ON PROJECTIVE MANIFOLDS WITH SEMI-POSITIVE HOLOMORPHIC SECTIONAL CURVATURE

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Abstract. We establish structure theorems for a smooth projective variety X with semi-positive holomorphic sectional curvature. We first prove that X is rationally connected if X has no truly flat tangent vectors at some point (which is satisfied when the holomorphic sectional curvature is quasi-positive). This result solves Yau's conjecture on positive holomorphic sectional curvature in a strong form. Moreover, we prove that X admits a locally trivial morphism $\phi : X \to Y$ such that the fiber F is rationally connected and the image Y has a finite étale cover $A \to Y$ by an abelian variety A. We also show that the universal cover of X is biholomorphic and isometric to the product $\mathbb{C}^m \times F$ of the complex Euclidean space $\mathbb{C}^m$ with the flat metric and the rationally connected fiber F with the induced Kähler metric. Our structure theorem is a natural generalization of the structure theorem established by Howard-Smyth-Wu and Mok for holomorphic bisectional curvature.

1. Introduction

One of the significant problems in differential geometry is to establish structure theorems or classifications for varieties satisfying certain curvature conditions. The Frankel conjecture, which had been solved by Siu-Yau in [SY80] and by Mori in [Mor79], says that any smooth projective variety with "positive" holomorphic bisectional curvature is isomorphic to the projective space (see [Mor79] for the Hartshorne conjecture). As one of the extensions of the Frankel conjecture, Howard-Smyth-Wu and Mok established the structure theorem for a compact Kähler manifold M with "semi-positive" holomorphic *bisectional* curvature in [HSW81] and [Mok88] (see also [CG71] and [CG72]). More precisely, [HSW81] proved that M admits a locally trivial morphism $f : M \to B$ so that the image B has a finite étale cover $T \to B$ by a complex torus T and the fiber F satisfies a certain quasi-positive condition (that is, its Ricci curvature is quasi-positive in addition to having semi-positive holomorphic bisectional curvature). Further, [Mok88] proved that F satisfying the above quasi-positive condition is a Hermitian symmetric manifold.

This paper is devoted to the study of "semi-positive" holomorphic *sectional* curvature, motivated by Howard-Smyth-Wu's structure theorem and Mok's result. Positivity of holomorphic sectional curvature is much weaker than that of holomorphic bisectional curvature. For example, even if a smooth projective variety X has positive holomorphic sectional curvature, the variety X can not be expected to be a Hermitian symmetric manifold and even to have the nef anti-canonical bundle (see [AHZ18, Theorem 1.1], [Hit75], [Yan16, Example 3.6]). Nevertheless, holomorphic sectional curvature actually determines holomorphic bisectional curvature (more generally, the

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curvature tensor). Hence it is interesting and natural to investigate a relation or an analogy between holomorphic sectional curvature and bisectional curvature. Particularly, from the viewpoint of geometry, it is one of the most important problems to establish structure theorems for holomorphic sectional curvature, comparing with the case of holomorphic bisectional curvature.

The first contribution of this paper concerns with Yau's conjecture. Yau's conjecture, which is one of the motivations in the studies of positive holomorphic sectional curvature, can be seen as an analogy of Mok's result and a relation between the "strict" positivity and the geometry of X (so-called rational connectedness). In their paper [HW15], Heier-Wong solved Yau's conjecture for quasi-positive holomorphic sectional curvature (that is, semi-positive everywhere and positive at some point). We emphasize that it is essentially important to consider quasi-positivity from the viewpoint of structure theorems since the fiber F appearing in Howard-Smyth-Wu's structure theorem satisfies a certain quasi-positivity, but it is not necessarily positive everywhere. In his paper [Yan18a], Yang affirmatively solved Yau's conjecture even for the case of compact Kähler manifolds by introducing the notion of RC positivity (see [Yan18c] and references therein for recent progress of RC positivity), but it seems to be quite difficult to apply his method to the case of quasi-positive holomorphic sectional curvature.

Conjecture 1.1 (Yau's conjecture for smooth projective varieties)**.** *If a smooth projective variety X admits a Kähler metric with positive holomorphic sectional curvature, then X is rationally connected* (*that is, two arbitrary points can be connected by a rational curve*)*.*

This paper solves Yau's conjecture in a strong form (Theorem 1.2) and gives a more precise relation than Yau's conjecture between the rational connectedness of X and positivity of holomorphic sectional curvature from the viewpoint of structure theorems. Our result can be seen as a version of Mok's result for holomorphic sectional curvature, and also a generalization of Yau's conjecture, Heier-Wong's result, and Yang's result.

Theorem 1.2. *Let X be a smooth projective variety equipped with a Kähler metric g with semi-positive holomorphic sectional curvature. Let $\phi : X \dashrightarrow Y$ be an MRC fibration of X. Then we have*

$$\dim X - \dim Y \geq n_{\mathrm{tf}}(X, g).$$

In particular, the manifold X is rationally connected if $n_{\mathrm{tf}}(X, g) = \dim X$ (*which is satisfied when the holomorphic sectional curvature is quasi-positive*)*.*

Here the invariant $n_{\mathrm{tf}}(X, g)$ is defined by

$$n_{\mathrm{tf}}(X, g) := \dim X - \inf_{p \in X} \dim V_{\mathrm{flat},p},$$

where $V_{\mathrm{flat},p}$ is the subspace of the tangent space $T_{X,p}$ at p consisting of all the truly flat tangent vectors (see Subsection 2.2 for the precise definition). The invariant $n_{\mathrm{tf}}(X, g)$ can be seen as an analog of the numerical Kodaira dimension in terms of truly flat tangent vectors introduced in [HLWZ18a], and it measures positivity of holomorphic sectional curvature. On the other hand, the fiber dimension of maximal rationally connected (MRC for short) fibrations measures how far X is from rationally connected varieties. See [Cam92] and [KoMM92] for MRC fibrations. The condition of $n_{\mathrm{tf}}(X, g) = \dim X$ (that is, there is no truly flat tangent vectors at some point) is a weaker assumption than quasi-positivity, but it works more flexibly from the viewpoint of our argument.

The second contribution of this paper is to establish the following structure theorem (Theorem 1.3) for smooth projective varieties with semi-positive holomorphic sectional curvature. Our theorem not only affirmatively solves [Mat21, Conjecture 1.2 and 1.3] in a stronger form, but also naturally generalizes the result of Howard-Smyth-Wu and Mok to holomorphic sectional curvature.

Theorem 1.3. *Let X be a smooth projective variety equipped with a Kähler metric g with semi-positive holomorphic sectional curvature. Then the following statements hold:*

(1) *There exists a surjective morphism $\phi : X \to Y$ to a smooth projective variety Y with the following properties:*

- *The morphism $\phi : X \to Y$ is a locally trivial morphism (that is, all the fibers are isomorphic each other).*
- *The image Y is a smooth projective variety with a flat metric. In particular, there exists a finite étale cover $A \to Y$ by an abelian variety A.*
- *The fiber F is a rationally connected manifold. In particular, the morphism $\phi : X \to Y$ is an MRC (maximal rationally connected) fibration of X.*

In particular, the fiber product $X^* := A \times_Y X$ admits the locally trivial Albanese map $X^* \to A$ to the abelian variety A with the rationally connected fiber F.

(2) *We obtain the isomorphism*

$$X_{\mathrm{univ}} \cong \mathbb{C}^m \times F,$$

where X_{univ} is the universal cover of X, and F is the rationally connected fiber of ϕ. We have the following commutative diagram:

$$\begin{array}{ccccc} X_{\mathrm{univ}} \cong \mathbb{C}^m \times F & \longrightarrow & X^* := A \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \phi \\ \mathbb{C}^m & \longrightarrow & A & \longrightarrow & Y. \end{array}$$

Moreover, there exists a representation $\rho : \pi_1(Y) \to \mathrm{Aut}(F)$ such that X is isomorphic to $\mathbb{C}^m \times F/\pi_1(Y)$.

(3) *There exist a Kähler metric g_F on the fiber F and a Kähler metric g_Y on Y with the following properties:*

- *The holomorphic sectional curvature of g_F is semi-positive.*
- *The Kähler metric g_Y is flat.*
- *The above isomorphism $X_{\mathrm{univ}} \cong \mathbb{C}^m \times F$ is not only biholomorphic but also isometric with respect to the Kähler metrics $\mu^* g$, $\pi^* g_Y$, and g_F.*

Here π and μ respectively denote the universal cover $\pi : \mathbb{C}^m \to Y$ of Y and the universal cover $\mu : X_{\mathrm{univ}} \to X$ of X.

Corollary 1.4. *Let X be a smooth projective variety equipped with a Kähler metric g with semi-positive holomorphic sectional curvature. Then the fundamental group of X is almost abelian.*

The naive strategy of the proof is to show an MRC fibration $\phi : X \dashrightarrow Y$ of X satisfies the desired properties. Recent breakthroughs in the related fields can achieve our strategy. Specifically, we combine the theory of (holomorphic) foliations and the idea of [CH19] to choose a "holomorphic" MRC fibration with the techniques to solve Yau's conjecture. It is worth mentioning that a part of our methods is strongly influenced by the idea in [HW15] and the notion of RC positivity in [Yan18a] (see also [Yan18b] and [Yan18c] for RC positivity).

There are the following two main difficulties in our strategy.

The first difficulty lies in the ambiguities of the choice of MRC fibrations. An MRC fibration is an almost holomorphic map (that is, rational map whose general fiber is compact), but not always a holomorphic map (see [EIM20, Example 6.5]). Hence we need to find a holomorphic MRC fibration from the ambiguities in the choices of the birational models of its image Y. For this purpose, we will apply the idea of [CH19, Theorem 1.2], in which a holomorphic MRC fibration of Z can be constructed under the different situation where the anti-canonical bundle K_Z of a smooth projective variety Z is nef. We briefly explain the idea of [CH19] and critical differences from our situation (see [CCM21, HIM21] for another approach). One of the key points is to reduce the problem to the case of Z being simply connected. In this step, Cao-Höring used the fact that the fundamental group of Z is almost abelian (see [Pău97]) and the structure theorem of the Albanese map of Z (see [Cao19]). Then they proved that Z is "birationally" a product, which leads to the splitting of the (holomorphic) tangent bundle T_Z. This splitting theorem implies the existence of holomorphic MRC fibrations thanks to the theory of foliations.

Our situation is quite different from [CH19]. For example, our object X has semi-positive holomorphic sectional curvature, but does not necessarily have the nef anti-canonical bundle. Further, we know nothing about the fundamental group of X, and we can not expect that X is birationally a product. Note that, interestingly enough, Theorem 1.3 conversely implies that the fundamental group of X is almost abelian (see Corollary 1.4). Nevertheless the tangent bundle T_X can be proved to split into locally free sheaves (see Theorem 3.1). Then we can construct a holomorphic MRC fibration of X by the theory of foliations with rationally connected leaves developed in [Hör07] (see Theorem 1.5).

Theorem 1.5. *Let X be a smooth projective variety admitting a Kähler metric with semi-positive holomorphic sectional curvature.*

Then we can choose an MRC fibration $X \to Y$ of X to be a morphism (*without indeterminacy locus*) *to a smooth projective variety Y.*

The second difficulty of Theorem 1.3 is that positivity of holomorphic sectional curvature is much weaker and partial, compared to holomorphic bisectional curvature. To overcome this difficulty, we observe truly flat tangent vectors (see [HLWZ18a] and Subsection 2.2), which well behave not only for holomorphic sectional curvature but also for bisectional curvature. This observation implies that each direct summand of T_X in Theorem 3.1 determines an integrable foliation. Then, by using Ehresmann's theorem (see Lemma 2.5) for foliations and truly flat tangent vectors again, we can obtain the structure theorem.

