UPPER AND LOWER BOUNDS FOR RICH LINES IN GRIDS

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Abstract. We prove upper and lower bounds for the number of lines in general position that are rich in a Cartesian product point set. This disproves a conjecture of Solymosi and improves work of Elekes, Borenstein and Croot, and Amirkhanyan, Bush, Croot, and Pryby.

The upper bounds are based on a version of the asymmetric Balog-Szemerédi-Gowers theorem for group actions combined with product theorems for the affine group. The lower bounds are based on a connection between rich lines in Cartesian product sets and amenability (or expanding families of graphs in the finite field case).

As an application of our upper bounds for rich lines in grids, we give a geometric proof of the asymmetric sum-product estimates of Bourgain and Shkredov.

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1. Introduction

Let $\mathbb{F}$ be a field and let $0 < \alpha \leq 1$ be a real number.

A line $\ell$ in the plane $\mathbb{F}^2$ is $\alpha$-rich in a Cartesian product point set $Y \times Y \subseteq \mathbb{F}^2$ if

$$|\ell \cap (Y \times Y)| \geq \alpha |Y|.$$  

For short, we call $Y \times Y$ a grid. Any line contains at most $N$ points of a $N \times N$ grid, so a line is $\alpha$-rich if it contains $\alpha$-percent of the maximum possible points of incidence. The parameter $\alpha$ may be a constant independent of $N$, or may be some small power of $1/N$.

There are two questions we wish to answer about rich lines in grids:

(1) how many $\alpha$-rich lines can a $N \times N$ grid support?

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(2) if a grid supports many rich lines, must these lines have some structure?

The first question was answered for $F = \mathbb{R}$ by Szemerédi-Trotter [37]: a $N \times N$ grid has at most $O(\alpha^{-3}N)$ $\alpha$-rich lines; this is sharp. Below, we discuss extensions of this upper bound and the examples that provide lower bounds. (We use standard asymptotic notation; see Section 1.4 for definitions.)

The second question is an inverse problem for point-line incidences. The inverse problem for the Szemerédi-Trotter theorem is to show that if $n$ points and $n$ lines in $\mathbb{R}^2$ have $\Omega(n^{4/3})$ incidences, then the point set has some structure [11, Problem 5.7]; sharpness examples suggest that the point set might contain a large Cartesian product of arithmetic progressions. Even under the assumption that the point set is a Cartesian product, little is known about the inverse problem for Szemerédi-Trotter. Question 2 has the further simplifying assumption that the lines are rich; in this case, it is possible to give a precise description of the set of lines and the point set [14, 15].

Solymosi conjectured that in the absence of structure, a grid can support at most a constant number of $\alpha$-rich lines [15, Conjecture 3.10]. A generic collection of lines contains no two parallel lines and no three lines through a common point; such a set of lines is said to be in general position. Solymosi’s conjecture is the following.

**Conjecture 1** ([15, Conjecture 3.10]). Among the lines that are $\alpha$-rich in a $N \times N$ Cartesian product, at most $C = C(\alpha) > 0$ can be in general position.

In [15], Conjecture 1 is stated for lines defined over $\mathbb{R}$ or $\mathbb{C}$. Solymosi’s conjecture is supported by the sharpness examples for the Szemerédi-Trotter incidence bound, and also implies a plausible conjecture of Elekes [15, Problem 3.9]; see Section 9 of [2] for a discussion.

Despite this evidence, Conjecture 1 is false: we disprove it with explicit examples over $\mathbb{Q}$, $\mathbb{C}$, and $\mathbb{F}_p$, the finite field with prime cardinality $p$. The examples we give are quite different from Cartesian products of arithmetic (or geometric) progressions, which show that the Szemerédi-Trotter incidence bound is sharp and motivate the sum-product conjecture.

Let $RLGP(F, N, \alpha)$ denote the maximum over all $Y \subseteq F$ with $|Y| = N$ of the maximum number of lines that are $\alpha$-rich in $Y \times Y$ and in general position. Explicitly, if $RLGP(Y, \alpha)$ is the maximum number of $\alpha$-rich lines in $Y \times Y$ that are in general position, then

$$RLGP(F, N, \alpha) := \max_{Y \subseteq F, |Y| = N} RLGP(Y, \alpha).$$

Conjecture 1 posits that for $F = \mathbb{R}$ (or $F = \mathbb{C}$) and for all $0 < \alpha < 1$, there is a constant $C(\alpha) > 0$ depending on $\alpha$ such that

$$RLGP(F, N, \alpha) \leq C(\alpha).$$

For $F = \mathbb{Q}$, we prove a lower bound for $RLGP(Q, N, \alpha)$ that is nearly logarithmic in $N$. In particular, this disproves Conjecture 1 for $F = \mathbb{R}$ and $F = \mathbb{C}$. 
Theorem 2. There is an absolute constant $C > 0$ such that for any $0 < \alpha < 1$

$$RLGP(Q, N, \alpha) \geq C(1 - \alpha) \frac{\log N}{\log \log N}.$$  

For $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{F}_p$, we prove upper and lower bounds for $RLGP(\mathbb{F}, N, \alpha)$ whose logarithms differ by a square root.

Theorem 3. Let $\mathbb{F}$ denote $\mathbb{C}$ or $\mathbb{F}_p$. For every $0 < \alpha < 1$, there is a constant $C_\alpha > 0$ such that

$$\frac{1}{C_\alpha} \sqrt{\frac{\log N}{\log \log N}} \leq \log RLGP(\mathbb{F}, N, \alpha) \leq C_\alpha \frac{\log N}{\log \log N}.$$  

If $\mathbb{F} = \mathbb{F}_p$, the upper bound holds only if $N^{1+\log(2/\alpha)/\log \log N} \leq p$.

The upper bound in Theorem 3 applies when $\mathbb{F} = \mathbb{R}$, since we may consider points and lines defined over $\mathbb{R}$ to be contained in $\mathbb{C}^2$. It would be interesting to know the true order of magnitude of $RLGP(\mathbb{F}, N, \alpha)$ for $\mathbb{F} = \mathbb{C}, \mathbb{F}_p$.

The upper bound in Theorem 3 is a special case of the following general structure theorem for rich lines in grids over $\mathbb{C}$ and $\mathbb{F}_p$.

Theorem 4. There is an absolute constant $C > 0$ such that the following holds. Let $Y$ be a finite subset of $\mathbb{F}$ and let $L$ be a set of $\alpha$-rich lines in $Y \times Y$. Let $J > 0$ be an integer such that $(\alpha/2)^{2J} \geq 1/|Y|$.

If $\mathbb{F} = \mathbb{C}$, then there is a subset $L' \subseteq L$ such that

1. the lines of $L'$ are either parallel or concurrent, and
2. $|L'| \gg \left(\frac{\alpha}{2}\right)^{2J} |Y|^{-C/J} |L|.$

If $\mathbb{F} = \mathbb{F}_p$, then the same conclusion holds, provided that $|Y| \leq (\alpha/2)^{2J} p$.

The key point is that by taking $J$ sufficiently large, the factor $|Y|^{-C/J}$ becomes negligible. Theorem 4 is a consequence of a version of the asymmetric Balog-Szemerédi-Gowers theorem for group actions, proved in [28], combined with a product theorem for the affine group.

The lower bounds in Theorems 2 and 3 follow from explicit constructions; see Theorems 14, 15, and 19 in Section 2. If Conjecture 1 were true over $\mathbb{R}$, then subgroups of $\text{Aff}(1, \mathbb{R})$ generated by a finite set of affine transformations in general position would not be amenable, however finitely generated solvable groups are amenable; Theorem 14 proves this quantitatively. Heuristically, if Conjecture 1 were true over $\mathbb{F}_p$, it might be possible to make an expanding family of Schreier graphs for $\text{Aff}(1, \mathbb{F}_p) \curvearrowright \mathbb{F}_p$ with bounded degree (following a similar strategy to Bourgain and Gamburd [6]), however this is known to be false by a theorem of Lubotzky and Weiss [27, 26].

To prove the lower bound in Theorem 3, we use a construction of Klawe [24, 25], which gives a quantitative proof of Lubotzky and Weiss’ theorem for $\text{Aff}(1, \mathbb{F}_p) \curvearrowright \mathbb{F}_p$; using a theorem of Grosu [20], we embed our counterexample into $\mathbb{C}^2$.

In Section 2 we construct examples of grids that support many $\alpha$-rich lines, and in Section 3 we prove upper bounds the number of $\alpha$-rich lines supported by a $N \times N$ grid. These sections are completely independent. The
remainder of the introduction contains background on rich lines in grids and
some positive results towards Conjecture 1, as well as an explanation of the
connection between rich lines and grids and sum-product problems.

For completeness, we sketch the proof of the group action version of the
Balog-Szemerédi-Gowers theorem and prove the necessary product theorems
for affine transformations in Appendices A and B. In particular, Appendix A
gives a proof of Elekes’ Theorem 5 and compares it with the proof of Theo-
rem 4.

1.1. Background on rich lines in grids. As mentioned, the Szemerédi-
Trotter theorem [37] implies that $O(\alpha^{-3}N)$ lines may be $\alpha$-rich in a $N \times N$
grid in $\mathbb{R}^2$. This lower bound is attained by two simple examples, up to
factors of $\alpha$.

(1) If $Y = \{1, \ldots, N\}$, then the parallel lines $\ell(x) = x + b$ are $\alpha$-rich for
$b \ll (1 - \alpha)N$, thus $Y \times Y$ supports roughly $N$ parallel $\alpha$-rich lines.

(2) If $Y = \{1, 2, \ldots, 2^{N-1}\}$, then the lines $\ell(x) = 2^jx$ through the ori-
gin are $\alpha$-rich for $j \ll (1 - \alpha)N$, thus $Y \times Y$ supports roughly $N$
concurrent $\alpha$-rich lines.

A more elaborate example, due to Erdős, achieves the correct power of $\alpha$.

Example. Let $N$ be a large positive integer, let $Y = \{n \in \mathbb{Z}: |n| \leq N\}$ and
let $P = Y \times Y$. For coprime integers $a < b$, define a set of lines

$$L_{a,b} = \{y - j = a \left(\frac{x}{b} - i\right): 1 \leq i \leq b, 1 \leq j \leq \frac{N}{2}\}.$$ 

Each line in $L_{a,b}$ is incident to at least $\frac{N}{2b} - 1$ points of $P$, and thus is $\alpha$-rich
in $P$ for $b \leq \left\lfloor \frac{1}{3\alpha} \right\rfloor$. On the other hand, the number of such lines is $\Theta(\alpha^{-3}N)$.
See [33] for details.

The following theorem of Elekes [12, 15] says that combinations of exam-
pies (1) and (2) are essentially the only possibilities.

**Theorem 5 (Elekes).** Let $0 < \alpha \leq 1$ be a constant. If $N$ lines are $\alpha$-rich
in an $N \times N$ grid in $\mathbb{R}^2$, then either

(1) $Ca^CN$ lines are parallel, or

(2) $Ca^CN$ lines are concurrent (incident to a common point),

where $C > 0$ is a constant independent of $\alpha$ and $N$.

By applying Freiman’s theorem, Elekes concludes that the family of parallel
lines obtained in Theorem 5 have $y$-intercepts in a generalized arithmetic
progression (similarly, if the lines are concurrent, then their slopes are in a
generalized geometric progression) [14, 15].

