Bounding diameter of singular Kähler metric

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Abstract

In this paper we investigate the differential geometric and algebro-geometric properties of the noncollapsing limit in the continuity method that was introduced by the first two named authors in [21].

Contents

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1 Introduction

The Analytic Minimal Model Program through Kähler-Ricci flow was initiated by the second named author and his collaborators ([44], [30, 31, 32], [39]). Many progresses have been made on the program (see [14], [26, 27, 28, 29], [33, 34, 36], [37], [50] et al) in addition to what mentioned above. However, there are some serious difficulties in studying the singularity formation of the Kähler-Ricci flow because we do not know how to bound the Ricci curvature from below along the flow. To overcome these difficulties, in [21], the first and second named authors introduced a new continuity method. It provides an alternative way of studying the Analytic Minimal Model Program and has the advantage of having Ricci curvature bounded from below along the deformation, so many analytic tools become available.

In this paper we investigate the differential geometric and algebro-geometric properties of the limit in the continuity method. We will focus on the noncollapsing case and confirm some conjectures proposed in [21] under the algebraic assumption of the Kähler class. This may be seen as the first step toward the Minimal Model Program through the continuity equation.

The method to study the algebraic structure of the limit space is provided by the partial C^0 estimate for a polarized line bundle on a Kähler manifold, following the recent work of Donaldson-Sun [12] and Tian [40] (also see [41]). In our case, however, the considered will be a "limit line bundle" of polarized bundles, so it is not canonically related to the Kähler metric on the underlying manifold. In general, this may cause some problem in finding numerous of holomorphic peak sections. Similar problem also appears in Kähler-Ricci flow [28]. In [28], Song proved the convergence of the normalized Kähler-Ricci flow on a minimal model of general type in the Cheeger-Gromov sense to a compact singular Kähler-Einstein metric on its canonical model. Song developed some analysis on the limit space to prove the theorem, namely the gradient estimate to the limit potential function, the gradient estimate to the holomorphic sections of canonical line bundle, the Hörmander L^2 estimate and a diameter estimate to the singular Kähler-Einstein metric. These depend on the property of (real) codimension 4 singularity in the Cheeger-Gromov limit space so that one can construct good cut-offs on both the limit space and its tangent cones. In our case, the Hörmander L^2 estimate holds automatically for the continuity method. The main difficulty in the continuity method is that the Cheeger-Gromov limit as well as its tangent cones may have (real) codimension 2 singularities. However, one can show by recursion that if this happens at a point in the tangent cone, then the singular set around that point must be a locally analytical set modeled by taking the limit of a divisor (ample locus of the limit Kähler class) on the original manifold. The picture fits into the case of convergence of conical Kähler-Einstein manifolds as considered in [41].

We begin with a compact projective manifold M with a Kähler metric $\omega_0 \in qc_1(L')$ where L' is a line bundle, q is a positive rational number. Consider a 1-parameter family of equations [21]

$$\omega = \omega_0 - t\,\mathrm{Ric}(\omega). \tag{1.1}$$

The equation is solvable up to

$$T =: \sup\{t|[\omega_0] - tc_1(M) > 0\}. \tag{1.2}$$

In the special case when the limit does not collapse, we have the following theorem.

Theorem 1.1 ([21]). *Assume $T < \infty$ and $([\omega_0] - Tc_1(M))^n > 0$, then ω_t converge to a unique weakly Kähler metric ω_T which is smooth outside of a subvariety $\mathcal{S}_M$ and satisfies*

$$\omega_T = \omega_0 - T\,\mathrm{Ric}(\omega_T), \quad \text{on } M\backslash\mathcal{S}_M. \tag{1.3}$$

Here, the subvariety $\mathcal{S}_M$ equals the non-ample locus of the cohomological class

$$\mathcal{S}_M = \bigcap\{D|D \text{ is a divisor satisfying } [\omega_0] - Tc_1(M) - \epsilon[D] > 0 \text{ for some } \epsilon > 0\}. \tag{1.4}$$

It is conjectured in [21, Conjecture 4.1] that the limit space has more regular properties, such as metric structure and algebraic structure, and the convergence of (M, ω_t) takes place in the Cheeger-Gromov topology. In this note we confirm this conjecture partially.

Theorem 1.2. *Assume as in above theorem. Then*

(1) *(M, ω_t) converges in the Cheeger-Gromov topology to a compact path metric space (M_T, d_T) which is the metric completion of $(M\backslash\mathcal{S}_M, \omega_T)$;*

(2) *M_T has regular/singular decomposition $M_T = \mathcal{R} \cup \mathcal{S}$, a point $x \in \mathcal{R}$ iff the tangent cone at x is $\mathbb{C}^n$;*

(3) *$\mathcal{S}$ is closed and has real codimension ≥ 2, $\mathcal{R}$ is geodesically convex;*

(4) *M_T is homeomorphic to a normal projective variety with $\mathcal{S}$ corresponding to a subvariety.*

Remark 1.3. *Our proof here used the Kawamata base point free theorem. We hope to remove this constraint eventually and prove the Kawamata base point free theorem for a general line bundle by applying a refinement of our method here together with some arguments in* [29].

Remark 1.4. *The next step toward the Minimal Model Program is to construct flips and deform the continuity equation* (1.1) *through the singular time T. This is basically the same as the Kähler-Ricci flow in* [32]. *We will leave the discussion in a forthcoming paper.*

Acknowledgement: The third named author is visiting the Math Department of Princeton University. He would like to thank the department for the nice working environment. He also want to thank Jian Song for his discussion on the Proposition 3.22 and Valentino Tosatti for the discussion of ample locus.

2 A prior estimate to the continuity equation

In this section, we present some estimate to the continuity equation (1.1). By a dilation of the cohomological class $[\omega_0]$ and the time parameter, we may assume that $[\omega_0] = c_1(L^{'})$ for some line bundle $L^{'}$. The rationality theorem of Kawamata [18] says that $T \in \mathbb{Q}$. Take a positive integer ℓ_0 such that $T\ell_0 \in \mathbb{Z}$, and define the"limit line bundle" $L = \ell_0(L^{'} + TK_M)$.

Let ω_t, $t \in [0, T)$, be the solution to (1.1) with initial metric ω_0. Since the limit class $[\omega_0] + TK_M$ is nef and big, according to the base point free theorem [19], we may assume ℓ_0 is chosen such that L has no base points. Let H_0 be any Hermitian metric on L. An orthonormal basis of $H^0(M, L)$ gives a holomorphic map

$$\Phi : M \to \mathbb{C}P^N \tag{2.1}$$

where $N = \dim H^0(M; L) - 1$. Denote by ω_{FS} the Fubini-Study metric of $\mathbb{C}P^N$, then the pull back form $\eta_T = \frac{1}{\ell_0}\Phi^*\omega_{FS}$ defines a smooth metric on M_{reg}, the set of regular points of Φ.

By putting $\eta_t = \frac{T-t}{T}\omega_0 + \frac{t}{T}\eta_T$, a family of background metrics, the solution ω_t can be written as

$$\omega_t = \eta_t + \sqrt{-1}\partial\bar{\partial}u_t. \tag{2.2}$$

Since $\frac{1}{T}(\omega_0 - \eta_T) \in c_1(M)$, there is a smooth volume form Ω on M such that

$$\text{Ric}(\Omega) = \frac{1}{T}(\omega_0 - \eta_T). \tag{2.3}$$

Then, the continuity equation (1.1) is equivalent to

$$\left(\eta_t + \sqrt{-1}\partial\bar{\partial}u_t\right)^n = e^{\frac{u_t}{t}}\Omega. \tag{2.4}$$

By [21], the solution (M, ω_t) converges smoothly outside $\mathcal{S}_M$ to a limit metric ω_T satisfying

$$\text{Ric}(\omega_T) = \frac{1}{T}(\omega_0 - \omega_T), \text{ on } M\backslash\mathcal{S}_M. \tag{2.5}$$

Lemma 2.1. *There is a constant C independent of t such that*

$$\|u_t\|_{C^0} \leq C, \forall t \in [\frac{T}{2}, T). \tag{2.6}$$

Proof. The uniform upper bound of u_t is trivial consequence of the maximum principle. The L^∞ bound follows from the capacity calculation of Kołodziej [20]; see also [49] or [13] for exactly our case when u_t has a uniform upper bound. □

Corollary 2.2. *There exists C independent of t such that*

$$C^{-1}\Omega \leq \omega_t^n \leq C\Omega, \forall t \in [\frac{T}{2}, T). \tag{2.7}$$

Lemma 2.3. *There exists C independent of t such that*

$$u^{'}, u^{''} \leq C, \forall \frac{T}{2} \leq t < T \tag{2.8}$$

where $'$ is the derivative in time t.

Proof. If there is no confusion, we simply denote ω_t by ω. Differentiating the Monge-Ampère equation (2.4) to get

$$\mathrm{tr}_\omega \omega^{'} = \frac{1}{t^2}\big(tu^{'} - u\big) \tag{2.9}$$

where

$$\omega^{'} = \frac{1}{T}(\eta_T - \omega_0) + \sqrt{-1}\partial\bar{\partial}u^{'} = \frac{1}{t}\big(\omega - \omega_0 - \sqrt{-1}\partial\bar{\partial}u\big) + \sqrt{-1}\partial\bar{\partial}u^{'}. \tag{2.10}$$

Thus,

$$\triangle(tu^{'} - u) = \mathrm{tr}_\omega\,\big(t\omega^{'} - \omega + \omega_0\big) = \frac{1}{t}\big(tu^{'} - u\big) + \mathrm{tr}_\omega\,\omega_0 - n. \tag{2.11}$$

Then applying the maximum principle we derive $tu^{'} - u \le C$. To get the upper bound of $u^{''}$ we first observe that (2.9) gives $tu^{'} - u = t^2\,\mathrm{tr}_\omega\,\omega^{'}$. Taking derivation,

$$tu^{''} = 2t\,\mathrm{tr}_\omega\,\omega^{'} + t^2\triangle u^{''} - t^2|\omega^{'}|^2 = t^2\triangle u^{''} - |\omega - t\omega^{'}|^2 + n.$$

So,

$$t^2\triangle u^{''} = tu^{''} + |\omega - t\omega^{'}|^2 - n. \tag{2.12}$$

Then apply the maximum principle. □

Lemma 2.4. *The function u_t converges uniformly to a bounded function u_T satisfying*

$$\big(\eta_T + \sqrt{-1}\partial\bar{\partial}u_T\big)^n = e^{\frac{u_T}{T}}\,\Omega \tag{2.13}$$

in the current sense.

Proof. By (2.11),

$$t\triangle\big(\frac{u}{t}\big)^{'} = \big(\frac{u}{t}\big)^{'} + \frac{1}{t}\big(\,\mathrm{tr}_\omega\,\omega_0 - n\big) \ge \big(\frac{u}{t}\big)^{'} - \frac{n}{t}.$$

Thus,

$$t\triangle\big(\frac{u}{t} - n\log t\big)^{'} \ge \big(\frac{u}{t} - n\log t\big)^{'}.$$

So, $\big(\frac{u}{t} - n\log t\big)$ is monotone decreasing, since by maximum principle we have $\sup_M \big(\frac{u}{t} - n\log t\big)^{'} \le 0$. Consequently, u_t converges uniformly to a unique limit u_T. It is obvious that u_T is smooth outside $\mathcal{S}_M$ and (2.13) holds in the current sense. □

Remark 2.5. *Similarly, the formula* (2.12) *imply that* $\big(u^{'} - n\log t\big)^{'} \le 0$. *In particular,* $\big(u^{'} - n\log t\big)$ *decrease to unique functions on M which is locally bounded on $M\backslash\mathcal{S}_M$.*

Remark 2.6. *The function u_T is the unique solution to the Monge-Ampère equation* (2.13) *in the big class $[\omega_0] + TK_M$, cf.* [2].

Proposition 2.7. *There exists C independent of t such that*

$$\eta_T \le C\omega_t,\ \forall t \in [\frac{T}{2}, T). \tag{2.14}$$

Proof. By Yau's Schwarz lemma [48], using $\mathrm{Ric}(\omega_t) \geq -\frac{1}{t}\omega_t$,

$$\triangle_\omega \log \mathrm{tr}_\omega \eta_T \geq -\frac{n}{t} - n\,\mathrm{tr}_\omega \eta_T.$$

On the other hand, $\eta_t \geq \epsilon \eta_T$ for some $\epsilon > 0$ independent of t, so

$$\triangle u = n - \mathrm{tr}_\omega \eta_t \leq n - \epsilon\,\mathrm{tr}_\omega \eta_T.$$

Hence,

$$\triangle_\omega(\log \mathrm{tr}_\omega \eta_T - \frac{2n}{\epsilon}u) \geq -\frac{C(n,T)}{\epsilon} + n\,\mathrm{tr}_\omega \eta_T,\ \forall t \in [\frac{T}{2}, T).$$

Notice that the maximal of $\big(\log \mathrm{tr}_\omega \eta_T - \frac{2n}{\epsilon}u\big)$ is achieved at some point x_0 where $\eta_T \neq 0$. At this maximum point, $\mathrm{tr}_\omega \eta_T(x_0) \leq C$. The maximum principle gives the desired estimate $\mathrm{tr}_\omega \eta_T \leq C$ for any $\frac{T}{2} \leq t < T$. □

Corollary 2.8. *The limit metric ω_T is smooth on the regular set M_{reg}.*

Proof. On any compact subset $K \subset M_{\mathrm{reg}}$ the measure η_T^n is uniformly equivalent to Ω. So, by (2.13) and (2.14)

$$C^{-1}\eta_T \leq \omega_T \leq C_K \eta_T, \text{ on } K,$$

for some constant C_K. In particular $n + \triangle_{\eta_T} u_T \leq C_K$ on K. Then applying a bootstrap argument we get the higher derivative bound $\|u_T\|_{C^k(K)} \leq C_{k,K}$ on K. □

Remark 2.9. *In Section 3 we will show that ω_t converges smoothly to ω_T on M_{reg} and that $M\backslash\mathcal{S}_M \subset M_{\mathrm{reg}}$. Due to the results of [10], we actually have that $M\backslash\mathcal{S}_M = M_{\mathrm{reg}}$.*

Inspired by Kähler-Ricci flow we define $w = (T-t)u' + u$ which satisfies

$$\triangle w = \frac{1}{t}w + n - \mathrm{tr}_\omega \eta_T - \frac{T}{t^2}u. \tag{2.15}$$

This can be seen by combining

$$\triangle u = n - \frac{T-t}{T}\mathrm{tr}_\omega \omega_0 - \frac{t}{T}\mathrm{tr}_\omega \eta_T \tag{2.16}$$

and

$$\triangle u' = \frac{1}{t^2}(tu' - u) + \frac{1}{T}\mathrm{tr}_\omega(\omega_0 - \eta_T). \tag{2.17}$$

A direct corollary of (2.15) is

$$\|w\|_{C^0} + \|\triangle w\|_{C^0} \leq C,\ \forall t \in [\frac{T}{2}, T). \tag{2.18}$$

Combining with the C^0 bound of u we also have

$$-\frac{C}{T-t} \leq u' \leq C,\ \forall t \in [\frac{T}{2}, T). \tag{2.19}$$

Finally we present a gradient estimate to the limit u_T, which may have potential applications. The authors learned the estimate from J. Song.