The following theorem reveals detailed properties of a morphism $\phi : X \to Y$ to a compact Kähler manifold Y with pseudo-effective canonical bundle. By applying Theorem 1.6 to the MRC fibration $\phi : X \to Y$ in Theorem 1.5, we can obtain all the conclusions in Theorem 1.3. In our situation, the fiber F is rationally connected (in particular, simply connected), and thus it is sufficient for Theorem 1.3 to consider only the restricted case. However we formulate Theorem 1.6 in a more general statement, since it seems to be important to treat Albanese maps (instead of MRC fibrations) to generalize Theorem 1.3 to non-projective varieties.

Theorem 1.6. *Let (X, g) be a compact Kähler manifold with semi-positive holomorphic sectional curvature, and let $\phi : X \to Y$ be a surjective morphism to a compact Kähler manifold Y with pseudo-effective canonical bundle.*

(1) *The following statements hold*:

- *The morphism $\phi : X \to Y$ is smooth and locally trivial.*
- *The exact sequence of vector bundles*

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \xrightarrow{d\phi_*} \phi^* T_Y \longrightarrow 0$$

  admits the holomorphic orthogonal splitting, that is, the orthogonal complement $T_{X/Y}^{\perp}$ is a holomorphic vector bundle and there exists an isomorphism $j : \phi^* T_Y \to T_{X/Y}^{\perp}$ such that it gives the holomorphic orthogonal decomposition

$$T_X = T_{X/Y} \oplus j(\phi^* T_Y) \cong T_{X/Y} \oplus \phi^* T_Y.$$

- *The image Y admits a Kähler metric g_Y such that $g_Q = \phi^* g_Y$ and the holomorphic sectional curvature of g_Y is identically zero. In particular, the image Y has a finite étale cover $T \to Y$ by a complex torus T. Here g_Q is the hermitian metric on $\phi^* T_Y$ induced by the above exact sequence and the metric g.*

(2) *Let F be a fiber of $\phi : X \to Y$ and F_{univ} be its universal cover. We obtain the isomorphism*

$$X_{\mathrm{univ}} \cong \mathbb{C}^m \times F_{\mathrm{univ}}.$$

Here m is the dimension of Y.

(3) *There exists a Kähler metric $g_{F_{\mathrm{univ}}}$ on F_{univ} such that the holomorphic sectional curvature of $g_{F_{\mathrm{univ}}}$ is semi-positive and the above isomorphism $X_{\mathrm{univ}} \cong \mathbb{C}^m \times F_{\mathrm{univ}}$ is isometric with respect to the Kähler metrics induced by g, g_Y, and $g_{F_{\mathrm{univ}}}$.*

The organization of the paper is as follows: In Section 2, we will summarize some tools and basic results on curvature or truly flat tangent vectors. In Section 3, we will prove the splitting theorem of the tangent bundle (see Theorem 3.1) and Theorem 1.2. In Section 4, we will prove Theorem 1.3, 1.5, and 1.6. In Section 5, we will discuss open problems related to the geometry of semi-positive holomorphic sectional curvature.

This paper is an expanded version of [Mat18b] and [Mat18c]. [Mat18b] and [Mat18c] shall not be published as an independent paper.

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2. Preliminaries

In this section, we shortly explain some formulas and properties of curvature tensors (in particular, holomorphic sectional curvature), truly flat tangent vectors, and (holomorphic) foliations, as we fix the notation and give references.

2.1. Curvature of vector bundles.

In this subsection, we recall several curvature formulas of induced hermitian metrics and the Gauss-Codazzi type formula for exact sequences of (holomorphic) vector bundles.

Let (E, g) be a (holomorphic) vector bundle on a complex manifold X equipped with a (smooth) hermitian metric g. Throughout this paper, we denote the dual vector bundle of E by the notation $E^\vee$ and the inner product with respect to g by the notation $\langle \bullet, \bullet \rangle_g$. The Chern curvature of (E, g)

$$\sqrt{-1}\Theta_g := \sqrt{-1}\Theta_g(E) \in C^\infty(X, \Lambda^{1,1} \otimes \mathrm{End}(E)),$$

defines the curvature tensor

$$R_g := R_{(E,g)} \in C^\infty(X, \Lambda^{1,1} \otimes E^\vee \otimes \bar{E}^\vee)$$

to be

$$R_g(v, \bar{w}, e, \bar{f}) := \big\langle \sqrt{-1}\Theta_g(v, \bar{w})(e), f \big\rangle_g$$

for tangent vectors $v, w \in T_X$ and vectors $e, f \in E$. The metric g induces the hermitian metric $\wedge^m g$ on the vector bundle $\wedge^m E$ of the m-th exterior product. Then we have

$$\begin{aligned} &\big\langle \sqrt{-1}\Theta_{\wedge^m g}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m), e_1 \wedge e_2 \wedge \cdots \wedge e_m \big\rangle_{\wedge^m g} \\ &= \sum_{k=1}^{m} \big\langle \sqrt{-1}\Theta_g(v, \bar{v})(e_k), e_k \big\rangle_g \end{aligned} \tag{2.1}$$

for a tangent vector $v \in T_X$ and vectors $\{e_k\}_{k=1}^m$ in E with $\langle e_i, e_j \rangle_g = \delta_{ij}$. In particular, the curvature $\sqrt{-1}\Theta_{\det g}$ of the determinant bundle $\det E := \wedge^{\mathrm{rk} E} E$ with the induced metric $\det g := \wedge^{\mathrm{rk} E} g$ satisfies that

$$\begin{aligned} \sqrt{-1}\Theta_{\det g}(v, \bar{v}) &= \big\langle \sqrt{-1}\Theta_{\det g}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_{\mathrm{rk} E}), e_1 \wedge e_2 \wedge \cdots \wedge e_{\mathrm{rk} E} \big\rangle_{\det g} \\ &= \sum_{k=1}^{\mathrm{rk} E} \big\langle \sqrt{-1}\Theta_g(v, \bar{v})(e_k), e_k \big\rangle_g \end{aligned}$$

for an orthonormal basis $\{e_k\}_{k=1}^{\mathrm{rk} E}$ of E.

For a subbundle S of E and its quotient vector bundle $Q := E/S$, we consider the exact sequence

$$0 \longrightarrow (S, g_S) \longrightarrow (E, g) \xrightarrow{\ p\ } (Q, g_Q) \longrightarrow 0.$$

Let $S^\perp$ be the orthogonal complement of S in E with respect to g, which is a C^∞-bundle but not always a holomorphic vector bundle. The C^∞-bundle $S^\perp$ is isomorphic to the quotient bundle Q as C^∞-bundles. More precisely, there exists the C^∞-bundle isomorphism $j : Q \cong S^\perp$ such that $p \circ j = \mathrm{id}_Q$. In this paper, we often identify the orthogonal complement $S^\perp$ with the quotient bundle Q under the isomorphism j. Then we have the orthogonal decomposition

$$E = S \oplus S^\perp = S \oplus j(Q) \cong S \oplus Q \text{ as } C^\infty\text{-bundles.}$$

From this orthogonal decomposition, the hermitian metric g_S (resp. g_Q) on S (resp. $Q \cong S^{\perp}$) is induced. For the hermitian vector bundles (E, g), (S, g_S), (Q, g_Q), we respectively denote their Chern connection by D, D_S, D_Q and their Chern curvature by $\sqrt{-1}\Theta_g$, $\sqrt{-1}\Theta_{g_S}$, $\sqrt{-1}\Theta_{g_Q}$. Then we can define

$$A \in C^{\infty}(X, \Lambda^{1,0} \otimes \mathrm{Hom}(S, S^{\perp})) \quad \text{and} \quad B \in C^{\infty}(X, \Lambda^{0,1} \otimes \mathrm{Hom}(S^{\perp}, S))$$

to be

$$D(f) = D_S(f) + A(f) \quad \text{and} \quad D(e) = B(e) + D_Q(e)$$

for a (local) section e of $S^{\perp}$ and a section f of S. It is known that A (equivalently B) is identically zero if and only if the above exact sequence admits the holomorphic orthogonal splitting, which means that $S^{\perp}$ is a holomorphic vector bundle and the above orthogonal decomposition gives the holomorphic decomposition of E. Further we have the following formulas:

$$\begin{aligned} \left\langle \sqrt{-1}\Theta_g(v,\bar{v})(e), e \right\rangle_g + \left\langle B_{\bar{v}}(e), B_{\bar{v}}(e) \right\rangle_{g_S} &= \left\langle \sqrt{-1}\Theta_{g_Q}(v,\bar{v})(e), e \right\rangle_{g_Q}, \\ \left\langle \sqrt{-1}\Theta_g(v,\bar{v})(f), f \right\rangle_g - \left\langle A_v(f), A_v(f) \right\rangle_{g_Q} &= \left\langle \sqrt{-1}\Theta_{g_S}(v,\bar{v})(f), f \right\rangle_{g_S} \end{aligned} \tag{2.2}$$

for a tangent vector $v \in T_X$, a vector $e \in S^{\perp}$, and a vector $f \in S$.

In the rest of this subsection, we summarize the notion of singular hermitian metrics on a line bundle L (see [Dem] for more details). A hermitian metric h on L is said to be a *singular hermitian metric*, if $\log |e|_h$ is an L^1_{loc}-function for any local frame e of L. Then the curvature current $\sqrt{-1}\Theta_h$ of (L, h) is defined by

$$\sqrt{-1}\Theta_h := \sqrt{-1}\Theta_h(L) := -\sqrt{-1}\partial\overline{\partial} \log |e|_h^2$$

in the sense of distributions. The singular hermitian metric h is said to have *neat analytic singularities*, if there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that the function $-\log |e|_h^2$ can be locally written as

$$-\log |e|_h^2 = c \log \left(|f_1|^2 + |f_2|^2 + \cdots + |f_k|^2\right) + \text{smooth function},$$

where c is a positive real number and $f_1, \ldots, f_k$ are local generators of $\mathcal{I}$. We say that h has *divisorial singularities* when the ideal sheaf $\mathcal{I}$ is defined by an effective divisor. The dual singular hermitian metric $h^{\vee}$ on the dual line bundle $L^{\vee}$ can be defined to be $|e^{\vee}|_{h^{\vee}} := |e|_h^{-1}$ for the dual local frame $e^{\vee}$. Further, for a morphism $f : Z \to X$, the singular hermitian metric f^*h on the pull-back f^*L can also be defined to be $|f^*e|_{f^*h} := f^*(|e|_h)$ for the local frame f^*e of f^*L. Then we have

$$\sqrt{-1}\Theta_h = -\sqrt{-1}\Theta_{h^{\vee}} = \sqrt{-1}\partial\overline{\partial} \log |e^{\vee}|_{h^{\vee}}^2 \quad \text{and} \quad f^*\sqrt{-1}\Theta_h := \sqrt{-1}\Theta_{f^*h}.$$

2.2. **Holomorphic sectional curvature and truly flat tangent vectors.** In this subsection, we summarize some properties of holomorphic sectional curvature and truly flat tangent vectors. For a hermitian metric g on the (holomorphic) tangent bundle T_X, the holomorphic sectional curvature H_g is defined to be

$$H_g([v]) := \frac{R_g(v, \bar{v}, v, \bar{v})}{|v|_g^4} = \frac{\left\langle \sqrt{-1}\Theta_g(v,\bar{v})(v), v \right\rangle_g}{|v|_g^4}$$

for a non-zero tangent vector $v \in T_X$. The holomorphic sectional curvature H_g is said to be *positive* (resp. *semi-positive*) if $H_g([v]) > 0$ (resp. $H_g([v]) \geq 0$) holds for any non-zero tangent vector $v \in T_X$.