Elekes reduces the proof of Theorem 5 to a product theorem. If $A$ and $B$
are finite sets of real affine transformations, then we define their composition
set by

$$A \circ B := \{\ell_a \circ \ell_b: \ell_a \in A, \ell_b \in B\}.$$ 

The collection of affine transformations is a group with product given by
composition of functions, so $A \circ B$ is the just the product set of $A$ and $B$. 


**Theorem 6** (Elekes [12, Theorem 1]). For every $K > 0$ there is a constant $\rho = \rho(K) > 0$ depending on $K$ with the following property.

Suppose $A, B$ are finite sets of real affine transformations with $|A|, |B| \geq N$ and

$$|A \circ B| \leq KN$$

Then there exist subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'|, |B'| \geq \rho N$ such that either

(1) both $A'$ and $B'$ consist of parallel lines, or
(2) both $A'$ and $B'$ consist of concurrent lines.

Though it is not explicit in Elekes’ work, $\rho$ depends polynomially on $K$. Parallel and concurrent lines correspond to cosets of abelian subgroups of the affine group, thus Theorem 6 is perhaps the first instance of a product theorem for a non-commutative group. Such theorems have now been studied extensively [7, 9, 10, 18, 21, 22, 31].

The assumption that $|L| \approx |Y|$ is essential for Elekes’ reduction of Theorem 5 to Theorem 6. Borenstein and Croot [4] made the first step towards removing this restriction. Building on [4], Amirkhanyan, Bush, Croot, and Pryby [2] proved an analog of Conjecture 1 where $\alpha = N^{-\delta}$ for some small $\delta > 0$.

**Theorem 7** (Amirkhanyan, Bush, Croot, and Pryby). For all $\epsilon > 0$ there exists a $\delta > 0$ such that the following holds for all sufficiently large positive integers $N$:

If $L$ is a set of $N^\epsilon$ lines in $\mathbb{R}^2$ that are $\alpha = N^{-\delta}$-rich in an $N \times N$ grid, then the lines of $L$ are not in general position.

Theorem 7 implies that for all $\epsilon > 0$, if $N$ is sufficiently large, then

$$RLGP(\mathbb{R}, N, \alpha) \leq N^\epsilon.$$ 

In [2, 4], the relationship between $\epsilon$ and $\delta$ is not explicit, so it is unclear how strong of a bound this method can achieve.

Borenstein and Croot roughly follow Elekes’ method: they reduce to the case of small product set, then contradict structural hypotheses about the initial set of lines. They do not use Theorem 6 (or a similar theorem), but instead use sum-product results, some of which are unique to $\mathbb{R}$. In particular, it is not clear that their methods should extend to $\mathbb{F}_p$ or to other questions about rich transformations for other groups, such as linear fractional transformations [16].

We use a *group action* version of the Balog-Szemerédi-Gowers theorem [28] to reduce the proof of Theorem 4 to a product theorem for the affine group; in particular, over $\mathbb{R}$ we could use Elekes’ Theorem 6. The group action Balog-Szemerédi-Gowers theorem is a generalization of Tao and Vu’s asymmetric Balog-Szemerédi-Gowers theorem [39, Theorem 2.35]. Helfgott pointed out that Borenstein and Croot’s method is similar to Tao and Vu’s method [4]. The group action Balog-Szemerédi-Gowers theorem is a common generalization of these methods.
1.2. **Connection to the sum-product problem.** Theorem 5 implies a non-trivial sum-product estimate. A sum-product estimate is a lower bound of the form

$$|A + A| + |AA| \gg |A|^{1+c}$$

where $A$ is a finite subset of $\mathbb{R}$ (or more generally, a ring), $c > 0$, and $A + A$ and $AA$ are the sets of pairwise sums and pairwise products of elements in $A$, respectively. Erdős and Szemerédi [17] conjectured that $c$ can be taken arbitrarily close to 1. Elekes [13] gave a beautiful geometric proof of a sum-product estimate with $c = 1/4$, based on the Szemerédi-Trotter bound.

The following sum-product estimate follows from Theorem 5, using the method of [13].

**Corollary 8.** Let $A, B,$ and $C$ be finite subsets of $\mathbb{R}$ with $|B||C| = |A|$. There is an absolute constant $c > 0$ such that if $|B|, |C| \geq |A|^\varepsilon$ for some $\varepsilon > 0$, then

$$|A + B| + |AC| \gg |A|^{1+c\varepsilon}.$$  

**Proof.** Let $\ell_{b,c}(x) = c(x - b)$ and let $L$ denote the set of $\ell_{b,c}$ with $b \in B$ and $c \in C$. Since $|B|, |C| \geq |A|^{\varepsilon}$, at most $|A|^{1-\varepsilon}$ lines of $L$ are parallel or concurrent.

Set $Y = (A + B) \cup (AC)$. Each line of $L$ is incident to at least $|A|$ points of $Y \times Y$.

If

$$|A + B| + |AC| \leq K|A|,$$

then each line of $L$ is $\alpha$-rich in $Y \times Y$ with $\alpha = 1/2K$.

By Theorem 5, at least $C_0\alpha^{2/3}|L|$ lines of $L$ are parallel or concurrent, thus we have

$$K^{-C_0}|A| \ll |A|^{1-\varepsilon} \implies K \gg |A|^{\varepsilon/C_0}.$$  

By choosing $K$ to be a sufficiently small power of $|A|$, we have a contradiction. \qed

The stronger conclusion of Theorem 4 over Theorem 7 allows us to give a geometric proof of Bourgain’s asymmetric sum-product estimate [5].

**Theorem 9 (Asymmetric sum-product estimate).** Let $A, B,$ and $C$ be finite subsets of a field $\mathbb{F}$.

If $\mathbb{F} = \mathbb{C}$ and there is an $\varepsilon > 0$ such that $|B|, |C| \geq |A|^\varepsilon$, then there exists a constant $c = c(\varepsilon) > 0$ such that

$$|A + B| + |AC| \gg |A|^{1+c\varepsilon}.$$  

If $\mathbb{F} = \mathbb{F}_p$, the same result holds provided that $|A| \ll p^{1-O(\varepsilon)}$.

In fact, we achieve estimates comparable to those of Shkredov [34]: we may take $c = 1/(J2^J)$ for $J \approx \gamma \varepsilon$.

**Theorem 10 (Asymmetric sum-product theorem).** Suppose that $A, B, C \subseteq \mathbb{F}$ are finite. Let $J > 0$ be a positive integer and let $0 < K \leq \frac{1}{2}|A|^{1/2J}$ be a parameter.

If $\mathbb{F} = \mathbb{C}$, then either

(1)  

$$|AC| + |A + B| > K|A|,$$  

or

$$|AC| + |A + B| \gg |A|^{1+c\varepsilon}.$$  

If $\mathbb{F} = \mathbb{F}_p$, the same result holds provided that $|A| \ll p^{1-O(\varepsilon)}$.  

$$|AC| + |A + B| \gg |A|^{1+c\varepsilon}.$$  

In fact, we achieve estimates comparable to those of Shkredov [34]: we may take $c = 1/(J2^J)$ for $J \approx \gamma \varepsilon$.  

(1)  

$$|AC| + |A + B| > K|A|,$$  

or

$$|AC| + |A + B| \gg |A|^{1+c\varepsilon}.$$  

If $\mathbb{F} = \mathbb{F}_p$, the same result holds provided that $|A| \ll p^{1-O(\varepsilon)}$.  

(1)
or
\[
\min(|B|, |C|) \ll K^{C^2J} |A|^{C/J}.
\]

If \( \mathbb{F} = \mathbb{F}_p \), the same dichotomy holds, provided that \(|A| \leq (2K)^{-2J} p \).

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1.4. Notation. We use standard asymptotic notation: \( f = O(g) \) means that there is a constant \( C > 0 \) such that \( |f(x)| \leq Cg(x) \) for all \( x \); \( f \ll g \) means the same as \( f = O(g) \), \( f = \Omega(g) \) and \( f \gg g \) mean the same as \( g \ll f \). The notation \( f \approx g \) means that \( f \ll g \) and \( g \ll f \); \( f = \Theta(g) \) means \( f \approx g \). We abuse asymptotic notation slightly for stating hypotheses: a condition of the form \( f \ll g \) means that there exists a constant \( C \) such that if \( |f| \leq Cg \), then the theorem holds. Notation such as \( f \ll_{\alpha} g \) or \( f = O_{\varepsilon}(g) \) means that the implicit constant \( C \) depends on the parameter in the subscript.

Unless otherwise stated, we use the following notation throughout:

- \( \alpha \) denotes a real number in \((0,1]\),
- lower case Greek letters denote (typically small) real parameters,
- \( C \) denotes a positive constant, which may change from line to line,
- \( \mathbb{F} \) denotes a field, which may be \( \mathbb{R}, \mathbb{C}, \mathbb{Q} \), or \( \mathbb{F}_p \), the finite field with prime cardinality \( p \),
- \( Y \) denotes a finite subset of \( \mathbb{F} \), and \( N \) denotes \( |Y| \),
- \( L \) denotes a finite set of lines in \( \mathbb{F}^2 = \mathbb{F} \times \mathbb{F} \),
- \( G \) denotes the group \( \text{Aff}(1, \mathbb{F}) \) of affine transformations of \( \mathbb{F} \); we represent elements of \( \text{Aff}(1, \mathbb{F}) \) by linear functions \( x \mapsto ax + b \) with \( a, b \in \mathbb{F}, a \neq 0 \), with composition as the group operation,
- \( A \) denotes a finite subset of \( G = \text{Aff}(1, \mathbb{F}) \).
- for \( Y \subseteq \mathbb{F} \) and \( 0 < \alpha \leq 1 \), we use \( \text{Sym}_\alpha(Y) \) to denote the set of \( g \in G \) such that \( |Y \cap gY| \geq \alpha |Y| \); this is called a symmetry set of \( Y \).

2. Lower bounds for rich lines in grids

In this section, we disprove Conjecture 1, which we recall here.

Conjecture 1. Among the lines in \( \mathbb{F}^2 \) that are \( \alpha \)-rich in an \( N \times N \) Cartesian product set, at most \( C = C(\alpha) > 0 \) lines are in general position.

In Section 2.2, we disprove Conjecture 1 over \( \mathbb{Q} \) with an explicit construction. In Section 2.3, we give an explicit construction of a large set of lines in general position in \( \mathbb{F}_p^2 \). In Section 2.4, we embed the counter-examples from the previous section into \( \mathbb{C}^2 \).
2.1. Qualitative lower bound based on amenability.

**Theorem 11.** Let $\mathbb{F}$ be an infinite field, and let $L$ be an arbitrary set of lines in $\mathbb{F}^2$. For all $\varepsilon > 0$ and all $N > 0$, there exists a subset $Y \subseteq \mathbb{F}$ such that $|Y| \geq N$ and $|(Y \times Y) \cap \ell| \geq (1 - \varepsilon)|Y|$ for all $\ell \in L$.

This shows that *any* set of lines, regardless of their structure, are $(1 - \varepsilon)$-rich in some grid, provided that the number of lines is sufficiently small compared to the size of the grid. This result is trivial if $\mathbb{F}$ is finite, since we may take $Y = \mathbb{F}$.

The proof of Theorem 11 uses a strategy of Lubotzky and Weiss [27] based on the amenability of finitely generate solvable groups, such as finitely generated subsets of $\text{Aff}(1, \mathbb{R})$.

We recall some basic facts about amenable groups.

