the proof of Theorem 2. We first note that the real hypersurface M_0, associated with the ODE $\mathcal{E}_0$, coincides with the real hypersurface $M_0^4 \subset \mathbb{C}^2$, considered in [44] (while all M_γ with $\gamma \neq 0$ are different from the hypersurfaces, considered in [44]). A detailed computation, provided in [44] (see Section 5 there), shows that the single-valued elementary mapping $\Lambda'\colon (z,w) \mapsto \left(\sqrt{2}z, e^{-\frac{2i}{3}w^{-3}}\right)$ takes $M_0 \setminus X$ into the compact sphere $S^3 = \{|Z|^2 + |W|^2 = 1\} \subset \mathbb{C}^2$ (where and $X = \{w = 0\}$). It follows from (4.20) and (4.12) that, for any fixed γ, a mapping of $M_\gamma \setminus X$ into S^3 has (locally) the form $\Lambda\colon (z,w) \mapsto (z\mu(w), \nu(w))$. According to the globalization result [43], $\mu(w), \nu(w)$ extend to analytic mappings $\mathbb{C} \setminus \{0\} \to \mathbb{CP}^1$. Since the ODE $\mathcal{E}_\gamma$ has a trivial monodromy, so does the mapping Λ and we conclude that the extensions of both $\mu(w)$ and $\nu(w)$ are single-valued.

Proof of Theorem 2. For $n = k = 1$ the result is just the one proved in Theorem 32. For $k = 1$ and $n > 1$ (which corresponds to hypersurfaces in $\mathbb{C}^{n+1}$) we consider the above hypersurfaces $M_\gamma, M_0 \subset \mathbb{C}^2$ and write them near the origin as $v = \Theta_\gamma(z\bar{z}, u)$ and $v = \Theta_0(z\bar{z}, u)$ respectively (here $w = u + iv$). The mapping $F(z,w) = (z\chi(w), \tau(w))$, as in (4.20), provides a C^∞ CR-equivalence between $(M_\gamma, 0)$ and $(M_0, 0)$. We now define

$$M := \{v = \Theta_\gamma(z_1\bar{z}_1 + ... + z_n\bar{z}_n, u)\} \subset \mathbb{C}^{n+1}, \quad M' := \{v = \Theta_0(z_1\bar{z}_1 + ... + z_n\bar{z}_n, u)\} \subset \mathbb{C}^{n+1}.$$

Then it is immediate that the mapping

$$H_n\colon (z_1, ..., z_n, w) \to (\chi(w)z_1, ..., \chi(w)z_n, \tau(w))$$

is a C^∞ CR-equivalence between M and M'.

We claim that $(M,0)$ and $(M',0)$ are biholomorphically inequivalent. As in the case of $\mathbb{C}^2$, the mapping $\Lambda_n'\colon (z_1,...,z_n,w) \to \left(\sqrt{2}z_1,...,\sqrt{2}z_n, e^{-\frac{2i}{3}w^{-3}}\right)$ maps $M'\setminus\{w=0\}$ into the sphere $S^{2n-1} = \left\{|Z_1|^2+...+|Z_n|^2+|W|^2=1\right\}$, and the mapping $\Lambda_n\colon (z_1,...,z_n,w)\to (\mu(w)z_1,...,\mu(w)z_n,\nu(w))$ maps $M\setminus\{w=0\}$ into S^{2n-1}. The pullback of the Segre family of the sphere S^{2n-1} by the mapping Λ_n' provides an extension of the Segre family of M' consisting of horizontal hyperplanes $\{w=const\}$ and the $(n+1)$-parameter family of complex hypersurfaces

$$\left\{a_1z_1+...+a_nz_n+be^{-\frac{2i}{3}w^{-3}}+c=0,\ |a_1|^2+..+|a_n|^2\neq 0\right\}.$$

