EXISTENCE OF MINIMAL HYPERSURFACES WITH NON-EMPTY FREE BOUNDARY FOR GENERIC METRICS

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Abstract. For almost all Riemannian metrics (in the C^∞ Baire sense) on a compact manifold with boundary $(M^{n+1}, \partial M)$, $3 \le (n+1) \le 7$, we prove that, for any open subset V of ∂M, there exists a compact, properly embedded free boundary minimal hypersurface intersecting V.

1. Introduction

In 1960s, Almgren [1,2] initiated a variational theory to find minimal submanifolds in any compact manifolds with boundary. For a closed manifold M^{n+1}, the regularity of such hypersurfaces was improved by Pitts [20] for $n \le 5$, and Schoen-Simon [21] for $n = 6$. Very recently, Li and Zhou finished this program for a general compact manifold with nonempty boundary in [13], in which they proved that every compact manifold with boundary admits a smooth compact minimal hypersurface with (possibly empty) free boundary. This result left widely open the following well-known question:

Question 1.1. *Does every compact manifold with non-empty boundary admit a minimal hypersurface with non-empty free boundary?*

We point out that there are similar questions in any free boundary variational theory. In particular, in the mapping approach by Fraser [6], Lin-Sun-Zhou [14], and Lauren-Petradis [12], it was not known whether their free boundary minimal surfaces have nontrivial boundary.

In this paper, we solve this problem in generic scenarios and prove a much stronger property: M admits infinitely many embedded minimal hypersurfaces with non-empty free boundary.

Theorem 1.2. *Let $(M^{n+1}, \partial M)$ be a compact manifold of dimension $3 \le (n+1) \le 7$. Then for a C^∞-generic Riemannian metric g on $(M, \partial M)$, the union of boundaries of all smooth, embedded, free boundary minimal hypersurfaces is dense in ∂M.*

We remark that a compact manifold with non-negative Ricci curvature and strictly convex boundary has no closed minimal hypersurface by [5, Lemma 2.2]. Therefore, by Marques-Neves [17] and Li-Zhou [13], it is known that there exist infinitely many properly embedded free boundary minimal hypersurfaces in such ambient manifolds.

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For a generic metric on $(M, \partial M)$, the author together with Guang, Li and Zhou proved the denseness of free boundary minimal hypersurfaces in [8, Theorem 1.3]. Making use of a maximum principle by White [24], such denseness gives that M contains minimal hypersurfaces with non-empty boundary by merely assuming strict mean convexity at one point of the boundary ∂M for a generic metric; see [8]. However, without any topological or curvature assumptions, it is in general very difficult to prevent the free boundary components from degenerating in the limit process (see e.g. [3,9]). Our theorem in this paper greatly improves this result by dropping off mean convexity assumption at one point.

The denseness result in [8, Theorem 1.3] can be seen as a natural free boundary analog of [11]. The key ingredient of [11] by Irie, Marques and Neves is the Weyl law for the volume spectrum proved by Liokumovich, Marques and Neves in [15]. The volume spectrum of a compact Riemannian manifold with boundary $(M^{n+1}, \partial M, g)$ is a nondecreasing sequence of numbers $\{\omega_k(M; g) : k \in \mathbb{N}\}$ defined variationally by performing a min-max procedure for the area functional over multi-parameter sweepouts. The first estimates for these numbers were proven by Gromov [7] in the late 1980s (see also [10]). A direct corollary of Weyl Law they used is that, for k large enough,

$$\omega_k(M; g) \neq \omega_k(M; g')$$

whenever $\mathrm{Vol}(M, g) \neq \mathrm{Vol}(M, g')$.

Another observation by Irie, Marques and Neves is that such spectrum depends continuously on the metrics of M; see [11, Lemma 2.1] and [19, Lemma 1]. Applying this, they showed that continuous perturbations in an open set must create new minimal hypersurfaces intersecting that set.

In this paper, we also borrow the idea from Irie-Marques-Neves [11]. However, the original perturbation would only produce new free boundary minimal hypersurfaces intersecting *an open set*, but not an n-dimensional subset, that we need to consider here. To overcome this new challenge, we perturb the metric g around a boundary point in *a special way* so that a hypersurface whose boundary does not intersect the prescribed subset of ∂M can also be regarded as a hypersurface in $(M, \partial M, g)$. Recall that Weyl law in [15] gives that for large k, ω_k will change continuously if the volume of M is changed under the perturbation. From these two observations, we are able to prove that such a special perturbation would produce new minimal hypersurfaces with free boundary intersecting the prescribed subset of ∂M.

We finish the introduction with the idea of the construction of the special perturbation. Making use of the cut-off trick, the unit inward normal vector field of ∂M can be extended to the whole M. Also, by multiplying another cut-off function, we can always construct a vector field whose support is close to our prescribed open set of ∂M. Such a vector field would give a one-parameter family of diffeomorphisms (not surjective) of M. Then the pull back metric given by such family is the desired perturbation since it is isometric to a subset of M with the original metric. We refer to Proposition 3.1 for more details.