**Definition 12 (Amenability).** Let $\Gamma$ be a group generated by $S = \{\gamma_1, \ldots, \gamma_k\}$ and suppose that $\Gamma$ acts on a set $X$. We say that $\Gamma$ is amenable if there exists a sequence of finite subsets $F_n \subseteq X$ such that for all $\gamma \in S$

\begin{equation}
\lim_{n \to \infty} \frac{\left|\gamma F_n \Delta F_n\right|}{|F_n|} = 0.
\end{equation}

Here $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference. A sequence of subsets $\{F_n\}$ satisfying (3) is called a Følner sequence. If $X$ is infinite, then $|F_n| \to \infty$ as $n \to \infty$ (see [3, Top of p.23]). See [3, Lemma 3.6, Lemma 3.7, Theorem 3.23].

Any finite group $G$ is amenable and any finitely generated abelian group is amenable. In fact, any group of sub-exponential growth is amenable, as can be seen by taking $\{F_n\}$ to be a sufficiently sparse sequence of balls about the identity in the word metric. Though solvable groups are not necessarily of sub-exponential growth, finitely generated solvable groups are amenable.

**Proposition 13.** Finitely generated solvable groups are amenable.

This is because the property of amenability is preserved by taking extensions and solvable groups can be constructed by extensions by abelian groups [3, Proposition 7.1].

We now have the background needed to prove Theorem 11.

**Proof of Theorem 11.** Let $S \subseteq \text{Aff}(1, \mathbb{F})$ be a set of affine transformations corresponding to the lines in $L$ (that is, each $\ell$ in $L$ has equation $y = \gamma(x)$ for some $g \in S$). Fix a positive integer $N$. We will show that there is a subset $Y \subseteq \mathbb{F}$ such that $|Y| \geq N$ and $|\gamma(Y) \cap Y| \geq (1 - \varepsilon)|Y|$ for all $\gamma \in S$.

Let $\Gamma$ denote the subgroup of $\text{Aff}(1, \mathbb{F})$ generated by $S$. Since $\Gamma$ is solvable and finitely generated, it is amenable by Proposition 13. By Definition 12 there is a Folner sequence $\{F_n\}$ of subsets of $X$.

By (3), there is a positive integer $n_0$ such that for all $n \geq n_0$ and all $\gamma$ in $S$, we have

$$|\gamma F_n \Delta F_n| \leq 2\varepsilon|F_n|.$$

Since

$$|\gamma F_n \Delta F_n| = 2(|F_n| - |F_n \cap \gamma F_n|),$$

...
we have $|\gamma(F_n) \cap F_n| \geq (1 - \varepsilon)|F_n|$ for all $n \geq n_0$.

Since $X = F$ is infinite, $|F_n| \to \infty$ as $n \to \infty$. It follows that for some $n_1 \geq n_0$, if $Y = F_{n_1}$ then $|Y| \geq N$. \hfill \Box

Lubotzky [26, Proposition 3.3.6] has shown how to apply this strategy to $\text{Aff}(1, F_p)$ acting on $F_p$, which would give a qualitative theorem of the same sort for sequences of finite affine groups. Rather than take this approach, we give an explicit example in Section 2.3, based on work of Klawe [24].

2.2. Quantitative lower bounds over $\mathbb{R}$. In this section, we give an explicit construction of arbitrarily large finite sets $Y$ in $\mathbb{R}$ such that $Y \times Y$ supports a large number of affine transformations in general position. In fact, the construction is defined over the integers.

Theorem 14. For all $0 < \alpha \leq 1$ and all $N_0 > 0$ there exists a set $Y \subseteq \mathbb{Z}$ such that $|Y| \geq N_0$ and $Y \times Y$ supports a set $L$ of $\alpha$-rich lines in general position such that

$$|L| \gg (1 - \alpha) \frac{\log |Y|}{\log \log |Y|}.$$ 

The construction is based on the construction of explicit Følner sequences for $\text{Aff}(1, \mathbb{R})$ acting on $\mathbb{R}$ [19, 42]. We thank John Mackay for suggesting a simpler way of writing our original set $Y$.

Proof of Theorem 14. Fix $0 < \varepsilon < 1$ such that $2\varepsilon \leq 1 - \alpha$. Fix an integer $N > 0$ so that $N^{N+1} \geq N_0$.

Define a set of positive integers $Y \subseteq \mathbb{Z}$ of size $N^{N+1}$ by

$$Y := \bigcup_{k=0}^{N-1} N^k \cdot (N^N + [0, N^N) \cap \mathbb{Z}).$$

Since $|[0, N^N) \cap \mathbb{Z}| = N^N$ and the terms of the union are disjoint, we have $|Y| = N^{N+1}$.

Let $L$ denote the set of transformations defined by $\ell_k(x) := N^k x + k N^{N-1}$, where $k$ ranges over integers satisfying $0 < k < \varepsilon N$. We make two claims.

Claim 1. The transformations in $L$ are in general position.

Claim 2. For all $\ell$ in $L$, we have

$$|\ell(Y) \setminus Y| \leq 2\varepsilon |Y|,$$

hence $|\ell \cap (Y \times Y)| \geq \alpha |Y|$ by our choice of $\varepsilon$.

The proof is complete assuming these claims, since $|L| \approx \varepsilon N \approx \varepsilon \log |Y| / \log \log |Y|$.

To prove Claim 1, it suffices to show that if $0 < i < k < \varepsilon N$, then

$$\det \begin{pmatrix} 1 & 1 & 1 \\ N^i & N^j & N^k \\ i & j & k \end{pmatrix} \neq 0. \tag{4}$$

(We have factored the common term $N^{N-1}$ out of the bottom row.) The left-hand side of (4) is

$$(k - i)N^j - (j - i)N^k - (k - j)N^i < (k - i)N^j - (j - i)N^k,$$
which is strictly less than zero:

\[(k - i)N^j < \varepsilon N^{j+1} \leq \varepsilon N^k \leq (j - i)N^k.\]

To prove Claim 2, fix an element \(\ell_b\) in \(L\) and consider its action on a general element \(y = N^k(N^N + x)\) of \(Y\), where \(x\) is an integer in \([0, N^N)\):

\[\ell_b(y) = N^{b+k}(N^N + x) + bN^{N-1} = N^{b+k}(N^N + x + bN^{N-1-b-k}).\]

There are two cases where \(\ell_b(y) \notin Y:\)

1. \(b + k \geq N,\)
2. \(b + k < N,\) but \(x + N^{N-1-b-k}b \geq N.\)

At most \(b\) values of \(k\) satisfy (1), hence at most \(bN^N \leq \varepsilon|Y|\) elements of \(Y\) fall into the first case. At most \(N^{N-1-b-k}b \leq \varepsilon N^N\) values of \(x\) satisfy (2), hence at most \(N \cdot \varepsilon N^N \leq \varepsilon|Y|\) elements of \(Y\) fall into the second case. \(\square\)

2.3. Quantitative lower bounds over \(\mathbb{F}_p\). In this section, we prove the lower bound in Theorem 3 for \(\mathbb{F} = \mathbb{F}_p\).

**Theorem 15.** For any prime \(p\), any \(0 < \alpha < 1\), any \(\varepsilon > 0\), and any integer \(m\) satisfying \(1 \ll \varepsilon m \ll p^{1-\varepsilon}\), there exists a subset \(Y \subseteq \mathbb{F}_p\) with \(|Y| \approx_\alpha m\) and a set of lines \(S\) in general position that are \(\alpha\)-rich in \(Y \times Y\) such that

\[\log |S| \approx_\alpha \sqrt{\frac{\log |Y|}{\log \log |Y|}}.\]

The proof of Theorem 15 is based on a construction of Klawe [24, 25], which proves explicitly that Schreier graphs of \(\text{Aff}(1, \mathbb{Z}/n\mathbb{Z}) \curvearrowright \mathbb{Z}/n\mathbb{Z}\) cannot be made into an expander family of constant degree. (Lubotzky [26] gives a qualitative proof of this fact using the method of [27].)

Before we state Klawe’s theorem, we need some notation. Let \(Q = \{q_1, \ldots, q_k\}\) denote a set of \(k\) primes. We say that \(n\) is a \(Q\)-power if \(n = q_1^{\alpha_1} \cdots q_k^{\alpha_k}\) and in this case, we write \(\mu(n) = \alpha_1 + \cdots + \alpha_k\). We use \(\phi(n)\) to denote Euler’s totient function; that is, \(\phi(n)\) the number of positive integers less than and relatively prime to \(n\).

**Theorem 16** (Klawe). Let \(Q = \{q_1, \ldots, q_k\}\) be a set of \(k\) prime numbers and set \(q = q_1 \cdots q_k\). Let \(N, M, L, r,\) and \(s\) be positive integers such that \(N = Mq^s + r, 0 \leq r < q,\) and \(L < M/q^s\).

Then there exists a subset \(Y \subseteq \mathbb{Z}/N\mathbb{Z}\) such that

\[|Y| = s^kL\phi(q)q^{s-1}\]

and for all positive integers \(0 < a, b < N\) such that \(a\) is a \(Q\)-power

\[|(aY + b) \setminus Y| \leq \left(\frac{\mu(a)}{s} + \frac{ar + b}{L\phi(q)}\right)|Y|.

The proof of Theorem 16 uses a construction similar to that of Theorem 14, but uses wrap-around to allow a much larger set of “slopes” \(a\).

We use the following corollary of Theorem 16 to prove Theorem 15.
Corollary 17. Let $Q = \{q_1, \ldots, q_k\}$ be a set of $k$ prime numbers and set $q = q_1 \cdots q_k$. Let $p$ be a prime and let $M, L, r,$ and $s$ be positive integers such that $p = Mq^s + r$, $0 \leq r < q$, and $L < M/q^s$.

If

$$L \geq \frac{8q}{\varphi(q)} \max \left( q^{(1+\frac{1}{s})s}, \left( \frac{s}{4K}\right)^{2k} \right),$$

then there exists a subset $Y \subseteq \mathbb{F}_p$ satisfying (5) and a set $S$ of affine transformations in general position such that $|S| \geq (s/4k)^k$ and $|\ell(Y) \setminus Y| \leq \frac{1}{2}|Y|$ for all transformations $\ell$ in $S$.

Proof. Applying Theorem 16 with $N = p$ yields a set $Y \subseteq \mathbb{F}_p$ such that (5) holds and for all positive integers $a$ and $b$ such that $a$ is a $Q$-power (6) holds.

We wish to choose a collection of pairs $(a, b)$ such that the corresponding set of lines $\ell(x) = ax + b$ are in general position and satisfy

$$\frac{\mu(a)}{s} + \frac{ar + b}{L} \frac{q}{\varphi(q)} \leq \frac{1}{2}.$$  \hspace{1cm} (8)

First, we will find a large number of integers $a, b$ satisfying

$$\mu(a) \leq \frac{s}{4}$$ \hspace{1cm} (9)

and

$$ar + b \leq \frac{\phi(q)L}{4q}.$$ \hspace{1cm} (10)

Let $A$ denote the set of positive integers of the form $a = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$ where $0 \leq \alpha_i \leq s/4k$ for $i = 1, \ldots, k$. Then each element $a$ in $A$ satisfies (9) and further $1 \leq a \leq q^{s/4k}$.

If $a \in A$ and $b$ is a positive integer, then

$$ar + b < q^{(1+\frac{1}{s})s} + b.$$  \hspace{1cm} (11)

By (7), we have $\phi(q)L/4q \geq 2q^{(1+\frac{1}{s})s}$, so (10) is satisfied for all $0 \leq b \leq \phi(q)L/8q$.

Note that $q^{s/4k} < p$ and $\phi(q)L/8q < p$, so $a$ and $b$ are unique modulo $p$.