In this paper, we handle only the case of g being a Kähler metric (that is, the associated $(1,1)$-form ω_g is d-closed). In this case, the following symmetries hold :

$$R_g(e_i, \bar{e}_j, e_k, \bar{e}_\ell) = R_g(e_k, \bar{e}_\ell, e_i, \bar{e}_j) = R_g(e_k, \bar{e}_j, e_i, \bar{e}_\ell).$$

The above symmetries lead to the following lemmas.

Lemma 2.1 ([Yan18a, Lemma 6.1], [Mat21, Lemma 2.2] cf. [Bre, page 136], [BKT13, Lemma 1.4])**.**
Let g be a Kähler metric of X and V be a subspace of $T_{X,p}$ at a point $p \in X$. If a unit vector $v \in V$ is a minimizer of the holomorphic sectional curvature H_g on V, that it, v satisfies

$$|v|_g = 1 \text{ and } H_g([v]) = \min\{H_g([x]) \,|\, 0 \neq x \in V\},$$

then we have

$$2R_g(v, \bar{v}, x, \bar{x}) \geq (1 + |\langle v, x\rangle_g|^2) R_g(v, \bar{v}, v, \bar{v})$$

for any unit vector $x \in V$. In particular, if the holomorphic sectional curvature H_g is semi-positive, a minimizer v of H_g on V satisfies that

$$R_g(v, \bar{v}, x, \bar{x}) \geq 0$$

for any tangent vector $x \in V$.

The case where V in Lemma 2.1 coincides with the whole tangent space $T_{X,p}$ was proved in [Yan18a, Lemma 6.1]. It is easy to see that the same argument works even in the case of V being a subspace of $T_{X,p}$, and thus we omit the proof.

Now we define truly flat tangent vectors and the invariant $n_{\mathrm{tf}}(X, g)$, following in [HLWZ18a].

Definition 2.2 (Truly flat tangent vectors and the invariant $n_{\mathrm{tf}}(X, g)$)**.** Let (X, g) be a Kähler manifold.

- A tangent vector $v \in T_X$ at p is said to be *truly flat* with respect to g if v satisfies that
$$R_g(v, \bar{x}, y, \bar{z}) = 0 \text{ for any tangent vectors } x, y, z \in T_{X,p}.$$
- We define the subspace $V_{\mathrm{flat},p}$ of $T_{X,p}$ at p by
$$V_{\mathrm{flat},p} := \{v \in T_{X,p} \,|\, v \text{ is a truly flat tangent vector in } T_{X,p}\}.$$
- We define the invariants $n_{\mathrm{tf}}(X, g)_p$ and $n_{\mathrm{tf}}(X, g)$ by
$$n_{\mathrm{tf}}(X, g)_p := \dim X - \dim V_{\mathrm{flat},p} \quad \text{and} \quad n_{\mathrm{tf}}(X, g) := \dim X - \inf_{p \in X} \dim V_{\mathrm{flat},p}.$$

It is easy to see that the invariant $n_{\mathrm{tf}}(X, g)_p$ is lower semi-continuous with respect to $p \in X$ in the classical topology. In particular, if we have the equality $n_{\mathrm{tf}}(X, g)_p = n_{\mathrm{tf}}(X, g)$ at p, the same equality holds on a neighborhood of p. The following lemma characterizes truly flat tangent vectors in terms of holomorphic sectional curvature and bisectional curvature.

Lemma 2.3 (cf. [HLWZ18a, Lemma 2.1])**.** *Let g be a Kähler metric of X with semi-positive holomorphic sectional curvature and V be a subspace of $T_{X,p}$ at a point $p \in X$. If a tangent vector $v \in T_X$ satisfies that*

$$H_g([v]) = 0 \quad \text{and} \quad R_g(v, \bar{v}, w, \bar{w}) = 0 \text{ for any tangent vector } w \in V,$$

then v satisfies that

$$R_g(v, \bar{x}, y, \bar{z}) = 0 \text{ for any tangent vectors } x, y, z \in V.$$

In particular, if v satisfies the above assumptions for any tangent vector $w \in T_{X,p}$, then v is a truly flat tangent vector at p.

Proof. When the holomorphic sectional curvature is semi-negative and the subspace V coincides with the tangent space $T_{X,p}$, the same conclusion was proved in [HLWZ18a, Lemma 2.1]. For reader's convenience, we will give a sketch of the proof.

For an arbitrary complex number $re^{\sqrt{-1}\theta}$, we obtain that

$$\begin{aligned} 0 &\leq H([v + re^{\sqrt{-1}\theta}w])|v + re^{\sqrt{-1}\theta}w|_g^4 \\ &= 2\Re(e^{\sqrt{-1}\theta}R_g(v,\bar{v},v,\bar{w}))r^3 + 2\Re(e^{\sqrt{-1}\theta}R_g(v,\bar{w},w,\bar{w}))r + R_g(w,\bar{w},w,\bar{w}) \end{aligned}$$

from the assumptions $R_g(v,\bar{v},v,\bar{v}) = 0$ and $R_g(v,\bar{v},w,\bar{w}) = 0$. Here we used the symmetries obtained from Kähler metrics. If $R_g(v,\bar{v},v,\bar{w})$ is not zero, we have a contradiction by suitably choosing θ such that $\Re(e^{\sqrt{-1}\theta}R_g(v,\bar{v},v,\bar{w})) < 0$ and by taking a sufficiently large $r > 0$. Hence we obtain $R_g(v,\bar{v},v,\bar{w}) = 0$. By repeating the same argument as above for $e^{\sqrt{-1}\theta}R_g(v,\bar{w},w,\bar{w})$, we can see that $R_g(v,\bar{w},w,\bar{w}) = 0$ for any tangent vector $w \in T_X$. Then we can easily check the desired equality by the standard polarization argument. □

It is worth emphasizing that the assumption that g is a Kähler metric plays a crucial role in the proof of the above lemmas. The holomorphic sectional curvature of non-Kähler metrics is also an interesting topic, but the technique displayed in this paper does not work in the non-Kähler case.

2.3. **Holomorphic foliations.** The theory of (holomorphic) foliation plays an important role in the proof of the main theorem. In this subsection, we recall some results of foliations needed later. Let T_X be the (holomorphic) tangent bundle on a complex manifold X and let V be a (holomorphic) subbundle of T_X. The subbundle $V \subset T_X$ is called an *integrable foliation* if the space of (local) sections of V is closed under the Lie bracket

$$[\bullet,\bullet] : T_X \times T_X \to T_X.$$

Frobenius' theorem asserts the integrability of V is equivalent to the condition that an arbitrary point in X has an open neighborhood U and a smooth morphism $\phi : U \to \mathbb{C}^m$ such that V coincides with the relative tangent bundle of ϕ (that is, $V = T_{U/\mathbb{C}^m} := \operatorname{Ker} d\phi_*$), where $m := \dim X - \operatorname{rank} V$. For the subbundle $V \subset T_X$, the Lie bracket determines the $\mathcal{O}_X$-module morphism $\wedge^2 V \to T_X/V$, and thus we obtain the holomorphic section

$$F_V := [\bullet,\bullet] \in H^0(X, \operatorname{Hom}(\wedge^2 V, T_X/V))$$

of the sheaf $\operatorname{Hom}(\wedge^2 V, T_X/V) \cong (\wedge^2 V)^* \otimes (T_X/V)$. The subbundle V is integrable if and only if the above section F_V is identically zero. If the subbundle V is integrable on a (non-empty) Zariski open set of X, then V is integrable on the whole space X. Indeed, the section F_V is vanishing on the Zariski open set in this case, and thus it follows that F_V should be identically zero on X since the sheaf $\operatorname{Hom}(\wedge^2 V, T_X/V)$ is locally free (in particular, torsion-free).

In the case of V being integrable, we can obtain the quotient map $X \to X/\!\sim$ by using the equivalence relation defined as follows: a point p is equivalent to a point q if and only if p and q can be connected by fibers of local morphisms $U \to \mathbb{C}^m$. A fiber of the quotient map $X \to X/\!\sim$ is called a *leaf* of the integrable foliation $V \subset T_X$. We remark that leaves are not necessarily compact and the quotient space $X/\!\sim$ does not necessarily admit a complex structure. Nevertheless, under some additional assumptions, it can be shown that $X/\sim$ is a complex space such that the quotient map $X \to X/\sim$ is a holomorphic map.

We will use Lemma 2.4 in the proof of Theorem 3.1 and use Lemma 2.5 (so-called Ehresmann's theorem) in the proof of Theorem 1.6.

Lemma 2.4 ([Hör07, Corollary 2.11]). *Let X be a compact Kähler manifold and $W \subset T_X$ be an integrable foliation of X. Assume that at least one leaf of the foliation W is compact and rationally connected.*

Then there exists a smooth morphism $X \to Y$ such that $W = T_{X/Y}$.

Lemma 2.5 ([CL, Section V] cf. [Hör07, Theorem 3.17]). *Let X and Y be compact complex manifolds, and let $\phi : X \to Y$ be a smooth morphism with connected fibers. Assume that there exists an integrable foliation $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$.*

Then $\phi : X \to Y$ is a locally trivial morphism. More precisely, there exists a representation $\rho : \pi_1(Y) \to \mathrm{Aut}(F)$ such that X is isomorphic to $Y_{\mathrm{univ}} \times F/\pi_1(Y)$, where F is a fiber of $\phi : X \to Y$. Further the morphism

$$\mu : Y_{\mathrm{univ}} \times F_{\mathrm{univ}} \to Y_{\mathrm{univ}} \times F/\pi_1(Y) \cong X$$

is the universal cover of X and we have

$$\mu^* V = \mathrm{pr}_1^*(T_{Y_{\mathrm{univ}}}) \text{ and } \mu^* T_{X/Y} = \mathrm{pr}_2^*(T_{F_{\mathrm{univ}}}).$$

Here Y_{univ} and F_{univ} respectively denote the universal cover of Y and F, and also pr_i denotes the projection of the product $Y_{\mathrm{univ}} \times F_{\mathrm{univ}}$ to the i-th component.

In order to apply Lemma 2.4, we will construct an integrable foliation by using the following lemmas and Theorem 3.1.

Lemma 2.6 ([Har, Ex. 5.8]). *Let $\mathcal{H}$ be a coherent sheaf on a complex manifold X. Then the function*

$$X \ni p \longmapsto \dim \mathcal{H}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p}/\mathfrak{m}_p \in \mathbb{Z}$$

is upper semi-continuous with respect to $p \in X$. Further, when the above function is constant, the sheaf $\mathcal{H}$ is a locally free sheaf.

