

VOLUME PINCHING THEOREMS FOR CAT(1) SPACES

KOICHI NAGANO

ABSTRACT. We examine volume pinching problems of CAT(1) spaces. We characterize a class of compact geodesically complete CAT(1) spaces of small specific volume. We prove a sphere theorem for compact CAT(1) homology manifolds of small volume. We also formulate a criterion of manifold recognition for homology manifolds on volume growths under an upper curvature bound.

1. INTRODUCTION

1.1. **Backgrounds.** Many problems of pinching theorems, including sphere theorems, on various metric invariants have attracted our interests in global Riemannian geometry. In this paper, we examine volume pinching problems of CAT(1) spaces as a subsequent study of the series of the works of Lytchak and the author [LN1] and [LN2].

For every metric space with an upper curvature bound, all the spaces of directions are CAT(1). Lytchak and the author have proved a *local topological regularity theorem* [LN2, Theorem 1.1]: A locally compact metric space with an upper curvature bound is a topological n -manifold if and only if all the spaces of directions are homotopy equivalent to an $(n - 1)$ -sphere. Once we would establish a sphere theorem for CAT(1) spaces, we could obtain an infinitesimal characterization of topological manifolds for spaces with an upper curvature bound.

We say that a triple of points in a CAT(1) space is a *tripod* if the three points have pairwise distance at least π . Lytchak and the author have invented a *capacity sphere theorem* [LN2, Theorem 1.5] for CAT(1) spaces: If a compact, geodesically complete CAT(1) space admits no tripod, then it is homeomorphic to a sphere.

Throughout this paper, we denote by \mathbb{S}^n the standard unit n -sphere, and by T the discrete metric space consisting of three points with pairwise distance π . For instance, the spherical join $\mathbb{S}^{n-1} * T$ is a compact, geodesically complete CAT(1) space containing the tripod T .

Date: January 15, 2021.

2010 Mathematics Subject Classification. 53C20, 53C23, 57P05.

Key words and phrases. CAT(1) space, Sphere theorem, Homology manifold.

Partially supported by JSPS KAKENHI Grant Numbers 26610012, 21740036, 18740023, and by the 2004 JSPS Postdoctoral Fellowships for Research Abroad.

We say that a separable metric space is *purely n -dimensional* if every non-empty open subset has finite (Lebesgue) covering dimension n . We denote by \mathcal{H}^n the n -dimensional Hausdorff measure. If X is a purely n -dimensional, compact, geodesically complete CAT(1) space, then $\mathcal{H}^n(X) \geq \mathcal{H}^n(\mathbb{S}^n)$; the equality holds if and only if X is isometric to \mathbb{S}^n ([N1, Lemma 3.1 and Proposition 7.1]); moreover, if $\mathcal{H}^n(X)$ is sufficiently close to $\mathcal{H}^n(\mathbb{S}^n)$, then X is bi-Lipschitz homeomorphic to \mathbb{S}^n ([N1, Theorem 1.10]). Lytchak and the author have proved a *volume sphere theorem* [LN2, Theorem 8.3] for CAT(1) spaces: If a purely n -dimensional, compact, geodesically complete CAT(1) space X satisfies

$$(*) \quad \mathcal{H}^n(X) < \frac{3}{2} \mathcal{H}^n(\mathbb{S}^n),$$

then X is homeomorphic to \mathbb{S}^n .

In the volume sphere theorem [LN2, Theorem 8.3], the pureness on the dimension is essential since we can construct counterexamples possessing lower dimensional subsets. The assumption $(*)$ of \mathcal{H}^n is optimal since the spherical join $\mathbb{S}^{n-1} * T$ satisfies $\mathcal{H}^n(\mathbb{S}^{n-1} * T) = (3/2) \mathcal{H}^n(\mathbb{S}^n)$.

1.2. Main results. We construct a CAT(1) n -sphere admitting a tripod whose n -dimensional Hausdorff measure is equal to $(3/2) \mathcal{H}^n(\mathbb{S}^n)$.

Example 1.1. The spherical join $\mathbb{S}^{n-2} * T$ can be represented by the quotient metric space $\bigsqcup_{i=1,2,3} \mathbb{S}_{+,i}^{n-1} / \sim$ obtained by gluing three closed unit $(n-1)$ -hemispheres $\mathbb{S}_{+,i}^{n-1}$ along their boundaries $\partial \mathbb{S}_{+,i}^{n-1} = \partial \mathbb{S}_{+,j}^{n-1}$. For $i = 1, 2, 3$, $3 + 1 = 1$, let Σ_i^{n-1} be the isometrically embedded unit $(n-1)$ -spheres $\mathbb{S}_{+,i}^{n-1} \sqcup \mathbb{S}_{+,i+1}^{n-1} / \sim$ in $\mathbb{S}^{n-2} * T$ obtained by the relation $\partial \mathbb{S}_{+,i}^{n-1} = \partial \mathbb{S}_{+,i+1}^{n-1}$. We take three copies of closed unit n -hemispheres $\mathbb{S}_{+,i}^n$, $i = 1, 2, 3$. Let X be the quotient metric space obtained as

$$X := (\mathbb{S}^{n-2} * T) \sqcup \left(\bigsqcup_{i=1,2,3} \mathbb{S}_{+,i}^n \right) / \sim$$

by attaching $\mathbb{S}_{+,i}^n$ to $\mathbb{S}^{n-2} * T$ along $\Sigma_i^{n-1} = \partial \mathbb{S}_{+,i}^n$ for each $i \in \{1, 2, 3\}$. We call X the *n -triplex*. The n -triplex X is a purely n -dimensional, compact, geodesically complete CAT(1) space that is homeomorphic to \mathbb{S}^n . This space has a tripod and satisfies $\mathcal{H}^n(X) = (3/2) \mathcal{H}^n(\mathbb{S}^n)$. We notice that the 1-triplex is by definition a circle of length 3π .

As one of the main results, we prove the following characterization:

Theorem 1.1. *Let X be a purely n -dimensional, compact, geodesically complete CAT(1) space. If X satisfies*

$$(1.1) \quad \mathcal{H}^n(X) = \frac{3}{2} \mathcal{H}^n(\mathbb{S}^n),$$

*then X is either homeomorphic to \mathbb{S}^n or isometric to $\mathbb{S}^{n-1} * T$. If in addition X has a tripod, then X is isometric to either the n -triplex or $\mathbb{S}^{n-1} * T$.*

Theorem 1.1 for the case of $n \leq 2$ was proved in [N2].

For CAT(1) homology manifolds, one can hope that Theorem 1.1 enables us to relax the condition (*) in the volume sphere theorem [LN2, Theorem 8.3]. We note that every CAT(1) homology manifold (without boundary) is geodesically complete.

The other main result is the following volume sphere theorem for CAT(1) homology manifolds:

Theorem 1.2. *For every positive integer n , there exists a sufficiently small positive number $\delta \in (0, \infty)$ depending only on n such that if a compact CAT(1) homology n -manifold X satisfies*

$$(1.2) \quad \mathcal{H}^n(X) < \frac{3}{2} \mathcal{H}^n(\mathbb{S}^n) + \delta,$$

then X is homeomorphic to \mathbb{S}^n .

Theorem 1.2 is new even for Riemannian manifolds. We notice that a complete Riemannian manifold is CAT(1) if and only if it has sectional curvature ≤ 1 and injectivity radius $\geq \pi$.

Remark 1.1. Let M be a simply connected, compact, $(2n)$ -dimensional Riemannian manifold of positive sectional curvature ≤ 1 . Due to the Klingenberg estimate of injectivity radii, we see that M has injectivity radius $\geq \pi$; in particular, M is CAT(1). By the sphere theorem of Coghlan and Itokawa [CI], we know that if $\mathcal{H}^{2n}(M) \leq (3/2) \mathcal{H}^{2n}(\mathbb{S}^{2n})$, then M is homeomorphic to \mathbb{S}^{2n} . In the sphere theorem of Coghlan and Itokawa [CI], the condition on the volume was relaxed by Wu [Wu], and by Wen [We1, We2] under lower sectional curvature bounds. In the proofs in [Wu] and in [We1, We2], the assumptions of the lower sectional curvature bounds for Riemannian manifolds is essential.