Proposition 2.10. *There exists C independent of t such that*

$$\|\nabla w\|_{C^0} \le C, \, \forall t \in [\frac{T}{2}, T). \tag{2.20}$$

In particular, since u' converges to a locally bounded function on $M \backslash \mathcal{S}_M$,

$$\|\nabla u_T\|_{C^0} \le C, \ \text{on } M\backslash \mathcal{S}_M. \tag{2.21}$$

Proof. It is inspired by the calculation in Kähler-Ricci flow; see [50] and [28]. Recall that the continuity equation (1.1) gives $Ric(\omega) = \frac{1}{t}(\omega_0 - \omega)$, so by the Bochner formula,

$$\triangle|\nabla w|^2 = |\nabla\nabla w|^2 + |\nabla\bar{\nabla} w|^2 + \nabla_i \triangle w \nabla_{\bar{i}} w + \nabla_{\bar{i}} \triangle w \nabla_i w + \frac{1}{t}\langle \omega_0 - \omega, \partial w \otimes \bar{\partial} w\rangle$$

where, by (2.15),

$$\nabla_i \triangle w \nabla_{\bar{i}} w + \nabla_{\bar{i}} \triangle w \nabla_i w = \frac{2}{t}|\nabla w|^2 - 2\,\mathrm{Re}\left(\nabla_i \operatorname{tr}_\omega \eta_T \nabla_{\bar{i}} w\right) - \frac{2T}{t^2}\,\mathrm{Re}\left(\nabla_i u \nabla_{\bar{i}} w\right).$$

So,

$$\begin{aligned} \triangle|\nabla w|^2 &\ge |\nabla\nabla w|^2 + |\nabla\bar{\nabla} w|^2 + \frac{1}{t}|\nabla w|^2 - 2\,\mathrm{Re}\left(\nabla_i \operatorname{tr}_\omega \eta_T \nabla_{\bar{i}} w\right) - \frac{2T}{t^2}\,\mathrm{Re}\left(\nabla_i u \nabla_{\bar{i}} w\right) \\ &\ge \frac{1}{2t}|\nabla w|^2 - 4t|\nabla \operatorname{tr}_\omega \eta_T|^2 - \frac{16T^2}{t^3}|\nabla u|^2. \end{aligned}$$

To estimate the term $|\nabla \operatorname{tr} \eta_T|^2$ recall that by the Schwarz lemma [48]

$$\triangle \operatorname{tr}_\omega \eta_T \ge \operatorname{tr}_\omega \eta_T \left(-\frac{n}{t} - A \operatorname{tr}_\omega \eta_T \right) + \frac{1}{\operatorname{tr}_\omega \eta_T}|\nabla \operatorname{tr}_\omega \eta_T|^2 \ge -C_1 + C_2^{-1}|\nabla \operatorname{tr}_\omega \eta_T|^2$$

where we used the C^0 bound of $\operatorname{tr}_\omega \eta_T$; to estimate the term $|\nabla_i u|^2$ we use

$$\triangle(-u) = -n + \frac{T-t}{T}\operatorname{tr}_\omega \omega_0 + \frac{t}{T}\operatorname{tr}_\omega \eta_T \ge \frac{T-t}{T}\operatorname{tr}_\omega \omega_0 - C_3$$

and

$$\triangle u^2 = 2u\triangle u + 2|\nabla u|^2 \ge 2|\nabla u|^2 - C_4\frac{T-t}{T}\operatorname{tr}_\omega \omega_0 - C_5.$$

The constants C_i here do not depend on $t \in [\frac{T}{2}, T)$. A combination gives

$$\triangle\left(|\nabla w|^2 + 4tC_2 \operatorname{tr}_\omega \eta_T + \frac{8T^2}{t^3}u^2 - \frac{8T^2}{t^3}C_4 u\right) \ge \frac{1}{2t}|\nabla w|^2 - C_6.$$

By maximum principle we get a uniform upper bound of $|\nabla w|$ when $t \in [\frac{T}{2}, T)$. □

Remark 2.11. *In §3.2, we will show that $M\backslash\mathcal{S}_M$ is dense in the Gromov-Hausdorff limit, so u_T is globally Lipschitz.*

3 Algebraic structure of the limit space

3.1 Preliminaries

We start with the Bochner and Weitzenböch formulas on a general line bundle. Let (M,ω) be a Kähler manifold of dimension n and (L,h) be a Hermitian line bundle over M. Let Θ_h be the Chern curvature form of h. Let ∇ and $\bar\nabla$ denote the $(1,0)$ and $(0,1)$ part of a connection respectively. The connection appeared in this paper is usually known as the Chern connection or Levi-Civita connection.

For a smooth section of L we write for simplicity

$$|\varsigma| = |\varsigma|_h,\ |\nabla\varsigma|^2 = \sum_i |\nabla_i\varsigma|^2,\ |\bar\nabla\varsigma|^2 = \sum_j |\nabla_{\bar j}\varsigma|^2,$$

and

$$|\nabla\nabla\varsigma|^2 = \sum_{i,j} |\nabla_i\nabla_j\varsigma|^2,\ |\bar\nabla\nabla\varsigma|^2 = \sum_{i,j} |\nabla_{\bar i}\nabla_j\varsigma|^2.$$

We will also use $\langle,\rangle$ to denote the inner product defined by h such that

$$|\varsigma|_h^2 = \langle\varsigma,\bar\varsigma\rangle$$

for any section ς. By direct computation we have (see Appendix for a proof)

Lemma 3.1 (Bochner formulas). *For any $\varsigma\in H^0(M,L)$ we have*

$$\triangle_\omega|\varsigma|^2 = |\nabla\varsigma|^2 - |\varsigma|^2\cdot \mathrm{tr}_\omega\,\Theta, \tag{3.1}$$

and

$$\begin{aligned}\triangle_\omega|\nabla\varsigma|^2 &= |\bar\nabla\nabla\varsigma|^2 + |\nabla\nabla\varsigma|^2 - \nabla_i(\mathrm{tr}_\omega\,\Theta)\langle\varsigma,\nabla_{\bar i}\bar\varsigma\rangle - \nabla_{\bar i}(\mathrm{tr}_\omega\,\Theta)\langle\nabla_i\varsigma,\bar\varsigma\rangle\\ &\quad + R_{i\bar j}\langle\nabla_j\varsigma,\nabla_{\bar i}\bar\varsigma\rangle - 2\Theta_{i\bar j}\langle\nabla_j\varsigma,\nabla_{\bar i}\bar\varsigma\rangle - |\nabla\varsigma|^2\cdot\mathrm{tr}_\omega\,\Theta,\end{aligned} \tag{3.2}$$

where $R_{i\bar j}$ is the Ricci curvature of ω.

Lemma 3.2 (Weitzenböch formulas). *For any smooth section $\xi\in\Gamma(T^{0,1}M\otimes L)$ we have*

$$(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)\xi = \bar\nabla^*\bar\nabla\xi + \big(\Theta + \mathrm{Ric}(\omega)\big)(\xi,\cdot), \tag{3.3}$$

and

$$(\bar\partial^*\bar\partial + \bar\partial\bar\partial^*)\xi = \nabla^*\nabla\xi + \Theta(\xi,\cdot) - \big(\mathrm{tr}_\omega\,\Theta\big)\xi \tag{3.4}$$

where $\mathrm{Ric}(\xi,\cdot)$ *(similar to $\Theta(\xi,\cdot)$) is defined by, for $\xi = \alpha_{\bar i}d\bar z^i\otimes\varsigma$ in local normal coordinate,*

$$\mathrm{Ric}(\xi,\cdot) = R_{i\bar j}\alpha_{\bar i}d\bar z^j\otimes\varsigma.$$

We also need a special version of the effective finite generation property of a line bundle. Suppose the line bundle L satisfies in addition that (i) L is base point free and (ii) $L-K_M$ is ample, then by Skoda division theorem, we have the following:

Theorem 3.3. *[22, Proposition 7] Let $s_0, \cdots, s_N$ be an orthonormal basis of $H^0(M;L)$ with respect to the L^2 metric, then for any $s \in H^0(M;L^k)$, $k > n+1$, we have the decomposition*

$$s = \sum_{\alpha \in \mathbb{N}^N, |\alpha| = k-n-1} e_\alpha s_0^{\alpha_0} \cdots s_N^{\alpha_N} \tag{3.5}$$

where $e_\alpha \in H^0(M; L^{n+1})$ satisfying

$$\int_M |e_\alpha|^2_{h^{n+1}} \omega^n \le C(k,h,\omega) \int_M |s|^2_{h^k} \omega^n, \tag{3.6}$$

the constant $C(k,h,\omega)$ depends on the metric h, ω, the power k and the upper and lower bound of the Bergman kernel $\rho_0(x) = \sum_i |s_i(x)|_h^2$.

3.2 Cheeger-Gromov convergence: global convergence

From now on, let $\omega_t, t \in [0,T)$, be the maximal solution to the continuity equation (1.1) on a Kähler manifold M. The metric ω_t converges smoothly on the ample locus $M_{\mathrm{amp}} = M \backslash \mathcal{S}_M$. In addition, since $\mathrm{Ric}(\omega_t) \ge -\frac{1}{t}\omega_t$, by Gromov precompactness theorem, for any sequence $t_i \to T$ and fixed $x_0 \in M_{\mathrm{amp}}$, we may assume that

$$(M, \omega_{t_i}, x_0) \xrightarrow{d_{GH}} (M_T, d_T, x_T) \tag{3.7}$$

after passing to a subsequence if necessary. The limit (M_T, d_T) is a complete length metric space, maybe noncompact in a prior. It has a regular/singular decomposition $M_T = \mathcal{R} \cup \mathcal{S}$, a point $x \in \mathcal{R}$ iff the tangent cone at x is the Euclidean space $\mathbb{R}^{2n}$. The proof of the following lemma is exactly same as [25, Proposition 8] so we omit it.

Lemma 3.4. *There is a constant $\delta > 0$ such that for any $\frac{T}{2} \le t < T$, if a metric ball $B_{\omega_t}(x,r)$ satisfies*

$$\mathrm{vol}_{\omega_t}\left(B_{\omega_t}(x,r)\right) \ge (1-\delta)\,\mathrm{vol}(B_r^0) \tag{3.8}$$

where $\mathrm{vol}(B_r^0)$ is the volume of a metric ball of radius r in $2n$-Euclidean space, then

$$\mathrm{Ric}(\omega_t) \le (2n-1) r^{-2} \omega_t, \;\; in \; B_{\omega_t}(x, \delta r). \tag{3.9}$$

If $x \in \mathcal{R}$, then, by Colding's volume convergence theorem [9], there is $r = r(x) > 0$ such that $\mathcal{H}^{2n}(B_{d_T}(x,r)) \ge (1-\frac{\delta}{2})\,\mathrm{vol}(B_r^0)$, where $\mathcal{H}^{2n}$ denotes the Hausdorff measure. Suppose $x_i \in M$ satisfying $x_i \xrightarrow{d_{GH}} x$, then by the volume convergence theorem again, $\mathrm{vol}_{\omega_{t_i}}(B_{\omega_{t_i}}(x_i, r)) \ge (1-\delta)\,\mathrm{vol}(B_r^0)$ for i sufficiently large. According to above lemma, together with Anderson's harmonic radius estimate [1], there is $\delta' = \delta'(\alpha) > 0$ for any $0 < \alpha < 1$ such that the $C^{1,\alpha}$ harmonic radius at x_i is bigger than $\delta' \delta r$. Passing to the limit, it gives a $C^{1,\alpha}$ harmonic coordinate on $B_{d_T}(x,r)$. This implies in particular that $B_{d_T}(x,r) \subset \mathcal{R}$. So $\mathcal{R}$ is open with a $C^{1,\alpha}$ Kähler metric, denoted by $\bar{\omega}_T$; moreover the metric ω_{t_i} converges in $C^{1,\alpha}$ topology to $\bar{\omega}_T$ on $\mathcal{R}$ for any $\alpha \in (0,1)$.

For any metric ω let d_ω be the length metric induced by ω.

Lemma 3.5. $(M_T, d_T) = \overline{(\mathcal{R}, d_{\bar{\omega}_T})}$*, the metric completion of* $(\mathcal{R}, d_{\bar{\omega}_T})$.

Proof. There is an exhaustion of $\mathcal{R}$ by compact subsets K_i with $K_i \subset K_{i+1}$ and a sequence of embeddings $\phi_i : K_i \to M$ such that (1) $\phi_i(x_T) = x_0$, (2) $\phi_i^* \omega_{t_i} \xrightarrow{C^{1,\alpha}} \bar{\omega}_T$ on $\mathcal{R}$ and (3) ϕ_i defines a Gromov-Hausdorff approximation of the convergence $(M, \omega_{t_i}, x_0) \xrightarrow{d_{GH}} (M_T, d_T, x_T)$. The third fact follows from a standard argument using $\operatorname{Codim} \mathcal{S} \geq 2$, cf. [6]; see also [24] or [51]. Moreover, (3) together with (1) implies (M, ω_{t_i}, x_0) converges in the Gromov-Hausdorff topology to $\overline{(\mathcal{R}, d_{\bar{\omega}_T})}$. By the uniqueness of the complete limit space we have $(M_T, d_T) = \overline{(\mathcal{R}, d_{\bar{\omega}_T})}$. □

Lemma 3.6. *$\mathcal{R}$ is geodesically convex in M_T in the sense that any minimal geodesic with endpoints in $\mathcal{R}$ lies in $\mathcal{R}$.*

Proof. It is simply a consequence of Colding-Naber's Hölder continuity of tangent cones along a geodesic in M_T [11]. Actually, if $x, y \in \mathcal{R}$, then for any minimal geodesic connecting x and y a neighborhood of endpoints lies in $\mathcal{R}$, so the geodesic will never touch the singular set $\mathcal{S}$. □

Let D be any divisor such that $\mathcal{S}_M \subset D$. Define the Gromov-Hausdorff limit of D

$$D_T =: \{x \in M_T | \text{ there exists } x_i \in D \text{ such that } x_i \xrightarrow{d_{GH}} x.\}$$

Proposition 3.7. *(M_T, d_T) is isometric to $\overline{(M \backslash D, d_{\omega_T})}$.*

Proof. First observe that, by the smooth convergence outside D, $M_T \backslash D_T \subset \mathcal{R}$ and $(M_T \backslash D_T, \bar{\omega}_T)$ is isometric to $(M \backslash D, \omega_T)$. We make the following

Claim 3.8. *$D_T \backslash \mathcal{S} \subset \mathcal{R}$ is a subvariety of dimension $(n-1)$ if it is not empty.*

Proof of the Claim. Let $x \in D_T \backslash \mathcal{S}$ and $x_i \in T$ converges to x. By the $C^{1,\alpha}$ convergence of ω_{t_i} around x, there are $C, r > 0$ independent of i and a sequence of harmonic coordinates in $B_{\omega_{t_i}}(x_i, r)$ such that $C^{-1}\omega_E \leq \omega_{t_i} \leq C\omega_E$ where ω_E is the Euclidean metric in the coordinates. Since the total volume of D is uniformly bounded for any ω_t, the local analytic sets $D \cap B_{\omega_{t_i}}(x_i, r)$ have a uniform bound of degree and so converge to an analytic set $D_T \cap B_{d_T}(x, r)$. □

It follows the Hausdorff dimension of $D_T = \mathcal{S} \cup (D_T \backslash \mathcal{S})$ is less than or equal to $2n-2$. Then as in Lemma 3.5, following the discussion in [6], one can show that the length metric $d_{\bar{\omega}_T}$ on $M_T \backslash D_T$ is same as d_T. It infers the required isometry

$$(M_T, d_T) = \overline{(M_T \backslash D_T, d_{\bar{\omega}_T})} \overset{\text{iso}}{\cong} \overline{(M \backslash D, d_{\omega_T})}.$$

□

A direct corollary is

Corollary 3.9. *(M, ω_t, x_0) converges globally to (M_T, d_T, x_T) in the Cheeger-Gromov sense as $t \to T$.*

Let M_{sing} be the subvariety of critical points of Φ and $M_{\text{reg}} = M \backslash M_{\text{sing}}$. We have shown that ω_T is a smooth metric on M_{reg}. Another corollary is

Corollary 3.10. *(M_T, d_T) is isometric to $\overline{(M_{\rm reg}, d_{\omega_T})}$.*

Proof. Notice that $M_{\rm reg}\backslash(M\backslash D) = M_{\rm reg} \cap D$ has codimension 2 in $(M_{\rm reg}, \omega_T)$. Thus, the length metric d_{ω_T} on $M\backslash D$ equals to the restricted extrinsic metric from $(M_{\rm reg}, d_{\omega_T})$. Since $M\backslash D$ is dense in $M_{\rm reg}$, we conclude the desired result

$$(M_T, d_T) \overset{\rm iso}{\cong} \overline{(M\backslash D, d_{\omega_T})} = \overline{(M_{\rm reg}, d_{\omega_T})|_{M\backslash D}} = \overline{(M_{\rm reg}, d_{\omega_T})}.$$

□

Lemma 3.11. *The identity map $\mathrm{id} : M_{\rm reg} \to M$ gives a Gromov-Hausdorff approximation representing the convergence $(M, \omega_t) \to (M_T, d_T)$ as $t \to T$.*

Proof. First observe that $(M\backslash D, d_T)$ is dense in $(M_{\rm reg}, d_T)$ and $(M\backslash D, d_T) = (M\backslash D, d_{\omega_T})$. So, it suffices to show that $\mathrm{id} : (M\backslash D, d_{\omega_T}) \to (M, \omega_t)$ defines a Gromov-Hausdorff approximation. This follows from the same argument as in the proof of Lemma 3.5. □

Therefore, the identity map id extends to an isometry

$$\overline{\mathrm{id}} : \overline{(M_{\rm reg}, d_{\omega_T})} \to (M_T, d_T).$$

Proposition 3.12. (1) *ω_t converges smoothly on $M_{\rm reg}$ to ω_T.*

(2) *$\overline{\mathrm{id}}(M_{\rm reg}) = \mathcal{R}$, the regular set of M_T.*

Proof. (1) For any compact subset $K \subset M_{\rm reg}$, there is $r = r_K > 0$ such that $\mathrm{vol}_{\omega_T}\big(B_{d_T}(x, r)\big) \geq (1 - \frac{\delta}{2})\,\mathrm{vol}(B^0_r)$ for any $x \in K$, where δ is the constant in Lemma 3.4. Then, since the identity map represents the Gromov-Hausdorff convergence, we have $\mathrm{vol}_{\omega_{t_i}}\big(B_{\omega_t}(x, r)\big) \geq (1 - \delta)\,\mathrm{vol}(B^0_r)$ for any $x \in K$ and t sufficiently close to T. By Lemma 3.4, the Ricci curvature $\mathrm{Ric}(\omega_t) \leq C\omega_t$ uniformly on K for some constant $C = C(K)$. Together with the uniform L^∞ bound of u_t, the continuity equation (1.1) then shows

$$C^{-1}\omega_{t_0} \leq \omega_t \leq C\omega_{t_0}, \ \text{ on } K$$

for some $C = C(K)$ independent of $t < T$. Then by a standard bootstrap to the complex Monge-Ampère equation (2.4) we get the uniform C^k estimate of the metrics ω_t on K, for any $k \geq 1$. This is sufficient to prove the smooth convergence of ω_t on K.