To form our set of lines $L$, we will choose slopes $a$ from $A$ one at a time, choosing $y$-intercepts $0 \leq b \leq \phi(q)L/8q$ so that the line $\ell(x) = ax + b$ intersects each previous line in a distinct point; this guarantees that no three lines in $L$ are incident to a common point. Since all of the lines in $L$ have distinct slopes, the resulting set of lines $L$ will be in general position.

If we have chosen $x$ lines by this process, then we must avoid $\binom{x}{2}$ points; this is always possible if we have more than $\binom{x}{2}$ choices for $b$. By (7),

$$\#(\text{choices for for } b) \geq \frac{\phi(q)L}{8q} \geq \left( \frac{s}{4K}\right)^{2k} > \binom{x}{2} \text{ for all } 0 \leq x \leq |A|. \hspace{1cm} \square$$

We want to take $k$ as large as possible relative to $q$; the following lemma gives $k \approx \log q/ \log \log q$.  \hspace{1cm} (12)
Lemma 18. Given \( x > 0 \), let \( Q = \{q_1, \ldots, q_k\} \) denote the set of primes less than or equal to \( x \), and let \( q = q_1 \cdots q_k \). We have the following estimates:

\[
(11) \quad k = |Q| = \frac{x}{\log x} \left( 1 + O \left( \frac{1}{\log x} \right) \right),
\]

\[
(12) \quad q = e^x \left( 1 + O \left( \frac{1}{\log x} \right) \right),
\]

and

\[
(13) \quad \frac{\phi(q)}{q} = e^{-\gamma} \left( 1 + O \left( \frac{1}{\log x} \right) \right),
\]

where \( \gamma \) is Euler’s constant.

Proof. Equation (11) is the Prime Number Theorem. Equation (12) follows from asymptotic estimates for Chebyshev’s function \( \vartheta(x) \):

\[
\vartheta(x) = \sum_{p \leq x} \log p = x \left( 1 + O \left( \frac{1}{\log x} \right) \right).
\]

Equation (13) is Merten’s formula \([23, \text{Equation (2.16)}]\). \( \Box \)

For simplicity, we will prove Theorem 15 for the case \( \alpha = \frac{1}{2} \); the general case follows in the same way, with implicit constants depending on \( \alpha \).

Proof of Theorem 15. Let \( x \) be a positive real number and let \( Q = \{q_1, \ldots, q_k\} \) denote the set of primes less than or equal to \( x \). Let \( q = q_1 \cdots q_k \) and let \( s = \lceil 4e \cdot k \rceil \). For convenience, let \( \delta = 1/4k \).

Set

\[
L = \frac{8q}{\phi(q)} q^{(1+\delta)s}. 
\]

Condition (7) of Corollary 17 holds if \( q^{(1+\delta)s} \geq (s/4k)^{2k} \). By Lemma 18,

\[
q^{(1+\delta)s} \geq q^{4k} = e^{2k \cdot 2x \left( 1 + O \left( \frac{1}{\log x} \right) \right)},
\]

while

\[
\left( \frac{s}{4k} \right)^{2k} \leq \left( e + \frac{1}{4k} \right)^{2k} \ll e^{2k},
\]

thus condition (7) holds if \( x \gg 1 \).

Write \( p = Mq^s + r \), where \( 0 \leq r < q^s \) and \( M > q^sL \). By Corollary 17 there is a set \( Y \subseteq \mathbb{F}_p \) such that

\[
|Y| = s^k L \phi(q) q^{s-1} = 8s^k q^{(2+\delta)s} 
\]

and a set \( S \) of lines in general position that are \( \frac{1}{k} \)-rich in \( Y \times Y \) such that

\[
|S| \geq \left( \frac{s}{4k} \right)^k \geq e^k.
\]

By Lemma 18,

\[
\log |S| \geq k \sim \frac{x}{\log x},
\]

while

\[
\log |Y| \approx k \log s + (2 + \delta)s \log q \approx \frac{x^2}{\log x}.
\]
Thus
\[ \log |S| \approx \sqrt{\frac{\log |Y|}{\log \log |Y|}}, \]
as desired.

Now we will derive constraints on \( m = |Y| \). Since \( x \gg 1 \), we have \( m \gg 1 \).

On the other hand, we must have
\[ p \geq Mq^s \geq q^{2s}L = \frac{8q}{\phi(q)} q^{(3+\delta)s} \sim 8e^{\gamma} \log x q^{(3+\delta)s} \approx (\log \log q) q^{(3+\delta)s}. \]

Since \( k \geq \frac{x}{\log x} \left( 1 - \frac{C}{\log x} \right) \) and \( q \approx e^x \), we have
\[ s^k \gg k^k \gg \left( \frac{x}{\log x} \right)^k \gg \frac{q^{C/\log \log q}}{q^{(3+\delta)s}}. \]

Thus
\[ |Y| \gg \frac{q^{(3+\delta)s}}{q^{C/\log \log q}}. \]

Thus to ensure \((\log \log q) q^{(3+\delta)s} \ll p\), it suffices to take \( |Y| \leq p^{1-\varepsilon} \) for any \( \varepsilon > 0 \).

2.4. Quantitative lower bounds over \( \mathbb{C} \). In this section, we prove the lower bound in Theorem 3 for \( \mathbb{F} = \mathbb{C} \).

**Theorem 19.** For all \( 0 < \alpha < 1 \) there exists an absolute constant \( N_0 \geq 0 \) such that for all \( N \geq N_0 \) there is a subset \( Y \subseteq \mathbb{C} \) with \( |Y| \geq N \) and a set \( S \) of lines in general position that are \( \alpha \)-rich in \( Y \times Y \) and satisfy
\[ \log |S| \approx_{\alpha} \sqrt{\frac{\log |Y|}{\log \log |Y|}}. \]

The proof of Theorem 19 is an application of a rectification theorem of Grosu [20], which allows us to embed small subsets of \( \mathbb{F}_p \) into \( \mathbb{C} \) while preserving algebraic equations of low complexity. In particular, Grosu’s theorem allows up to embed the counterexamples constructed in Theorem 15 into \( \mathbb{C}^2 \). This seems to be the first time that Grosu’s theorem has been used to prove a counterexample to a statement over \( \mathbb{C} \), rather than to prove a positive statement for very small subsets of \( \mathbb{F}_p \).

Before we state Grosu’s theorem, we need some definitions. A polynomial \( f \in \mathbb{Z}[x_1, \ldots, x_n] \) is \( k \)-bounded if \( \deg(f) \leq k \) and the sum of the absolute values of the coefficients of \( f \) are bounded by \( k \). Given rings \( R_1 \) and \( R_2 \) and subsets \( A = \{a_1, \ldots, a_n\} \subseteq R_1 \) and \( B \subseteq R_2 \), we call a bijection \( \phi : A \to B \) a Freiman ring isomorphism of order \( k \) (or \( F_k \)-ring isomorphism) if for any \( k \)-bounded \( f \in \mathbb{Z}[x_1, \ldots, x_n] \) we have
\[ f(a_1, \ldots, a_n) = 0 \iff f(\phi(a_1)), \ldots, f(\phi(a_n)) = 0. \]

**Theorem 20** (Grosu). Let \( k \geq 2 \) be an integer, let \( p \) be a prime, and let \( A \) be a subset of \( \mathbb{F}_p \). If \( |A| < \log_2 \log_2 \log_2 p - 1 \), then there exists a subset \( A' \subseteq \mathbb{C} \), and a homomorphism \( \phi_p : \mathbb{Z}[A'] \to \mathbb{F}_p \) such that \( \phi_p \) is an \( F_k \)-ring homomorphism between \( A' \) and \( A \).
Grosu used Theorem 20 to prove that incidence bounds for points and lines in $\mathbb{C}^2$ can be applied to small sets of points and lines in $\mathbb{F}_p^2$ [20, Theorem 10]. We give a variation on this argument that guarantees that lines in general position in $\mathbb{C}^2$ correspond to lines in general position in $\mathbb{F}_p^2$.

**Corollary 21.** Let $p$ be a prime, let $Y$ be a subset of $\mathbb{F}_p$, and let $S$ be a set of lines in $\mathbb{F}_p^2$ in general position that are $\alpha$-rich in $Y \times Y$ for some $0 < \alpha < 1$.

If $|Y| + 2|S| + \binom{|S|}{3} < \log_2 \log_{14} \log_{98} p - 2$, then there exists a subset $Y' \subseteq \mathbb{C}$ and a set of lines $S'$ in $\mathbb{C}^2$ that are in general position and $\alpha$-rich in $Y' \times Y'$.

**Proof.** We will show that it suffices to construct a $F_7$-ring isomorphism between a certain subset $A \subseteq \mathbb{F}_p$ and some subset $A' \subseteq \mathbb{C}$.

Suppose that the elements of $S$ have the form $\ell_i(x) = a_i x + b_i$. If $\ell_i, \ell_j, \ell_k$ are distinct lines that intersect in a common point, then the matrix

$$
\begin{pmatrix}
a_i & b_i & 1 \\
a_j & b_j & 1 \\
a_k & b_k & 1
\end{pmatrix}
$$

is singular. By hypothesis, the lines of $S$ are in general position, so the numbers

$$
(16) \quad d_{ijk} := \det \begin{pmatrix} a_i & b_i & 1 \\ a_j & b_j & 1 \\ a_k & b_k & 1 \end{pmatrix}
$$

are non-zero.

Let $A$ be the union of $Y, \{a_i\}, \{b_i\}, \{d_{ijk}\}$, and $\{0\}$. Then by hypothesis

$$
(17) \quad |A| \leq |Y| + 2|S| + \binom{|S|}{3} + 1 < \log_2 \log_{14} \log_{98} p - 1.
$$

For each line $\ell_i$, we have at least $\alpha|Y|$ solutions to

$$
(18) \quad y' = a_i y + b
$$

with $y, y'$ in $Y$. This equation is 3-bounded. The equation (16) is 7-bounded.

By (17), we may apply Theorem 20 to $A$ to find a subset $A' \subseteq \mathbb{C}$ and a $F_7$-ring homomorphism $\phi_p : \mathbb{Z}[A'] \to \mathbb{F}_p$ from $A'$ to $A$.

Let $Y' \subseteq \mathbb{C}$ denote the set of elements in $A'$ that map to $Y$ under $\phi_p$ and let $S'$ denote the set of lines defined by $\ell'_i(x) = a'_i x + b'_i$ where $\phi_p(a'_i) = a_i$ and $\phi_p(b'_i) = b_i$. Since $\phi_p$ is a bijection from $A'$ to $A$, we have $|Y'| = |Y|$ and $|S'| = |S|$.

Since $\phi_p$ preserves 16 and (18), the lines of $S'$ are in general position (since by bijectivity, no $d_{ijk}$ is mapped to 0), and each line in $S'$ is incident to at least $\alpha|Y'|$ points of $Y' \times Y'$.

We are now ready to prove Theorem 19.