In the proof of Theorem 1.2, we need a criterion of manifold recognition for homology manifolds on volume growths. For $\kappa \in \mathbb{R}$, we denote by M_κ^n the simply connected, complete Riemannian n -manifold of constant curvature κ . Let D_κ denote the diameter of M_κ^n . For $r \in (0, D_\kappa)$, we denote by $\omega_\kappa^n(r)$ the n -dimensional Hausdorff measure of any metric ball in M_κ^n of radius r if $n \geq 2$, and by $\omega_\kappa^1(r)$ the 1-dimensional Hausdorff measure of $[-r, r]$. From the local topological regularity theorem [LN2, Theorem 1.1] and the volume sphere theorem [LN2, Theorem 8.3], we deduce a *local topological regularity theorem on volume growths*: Let X be a locally compact, geodesically complete CAT(κ) space, and let W be a purely n -dimensional open subset of X . If for every $x \in W$ there exists $r \in (0, D_\kappa)$ satisfying $\mathcal{H}^n(B_r(x))/\omega_\kappa^n(r) < 3/2$, then W is a topological n -manifold, where $B_r(x)$ is the closed metric ball of radius r centered at x (see Theorem 3.7).

As one of the key ingredients in the proof of Theorem 1.2, we provide the following criterion of manifold recognition:

Theorem 1.3. *For every positive integer n , there exists a sufficiently small positive number $\delta \in (0, \infty)$ depending only on n with the following property: Let X be a $\text{CAT}(\kappa)$ homology n -manifold, and let W be an open subset of X . If for every $x \in W$ there exists $r \in (0, D_\kappa)$ satisfying*

$$(1.3) \quad \frac{\mathcal{H}^n(B_r(x))}{\omega_\kappa^n(r)} < \frac{3}{2} + \delta,$$

then W is a topological n -manifold.

1.3. Outline. The organization of this paper is as follows: In Section 2, we recall the known basic properties of metric spaces with an upper curvature bound. In Section 3, we deduce the local topological regularity theorem (Theorem 3.7) on volume growths mentioned above.

In Section 4, we prove Theorem 1.1. Due to the capacity sphere theorem for $\text{CAT}(1)$ spaces [LN2], it suffices to consider the case where X is a purely n -dimensional, compact, geodesically complete $\text{CAT}(1)$ space with (1.1) admitting a tripod. By the volume rigidity of Bishop-Günther type [N1], the space X consists of three unit n -hemispheres. Observing how the hemispheres meet each other, we obtain the conclusion. When we determine the geometric structure, we use the volume sphere theorem for $\text{CAT}(1)$ spaces [LN2].

In Section 5, we prove Theorems 1.2 and 1.3 by contradiction. To achieve the tasks, we use Theorem 1.1, the local topological regularity theorem [LN2], and the volume convergence theorem [N1].

1.4. Problem. As a natural question beyond Theorem 1.2, we pose the following volume pinching problem for $\text{CAT}(1)$ spaces:

Problem 1.1. Let $n \geq 2$. Let R_n be the supremum of $R \in (3/2, \infty)$ for which every $\text{CAT}(1)$ homology n -manifold X with $\mathcal{H}^n(X)/\mathcal{H}^n(\mathbb{S}^n) \leq R$ is homeomorphic to \mathbb{S}^n .

- (1) Find the concrete value R_n .
- (2) Describe all compact $\text{CAT}(1)$ homology n -manifolds X satisfying $\mathcal{H}^n(X)/\mathcal{H}^n(\mathbb{S}^n) = R_n$ in the maximal critical case.

This problem seems to be interesting even for Riemannian manifolds.

A Riemannian manifold M is said to be a C_l -manifold if every geodesic in M is contained in a periodic closed geodesic of length l ; in this case, the Riemannian metric of M is called a C_l -metric. For any n -dimensional $C_{2\pi}$ -manifold M , the volume ratio $\mathcal{H}^n(M)/\mathcal{H}^n(\mathbb{S}^n)$ is an integer ([W, Theorem A]), called the *Weinstein integer* for M (see [Bes, Theorem 2.21]). We know the concrete values of the Weinstein integers for compact symmetric spaces of rank one with the standard $C_{2\pi}$ -metric (see [Be, VI.7] and [Bes, 2.23]).

Every compact symmetric space of rank one with the standard $C_{2\pi}$ -metric is $\text{CAT}(1)$. The number R_n in Problem 1.1 is not greater than the Weinstein integers for the projective spaces with the $C_{2\pi}$ -metric.

Acknowledgments. The author would like to express his gratitude to Alexander Lytchak for many valuable discussions and comments. The author would like to thank an anonymous referee for providing the proof of Proposition 4.1 much shorter than a proof in a previous manuscript of this paper (see Remark 4.1).

2. PRELIMINARIES

We refer the readers to [AKP], [ABN], [B], [BH], [BBI], [BS] for the basic facts on metric spaces with an upper curvature bound.

2.1. Metric spaces. We denote by d the metrics on metric spaces. For $r \in (0, \infty)$, and for a point p in a metric space, we denote by $U_r(p)$, $B_r(p)$, and $\partial B_r(p)$ the open metric ball of radius r centered at p , the closed one, and the metric sphere, respectively. A metric space is said to be *proper* if every closed metric ball is compact.

For a metric space X , we denote by $C(X)$ the Euclidean cone over X . For metric spaces Y and Z , we denote by $Y * Z$ the spherical join of Y and Z . Note that $C(Y * Z)$ is isometric to the ℓ^2 -direct product $C(Y) \times C(Z)$. The spherical join $\mathbb{S}^{m-1} * \mathbb{S}^{n-1}$ is isometric to \mathbb{S}^{m+n-1} .

For $r \in (0, \infty]$, a metric space X is said to be *r -geodesic* if every pair of points p, q with distance $< r$ can be joined by a geodesic pq in X , where a *geodesic* pq means the image of an isometric embedding $\gamma: [a, b] \rightarrow X$ from a closed interval $[a, b]$ with $\gamma(a) = p$ and $\gamma(b) = q$. A metric space is *geodesic* if it is ∞ -geodesic. A geodesic space is *proper* if and only if it is complete and locally compact.

For $r \in (0, \infty]$, a subset C in a metric space is said to be *r -convex* if C itself is r -geodesic as a metric subspace, and if every geodesic joining two points in C is contained in C . A subset C in a metric space is *convex* if C is ∞ -convex.

2.2. CAT(κ) spaces. For $\kappa \in \mathbb{R}$, a complete metric space X is said to be CAT(κ) if X is D_κ -geodesic, and if every geodesic triangle in X with perimeter $< 2D_\kappa$ is not thicker than the comparison triangle in M_κ^2 . Our CAT(κ) spaces are assumed to be complete. A metric space *has an upper curvature bound* κ if every point has a CAT(κ) neighborhood.

Let X be a CAT(κ) space. Every pair of points in X with distance $< D_\kappa$ can be uniquely joined by a geodesic. Let $p \in X$ be arbitrary. For every $r \in (0, D_\kappa/2]$, the ball $B_r(p)$ is convex. Along the geodesics emanating from p , for every $r \in (0, D_\kappa)$ the ball $B_r(p)$ is contractible inside itself. Every open subset of X is an ANR (absolute neighborhood retract) ([O], [Kr]). For $x, y \in U_{D_\kappa}(p) - \{p\}$, we denote by $\angle_p(x, y)$ the angle at p between px and py . Put $\Sigma'_p X := \{px \mid x \in U_{D_\kappa}(p) - \{p\}\}$. The angle \angle_p at p is a pseudo-metric on $\Sigma'_p X$. The *space of directions* $\Sigma_p X$ at p is defined as the \angle_p -completion of the quotient metric space $\Sigma'_p X / \angle_p = 0$. For $x \in U_{D_\kappa}(p) - \{p\}$, we denote by $x'_p \in \Sigma_p X$ the starting direction of px at p . The *tangent space* $T_p X$ at p is defined as

the Euclidean cone $C(\Sigma_p X)$ over $\Sigma_p X$. We denote by $o_p \in T_p X$ the vertex of the cone $T_p X$. The space $\Sigma_p X$ is CAT(1), and the space $T_p X$ is CAT(0). In fact, for a metric space Σ , the Euclidean cone $C(\Sigma)$ is CAT(0) if and only if Σ is CAT(1). For metric spaces Y and Z , the spherical join $Y * Z$ is CAT(1) if and only if Y and Z are CAT(1).