(2) Since $M_{\rm reg}$ has smooth structure in M_T we have immediately $\overline{\mathrm{id}}(M_{\rm reg}) \subset \mathcal{R}$. Next we show the converse, namely $\overline{\mathrm{id}}(M_{\rm reg})$ is the maximal regular subset of M_T. The idea follows from [24]; see also [28]. We argue by contradiction. Suppose we have a point $p \in \mathcal{R}\backslash\overline{\mathrm{id}}(M_{\rm reg})$, then there exists a family of points $p_t \in M_{\rm sing}$ such that $p_t \to p$. Denote $m = \dim_{\mathbb{C}} M_{\rm sing}$. By $C^{1,\alpha}$ convergence on $\mathcal{R}$, there exist $C, r > 0$ independent of t and a sequence of harmonic coordinates on $B_{\omega_t}(p_t, r)$ such that $C^{-1}\omega_E \leq \omega_t \leq C\omega_E$ where ω_E is the Euclidean metric in this coordinate. Then

$$\mathrm{vol}_{\omega_t}\big(M_{\rm sing} \cap B_{\omega_t}(p_t, r)\big) = \int_{M_{\rm sing}\cap B_{\omega_t}(p_t,r)} \omega_t^m \geq \int_{M_{\rm sing}\cap B_{\omega_E}(C^{-1/2}r)} (C^{-1}\omega_E)^m$$

which has a uniform lower bound $C^{-2m}c(m)r^{2m}$ where $c(m)$ is the volume of unit sphere in $\mathbb{C}^m$. This follows from the classical analysis of the lower volume estimate or multiplicity estimate of an analytical set in the Euclidean space. However, this contradicts with the degeneration of the limit metric η_T along M_{sing}:

$$\text{vol}_{\omega_t}\big(M_{\text{sing}} \cap B_{\omega_t}(p_t, r)\big) \leq \text{vol}_{\omega_t}(M_{\text{sing}}) = \int_{M_{\text{sing}}} \omega_t^m = \int_{M_{\text{sing}}} \eta_t^m = \left(\frac{T-t}{T}\right)^m \int_{M_{\text{sing}}} \omega_0^m$$

which tends to 0 as $t \to T$. So we have $\overline{\text{id}}(M_{\text{reg}}) \supset \mathcal{R}$ as desired. □

It follows that, for any Hermitian line bundle (L', h') and $k \in \mathbb{Z}$, the twisted line bundle $(L' \otimes K_M^k, h' \otimes (\omega_{t_i}^{-n})^k)$ converges smoothly to a limit Hermitian line bundle $(L' \otimes K_{\mathcal{R}}^k, h' \otimes (\bar{\omega}_T^{-n})^k)$ on $\mathcal{R}$. Another corollary is $M_{\text{amp}} \subset M_{\text{reg}}$.

3.3 L^2 estimate to $\bar{\partial}$-operator

Let $L = \ell_0(L^{'} + TK_M)$ be the limit line bundle. Up to raising the power ℓ_0, we assume that (i) L is semi-ample and (ii) $L - K_M$ is ample. The later follows from $L - K_M = \ell_0(L^{'} + \tau K_M)$ where $L' + \tau K_M$ is ample because $\tau = T - \frac{1}{\ell_0} < T$.

Let η_T, u_t be defined as in the previous section. Choose a Hermitian metric $h_{L'}$ on $L^{'}$ whose curvature form $\Theta_{h_{L'}} = \omega_0$ and put $h_t = h_{L'}^{\ell_0} \otimes \big(\omega_t^{-n}\big)^{\ell_0 T}$, a family of Hermitian metric on L for any $0 \leq t < T$. The curvature form of h_t is

$$\Theta_{h_t} = \ell_0 \frac{T}{t}\omega_t - \ell_0 \frac{T-t}{t}\omega_0. \tag{3.10}$$

Lemma 3.13. *For any $k \geq 1$ and smooth section $\xi \in \Gamma(T^{0,1} \otimes L^k)$ we have*

$$\int_M \big(|\bar{\partial}\xi|^2 + |\bar{\partial}^*\xi|^2\big)\omega_t^n \geq \frac{k\ell_0 T - 1}{t} \int_M |\xi|^2 \omega_t^n,\ \forall t \in [T - \frac{1}{k\ell_0}, T). \tag{3.11}$$

Proof. Combining with the curvature formulas (3.10) and $\text{Ric}(\omega_t) = \frac{\omega_0 - \omega_t}{t}$ derives

$$\Theta_{h_t^k} + \text{Ric}(\omega_t) = \frac{\omega_0}{t}\big(1 - k\ell_0(t-T)\big) + \frac{\omega_t}{t}(k\ell_0 T - 1).$$

Then apply the Weitzenböch formula (3.3). □

Proposition 3.14 (L^2 estimate)**.** *For any $k \geq \frac{2}{\ell_0 T}$ and $t \in [T - \frac{1}{k\ell_0}, T)$ and $\xi \in C^\infty(M, T^{1,0}M \otimes L^k)$ with $\bar{\partial}\xi = 0$ we can find a solution $\bar{\partial}\varsigma = \xi$ which satisfies*

$$\int_M |\varsigma|^2_{h_t^k} \omega_t^n \leq \frac{2}{k} \int_M |\xi|^2_{h_t^k \otimes \omega_t} \omega_t^n. \tag{3.12}$$

Proof. By above corollary, the Hodge Laplacian $\triangle_{\bar{\partial}}$ on $T^{1,0}M \otimes L^k$ is strictly positive, in fact $\triangle_{\bar{\partial}} \geq \frac{k\ell_0 T - 1}{t} \geq \frac{k}{2}$, when $k \geq \frac{2}{\ell_0 T}$. This in turn implies the first positive eigenvalue of $\triangle_{\bar{\partial}}$ on L is bigger than $\frac{k}{2}$. The solvability of $\bar{\partial}\varsigma = \xi$ and the L^2 estimate of the solution are easy from classical Hodge theory. □

3.4 L^∞ estimate to holomorphic sections

Recall that the curvature of h_t

$$\Theta_{h_t} = \ell_0 \frac{T}{t}\omega_t - \ell_0 \frac{T-t}{t}\omega_0 \le \ell_0 \frac{T}{t}\omega_t.$$

So, by the Bochner formula (3.1) we have

$$\triangle_{\omega_t} |\varsigma|^2_{h_t^k} \ge |\nabla \varsigma|^2_{h_t^k \omega_t} - k\ell_0 \frac{T}{t} |\varsigma|^2_{h_t^k}, \ \forall \varsigma \in H^0(M; L^k). \tag{3.13}$$

Also recall that we have the following well-known Sobolev inequality: for any $R > 0$, there is $C(R)$ independent of t such that

$$\left(\int_{B_{\omega_t}(x_0,R)} f^{\frac{2n}{n-2}} \omega_t^n \right)^{\frac{n-1}{n}} \le C(R) \int_{B_{\omega_t}(x_0,R)} \big(f^2 + |\nabla f|^2_{\omega_t}\big)\omega_t^n,$$

for all $f \in C_0^1\big(B_{\omega_t}(x_0,R)\big)$.

By a standard iteration argument we have

Lemma 3.15. *For any $R > 0$, there exists $C(R)$ independent of t and $k \ge 1$ such that for any $\frac{T}{2} \le t < T$ and $B_{\omega_t}(x,2r) \subset B_{\omega_t}(x_0,R)$, if $\varsigma \in H^0(B_{\omega_t}(x,2r); L^k)$, then*

$$\sup_{B_{\omega_t}(x,r)} |\varsigma|^2_{h_t^k} \le C(R) \cdot r^{-2n} \cdot k^n \int_{B_{\omega_t}(x,2r)} |\varsigma|^2_{h_t^k} \omega_t^n. \tag{3.14}$$

Recall the Cheeger-Gromov convergence

$$(M, \omega_t, x_0) \xrightarrow{d_{GH}} (M_T, d_T, x_T). \tag{3.15}$$

Define the Hermitian line bundle (L_T, h_T) on the regular set $\mathcal{R} \subset M_T$ by

$$L_T = \ell_0(L' + TK_{\mathcal{R}}),\ h_T = h_{L'}^{\ell_0} \otimes \bar{\omega}_T^{-n\ell_0 T}.$$

Under the isometry $\overline{\mathrm{id}} : \overline{(M_{\mathrm{reg}}, d_{\omega_T})} \to (M_T, d_T)$ constructed in Subsection 3.2, the Hermitian line bundle is isometric to $(L, h_{L'}^{\ell_0} \otimes \omega_T^{-n\ell_0 T})$ on M_{reg}. Furthermore, the Hermitian line bundles (L, h_t) converges smoothly to (L_T, h_T) on $\mathcal{R}$.

Corollary 3.16. *Let $R > 0$ be any constant, $t_i \to T$ be any subsequence of times and ς_i be a sequence of holomorphic sections of L^k, $k \ge 1$, satisfying*

$$\int_M |\varsigma_i|^2_{h_{t_i}^k} \omega_{t_i}^n \le 1. \tag{3.16}$$

Then, passing to a subsequence if necessary, ς_i converges to a locally bounded holomorphic section ς_∞ of L_T^k over $\mathcal{R}$ which satisfies

$$\sup_{B_{d_T}(x,r)\cap\mathcal{R}} |\varsigma_\infty|^2_{h_T^k} \le C(R) \cdot r^{-2n} \cdot k^n \int_{B_{d_T}(x,2r)\cap\mathcal{R}} |\varsigma_\infty|^2_{h_T^k} \bar{\omega}_T^n. \tag{3.17}$$

whenever $B_{d_T}(x,r) \subset B_{d_T}(x_T,R)$.

3.5 Gradient estimate to holomorphic sections: a prior L^∞ finiteness

Next we derive the gradient estimate to holomorphic sections of L^k. Recall that by the Bochner formula (3.2) we get, for any holomorphic section $\varsigma \in H^0(M; L^k)$ and time $t < T$,

$$\begin{aligned}\triangle_{\omega_t}|\nabla\varsigma|^2_{h_t^k\otimes\omega_t} \geq \quad & |\bar{\nabla}^{h_t}\nabla^{h_t}\varsigma|^2_{h_t^k\otimes\omega_t} + |\nabla^{h_t}\nabla^{h_t}\varsigma|^2_{h_t^k\otimes\omega_t} - \frac{(n+2)k\ell_0 T+1}{t}|\nabla\varsigma|^2_{h_t^k\otimes\omega_t} \\ & + k\ell_0(T-t)\,\mathrm{Re}\left(\nabla_i s\langle\varsigma, \nabla_{\bar{i}}\bar{\varsigma}\rangle_{h_t^k\otimes\omega_t}\right) \qquad (3.18)\end{aligned}$$

where s is the scalar curvature of ω_t, all terms are calculated with respect to ω_t and h_t^k. To verify this, just notice that $\mathrm{Ric} = \frac{1}{t}(\omega_0 - \omega_t)$ and

$$-\nabla_j\Theta_{i\bar{j}} = k\ell_0(T-t)\nabla_j R_{i\bar{j}} = k\ell_0(T-t)\nabla_i s.$$

In order to derive a uniform gradient estimate from the formula (3.18) we need a Type I estimate to the scalar curvature under the continuity equation, namely $(T-t)s$ admits a uniform bound. Motivated by the Kähler-Ricci flow this should be true in general. However, as for Kähler-Ricci flow, it is difficult to show the Type I property right now.

Another approach is to consider the gradient estimate to the limit sections of L_T^k on $\mathcal{R}$. However, a prior L^∞ bound of holomorphic sections will be necessary for the limit process. To do this from now on in this subsection we introduce a new Hermitian metric on L, namely,

$$h_{FS} = h_{L'}^{\ell_0} \otimes \Omega^{-\ell_0 T} \qquad (3.19)$$

where $h_{L'}$ is the Hermitian metric on L' whose curvature $\Theta_{h_{L'}} = \omega_0$, Ω is the volume form on M whose curvature $\Theta_\Omega = \frac{1}{T}(\omega_0 - \eta_T)$. The metric h_{FS} has curvature

$$\Theta_{h_{FS}} = \ell_0\eta_T, \qquad (3.20)$$

where η_T is the induced Fubini-Study metric which satisfies $\eta_T \leq C\omega_t$ for some C independent of t; see Section 2. An easy calculation shows that for any $0 < t < T$,

$$h_t = e^{-\ell_0\frac{T}{t}u_t} h_{FS}, \qquad (3.21)$$

so h_{FS} is uniformly equivalent to h_t for any $\frac{T}{2} \leq t < T$.

In the following computation we denote $\nabla\varsigma = \nabla^{h_{FS}^k}\varsigma$, $\bar{\nabla}\nabla\varsigma = \bar{\nabla}^{h_{FS}^k}\nabla^{h_{FS}^k}\varsigma$ and $|\nabla\varsigma| = |\nabla^{h_{FS}^k}\varsigma|_{h_{FS}^k\otimes\omega_t}$, etc., for any $\varsigma \in H^0(M; L^k)$, $k \geq 1$.

Lemma 3.17. *For any $\frac{T}{2} \leq t < T$ and $\varsigma \in H^0(M, L^k)$, $k \geq 1$, we have*

$$\triangle|\varsigma|^2 \geq |\nabla\varsigma|^2 - Ck|\varsigma|^2, \qquad (3.22)$$

and

$$\triangle|\nabla\varsigma|^2 \geq |\bar{\nabla}\nabla\varsigma|^2 + |\nabla\nabla\varsigma|^2 - k\ell_0\nabla_j(\eta_T)_{i\bar{j}}\langle\varsigma, \nabla_{\bar{i}}\bar{\varsigma}\rangle - k\ell_0\nabla_{\bar{j}}(\mathrm{tr}_\omega\,\eta_T)\langle\nabla_j\varsigma, \bar{\varsigma}\rangle - Ck|\nabla\varsigma|^2, \qquad (3.23)$$

where C does not depend on t, k and ς.