Similarly, for M we get the $(n+1)$-parameter family of complex hypersurfaces

$$\left\{\mu(w)(a_1z_1+...+a_nz_n)+b\nu(w)+c=0,\ |a_1|^2+..+|a_n|^2\neq 0\right\}, \tag{4.21}$$

where $\mu(w),\nu(w)$ are as above. In particular, for the real hypersurface M' and $b=0$ we get, for appropriate values of $(a_1,...,a_n)$, an n-parameter family of complex hypersurfaces (in fact, complex hyperplanes), defined in an open neighborhood of the origin and intersecting the complex locus $X=\{w=0\}$ of M' transversally. In case M and M' are locally biholomorphically equivalent at the origin, a similar n-parameter family must exist for M as well. However, from the form of (4.21) this is possible only if one of the functions $\frac{1}{\mu(w)}$ or $\frac{\nu(w)}{\mu(w)}$ extends to $w=0$ holomorphically (as a mapping into $\mathbb{C}$). In both cases we conclude that the extended Segre family of the above real hypersurface $M_\gamma\subset\mathbb{C}^2$ contains a graph $z=f(w)$, $f(w)\not\equiv 0$, with f holomorphic near the origin, and hence the ODE $\mathcal{E}_\gamma$ has a non-zero holomorphic solution, which is a contradiction. This completes the proof for $k=1, n>1$.

Finally, for the case $k>1$ and CR-dimension $n\geq 1$ we argue similarly to the proof of the analogous statement in [44] in the case $k>1$ and consider the holomorphically nondegenerate CR-submanifolds $P=M\times\Pi_{k-1}$ and $P'=M'\times\Pi_{k-1}$, where $M,M'\subset\mathbb{C}^{n+1}$ are chosen from the hypersurface case and $\Pi_{k-1}\subset\mathbb{C}^{k-1}$ is the totally real plane $\operatorname{Im}W=0, W\in\mathbb{C}^{k-1}$. Then the direct product of the above mapping H_n and the identity map gives a C^∞ CR-equivalence between P and P'. To show that P and P' are inequivalent holomorphically, we denote the coordinates in $\mathbb{C}^{n+k}$ by (Z,W), $Z\in\mathbb{C}^{n+1}, W\in\mathbb{C}^{k-1}$ and note that, since Π is totally real, for each holomorphic equivalence

$$(\Phi(Z,W),\Psi(Z,W)):\ (M\times\Pi_{k-1},0)\longrightarrow(M'\times\Pi_{k-1},0),$$

one has $\Psi(Z,W)=\Psi(W)$ for a vector power series $\Psi(W)$ with real coefficients such that $\Psi(0)=0$. Since the initial mapping $(\Phi(Z,W),\Psi(Z,W))$ is invertible at 0, we conclude that the mapping $\Phi(Z,0):(\mathbb{C}^n,0)\longrightarrow(\mathbb{C}^n,0)$ is invertible at 0 as well, and since $(\Phi(Z,W),\Psi(W)):(M\times\Pi_{k-1},0)\longrightarrow(M'\times\Pi_{k-1},0)$, the map $\Phi(Z,0)$ is a local equivalence between $(M,0)$ and $(M',0)$. Now the desired statement is obtained from the hypersurface case. □

5. Applications to CR-automorphisms

From the results of the previous section we are able to obtain various somehow paradoxical phenomena for CR-automorphisms of nonminimal real-analytic hypersurfaces. We start by recalling the form of the infinitesimal automorphism algebra of a point in the sphere S^3. This

8-dimensional real Lie algebra is spanned by the vector fields

$$\begin{aligned} X_1 &= iZ\frac{\partial}{\partial Z}, \\ X_2 &= iW\frac{\partial}{\partial W}, \\ X_3 &= W\frac{\partial}{\partial Z} - Z\frac{\partial}{\partial W}, \\ X_4 &= iW\frac{\partial}{\partial Z} + iZ\frac{\partial}{\partial W}, \\ X_5 &= (1-Z^2)\frac{\partial}{\partial Z} - ZW\frac{\partial}{\partial W}, \\ X_6 &= i(1+Z^2)\frac{\partial}{\partial Z} + iZW\frac{\partial}{\partial W}, \\ X_7 &= -ZW\frac{\partial}{\partial Z} + (1-W^2)\frac{\partial}{\partial W}, \\ X_8 &= iZW\frac{\partial}{\partial Z} + i(1+W^2)\frac{\partial}{\partial W}. \end{aligned} \tag{5.1}$$