2.3. Geodesically complete CAT(κ) spaces. We refer the readers to [LN1] for the basic properties of GCBA spaces, that is, locally compact, separable, locally geodesically complete spaces with an upper curvature bound. Recall that a CAT(κ) space is said to be *locally geodesically complete* (or *has geodesic extension property*) if every geodesic defined on a compact interval can be extended to a local geodesic beyond endpoints. A CAT(κ) space is said to be *geodesically complete* if every geodesic can be extended to a local geodesic defined on \mathbb{R} . Every locally geodesically complete CAT(κ) space is geodesically complete. The geodesic completeness for compact (resp. proper) CAT(κ) spaces is preserved under the (resp. pointed) Gromov-Hausdorff limit.

Let X be a proper, geodesically complete CAT(κ) space. For every $p \in X$, the space $\Sigma_p X$ coincides with the set $\Sigma'_p X$ of all starting directions at p . Moreover, $\Sigma_p X$ is compact and geodesically complete, and $T_p X$ is proper and geodesically complete. In fact, for a CAT(1) space Σ , the cone $C(\Sigma)$ is geodesically complete if and only if Σ is geodesically complete and not a singleton. For CAT(1) spaces Y and Z , the join $Y * Z$ is geodesically complete if and only if so are Y and Z .

2.4. Dimension of CAT(κ) spaces. Let X be a separable CAT(κ) space. The (Lebesgue) covering dimension $\dim X$ satisfies

$$\dim X = 1 + \sup_{p \in X} \dim \Sigma_p X = \sup_{p \in X} \dim T_p X$$

([K]). Let X be proper and geodesically complete. Every relatively compact open subset of X has finite covering dimension (see [LN1, Subsection 5.3]). The dimension $\dim X$ is equal to the Hausdorff dimension of X ; moreover, $\dim X$ is equal to the supremum of m such that X admits an open subset U homeomorphic to the Euclidean m -space \mathbb{R}^m ([LN1, Theorem 1.1]). If $\dim X = n$, then the support of \mathcal{H}^n coincides with the set of all points $x \in X$ with $\dim \Sigma_x X = n - 1$ ([LN1, Theorem 1.2]).

From the studies in [LN1, Subsection 11.3] on the stability of dimension, we can immediately derive the following three lemmas:

Lemma 2.1. *Let (X_i, p_i) , $i = 1, 2, \dots$, be a sequence of pointed proper geodesically complete CAT(κ) spaces. Assume that (X_i, p_i) converges to some (X, p) in the pointed Gromov-Hausdorff topology. Then*

$$\dim X \leq \liminf_{i \rightarrow \infty} \dim X_i.$$

Proof. Assume that for some positive integer n there exists $x_n \in X$ with $\dim \Sigma_{x_n} X = n - 1$. In this case, we have $\dim X \geq n$. We can take a sequence $x_{n,i} \in X_i$, $i = 1, 2, \dots$, converging to the point $x_n \in X$. Since $\dim \Sigma_{x_n} X = n - 1$, there exists $r_n \in (0, D_\kappa)$ such that we have $\dim U_{r_n}(x_{n,i}) = n$ for all sufficiently large i ([LN1, Lemma 11.5]). This implies $n \leq \liminf_{i \rightarrow \infty} \dim X_i$, and the lower semi-continuity. \square

On the Gromov-Hausdorff topology, we have:

Lemma 2.2. *Let X_i , $i = 1, 2, \dots$, be a sequence of compact geodesically complete CAT(κ) spaces. Assume that X_i converges to some X in the Gromov-Hausdorff topology, then*

$$\lim_{i \rightarrow \infty} \dim X_i = \dim X.$$

Proof. By Lemma 2.1, it is enough to show the upper semi-continuity $\limsup_{i \rightarrow \infty} \dim X_i \leq \dim X$. We may assume that $\dim X$ is finite. Set $n = \dim X$. Then all the spaces of directions in X have dimension $\leq n - 1$. Suppose that the sequence X_i , $i = 1, 2, \dots$, has a subsequence X_j , $j = 1, 2, \dots$, such that $\dim X_j \geq n + 1$ for all j . Then we can take a sequence $x_j \in X_j$, $j = 1, 2, \dots$, such that $\dim \Sigma_{x_j} X_j \geq n$ for all j , and a point $x \in X$ to which the sequence $x_j \in X_j$, $j = 1, 2, \dots$, converges. Since $\dim \Sigma_x X \leq n - 1$, we have $\dim \Sigma_{x_j} X_j \leq n - 1$ for all sufficiently large j ([LN1, Lemma 11.5]). This is a contradiction, and proves the upper semi-continuity. \square

On the pureness on the dimension, we have:

Lemma 2.3. *Let (X_i, p_i) , $i = 1, 2, \dots$, be a sequence of pointed proper geodesically complete CAT(κ) spaces. Assume that (X_i, p_i) converges to some (X, p) in the pointed Gromov-Hausdorff topology. If each X_i is purely n -dimensional, then so is X .*

Proof. Assume that each X_i is purely n -dimensional. Then all the spaces of directions in X_i have dimension $n - 1$. From Lemma 2.1 we derive $\dim X \leq n$, so that all the spaces of directions in X have dimension $\leq n - 1$. Moreover, we see $\dim X = n$. Indeed, if we would have $\dim X \leq n - 1$, then we could find a point $x_0 \in X$ with $\dim \Sigma_{x_0} X \leq n - 2$, and a sequence $x_i \in X_i$, $i = 1, 2, \dots$, converging to the point $x_0 \in X$, so that $\dim \Sigma_{x_i} X_i \leq n - 2$ for all sufficiently large i ([LN1, Lemma 11.5]). Similarly, we see that for every $x \in X$ we have $\dim \Sigma_x X = n - 1$. Therefore X is purely n -dimensional too. \square

We say that a separable metric space is *pure-dimensional* if it is purely n -dimensional for some n .

We have the following characterization ([LN2, Proposition 8.1]):

Proposition 2.4. ([LN2]) *Let X be a proper, geodesically complete, geodesic CAT(κ) space. Let W be a connected open subset of X . Then the following are equivalent:*

- (1) W is pure-dimensional;
- (2) for every $p \in W$ the space $\Sigma_p X$ is pure-dimensional;
- (3) for every $p \in W$ the space $T_p X$ is pure-dimensional.

3. TOPOLOGICAL REGULARITY ON VOLUME GROWTHS

In this section, we discuss direct consequences of the study in [N1] and the studies in [LN1] and [LN2].

3.1. Volume comparisons of CAT(κ) spaces. We recall that for every proper, geodesically complete CAT(κ) space X of $\dim X = n$, the support of \mathcal{H}^n coincides with the set of all points $x \in X$ with $\dim \Sigma_x X = n-1$ ([LN1, Theorem 1.2]). We can reformulate the volume comparisons studied in [N1] in the following way.