Proof. (3.22) is a direct consequence of the Bochner formula (3.1). (3.23) follows from the Bochner formula (3.2)

$$\begin{aligned}\triangle|\nabla\varsigma|^2 &= |\bar\nabla\nabla\varsigma|^2+|\nabla\nabla\varsigma|^2-k\ell_0\nabla_j(\eta_T)_{i\bar j}\langle\varsigma,\nabla_{\bar i}\bar\varsigma\rangle-k\ell_0\nabla_{\bar j}(\mathrm{tr}_\omega\,\eta_T)\langle\nabla_j\varsigma,\bar\varsigma\rangle\\ &\quad +R_{i\bar j}\langle\nabla_j\varsigma,\nabla_{\bar i}\bar\varsigma\rangle-2k\ell_0(\eta_T)_{i\bar j}\langle\nabla_j\varsigma,\nabla_{\bar i}\bar\varsigma\rangle-k\ell_0|\nabla\varsigma|^2\cdot\mathrm{tr}_\omega\,\eta_T.\end{aligned}$$

and $\mathrm{Ric}(\omega_t)=\frac{1}{t}(\omega_0-\omega_t)\geq-\frac{1}{t}\omega_t$. □

Proposition 3.18. *For any $R>0$, there exists $C(R)$ independent of t and $k\geq 1$ such that for any $\frac{T}{2}\leq t<T$ and $B_{\omega_t}(x,2r)\subset B_{\omega_t}(x_0,R)$, if $\varsigma\in H^0(B_{\omega_t}(x,2r);L^k)$, then*

$$\sup_{B_{\omega_t}(x,r)}|\varsigma|^2_{h^k_{FS}}\leq C(R)\cdot r^{-2n}\cdot k^n\int_{B_{\omega_t}(x,2r)}|\varsigma|^2_{h^k_{FS}}\omega_t^n. \tag{3.24}$$

$$\sup_{B_{\omega_t}(x,r)}|\nabla^{h_{FS}}\varsigma|^2_{h^k_{FS}\otimes\omega_t}\leq C(R)\cdot r^{-2n-2}\cdot k^{n+1}\int_{B_{\omega_t}(x,2r)}|\varsigma|^2_{h^k_{FS}}\omega_t^n. \tag{3.25}$$

Proof. The proof uses Nash-Moser iteration. We only prove the gradient estimate, the L^∞ estimate is obvious by (3.22). For simplicity we may assume

$$\int_{B_{\omega_t}(x,2r)}|\varsigma|^2\omega_t^n=1.$$

Then the L^∞ estimate of ς shows

$$\sup_{B_{\omega_t}(x,\frac{7}{4}r)}|\varsigma|\leq C(R)\cdot r^{-n}k^{\frac{n}{2}}.$$

Keep in mind that $\eta_T\leq C\omega_t$ uniformly for any $\frac{T}{2}\leq t<T$. Now, for any $p\geq\frac{n}{n-1}$ and cut-off $\rho\in C_0^\infty(B_{\omega_t}(x,2r))$ with $0\leq\rho\leq 1$, we have by (3.23) and integration by parts,

$$\begin{aligned}\int_M\rho^2\big|\nabla|\nabla\varsigma|^p\big|^2 &= \frac{p^2}{4(p-1)}\int\rho^2\nabla_i|\nabla\varsigma|^{2(p-1)}\nabla_{\bar i}|\nabla\varsigma|^2\\ &= \frac{p^2}{4(p-1)}\int\Big(-\rho^2|\nabla\varsigma|^{2(p-1)}\triangle|\nabla\varsigma|^2-2\rho|\nabla\varsigma|^{2(p-1)}\nabla_i\rho\nabla_{\bar i}|\nabla\varsigma|^2\Big)\\ &\leq \frac{p^2}{4(p-1)}\int\Big(-\rho^2|\nabla\varsigma|^{2(p-1)}\big(|\bar\nabla\nabla\varsigma|^2+|\nabla\nabla\varsigma|^2\big)\\ &\quad +k\ell_0\rho^2|\nabla\varsigma|^{2(p-1)}\big(\nabla_j(\eta_T)_{i\bar j}\langle\varsigma,\nabla_{\bar i}\bar\varsigma\rangle+\nabla_{\bar j}(\mathrm{tr}_\omega\,\eta_T)\langle\nabla_j\varsigma,\bar\varsigma\rangle\big)\\ &\quad +Ck\rho^2|\nabla\varsigma|^{2p}-2\rho|\nabla\varsigma|^{2(p-1)}\nabla_i\rho\nabla_{\bar i}|\nabla\varsigma|^2\Big).\end{aligned}$$

The term $\int\rho^2|\nabla\varsigma|^{2(p-1)}\nabla_j(\eta_T)_{i\bar j}\langle\varsigma,\nabla_{\bar i}\bar\varsigma\rangle$ can be estimated by integration by parts as follows

$$
\begin{aligned}
& \int k\ell_0\rho^2|\nabla\varsigma|^{2(p-1)}\nabla_j(\eta_T)_{i\bar{j}}\langle\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle \\
= \;& -k\ell_0\int(\eta_T)_{i\bar{j}}\nabla_j\big(\rho^2|\nabla\varsigma|^{2(p-1)}\langle\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle\big) \\
= \;& -k\ell_0\int(\eta_T)_{i\bar{j}}\Big(\rho^2|\nabla\varsigma|^{2(p-1)}\big(\langle\nabla_j\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle+\langle\varsigma,\nabla_j\nabla_{\bar{i}}\bar{\varsigma}\rangle\big) \\
& +(p-1)\rho^2|\nabla\varsigma|^{2(p-2)}\nabla_j|\nabla\varsigma|^2\langle\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle+2\rho|\nabla\varsigma|^{2(p-1)}\nabla_j\rho\langle\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle\Big) \\
\leq \;& \frac{1}{8}\int\rho^2|\nabla\varsigma|^{2(p-1)}\big(|\bar{\nabla}\nabla\varsigma|^2+|\nabla\nabla\varsigma|^2\big)+C(p-1)^2k^2\int\rho^2|\varsigma|^2|\nabla\varsigma|^{2(p-1)} \\
& \qquad +Ck\int\Big(|\nabla\rho|^2|\varsigma|^2|\nabla\varsigma|^{2(p-1)}+\rho^2|\nabla\varsigma|^{2p}\Big),
\end{aligned}
$$

where we used in the last inequality that $0\leq\eta_T\leq C\omega_t$, cf. (2.14), and the Cauchy-Schwarz inequality, such as $|(\eta_T)_{i\bar{j}}\langle\nabla_j\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle|\leq C|\nabla\varsigma|^2$; similar estimate remains valid for $\int k\ell_0\rho^2|\nabla\varsigma|^{2(p-1)}\nabla_{\bar{j}}(\mathrm{tr}_\omega\,\eta_T)\langle\nabla_j\varsigma,\bar{\varsigma}\rangle$; moreover,

$$
\begin{aligned}
& -2\int\rho|\nabla\varsigma|^{2(p-1)}\nabla_i\rho\nabla_{\bar{i}}|\nabla\varsigma|^2 \\
\leq \;& \frac{1}{4}\int\rho^2|\nabla\varsigma|^{2(p-1)}\big(|\bar{\nabla}\nabla\varsigma|^2+|\nabla\nabla\varsigma|^2\big)+\int|\nabla\rho|^2|\nabla\varsigma|^{2p}.
\end{aligned}
$$

Summing up the estimates we get

$$
\begin{aligned}
\int\rho^2\big|\nabla|\nabla\varsigma|^p\big|^2 \;\leq\; & Cp^3k\int\Big(\rho^2|\nabla\varsigma|^{2p}+|\nabla\rho|^2|\varsigma|^2|\nabla\varsigma|^{2(p-1)} \\
& +|\nabla\rho|^2|\nabla\varsigma|^{2p}+k\rho^2|\varsigma|^2|\nabla\varsigma|^{2(p-1)}\Big).
\end{aligned}
$$

Applying the Sobolev inequality we have

$$
\begin{aligned}
\Big(\int(\rho|\nabla\varsigma|^p)^{\frac{2n}{n-1}}\Big)^{\frac{n-1}{n}} \;\leq\; & Cp^3k\int\Big(\rho^2|\nabla\varsigma|^{2p}+|\nabla\rho|^2|\varsigma|^2|\nabla\varsigma|^{2(p-1)} \\
& +|\nabla\rho|^2|\nabla\varsigma|^{2p}+k\rho^2|\varsigma|^2|\nabla\varsigma|^{2(p-1)}\Big),
\end{aligned}
$$

where $C=C(R)$. We next consider three independent cases to run the iteration. Put $p_j=\nu^{j+1}$, $j\geq 0$, where $\nu=\frac{n}{n-1}$. Define a family of radius inductively by $r_0=\frac{3}{2}r$ and $r_j=r_{j-1}-2^{-j-1}r$, and a family of cut-offs $\rho_j\in C_0^\infty(B_j)$ where $B_j=B_{\omega_t}(x_0,r_j)$ such that

$$0\leq\rho_j\leq 1,\ |\nabla\rho_j|\leq 2^{j+2}r^{-1} \text{ and } \rho_j=1 \text{ on } B_{j+1}.$$

Above formula finally gives, by setting $\rho=\rho_j$,

$$
\Big(\int_{B_{j+1}}|\nabla\varsigma|^{2p_{j+1}}\Big)^{\frac{n-1}{n}}\leq Cp_j^3 2^{2j}kr^{-2}\int_{B_j}\big(|\nabla\varsigma|^{2p_j}+k|\varsigma|^2|\nabla\varsigma|^{2(p_j-1)}\big),\ \forall j\geq 0. \tag{3.26}
$$

Case 1: $\left(\int_{B_j} |\nabla\varsigma|^{2p_j}\right)^{\frac{1}{p_j}} \geq k\left(\int_{B_j} |\varsigma|^{2p_j}\right)^{\frac{1}{p_j}}$, for all $j \geq 0$. Then,

$$\int_{B_j} k|\varsigma|^2|\nabla\varsigma|^{2(p_j-1)} \leq k\left(\int_{B_j} |\nabla\varsigma|^{2p_j}\right)^{1-\frac{1}{p_j}}\left(\int_{B_j} |\varsigma|^{2p_j}\right)^{\frac{1}{p_j}} \leq \int_{B_j} |\nabla\varsigma|^{2p_j}.$$

Then (3.26) gives

$$\left(\int_{B_{j+1}} |\nabla\varsigma|^{2p_{j+1}}\right)^{\frac{1}{p_{j+1}}} \leq (Ckr^{-2})^{\frac{1}{p_j}} p_j^{\frac{3}{p_j}} 4^{\frac{j}{p_j}} \left(\int_{B_j} |\nabla\varsigma|^{2p_j}\right)^{\frac{1}{p_j}}, \forall j \geq 0.$$

By an easy iteration,

$$\left\||\nabla\varsigma|^2\right\|_{L^{p_j}(B_j)} \leq (Ckr^{-2})^{\sum \nu^{-i}} \cdot \Pi p_i^{\frac{3}{p_i}} \cdot 4^{\sum i\nu^{-i}} \cdot \left\||\nabla\varsigma|^2\right\|_{L^{p_0}(B_0)}, \forall j \geq 0.$$

In particular by letting $j \to \infty$ we obtain

$$\sup_{B_{\omega_t}(x,r)} |\nabla\varsigma|^2 \quad \leq \quad (Ckr^{-2})^{n-1} \cdot C(n) \cdot \left(\int_{B_0} |\nabla\varsigma|^{2\nu}\right)^{\frac{1}{\nu}}$$

Finally, a cut-off argument gives

$$\left(\int_{B_0} |\nabla\varsigma|^{2\nu}\right)^{\frac{1}{\nu}} \leq C(R)\int_{B_{\omega_t}(x,\frac{7}{4}r)\setminus D} \left(|\bar{\nabla}\nabla\varsigma|^2 + |\nabla\nabla\varsigma|^2 + r^{-2}|\nabla\varsigma|^2\right) \leq C(R)k^{-2}r^{-4}.$$

In the last inequality we used the lemma below. So we have

$$\sup_{B_{\omega_t}(x,r)} |\nabla\varsigma|^2 \leq C(R)(kr^{-2})^{n+1}.$$

Case 2: There exists $j_0 \geq 0$ such that $\left(\int_{B_j} |\nabla\varsigma|^{2p_j}\right)^{\frac{1}{p_j}} \geq k\left(\int_{B_j} |\varsigma|^{2p_j}\right)^{\frac{1}{p_j}}$ for all $j > j_0$ but

$$\left(\int_{B_{j_0}} |\nabla\varsigma|^{2p_{j_0}}\right)^{\frac{1}{p_{j_0}}} < k\left(\int_{B_{j_0}} |\varsigma|^{2p_{j_0}}\right)^{\frac{1}{p_{j_0}}}.$$

Then,

$$\int_{B_{j_0}} k|\varsigma|^2|\nabla\varsigma|^{2(p_{j_0}-1)} \leq k^{p_{j_0}} \int_{B_{j_0}} |\varsigma|^{2p_{j_0}}.$$

By a same iteration for $j \geq j_0 + 1$ as above we have,

$$\begin{aligned}
\sup_{B_{\omega_t}(x,r)\setminus D} |\nabla\varsigma|^2 \quad &\leq \quad (Ckr^{-2})^{\frac{1}{p_{j_0+1}}\sum \nu^{-i}} \cdot C(n) \cdot \left(\int_{B_{j_0+1}} |\nabla\varsigma|^{2p_{j_0+1}}\right)^{\frac{1}{p_{j_0+1}}} \\
&\leq \quad (Ckr^{-2})^{\frac{1}{p_{j_0}}\sum \nu^{-i}} \cdot \left(\int_{B_{j_0}} |\nabla\varsigma|^{2p_{j_0}} + k|\varsigma|^2|\nabla\varsigma|^{2(p_{j_0}-1)}\right)^{\frac{1}{p_{j_0}}} \\
&\leq \quad (Ckr^{-2})^{\frac{n}{p_{j_0}}} \cdot k \cdot \left(\int_{B_{j_0}} |\varsigma|^{2p_{j_0}}\right)^{\frac{1}{p_{j_0}}}.
\end{aligned}$$

The supremum estimate of $|\nabla\varsigma|$ follows from

$$\Big(\int_{B_{j_0}}|\varsigma|^{2p_{j_0}}\Big)^{\frac{1}{p_{j_0}}}\leq\big(\sup_{B_{j_0}}|\varsigma|\big)^{\frac{2(p_{j_0}-1)}{p_{j_0}}}\Big(\int_{B_{j_0}}|\varsigma|^2\Big)^{\frac{1}{p_{j_0}}}\leq(Ckr^{-2})^{n-\frac{n}{p_{j_0}}}.$$

Case 3: $\big(\int_{B_j}|\nabla\varsigma|^{2p_j}\big)^{\frac{1}{p_j}}\leq k\big(\int_{B_j}|\varsigma|^{2p_j}\big)^{\frac{1}{p_j}}$ for all $j\geq 0$. It is obvious that

$$\sup_{B_{\omega_t}(x,r)}|\nabla\varsigma|\leq k\sup_{B_{\omega_t}(x,r)}|\varsigma|\leq Ck^{n+1}r^{-2n}.$$

Summing up the three cases we conclude the gradient estimate on $B_{\omega_t}(x,r)$. □

Lemma 3.19. *Assume as in above proposition. We have*

$$\int_{B_{\omega_t}(x,\frac{7}{4}r)}|\nabla\varsigma|^2\omega_t^n\leq Ckr^{-2}\int_{B_{\omega_t}(x,2r)}|\varsigma|^2\omega_t^n,\tag{3.27}$$

and

$$\int_{B_{\omega_t}(x,\frac{7}{4}r)}(|\bar\nabla\nabla\varsigma|^2+|\nabla\nabla\varsigma|^2)\omega_t^n\leq Ck^2r^{-4}\int_{B_{\omega_t}(x,2r)}|\varsigma|^2_{h_t}\omega_t^n.\tag{3.28}$$

where C is a constant independent of t, k and $\varsigma\in H^0(B_{\omega_t}(x,2r),L^k)$.

Proof. Let $\rho\in C_0^\infty(B_{\omega_t}(x,\frac{15}{8}r))$ be any cut-off such that $0\leq\rho\leq 1$, $|\nabla\rho|\leq 100r^{-2}$ and $\rho=1$ on $B_{\omega_t}(x,\frac{7}{4}r)$, then by (3.22),

$$\int\rho^2|\nabla\varsigma|^2\leq Ck\int\rho^2|\varsigma|^2+\int\rho^2\triangle|\varsigma|^2$$

where

$$\int\rho^2\triangle|\varsigma|^2=-2\int\rho\nabla_{\bar i}\rho\langle\nabla_i\varsigma,\bar\varsigma\rangle\leq\frac{1}{2}\int\rho^2|\nabla\varsigma|^2+2\int|\nabla\varsigma|^2|\varsigma|^2.$$

Thus,

$$\int\rho^2|\nabla\varsigma|^2\leq Ckr^{-2}\int_{B_{\omega_t}(x,\frac{15}{8}r)}|\varsigma|^2.$$

The first estimate follows. Then, by (3.23),

$$\begin{aligned}&\int\rho^2\big(|\bar\nabla\nabla\varsigma|^2+|\nabla\nabla\varsigma|^2\big)\\ \leq\ &\int\rho^2\Big(\triangle|\nabla\varsigma|^2+k\ell_0\nabla_j(\eta_T)_{i\bar j}\langle\varsigma,\nabla_{\bar i}\bar\varsigma\rangle+k\ell_0\nabla_{\bar j}(\mathrm{tr}_\omega\,\eta_T)\langle\nabla_j\varsigma,\bar\varsigma\rangle\Big)+Ck\int\rho^2|\nabla\varsigma|^2,\end{aligned}$$

where

$$\int\rho^2\triangle|\nabla\varsigma|^2=-2\int\rho\nabla_i\rho\nabla_{\bar i}|\nabla\varsigma|^2\leq\frac{1}{4}\int\rho^2\big(|\bar\nabla\nabla\varsigma|^2+|\nabla\nabla\varsigma|^2\big)+8\int|\nabla\rho|^2|\nabla\varsigma|^2,$$

$$\begin{aligned}\int k\ell_0\rho^2\nabla_j(\eta_T)_{i\bar{j}}\chi^2\langle\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle &= -\int k\ell_0\rho^2(\eta_T)_{i\bar{j}}\big(\langle\nabla_j\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle+\langle\varsigma,\nabla_j\nabla_{\bar{i}}\bar{\varsigma}\rangle\big)\\ &\quad -2k\ell_0\int\rho(\eta_T)_{i\bar{j}}\nabla_j\rho\langle\varsigma,\nabla_{\bar{i}}\bar{\varsigma}\rangle\\ &\le \frac{1}{4}\int\rho^2|\bar{\nabla}\nabla\varsigma|^2+Ck^2\int\rho^2|\varsigma|^2+C\int|\nabla\rho|^2|\nabla\varsigma|^2,\end{aligned}$$

the estimate to the integrand $\nabla_{\bar{j}}(\mathrm{tr}_\omega\,\eta_T)\langle\nabla_j\varsigma,\bar{\varsigma}\rangle$ has the same form. Summing up these we get

$$\int_{B_{\omega_t}(x,\frac{7}{4}r)}\big(|\bar{\nabla}\nabla\varsigma|^2+|\nabla\nabla\varsigma|^2\big)\le Ckr^{-2}\int_{B_{\omega_t}(x,\frac{15}{8}r)}|\nabla\varsigma|^2+Ck^2\int_{B_{\omega_t}(x,2r)}|\varsigma|^2.$$