Consider now a hypersurface M_γ with $\gamma \neq 0$ as well as the hypersurface M_0 (see Section 4). Denote by $F(z,w) = (z\chi(w), \tau(w))$ the non-analytic C^∞ CR-equivalence (4.20) between $(M_\gamma, 0)$ and $(M_0, 0)$, and by Λ and Λ' the above described single-valued mappings of $M_\gamma \setminus X$ and $M_0 \setminus X$ respectively into the sphere S^3 (see the paragraph just before the proof of Theorem 2 in Section 4). Substituting the elementary mapping Λ' into (5.1) for $w \neq 0$, it is straightforward to check that the vector fields $\Lambda'_* X_1, \Lambda'_* X_2, \Lambda'_* X_5, \Lambda'_* X_6$ extend to elements of $\mathfrak{hol}^\omega(M_0, 0)$ and, moreover,

$$\Lambda'_* X_1 = iz\frac{\partial}{\partial z},\ \Lambda'_* X_2 = 2w^4\frac{\partial}{\partial w}, \tag{5.2}$$
$$\Lambda'_* X_5 = \frac{1}{\sqrt{2}}(1-2z^2)\frac{\partial}{\partial z} + 2\sqrt{2}izw^4\frac{\partial}{\partial w},\ \Lambda'_* X_6 = \frac{i}{\sqrt{2}}(1+2z^2)\frac{\partial}{\partial z} + 2\sqrt{2}zw^4\frac{\partial}{\partial w}$$

It is also straightforward that no non-zero linear combination of $\Lambda'_* X_3, \Lambda'_* X_4, \Lambda'_* X_7, \Lambda'_* X_8$ extends neither to an element of $\mathfrak{hol}^\omega(M_0, 0)$ nor to an element of $\mathfrak{hol}^\infty(M_0, 0)$, so that

$$\mathfrak{hol}^\omega(M_0,0) = \mathfrak{hol}^\infty(M_0,0) = \mathrm{span}_{\mathbb{R}} < \Lambda'_* X_1, \Lambda'_* X_2, \Lambda'_* X_5, \Lambda'_* X_6 > .$$

Since the mapping $F(z,w)$ provides a C^∞ CR-equivalence between $(M_\gamma, 0)$ and $(M_0, 0)$, we have $\mathfrak{hol}^\infty(M_\gamma, 0) = F_*(\mathfrak{hol}^\infty(M_0, 0))$. Substitution of F into $\mathrm{span}_{\mathbb{R}} < \Lambda'_* X_1, \Lambda'_* X_2, \Lambda'_* X_5, \Lambda'_* X_6 >$ gives $\mathrm{span}_{\mathbb{R}} < Y_1, Y_2, Y_5, Y_6 >$, where

$$\begin{aligned} Y_1 &= iz\frac{\partial}{\partial z}, \\ Y_2 &= -\frac{2\tau^4\chi'}{\chi\tau'}z\frac{\partial}{\partial z} + \frac{2\tau^4}{\tau'}\frac{\partial}{\partial w}, \\ Y_5 &= \left(\frac{1}{\sqrt{2}}\frac{1}{\chi} - 2\chi z^2 - 2\sqrt{2}i\frac{\chi'\tau^4}{\tau'}z^2\right)\frac{\partial}{\partial z} + 2\sqrt{2}i\frac{\chi\tau^4}{\tau'}z\frac{\partial}{\partial w}, \\ Y_6 &= \left(\frac{i}{\sqrt{2}}\frac{1}{\chi} + 2i\chi z^2 - 2\sqrt{2}\frac{\chi'\tau^4}{\tau'}z^2\right)\frac{\partial}{\partial z} + 2\sqrt{2}\frac{\chi\tau^4}{\tau'}z\frac{\partial}{\partial w} \end{aligned} \tag{5.3}$$

and $\chi = \chi(w)$, $\tau = \tau(w)$.