Let X be a proper, geodesically complete CAT(κ) space. Let $p \in X$ be a point with $\dim \Sigma_p X = n-1$. Then there exists $u \in \Sigma_p X$ such that \mathbb{S}^{n-2} is isometrically embedded into $\Sigma_u \Sigma_p X$ ([K, Theorem B], [LN1, Theorem 1.3]). Since $\Sigma_p X$ is geodesically complete, there exists a surjective 1-Lipschitz map φ_p from $\Sigma_p X$ onto the unit tangent sphere $\Sigma_o M_\kappa^n$ at a point $o \in M_\kappa^n$ with $d(\varphi_p(u), \varphi_p(v)) = d(u, v)$ ([N1, Lemma 3.1], [L, Lemma 2.2], [LN1, Proposition 11.3]). For every $r \in (0, D_\kappa)$, there exists a surjective 1-Lipschitz map $\Phi_p: B_r(p) \rightarrow B_r(o)$ defined by $\Phi_p(x) := \exp_o d(p, x) \varphi_p(x'_p)$, where \exp_o is the exponential map at o . The map Φ_p gives us an absolute volume comparison of Bishop-Günther type. If in addition X is purely n -dimensional, then we see a volume rigidity ([N1, Proposition 6.1]). Namely, we have:

Proposition 3.1. ([N1]) *Let X be a proper, geodesically complete CAT(κ) space, and let $p \in X$ be a point with $\dim \Sigma_p X = n-1$. Then for every $r \in (0, D_\kappa)$ we have*

$$\mathcal{H}^n(B_r(p)) \geq \omega_\kappa^n(r).$$

Moreover, if in addition X is purely n -dimensional, then the equality holds if and only if the pair $(B_r(p), p)$ is isometric to $(B_r(o), o)$ for any point $o \in M_\kappa^n$.

Furthermore, we have the following relative volume comparison of Bishop-Gromov type ([N1, Proposition 6.3]):

Proposition 3.2. ([N1]) *Let X be a proper, geodesically complete CAT(κ) space, and let $p \in X$ be a point with $\dim \Sigma_p X = n-1$. Then the function $f: (0, D_\kappa) \rightarrow [1, \infty]$ defined as*

$$f(t) := \frac{\mathcal{H}^n(B_t(p))}{\omega_\kappa^n(t)}$$

is monotone non-decreasing.

3.2. Volume convergence of CAT(κ) spaces. Let X_i , $i = 1, 2, \dots$, be a sequence of compact geodesically complete CAT(κ) spaces of $\dim X_i = n$. Assume that X_i converges to some compact metric space X in the Gromov-Hausdorff topology. By Lemmas 2.2 and 2.3, the compact, geodesically complete CAT(κ) space X satisfies $\dim X = n$; if in addition each X_i is purely n -dimensional, then so is X .

We can quote the volume convergence theorem for CAT(κ) spaces in [N1, Theorem 1.1] in the following form:

Theorem 3.3. ([N1]) *Let X_i , $i = 1, 2, \dots$, be a sequence of compact, geodesically complete CAT(κ) spaces of $\dim X_i = n$. If X_i converges to some compact metric space X in the Gromov-Hausdorff topology, then*

$$\mathcal{H}^n(X) = \lim_{i \rightarrow \infty} \mathcal{H}^n(X_i).$$

From Proposition 3.1 we deduce the following ([N2, Proposition 6.5]):

Proposition 3.4. ([N2]) *Let $c \in (0, \infty)$. Then every isometry class of purely n -dimensional compact geodesically complete CAT(κ) spaces whose n -dimensional Hausdorff measures are bounded above by c are precompact in the Gromov-Hausdorff topology.*

We have the following infinitesimal regularity of Hausdorff measures on CAT(κ) spaces ([N1, Theorem 1.4]):

Theorem 3.5. ([N1]) *Let X be a proper, geodesically complete CAT(κ) space, and let $p \in X$ be a point with $\dim \Sigma_p X = n - 1$. Then*

$$\lim_{t \rightarrow 0} \frac{\mathcal{H}^n(B_t(p))}{t^n} = \mathcal{H}^n(B_1(o_p)),$$

where $B_1(o_p)$ is the unit ball centered at the vertex o_p in $T_p X$.

Remark 3.1. Lytchak and the author have generalized Theorems 3.3, 3.5, and Proposition 3.4 for a canonical geometric volume measure in [LN1, Theorems 1.4, 1.5, and Subsection 12.5].

3.3. Topological regularity. In what follows, we will use:

Lemma 3.6. *Let X be a proper, geodesically complete CAT(κ) space, and let $p \in X$ be a point with $\dim \Sigma_p X = n - 1$. Then for every $r \in (0, D_\kappa)$ we have*

$$\frac{\mathcal{H}^{n-1}(\Sigma_p X)}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \leq \frac{\mathcal{H}^n(B_r(p))}{\omega_\kappa^n(r)}.$$

Proof. From Theorem 3.5 and Proposition 3.2, we derive

$$\frac{\mathcal{H}^{n-1}(\Sigma_p X)}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} = \frac{\mathcal{H}^n(B_1(o_p))}{\omega_0^n(1)} = \lim_{t \rightarrow 0} \frac{\mathcal{H}^n(B_t(p))}{\omega_\kappa^n(t)} \leq \frac{\mathcal{H}^n(B_r(p))}{\omega_\kappa^n(r)}$$

(cf. [N2, Remark 2.10]), as required. \square

Now we prove the following regularity:

Theorem 3.7. *Let X be a proper, geodesically complete $\text{CAT}(\kappa)$ space, and let W be a purely n -dimensional open subset of X . If for every $x \in W$ there exists $r \in (0, D_\kappa)$ satisfying*

$$(3.1) \quad \frac{\mathcal{H}^n(B_r(x))}{\omega_\kappa^n(r)} < \frac{3}{2},$$

then W is a topological n -manifold.

Proof. By Lemma 3.6, for every $x \in W$ the condition (3.1) leads to $\mathcal{H}^{n-1}(\Sigma_x X) < (3/2) \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$. From Proposition 2.4 it follows that $\Sigma_x X$ is a purely $(n-1)$ -dimensional, compact, geodesically complete $\text{CAT}(1)$ space. Due to the volume sphere theorem [LN2, Theorem 8.3], the space $\Sigma_x X$ is homeomorphic to \mathbb{S}^{n-1} . Applying the local topological regularity theorem [LN2, Theorem 1.1] to W , we conclude that W is a topological n -manifold. \square

Remark 3.2. The assumption (3.1) in Theorem 3.7 is optimal.

Example 3.1. For $\kappa \in (0, \infty)$, let X be the $(1/\sqrt{\kappa})$ -rescaled space $(1/\sqrt{\kappa})(\mathbb{S}^{n-1} * T)$ of the spherical join $\mathbb{S}^{n-1} * T$. The space X is a purely n -dimensional, compact, geodesically complete $\text{CAT}(\kappa)$ space, and not a topological n -manifold at any point in the spherical factor \mathbb{S}^{n-1} . For every point $x \in X$ in the spherical factor \mathbb{S}^{n-1} of X , and for every $r \in (0, D_\kappa)$, we have $\mathcal{H}^n(B_r(x))/\omega_\kappa^n(r) = 3/2$.

Example 3.2. For $\kappa \in (-\infty, 0]$, let X be the κ -cone $C_\kappa(\mathbb{S}^{n-2} * T)$ over $\mathbb{S}^{n-2} * T$ (see [BH, Definition I.5.6]). Since $\mathbb{S}^{n-2} * T$ is $\text{CAT}(1)$, the κ -cone X is a purely n -dimensional, proper, geodesically complete $\text{CAT}(\kappa)$ space, and not a topological n -manifold at the vertex o of the cone X . For every $r \in (0, D_\kappa)$, we have $\mathcal{H}^n(B_r(o))/\omega_\kappa^n(r) = 3/2$.

4. A CLASSIFICATION OF $\text{CAT}(1)$ SPACES OF SMALL VOLUME

4.1. Spherical convex subsets. We begin with the following:

Proposition 4.1. *Let X be a compact, geodesically complete $\text{CAT}(1)$ space of $\dim X = n$ with decomposition $X = \bigcup_{i=1}^3 \Sigma_i$ for some subsets $\Sigma_1, \Sigma_2, \Sigma_3$ of X satisfying the following:*

- (1) Σ_i is a closed π -convex subset in X that is isometric to \mathbb{S}^n for all $i \in \{1, 2, 3\}$;
- (2) $\Sigma_i \subset \Sigma_j \cup \Sigma_k$ for all $i, j, k \in \{1, 2, 3\}$.

*Then X is isometric to either $\mathbb{S}^{n-1} * T$ or \mathbb{S}^n .*

Proof. If for some distinct $i, j \in \{1, 2, 3\}$ we have $\Sigma_i \subset \Sigma_j$, then by (1) we have $\Sigma_i = \Sigma_j$, and hence by (2) we see that X is isometric to \mathbb{S}^n .