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2. Preliminaries

Let (M^{n+1}, g) be a smooth compact connected Riemannian manifold with nonempty boundary ∂M and $3 \leq (n+1) \leq 7$. Moreover, M can always be embedded to a closed Riemannian manifold $\widetilde{M}$ which has the same dimension with M. We can also assume that $\widetilde{M}$ is isometrically embedded in some $\mathbb{R}^L$ for L large enough.

2.1. **Geometric measure theory.**

We now recall some basic notations in geometric measure theory; see [13].

We use $\mathcal{V}_k(M)$ to denote the closure of the space of k-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support contained in M. Let $\mathcal{R}_k(M; \mathbb{Z}_2)$ (resp. $\mathcal{R}_k(\partial M; \mathbb{Z}_2)$) be the space of k-dimensional modulo two flat chains of finite mass in $\mathbb{R}^L$ which are supported in M (resp. in ∂M). Denote by $\operatorname{spt} T$ the support of $T \in \mathcal{R}_k(M; \mathbb{Z}_2)$. Given any $T \in \mathcal{R}_k(M; \mathbb{Z}_2)$, denote by $|T|$ and $\|T\|$ the integer rectifiable varifold and the Radon measure in M associated with T, respectively. The mass norm and the flat metric on $\mathcal{R}_k(M; \mathbb{Z}_2)$ are denoted by $\mathbf{M}$ and $\mathcal{F}$ respectively; see [4]. Set

$$Z_k(M, \partial M; \mathbb{Z}_2) = \{T \in \mathcal{R}_k(M; \mathbb{Z}_2) : \operatorname{spt}(\partial T) \subset \partial M\}.$$

We say that $T, S \in Z_k(M, \partial M; \mathbb{Z}_2)$ are equivalent if $T - S \in \mathcal{R}_k(\partial M; \mathbb{Z}_2)$. We use $\mathcal{Z}_k(M, \partial M; \mathbb{Z}_2)$ to denote the space of all such equivalence classes; see [8, Section 3] for the equivalence with the formulation using integer rectifiable currents in [13] .

The flat metric and the mass norm in the space of relative cycles are defined, respectively, as

$$\mathcal{F}(\tau) = \inf\{\mathcal{F}(T) : T \in \tau\}, \quad \mathbf{M}(\tau) = \inf\{\mathbf{M}(T) : T \in \tau\}.$$

The connected component of $\mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ containing 0 is weakly equivalent to $\mathbb{RP}^\infty$ by Almgren [1] (see also [15, §2.5] and [8, Section 3]). Denote by $\bar{\lambda}$ the generator of $H^1(\mathcal{Z}_n(M, \partial M; \mathbb{Z}_2); \mathbb{Z}_2) = \mathbb{Z}_2$.

2.2. **Auxiliary Lemmas.**

In this part, we introduce some Lemmas in [8, 11, 19].

Let X be a finite dimensional simplicial complex. A continuous map $\Phi : X \to \mathcal{Z}_n(M, \partial M; \mathbb{Z}_2)$ is called a *k-sweepout* if

$$\Phi^*(\bar{\lambda}^k) \neq 0 \in H^k(X; \mathbb{Z}_2).$$

Denote by $\mathcal{P}_k(M)$ the set of all k-sweepouts that have *no concentration of mass*, meaning that

$$\lim_{r \to 0} \sup\{\mathbf{M}(\Phi(x) \cap B_r(p)) : x \in X, p \in M\} = 0.$$

Definition 2.1. The *k-width of* $(M, \partial M, g)$ is defined as

$$\omega_k(M) := \inf_{\Phi \in \mathcal{P}_k(M)} \sup\{\mathbf{M}(\Phi(x)) : x \in \operatorname{dmn}(\Phi)\},$$

where $\operatorname{dmn}(\Phi)$ is the domain of Φ.

For any compact Riemannian manifold with boundary $(M, \partial M, g)$, the sequence $\{\omega_p(M)\}$ satisfies Weyl Law, which is proven by Liokumovich, Marques and Neves.

Theorem 2.2 (Weyl Law for the Volume Spectrum; [15])**.** *There exists a constant $\alpha(n)$ such that, for every compact Riemannian manifold with (possibly empty) boundary $(M^{n+1}, \partial M, g)$, we have*

$$\lim_{k\to\infty} \omega_k(M;g)k^{-\frac{1}{n+1}} = \alpha(n)\mathrm{Vol}(M,g)^{\frac{n}{n+1}}.$$

Irie-Marques-Neves [11, Lemma 2.1] proved that $\omega_k(M;g)$ depends continuously on metrics. The following is an improved version given by Marques, Neves and Song.

Lemma 2.3 ([11, Lemma 2.1; 19, Lemma 1])**.** *Let g_0 be a C^2 Riemannian metric on $(M, \partial M)$, and let $C_1 < C_2$ be positive constants. Then there exists $K = K(g_0, C_1, C_2) > 0$ such that*

$$\left|p^{-\frac{1}{n+1}}\omega_p(M;g) - p^{-\frac{1}{n+1}}\omega_p(M;g')\right| \leq K \cdot |g - g'|_{g_0}$$

for any C^2 metric $g, g' \in \{h : C_1 g_0 \leq h \leq C_2 g_0\}$ and any $p \in \mathbb{N}$.