Lemma 2.7 ([Har80, Proposition 1.6, Corollary 1.4, Corollary 1.7]). *Let $\mathcal{F}$ and $\mathcal{G}$ be reflexive coherent sheaves on a normal variety X. If there exists a Zariski open set $X_o \subset X$ such that $\mathrm{codim}(X \setminus X_o) \geq 2$ and $\mathcal{F} \cong \mathcal{G}$ on X_o, then we have $\mathcal{F} \cong \mathcal{G}$ on the ambient space X.*

3. Splitting Theorem of Tangent Bundles and Yau's Conjecture

In this section, we first prove the following splitting theorem of the tangent bundle. This splitting theorem is the core of the proof of Theorem 1.5 and Theorem 1.2. Theorem 1.2 can be obtained from the proof of Theorem 3.1.

Theorem 3.1. *Let (X, g) be a compact Kähler manifold with semi-positive holomorphic sectional curvature, and let $\phi : X \dashrightarrow Y$ be an almost holomorphic map (that is, meromorphic map whose general fiber is compact) to a compact Kähler manifold Y with pseudo-effective canonical bundle K_Y. For the indeterminacy locus Z_o of ϕ, we consider the Zariski open set X_o defined by $X_o := X \setminus Z_o$.*

Then the morphism ϕ is a smooth morphism on X_o. Moreover the standard exact sequence

$$0 \to T_{X/Y} \to T_X \to \phi^* T_Y \to 0 \text{ on } X_o$$

admits the holomorphic orthogonal splitting, that is, the orthogonal complement $T_{X/Y}^{\perp}$ is a holomorphic vector bundle on X_o and there exists an isomorphism $j : \phi^* T_Y \cong T_{X/Y}^{\perp}$ on X_o such that

$$T_X = T_{X/Y} \oplus j(\phi^* T_Y) \cong T_{X/Y} \oplus \phi^* T_Y \text{ on } X_o.$$

Proof of Theorem 3.1. Our first purpose is to prove that ϕ is actually a smooth morphism on X_o. For this purpose, from a given Kähler metric g of X with semi-positive holomorphic sectional curvature, we construct a possibly singular hermitian metric G on the extension of the line bundle $\phi^* K_Y^{\vee}$ defined on X_o to the ambient space X such that the singularities of G corresponds to the non-smooth locus of ϕ on X_o. This enables us to reduce our first purpose to observe the singularities of G.

Now we take a resolution $\tau : \Gamma \to X$ of the indeterminacy locus Z_o of ϕ and denote by the notation $\bar{\phi} : \Gamma \to Y$ the morphism satisfying the following commutative diagram:

$$\begin{array}{ccccc} & & \Gamma & & \\ & & \downarrow \tau & \searrow \bar{\phi} & \\ \bar{X} & \xrightarrow{\pi} & X & \overset{\phi}{\dashrightarrow} & Y. \end{array}$$

We define the coherent sheaf L on X by

$$L := \big(\tau_* \mathcal{O}_X(\bar{\phi}^* K_Y)\big)^{\vee\vee}.$$

Note that L is actually an invertible sheaf since it is a reflexive sheaf of rank one (see [Har80, Corollary 1.2, Proposition 1.9]). By the definition, the line bundle L coincides with the usual pull-back $\phi^* K_Y$ on the Zariski open set X_o, and thus L can be seen as the extension of the pull-back $\phi^* K_Y$. From the sheaf morphism

$$\bar{\phi}^* K_Y \xrightarrow{\ d\bar{\phi}^*\ } \wedge^m \Omega_\Gamma = \wedge^m T_\Gamma^{\vee} \quad \text{on } \Gamma$$

defined by the pull-back $d\bar{\phi}^*$, we can get the injective sheaf morphism

$$L \xrightarrow{\ f\ } \wedge^m \Omega_X = \wedge^m T_X^{\vee} \quad \text{on } X$$

by taking the push-forward τ_* with the birational morphism τ and by using the formula $\tau_* \mathcal{O}_\Gamma(\wedge^m \Omega_\Gamma) = \mathcal{O}_X(\wedge^m \Omega_X)$. Here m is the dimension of Y.

By using the (smooth) hermitian metric $\wedge^m h$ on $\wedge^m \Omega_X$ induced by the dual metric $h := g^{\vee}$ on the cotangent bundle Ω_X, we can define the singular hermitian metric H on L to be $|e_L|_H := |f(e_L)|_{\wedge^m h}$ for a non-vanishing local section e_L of L, where f is the above sheaf morphism. The dual singular hermitian line bundle

$$(L^{\vee}, G := H^{\vee} := H^{-1})$$

plays a central role rather than (L, H) in the proof (see [Dem] or Subsection 2.1 for singular hermitian metrics). For a local coordinate $(t_1, t_2, \dots, t_m)$ of Y, the m-form $dt := dt_1 \wedge dt_2 \wedge \cdots \wedge dt_m$ naturally determines the local frame of $L = \phi^* K_Y$ on X_o, which we denote by the same notation

dt. By the definitions of the curvature and the dual metric, the curvature current of $(L^\vee, G)$ on X_o can be locally written as

$$\sqrt{-1}\Theta_G := \sqrt{-1}\Theta_G(L^\vee) = \sqrt{-1}\partial\overline{\partial}\log|\phi^* dt|^2_{\wedge^m h} \text{ on } X_o, \tag{3.1}$$

where $\phi^* dt$ is the pull-back of the m-form dt by ϕ. We can easily see that the sheaf morphism f is a bundle morphism (that is, the section $f(e_L)$ is also non-vanishing in our case) if and only if G (equivalently H) is a smooth metric. Further, by the above expression and $f = d\phi^*$ on X_o, we can see that f is a bundle morphism on X_o if and only if ϕ is smooth on X_o. Therefore we can conclude that ϕ is a smooth morphism on X_o if G can be shown to be smooth. This is our first purpose.

We take a resolution $\pi : \bar{X} \to X$ of the degenerate ideal of f. Then we obtain the following claim:

Claim 3.2. *Let Z be the subvariety defined by the degenerate ideal of f. Then the singular hermitian metric $\pi^* G$ on $\pi^* L^\vee$ has neat divisorial singularities along $\pi^{-1}(Z)$. More precisely, the pull-back $\pi^*\sqrt{-1}\Theta_G$ of the curvature current $\sqrt{-1}\Theta_G$ can be written as*

$$2\pi c_1(\pi^* L^\vee) \ni \pi^*\sqrt{-1}\Theta_G = \gamma + [E], \tag{3.2}$$

where γ is a smooth $(1,1)$-form on $\bar{X}$ and $[E]$ is the integration current defined by an effective divisor E.

Proof of Claim 3.2. If the metric G itself has neat analytic singularities, the conclusion is obvious. However we do not know whether or not the metric G has neat analytic singularities (see Remark 3.3 for more details).

For a given point $p \in \bar{X}$, we take a non-vanishing section e_L of L and a local frame $\{s_i\}_{i=1}^N$ of $\wedge^m \Omega_X$ on a neighborhood of $\pi(p)$. Here we put $N := \binom{n}{m}$ for simplicity. The holomorphic m-form $f(e_L)$ can be locally written as

$$f(e_L) = \sum_{i=1}^N f_i s_i \text{ on a neighborhood of } \pi(p)$$

for some holomorphic functions $\{f_i\}_{i=1}^N$. The degenerate ideal $\mathcal{I}$ of f is generated by $\{f_i\}_{i=1}^N$ and the ideal $\pi^{-1}\mathcal{I} = \mathcal{I} \cdot \mathcal{O}_{\bar{X}}$ is the ideal sheaf associated to some effective divisor E. Let t be a local holomorphic function such that t determines the effective divisor E. Then it follows that $g_i := \pi^* f_i / t$ is a holomorphic function and the common zero locus $\cap_{i=1}^N g_i^{-1}(0)$ is empty from the choice of π. Therefore a simple computation yields

$$\log \pi^*(|e_L|^2_H) = \log \pi^*(|f(e_L)|^2_{\wedge^m h}) = \log|t|^2 + \log \sum_{i,j=1}^N g_i \bar{g}_j \pi^* \langle s_i, s_j \rangle_{\wedge^m h}.$$

It can be proven that the Levi form of the first term is equal to the integration current $[E]$ by the Poincaré-Lelong formula. On the other hand, it follows that the Levi form of the second term determines a smooth $(1,1)$-form γ, since it is easy to see that the function

$$\sum_{i,j=1}^N g_i \bar{g}_j \pi^* \langle s_i, s_j \rangle_{\wedge^m h}$$

is a non-vanishing smooth function. □

Remark 3.3. • It follows that the smooth form γ can be identified with the curvature $\sqrt{-1}\Theta_G$ under the isomorphism $\pi : \bar{X} \setminus \pi^{-1}(Z) \cong X \setminus Z$.

• The metric G itself may not have neat analytic singularities although the pull-back $\pi^* G$ has divisorial singularities. For example, in the case of $n = 2$ and $m = 1$, we consider the following situation:

$$\phi^* dt = z_1 s_1 + z_2 s_2 \text{ and } h = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{with respect to a local frame } (s_1, s_2) \text{ of } \Omega_X.$$

Here (z_1, z_2) is a local coordinate of X. Then we can see that the function

$$\frac{|\phi^* dt|_h^2}{|z_1|^2 + |z_2|^2} = \frac{2|z_1|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 + 2|z_2|^2}{|z_1|^2 + |z_2|^2}$$

can not be extended to a smooth function defined at the origin. Of course, when we take a resolution of the degenerate ideal (which is just one point blow-up in this case), we can easily check that the pull-back of the above function is a non-vanishing smooth function.

Our purpose is to prove that the first Chern class of L is zero and γ is a semi-positive $(1, 1)$-form, which implies that $E = 0$ (in particular ϕ is smooth on X_o).

Now Y has the pseudo-effective canonical bundle, and thus the line bundle L can be shown to be pseudo-effective. Indeed, since L coincides with $\phi^* K_Y$ on X_o by the definition, there exists a singular hermitian metric on $L|_{X_o}$ with semi-positive curvature, which can be extended by $\mathrm{codim}(X \setminus X_o) \geq 2$. Then, for the Kähler form ω associated to the metric g, it follows that the intersection number $c_1(\pi^* L) \cdot \{\pi^* \omega\}^{n-1}$ is non-negative since $\pi^* L$ is also pseudo-effective and $\pi^* \omega$ is a smooth semi-positive form. Therefore we obtain

$$0 \geq 2\pi \int_{\bar{X}} c_1(\pi^* L^\vee) \wedge \pi^* \omega^{n-1} = \int_{\bar{X}} \gamma \wedge \pi^* \omega^{n-1} + \int_E \pi^* \omega^{n-1} \tag{3.3}$$

from (3.2) by taking the wedge product with $\pi^* \omega^{n-1}$ and the integration on $\bar{X}$. The second term of the right hand side is non-negative. We will show that the first term is also non-negative, which implies that the first Chern class of L is zero.

We can take a (non-empty) Zariski open set Y_1 such that the restriction $\phi : X_1 := \phi^{-1}(Y_1) \to Y_1$ is a proper smooth morphism since ϕ is an almost holomorphic map. Throughout this paper, we say that a tangent vector $v \in T_X$ is in the *horizontal direction* (resp. in the *vertical direction*) in the case of $v \in (T_{X/Y})^\perp$ (resp. $v \in T_{X/Y}$). Here $(T_{X/Y})^\perp$ is the orthogonal complement of $T_{X/Y}$ in T_X with respect to g and it is identified with $\phi^* T_Y$ at p. Then we obtain the following claim:

Claim 3.4. *For a point $p \in X_1$, there exists an orthonormal basis $\{e_k\}_{k=1}^n$ of T_X at p with the following properties:*

(1) $\{e_i\}_{i=1}^m$ *is an orthonormal basis of* $(T_{X/Y})^\perp$ *at* p.