Assume now that for all distinct $i, j \in \{1, 2, 3\}$ we have $\Sigma_i \not\subset \Sigma_j$. Put $\Sigma_{ij} := \Sigma_i \cap \Sigma_j$. Then Σ_{ij} is isometric to a non-empty, proper,

closed π -convex subset of \mathbb{S}^n , and hence contained in a closed unit n -hemisphere. Therefore we have $\mathcal{H}^n(\Sigma_{ij}) \leq (1/2)\mathcal{H}^n(\mathbb{S}^n)$. From (2) we derive $X = \Sigma_{12} \cup \Sigma_{23} \cup \Sigma_{31}$. Hence we obtain

$$\mathcal{H}^n(X) \leq \mathcal{H}^n(\Sigma_{12}) + \mathcal{H}^n(\Sigma_{23}) + \mathcal{H}^n(\Sigma_{31}) \leq \frac{3}{2}\mathcal{H}^n(\mathbb{S}^n).$$

From the present assumption it follows that X is not a topological n -manifold; indeed, for all points in $\Sigma_i - \Sigma_j$, the closest points on Σ_{ij} possess no neighborhoods homeomorphic to \mathbb{R}^n . Since X is not homeomorphic to \mathbb{S}^n , the volume sphere theorem [LN2, Theorem 8.3] implies $\mathcal{H}^n(X) \geq (3/2)\mathcal{H}^n(\mathbb{S}^n)$. Thus for all distinct i, j we have $\mathcal{H}^n(\Sigma_{ij}) = (1/2)\mathcal{H}^n(\mathbb{S}^n)$; in particular, Σ_{ij} is isometric to a closed unit n -hemisphere. Since by (2) we have $X = \Sigma_i \cup \Sigma_j$ for all distinct i, j , we conclude that X is isometric to $\mathbb{S}^{n-1} * T$. \square

Remark 4.1. An anonymous referee tells us the proof of Proposition 4.1 discussed above. In a previous manuscript of this paper, when we proved Proposition 4.1, we used the lune lemma of Ballmann and Brin [BB, Lemma 2.5], the metric characterizations of spherical buildings of Balser and Lytchak [BL, Theorems 1.1 and 1.4], and the spherical join decomposition theorem of Lytchak [L, Corollary 1.2].

4.2. Proof of Theorem 1.1. If X is a purely 1-dimensional, compact, geodesically complete CAT(1) space with $\mathcal{H}^1(X) = (3/2)\mathcal{H}^1(\mathbb{S}^1)$, and if X has a tripod, then X is isometric to either the 1-triplex or $\mathbb{S}^0 * T$. For the 2-dimensional case, we know the following ([N2, Theorem B]):

Proposition 4.2. ([N2]) *Let X be a purely 2-dimensional, compact, geodesically complete CAT(1) space with $\mathcal{H}^2(X) = (3/2)\mathcal{H}^2(\mathbb{S}^2)$. If X has a tripod, then X is isometric to either the 2-triplex or $\mathbb{S}^1 * T$.*

To finish the proof of Theorem 1.1, we show:

Proposition 4.3. *Let X be a purely n -dimensional, compact, geodesically complete CAT(1) space satisfying (1.1). If X has a tripod, then X is isometric to either the n -triplex or $\mathbb{S}^{n-1} * T$.*

Proof. By Proposition 4.2, we may assume $n \geq 3$. Let $p_1, p_2, p_3 \in X$ be elements of a tripod. By Proposition 3.1 and (1.1), we see that X has the decomposition $X = \bigcup_{i=1}^3 B_{\pi/2}(p_i)$ such that $B_{\pi/2}(p_i)$ is isometric to a closed unit n -hemisphere for each $i \in \{1, 2, 3\}$; in particular, $\partial B_{\pi/2}(p_i)$ is isometric to \mathbb{S}^{n-1} . Put $\Sigma_i := \partial B_{\pi/2}(p_i)$.

Let $Y := \bigcup_{i=1}^3 \Sigma_i$. Note that Σ_i is a closed π -convex subset in X for each $i \in \{1, 2, 3\}$. The geodesical completeness of X implies that Σ_i is contained in $\Sigma_j \cup \Sigma_k$ for all $i, j, k \in \{1, 2, 3\}$.

We show that $\Sigma_i \cup \Sigma_j$ is π -convex in X for all distinct $i, j \in \{1, 2, 3\}$. For distinct $i, j \in \{1, 2, 3\}$, take y_1, y_2 in $\Sigma_i \cup \Sigma_j$ with $d(y_1, y_2) < \pi$, and let $y_1 y_2$ be the geodesic joining them. We may assume that $y_1 \in \Sigma_i - \Sigma_j$ and $y_2 \in \Sigma_j - \Sigma_i$. By the geodesical completeness of X , the points y_1

and y_2 must belong to Σ_k for $k \in \{1, 2, 3\}$ distinct to i, j . The π -convexity of Σ_k in X implies that $y_1 y_2$ is contained in Σ_k , and hence $y_1 y_2$ is contained in $\Sigma_i \cap \Sigma_j$; indeed, the set Σ_k is contained in $\Sigma_i \cup \Sigma_j$. Hence $\Sigma_i \cup \Sigma_j$ is π -convex.

Since $\Sigma_i \cup \Sigma_j$ is π -convex in X for all distinct $i, j \in \{1, 2, 3\}$, so is the whole Y . Note that every closed π -convex subspace of a CAT(1) space is also CAT(1). Hence the subspace Y is a CAT(1) space of $\dim Y = n - 1$. The present assumption $n \geq 3$ implies $\dim Y \geq 2$. Observe that Y is geodesically complete, and Σ_i is also closed and π -convex in Y for all $i \in \{1, 2, 3\}$. By Proposition 4.1, we see that Y is isometric to either $\mathbb{S}^{n-2} * T$ or \mathbb{S}^{n-1} . If Y is $\mathbb{S}^{n-2} * T$, then X is isometric to the n -triplex. If Y is \mathbb{S}^{n-1} , then X is isometric to $\mathbb{S}^{n-1} * T$. \square

Proposition 4.3 and the capacity sphere theorem [LN2, Theorem 1.5] complete the proof of Theorem 1.1. \square

5. A SPHERE THEOREM FOR CAT(1) HOMOLOGY MANIFOLDS

5.1. Homology manifolds. Let H_* denote the singular homology with \mathbb{Z} -coefficients. A locally compact, separable metric space M is said to be a *homology n -manifold* if for every $p \in M$ the local homology $H_*(M, M - \{p\})$ at p is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\})$, where 0 is the origin of \mathbb{R}^n . A homology n -manifold M is a *generalized n -manifold* if M is an ANR of $\dim M < \infty$. Every generalized n -manifold has dimension n . Due to the theorem of Moore (see [Wi, Chapter IV]), for each $n \in \{1, 2\}$, every generalized n -manifold is a topological n -manifold.

Every homology n -manifold with an upper curvature bound is a geodesically complete generalized n -manifold. Thurston [T, Theorem 3.3] proved that every homology 3-manifold with an upper curvature bound is a topological 3-manifold. We refer the readers to [LN2] for advanced studies of homology manifolds with an upper curvature bound.

We recall the following ([LN2, Lemma 3.1 and Corollary 3.4]):

Proposition 5.1. ([LN2]) *Let X be a metric space with an upper curvature bound. A locally compact open subset M of X is a homology n -manifold if and only if for every $p \in M$ the space $\Sigma_p X$ has the same homology as \mathbb{S}^{n-1} ; in this case, $\Sigma_p X$ is a homology $(n - 1)$ -manifold and $T_p X$ is a homology n -manifold.*

Remark 5.1. Lytchak and the author [LN2, Theorems 1.2 and 6.5] have proved that for every homology n -manifold M with an upper curvature bound there exists a locally finite subset E of M such that $M - E$ is a topological n -manifold; moreover, every point in M has a neighborhood homeomorphic to some cone over a closed topological $(n - 1)$ -manifold.