**Proof of Theorem 19.** Without loss of generality, we may assume that $|S| < N^{1/3}$. Choose a prime $p$ so that

$$
(19) \quad 5N \leq \log_2 \log_{14} \log_{98} p - 2.
$$
Since $N_0 \leq N \leq p^{1/2}$, if $N_0$ is sufficiently large (depending on $\alpha$), then by Theorem 15 there is a subset $Y \subseteq \mathbb{F}_p$ of size $\approx \alpha N$ and a set $S$ of lines in $\mathbb{F}_p^2$ in general position and $\alpha$-rich in $Y \times Y$ such that

$$\log |S| \approx \alpha \sqrt{\frac{\log |Y|}{\log \log |Y|}}.$$ 

By (19) we have

$$|Y| + 2|S| + \left(\frac{|S|}{3}\right) \leq 5N \leq \log_2 \log_{14} \log_{98} p - 2,$$

so by Corollary 21 we may embed $Y$ into $\mathbb{C}$ and $S$ into $\mathbb{C}^2$. \hfill \Box

3. Upper bounds for rich lines in grids

In this section, we prove two upper bounds for the number of rich lines in a $N \times N$ grid in $\mathbb{F}^2$, and an asymmetric sum-product estimate over $\mathbb{F}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{F}_p$. If $\mathbb{F} = \mathbb{F}_p$, we need an additional constraint to rule out trivial counter-examples: the grid $\mathbb{F}_p \times \mathbb{F}_p$ has $p^2$ 1-rich lines, and $\mathbb{F}_p$ does not grow under addition or multiplication.

These theorems are all consequences of Theorem 4, which is a general inverse theorem for rich lines in grids. Theorem 4 is an immediate corollary of Theorem 24, which is an inverse theorem for rich affine transformations. The difference between Theorems 4 and 24 is a matter of language, and we give a dictionary between geometric and algebraic terminology in Section 3.1.

First we state the upper bound for $\alpha$-rich lines in a $N \times N$ grid where $\alpha = N^{-\delta}$, which generalizes Theorem 5 to sets of lines of size $N^\varepsilon$ for any $\varepsilon > 0$, as well as to points and lines defined over $\mathbb{C}$ or $\mathbb{F}_p$.

**Theorem 22** (Upper bound, polynomial density). For all $\varepsilon > 0$ and $0 < \gamma < 1$, there is a $\delta > 0$ such the following holds for all $N > 0$.

Let $L$ be a set of $N^\varepsilon$ lines in $\mathbb{F}^2$ that are $N^{-\delta}$-rich in an $N \times N$ grid.

- If $\mathbb{F} = \mathbb{C}$, then there is a subset $L' \subseteq L$ of size $|L'| \gg |L|^{1-\gamma}$ such that the lines of $L'$ are either parallel or concurrent.
- If $\mathbb{F} = \mathbb{F}_p$, the same conclusion holds, provided that $N \ll p^{-O(\gamma \varepsilon)}$.

Further, we may take $\delta = 1/(J 2^J)$, where $J \approx \gamma \varepsilon$.

Theorem 22 immediately implies the main theorem of [2] (Theorem 7), since if the lines of $L$ are in general position, then $|L'| \leq 2$, which yields a contradiction for $N$ sufficiently large.

Next we consider $\alpha$-rich lines in an $N \times N$ grid where $\alpha$ is fixed.

**Theorem 23** (Upper bound, constant density). For all $0 < \alpha < 1$ there is a constant $C = C(\alpha) > 0$ such that the following holds for all $N > 0$.

Let $L$ be a set of lines in $\mathbb{F}^2$ that are in general position and are $\alpha$-rich in an $N \times N$ grid.

- If $\mathbb{F} = \mathbb{C}$, then $|L| \ll_{\alpha} N C/\log \log N$.
- If $\mathbb{F} = \mathbb{F}_p$, then same conclusion holds, provided that $N^{1 + \log(2/\alpha)/\log \log N} \leq p$. 


Theorem 23 proves the upper bounds stated in Theorem 3.
We will prove the following asymmetric sum-product result, which immediately implies Theorem 9.

**Theorem 10.** Suppose that $A, B, C \subseteq \mathbb{F}$ are finite. Let $J > 0$ be a positive integer and let $0 < K \leq \frac{1}{2}|A|^{1/2J}$ be a parameter.

If $\mathbb{F} = \mathbb{C}$, then either

$$|AC| + |A + B| > K|A|,$$

or

$$\min(|B|, |C|) \ll K|A|^{C^2J}. \quad (21)$$

If $\mathbb{F} = \mathbb{F}_p$, the same dichotomy holds, provided that $|A| \leq (2K)^{-2J} p$.

Choosing $2K = |A|^{\frac{1}{2J}}$ proves Theorem 9 with $\varepsilon = 1/J$, since (2) cannot hold for this choice of $K$.

Theorems 22, 23, and 10 are special cases of the following general inverse theorem for rich lines in grids, which we stated in the introduction.

**Theorem 4.** There is an absolute constant $C > 0$ such that the following holds. Let $Y$ be a finite subset of $\mathbb{F}$ and let $L$ be a set of $\alpha$-rich lines in $Y \times Y$. Let $J > 0$ be an integer such that $(\alpha/2)^{2J} \geq 1/|Y|$.

If $\mathbb{F} = \mathbb{C}$, then there is a subset $L' \subseteq L$ such that

1. the lines of $L'$ are either parallel or concurrent, and
2. $|L'| \gg \left(\frac{\alpha}{2}\right)^{C^2J} |Y|^{-C^2J} |L|$.

If $\mathbb{F} = \mathbb{F}_p$, then the same conclusion holds, provided that $|Y| \leq (\alpha/2)^{2J} p$.

In turn, Theorem 4 is a simple translation of an algebraic inverse theorem for rich affine transformations.

We need some notation. If $Y$ is a finite subset of $\mathbb{F}$, we let $\text{Sym}_\alpha(Y)$ denote the set of transformations $g$ in $\text{Aff}(1, \mathbb{F})$ such that $|Y \cap gY| \geq \alpha|Y|$.

**Theorem 24** (Inverse theorem for $\text{Aff}(1, \mathbb{F}) \rtimes \mathbb{F}$). Let $\mathbb{F}$ denote $\mathbb{C}$ or $\mathbb{F}_p$. There exists an absolute constant $C > 0$ such that the following holds:

Suppose that $Y \subseteq \mathbb{F}$ is finite, $0 < \alpha < 1$, and $A \subseteq \text{Sym}_\alpha(Y)$.

Let $J \geq 0$ be an integer such that $(\alpha/2)^{2J} \geq 1/|Y|$, and if $\mathbb{F} = \mathbb{F}_p$, suppose that $J$ also satisfies $|Y| \leq \left(\frac{\alpha}{2}\right)^{2J} p$.

Then there is an element $g$ in $G$ and an abelian subgroup $H$ of $G$ such that

$$|A \cap gH| \gg \left(\frac{\alpha}{2}\right)^{C^2J} |Y|^{-C^2J}|A|.$$
3.1. **A geometric/algebraic dictionary and proof of Theorem 4.** As we have said, $G = \text{Aff}(1, \mathbb{F})$ consists of transformations $x \mapsto ax + b$ with $a, b \in \mathbb{F}$ and $a \neq 0$. The group $G$ acts on the affine line $X = \mathbb{F}$ by linear maps. If $g \in G$, $Y$ is a finite subset of $X$, and $|Y \cap gY| \geq \alpha|Y|$, we say that $g$ is an $\alpha$-approximate symmetry of $Y$. The collection of all $\alpha$-approximate symmetries of a set is called a symmetry set

$$\text{Sym}_\alpha(Y) = \{g \in G : |Y \cap gY| \geq \alpha|Y|\}.$$ 

Symmetry sets were first defined in additive combinatorics in [39, Section 2.7]; symmetry sets for a general action of a group $G$ on a set $X$ are discussed in more detail in [28].

Every affine transformation in $\text{Aff}(1, \mathbb{F})$ corresponds to a line (its graph) in $\mathbb{F}^2$. By convention, we ignore vertical lines, thus every line in $\mathbb{F}^2$ is the graph of a transformation in $\text{Aff}(1, \mathbb{F})$.

Several properties of rich lines correspond to properties of approximate symmetries:

1. collections of rich lines in grids correspond to symmetry sets,
2. collections of parallel lines correspond to cosets of the translation subgroup,
3. collections of concurrent lines correspond to cosets of homothety subgroups.

To prove (1), simply note that if a line $\ell$ has the equation $y = ax + b$ then

$$|\ell \cap (Y \times Y)| \geq \alpha|Y| \iff |Y \cap (aY + b)| \geq \alpha|Y|.$$ 

To prove (2) and (3), we need a bit of background on the subgroups of the affine group.

Let $\tau_b$ denote the transformation $x \mapsto x + b$. The translation subgroup $U := \{\tau_b : b \in \mathbb{F}\}$ is a normal subgroup of $G$ corresponding to translations of $\mathbb{F}$. ($U$ is for “unipotent”.)

Let $d_a$ denote the transformation $x \mapsto ax$. The dilation subgroup $T = \{d_a : a \in \mathbb{F}^*\}$ corresponds to dilations of $\mathbb{F}$ about 0. In general, the stabilizer of a point $x$ in $\mathbb{F}$ under the action of $\text{Aff}(1, \mathbb{F})$ has the form $\text{Stab}(x) = gTg^{-1}$, where $g(0) = x$. We call $\text{Stab}(x)$ the homothety subgroup of dilations about $x$.

The dilation subgroup and the homothety subgroups are the maximal abelian subgroups of $G$. (If $H$ is abelian, either $H \subseteq U$ or there is an element $x \in H \setminus U$, and $H$ is contained in the centralizer of $x$, which is a homothety subgroup.) We will usually say “abelian subgroup” rather than saying “dilation or homothety subgroup”.

For $x, y \in \mathbb{F}$, the set of transformations $\text{Trans}(x, y)$ sending $x$ to $y$ has the form $\text{Trans}(x, y) = gTh$, where $h(y) = 0$ and $g(0) = x$ ($g$ and $h$ are not unique). We call $\text{Trans}(x, y)$ the transporter of $x$ to $y$; it is a left coset of $\text{Stab}(x)$ and a right coset of $\text{Stab}(y)$.

If $L$ is a set of (non-vertical) lines in $\mathbb{F}^2$, let $A_L$ denote corresponding set of affine transformations.
• Property (2) holds since the lines of lines $L$ have common slope $a$ if and only if the corresponding set of affine transformations $A_L$ is contained in $d_U$, and

• Property (3) holds since the lines of $L$ are incident to a common point $(x, y)$ in $\mathbb{F}^2$ if and only if $A_L$ is contained in $\text{Trans}(x, y)$, which is a coset of a homothety subgroup.

Now we derive Theorem 4 from Theorem 24.

Proof of Theorem 4. Let $L$ be a set of $\alpha$-rich lines in $Y \times Y$ and let $A$ denote the set of affine transformations corresponding to $L$.

By Theorem 24, if $(\alpha/2)^{2J} \geq 1/|Y|$, and $|Y| \leq (\alpha/2)^{2J} p$ in the case $\mathbb{F} = \mathbb{F}_p$, then there is an abelian subgroup $S \leq G$ and an element $g$ in $G$ such that

$$|A \cap gS| \gg (\alpha/2)^{C^2J} |A|.$$  

Let $L'$ denote the set of lines in $L$ that correspond to elements of $A \cap gS$. By Properties 2 and 3, the lines of $L'$ are either parallel or concurrent, and since $|L'| = |A \cap gS|$ and $|L| = |A|$, the desired lower bound holds. □

3.2. Proof of Theorems 22, 23, and 10. In this section we prove Theorems 22, 23, and 10 using Theorem 4. The proofs of Theorems 22 and 23 simply consist of choosing parameters and checking that the hypotheses of Theorem 4 are satisfied. The proof of Theorem 10 is essentially the same as the proof of Corollary 8 presented in the introduction.

Proof of Theorem 22. Let $N = |Y|$. Let $J$ be a positive integer such that $J > 2C/\gamma \varepsilon$, where $C$ is the constant from Theorem 4. Choose $\delta = 1/(J2^J)$.