Then use the first estimate once again on $B_{\omega_t}(x,\frac{15}{8}r)$ to get the second estimate. □

3.6 Gradient estimate to holomorphic sections: estimate to the limit sections

Lemma 3.20. *There is a family of cut-offs $\gamma_\epsilon\in C_0^\infty(\mathcal{R})$, $\epsilon>0$, with $0\le\gamma_\epsilon\le 1$ such that $\gamma_\epsilon^{-1}(1)$ forms an exhaustion of $\mathcal{R}$ and, moreover,*

$$\int_{M_T}|\bar{\partial}\gamma_\epsilon|^2\bar{\omega}_T^n\to 0,\quad as\ \epsilon\to 0. \tag{3.29}$$

Proof. Notice that by Proposition 3.12, the regular set $(\mathcal{R},\bar{\omega}_T)$ can be identified with $(M_{\mathrm{reg}},\omega_T)$ where M_{reg} is the regular set of the holomorphic map $\Phi: M\to\mathbb{C}P^N$. The metric $\omega_T=\eta_T+\sqrt{-1}\partial\bar{\partial}u_T$ where u_T is a bounded potential. Then the lemma is more or less standard in the pluripotential theory. For a detailed proof we refer to [41, Lemma 6.4]; see also [42] or [28, Lemma 3.7]. □

By a standard iteration we have (cf. [28])

Proposition 3.21. *Let $R>0$ be any constant, $t_i\to T$ be any subsequence of times and ς_i be a sequence of holomorphic sections of L^k, $k\ge 1$, satisfying*

$$\int_M|\varsigma_i|^2_{h^k_{t_i}}\omega^n_{t_i}\le 1. \tag{3.30}$$

Then, passing to a subsequence if necessary, ς_i converges to a locally bounded holomorphic section ς_∞ of L_T^k over $\mathcal{R}$ which satisfies

$$\sup_{B_{d_T}(x,r)\cap\mathcal{R}}|\nabla^{h_T}\varsigma_\infty|^2_{h_T^k\otimes\bar{\omega}_T}\le C(R)\cdot r^{-2n-2}\cdot k^{n+1}\int_{B_{d_T}(x,2r)\cap\mathcal{R}}|\varsigma_\infty|^2_{h_T^k}\bar{\omega}_T^n, \tag{3.31}$$

whenever $B_{d_T}(x,r)\subset B_{d_T}(x_T,R)$.

Proof. We present a brief discussion to the proof of the gradient estimate. Let $h_{FS}=e^{\ell_0u_T}h_T$ be another Hermitian metric on $(\mathcal{R},L_T)$, which is exactly the limit of h_{FS} on (M,L) under the Gromov-Hausdorff convergence. By Proposition 3.18, we have

$$|\nabla^{h_{FS}}\varsigma_\infty|_{h^k_{FS}\otimes\bar{\omega}_T}\le C,\ \text{ over } B_{d_T}(x,\frac{3}{2}r)\cap\mathcal{R}.$$

Then, apply the C^0 estimate and gradient estimate to u_T, cf. (2.21), we obtain a uniform gradient estimate of ς_∞ with respect to h_T,

$$|\nabla^{h_T}\varsigma_\infty|_{h_T^k\otimes\bar\omega_T} \le |\nabla^{h_{FS}}\varsigma_\infty|_{h_T^k\otimes\bar\omega_T} + \ell_0\cdot|du_T\otimes\varsigma_\infty|_{h_T^k\otimes\bar\omega_T} \le C$$

over $B_{d_T}(x,\frac{3}{2}r)\cap\mathcal{R}$. Recall that, by (3.18) and smooth convergence on $\mathcal{R}$, we have

$$\triangle_{\bar\omega_T}|\nabla^{h_T}\varsigma_\infty|^2_{h_T^k\otimes\bar\omega_T} \ge |\bar\nabla^{h_T}\nabla^{h_T}\varsigma_\infty|^2_{h_T^k\otimes\bar\omega_T} + |\nabla^{h_T}\nabla^{h_T}\varsigma_\infty|^2_{h_T^k\otimes\bar\omega_T} - C'\cdot k|\nabla^{h_T}\varsigma_\infty|^2_{h_T^k\otimes\bar\omega_T}$$

for some $C'=C'(n,\ell_0,T)$ over $\mathcal{R}$. Let γ_ϵ be the family of cut-offs as in above lemma. Then, for any function ρ supported on $B_{d_T}(x,\frac{3}{2}r)$ we have for any $p\ge 1$,

$$\begin{aligned}
&\int_{M_T}(\rho\gamma_\epsilon)^2\big|\nabla|\nabla\varsigma_\infty|^p\big|^2\\
=\;&\frac{p^2}{4(p-1)}\int(\rho\gamma_\epsilon)^2\nabla_i|\nabla\varsigma_\infty|^{2(p-1)}\nabla_{\bar i}|\nabla\varsigma_\infty|^2\\
=\;&\frac{p^2}{4(p-1)}\int\Big(-(\rho\gamma_\epsilon)^2|\nabla\varsigma_\infty|^{2(p-1)}\triangle|\nabla\varsigma_\infty|^2 - 2\rho\gamma_\epsilon|\nabla\varsigma_\infty|^{2(p-1)}\nabla_i(\rho\gamma_\epsilon)\nabla_{\bar i}|\nabla\varsigma_\infty|^2\Big)\\
\le\;&\frac{p^2}{4(p-1)}\int\Big(-(\rho\gamma_\epsilon)^2|\nabla\varsigma_\infty|^{2(p-1)}\big(|\bar\nabla\nabla\varsigma_\infty|^2+|\nabla\nabla\varsigma_\infty|^2\big)+C'k(\rho\gamma_\epsilon)^2|\nabla\varsigma_\infty|^{2p}\\
&+2\rho\gamma_\epsilon|\nabla\varsigma_\infty|^{2(p-1)}|\nabla(\rho\gamma_\epsilon)|\big(|\bar\nabla\nabla\varsigma_\infty|^2+|\nabla\nabla\varsigma_\infty|^2\big)|\nabla\varsigma_\infty|\Big)\\
\le\;&\frac{p^2}{4(p-1)}\int\Big(C'k(\rho\gamma_\epsilon)^2|\nabla\varsigma_\infty|^{2p}+|\nabla(\rho\gamma_\epsilon)|^2|\nabla\varsigma_\infty|^{2p}\Big).
\end{aligned}$$

Since $|\nabla\varsigma_\infty|$ is uniformly bounded on $B_{d_T}(x,\frac{3}{2}r)$, by letting $\epsilon\to 0$ we get

$$\int_{\mathcal{R}}\rho^2\big|\nabla|\nabla\varsigma_\infty|^p\big|^2 \le \frac{C'p^2}{4(p-1)}\int_{\mathcal{R}}\big(k\rho^2|\nabla\varsigma_\infty|^{2p}+|\nabla\rho|^2|\nabla\varsigma_\infty|^{2p}\big).$$

Then applying the standard iteration one can get the gradient estimate to ς_∞. □

3.7 Algebraic structure of M_T

Let $\Phi^k : M \to \mathbb{C}P^{N_\ell}$ be the holomorphic map defined by an orthonormal basis of $H^0(M,L^k)$ with respect to Hermitian metric H_0^k where $N_k = \dim H^0(M;L^k)-1$, H_0 is any fixed Hermitian metric. For any k and time t, the map

$$\Phi_t^k = \Phi^k : (M,\omega_t) \to (\Phi^k(M),\omega_{FS})$$

is Lipschitz with a uniform Lipschitz constant, since $(\Phi_t^k)^*\omega_{FS} = k\ell_0\eta_T \le Ck\ell_0\omega_t$ for any $t\le T$. Let $t_i\to T$ be a sequence of times and $(M,\omega_{t_i},x_0)\xrightarrow{d_{GH}}(M_T,d_T,x_T)$ be the Gromov-Hausdorff convergence considered in Subsection 3.2, then by a diagonalization argument, up to taking a subsequence, $\Phi_{t_i}^k$ converges to a Lipschitz map

$$\Phi_T^k = \lim_{t_i\to T}\Phi_{t_i}^k : (M_T,d_T)\to(\Phi^k(M),\omega_{FS}).$$

Proposition 3.22. Φ_T^{n+1} *is injective.*

If M_T is compact, then Φ_T^{n+1} is a homeomorphism. In general we have

Proposition 3.23. Φ_T^{n+1} *is a local homeomorphism.*

Therefore the limit M_T is locally algebraic. The procedure to prove the propositions are contained in the work of Donaldson-Sun [12] and Tian [40], [41]. We outline a proof following [41].

The Hermitian line bundles (L, h_{t_i}) have curvature $\Theta_{h_{t_i}} = \ell_0 \frac{T}{t_i}\omega_{t_i} - \ell_0 \frac{T-t_i}{t_i}\omega_0$. As $t_i \to T$, the Hermitian line bundles (L, h_{t_i}) converges smoothly to a limit Hermitian line bundle (L_T, h_T) over $\mathcal{R}$ whose curvature

$$\Theta_{h_T} = \ell_0 \bar{\omega}_T, \ \text{ on } \mathcal{R}. \tag{3.32}$$

Step 1: Construction of local approximating holomorphic sections.

Let $p \in M_T$ be any point. Let $r_j \to 0$ be a decreasing sequence of radius, $k_j = r_j^{-2} \in \mathbb{Z}$, and $\mathcal{C}_p = \lim_{j\to\infty}(M_T, r_j^{-1} d_T, p)$ be a tangent cone at p. By the regularity theory of Cheeger-Colding [5, 6] and Cheeger-Colding-Tian [7], we have

$\mathbf{T}_1$. $\mathcal{C}_p$ is smooth outside a closed subcone $\mathcal{S}_p$ of complex codimension at least 1 which is the singular set of $\mathcal{C}_p$;

$\mathbf{T}_2$. There is a Kähler Ricci-flat cone metric ω_p of the form $\sqrt{-1}\,\partial\bar{\partial}\rho^2$ on $\mathcal{C}_p \backslash \mathcal{S}_p$, where ρ denotes the distance function from the vertex of $\mathcal{C}_p$, denoted by o.

$\mathbf{T}_3$. Denote by L_p the trivial bundle $\mathcal{C}_p \times \mathbb{C}$ over $\mathcal{C}_p$ equipped with the Hermitian metric $e^{-\ell_0\rho^2}|\cdot|^2$. The curvature of this Hermitian metric is given by ω_p.

For any $\epsilon > 0$, we put

$$V(p;\epsilon) = \{\, y \in \mathcal{C}_p \,|\, y \in B_{\epsilon^{-1}}(o,\omega_p) \setminus \overline{B_\epsilon(o,\omega_p)},\ d(y,\mathcal{S}_p) > \epsilon \,\}.$$

For any $\epsilon > 0$ and $\delta > 0$, we can have a $j_0 = j_0(\epsilon,\delta)$ such that $r_{j_0} \le \epsilon^2$, and for each $j \ge j_0$, there is a diffeomorphism $\phi_j : V(p; \frac{\epsilon}{4}) \to M_T \backslash \mathcal{S}$, where $\mathcal{S}$ is the singular set of M_T, satisfying:

(i) $d(p, \phi_j(V(p;\epsilon))) < 10\,\epsilon r_j$ and $\phi_j(V(p;\epsilon)) \subset B_{(1+\epsilon^{-1})r_j}(p)$;

(ii) The Kähler metric ω_T on $M_T \backslash \mathcal{S}$ satisfies

$$||r_j^{-2}\phi_j^*\omega_T - \omega_p||_{C^6(V(p;\frac{\epsilon}{2}))} \le \delta, \tag{3.33}$$

where the norm is defined in terms of the metric ω_p.

Lemma 3.24. *Given* $\epsilon > 0$ *and any sufficiently small* $\delta > 0$*, there are a sufficiently large* j*, a diffeomorphism* $\phi_j : V(p;\frac{\epsilon}{4}) \to M_T \backslash \mathcal{S}$ *with properties* (i) *and* (ii) *above, and an isomorphism* ψ_j *from the trivial bundle* $\mathcal{C}_p \times \mathbb{C}$ *onto* L^{k_j} *over* $V(p;\epsilon)$ *commuting with* ϕ_j *satisfying:*

$$|\psi_j(1)|^2_{h_T} = e^{-\ell_0\rho^2} \quad \text{and} \quad ||\nabla\psi_j||_{C^6(V(p;\epsilon))} \le \delta, \tag{3.34}$$

where ∇ *denotes the covariant derivative with respect to the metrics* h_T *and* $e^{-\ell_0\rho^2}|\cdot|^2$.

We refer the readers to [41, Lemma 5.7] for its proof. The proof uses that the limit line bundle (L_T, h_T) is Hermitian Einstein, namely $\Theta_{h_T} = \ell_0\omega_T$. We also need the following lemma.

Lemma 3.25. *For any $\bar\epsilon > 0$, there is a smooth function $\gamma_{\bar\epsilon}$ on $\mathcal{C}_p$ satisfying:*

(1) $\gamma_{\bar\epsilon}(y) = 1$ *if* $d(y, \mathcal{S}_p) \geq \bar\epsilon$*;*

(2) $0 \leq \gamma_{\bar\epsilon} \leq 1$ *and* $\gamma_{\bar\epsilon}(y) = 0$ *in an neighborhood of* $\mathcal{S}_p$*;*

(3) $|\nabla\gamma_{\bar\epsilon}| \leq C$ *for some constant* $C = C(\bar\epsilon)$ *and*

$$\int_{B_{\bar\epsilon^{-1}}(o,\omega_p)} |\nabla\gamma_{\bar\epsilon}|^2\, \omega_p^n \leq \bar\epsilon.$$

This is exactly Lemma 5.8 of [41]. It holds trivially for two simple cases: (a) $\mathcal{S}_p$ is of codimension at least 4 and (b) the tangent cone $\mathcal{C}_p$ splits as $\mathbb{C}^{n-1} \times \mathcal{C}'_p$ where $\mathcal{C}'_p$ is a 2-dimensional flat cone. The general case when $\mathcal{S}_p$ is of codimension 2 can be proved by recursion based on the construction of peak holomorphic section in Step 2 below; see Appendix A of [41]. If $x \in \mathcal{C}_p$ has an iterated tangent cone of form $\mathbb{C}^{n-1} \times \mathcal{C}'_x$, then the singular set around x is a locally analytical set modeled by taking the limit of the ample locus M_{amp} on the original manifold.

Assuming the two lemmas, one can find for any $\bar\epsilon$ one $0 < \epsilon = \epsilon(\bar\epsilon) < \bar\epsilon$ such that $\mathrm{Supp}(\gamma_{\bar\epsilon}) \subset V(p;\epsilon)$ and then one $j = j(\bar\epsilon)$ satisfying Lemma 3.24 for ϵ. Then $\tau = \psi_j(\gamma_{\bar\epsilon}1)$ extends to a smooth section of $L_T^{k_j}$ on M_T which satisfies:

$$\int_{M_T} |\bar\partial\tau|^2_{h_T^{k_j}\otimes k_j\omega_T} (k_j\omega_T)^n \leq \bar\epsilon.$$

Step 2: Existence of holomorphic peak sections on M.