5.2. Locally geometrical contractivity. Following the terminology in [GPW] and [P], we say that a function $\rho: [0, r) \rightarrow [0, \infty)$ with $\rho(0) = 0$ is a *contractivity function* if ρ is continuous at 0, and if

$\rho \geq \text{id}_{[0,r]}$, where $\text{id}_{[0,r]}$ is the identity function on $[0, r)$. For a contractivity function $\rho: [0, r) \rightarrow [0, \infty)$, a metric space X is $\text{LGC}(\rho)$, *locally geometrically contractible with respect to ρ* , if for every $p \in X$ and for every $s \in (0, r)$ the ball $B_s(p)$ is contractible inside the concentric ball $B_{\rho(s)}(p)$. Every $\text{CAT}(\kappa)$ space is $\text{LGC}(\text{id}_{[0,D_\kappa)})$.

We recall the following, which is just a combination of the theorem of Grove, Petersen, and Wu [GPW, Theorem 2.1] and the theorems of Petersen [P, Theorem A, and Theorem in Section 5].

Theorem 5.2. ([GPW], [P]) *Let $\rho: [0, r) \rightarrow [0, \infty)$ be a contractivity function. If a sequence of compact $\text{LGC}(\rho)$ spaces X_i , $i = 1, 2, \dots$, of dimension $\leq n$ converges to some compact metric space X of finite dimension in the Gromov-Hausdorff topology, then*

- (1) X is an $\text{LGC}(\rho)$ space of $\dim X \leq n$;
- (2) X is homotopy equivalent to X_i for all sufficiently large i ;
- (3) if in addition each X_i is a topological n -manifold, then X is a generalized n -manifold.

For sequences of compact $\text{CAT}(\kappa)$ homology n -manifolds, we have the following ([LN2, Lemma 3.3]):

Lemma 5.3. ([LN2]) *If a sequence of compact $\text{CAT}(\kappa)$ homology n -manifolds X_i , $i = 1, 2, \dots$, converges to some compact metric space X in the Gromov-Hausdorff topology, then X is a homology n -manifold*

Remark 5.2. Lytchak and the author [LN2, Theorems 1.3 and 7.5] have proved that if a sequence of compact $\text{CAT}(\kappa)$ Riemannian n -manifolds X_i , $i = 1, 2, \dots$, converges to some compact metric space X in the Gromov-Hausdorff topology, then X is a topological n -manifold in which all iterated spaces of directions are homeomorphic to spheres; moreover, X is homeomorphic to X_i for all sufficiently large i .

5.3. The Lusternik-Schnirelmann category. Let X be a topological space X . The *Lusternik-Schnirelmann category* of X , denoted by $\text{cat } X$, is defined as the least non-negative integer k such that there exists an open covering of X consisting of $k + 1$ contractible subsets in X (possibly ∞ if such a finite covering does not exist). By definition, it follows that $\text{cat } X = 0$ if and only if X is contractible. Notice that $\text{cat } X$ depends only on the homotopy type of X .

The following seems to be well-known for experts.

Proposition 5.4. *If a connected, compact topological n -manifold M satisfies $\text{cat } M = 1$, then M is homotopy equivalent to \mathbb{S}^n ; in particular, M is homeomorphic to \mathbb{S}^n .*

The second half of Proposition 5.4 is derived from the first half and the resolutions of the Poincaré conjecture due to Perelman, and the generalized Poincaré conjecture due to Freedman and Smale. The first half is well-known in algebraic topology (see [LN2, Lemma 8.2]).

5.4. A sphere theorem for topological manifolds. Before showing Theorem 1.2, we prove a weaker one:

Proposition 5.5. *For every positive integer n , there exists a positive number $\delta \in (0, \infty)$ such that if a compact CAT(1) topological n -manifold X satisfies (1.2) for δ , then X is homeomorphic to \mathbb{S}^n .*

Proof. Suppose the contrary. By virtue of the volume sphere theorem [LN2, Theorem 8.3], we may suppose that there exists a sequence of compact CAT(1) topological n -manifolds X_i , $i = 1, 2, \dots$, with $\lim_{i \rightarrow \infty} \mathcal{H}^n(X_i) = (3/2) \mathcal{H}^n(\mathbb{S}^n)$ such that each X_i is not homeomorphic to \mathbb{S}^n . Due to the capacity sphere theorem [LN2, Theorem 1.5], we may assume that each X_i has a tripod. By Proposition 3.4, the sequence X_i , $i = 1, 2, \dots$, has a convergent subsequence X_j , $j = 1, 2, \dots$, tending to some compact metric space X in the Gromov-Hausdorff topology. By Lemma 2.3, the limit X is a purely n -dimensional, compact, geodesically complete CAT(1) space. Since each X_j has a tripod, so does X . From Theorem 3.3 we derive $\mathcal{H}^n(X) = (3/2) \mathcal{H}^n(\mathbb{S}^n)$.

Theorem 1.1 implies that X is isometric to either the n -triplex or $\mathbb{S}^{n-1} * T$. Theorem 5.2 tells us that X is a homology n -manifold, and hence it must be isometric to the n -triplex. Recall that the n -triplex X consists of $\mathbb{S}^{n-2} * T$ and the three copies of unit n -hemispheres (see Example 1.1). Hence we find a pair of two points p_1, p_2 in the π -convex subset \mathbb{S}^{n-2} of X with $d(p_1, p_2) = \pi$ such that $X = B_{\pi/2}(p_1) \cup B_{\pi/2}(p_2)$. Take $p_{j,k} \in X_j$, $k = 1, 2$, converging to $p_k \in X$ as $j \rightarrow \infty$. Then $X_j = U_{3\pi/4}(p_{j,1}) \cup U_{3\pi/4}(p_{j,2})$, provided j is large enough; in particular, $\text{cat } X_j = 1$. From Proposition 5.4 it follows that X_j is homeomorphic to \mathbb{S}^n . Thus we obtain a contradiction. This completes the proof. \square

5.5. A homotopy sphere theorem for homology manifolds. Similarly to Proposition 5.5, we obtain:

Proposition 5.6. *For every positive integer n , there exists a positive number $\delta \in (0, \infty)$ such that if a compact CAT(1) homology n -manifold X satisfies (1.2) for δ , then X is homotopy equivalent to \mathbb{S}^n .*

Proof. Suppose the contrary. Similarly to the proof of Proposition 5.5, we may suppose that there exists a sequence of compact CAT(1) homology n -manifolds X_i , $i = 1, 2, \dots$, with $\lim_{i \rightarrow \infty} \mathcal{H}^n(X_i) = (3/2) \mathcal{H}^n(\mathbb{S}^n)$ such that each X_i is not homotopy equivalent to \mathbb{S}^n . By Proposition 3.4, the sequence X_i , $i = 1, 2, \dots$, has a convergent subsequence X_j , $j = 1, 2, \dots$, tending to some X in the Gromov-Hausdorff topology. By Lemma 2.3, the limit X is a purely n -dimensional, compact, geodesically complete CAT(1) space. From Theorem 3.3 we derive $\mathcal{H}^n(X) = (3/2) \mathcal{H}^n(\mathbb{S}^n)$. Due to Theorem 1.1, we see that X is either homeomorphic to \mathbb{S}^n or isometric to $\mathbb{S}^{n-1} * T$. By Lemma 5.3, the limit X must be a homology n -manifold, and hence homeomorphic to

\mathbb{S}^n . It follows from Theorem 5.2 that X_j is homotopy equivalent to an n -sphere X for all sufficiently large j . This is a contradiction. \square

Remark 5.3. Combining Proposition 5.6 and the resolutions of the Poincaré conjecture also lead to Proposition 5.5.