(2) $R_g(e_i, \bar{e}_i, e_j, \bar{e}_j) \geq 0$ *for any* $1 \leq i, j \leq m$.

(3) $\sqrt{-1}\Theta_G(e_i, \bar{e}_i) \geq 0$ *for any* $i = 1, 2, \ldots, m$.

Proof of Claim 3.4. We remark that the curvature $\sqrt{-1}\Theta_G$ is a smooth $(1, 1)$-form on a neighborhood of p by $p \in X_1$. We first take an arbitrary orthonormal basis $\{e_k\}_{k=1}^n$ of T_X at p such that

$$(T_{X/Y})^\perp = \mathrm{Spn}\langle \{e_i\}_{i=1}^m \rangle \quad \text{and} \quad T_{X/Y} = \mathrm{Spn}\langle \{e_j\}_{j=m+1}^n \rangle.$$

By choosing a new orthonormal basis $\{e_i\}_{i=1}^m$ of $(T_{X/Y})^\perp$, we may assume that e_1 is the minimizer of H_g on $(T_{X/Y})^\perp = \mathrm{Spn}\langle\{e_k\}_{k=1}^m\rangle$, that is, the unit tangent vector e_1 satisfies that

$$H_g([e_1]) = \min\{H_g([v]) \,|\, 0 \neq v \in \mathrm{Spn}\langle\{e_k\}_{k=1}^m\rangle\}.$$

After we fix the tangent vector e_1 chosen as above, we choose an orthonormal basis $\{e_i\}_{i=2}^m$ of

$$(T_{X/Y} \oplus \mathrm{Spn}\langle e_1\rangle)^\perp = \mathrm{Spn}\langle\{e_k\}_{k=2}^m\rangle$$

such that e_2 is the minimizer of H_g on $\mathrm{Spn}\langle\{e_k\}_{k=2}^m\rangle$. By repeating this process, we can construct an orthonormal basis $\{e_i\}_{i=1}^m$ of $(T_{X/Y})^\perp$ satisfying that

$$H_g([e_i]) = \min\{H_g([v]) \,|\, 0 \neq v \in \mathrm{Spn}\langle\{e_k\}_{k=i}^m\rangle\}.$$

for any $i = 1, 2, \dots, m$.

Then, for this orthonormal basis, we can prove property (2) in Claim 3.4. Indeed, we may assume that $i \leq j$ by the symmetry

$$R_g(e_i, \bar{e_i}, e_j, \bar{e_j}) = R_g(e_j, \bar{e_j}, e_i, \bar{e_i}).$$

Further, for $i \leq j$, the tangent vector e_i is the minimizer of H_g on the subspace $\mathrm{Spn}\langle\{e_k\}_{k=i}^m\rangle$ which contains e_j. Therefore it follows that $R_g(e_i, \bar{e_i}, e_j, \bar{e_j})$ is non-negative from Lemma 2.1.

By applying the formulas (2.1) and (2.2) to the exact sequence

$$0 \longrightarrow \mathrm{Ker}\, d\phi_* \longrightarrow (\wedge^m T_X, \wedge^m g) \xrightarrow{d\phi_*} (\phi^* K_Y^\vee, G) \longrightarrow 0$$

on a neighborhood of p, we obtain that

$$\begin{aligned}
\sum_{k=1}^m R_g(v, \bar{v}, e_k, \bar{e_k}) &= \big\langle \sqrt{-1}\Theta_{\wedge^m g}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m), e_1 \wedge e_2 \wedge \cdots \wedge e_m \big\rangle_{\wedge^m g} \\
&\leq \big\langle \sqrt{-1}\Theta_G(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m), e_1 \wedge e_2 \wedge \cdots \wedge e_m \big\rangle_G \\
&= \sqrt{-1}\Theta_G(v, \bar{v}) |e_1 \wedge e_2 \wedge \cdots \wedge e_m|_G^2 \\
&= \sqrt{-1}\Theta_G(v, \bar{v})
\end{aligned} \tag{3.4}$$

for any tangent vector $v \in T_X$. Note that G (defined by the dual metric of H) is equal to the quotient metric induced by $\wedge^m g$ since p is a smooth point of ϕ. When we consider the above formula in the case of $v = e_i$, we can see that the left hand side is non-negative by property (2) in Claim 3.4. Therefore we can conclude that $\sqrt{-1}\Theta_G(e_i, \bar{e}_i)$ is non-negative for any $i = 1, 2, \dots, m$. □

We decompose the first term of (3.3) into the vertical part and the horizontal part, in order to prove that the right hand side is non-negative by using Claim 3.4. The integration of $\gamma \wedge \pi^*\omega^{n-1}$ on $\bar{X}$ is equal to the integration on a Zariski open set since γ and $\pi^*\omega$ are smooth differential forms. Further $\bar{X} \setminus \pi^{-1}(Z)$ is isomorphic to $X \setminus Z$ by the morphism π and the equality $\gamma = \pi^*\sqrt{-1}\Theta_G$ holds on the Zariski open set $\bar{X} \setminus \pi^{-1}(Z)$ by $\mathrm{Supp}\, E = \pi^{-1}(Z)$ (cf. Remark 3.3). Moreover, we have the inclusion $X_1 \subset X \setminus Z$ and G is smooth on $X \setminus Z$. From the above observations, we can obtain that

$$\begin{aligned}
\int_{\bar{X}} \gamma \wedge \pi^*\omega^{n-1} &= \int_{\bar{X}\setminus\pi^{-1}(Z)} \gamma \wedge \pi^*\omega^{n-1} \\
&= \int_{\bar{X}\setminus\pi^{-1}(Z)} \pi^*(\sqrt{-1}\Theta_G \wedge \omega^{n-1}) \\
&= \int_{X\setminus Z} \sqrt{-1}\Theta_G \wedge \omega^{n-1} \\
&= \int_{X_1} \sqrt{-1}\Theta_G \wedge \omega^{n-1}.
\end{aligned} \tag{3.5}$$

On the other hand, for a given point $p \in X_1$, we take an orthonormal basis $\{e_k\}_{k=1}^n$ of T_X at p satisfying the properties in Claim 3.4. Then we have $\omega = (\sqrt{-1}/2)\sum_{k=1}^n e_k^\vee \wedge \bar{e}_k^\vee$ at p, and thus we obtain

$$\frac{n}{2}\sqrt{-1}\Theta_G \wedge \omega^{n-1} = \sum_{i=1}^m \sqrt{-1}\Theta_G(e_i,\bar{e}_i)\,\omega^n + \sum_{j=m+1}^n \sqrt{-1}\Theta_G(e_j,\bar{e}_j)\,\omega^n$$

from straightforward computations of the trace of $\sqrt{-1}\Theta_G$ with respect to ω. Note that the first term (and also second term) do not depend on the choice of the orthonormal basis $\{e_i\}_{i=1}^n$. The first term of the right hand side is non-negative by property (3) in Claim 3.4. The second term (that is, the vertical part) can be shown to be actually non-negative later. Nevertheless it seems to be quite difficult to directly check this fact. For this reason, we first consider the integral of each term over X_1, and further we show that the integral of the second term is non-negative by using Stokes's theorem and Fubini's theorem (see Claim 3.5). As a result, we can obtain

$$\begin{aligned}
&\frac{n}{2}\int_{X_1} \sqrt{-1}\Theta_G \wedge \omega^{n-1} \\
&= \int_{X_1} \sum_{i=1}^m \sqrt{-1}\Theta_G(e_i,\bar{e}_i)\,\omega^n + \int_{X_1} \sum_{j=m+1}^n \sqrt{-1}\Theta_G(e_j,\bar{e}_j)\,\omega^n.
\end{aligned} \tag{3.6}$$

Indeed, the left hand side converges by (3.5) and each term in the right hand side is non-negative thanks to Claim 3.5 and property (3) in Claim 3.4. Hence each integral in the right hand side can be proven to converge.

Claim 3.5. *The second term of (3.6) is non-negative, that is,*

$$\int_{X_1} \sum_{j=m+1}^n \sqrt{-1}\Theta_G(e_j,\bar{e}_j)\,\omega^n \geq 0.$$

Proof of Claim 3.5. Let ω_Y be a Kähler form on Y. Then, for a given local coordinate $(t_1, t_2, \ldots, t_m)$ of Y_1, there exists a smooth positive function f defined on an open set in Y_1 such that

$$\omega^n = \frac{1}{\phi^*f \cdot |\phi^*dt|^2_{\wedge^m h}}\,\phi^*\omega_Y^m \wedge \omega^{n-m},$$

where $dt := dt_1 \wedge dt_2 \wedge \cdots \wedge dt_m$. We remark that ϕ^*f and $|\phi^*dt|^2_{\wedge^m h}$ depend on the choice of local coordinates, but the product is independent of the coordinates and it is globally defined on X_1.

Indeed, it can be seen that

$$\langle \phi^* dt_\ell, e_j^\vee \rangle_h = \langle \phi^* dt_\ell, e_j \rangle_{\text{pair}} = \langle dt_\ell, \phi_* e_j \rangle_{\text{pair}} = 0$$

for any $j = m+1, \ldots, n$ since e_j is in the kernel of $d\phi_*$. Here the notation $\langle \bullet, \bullet \rangle_{\text{pair}}$ denotes the natural pairing. Therefore we obtain

$$\phi^* dt_\ell = \sum_{k=1}^{n} \langle \phi^* dt_\ell, e_k^\vee \rangle_h \, e_k^\vee = \sum_{i=1}^{m} \langle \phi^* dt_\ell, e_i^\vee \rangle_h \, e_i^\vee.$$

Further we obtain

$$\phi^* dt = \det[\langle \phi^* dt_\ell, e_i^\vee \rangle_h] \, e_1^\vee \wedge e_2^\vee \wedge \cdots \wedge e_m^\vee \quad \text{ and } \quad |\phi^* dt|^2_{\wedge^m h} = \big| \det[\langle \phi^* dt_\ell, e_i^\vee \rangle_h] \big|^2$$

by straightforward computations. On the other hand, the Kähler form ω_Y can be locally written as

$$\omega_Y = \sqrt{-1} \sum_{i,j=1}^{m} f_{ij} dt_i \wedge d\bar{t}_j$$

in terms of the given local coordinate $(t_1, t_2, \ldots, t_m)$. From this local expression, we can easily show that

$$\begin{aligned}\phi^* \omega_Y^m \wedge \omega^{n-m} &= c_m \, \phi^*(\det[f_{ij}]) \, \phi^*(dt \wedge \bar{d}t) \wedge \omega^{n-m} \\ &= d_{n,m} \, \phi^*(\det[f_{ij}]) \, \big| \det[\langle \phi^* dt_\ell, e_i^\vee \rangle_h] \big|^2 \, \omega^n,\end{aligned}$$

where $c_{n,m}$ and $d_{n,m}$ are the universal constants depending only on n and m. Therefore it can be seen that $f := d_{n,m} \det[f_{ij}]$ satisfies the desired equality.