To apply Theorem 4 for $\alpha = N^{-\delta}$, we must check the constraints on $\alpha$ and $J$. Since

$$\left(\frac{\alpha}{2}\right)^{2J} = \left(\frac{1}{2N^J}\right)^{2J} = \frac{1}{2^{2J} N^{1/J}},$$

for $N$ sufficiently large, we have $(\alpha/2)^{2J} \geq 1/N = 1/|Y|$. If $\mathbb{F} = \mathbb{F}_p$, we must check the additional constraint $|Y| \leq (\alpha/2)^{2J} p$. Since

$$\left(\frac{\alpha}{2}\right)^{2J} p \gg \gamma \varepsilon N^{-\gamma \varepsilon/2C} p,$$

the additional constraint follows from the addition hypothesis $N \ll p^{1-O(\gamma \varepsilon)}$ when $\mathbb{F} = \mathbb{F}_p$.

Thus in either case, we may apply Theorem 4 to find a subset $L' \subseteq L$ of either parallel or concurrent lines such that

$$|L'| \gg \left(\frac{\alpha}{2}\right)^{C^2J} |Y|^{-C/J} |L| \gg J^J N^{-C\delta 2^J} N^{-C/J} |L|.$$  

To complete the proof, we must show that $|L'| \gg |L|^{1-\gamma}$, which follows from our choice of $J$ and $\delta$:

$$N^{C\delta 2^J} N^{C/J} \ll N^{\gamma \varepsilon} \leq |L|^{\gamma}.$$  

□
Proof of Theorem 23. Let \( N = |Y| \) and set
\[
J = \log_2 \left( \frac{\log_2 N}{\log_2 \log_2 N} \right).
\]
Then \( N^{1/2^J} = \log_2 N \), so \( (\alpha/2)^{2^J} \geq 1/|Y| \) for \( N \) sufficiently large. Since
\[
(24) \quad \left(\frac{\alpha}{2}\right)^{2^J} = N^{-\log_2(2/\alpha)/\log_2 \log_2 N},
\]
the constraint \( |Y| \leq (\alpha/2)^{2^J} \) follows from the condition
\[
N^{1-\log_2(2/\alpha)/\log_2 \log_2 N} \leq p.
\]
Thus we may apply Theorem 4 to find a subset \( L' \subseteq L \) of either parallel or concurrent lines such that
\[
|L'| \gg \left(\frac{\alpha}{2}\right)^{C 2^J} |Y|^{-C/J}|L|.
\]
Since the lines of \( L \) are in general position, we have \( |L'| \leq 2 \). Thus
\[
|L| \ll \left(\frac{\alpha}{2}\right)^{C 2^J} N^{C/J}. \]
For \( N \) sufficiently large, \( J \gg \log_2 \log_2 N \), by (24) we have
\[
|L| \ll N^{C \frac{1-\log(\alpha)}{\log \log N}}. \]
\[\square\]

Proof of Theorem 10. Suppose that (1) is false. Let \( Y = AC \cup (A + B) \). Then \( |Y| \leq K|A| \).

Let \( L \) denote the set of lines of the form \( y = c(x - b) \) with \( b \in B \) and \( c \in C \). Each line \( \ell \) in \( L \) satisfies \( |Y \cap \ell(Y)| \geq |A| \geq \frac{1}{K}|Y| \), hence \( L \) is a set of \( \alpha \)-rich lines in \( Y \times Y \) with \( \alpha = 1/K \).

The constraints on \( K \) imply that \( (\alpha/2)^{2^J} \geq 1/|Y| \) and \( |Y| \leq (\alpha/2)^{2^J} p \), if \( \mathbb{F} = \mathbb{F}_p \). Thus by Theorem 24, there is subset \( L' \subseteq L \) consisting of either parallel or concurrent lines with size
\[
|L'| \gg \left(\frac{\alpha}{2}\right)^{C 2^J} |Y|^{-C/J}|L|.
\]
Since \( L \) contains at most \( |B| \) parallel lines and at most \( |C| \) concurrent lines, we have
\[
\max(|B|, |C|) \gg \left(\frac{\alpha}{2}\right)^{C 2^J} |Y|^{-C/J}|B||C|,
\]
and hence
\[
\min(|B|, |C|) \ll (2K)^{C 2^J} |Y|^{C/J}.
\]
\[\square\]

Remark. Theorem 10 can be proved directly from Theorem 24 by noting that the transformations \( x \mapsto c(x - b) \) are contained in \( \text{Sym}_\alpha(Y) \) for \( \alpha = 1/K \).
3.3. Proof of Theorem 24. Theorem 24 follows from a general inverse theorem for groups actions, which is a group action version of (asymmetric) Balog-Szemerédi-Gowers theorem [28]. In addition to this general inverse theorem, we need two other inputs specific to the action of Aff(1, F) on F for F = C and F = F_p:

1. a product theorem for Aff(1, F), and
2. bounds for the size of Sym_α(Y).

3.3.1. Group action version of the (asymmetric) Balog-Szemerédi-Gowers theorem. First, we state the group action version of the Balog-Szemerédi-Gowers theorem from [28]. We simplify the statement slightly, and specialize to Aff(1, F) acting on F.

**Theorem 25.** There is an absolute constant C > 0 such that the following holds.

Let Y be a finite subset of F and let A be a finite subset of Aff(1, F). Given a number 0 < α < 1 and an integer J ≥ 0, define

\[ \alpha_J = 2 \left( \frac{\alpha}{2} \right)^{2^J} \text{ and } K = \left( \frac{|\text{Sym}_{\alpha_J}(Y)|}{|A|} \right)^{1/J}. \]

If A ⊆ Sym_α(Y), then

1. there is an element g_ in G and a finite subset A_ ⊆ G such that

\[ g_ - 1 A_ ⊆ \text{Sym}_{\alpha_J}(Y) \]

and

\[ |A_3| \ll \left( \frac{K}{\alpha_J} \right)^C |A_|. \]

2. for any subset S ⊆ G there is an element g in G such that

\[ |A ∩ gS| \gg \left( \frac{\alpha_J}{K} \right)^C \frac{|S ∩ A_3|}{|A_3|} |A|. \]

Part (1) of Theorem 25 says that some symmetry set of Y contains a set A_ with small tripling, which will allow us to apply the product theorems, stated next, to find a coset S of an abelian subgroup such that |A_ ∩ S| is large. Part (2) of Theorem 25 then says that |A ∩ gS| is large as well, which gives us the desired structure in A.

3.3.2. Product theorems for Aff(1, C) and Aff(1, F_p). The following product theorem is a special case of a product theorem for solvable groups of GL_n(C), due to Breuillard and Green [8, Theorem 1.4].

**Theorem 26 (Product theorem for Aff(1, C)).** Fix K ≥ 1. If A is a finite subset of Aff(1, C) such that |A^3| ≤ K|A|, there is a subset A' ⊆ A with size |A'| ≥ K^{-C}|A| that is contained in a coset of an abelian subgroup of Aff(1, C).

Over F_p, Helfgott has proved a similar theorem [22, Proposition 4.8].
Theorem 27 (Product theorem for \( \text{Aff}(1, \mathbb{F}_p) \)). Let \( G = \text{Aff}(1, \mathbb{F}_p) \), let \( U \) be the translation subgroup, and let \( \pi : G \to G/U \) be the quotient map.

For a subset \( A \subseteq G \), if there is a constant \( K \geq 1 \) such that \( |A^3| \leq K|A| \), then for an absolute constant \( C > 0 \) we have either

\[ |A \cap T| \geq \frac{1}{3}|A| \]

for some torus \( T \),

\[ |\pi(A)| \ll K^C, \]

or

\[ K^C|A| \gg |\pi(A)|. \]

Theorems 26 and 27 can be proved by combining the orbit-stabilizer theorem for sets [22, Lemma 4.1] with a pivot argument or sum-product theorem. For completeness, we include proofs of Theorems 26 and 27 in Appendix B, using the sum-product theorems from [32, 1].

Since \( |\pi(A)| \) is the number of cosets of \( U \) needed to cover \( A \), if (30) holds, then there is an element \( g \) in \( G \) such that \( |A \cap gU| \gg K^{-C}|A| \). We also know that \( |A| \ll |\text{Sym}_{\alpha J}(Y)| \), and we will use this to draw a similar conclusion from (31) using the upper bounds for \( |\text{Sym}_{\alpha}(Y)| \).

3.3.3. Upper bounds for \( |\text{Sym}_{\alpha}(Y)| \). Finally, we quote upper bounds for the symmetry sets for the action of \( \text{Aff}(1, \mathbb{F}) \) on \( \mathbb{F} \).

Theorem 28. Let \( Y \) be a finite subset of \( \mathbb{F} \) and let \( \alpha \) be greater than \( 2/|Y| \).

If \( \mathbb{F} = \mathbb{C} \), then

\[ |\text{Sym}_{\alpha}(Y)| \ll \alpha^{-3}|Y|. \]

If \( \mathbb{F} = \mathbb{F}_p \) and \( |Y| \leq \frac{\alpha}{2}p \), then

\[ |\text{Sym}_{\alpha}(Y)| \ll \alpha^{-4}|Y|. \]

The first bound follows immediately from the Szemerédi-Trotter theorem [37, 40, 43, 35], and the second bound follows from the Stevens-de Zeeuw bound [36] combined with some additional arguments [30].

Remark. Weaker bounds than those of Theorem 28 suffice for the proof of Theorem 24. We give specifics after the proof. This is in constrast to Elekes’ proof of Theorem 5, which depends crucially on having bounds for \( |\text{Sym}_{\alpha}(Y)| \) that are linear in \( |Y| \).

3.3.4. Proof of Theorem 24.

Proof of Theorem 24. The condition \((\alpha/2)^{2^J} \geq 1/|Y|\) implies that \( \alpha_J \geq 2/|Y| \), and the condition \( |Y| \leq (\alpha/2)^{2^J}p \) implies that \( |Y| \leq \frac{1}{2}\alpha p \). Hence by Theorem 28 we have

\[ K \leq |\text{Sym}_{\alpha J}(Y)|^{1/J} \ll \left( \frac{\alpha}{2} \right)^{-C} |Y|^{1/J}. \]

By Theorem 25, there is a constant \( C > 0 \), an element \( g_* \) in \( G \), and a subset \( A_* \) of \( g_* \text{Sym}_{\alpha J}(Y) \) such that

\[ |A_*^3| \ll (\alpha_J^{-1}K)^C|A_*|. \]
Now, suppose that $F = \mathbb{C}$. By (33) and Theorem 26, there is an element $g$ in $G$ and an abelian subgroup $H$ of $G$ such that

$$|A_s \cap gH| \gg (\alpha J^{-1})^{-C} |A_s|.$$  

By equation (28) of Theorem 25, there is an element $g'$ in $G$ such that

$$|A \cap g'H| \gg \alpha^2 (\alpha^{-1} J^{-1})^{-C} \frac{|A_s \cap gS|}{|A_s|} |A_0| \gg \alpha^C |Y|^{-C/J} |A_0|.$$  

Since $\alpha J = 2(\alpha / 2)^{2J}$, the proof is complete.