5.6. Proof of Theorem 1.3. Let $\delta \in (0, \infty)$ be sufficiently small. Let X be a CAT(κ) homology n -manifold, and let W be an open subset of X . Assume that for every $x \in W$ there exists $r \in (0, D_\kappa)$ satisfying (1.3). By Lemma 3.6, for every $x \in W$ the condition (1.3) leads to

$$\frac{\mathcal{H}^{n-1}(\Sigma_x X)}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} < \frac{3}{2} + \delta.$$

From Proposition 5.1 it follows that $\Sigma_x X$ is a compact CAT(1) homology $(n-1)$ -manifold. By Proposition 5.6, the space $\Sigma_x X$ is homotopy equivalent to \mathbb{S}^{n-1} , provided δ is small enough. Due to the local topological regularity theorem [LN2, Theorem 1.1], we conclude that W is a topological n -manifold. Thus we obtain Theorem 1.3. \square

5.7. Proof of Theorem 1.2. Let $\delta \in (0, \infty)$ be sufficiently small. Let X be a compact CAT(1) homology n -manifold satisfying (1.2). From Proposition 3.2 and (1.2), for every $x \in X$ we derive

$$\frac{\mathcal{H}^n(B_r(x))}{\omega_1^n(r)} \leq \frac{\mathcal{H}^n(X)}{\mathcal{H}^n(\mathbb{S}^n)} < \frac{3}{2} + \frac{\delta}{\mathcal{H}^n(\mathbb{S}^n)}$$

for any $r \in (0, \pi)$. From Theorem 1.3 we deduce that X is a topological n -manifold, provided δ is small enough. This together with Proposition 5.5 completes the proof of Theorem 1.2. \square

APPENDIX A. THREE-MANIFOLD RECOGNITION REVISITED

One of the key points in the proof of Theorem 1.2 is to prove Proposition 5.5. In the proof of Proposition 5.5 discussed above, we rely on the resolutions of the (generalized) Poincaré conjecture when we use Proposition 5.4. As explained below, we can prove Proposition 5.5 in the 3-dimensional case without relying on the Poincaré conjecture.

A locally compact, separable metric space M is said to be a *homology n -manifold with boundary* if for every $p \in M$ there exists $x \in D^n$ such that $H_*(M, M - \{p\})$ coincides with $H_*(D^n, D^n - \{x\})$, where D^n is the Euclidean closed unit n -disk centered at the origin in \mathbb{R}^n ; the *boundary* ∂M of M is defined as the set of all points $p \in M$ at which the local homologies $H_*(M, M - \{p\})$ are trivial. A homology n -manifold M with boundary is a *generalized n -manifold with boundary* if M is an ANR of $\dim M < \infty$. If M is a generalized n -manifold with boundary, then $\dim M = n$, and ∂M is closed and nowhere dense in M (see e.g., [M, Lemma 2]). From the theorem of Mitchell [M, Theorem], it follows that if M is a generalized n -manifold with boundary, then ∂M is either empty or a generalized $(n-1)$ -manifold without boundary.

Thurston showed in [T, Proposition 2.7] that if X is a $\text{CAT}(\kappa)$ homology n -manifold, then for every $r \in (0, D_\kappa/2)$, and for every $p \in X$, the compact contractible metric ball $B_r(x)$ is a generalized n -manifold with boundary $\partial B_r(p)$; in particular, by the theorem of Mitchel [M, Theorem], and the Poincaré duality for homology manifolds (see e.g., [Br]), the metric sphere $\partial B_r(p)$ is a generalized $(n-1)$ -manifold with the same homology as \mathbb{S}^{n-1} (see also [LN2, Lemma 3.2]). This property holds true for any $r \in (0, D_\kappa)$ beyond $D_\kappa/2$.

Lemma A.1. (cf. [T]) *If X is a $\text{CAT}(\kappa)$ homology n -manifold, then for every $r \in (0, D_\kappa)$, and for every $p \in X$, the ball $B_r(p)$ is a generalized n -manifold with boundary $\partial B_r(p)$; in particular, $\partial B_r(p)$ is a generalized $(n-1)$ -manifold with the same homology as \mathbb{S}^{n-1} .*

Proof. For every $x \in \partial B_r(p)$, the set $B_r(p) - \{x\}$ is contractible to p inside itself along the geodesics from p , and hence the reduced homology $\tilde{H}_*(B_r(p), B_r(p) - \{x\})$ is trivial since we have the exact sequence

$$\begin{aligned} \tilde{H}_k(B_r(p)) &\rightarrow \tilde{H}_k(B_r(p), B_r(p) - \{x\}) \\ &\rightarrow \tilde{H}_{k-1}(B_r(p) - \{x\}) \rightarrow \tilde{H}_{k-1}(B_r(p)) \end{aligned}$$

for all $k \in \mathbb{N}$. Hence $\partial B_r(p)$ is the boundary of $B_r(p)$ as generalized manifolds. The theorem of Mitchel [M, Theorem] together with the Poincaré duality leads to the second half of the lemma. \square

From now on, we focus on the 3-dimensional case. Thurston proved in [T, Theorem 3.3] that if p is a point in a $\text{CAT}(\kappa)$ homology 3-manifold, then $U_r(p)$ is homeomorphic to \mathbb{R}^3 for any $r \in (0, D_\kappa/2)$ whose upper bound $D_\kappa/2$ guarantees the strong convexity of $U_r(p)$. By the same arguments as in [T], we can prove the following:

Theorem A.2. (cf. [T]) *Let X be a $\text{CAT}(\kappa)$ homology 3-manifold. Then for every $p \in X$, and for every $r \in (0, D_\kappa)$, the ball $U_r(p)$ is homeomorphic to \mathbb{R}^3 .*

Reviewing the arguments discussed in [T], we sketch the proof.

Let Y and Z be topological spaces. A map $f: Y \rightarrow Z$ is said to be *approximable by homeomorphisms*, abbreviated as ABH, if for every open covering \mathcal{U} of Z there exists a homeomorphism $h: Y \rightarrow Z$ such that for each $y \in Y$ we find $U \in \mathcal{U}$ with $f(y) \in U$ and $h(y) \in U$. By the Daverman-Preston sliced shrinking theorem [DP, Theorem], we already know that if $f: Y \times \mathbb{R} \rightarrow Z$ is a proper, surjective continuous map such that each fiber $f^{-1}(z)$ is contained in some slice $Y \times \{t\}$, and if each of the level maps of f is ABH, then f is ABH too.

Let \tilde{H}^* denote the reduced Čech cohomology with \mathbb{Z} -coefficients. A proper surjective map $c: Y \rightarrow Z$ between locally compact Hausdorff spaces is said to be *acyclic* if $\tilde{H}^*(c^{-1}(z))$ is trivial for all $z \in Z$.

Proof of Theorem A.2. Let X be a CAT(κ) homology 3-manifold. Let $p \in X$ be arbitrary. From Lemma A.1 we see that for each $t \in (0, D_\kappa)$ the metric sphere $\partial B_t(p)$ is homeomorphic to \mathbb{S}^2 since every generalized 2-manifold is a topological 2-manifold.

Take arbitrary $s, r \in (0, D_\kappa)$ with $s < r$. Let $c_{r,s}: \partial B_r(p) \rightarrow \partial B_s(p)$ be the continuous surjective map defined as $c_{r,s}(x) := \gamma_{px}(s)$, where $\gamma_{px}: [0, d(p, x)] \rightarrow X$ is the geodesic from p to x . Choose a point $z \in \partial B_s(p)$. Let $\Gamma_z c_{r,s}^{-1}(z)$ be the geodesic cone in X defined as

$$\Gamma_z c_{r,s}^{-1}(z) := \bigcup \{ zy \mid y \in c_{r,s}^{-1}(z) \}.$$

By definition, any point in $\Gamma_z c_{r,s}^{-1}(z)$ lies on a geodesic joining p and some point in $\partial B_r(p)$. Hence $\Gamma_z c_{r,s}^{-1}(z)$ is contained in $B_r(p)$.

Observe that $B_r(p) - \Gamma_z c_{r,s}^{-1}(z)$ is contractible to p inside itself along the geodesics from p . Following the same way as in [T, Corollary 2.10], by Lemma A.1 and the Alexander-Lefschetz duality for homology manifolds (see e.g., [T, Proposition 2.8]), we see that $\tilde{H}^*(c_{r,s}^{-1}(z))$ is trivial. Since z is arbitrary in $\partial B_s(p)$, the map $c_{r,s}$ is acyclic; in particular, each of the fibers of $c_{r,s}$ fails to separate the 2-sphere $\partial B_r(p)$. This implies that the map $c_{r,s}$ is ABH for any $s, r \in (0, D_\kappa)$ with $s < r$.