Let $p \in M_T$ satisfies $d_T(p, x_T) \leq R$. Suppose $p_i \in M$ satisfies $p_i \xrightarrow{d_{GH}} p$ under the Cheeger-Gromov convergence. By the smooth convergence on the regular set $\phi_j(V(p;\epsilon))$, the approximating holomorphic section τ on $L_T^{k_j}$ descends to a family of smooth section of L^{k_j} on M, denoted τ_i, via a family of smooth maps $f_i : \mathcal{R} \to M$ representing the Gromov-Hausdorff convergence $(M, \omega_{t_i}, x_0) \xrightarrow{d_{GH}} (M_T, d_T, x_T)$, which satisfies:

$$\mathrm{Supp}(\tau_i) \subset f_i\big(\phi_j(V(p;\epsilon))\big) \subset B_{k_j\omega_{t_i}}(p_i, 2\sqrt{k_j}\bar\epsilon), \tag{3.35}$$

$$\left| |\tau_i|_{h_{t_i}^{k_j}}(x) - e^{-\ell_0 d^2_{k_j\omega_{t_i}}(x,p_i)} \right| \leq \bar\epsilon, \ \text{ on } f_i\big(\phi_j(V(p;\bar\epsilon))\big), \tag{3.36}$$

and

$$C^{-1} \leq \int_M |\tau_i|^2_{h_{t_i}^{k_j}\otimes k_j\omega_{t_i}} (k_j\omega_{t_i})^n \leq C, \tag{3.37}$$

for some constant $C = C(R)$ depending on the volume ration of the tangent cone $\mathcal{C}_p$, and

$$\int_M |\bar\partial\tau_i|^2_{h_{t_i}^{k_j}\otimes k_j\omega_{t_i}} (k_j\omega_{t_i})^n \leq 2\bar\epsilon, \tag{3.38}$$

for any i sufficiently large. We may assume that

$$f_i\big(\phi_j(V(p;2\bar{\epsilon}^{\frac{1}{4n}}))\big)\cap B_{k_j\omega_{t_i}}(p_i,4\bar{\epsilon}^{\frac{1}{4n}})\neq\emptyset, \tag{3.39}$$

$$B_{k_j\omega_{t_i}}(p_i',\frac{1}{2}\bar{\epsilon}^{\frac{1}{4n}})\subset f_i\big(\phi_j(V(p;\bar{\epsilon}))\big), \tag{3.40}$$

for some $p_i'\in M$ with $d_{k_j\omega_{t_i}}(p_i,p_i')\leq 2\bar{\epsilon}^{\frac{1}{4n}}$. When i is large enough, $T-t_i\leq\frac{1}{2\ell_0 k_j}$, so by the L^2 estimate in Lemma 3.14, there exists a smooth section v_i solving $\bar{\partial}v_i=\bar{\partial}\tau_i$ with

$$\int_M |v_i|^2_{h_{t_i}^{k_j}}(k_j\omega_{t_i})^n\leq C\cdot\bar{\epsilon} \tag{3.41}$$

for some C independent of i. Noticing that the metric ball $B_{k_j\omega_{t_i}}(p_i',\frac{1}{2}\bar{\epsilon}^{\frac{1}{4n}})$ has volume bigger than $C(R)^{-1}\bar{\epsilon}^{\frac{1}{2}}$, by the mean value property we can slightly justify p_i' such that

$$|v_i|^2_{h_{t_i}^{k_j}}(p_i')\leq C\bar{\epsilon}^{\frac{1}{2}}, \tag{3.42}$$

where $C=C(R)$. Therefore, $\sigma_i=\tau_i-v_i$ defines a holomorphic section of L^k satisfying

$$|\sigma_i|_{h_{t_i}^{k_j}}(p_i')\geq e^{-\ell_0 d^2_{k_j\omega_{t_i}}(p_i,p_i')}-\bar{\epsilon}-C\bar{\epsilon}^{\frac{1}{4}}\geq\frac{1}{2} \tag{3.43}$$

once $\bar{\epsilon}=\epsilon(p)$ is chosen sufficiently small. On the other hand, $\sigma_i=v_i$ outside $f_i\big(\phi_j(V(p;\epsilon))\big)$, a domain satisfying

$$\sup_{x\in f_i\left(\phi_j(V(p;\epsilon))\right)} d_{k_j\omega_{t_i}}(p_i,x)\leq\sup_{x\in\phi_j(V(p;\epsilon))} d_{k_jd_T}(p,x)+1\leq\epsilon^{-1}+2.$$

Therefore, for any $x\in M$ with $\epsilon^{-1}+3\leq d_{k_j\omega_{t_i}}(x,p_i)\leq 2k_j^{\frac{1}{2}}R$, an iteration gives

$$|\sigma_i|^2_{h_{t_i}^{k_j}}(x)\leq C\int_{B_{k_j\omega_{t_i}}(x,1)}|\sigma_i|^2_{h_{t_i}^{k_j}}(k_j\omega_{t_i})^n\leq C\cdot\bar{\epsilon}$$

where $C=C(R)$. Noticing that $k_j=r_j^{-2}$ and $r_j\leq\epsilon^2$, we conclude

$$|\sigma_i|_{h_{t_i}^{k_i}}\leq C\bar{\epsilon}^{\frac{1}{2}},\ \text{ on } B_{\omega_{t_i}}(x,2R)\backslash B_{\omega_{t_i}}(p_i,2\epsilon). \tag{3.44}$$

Besides (3.43) and (3.44) the section σ_i also satisfies

$$C^{-1}\leq\int_{B_{\omega_{t_i}}(p_i,2\bar{\epsilon})}|\sigma_i|^2_{h_{t_i}^{k_j}}(k_j\omega_{t_i})^n\leq C \tag{3.45}$$

and

$$\int_{M\backslash B_{\omega_{t_i}}(p_i,2\bar{\epsilon})}|\sigma_i|^2_{h_{t_i}^{k_j}}(k_j\omega_{t_i})^n\leq C\cdot\bar{\epsilon} \tag{3.46}$$

for some $C = C(R)$ independent of the specified i, j and ς.

Passing to a subsequence if necessary, the sequence of points p_i' converge to a point $p' \in \mathcal{R}$ with $d_T(p, p') \leq 2k_j^{-\frac{1}{2}} \bar{\epsilon}^{\frac{1}{4n}}$, the sections $\sigma_i \in H^0(M; L^k)$ converges to a holomorphic section $\sigma_\infty \in H^0(\mathcal{R}; L_T^k)$ such that

$$|\sigma_\infty|_{h_T^{k_j}}(p') \geq \frac{1}{2}, \tag{3.47}$$

$$|\sigma_\infty|_{h_T^{k_i}} \leq C \cdot \bar{\epsilon}^{\frac{1}{2}}, \ \text{ on } B_{d_T}(x_T, 2R) \backslash B_{d_T}(p, 2\epsilon), \tag{3.48}$$

$$C^{-1} \leq \int_{B_{d_T}(p,2\bar{\epsilon})\cap\mathcal{R}} |\sigma_\infty|^2_{h_T^{k_j}} (k_j \bar{\omega}_T)^n \leq C \tag{3.49}$$

and

$$\int_{\mathcal{R}\backslash B_{d_T}(p,2\bar{\epsilon})} |\sigma_\infty|^2_{h_T^{k_j}} (k_j \bar{\omega}_T)^n \leq C \cdot \bar{\epsilon} \tag{3.50}$$

for some $C = C(R)$. By the gradient estimate in Proposition 3.21 we have

$$|\sigma_\infty|_{h_T^{k_j}}(p) \geq \frac{1}{2} - C\bar{\epsilon}^{\frac{1}{4n}} \geq \frac{1}{4} \tag{3.51}$$

if $\bar{\epsilon}$ is chosen sufficiently small.

Step 3: Φ_T^{n+1} is injective.

For any $R > 0$, $p, q \in B_{d_T}(x_T, R)$, and any $\bar{\epsilon} << d_T(p, q)$, there is an integer $k = k(p, q)$ and sections $\sigma_p \in H^0(\mathcal{R}; L_T^k)$, $\sigma_q \in H^0(\mathcal{R}; L_T^k)$ such that (3.48)-(3.51) hold respectively at p_i, q_i, for $C = C(R)$. Notice that by Schwarz inequality,

$$\begin{aligned}
\left| \int_{\mathcal{R}} \langle \sigma_p, \bar{\sigma}_q \rangle_{h_T^k} \bar{\omega}_T^n \right| &\leq \int_{\mathcal{R}\backslash B_{d_T}(q,2\bar{\epsilon})} |\sigma_p||\sigma_q| \bar{\omega}_T^n + \int_{\mathcal{R}\backslash B_{d_T}(p,2\bar{\epsilon})} |\sigma_p||\sigma_q| \bar{\omega}_T^n \\
&\leq Ck^{-\frac{n}{2}} \left(\Big(\int_{\mathcal{R}\backslash B_{d_T}(q,2\bar{\epsilon})} |\sigma_q|^2 \bar{\omega}_T^n \Big)^{\frac{1}{2}} + \Big(\int_{\mathcal{R}\backslash B_{d_T}(p,2\bar{\epsilon})} |\sigma_p|^2 \bar{\omega}_T^n \Big)^{\frac{1}{2}} \right) \\
&\leq Ck^{-n} \bar{\epsilon}^{\frac{1}{2}}.
\end{aligned}$$

Let $\tilde{\sigma}_p$ and $\tilde{\sigma}_q$ respectively be the unit normalization of σ_p and $\sigma_q - \langle \sigma_p, \bar{\sigma}_q \rangle_{h_t^k} \sigma_q$, then $\tilde{\sigma}_p$ is orthogonal to $\tilde{\sigma}_q$, and

$$|\tilde{\sigma}_p|_{h_T^k}(p) \geq \frac{1}{4}, \ |\tilde{\sigma}_p|_{h_T^k}(q) \leq C\bar{\epsilon}^{\frac{1}{2}},$$

$$|\tilde{\sigma}_q|_{h_T^k}(p) \leq C\bar{\epsilon}^{\frac{1}{2}}, \ |\tilde{\sigma}_q|_{h_T^k}(q) \geq \frac{1}{4} - C\bar{\epsilon}^{\frac{1}{2}},$$

where $C = C(R)$. Thus,

$$\left| \frac{\tilde{\sigma}_p(p)}{\tilde{\sigma}_q(p)} \right| \geq \frac{1}{4C\bar{\epsilon}^{\frac{1}{2}}}, \text{ and, } \left| \frac{\tilde{\sigma}_p(q)}{\tilde{\sigma}_q(q)} \right| \leq \frac{C\bar{\epsilon}^{\frac{1}{2}}}{8}.$$

Let $p_i \to p$, $q_i \to q$ under the Gromov-Hausdorff convergence. Then

$$\lim_{t_i \to T} d_{FS}(\Phi_{t_i}^k(p_i), \Phi_{t_i}^k(q_i)) \geq C_5^{-1}$$

for some $C_5 = C_5(R) > 0$. By the uniform equivalence of h_t, we have

$$\lim_{t_i \to T} d_{FS}(\Phi_0^k(p_i), \Phi_0^k(q_i)) \geq C_6^{-1} \tag{3.52}$$

for some $C_6 = C_6(R, k) > 0$. Then applying the effective version of the finite generation of the canonical ring $\bigoplus_{k\geq 0} H^0(M; L^k)$, see Theorem 3.3, we have

Claim 3.26. $\lim_{t_i\to T} d_{FS}(\Phi_0^{n+1}(p_i), \Phi_0^{n+1}(q_i)) \geq C_7^{-1}$ *for some* $C_7 = C_7(R, p, q) > 0$.

Proof of the Claim. Let σ_i be a sequence of holomorphic sections of (M, L^k) which converges to σ_p. By the uniform equivalence of the Hermitian metric h_t and volume form ω_t^n, we may assume

$$|\sigma_i|_{h_0^k}(p_i) \geq C^{-1} \text{ and } \int_M |\sigma_i|^2_{h_0^k} \omega_0^n \leq C.$$

Applying the L^∞ estimate we see $\sup_M |\sigma_i|_{h_0^k} \leq C$. Then, by Theorem 3.3, there must be a family of sections $\varsigma_i \in H^0(M, L)$ such that

$$|\varsigma_i|_{h_0}(p_i) \geq C^{-1} \text{ and } \sup_M |\varsigma_i|_{h_0} \leq C.$$

All the constants C above depend on k, h_0 and ω_0, but not depend on i. By Theorem 3.3 again, for any unit section $\sigma' \in H^0(M, L^k)$, we have $\sigma' = \sum e'_\alpha s_0^{\alpha_0} \cdots s_N^{\alpha_N}$ where $s_0, \cdots, s_N$ is an orthonormal basis of $H^0(M, L)$ with respect to Hermitian metric H_0 and $\alpha = (\alpha_0, \cdots, \alpha_N)$ is an element of $\mathbb{N}^{N+1}$ with $|\alpha| = k - n - 1$ and $e'_\alpha \in H^0(M, L^{n+1})$ satisfies $\int_M |e'_\alpha|_{h_0^{n+1}} \omega_0^n \leq C$. In particular we have $\sup_M |\sigma'|_{h_0^k} \leq C$ and $\sup_M |e'_\alpha|_{h_0^{n+1}} \leq C$ for any α. Thus, with respect to the Fubini-Study metric on $\mathbb{C}P^1$,

$$\begin{aligned}
& d_{FS}\big([\sigma'(p_i) : \varsigma_i^k(p_i)], [\sigma'(q_i) : \varsigma_i^k(q_i)]\big) \\
\leq\ & C \sum_\alpha d_{FS}\big([e'_\alpha s_0^{\alpha_0} \cdots s_N^{\alpha_N}(p_i) : \varsigma_i^k(p_i)], [e'_\alpha s_0^{\alpha_0} \cdots s_N^{\alpha_N}(q_i) : \varsigma_i^k(q_i)]\big) \\
\leq\ & C \sum_\alpha \big(d_{FS}([e'_\alpha(p_i) : \varsigma_i^{n+1}(p_i)], [e'_\alpha(q_i) : \varsigma_i^{n+1}(q_i)]) \\
& \quad + (k-n-1) \sup_\ell d_{FS}([s_\ell(p_i) : \varsigma_i(p_i)], [s_\ell(q_i) : \varsigma_i(q_i)])\big) \\
\leq\ & C(k-n)^{N+1}\big(d_{FS}(\Phi_0^{n+1}(p_i), \Phi_0^{n+1}(q_i)) + (k-n-1) d_{FS}(\Phi_0^1(p_i), \Phi_0^1(q_i))\big)
\end{aligned}$$

where in the last inequality we used that

$$\sup_\ell d_{FS}([s_\ell(p_i) : \varsigma_i(p_i)], [s_\ell(q_i) : \varsigma_i(q_i)]) \leq d_{FS}(\Phi_0^1(p_i), \Phi_0^1(q_i)),$$

$$d_{FS}([e'_\alpha(p_i) : \varsigma_i^{n+1}(p_i)], [e'_\alpha(q_i) : \varsigma_i^{n+1}(q_i)]) \leq d_{FS}(\Phi_0^{n+1}(p_i), \Phi_0^{n+1}(q_i)).$$

Now, if $\limsup_{t_i\to T} d_{FS}(\Phi_0^{n+1}(p_i), \Phi_0^{n+1}(q_i)) = \delta$ for some $\delta \geq 0$, then

$$\limsup_{t_i \to T} d_{FS}(\Phi_0^1(p_i), \Phi_0^1(q_i)) \leq K \delta^{\frac{1}{n+1}}$$

for some $K > 0$ depending on the initial Hermitian metric H_0 on L. It is trivial that

$$\lim_{t_i \to T} d_{FS}(\Phi_0^k(p_i), \Phi_0^k(q_i)) \le (N_k+1)^2 \cdot \lim_{t_i \to T} \sup_{s_1, s_2} d_{FS}\big([s_1(p_i) : s_2(p_i)], [s_1(q_i) : s_2(q_i)]\big)$$

where s_1, s_2 are orthogonal unit holomorphic sections of $H^0(M; L^k)$, the metric d_{FS} on the right hand side is induced by the Fubini-Study metric on $\mathbb{C}P^1$. It follows that

$$\begin{aligned}
\lim_{t_i \to T} d_{FS}(\Phi_0^k(p_i), \Phi_0^k(q_i)) &\le 2(N_k+1)^2 \lim_{t_i \to T} \sup_{\sigma'} d_{FS}\big([\sigma'(p_i) : \varsigma_i(p_i)], [\sigma'(q_i) : \varsigma_i(q_i)]\big) \\
&\le C \limsup_{t_i \to T} \big(d_{FS}(\Phi_0^{n+1}(p_i), \Phi_0^{n+1}(q_i)) + d_{FS}(\Phi_0^1(p_i), \Phi_0^1(q_i))\big) \\
&\le C\big(\delta + K\delta^{\frac{1}{n+1}}\big).
\end{aligned}$$

Then (3.52) gives a lower bound of δ. □

Applying the uniform equivalence of h_t again we obtain

$$d_{FS}(\Phi_T^{n+1}(p), \Phi_T^{n+1}(q)) \ge C_8^{-1}.$$

Therefore, the map Φ_T^{n+1} is an injection.

Step 4: Φ_T^{n+1} is a local homeomorphism.

By the discussion in Step 3, together with the relative C^0 estimate in Proposition 3.18, for any $p \in B(x_T, R)$ and q with $d_T(p, q) = 1$, there exists $\delta(p, q) > 0$ and $r(p, q) > 0$ such that

$$d_{FS}(\Phi_T^{n+1}(x), \Phi_T^{n+1}(p)) \ge \delta(p, q),\ \forall x \in B(q, r(p, q)).$$

Since $\partial B(p, 1)$ is compact, we can find a uniform $\delta > 0$ such that

$$d_{FS}(\Phi_T^{n+1}(q), \Phi_T^{n+1}(p)) \ge \delta,\ \forall q \in \partial B(p, 1). \tag{3.53}$$

It follows that Φ_T^{n+1} is an open map. Since Φ_T^{n+1} is also an injection, it must be a local homeomorphism.