In what follows we denote by $\mathcal{O}_0$ the space of germs of holomorphic functions at the origin. Recall that the restrictions $\chi^{\pm},\tau^{\pm}$ of the functions $\chi(w),\tau(w)$ to the sectors $S^{\pm}_{\frac{\pi}{3}}$ respectively have the asymptotic representations $\hat\chi(w)=1+O(w)$, $\hat\tau(w)=w+O(w^5)$ (see the proof of Proposition 25). Note that at least one of the functions $\chi(w),\tau(w)$ does not belong to $\mathcal{O}_0$, because the CR-equivalence $F(z,w)$ is not holomorphic at the origin. Also note that the vector field Y_1 extends to the origin holomorphically. We now consider three cases.

Assume first that $\tau(w)\notin\mathcal{O}_0,\chi(w)\in\mathcal{O}_0$. Then the function $\frac{\tau^4}{\tau'}=-1/\left(3(\tau^{-3})'\right)\notin\mathcal{O}_0$ and, considering the $\frac{\partial}{\partial w}$ components of the three vector fields Y_2,Y_3,Y_4, we conclude that no nontrivial real linear combinations of Y_2,Y_3,Y_4 extends to the origin holomorphically. Thus $\mathfrak{hol}^{\omega}(M_\gamma,0)=\mathrm{span}_{\mathbb{R}}<Y_1>$.

Assume next $\tau(w)\in\mathcal{O}_0,\chi(w)\notin\mathcal{O}_0$. Then the functions $\frac{1}{\chi},\frac{\chi'}{\chi}\notin\mathcal{O}_0$ and, considering the $\frac{\partial}{\partial z}$ components of the three vector fields Y_2,Y_3,Y_4, we conclude that all real non-zero linear combinations of Y_2,Y_3,Y_4 do not extend to the origin holomorphically. Thus $\mathfrak{hol}^{\omega}(M_\gamma,0)=\mathrm{span}_{\mathbb{R}}<Y_1>$.

Finally, assume $\tau(w)\notin\mathcal{O}_0,\chi(w)\notin\mathcal{O}_0$. Then $\frac{\tau^4}{\tau'},\frac{1}{\chi},\frac{\chi'}{\chi}\notin\mathcal{O}_0$ and, considering the $\frac{\partial}{\partial w}$ component for Y_2 and the $\frac{\partial}{\partial z}$ component for Y_3,Y_4, we also come to the conclusion $\mathfrak{hol}^{\omega}(M_\gamma,0)=\mathrm{span}_{\mathbb{R}}<Y_1>$.

We summarize our arguments in

Theorem 34. *For $\gamma\neq0$ the hypersurface M_γ defined above satisfies*

$$dim\,\mathfrak{hol}^{\omega}(M_\gamma,0)=1,\ dim\,\mathfrak{aut}^{\omega}(M_\gamma,0)=1,$$

while

$$dim\,\mathfrak{hol}^{\infty}(M_\gamma,0)=4,\ dim\,\mathfrak{aut}^{\infty}(M_\gamma,0)=2.$$

We are now in the position to prove our second main result.