By Fubini's theorem, we have

$$\begin{aligned}&\int_{X_1} \sum_{j=m+1}^{n} \sqrt{-1} \Theta_G(e_j, \bar{e}_j) \, \omega^n \\ =& \int_{Y_1} \frac{1}{f} \, \omega_Y^m \int_{X_y} \frac{1}{|\phi^* dt|^2_{\wedge^m h}} \sum_{j=m+1}^{n} \sqrt{-1} \Theta_G(e_j, \bar{e}_j) \, \omega^{n-m} \\ =& \frac{n-m}{2} \int_{Y_1} \frac{1}{f} \, \omega_Y^m \int_{X_y} \frac{1}{|\phi^* dt|^2_{\wedge^m h}} \sqrt{-1} \Theta_G \wedge \omega^{n-m-1},\end{aligned}$$

where X_y is the fiber of ϕ at $y \in Y_1$. Here we used the equality

$$\sum_{j=m+1}^{n} \sqrt{-1} \Theta_G(e_j, \bar{e}_j) \, \omega^{n-m} = \frac{n-m}{2} \sqrt{-1} \Theta_G \wedge \omega^{n-m-1}$$

of the scalar curvature on the fiber X_y. We finally prove that the fiber integral in the above equality is non-negative. For simplicity, we put $F := |\phi^* dt|^2_{\wedge^m h}$. Then, by the expression of the curvature $\sqrt{-1}\Theta_G$ on $X_1 \subset X_o$ (see (3.1)), we can show that

$$\begin{aligned}
&\int_{X_y} \frac{1}{|\phi^* dt|^2_{\wedge^m h}} \sqrt{-1}\Theta_G \wedge \omega^{n-m-1} \\
=&\int_{X_y} \frac{1}{F} \sqrt{-1}\partial\overline{\partial} \log F \wedge \omega^{n-m-1} \\
=&\sqrt{-1}\int_{X_y} \partial\Big(\frac{1}{F}\,\overline{\partial}\log F \wedge \omega^{n-m-1}\Big) - \sqrt{-1}\int_{X_y} \partial\Big(\frac{1}{F}\Big) \wedge \overline{\partial}\log F \wedge \omega^{n-m-1} \\
=&\int_{X_y} \frac{1}{F^3} \sqrt{-1}\partial F \wedge \overline{\partial} F \wedge \omega^{n-m-1}.
\end{aligned}$$

The last equality follows from Stokes's theorem. The integrand of the right hand side is non-negative, and thus the desired inequality can be obtained. □

By Claim 3.5, we can conclude that all the terms appearing in (3.3) and (3.6) should be equal to zero. Hence we can see that $\Theta_G(e_i, \bar{e}_i) = 0$ for any $i = 1, 2, \ldots, m$ since the first term in (3.6) is zero and its integrand is non-negative by property (3) in Claim 3.4. By applying the formula (2.2) to the exact sequence

$$0 \longrightarrow \operatorname{Ker} d\phi_* \longrightarrow \wedge^m T_X \xrightarrow{d\phi_*} L^\vee = \phi^* K_Y^\vee \longrightarrow 0$$

on a neighborhood of $p \in X_1$, we can obtain

$$\sum_{k=1}^m R_g(e_i, \bar{e}_i, e_k, \bar{e_k}) \leq \sqrt{-1}\Theta_G(e_i, \bar{e}_i) = 0$$

for any $i = 1, 2, \ldots, m$. Each term in the left hand side is non-negative by property (2) in Claim 3.4, and thus we can see that

$$H_g([e_i]) = R_g(e_i, \bar{e}_i, e_i, \bar{e}_i) = 0 \text{ for any } i = 1, 2, \ldots, m.$$

In particular, the tangent vector e_i is the minimizer of the semi-positive holomorphic sectional curvature H_g. Hence it follows that $R_g(v, \bar{v}, e_i, \bar{e}_i)$ is non-negative for any tangent vector $v \in T_X$ from Lemma 2.1. By applying the formula (2.2) to a tangent vector $v \in T_X$ again, we can see that

$$0 \leq \sum_{i=1}^m R_g(v, \bar{v}, e_i, \bar{e}_i) \leq \sqrt{-1}\Theta_G(v, \bar{v}).$$

This means that the curvature $\sqrt{-1}\Theta_G$ is semi-positive on X_1. On the other hand, we have $\gamma = \pi^*\sqrt{-1}\Theta_G$ on a (non-empty) Zariski open set of $\bar{X}$. Hence it follows that γ is a semi-positive $(1,1)$-form on $\bar{X}$ since γ is smooth on $\bar{X}$.

The first Chern class $c_1(\pi^* L^\vee)$ is represented by the sum of the semi-positive form γ and the positive current $[E]$ by (3.2) and the above argument. On the other hand, the first Chern class $c_1(\pi^* L^\vee)$ is numerically trivial on $\bar{X}$, since we have

$$\int_X c_1(L^\vee) \cdot \{\omega\}^{n-1} = \int_{\bar{X}} c_1(\pi^* L^\vee) \cdot \{\pi^*\omega\}^{n-1} = 0$$

and L is pseudo-effective. This implies that $\gamma = 0$ and $E = 0$ (namely, $\sqrt{-1}\Theta_G = 0$). Here we used the fact that there is no positive current (except for the zero current) representing the numerically trivial class since $\bar{X}$ is compact. Therefore we can conclude that G is a smooth metric. In particular, the morphism ϕ is a smooth morphism on X_o.

In the rest of the proof, we will show that

$$B \in C^{\infty}(X, \Lambda^{0,1} \otimes \mathrm{Hom}(T_{X/Y}^{\perp}, T_{X/Y}))$$

defined for the below exact sequence is identically zero on X_o, which leads to the desired decomposition of the tangent bundle T_X (see Subsection 2.1). By the above argument, we have already proved that $R_g(v, \bar{v}, e_i, \bar{e}_i)$ is non-negative for any $i = 1, 2, \ldots, m$ and any tangent vector $v \in T_X$. For the metrics g_S and g_Q on $T_{X/Y}$ and $\phi^* T_Y$ induced by the standard exact sequence

$$0 \longrightarrow (T_{X/Y}, g_S) \longrightarrow (T_X, g) \longrightarrow (\phi^* T_Y, g_Q) \longrightarrow 0 \text{ on } X_o,$$

we can obtain

$$0 \leq R_g(v, \bar{v}, e_i, \bar{e}_i) + \big\langle B_{\bar{v}}(e_i), B_{\bar{v}}(e_i) \big\rangle_{g_S} = \big\langle \sqrt{-1}\Theta_{g_Q}(v, \bar{v})(e_i), e_i \big\rangle_{g_Q} \tag{3.7}$$

by applying the formula (2.2) to the above exact sequence. On the other hand, the induced metric $\det g_Q$ on $\phi^* K_Y^{\vee} = \det \phi^* T_Y$ coincides with the metric G on X_o by the definition of G, and further the curvature $\sqrt{-1}\Theta_G$ of $(\phi^* K_Y^{\vee}, \det g_Q = G)$ is flat on X_o by the above argument. Therefore we have

$$\begin{aligned} &\sum_{i=1}^{m} \big\langle \sqrt{-1}\Theta_{g_Q}(v, \bar{v})(e_i), e_i \big\rangle_{g_Q} \\ =&\big\langle \sqrt{-1}\Theta_{\det g_Q}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m), e_1 \wedge e_2 \wedge \cdots \wedge e_m \big\rangle_{\det g_Q} \\ =&0. \end{aligned}$$

By combining it with the inequality (3.7), we can obtain that

$$R_g(v, \bar{v}, e_i, \bar{e}_i) = 0 \quad \text{and} \quad \big\langle B_{\bar{v}}(e_i), B_{\bar{v}}(e_i) \big\rangle_{g_S} = 0 \tag{3.8}$$

for any $i = 1, 2, \ldots, m$ and any tangent vector $v \in T_X$. Here we used the fact that $\langle B_{\bar{v}}(\bullet), B_{\bar{v}}(\bullet) \rangle_{g_S}$ is a semi-positive definite quadratic form on $T_{X/Y}^{\perp}$. We can see that B is identically zero on X_o since the trace $\sum_{i=1}^{m} \langle B_{\bar{v}}(e_i), B_{\bar{v}}(e_i) \rangle_{g_S}$ is zero. Therefore we obtain the desired decomposition on X_o (see Subsection 2.1). □

Remark 3.6. By (3.8) and Lemma 2.3, it can be seen that e_i is a truly flat tangent vector any $i = 1, 2, \ldots, m$. Hence all the tangent vectors in the horizontal direction are truly flat.

In the rest of this section, we deduce Theorem 1.2 from the proof of Theorem 3.1. Note that we do not need the assumption that a general fiber of $\phi : X \dashrightarrow Y$ is rationally connected in the proof. This assumption will be essentially used in the proof of Theorem 1.5. Hence the inequality in Theorem 1.2 holds under the same setting as in Theorem 3.1.

Proof of Theorem 1.2. In the proof, we will use the arguments and the notation in the proof of Theorem 3.1. We take a Zariski open set Y_1 in Y such that $\phi : X_1 = \phi^{-1}(Y_1) \to Y_1$ is a morphism (over which ϕ is actually smooth by Theorem 3.1). The invariant $n_{\mathrm{tf}}(X, g)_p$ is lower semi-continuous with respect to $p \in X$ in the classical topology (see Definition 2.2). In particular, the condition $n_{\mathrm{tf}}(X, g)_p = n_{\mathrm{tf}}(X, g)$ is an open condition. Hence we can find a point p such that

$$n_{\mathrm{tf}}(X, g)_p = \max_{p \in X} n_{\mathrm{tf}}(X, g)_p \quad \text{and} \quad p \in X_1.$$

It follows that tangent vectors $\{e_i\}_{i=1}^m$ in the horizontal direction at p are truly flat from the proof of Theorem 3.1 (see Remark 3.6). In particular, the vector space $\phi^* T_Y = \mathrm{Spn}\langle \{e_i\}_{i=1}^m \rangle$ at p is contained in $V_{\mathrm{flat},p}$. Therefore we obtain the desired inequality $m \leq n - n_{\mathrm{tf}}(X, g)$. □

4. Proof of the structure theorems

In this section, we give a proof of Theorem 1.3. For this purpose, we first prove Theorem 1.5 and Theorem 1.6. Theorem 1.5 can be obtained as an application of Theorem 3.1 and Lemma 2.4.