4 Metric structure of the limit space

Let (M, ω_t) be the solution to the continuity equation (1.1) and L be the limit line bundle defined in the previous section.

4.1 Diameter bound of the singular Kähler metric

In [28], Song developed a method to prove the diameter bound of a singular Kähler-Einstein metric. In this subsection, we follow his idea (see also [16]) to show the diameter bound of $(M \backslash D, \omega_T)$ where D is any divisor such that $[\omega_0] - Tc_1(M) - \varepsilon[D] > 0$ for some $\epsilon > 0$. A bit difference is that the metric ω_T is a twisted Kähler-Einstein metric.

More precisely, let $p \in D$ be any point, $\pi : \widetilde{M} \to M$ be the blow-up at p with exceptional divisor $\pi^{-1}(p) = E$. Then

$$K_{\widetilde{M}} = \pi^* K_M + (n-1)E.$$

Let h_E be the Hermitian metric on L_E, the associated line bundle of E, and σ_E be a defining section. We denote by $\widetilde{D}$ the proper transformation of D and $h_{\widetilde{D}} = \pi^* h_D$, Hermitian metric on $L_{\widetilde{D}}$. Let $\sigma_{\widetilde{D}}$ be a defining section of $\widetilde{D}$. Let χ be a fixed Kähler metric on $\widetilde{M}$. We may assume that

$$\pi^*\eta_T + \varepsilon\sqrt{-1}\partial\bar\partial \log \|\sigma_E\|^2_{h_E} + \delta_1\sqrt{-1}\partial\bar\partial \log \|\sigma_{\widetilde{D}}\|^2_{h_{\widetilde{D}}} \geq \delta_2 \chi$$

for some $\delta_1, \delta_2 > 0$. Observe that

$$\widetilde{\Omega} = \|\sigma_E\|^{-2(n-1)}_{h_E} \pi^*\Omega$$

defines a smooth volume form on $\widetilde{M}$. Consider the following family of Monge-Ampère equations on $\widetilde{M}$, for $0 < \epsilon \leq 1$,

$$\left(\pi^*\eta_T + \epsilon\chi + \sqrt{-1}\partial\bar\partial\tilde\varphi_\epsilon\right)^n = e^{\frac{1}{T}\tilde\varphi_\epsilon}\left(\|\sigma_E\|^2_{h_E} + \epsilon^2\right)^{n-1}\widetilde{\Omega}. \tag{4.1}$$

By Yau's solution to Calabi problem [47], the equation has a unique smooth solution $\tilde\varphi_\epsilon$, for all $0 < \epsilon \leq 1$; moreover,

$$\tilde\omega_\epsilon =: \pi^*\eta_T + \epsilon\chi + \sqrt{-1}\partial\bar\partial\tilde\varphi_\epsilon \tag{4.2}$$

is a smooth Kähler metric on $\widetilde{M}$.

Lemma 4.1. *There exists C independent of ϵ such that*

$$\tilde\varphi_\epsilon \leq C - (n-1)T\log\left(\|\sigma_E\|^2_{h_E} + \epsilon^2\right). \tag{4.3}$$

Proof. The proof uses simply the maximum principle. We rewrite the Monge-Ampère equation (4.1) as

$$\left(\theta + \sqrt{-1}\partial\bar\partial(\tilde\varphi_\epsilon + f_\epsilon)\right)^n = e^{\frac{1}{T}(\tilde\varphi_\epsilon + f_\epsilon)}\widetilde{\Omega}, \text{ on } \widetilde{M}$$

where $f_\epsilon = (n-1)T\log\left(\|\sigma_E\|^2_{h_E} + \epsilon^2\right)$ and $\theta = \pi^*\eta_T + \epsilon\chi - \sqrt{-1}\partial\bar\partial f_\epsilon$. By direct calculation, cf. [28, Lemma 4.9],

$$\sqrt{-1}\partial\bar\partial\log\left(\|\sigma_E\|^2_{h_E} + \epsilon^2\right)^{n-1} \geq -C\chi.$$

Therefore, $\theta \leq C\chi$. By maximum principle we get $\sup(\tilde\varphi_\epsilon + f_\epsilon) \leq C$. □

Lemma 4.2. *There exist λ and C independent of ϵ such that*

$$\|\tilde\varphi_\epsilon\|_{C^0} \leq C \tag{4.4}$$

and

$$\tilde\omega_\epsilon \leq \frac{C}{\|\sigma_E\|^{2\lambda}_{h_E}\|\sigma_{\widetilde{D}}\|^{2\lambda}_{h_{\widetilde{D}}}}\chi. \tag{4.5}$$

Moreover, for any compact subset $K \subset \widetilde{M}\backslash\widetilde{D}$ and integer k there exists $C_{K,k}$ independent of ϵ such that

$$\|\tilde\omega_\epsilon\|_{C^k(K)} \leq C_{K,k}. \tag{4.6}$$

Proof. The upper bound of $\tilde{\varphi}_\epsilon$ implies that $\tilde{\omega}_\epsilon^n \le C\chi^n$ for some C independent of ϵ. The C^0 estimate of $\tilde{\varphi}_\epsilon$ follows from [49] or [13]. The proof of the next two estimates are standard; see [28] for details. □

Furthermore, as in [28], the estimate of $\tilde{\omega}_\epsilon$ can be improved in the "normal direction" along the proper transformation of D, say F, which is the closure of $\pi^{-1}(D\backslash\{p\})$. Locally, let B be a disk centered at p and $\widetilde{B} = \pi^{-1}(B)$. Let $f_1, \cdots, f_N$ be the defining functions of the divisor F in $\widetilde{B}$. Then, totally as in [28] (see also [34] for a special case of blow-down models), we can prove

Proposition 4.3. *There exists $\delta > 0$, $\lambda > 0$ and $C > 0$ independent of ϵ such that*

$$\tilde{\omega}_\epsilon \le \frac{C}{\|\sigma_E\|_{h_E}^{2(1-\delta)} \prod |f_i|^{2\lambda}}\chi, \quad \text{in } \widetilde{B}. \tag{4.7}$$

Proof. We sketch a proof following Proposition 4.7 of [28]. First of all, we have the estimate

$$\triangle_{\tilde{\omega}_\epsilon} \log \big(\|\sigma_E\|_{h_E}^2 \|\sigma_{\widetilde{D}}\|_{h_{\widetilde{D}}}^{\frac{2\delta_1}{\varepsilon}} \big) \ge C_0^{-1} \operatorname{tr}_{\tilde{\omega}_\epsilon} \chi - C_0 \operatorname{tr}_{\tilde{\omega}_\epsilon}(\pi^* \eta_T)$$

over $\widetilde{B}$, for a constant $C_0 > 0$. Let ω_F be the pull-back of the Euclidean metric on $\widetilde{B}$ which satisfies the two-sided bound $C_1^{-1}\omega_F \le \chi \le C_1 \|\sigma_E\|_{h_E}^{-2}\omega_F$ for some C_1. A simple calculation shows

$$\operatorname{Ric}(\tilde{\omega}_\epsilon) \le -\frac{1}{T}\tilde{\omega}_\epsilon + C_2\chi$$

for some $C_2 > 0$ depending on the Hermitian metric h_E and χ. Thus, we have

$$\triangle_{\tilde{\omega}_\epsilon} \log \operatorname{tr}_\chi \tilde{\omega}_\epsilon \ge -\frac{\operatorname{tr}_\chi \operatorname{Ric}(\tilde{\omega}_\epsilon)}{\operatorname{tr}_\chi \tilde{\omega}_\epsilon} - C_3 \operatorname{tr}_{\tilde{\omega}_\epsilon} \chi \ge \frac{1}{T} - \frac{C_2 n}{\operatorname{tr}_\chi \tilde{\omega}_\epsilon} - C_3 \operatorname{tr}_{\tilde{\omega}_\epsilon} \chi$$

and

$$\triangle_{\tilde{\omega}_\epsilon} \log \operatorname{tr}_{\omega_F} \tilde{\omega}_\epsilon \ge -\frac{\operatorname{tr}_{\omega_F} \operatorname{Ric}(\tilde{\omega}_\epsilon)}{\operatorname{tr}_{\omega_F} \tilde{\omega}_\epsilon} \ge \frac{1}{T} - \frac{C_1 C_2}{\|\sigma_E\|_{h_E}^2 \operatorname{tr}_{\omega_F} \tilde{\omega}_\epsilon}$$

in $\widetilde{B}\backslash(E \cup F)$. On the other hand,

$$\triangle_{\tilde{\omega}_\epsilon} \tilde{\varphi}_\epsilon = n - \operatorname{tr}_{\tilde{\omega}_\epsilon}(\pi^*\eta_T + \epsilon\chi) \le n - \operatorname{tr}_{\tilde{\omega}_\epsilon}(\pi^*\eta_T).$$

Finally notice that both $\|\sigma_E\|_{h_E}^2 \|\sigma_{\widetilde{D}}\|_{h_{\widetilde{D}}}^{\frac{2\delta_1}{\varepsilon}} \prod |f_i|^{2\lambda} \operatorname{tr}_{\omega_F} \tilde{\omega}_\epsilon$ and $\prod |f_i|^{2\lambda} \operatorname{tr}_\chi \tilde{\omega}_\epsilon$ are well defined in $\widetilde{B}$ and uniformly bounded on $\partial\widetilde{B}$ by Lemma 4.2. Then the function

$$H_\epsilon = \log \big(\|\sigma_E\|_{h_E}^2 \|\sigma_{\widetilde{D}}\|_{h_{\widetilde{D}}}^{\frac{2\delta_1}{\varepsilon}} \prod |f_i|^{2\lambda} \operatorname{tr}_{\omega_F} \tilde{\omega}_\epsilon \big) + \frac{1}{2C_0C_3} \log \big(\prod |f_i|^{2\lambda} \operatorname{tr}_\chi \tilde{\omega}_\epsilon \big) - C_0\tilde{\varphi}_\epsilon,$$

satisfies

$$\triangle_{\tilde{\omega}_\epsilon} H_\epsilon \ge \frac{1}{2C_0} \operatorname{tr}_{\tilde{\omega}_\epsilon} \chi - \frac{C_1 C_2}{\|\sigma_E\|_{h_E}^2 \operatorname{tr}_{\omega_F} \tilde{\omega}_\epsilon} - \frac{C_4}{\operatorname{tr}_\chi \tilde{\omega}_\epsilon} - C_5$$

where $C_0, \cdots, C_5$ are constants independent of ϵ. By maximum principle one can get a uniform upper bound of H_ϵ which gives (4.7). For more details see [28]. □

Proposition 4.4. *$\tilde{\omega}_\epsilon$ converges to $\pi^*\omega_T$ as $\epsilon \to 0$ in the current sense. Moreover, the convergence takes place smoothly on $\widetilde{M}\backslash\widetilde{D}$.*

Proof. It suffices to show that for any sequence $\tilde{\omega}_i = \tilde{\omega}_{\epsilon_i}$ with $\epsilon_i \to 0$, if $\tilde{\omega}_i \to \tilde{\omega}_0$ in the current sense, then $\tilde{\omega}_0 = \pi^*\omega_T$.

Write $\tilde{\omega}_i = \pi^*\eta_T + \epsilon_i\chi + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_i$, then $\tilde{\varphi}_i \to \tilde{\varphi}_0$ for some $\pi^*\eta_T$-plurisubharmonic function $\tilde{\varphi}_0$ such that $\tilde{\omega}_0 = \pi^*\eta_T + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_0$. It is trivial that

$$\left(\pi^*\eta_T + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_0\right)^n = e^{\frac{\tilde{\varphi}_0}{T}}\pi^*\Omega.$$

Observe that π^*u_T satisfies the same Monge-Ampère equation

$$\left(\pi^*\eta_T + \sqrt{-1}\partial\bar{\partial}\pi^*u_T\right)^n = e^{\frac{\pi^*u_T}{T}}\pi^*\Omega.$$

Furthermore, both $\tilde{\varphi}_0$ and π^*u_T are uniformly bounded, both the volume forms $e^{\frac{\tilde{\varphi}_0}{T}}\pi^*\Omega$ and $e^{\frac{\pi^*u_T}{T}}\pi^*\Omega$ have full measure in the big cohomological class $[\omega_0] + TK_M$. One can use the comparison principle [2, Corollary 2.3] to conclude that $\tilde{\varphi}_0 = \pi^*u_T$. In other words, $\tilde{\omega}_0 = \pi^*\omega_T$. $\square$

Corollary 4.5. *Assume as above. There exist $\delta > 0$, $\lambda > 0$ and $C > 0$ such that*

$$\pi^*\omega_T \le \frac{C}{\|\sigma_E\|_{h_E}^{2(1-\delta)}\prod|f_i|^{2\lambda}}\chi, \quad \text{in } \widetilde{B}. \tag{4.8}$$

This implies that any point of D is a finite point in the metric completion of $(M\backslash D, \omega_T)$. However, it can not be concluded from this the diameter bound of $(M\backslash D, \omega_T)$.

From now on we turn to the Gromov-Hausdorff convergence. Let $t_i \to T$ be a sequence of times and $x_0 \in M\backslash\mathcal{S}_M$ such that

$$(M, \omega_{t_i}, x_0) \overset{d_{GH}}{\longrightarrow} (M_T, d_T, x_T). \tag{4.9}$$

Let $\mathcal{R}$ be the regular set of M_T with a $C^{1,\alpha}$ metric $\bar{\omega}_T$ for any $\alpha > 0$ such that $\omega_{t_i} \overset{C^{1,\alpha}}{\longrightarrow} \bar{\omega}_T$. Let $\Phi^{n+1} : M \to \mathbb{C}P^{N_{n+1}}$ be the holomorphic map via an orthonormal basis of (L^{n+1}, h_0^{n+1}) defined in the previous section, where $N_{n+1} = \dim H^0(M; L^{n+1}) - 1$. After choosing a subsequence, the map $\Phi_{t_i}^{n+1} = \Phi^{n+1} : (M, \omega_{t_i}) \to (\Phi^{n+1}(M), \omega_{FS})$ converges to a Lipschitz map

$$\Phi_T^{n+1} : (M_T, d_T) \to (\Phi^{n+1}(M), \omega_{FS})$$

by putting $\Phi_T^{n+1}(x) = \lim \Phi_{t_i}^{n+1}(x_i)$ where $x_i \in M$ is an sequence satisfying $x_i \overset{d_{GH}}{\longrightarrow} x$. Since $M\backslash D \subset M\backslash\mathcal{S}_M \subset M_{\text{reg}}$, the metric ω_{t_i} converges smoothly to ω_T on $M\backslash D$. Recall

$$D_T = \{x \in M_T | \exists\, x_i \in D \text{ such that } x_i \overset{d_{GH}}{\longrightarrow} x.\}.$$

The singular set $\mathcal{S} \subset D_T$; moreover, $(M\backslash D, \omega_T)$ is isometric to $(M_T\backslash D_T, \bar{\omega}_T)$.