For a fixed $r \in (0, D_\kappa)$, we consider the proper, continuous surjective map $f_r: \partial B_r(p) \times (0, r) \rightarrow U_r(p) - \{p\}$ defined by $f_r(y, s) := c_{r,s}(y)$. Each fiber $f_r^{-1}(z)$ is contained in some slice $\partial B_r(p) \times \{s\}$. Due to the Daverman-Preston sliced shrinking theorem [DP, Theorem], we obtain a homeomorphism between $\partial B_r(p) \times (0, r)$ and $U_r(p) - \{p\}$, and hence between \mathbb{R}^3 and $U_r(p)$. Thus we conclude Theorem A.2. \square

We give another proof of Proposition 5.5 in the 3-dimensional case without using the resolution of the Poincaré conjecture.

Proof of Proposition 5.5 in the 3-dimensional case. Suppose now that there exists a sequence of compact CAT(1) topological 3-manifolds X_i , $i = 1, 2, \dots$, with $\lim_{i \rightarrow \infty} \mathcal{H}^3(X_i) = (3/2) \mathcal{H}^3(\mathbb{S}^3)$ such that each X_i is not homeomorphic to \mathbb{S}^3 . Similarly to the proof in Subsection 5.4, we see that the sequence X_i , $i = 1, 2, \dots$, has a convergent subsequence X_j , $j = 1, 2, \dots$, tending to the 3-triplex X in the Gromov-Hausdorff topology; moreover, we find a pair of two points $p_1, p_2 \in X$ with $d(p_1, p_2) = \pi$ such that $X = B_{\pi/2}(p_1) \cup B_{\pi/2}(p_2)$. Take $p_{j,k} \in X_j$, $k = 1, 2$, converging to $p_k \in X$ as $j \rightarrow \infty$. Then $X_j = U_{3\pi/4}(p_{j,1}) \cup U_{3\pi/4}(p_{j,2})$, provided j is large enough. From Theorem A.2 it follows that the balls $U_{3\pi/4}(p_{j,1})$ and $U_{3\pi/4}(p_{j,2})$ are homeomorphic to \mathbb{R}^3 . The generalized Schoenflies theorem (see e.g., [R, Theorem 1.8.2]) implies that X_j is homeomorphic to \mathbb{S}^3 , and leads to a contradiction. This completes the proof. \square

REFERENCES

- [AKP] S. Alexander, V. Kapovitch, and A. Petrunin, *Alexandrov geometry*: preliminary version no. 1, preprint, 2019: arXiv:1903.08539v1.
- [ABN] A. D. Alexandrov, V. N. Berestovskii, and I. G. Nikolaev, *Generalized Riemannian spaces* [Russian], Uspekhi Mat. Nauk **41** (1986), no. 3 (249), 3–44, 240; translation in Russian Math. Surveys **41** (1986), no. 3, 1–54.
- [B] W. Ballmann, *Lectures on Spaces of Nonpositive Curvature*, DMV Seminar, Band 25, Birkhäuser, 1995.
- [BB] W. Ballmann and M. Brin, *Diameter rigidity of spherical polyhedra*, Duke Math. J. **97** (1999), no. 2, 235–259.
- [BL] A. Balsler and A. Lytchak, *Building-like spaces*, J. Math. Kyoto Univ. **46** (2006), no. 4, 789–804.
- [Be] M. Berger, *Lectures on Geodesics in Riemannian Geometry*, Tata Institute of Fundamental Research, Volume 33, Bombay, 1965.
- [Bes] A. L. Besse, *Manifolds all of whose Geodesics are Closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume 93, Springer-Verlag, 1978.
- [Br] G. E. Bredon, *Sheaf Theory: 2nd edition*, Graduate Texts in Mathematics, Volume 170, Springer-Verlag, 1997.
- [BH] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften, Volume 319, Springer-Verlag, 1999.
- [BBI] D. Burago, Yu. D. Burago, and S. Ivanov, *A Course in Metric Geometry*, Graduate Studies in Mathematics, Volume 33, Amer. Math. Soc., 2001.
- [BS] S. V. Buyalo and V. Schroeder, *Spaces of curvature bounded above*, Surveys in Differential Geometry, Volume XI, Metric and Comparison Geometry (J. Cheeger and K. Grove, eds.), International Press, 2007, pp. 295–327.
- [CI] L. Coghlan and Y. Itokawa, *A sphere theorem for reverse volume pinching on even-dimensional manifolds*, Proc. Amer. Math. Soc. **111** (1991), no. 3, 815–819.
- [DP] R. J. Daverman and D. K. Preston, *Shrinking certain sliced decompositions of E^{n+1}* , Proc. Amer. Math. Soc. **79** (1980), no. 3, 477–483.
- [GPW] K. Grove, P. Petersen, and J. Y. Wu, *Gromov-Hausdorff convergence via controlled topology*, Invent. Math. **99** (1990), no. 1, 205–213; *Erratum*: Invent. Math. **104** (1991), no. 1, 221–222.
- [K] B. Kleiner, *The local structure of length spaces with curvature bounded above*, Math. Z. **231** (1999), no. 3, 409–456.
- [Kr] L. Kramer, *On the local structure and the homology of $\text{CAT}(\kappa)$ spaces and euclidean buildings*, Adv. in Geom. **11** (2011), no. 2, 347–369.
- [L] A. Lytchak, *Rigidity of spherical buildings and spherical joins*, Geom. Funct. Anal. **15** (2005), no. 3, 720–752.
- [LN1] A. Lytchak and K. Nagano, *Geodesically complete spaces with an upper curvature bound*, Geom. Funct. Anal. **29** (2019), no. 1, 295–342.
- [LN2] A. Lytchak and K. Nagano, *Topological regularity of spaces with an upper curvature bound*, preprint, 2018: arXiv:1809.06183; to appear in J. Eur. Math. Soc. (JEMS).
- [M] W. J. R. Mitchell, *Defining the boundary of a homology manifolds*, Proc. Amer. Math. Soc. **110** (1990), no. 2, 509–513.
- [N1] K. Nagano, *A volume convergence theorem for Alexandrov spaces with curvature bounded above*, Math. Z. **241** (2002), no. 1, 127–163.
- [N2] K. Nagano, *A sphere theorem for 2-dimensional $\text{CAT}(1)$ -spaces*, Pacific J. Math. **206** (2002), no. 2, 401–423.

- [O] P. Ontaneda, *Cocompact CAT(0) spaces are almost geodesically complete*, *Topology* **44** (2005), no. 1, 47–62.
- [P] P. Petersen, *A finiteness theorem for metric spaces*, *J. Differential Geom.* **31** (1990), no. 2, 387–395.
- [R] T. B. Rushing, *Topological Embeddings*, Pure and Applied Mathematics, Volume 52, Academic Press, 1973.
- [T] P. Thurston, *CAT(0) 4-manifolds possessing a single tame point are Euclidean*, *J. Geom. Anal.* **6** (1996), no. 3, 475–494.
- [W] A. Weinstein, *On the volume of manifolds all of whose geodesics are closed*, *J. Differential Geom.* **9** (1974), no. 4, 513–517.
- [We1] Y. Wen, *A note on pinching sphere theorem*, *C. R. Math. Acad. Sci. Paris* **338** (2004), no. 3, 229–234.
- [We2] Y. Wen, *A topological sphere theorem for arbitrary-dimensional manifolds*, *Pacific J. Math.* **244** (2010), no. 1, 193–200.
- [Wi] R. Wilder, *Topology of Manifolds*, American Mathematical Society Colloquium Publications, Volume 32, Amer. Math. Soc., 1979.
- [Wu] J. Y. Wu, *Hausdorff convergence and sphere theorems*, *Differential Geometry: Riemannian Geometry*, Proceedings of Symposia in Pure Mathematics, Volume 54, Part 3, Amer. Math. Soc., 1993, pp. 685–692.

(Koichi Nagano)

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA

TENNODAI 1-1-1, TSUKUBA, IBARAKI, 305-8571, JAPAN

Email address: nagano@math.tsukuba.ac.jp