Proof of Theorem 5. The strategy of the proof is similar to that for Theorem 2. For $N=2$ the result is contained in Theorem 34. For $N>1$ we consider a hypersurface $M_\gamma\subset\mathbb{C}^2$, $\gamma\neq0$ and write it up near the origin as $v=\Theta_\gamma(z\bar z,u)$ (here $w=u+iv$). Set

$$M:=\{v=\Theta_\gamma(z_1\bar z_1+...+z_{N-1}\bar z_{N-1},u)\}\subset\mathbb{C}^N$$

(here we denote by $(z_1,...,z_{N-1},w)$ the coordinates in $\mathbb{C}^N$). Then M is a real-analytic holomorphically nondegenerate hypersurface. It follows from the fact that $Y_2=-\frac{2\tau^4\chi'}{\chi\tau'}z\frac{\partial}{\partial z}+\frac{2\tau^4}{\tau'}\frac{\partial}{\partial w}\in\mathfrak{aut}^{\infty}(M_\gamma,0)$ that the vector field

$$Y=-\frac{2\tau^4\chi'}{\chi\tau'}\left(z_1\frac{\partial}{\partial z_1}+...+z_{N-1}\frac{\partial}{\partial z_{N-1}}\right)+\frac{2\tau^4}{\tau'}\frac{\partial}{\partial w}\in\mathfrak{aut}^{\infty}(M,0).$$

Then arguments identical to the ones used for the proof of Theorem 34 show that $Y\notin\mathfrak{hol}^{\omega}(M,0)$, i.e., $\mathfrak{aut}^{\omega}(M,0)\subsetneq\mathfrak{aut}^{\infty}(M,0)$ and $\mathfrak{hol}^{\omega}(M,0)\subsetneq\mathfrak{hol}^{\infty}(M,0)$. This proves the theorem. □

We shall now introduce a new notion of homogeneity for CR-manifolds, relevant to the non-minimal situation. For that, we employ the notion of a *CR-orbit* (see [3] for the details of the concept). We say that a real-analytic CR-submanifold $M\subset\mathbb{C}^N$ is *orbitally homogeneous,* if for any point $p\in M$ we can choose a connected neighborhood U of p in M such that for the unique CR-orbit P of U through p, the image of $\mathfrak{hol}^{\omega}(M,0)$ under the evaluation mapping

$e_p : L \mapsto L|_p$, $L \in \mathfrak{hol}\,(M,0)$ contains T_pP. We say that a real-analytic CR-submanifold $M \subset \mathbb{C}^N$ is *orbitally CR-homogeneous,* if in the above definition $\mathfrak{hol}^\omega\,(M,0)$ is replaced by $\mathfrak{hol}^\infty\,(M,0)$. For an orbitally homogeneous (resp. orbitally CR-homogeneous) CR-manifold its germs at any two points, belonging to the same CR-orbit, are holomorphically (resp. C^∞ CR) equivalent. For minimal CR-manifolds, when the CR-orbit of any point coincides with the whole M, the orbital homogeneity is equivalent to the standard local homogeneity (see Zaitsev [68] for details of the concept). By combining [6, 68], we get that in the minimal case the local CR-homogeneity and the local (holomorphic) homogeneity coincide. However, it turns out that in the nonminimal settings the concepts of orbital and CR-orbital homogeneities respectively are distinct, even for the case of holomorphically nondegenerate hypersurfaces.

Theorem 35. *Any hypersurface M_γ with $\gamma \neq 0$ as above is orbitally CR-homogeneous, but not orbitally homogeneous.*

Proof. Clearly, the orbit of the origin under the action of the Lie algebra $\mathfrak{hol}^\infty\,(M_\gamma,0)$ of CR-vector fields coincides with the 2-dimensional CR-orbit $X = \{w = 0\}$, and we obtain the orbital CR-homogeneity of X. The local homogeneity of the maximal-dimensional CR-orbits $M_\gamma^\pm = M_\gamma \cap \{\pm \mathrm{Re}\, w > 0\}$ follows from their sphericity at each point. Thus M_γ is orbitally CR-homogeneous. The fact that M_γ is not orbitally homogeneous follows from Theorem 34. □

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Department of Mathematics, Masaryk University - Brno
E-mail address: kossovskiyi@math.muni.cz

Department of Mathematics, University of Vienna
E-mail address: bernhard.lamel@univie.ac.at