If $F = \mathbb{F}_p$, then we apply Theorem 27 in place of Theorem 26. If (29) or (30) hold, then the proof is the same as in the case of $F = \mathbb{C}$, so suppose that (31) holds:

$$\left( \frac{K}{\alpha J} \right)^C |A_s| \gg |\pi(A_s)| p.$$  

Since $A_s \subseteq g_s \operatorname{Sym}_{\alpha J}(Y)$, by Theorem 28 we have

$$|A_s| \leq |\operatorname{Sym}_{\alpha J}(Y)| \ll \alpha_j^{-4} |Y| \ll \alpha_j^{-3} p.$$  

Combining this with (35) we have

$$|\pi(A_s)| \ll \left( \frac{K}{\alpha J} \right)^C,$$

which implies that there is an affine transformation $g$ such that

$$|A_s \cap gU| \gg \left( \frac{\alpha J}{K} \right)^C |A_s|.$$  

The rest of the proof is the same as in (34). \hfill \Box

Remark. Instead of using Theorem 28 to prove (32), we could have used the bound $|\operatorname{Sym}_\alpha(Y)| \ll \alpha^{-2} |Y|^2$, which follows from the Cauchy-Schwarz inequality and holds for $\text{Aff}(1, \mathbb{F}) \cap \mathbb{F}$ for any field $\mathbb{F}$, or even the trivial bound $|\operatorname{Sym}_\alpha(Y)| \leq |Y|^4$, which holds because $|Y|^2$ points support at most $|Y|^4$ lines containing at least two elements of the point set.

Equation (36) could be proved using Vinh's incidence bound [41], which can also be proved using only Cauchy-Schwarz [29].

APPENDIX A. PROOF OF GROUP ACTION BALOG-SZEMERÉDI-GOWERS

In this section, we sketch the proof of the group action version the asymmetric Balog-Szemeredi-Gowers theorem, which we recall here. This theorem is proved in more generality (and in full detail) in [28].

**Theorem 25.** There is an absolute constant $C > 0$ such that the following holds.

Let $Y$ be a finite subset of $\mathbb{F}$ and let $A$ be a finite subset of $\text{Aff}(1, \mathbb{F})$. Given a number $0 < \alpha < 1$ and an integer $J \geq 0$, define

$$\alpha J = 2 \left( \frac{\alpha}{2} \right)^{2J} \quad \text{and} \quad K = \left( \frac{|\operatorname{Sym}_\alpha(Y)|}{|A|} \right)^{1/J}.$$  

If $A \subseteq \operatorname{Sym}_\alpha(Y)$, then
(1) there is an element $g_*$ in $G$ and a finite subset $A_* \subseteq G$ such that
(38) $g_*^{-1}A_* \subseteq \text{Sym}_{\alpha J}(Y)$

and

(39) $|A_*^3| \ll \left( \frac{K}{\alpha J} \right)^C |A_*|$, 

(2) for any subset $H \subseteq G$ there is an element $g$ in $G$ such that
(40) $|A \cap gH| \gg \left( \frac{\alpha J}{K} \right)^C \frac{|H \cap A_*|}{|A_*|} |A|$. 

To understand how our method works, we will first revisit Elekes’ proof of Theorem 5. The key idea is that symmetry sets behave weakly like groups. In fact, $\text{Sym}_1(Y)$ is a group: it is the stabilizer of $Y$ under the induced action of on subsets of $X$. For $\alpha < 1$, a weak form of multiplicative closure holds.

**Proposition 29 (Approximate multiplicative closure).** If $S$ is a non-empty subset of $\text{Sym}_\alpha(Y)$, then there exists a relation $E \subseteq S^{-1} \times S$ such that

$$|E| \geq \frac{\alpha^2}{2} |S|^2 \quad \text{and} \quad S^{-1}E \cdot S \subseteq \text{Sym}_{\alpha^2/2}(Y).$$

Further, $(S^{-1}E)^{-1} = S^{-1}E.S$. 

This is [28, Proposition 10], which is a straightforward generalization of [39, Lemma 2.33], which follows easily from Cauchy-Schwarz.

To prove that Theorem 5 follows from a product theorem, such as Theorem 6, we combine Proposition 29 with the upper bounds of Theorem 28.

**Proposition 30.** Let $F$ be a field, and let $G = \text{Aff}(1,F)$ act on $X = F$ by affine transformations. Let $A \subseteq G$ and $Y \subseteq X$ be finite subsets such that $A \subseteq \text{Sym}_\alpha(Y)$ and $|A| \geq |Y|$. Then there is a subset $E \subseteq A \times A$ such that

$$|E| \geq \frac{\alpha^2}{2} |A|^2$$

and

$$A^{-1}E \cdot A \subseteq \text{Sym}_{\alpha^2/2}(Y).$$

By Theorem 28, if $F = \mathbb{C}$,

$$|A^{-1}E \cdot A| \leq |\text{Sym}_{\alpha^2/2}(Y)| \ll \alpha^{-6}|Y| \leq \alpha^{-6}|A|,$$

while if $F = \mathbb{F}_p$ and $|Y| \leq \frac{1}{2} \alpha p$,

$$|A^{-1}E \cdot A| \leq |\text{Sym}_{\alpha^2/2}(Y)| \ll \alpha^{-8}|Y| \leq \alpha^{-8}|A|. \square$$

Now we will prove the following theorem, in the spirit of Elekes’ Theorem 5.
Theorem 31. If $N$ lines are $\alpha$-rich in a $N \times N$ grid in $\mathbb{C}^2$, then either

1. $C^CN$ lines are parallel, or
2. $C^CN$ lines are concurrent,

where $C > 0$ is a constant independent of $\alpha$ and $N$.

The same holds for $\mathbb{F}_p^2$, provided that $N \leq \frac{1}{2} \alpha p$.

To prove Theorem 31, we need the Balog-Szemeredi-Gowers theorem. The following version [28, Lemma 12] is essentially contained in [38].

Theorem 32. If $A$ and $B$ are finite subsets of a group $G$ and $E \subseteq A \times B$ is a relation such that

$$|E| \geq \alpha |A||B| \quad \text{and} \quad |A^E B| \leq K |A|^{1/2} |B|^{1/2},$$

where $\alpha \in (0,1]$ and $K > 0$, then there is an element $a$ in $A$ and a subset $S \subseteq a^{-1}A$ such that

$$|S| \gg \left( \frac{\alpha}{K} \right)^C |A| \quad \text{and} \quad |S^3| \ll \left( \frac{K}{\alpha} \right)^C |S|,$$

where $C$ is an absolute constant.

Proof. Let $F$ denote $\mathbb{C}$ or $\mathbb{F}_p$, let $Y$ be a finite subset of $F$, and suppose that $L$ is a set of lines that are $\alpha$-rich in $Y \times Y$. Translating to algebraic language, we have $A \subseteq \text{Sym}_\alpha(Y)$, where $A = A_L$ is the set of affine transformations corresponding to the elements of $L$.

By Proposition 29, there is a subset $E \subseteq A^{-1} \times A$ such that

$$|E| \geq \frac{\alpha^2}{2} |A|^2 \quad \text{and} \quad |A^{-1} E A| \ll \alpha^{-8} |A|.$$

By Theorem 32, there is an element $a$ of $A$ and a subset $S$ of $\text{Aff}(1,F)$ such that $S \subseteq aA^{-1}$,

$$|S| \gg \alpha^{-C} |A|, \quad \text{and} \quad |S^3| \ll \alpha^{-C} |S|.$$

Now, as in the proof of Theorem 24, we may apply Theorem 26 or Theorem 27, depending on $F = \mathbb{C}$ or $F = \mathbb{F}_p$, to deduce that there is an abelian subgroup $H$ of $\text{Aff}(1,F)$ such that $|S \cap gH| \gg \alpha^{-C} |S|$ for some $g$ in $\text{Aff}(1,F)$. Since $S \subseteq aA^{-1}$ and $|S| \gg \alpha^{-C} |A|$, we have $|aA^{-1} \cap gH| \gg \alpha^{-C} |A|$, hence $|A \cap g^{-1}aH| \gg \alpha^{-C} |A|$ for some subgroup $H'$ conjugate to $H$.

To complete the proof, we translate to back geometric language, as in the proof of Theorem 4. \qed

It is a credit to Elekes’ ingenuity that he proved Theorem 5 without the Balog-Szemeredi-Gowers theorem.

The assumption $|A| \geq |Y|$ is necessary to compare $|A^{-1} E A|$ to $|A|$; if $|A| < |Y|$, then one may iterate Proposition 29 until we reach an iterated partial product set with small doubling. This strategy was used by Borenstein and Croot [4] to prove an analog of Elekes’ results for small sets of lines (affine transformations). The analogy between Borenstein and Croot’s work [4] and the asymmetric Balog-Szemeredi-Gowers theorem [39, Theorem 2.35], as observed by Helfgott [4], motivated Theorem 25.
To prove Theorem 25, we use a variation of Proposition 29. We use the notation $\preceq$ to hide logarithmic factors of $\alpha^{-1}$ and $|A|$, and for finite subsets $A$ and $B$ of a group and $E \subseteq A \times B$ we define

\[ A^E \cdot B = \{ab: (a, b) \in E\} \quad \text{and} \quad r_E(x) = |\{(a, b) \in E: ab = x\}|. \]

**Proposition 33** (Uniform approximate closure). If $A$ is a non-empty subset of $\text{Sym}_\alpha(Y)$ then there is a relation $E \subseteq A^{-1} \times A$ such that

\begin{align*}
(41) & |E| \gtrapprox \alpha^2 |A|^2, \\
(42) & r_E(x) \geq \frac{|E|}{2|A^{-1} \cdot E : A|} \quad \text{for all } x \text{ in } A^{-1} \cdot E, \\
(43) & A^{-1} \cdot E \subseteq \text{Sym}_{\alpha^2}(Y).
\end{align*}

The proof of Proposition 33 is essentially the same as the proof of [39, Lemma 2.34]: combine Proposition 29 with a dyadic pigeonholing argument. (While [39, Lemma 2.34] is stated only for abelian groups, the proof works verbatim for non-abelian groups.)

Proposition 33 implies that if a set $S$ is dense in the product set $A^{-1} \cdot E \cdot A$, then some translate of $S$ is dense in $A$. Thus, if we find a “structured” subset of the product set $A^{-1} \cdot E \cdot A$, we may bring that structure back to the original set $A$. More precisely, if $A$ is a finite subset of $G$ and $E \subseteq A^{-1} \times A$ satisfies (41) and (42), then for any subset $S$ of $G$, there is an element $a$ in $A$ such that

\[ |A \cap aS| \gtrapprox \alpha^2 |(A^{-1} \cdot E) \cap S| / |A^{-1} \cdot E : A|. \]

Now we sketch the proof of Theorem 25.