Lemma 4.6. *$\Phi_T^{n+1} : M_T\backslash D_T \to \Phi^{n+1}(M\backslash D)$ is a bijection.*

Proof. For any $x \in M_T \backslash D_T$, there exists $x' \in M \backslash D$ such that $x' \xrightarrow{d_{GH}} x$. Then $\Phi_T^{n+1}(x) = \lim \Phi_{t_i}^{n+1}(x') = \Phi^{n+1}(x') \in \Phi(M \backslash D)$, so $\Phi_T^{n+1}(M_T \backslash D_T) \subset \Phi^{n+1}(M \backslash D)$. Conversely, for any $x' \in M \backslash D$, we have $x' \xrightarrow{d_{GH}} x$ for some $x \in M_T \backslash D_T$. This is because $d_{\omega_{t_i}}(x', D) \geq \delta$ uniformly for some $\delta > 0$ independent of i. Thus, $\Phi^{n+1}(x') = \Phi_T^{n+1}(x)$, the map Φ_T^{n+1} is surjective onto $\Phi^{n+1}(M \backslash D)$. □

Lemma 4.7. *$\Phi_T^{n+1} : D_T \to \Phi^{n+1}(D)$ is surjective.*

Proof. Noticing that $\Phi^{n+1}(D)$ is compact, the limits of points in $\Phi^{n+1}(D)$ remains in this set, so $\Phi_T^{n+1}(D_T) \subset \Phi^{n+1}(D)$. On the other hand, by Corollary 4.5, for any $x' \in D$ there exists a curve $\gamma : [0, 1] \to M$ with $\gamma(0) = x'$ and $\gamma\big((0, 1]\big) \subset M \backslash D$ such that

$$L_{\omega_T}(\gamma) = \int_0^1 |\dot{\gamma}|_{\omega_T} dt < \infty.$$

Through an isometry from $(M \backslash D, \omega_T)$ to $(M_T \backslash D_T, \bar{\omega}_T)$, the curve $\gamma(t)$ gives a curve $\bar{\gamma}(t)$, $0 < t \leq 1$, which is bounded. Hence, there is a limit $x'' = \lim_{t\to 0} \bar{\gamma}(t)$ in M_T. Then, by the continuity of Φ_T^{n+1},

$$\Phi_T^{n+1}(x'') = \lim_{t\to 0} \Phi_T^{n+1}(\bar{\gamma}(t)) = \lim_{t\to 0} \lim_{i\to\infty} \Phi_{t_i}^{n+1}(\gamma(t)) = \lim_{t\to 0} \Phi^{n+1}(\gamma(t)) = \Phi^{n+1}(x').$$

Finally from above lemma we know that $x'' \in D_T$, i.e., $\Phi^{n+1}(x') \in \Phi_T^{n+1}(D_T)$. □

Corollary 4.8. *Φ_T^{n+1} is surjective.*

Combining with the Propositions 3.22 and 3.23 we conclude that

Proposition 4.9. *$\Phi_T^{n+1} : M_T \to \Phi^{n+1}(M)$ is a homeomorphism. As a consequence, the diameter of M_T is finite.*

Proof. It is clear that $\Phi^{n+1}(M)$ is compact, so M_T is compact and has finite diameter. □

Remark 4.10. *In* [16]*, Guo presented another proof of the proposition without use of the local homeomorphic property.*

Remark 4.11. *The continuity method provides an approach to construct Kähler currents in the big cohomological classes; as has shown, the Kähler currents are smooth on the ample locus. In general it is hard to detect the metric property of the Kähler currents.*

4.2 Cheeger-Gromov convergence: diameter finiteness

The uniform diameter bound of (M, ω_t) would be a consequence of the global Gromov-Hausdorff convergence and compactness of the limit space. Here we present a second argument depending on the smooth convergence of ω_t on the regular domain.

Lemma 4.12. *There exists C such that*

$$\operatorname{diam}(M, \omega_t) \leq C, \ \forall 0 < t < T. \tag{4.10}$$

Proof. It suffices to prove a uniform diameter bound of ω_t when t is close to T. This is simply a consequence of the relative volume comparison theorem.

Recall that by the formula (1.1) we may assume that $\mathrm{Ric}(\omega_t)$ is bounded below uniformly, say

$$\mathrm{Ric}(\omega_t) \geq -(n-1)\Lambda\omega_t,\ \forall \frac{T}{2} \leq t \leq T.$$

Denote by $R_0 = \mathrm{diam}_{d_T}(M_T)$. Let $\varepsilon > 0$ be any number and D be any divisor such that $\mathcal{S}_M \subset D$. Since the regular set $\mathcal{R}$ is geodesically convex, we can choose $K \subset M\backslash D$, a connected and compact subset such that $\mathrm{vol}_{\omega_T}(M\backslash K) \leq \varepsilon$ and $\mathrm{diam}_{d_{\omega_T}}(K) \leq 2R_0$. By smooth convergence on K, we have

$$\mathrm{vol}_{\omega_t}(M\backslash K) \leq 2\varepsilon$$

and

$$\mathrm{diam}_{d_{\omega_t}}(K) \leq 2\,\mathrm{diam}_{d_{\omega_T}}(K) \leq 2R_0$$

for t sufficiently close to T, where the diameter is measured with respect to the intrinsic length metric induced by ω_t on K.

Suppose $x_t \in M\backslash K$ achieves maximum distance to K in (M, ω_t) and put $R_1 = R_1(t) = d_{\omega_t}(x_t, K)$. Then by the relative volume comparison theorem we have, when t is chosen close to T,

$$\frac{\mathrm{vol}_{\omega_t}(K)}{\varepsilon} \leq \frac{\mathrm{vol}_{\omega_t}(B_{2R_0+R_1}(x_t)\backslash B_{R_1}(x_t))}{\mathrm{vol}_{\omega_t}(B_{R_1}(x_t))} \leq \frac{\int_{R_1}^{2R_0+R_1}\sinh^{n-1}\left(\sqrt{\Lambda}t\right)dt}{\int_0^{R_1}\sinh^{n-1}\left(\sqrt{\Lambda}t\right)dt}.$$

If $\varepsilon = \varepsilon(R_0, \Lambda, n, \mathrm{vol}_{\omega_t}(M))$ is chosen sufficiently small, this leads to a desired upper bound of R_1 in terms of $\mathrm{vol}_{\omega_t}(M)$, ε, n, Λ and R_0, which does not depend on the time t. □

Now, summing up the discussions in §3.2, §3.7 and this subsection, we end the proof of Theorem 1.2.

5 Some examples

5.1 Minimal models of general type

In the case when M is a smooth minimal model of general type, the continuity equation (1.1) is solvable for all $t > 0$. We normalize the equation as follows

$$(1+t)\omega = \omega_0 - t\,\mathrm{Ric}\,. \tag{5.1}$$

$\omega(t)$ is a family of solution to this equation iff $\frac{1}{1+t}\omega(\frac{t}{1+t})$ solves the initial equation (1.1). Therefore, 5.1 is solvable for all $t > 0$. Moreover, as in the case of Kähler-Ricci flow, cf. [44, 28], we have

Theorem 5.1. *When M is a smooth minimal model of general type, the solution $\omega(t)$ of* (5.1) *satisfies*

(1) *$\omega(t)$ converges as $t \to \infty$ in the current sense to a positive current $\omega_\infty \in -2\pi c_1(M)$ satisfying* $\mathrm{Ric}(\omega_\infty) = -\omega_\infty$,

(2) *$\omega(t)$ converges smoothly to ω_∞ outside the exceptional locus M_{exc} of the birational morphism to the canonical model of M,*

(3) *the metric completion of $(M \backslash M_{\mathrm{exc}}, \omega_\infty)$ is a compact length metric space, denoted (M_∞, d_∞), which is homeomorphic to the canonical model of M,*

(4) *$(M, \omega(t))$ converges in the Cheeger-Gromov sense to (M_∞, d_∞).*

In Kähler-Ricci flow, it is conjectured the solution globally Cheeger-Gromov converges to the singular Kähler-Einstein metric on the canonical model; see Conjecture 4.1 in [28]. When the Ricci curvature is bounded below, the conjecture is confirmed by Guo [16]. In [45], the last two named author proved the case of dimension less than or equal to three; in [17], Guo-Song-Weinkove gave an independent proof of the case of minimal surfaces. On the other hand, the analytic counterpart of the convergence of Kähler-Ricci flow on smooth minimal models of general type has been known by the work of Tsuji [46] and Tian-Zhang [44].

5.2 Algebraic surfaces

When M is an algebraic surface, the contractions are blow-downs. Hence, exactly as in Kähler-Ricci flow [34], we have

Theorem 5.2. *Let M be an algebraic surface and $\omega_0 \in 2\pi c_1(L')$ for some line bundle L'. Suppose $T < \infty$, then either*

(1) *$\omega(t)$ collapse at T in the sense that $([\omega_0] + TK_M)^2 = 0$, or*

(2) *$\omega(t)$ converges as $t \to T$ to a blow-down $\pi : M \to X$, along a finite number of disjoint exceptional curves, in the sense described in Theorem 1.2.*

6 Appendix: Proofs of Lemma 3.1 and 3.2

In the appendix we give the proofs of Lemma 3.1 and 3.2 for the reader's convenience. Recall that (M, ω) is a Kähler manifold and (L, h) is a Hermitian line bundle over M. Let ∇ and $\bar{\nabla}$ be the $(1,0)$ and $(0,1)$ part of the Chern connection. In the following calculation we denote for simplicity that $\nabla_i = \nabla_{\frac{\partial}{\partial z^i}}$ and $\nabla_{\bar{j}} = \bar{\nabla}_{\frac{\partial}{\partial \bar{z}^j}}$ in local coordinates. We also let $g_{i\bar{j}} = \omega(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})$ be the coefficient of the Hermitian metric. Then we have the basic formula,

$$(\nabla_i \nabla_{\bar{j}} - \nabla_{\bar{j}} \nabla_i)\varsigma = \Theta_{i\bar{j}}\varsigma,$$

for any section ς where Θ is the Chern curvature form of h.

Proof of Lemma 3.1. Let ς be any holomorphic section of L. Then, $\bar{\nabla}\varsigma = 0$, so

$$\triangle|\varsigma|_h^2 = g^{i\bar{j}}\nabla_i\nabla_{\bar{j}}\langle\varsigma, \bar{\varsigma}\rangle = g^{i\bar{j}}\nabla_i\langle\varsigma, \nabla_{\bar{j}}\bar{\varsigma}\rangle = g^{i\bar{j}}\langle\nabla_i\varsigma, \nabla_{\bar{j}}\bar{\varsigma}\rangle + g^{i\bar{j}}\langle\varsigma, \nabla_i\nabla_{\bar{j}}\bar{\varsigma}\rangle = |\nabla s|^2 - g^{i\bar{j}}\Theta_{i\bar{j}} \cdot |s|^2.$$

Similar calculation, in local normal coordinates, gives

$$\begin{aligned}\triangle|\nabla\varsigma|^2 &= g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i\nabla_{\bar{j}}\langle\nabla_k\varsigma,\nabla_{\bar{\ell}}\bar{\varsigma}\rangle = \nabla_i\nabla_{\bar{i}}\langle\nabla_k\varsigma,\nabla_{\bar{k}}\bar{\varsigma}\rangle\\ &= \nabla_i\langle\nabla_{\bar{i}}\nabla_k\varsigma,\nabla_{\bar{k}}\bar{\varsigma}\rangle + \nabla_i\langle\nabla_k\varsigma,\nabla_{\bar{i}}\nabla_{\bar{k}}\bar{\varsigma}\rangle\\ &= |\bar{\nabla}\nabla\varsigma|^2 - \langle\nabla_i\big(\Theta_{k\bar{i}}\varsigma\big),\nabla_{\bar{k}}\bar{\varsigma}\rangle + |\nabla\nabla\varsigma|^2 + \langle\nabla_k\varsigma,\nabla_i\nabla_{\bar{i}}\nabla_{\bar{k}}\bar{\varsigma}\rangle\\ &= |\bar{\nabla}\nabla\varsigma|^2 + |\nabla\nabla\varsigma|^2 - \nabla_i\Theta_{k\bar{i}}\langle\varsigma,\nabla_{\bar{k}}\bar{\varsigma}\rangle - \Theta_{k\bar{i}}\langle\nabla_i\varsigma,\nabla_{\bar{k}}\bar{\varsigma}\rangle + \langle\nabla_k\varsigma,\nabla_i\nabla_{\bar{i}}\nabla_{\bar{k}}\bar{\varsigma}\rangle.\end{aligned}$$

The term $\nabla_i\nabla_{\bar{i}}\nabla_{\bar{k}}\bar{\varsigma} = \overline{\nabla_{\bar{i}}\nabla_i\nabla_k\varsigma}$ can be computed as follows: its conjugation equals

$$\nabla_{\bar{i}}\nabla_i\nabla_k\varsigma = \nabla_i\nabla_{\bar{i}}\nabla_k\varsigma - R_{\bar{i}i\bar{\ell}k}\nabla_\ell\varsigma - \Theta_{i\bar{i}}\nabla_k\varsigma = -\nabla_i(\Theta_{k\bar{i}}\varsigma) + R_{k\bar{\ell}}\nabla_\ell\varsigma - \operatorname{tr}_\omega\Theta\nabla_k\varsigma.$$

Substituting into above formula gives

$$\begin{aligned}\triangle|\nabla\varsigma|^2 &= |\bar{\nabla}\nabla\varsigma|^2 + |\nabla\nabla\varsigma|^2 - \nabla_i\Theta_{k\bar{i}}\langle\varsigma,\nabla_{\bar{k}}\bar{\varsigma}\rangle - \Theta_{k\bar{i}}\langle\nabla_i\varsigma,\nabla_{\bar{k}}\bar{\varsigma}\rangle\\ &\quad +\langle\nabla_k\varsigma,\overline{-\nabla_i(\Theta_{k\bar{i}}\varsigma) + R_{k\bar{\ell}}\nabla_\ell\varsigma - \operatorname{tr}_\omega\Theta\nabla_k\varsigma}\rangle\\ &= |\bar{\nabla}\nabla\varsigma|^2 + |\nabla\nabla\varsigma|^2 + R_{\ell\bar{k}}\langle\nabla_k\varsigma,\nabla_{\bar{\ell}}\bar{\varsigma}\rangle - \operatorname{tr}_\omega\Theta|\nabla\varsigma|^2 - 2\Theta_{k\bar{i}}\langle\nabla_i\varsigma,\nabla_{\bar{k}}\bar{\varsigma}\rangle\\ &\quad -\nabla_i\Theta_{k\bar{i}}\langle\varsigma,\nabla_{\bar{k}}\bar{\varsigma}\rangle - \nabla_{\bar{i}}\Theta_{i\bar{k}}\langle\nabla_k\varsigma,\bar{\varsigma}\rangle.\end{aligned}$$

Finally, by Bianchi identity we have $\nabla_i\Theta_{k\bar{i}} = \nabla_k(\operatorname{tr}_\omega\Theta)$ and $\nabla_{\bar{i}}\Theta_{i\bar{k}} = \nabla_{\bar{k}}(\operatorname{tr}_\omega\Theta)$. Actually, from the curvature formula $\Theta = -\sqrt{-1}\partial\bar{\partial}\log|e|^2$ where e is any local holomorphic frame we have that

$$\nabla_i\Theta_{k\bar{i}} = -\sqrt{-1}\nabla_i\nabla_k\nabla_{\bar{i}}\log|e|^2 = \nabla_k(\operatorname{tr}_\omega\Theta).$$

The required Bochner formula now follows. □

Proof of Lemma 3.2. Recall the formulas

$$\bar{\partial} = d\bar{z}^j\wedge\nabla_{\bar{j}},\ \bar{\partial}^* = -\iota_{\frac{\partial}{\partial\bar{z}^j}}\circ\nabla_j$$

where ι is the inner product operator. Then, by a direct calculation, for $\xi\in\Gamma(T^{0,1}M\otimes L)$,

$$\bar{\partial}^*\bar{\partial}\xi = -\iota_{\frac{\partial}{\partial\bar{z}^k}}\nabla_k(d\bar{z}^j\wedge\nabla_{\bar{j}}\xi) = -\iota_{\frac{\partial}{\partial\bar{z}^k}}(d\bar{z}^j\wedge\nabla_k\nabla_{\bar{j}}\xi) = -\nabla_j\nabla_{\bar{j}}\xi + d\bar{z}^j\wedge\iota_{\frac{\partial}{\partial\bar{z}^k}}\nabla_k\nabla_{\bar{j}}\xi,$$

and

$$\bar{\partial}\bar{\partial}^*\xi = -d\bar{z}^j\wedge\nabla_{\bar{j}}(\iota_{\frac{\partial}{\partial\bar{z}^k}}\nabla_k\xi) = -d\bar{z}^j\wedge\iota_{\frac{\partial}{\partial\bar{z}^k}}\nabla_{\bar{j}}\nabla_k\xi.$$

Thus,

$$(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\xi = \bar{\nabla}^*\bar{\nabla}\xi + d\bar{z}^j\wedge\iota_{\frac{\partial}{\partial\bar{z}^k}}\big(\nabla_k\nabla_{\bar{j}}\xi - \nabla_{\bar{j}}\nabla_k\xi\big).$$

If we write $\xi = \alpha_{\bar{i}}d\bar{z}^i\otimes s$ for some one form $\alpha_{\bar{i}}d\bar{z}^i$ and section $s\in H^0(M,L)$ in local coordinate, then

$$\nabla_k\nabla_{\bar{j}}\xi - \nabla_{\bar{j}}\nabla_k\xi = -R_{k\bar{j}\ell\bar{i}}\alpha_{\bar{\ell}}d\bar{z}^i\otimes s + \Theta_{k\bar{j}}\alpha_{\bar{i}}d\bar{z}^i\otimes s$$

where $R_{k\bar{j}\ell\bar{i}}$ is the curvature tensor of the Kähler metric ω. So,

$$(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\xi = \bar{\nabla}^*\bar{\nabla}\xi + R_{\ell\bar{j}}\alpha_{\bar{\ell}}d\bar{z}^j\otimes s + \Theta_{k\bar{j}}\alpha_{\bar{k}}d\bar{z}^j\otimes s.$$

This is exactly the first formula (3.3). Moreover, noticing that

$$\bar{\nabla}^*\bar{\nabla}\xi = -\nabla_i\nabla_{\bar{i}}\xi = -\nabla_{\bar{i}}\nabla_i\xi - R_{\ell\bar{j}}\alpha_{\bar{\ell}}d\bar{z}^j\otimes s - (\operatorname{tr}_\omega\Theta)\xi,$$

we get the second formula (3.4). □

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