Proof of Theorem 1.5. We first take an arbitrary MRC fibration $\phi : X \dashrightarrow Y$ to a smooth projective variety Y. We will find a morphism $\phi' : X \to Y'$ (which may be different from the original one $\phi : X \dashrightarrow Y$) such that $\phi' : X \to Y'$ is one of MRC fibrations of X, by using Theorem 3.1, Lemma 2.4, and a similar way to [CH19, Theorem 2.1]. Let X_o be the maximal Zariski open set in X such that the restriction ϕ_o of ϕ

$$\phi_o := \phi|_{X_o} : X_o \to Y_o := \phi(X_o) \subset Y$$

is a morphism and let $i : X_o \hookrightarrow X$ be the natural inclusion. Note that the codimension of the Zariski closed set $X \setminus X_o$ is larger than or equal to two. In summary, we have the following commutative diagram:

$$\begin{array}{ccc} X_o & & \\ {\scriptstyle i}\downarrow & \searrow^{\phi_o} & \\ X & \overset{\phi}{\dashrightarrow} & Y. \end{array}$$

By Theorem 3.1, the orthogonal complement $T_{X_o/Y_o}^{\perp}$ is a holomorphic vector bundle and there exists the isomorphism $j : \phi^* T_{Y_o} \cong T_{X_o/Y_o}^{\perp}$ such that

$$T_{X_o} = T_{X_o/Y_o} \oplus j(\phi_o^* T_{Y_o}). \tag{4.1}$$

Now we respectively define the coherent sheaves V and W on X to be the double duals of

$$V := \Big(i_* \mathcal{O}_{X_o}\big(j(\phi_o^* T_{Y_o})\big)\Big)^{\vee\vee} \text{ and } W := \Big(i_* \mathcal{O}_{X_o}(T_{X_o/Y_o})\Big)^{\vee\vee}.$$

Note that V and W are reflexive sheaves on X since they are defined by the double dual. Then it can be seen that $T_X = V \oplus W$ holds on X_o by the holomorphic decomposition (4.1). By lemma 2.7 and $\mathrm{codim}(X \setminus X_o) \geq 2$, this decomposition can be extended to the ambient space X, that is, we obtain the decomposition

$$T_X \cong V \oplus W \text{ on } X.$$

Now we will show that the subsheaf $W \subset T_X$ is actually a subbundle of T_X and it determines an integrable foliation. The dimensions

$$\dim(W_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p}/\mathfrak{m}_p) \quad \text{and} \quad \dim(V_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p}/\mathfrak{m}_p)$$

are upper semi-continuous with respect to $p \in X$ by Lemma 2.6 and the sum is equal to $n = \dim X$. Hence they should be constant. Then it follows that V and W are actually locally free sheaves from Lemma 2.6. In order to check the integrality of $W \subset T_X$, we consider the section

$$F_W \in H^0(X, \mathrm{Hom}(\wedge^2 W, T_X/W))$$

obtained from the Lie bracket (see Subsection 2.3). The foliation $W \subset T_X$ is integrable on X_o since $\phi_o : X_o \to Y_o$ is smooth and $W = T_{X_o/Y_o}$ holds on X_o. In particular, the section F_W is vanishing on

X_o. Then it follows that F_W is identically zero on X since the sheaf $Hom(\wedge^2 W, T_X/W)$ is locally free. Hence we can see that the foliation $W \subset T_X$ is integrable.

A general fiber of ϕ is a leaf of the integrable foliation $W \subset T_X$. Hence a general leaf is compact since $\phi : X \dashrightarrow Y$ is almost holomorphic and it is rationally connected since $\phi : X \dashrightarrow Y$ is an MRC fibration. By applying Lemma 2.4 to our situation, we can find a smooth morphism $\phi' : X \to Y'$ such that $W = T_{X/Y'}$.

It remains to show that the morphism $\phi' : X \to Y'$ is also an MRC fibration. A general fiber of $\phi : X \dashrightarrow Y$ is not only a leaf of W but also a leaf of $T_{X/Y'}$. Therefore we can construct a birational map $\pi : Y \dashrightarrow Y'$ such that

$$\begin{array}{ccc} X & & \\ {\scriptstyle \phi}\downarrow & \searrow^{\phi'} & \\ Y & \overset{\pi}{\dashrightarrow} & Y'. \end{array}$$

This means that the morphism $\phi' : X \to Y'$ is an MRC fibration. □

We will prove Theorem 1.6 by using Lemma 2.5 and the proof of Theorem 3.1.

Proof of Theorem 1.6. By Theorem 3.1, we can see that ϕ is a smooth morphism and the standard exact sequence admits the holomorphic orthogonal splitting, that is, there exists the isomorphism $j : \phi^* T_Y \cong T_{X/Y}^{\perp} \subset T_X$ as holomorphic vector bundles such that it gives the orthogonal decomposition

$$T_X = T_{X/Y} \oplus T_{X/Y}^{\perp} = T_{X/Y} \oplus j(\phi^* T_Y).$$

We first show that there exists a Kähler metric g_Y on T_Y such that $g_Q = \phi^* g_Y$ holds and the holomorphic sectional curvature of (Y, g_Y) is identically zero. This implies that the metric g_Y is flat and the manifold Y has a finite étale cover $T \to Y$ by a complex torus T (see [HLW16, Proposition 2.2], [Igu54], and [Ber66]). Here g_Q is the metric on $\phi^* T_Y$ induced by g and the exact sequence

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \xrightarrow{d\phi_*} \phi^* T_Y \longrightarrow 0.$$

For a (local) holomorphic vector field v of T_Y defined on an open set U in Y, we consider the section $\phi^* v \in H^0(\phi^{-1}(U), \phi^* T_Y)$ defined by

$$\phi^{-1}(U) \ni p \to v(\phi(p)) \in T_{Y,\phi(p)} = (\phi^* T_Y)_p.$$

If the function $|\phi^* v|_{g_Q}$ is a constant on the fiber X_y, we can define the hermitian metric g_Y of Y by $|v|_{g_Y} := |\phi^* v|_{g_Q}$. Then we have $g = \phi^* g_Y$ by the definition.

If we can show that the restriction of $\sqrt{-1}\partial\overline{\partial} \log |\phi^* v|_{g_Q}$ to the fiber is a semi-positive $(1,1)$-form, it should be a constant by the maximal principle, since it is a psh function globally defined on the compact fiber. For this purpose, we consider the sub-line bundle L of $\phi^* T_Y$

$$(L := \mathrm{Spn}\langle \phi^* v \rangle, g_L) \subset (\phi^* T_Y, g_Q)$$

spanned by $\phi^* v$, where g_L is the induced metric on L from g_Q. By the definition of the curvature and the induced metric, we obtain

$$\sqrt{-1}\Theta_{g_L} := \sqrt{-1}\Theta_{g_L}(L) = -\sqrt{-1}\partial\overline{\partial} \log |\phi^* v|^2_{g_L} = -\sqrt{-1}\partial\overline{\partial} \log |\phi^* v|^2_{g_Q}.$$

By applying the formula (2.2) to the above injective bundle morphism, we obtain that

$$\sqrt{-1}\Theta_{g_L}(w, \bar{w}) |\phi^* v|^2_{g_L} = \left\langle \sqrt{-1}\Theta_{g_L}(w, \bar{w})(\phi^* v), \phi^* v \right\rangle_{g_L} \leq \left\langle \sqrt{-1}\Theta_{g_Q}(w, \bar{w})(\phi^* v), \phi^* v \right\rangle_{g_Q}$$

for a tangent vector $w \in T_X$. We have already shown that the tangent vectors $\{e_i\}_{i=1}^m$ in the horizontal direction are truly flat in the proof of Theorem 3.1 (see Remark 3.6). The vector $\phi^* v$ can be written as a linear combination of $\{e_i\}_{i=1}^m$, and thus it is also truly flat. On the other hand, by the holomorphic orthogonal decomposition $T_X = T_{X/Y} \oplus \phi^* T_Y$, the section $\phi^* v$ of $\phi^* T_Y$ determines the section of T_X, which we denote by the same notation $\phi^* v$. Then we obtain

$$\left\langle \sqrt{-1}\Theta_{g_Q}(w,\bar{w})(\phi^* v), \phi^* v \right\rangle_{g_Q} = \left\langle \sqrt{-1}\Theta_{g}(w,\bar{w})(\phi^* v), \phi^* v \right\rangle_{g} = 0.$$

The right equality follows from the truly flatness of $\phi^* v$. Therefore we can see that $\sqrt{-1}\Theta_{g_L}$ is semi-negative (in particular $|\phi^* v|_{g_Q}$ is a constant).

We will check that the holomorphic sectional curvature of g_Y is identically zero. For a given tangent vector $v \in T_Y$, the vector $\phi^* v$ satisfies $d\phi_*(\phi^* v) = v$. Hence we can easily see that

$$\begin{aligned} 0 = \left\langle \sqrt{-1}\Theta_{g}(\phi^* v,\bar{\phi}^* v)(\phi^* v), \phi^* v \right\rangle_{g} &= \left\langle \sqrt{-1}\Theta_{g_Q}(\phi^* v,\bar{\phi}^* v)(\phi^* v), \phi^* v \right\rangle_{g_Q} \\ &= \left\langle \sqrt{-1}\Theta_{g_Y}(v,\bar{v})(v), v \right\rangle_{g_Y} \end{aligned}$$

by $d\phi_*(\phi^* v) = v$, $g_Q = \phi^* g_Y$, and the truly flatness of $\phi^* v$.

By the above argument, we can see that it is enough for the statement (1) in Theorem 1.6 to check that ϕ is locally trivial. For this purpose, we will show that the holomorphic subbundle $j(\phi^* T_Y) = T_{X/Y}^{\perp} \subset T_X$ determines an integrable foliation. Since g is a Kähler metric, the Chern connection D of (T_X, g) corresponds to the Levi-Civita connection of the induced Riemannian metric. Hence, from the torsion-freeness, we obtain that

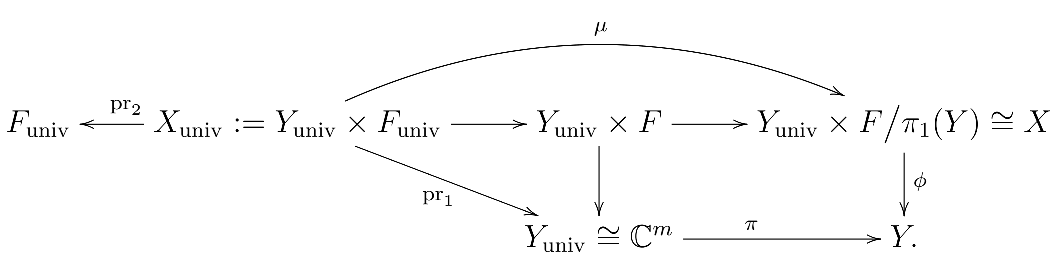
$$[u_1, u_2] = D_{u_1} u_2 - D_{u_2} u_1 \tag{4.2}$$

for any (local) vector fields u_1, u_2 on X. On the other hand, for a vector field u in the horizontal direction (that is, it is a section of $j(\phi^* T_Y) = T_{X/Y}^{\perp}$), we have

$$D(u) = B(u) + D_Q(u) = D_Q(u).$$

Here we used the fact that B is identically zero, which can be obtained from the holomorphic orthogonal splitting of the above exact sequence. In particular, we can see that $D_{u_1} u_2$ and $D_{u_2} u_1$ are also vector fields in the horizontal direction for any vector fields u_1 and u_2 in the horizontal direction. By the formula (4.2), we can conclude that the Lie bracket $[u_1, u_2]$ is in the horizontal direction. This means that the horizontal tangent bundle $j(\phi^* T_Y)$ is integrable. Therefore, by the Ehresmann theorem (see Lemma 2.5), we can see that ϕ is locally trivial. This verifies the statement (1).

Further there exists a representation $\rho : \pi_1(Y) \to \mathrm{Aut}(F)$ such that $X \cong Y_{\mathrm{univ}} \times F / \pi_1(Y)$. Then we have the following commutative diagram :



Moreover the Ehresmann theorem asserts that the étale cover

$$\mu : Y_{\rm univ} \times F_{\rm univ} \to Y_{\rm univ} \times F \to Y_{\rm univ} \times F/\pi_1(Y) \cong X$$

is the universal cover of X and we have

$$\mathrm{pr}_1^* T_{Y_{\rm univ}} = \mu^* j(\phi^* T_Y) \quad \text{and} \quad \mathrm{pr}_2^* T_{F_{\rm univ}} = \mu^*(T_{X/Y}). \tag{4.3}$$

This verifies the statement (2).