**Proof of Theorem 25.** Let $A_0 = A$ and $\alpha_0 = \alpha$. By Proposition 33, there is a subset $E_0 \subseteq A_0^{-1} \times A_0$ such that (41), (42), and (43) hold. Define $A_1 := A_0^{-1} \cdot E_0 \cdot A_0$. By (44), for any subset $S$ of $\text{Aff}(1, F)$, there is an element $a_0$ in $A_0$ such that

\[ |A_0 \cap a_0S| / |A_0| \gtrapprox \alpha_0^2 |A_1 \cap S| / |A_1|. \]

Since $A_1 \subseteq \text{Sym}_{\alpha_1}(Y)$, where $\alpha_1 = \alpha_0^2 / 2$, we may iterate this process to find a sequence of numbers

\[ \alpha_0 > \alpha_1 > \cdots > \alpha_J > 0 \]

such that $\alpha_{j+1} = \alpha_j^2 / 2$, and a sequence of sets $A_j \subseteq \text{Aff}(1, F)$ such that $A_j \subseteq \text{Sym}_{\alpha_j}(Y)$, and for any set $S$ in $\text{Aff}(1, F)$, there is an element $a_j$ in $A_j$ such that

\[ |A_j \cap a_jS| / |A_j| \gtrapprox \alpha_j^2 |A_j \cap S| / |A_j|. \]
Now for the key step: setting $K^J = |A_J|/|A_0|$, we have

$$K^J = \frac{|A_J|}{|A_0|} = \prod_{j=0}^{J-1} \frac{|A_{j+1}|}{|A_j|},$$

so by the pigeonhole principle, there is an index $0 \leq j \leq J - 1$ such that $|A_{j+1}| \leq K|A_j|$. That is,

$$|A_j^{-1} E_j A_j| \leq K|A_j|.$$

Since $|E_j| \gtrsim \alpha_j^2 |A_j|$, we can now apply the Balog-Szemerédi-Gowers theorem, as in the proof of Theorem 31, to find a subset $S$ of $a_j A_j^{-1}$ such that

$$|S \cap a_j A_j^{-1}| \gg \left( \frac{\alpha_j}{K} \right)^C |A_j| \quad \text{and} \quad |S^3| \ll \left( \frac{K}{\alpha_j} \right)^C |S|.$$

If we wished to prove Theorem 24 directly, we would now apply a product theorem to find an abelian subgroup of $\text{Aff}(1, \mathbb{F})$ with large overlap with $S$.

Instead, we simply assume that there is some set $H$ such that $|S \cap H| \gg (\alpha_j/K)^C |S|$. Since $S \subseteq a_j A_j^{-1}$, we have $|a_j^{-1} H \cap A_j^{-1}| \gg (\alpha_j/K)^C |A_j|$. Iterating (45) yields an element $g$ in $G$ such that

$$\frac{|A_0 \cap g a_j^{-1} H|}{|A_0|} \gtrsim (\alpha_0 \cdots \alpha_j)^2 \frac{|A_j \cap g a_j^{-1} H|}{|A_j|}.$$

Since $\alpha_j = 2(\alpha/2)^{2j} \geq 2(\alpha/2)^{2j}$, this completes the proof of Theorem 25. □

**Appendix B. Product theorems for $\text{Aff}(1, \mathbb{F})$**

Let $U$ be a subgroup of a group $G$, and let $\pi: G \to G/U$ be the quotient map. For a subset $A$ of $G$, let $A/U$ denote the image of $A$ under $\pi$; that is, $A/U$ is the set of left cosets of $U$ of the form $aU$ with $a$ in $A$.

Recall that if $G = \text{Aff}(1, \mathbb{F})$, then a maximal torus $T$ is a subgroup conjugate to the diagonal subgroup, and the unipotent subgroup $U$ consists of upper triangular matrices with 1’s on the diagonal. Every abelian subgroup of $\text{Aff}(1, \mathbb{F})$ is either contained in the unipotent subgroup $U$ or a maximal torus.

The following is a specialization of [8, Theorem 1.4] to $\text{Aff}(1, \mathbb{C})$.

**Theorem 34** (Product theorem for $\text{Aff}(1, \mathbb{C})$). If $A$ is a subset of $\text{Aff}(1, \mathbb{C})$ such that $|A^3| \leq K|A|$, then either $|A|/3$ elements of $A$ are contained in a torus, or

$$K^{10}|A| \gg |A/U|^{1/2}|A|,$$

hence there is an element $g$ in $G$ such that $|A \cap gU| \gg K^{-20}|A|$.

Theorem 34 says that if $A$ is not contained in a torus, then either $A$ is covered by a small number of cosets of $U$ (so that $A/U$ is small), or $A$ grows under multiplication: $|A^3| \gg |A/U|^{1/20}|A|$.

The next theorem is a slight quantitative improvement of the product theorem for $\text{Aff}(1, \mathbb{F}_p)$ that appears in [22].
**Theorem 35** (Product theorem for $\text{Aff}(1, \mathbb{F}_p)$). If $A$ is a subset of $\text{Aff}(1, \mathbb{F}_p)$ such that $|A^3| \leq K|A|$, then either $\geq |A|/3$ elements of $A$ are contained in a torus, or

$$K^{10}|A| \gg |A/U|^{1/2}|A|,$$

or

$$K^{10}|A| \gg |A/U|^p.$$

**B.1. Technical lemma.** The following lemma contains the common elements of the proofs of Theorem 34 and Theorem 35.

**Lemma 36.** If $\mathbb{F}$ is a field and $A$ is a finite subset of $\text{Aff}(1, \mathbb{F})$, then either more than one third of the elements of $A$ are contained in an abelian subgroup, or there exists an $x$ in $A$ such that $|x^A x^{-1}| = |[A, x]| > 1$.

Further there is an $a_0$ in $A$ such that if $S := x^A x^{-1} \subseteq U$ and $T := (a_0^{-1}A) \cap C(x)$ then

$$|S||T| \geq |A| \quad \text{and} \quad |T| \leq |A/U|.$$

In addition, if $|A^3| \leq K|A|$, then

$$K^{10}|A| \gg |A/U| \cdot |B - BC|,$$

where $|B| = |S|$ and $|C| = |T|$.

The proof of Lemma 36 requires the following version of the orbit-stabilizer theorem for sets, rather than for groups [22] and Ruzsa’s triangle inequality.

**Lemma 37.** Suppose $G \rtimes X$, $x \in X$, and $A \subseteq G$ is finite. Then there exists $a_0$ in $A$ such that

$$|(a_0^{-1}A) \cap \text{Stab}(x)| \geq \frac{|A|}{|A(x)|},$$

and for all finite sets $B \subseteq G$,

$$|BA| \geq |A \cap \text{Stab}(x)||B(x)|.$$

**Proposition 38** (Ruzsa triangle inequality). If $A, B, C$ are finite subsets of a group, then

$$|AC^{-1}| \leq \frac{|AB^{-1}||BC^{-1}|}{|B|}.$$

**Proof of Lemma 36.** Suppose that at most $|A|/3$ elements of $A$ are contained in an abelian subgroup. Then at least $2|A|/3$ element of $A$ are not contained in the unipotent group $U$, so without loss of generality, we may assume that $A$ does intersect the unipotent group $U$. (That is, we will use $A$ to denote $A \setminus U$.)

We still know that half of the elements of $A$ are not contained in an abelian subgroup, thus there exists an $x$ in $A$ such that

$$|x^A x^{-1}| = |[A, x]| > 1.$$

Otherwise, $axa^{-1}x^{-1} = e$ for all $a, x$ in $A$, which implies that the subgroup generated by $A$ is abelian.

The set $x^A = \{axa^{-1} : a \in A\}$ is the orbit of $x$ under the action of $G$ on itself by conjugation; the stabilizer of $x$ is denoted $C(x)$. Since $x \notin U$,
we know that $C(x)$ is conjugate to the diagonal subgroup of $\text{Aff}(1, \mathbb{F})$; in particular, the only element of $U$ fixed by $C(x)$ under conjugation is the identity element.

By Lemma 37, there is an element $a_0$ in $A$ such that

$$|(a_0^{-1}A) \cap C(x)| \geq \frac{|A|}{|xA|}.$$  

Let $T$ denote $a_0^{-1} \cap C(x)$.

Now, if $S = [A, x] = x^4 \cdot x^{-1}$, then $|S| = |x^4|$; by (48) we know $|S| > 1$, so $S$ contains an element of $U$ besides the identity. Since $|T| = |(a_0^{-1}A) \cap C(x)|$, the previous equation can be restated as

$$|S||T| \geq |A|.$$  

In addition, note that $|T| \leq |A/U|$, where $A/U$ is the image of $A$ under the quotient map $\pi : G \to G/U$. The inequality $|T| \leq |A/U|$ follows since $\pi$ is injective when restricted to a torus.

Let $S^T$ denote the image of $S$ under the action of $T$ by conjugation. Since $S \subseteq U$ and $U$ is preserved by conjugation, we have $S \cdot (S^T)^{-1} \subseteq U$.

Let $B = \{ z : (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \in S, z \neq 0 \}$ and let $C = \{ a : \exists b (\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}) \in T \}$.

Then

$$|S \cdot (S^T)^{-1} \cap U| = |S \cdot (S^T)^{-1}| \geq |B - BC|,$$

since conjugating $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ by $(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix})$ yields $(\begin{smallmatrix} 1 & az \\ 0 & 1 \end{smallmatrix})$.

Clearly, $|B| \geq |S| - 1$ and since $\pi : G \to G/U$ is injective when restricted to $T$, we have $|C| = |T|$.

Note that

$$S \cdot (S^T)^{-1} = x^4x^{-1}((x^4x^{-1})^T)^{-1} = x^4x^{-1}(x^{TA}x^{-1})^{-1} = x^A(x^{-1})^{TA}.$$

So that

$$S \cdot (S^T)^{-1} \subseteq AxA^{-1}A^2x^{-1}A^{-2} \subseteq A^2A^{-1}A^2A^{-3}.$$

By Lemma 37, we have

$$|A^3A^{-1}A^2A^{-3}| \geq |(A^2A^{-1}A^2A^{-3}) \cap U||A/U| \geq |S \cdot (S^T)^{-1}| |A/U|.$$  

By Proposition 38, if $|A^3| \leq K|A|$, then

$$|A^3A^{-1}A^2A^{-3}| \leq K^{10}|A|.$$  

All together, we have

$$K^{10}|A| \geq |S \cdot (S^T)^{-1}| |A/U| \geq |A/U| |B - BC|,$$

as desired. □

Note that if $B$ contains only 0, then $|B - BC| = 1$; where as if $B$ contains a non-zero element, then $|B - BC| \geq |C|$. 

B.2. **Proof of results over $C$.** The following theorem is an easy consequence of the Szemerédi-Trotter theorem, see [39, Exercise 8.3.3].

**Proposition 39.** If $A, B, C$ are finite subsets of $C$, then

$$|A + BC| \gg \sqrt{|A||B||C|}.$$ 

**Proof of Theorem 34.** If at least one third of the elements in $H$ are contained in an abelian subgroup, then we are done.

Otherwise, by Lemma 36 and Proposition 39, we have

$$K^{10}|H| \gg |H/U||B - BC| \gg |H/U||B||C|^{1/2} = |H/U||S||T|^{1/2}.$$

Since $|T| \leq |H/U|$ and $|S||T| \geq |H|$, we have

$$K^{10}|H| \gg |H/U|^{1/2}|S||T| \geq |H/U|^{1/2}|H|.$$ 

\[\square\]

B.3. **Proof of results over $\mathbb{F}_p$.** The following sum-product theorem is a slight improvement of a result of Roche-Newton, Rudnev, and Shkredov [32], due to Stevens and de Zeeuw [36, Corollary 10].

**Proposition 40.** If $A, B, C \subseteq \mathbb{F}_p$ where $p$ is prime, then

$$|A + BC| \gg \min\left(\sqrt{|A||B||C|}, p\right).$$

In particular,

$$|B \pm BC| \gg \min\left(|B||C|^{1/2}, p\right).$$

(49)

**Proof of the product theorem over $\mathbb{F}_p$.** If more than one third of $A$ is contained in an abelian subgroup, then we are done.

Otherwise, by Lemma 36 and Proposition 40, we have

$$K^{10}|A| \gg |A/U||B - BC| \gg |A/U|\min\left(|B||C|^{1/2}, p\right).$$

If the minimum is $|B||C|^{1/2}$, then as in the previous proof, we have

$$K^{10}|A| \gg |A/U|^{1/2}|A|.$$ 

If the minimum is $p$, then we have

$$K^{10}|A| \gg |A/U|p.$$ 

\[\square\]

**References**


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