It remains to show the statement (3) in Theorem 1.6. Now we consider the induced metric $\mu^* g$ on $\mu^* T_X = T_{X_{\rm univ}}$ and the holomorphic decomposition

$$T_{X_{\rm univ}} = \mathrm{pr}_1^* T_{Y_{\rm univ}} \oplus \mathrm{pr}_2^* T_{F_{\rm univ}} = \mu^* j(\phi^* T_Y) \oplus \mu^*(T_{X/Y}).$$

The relative tangent bundle $T_{X/Y}$ is orthogonal to $j(\phi^* T_Y) = T_{X/Y}^{\perp}$ with respect to the metric g, and thus the above decomposition is also an orthogonal decomposition with respect to $\mu^* g$. Therefore it is sufficient to prove that g_1 and g_2 are respectively obtained from the pull-back of some metric of $T_{Y_{\rm univ}}$ and $T_{F_{\rm univ}}$. Here g_1 (resp. g_2) is the metric on $\mathrm{pr}_1^* T_{Y_{\rm univ}}$ (resp. $\mathrm{pr}_2^* T_{F_{\rm univ}}$) induced by the metric $\mu^* g$ and the above decomposition.

We can easily prove that g_1 coincides with the pull-back of $\pi^* g_Y$ by pr_1. Indeed, we have $g_1 = \mu^* g_Q$ by $\mathrm{pr}_1^* T_{Y_{\rm univ}} = \mu^* j(\phi^* T_Y)$ and $g_Q = \phi^* g_Y$ by the property of g_Q. Hence, by the above commutative diagram, we can easily check that

$$g_1 = \mu^*(\phi^* g_Y) = \mathrm{pr}_1^*(\pi^* g_Y).$$

In the rest of the proof, we will show that the metric g_2 on $\mathrm{pr}_2^* T_{F_{\rm univ}}$ can be obtained from the pull-back of some metric on $F_{\rm univ}$, in other words, the metric on the fiber $\mathrm{pr}_1^{-1}(y) (\cong F_{\rm univ})$ defined by the restriction of $\mu^* g$ and $\mathrm{pr}_2^* T_{F_{\rm univ}}|_{\mathrm{pr}_1^{-1}(y)} \cong T_{F_{\rm univ}}$ is independent of $y \in Y_{\rm univ}$. For a local holomorphic vector field w on $F_{\rm univ}$, we consider the section $\widetilde{w} := \mathrm{pr}_2^* w$ of $\mathrm{pr}_2^* T_{F_{\rm univ}}$. Note that the section $\widetilde{w}$ can be seen as a vector field in the vertical direction by the inclusion $\mathrm{pr}_2^* T_{F_{\rm univ}} \subset T_{X_{\rm univ}}$. If the norm $|\widetilde{w}|_{\mu^* g} = |\widetilde{w}|_{g_2}$ is constant on a fiber $\mathrm{pr}_2^{-1}(p) (\cong Y_{\rm univ})$ of the second projection pr_2, we can easily see that the metric $g_{F_{\rm univ}}$ on $F_{\rm univ}$ defined by $|w|_{g_{F_{\rm univ}}} = |\widetilde{w}|_{\mu^* g}$ satisfies the desired property (that is, $g_2 = \mathrm{pr}_2^* g_{F_{\rm univ}}$). In order to check that $|\widetilde{w}|_{\mu^* g}$ is constant on a fiber $\mathrm{pr}_2^{-1}(p)$, we will show that the differential of $|\widetilde{w}|_{\mu^* g}$ by the vector field $\widetilde{v}$ is identically zero for any (local) vector v on $Y_{\rm univ}$. Here $\widetilde{v}$ is the vector field on $X_{\rm univ}$ defined by the section $\widetilde{v} := \phi^* v$ of $\mathrm{pr}_1^* T_{Y_{\rm univ}}$ and the inclusion $\mathrm{pr}_1^* T_{Y_{\rm univ}} \subset T_{X_{\rm univ}}$. By the formula $\partial |\widetilde{w}|^2_{\mu^* g} = \langle D\widetilde{w}, \widetilde{w} \rangle_{\mu^* g}$, we obtain

$$\langle \partial |\widetilde{w}|^2_{\mu^* g}, \widetilde{v} \rangle_{\rm pair} = \langle D_{\widetilde{v}} \widetilde{w}, \widetilde{w} \rangle_{\mu^* g}.$$

On the other hand, we have

$$[\widetilde{v}, \widetilde{w}] = D_{\widetilde{v}} \widetilde{w} - D_{\widetilde{w}} \widetilde{v}$$

by (4.2). The vector fields $\widetilde{v}$ and $\widetilde{w}$ can be locally written as

$$\widetilde{v} = \sum_{i=1}^{m} a_i(z) \frac{\partial}{\partial z_i} \quad \text{and} \quad \widetilde{w} = \sum_{j=m+1}^{n} b_i(w) \frac{\partial}{\partial w_j}$$

in terms of a local coordinate $z = (z_1, \ldots, z_m)$ of $Y_{\rm univ}$ and a local coordinate $w = (w_{m+1}, \ldots, w_n)$ of $F_{\rm univ}$, since $\widetilde{v}$ (resp. $\widetilde{w}$) is constructed from the pull-back by pr_1 (resp. pr_2). From the above local expression and straightforward computations, we can easily check that $[\widetilde{v}, \widetilde{w}] = 0$. Further $D\widetilde{v}$ and $D\widetilde{w}$ respectively preserve the horizontal and the vertical direction since the natural splitting $T_{X_{\rm univ}} = \mathrm{pr}_1^* T_{Y_{\rm univ}} \oplus \mathrm{pr}_2^* T_{F_{\rm univ}}$ is an orthogonal decomposition (see the proof of the statement (1)).

Hence we can conclude that $D_{\widetilde{v}}\widetilde{w} = 0$ and $D_{\widetilde{w}}\widetilde{v} = 0$. The differential of the norm $|\widetilde{w}|^2_{\mu^* g}$ in the horizontal direction is identically zero, and thus it is constant on a fiber of pr_2. This finishes the proof. □

We finally prove the main result of this paper, as a direct application of Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.3. When the holomorphic sectional curvature is identically zero, the variety X itself has a finite étale cover $A \to X$ by an abelian variety A by [Igu54] (see also [HLW16, Proposition 2.2] and [Ber66]). In this case, there is nothing to prove since the identity map of X satisfies the desired properties.

When the holomorphic sectional curvature is semi-positive but not identically zero, the canonical bundle K_X is not pseudo-effective. Indeed, we have the equality

$$\int_X c_1(K_X) \wedge \omega^{n-1} = -\frac{1}{n\pi} \int_X S\, \omega^n,$$

where ω is the Kähler form associated to g and S is the scalar curvature of g. In our case, the scalar curvature S is positive on a neighborhood of at least one point since S can be described as the integral of the holomorphic sectional curvature (for example, see the proof of [Yan18a, Proposition 6.4]). Therefore the right hand side is negative. This implies that X is uniruled by [BDPP13]. There is nothing to prove in the case of X being rationally connected. Therefore we may assume that X admits a non-trivial MRC fibration. In this case, by Theorem 1.5, we can take an MRC fibration $\phi : X \to Y$ of X to be a morphism to a smooth projective variety Y. Further it can be seen that the canonical bundle K_Y of Y is pseudo-effective by [GHS03] and [BDPP13]. Hence the assumptions in Theorem 1.6 are satisfied. We then obtain all the conclusions by applying Theorem 1.6 to the MRC fibration $\phi : X \to Y$ and by using the fact that rationally connected manifolds are simply connected. □

By the classification of surfaces, we can see that Theorem 1.3 holds for compact Kähler surfaces Indeed, let X be a compact Kähler surface such that the holomorphic sectional curvature is semi-positive, but not identically zero. Then, by the proof of Theorem 1.3, we can see that K_X is not pseudo-effective. From the classification of compact complex surfaces, it follows that X is a rational surface, a ruled surface over a curve of genus ≥ 1, or a minimal surface of class VII. However a minimal surface of class VII is not Kähler, and thus we can conclude that X is rationally connected or a ruled surface over a curve of genus ≥ 1. Hence X is automatically projective.

5. Open problems related to semi-positive holomorphic sectional curvature

In this section, we give open problems related to the geometry of semi-positive holomorphic sectional curvature.

The most important problem concerning the structure theorem is an extension of the main results in this paper to compact Kähler manifolds. The main difficulty of this problem lies in constructing MRC fibrations of (non-projective) compact Kähler manifolds. As the first step to this problem, it seems to be important to ask what conditions on holomorphic sectional curvature guarantee that compact Kähler manifolds are automatically projective. It follows that any compact Kähler manifolds with positive holomorphic sectional curvature are automatically projective from [Yan18a, Theorem 1.7]. Then, the following problem, which was posed by Yang in a private discussion, naturally arises as a generalization of Yang's criteria of projectivity. It is also an interesting problem to consider rational connectedness or holomorphic sectional curvature from the viewpoint of (uniform) RC positivity (see [Yan18b] for vanishing theorems and [Yan18c, Theorem 1.7, Conjecture 1.9] for rational connectedness).

Problem 5.1. *Let (X, g) be a compact Kähler manifold (or more generally a hermitian manifold) with the semi-positive holomorphic sectional curvature H_g. Assume that X has no truly flat vector at some point of X (or H_g is quasi-positive).*

(1) *Does it hold that $h^0(X, \Omega_X^q) = 0$ for any $q > 0$?*
(2) *Is X automatically projective?*

The MRC fibration appearing in Theorem 1.3 coincides with the Albanese map after we take the fiber product $X^* = A \times_Y X$ by a finite étale cover $A \to Y$. Hence, in the case of compact Kähler manifolds, it seems to be natural to focus on Albanese maps instead of MRC fibrations and expect that Albanese maps actually give MRC fibrations. Based this strategy, the fiber of Albanese maps is expected to have a certain quasi-positivity (see Problem 5.2). It can be seen that the MRC fibration coincides with the Albanese map if Problem 5.1 and 5.2 are shown to be true.

Problem 5.2. *Let (X, g) be a compact Kähler manifold with the semi-positive holomorphic sectional curvature H_g. We take a finite étale cover $X^* \to X$ such that the covering space X^* attains the augmented irregularity, that is,*

$$q(X^*) = \max\{q(X') \,|\, X' \to X \textit{ is a finite étale cover}\}$$

and consider its Albanese map $\alpha : X^* \to \mathrm{Alb}(X^*)$.

(1) *Is the fiber F of the Albanese map automatically projective?*
(2) *Does the fiber F admit a Kähler metric g_F such that $n_{\mathrm{tf}}(F, g_F) = \dim F$?*
(3) *Does the equality $\dim X = \dim Y + n_{\mathrm{tf}}(X, g)$ hold?*

As an analog of semi-positive holomorphic bisectional curvature, there is the notion of nef tangent bundles in algebraic geometry. It is interesting to consider an analog of semi-positive holomorphic sectional curvature in algebraic geometry. From the viewpoint of algebraic geometry, it is also an interesting problem to classify all varieties of dimension two or three admitting semi-positive holomorphic sectional curvature by applying the structure theorem.

Problem 5.3.

(1) *Can we classify all the compact surfaces (or the 3-folds) with semi-positive (or positive) holomorphic sectional curvature?*
(2) *Can we find an analog of semi-positive (or positive) holomorphic sectional curvature?*

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