Inspired by Marques-Neves [16], the author with Guang, Li, and Zhou (see [8, Theorem 2.1]) gave a general index estimate for min-max minimal hypersurfaces with free boundary. Combining with a compactness theorem in [9] by the author and Guang and Zhou, we also proved in [8] that the k-width is realized by the area (counting multiplicities) of *almost properly embedded free boundary minimal hypersurfaces* (see [13, Definition 2.6]).

Proposition 2.4 ([8, Proposition 7.3; 11, Proposition 2.2])**.** *Suppose $3 \leq (n+1) \leq 7$. Then for each $k \in \mathbb{N}$, there exist a finite disjoint collection $\{\Sigma_1, \cdots, \Sigma_N\}$ of almost properly embedded free boundary minimal hypersurfaces in $(M, \partial M, g)$, and integers $\{m_1, \cdots, m_N\} \subset \mathbb{N}$, such that*

$$\omega_k(M;g) = \sum_{j=1}^{N} m_j \mathrm{Area}_g(\Sigma_j) \quad \text{and} \quad \sum_{j=1}^{N} \mathrm{index}(\Sigma_j) \leq k.$$

Remark 2.5. In a recent exciting work [25], X. Zhou proved that, for a bumpy metric on a closed manifold, each m_j equals to 1, which is conjectured by Marques and Neves in [16]. Based on this Multiplicity One Theorem, Marques-Neves [18] proved that the index is in fact equals to k for min-max minimal hypersurfaces realizing ω_k.

3. Proof of Theorem 1.2

Let $(M^{n+1}, \partial M)$ be a compact manifold with boundary and $3 \leq (n+1) \leq 7$. Let $\mathcal{M}$ be the space of all smooth Riemannian metrics on M, endowed with the smooth topology. Suppose that $V \subset \partial M$ is a non-empty open set. Let $\mathcal{M}_V$ be the set of metrics $g \in \mathcal{M}$ such that there exists a non-degenerate, properly embedded free boundary minimal hypersurface Σ in $(M, \partial M, g)$ whose boundary intersects V.

We approach the theorem by proving the following proposition.

Proposition 3.1. *For any compact manifold $(M, \partial M)$ and any open subset $V \subset \partial M$, $\mathcal{M}_V$ is open and dense in $\mathcal{M}$ in the smooth topology.*

Proof. Let $g \in \mathcal{M}_V$ and Σ be like in the statement of the proposition. Following the step by Irie-Marques-Neves in [11], we first show the openness of $\mathcal{M}_V$. Since Σ is a properly embedded, then the Structure Theorem proven by White [23, Theorem 2.1] (see [3, Theorem 35] for a version on free boundary minimal hypersurfaces) also gives that for every Riemannian metric g' sufficiently close to g, there exists a unique nondegenerate properly embedded free boundary minimal hypersurface Σ' which is diffeomorphic and close to Σ. Moreover, the boundary of Σ' intersects V. This implies $\mathcal{M}_V$ is open.

It remains to show the set $\mathcal{M}_V$ is dense. Let g be an arbitrary smooth Riemannian metric on $(M, \partial M)$ and $\mathcal{V}$ be an arbitrary neighborhood of g in the C^∞ topology. By the Bumpy Metrics Theorem ([3, Theorem 9; 23, Theorem 2.1]), there exists $g' \in \mathcal{V}$ such that every compact, almost properly embedded free boundary minimal hypersurface with respect to g' is nondegenerate.

Since g' is bumpy, then by [8, Proposition 5.3] (see also [9, 22]), the space of almost embedded free boundary minimal hypersurfaces with Area $\leq \Lambda$ and index $\leq I$ is countable with respect to g' for all $\Lambda > 0$ and $I \geq 0$. Therefore, the set

$$\mathcal{C} := \left\{ \sum_{j=1}^{N} m_j \mathrm{Area}_{g'}(\Sigma_j) \;\middle|\; \begin{array}{l} N \in \mathbb{N}, \{m_j\} \subset \mathbb{N}, \{\Sigma_j\} \text{ disjoint collection of} \\ \text{almost properly embedded free boundary} \\ \text{minimal hypersurfaces in } (M, \partial M, g') \end{array} \right\}$$

is countable.

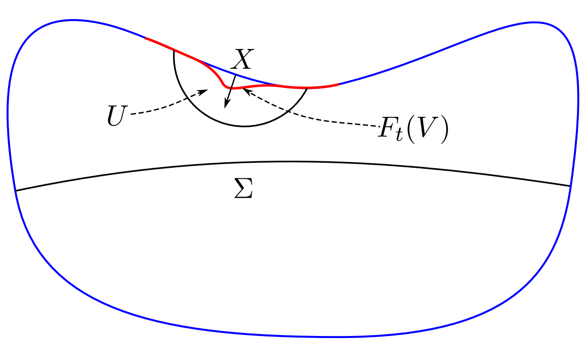

FIGURE I. Perturbing ∂M in V.

Let U be an open set of M such that $\overline{U} \cap \partial M \subset V$ is non-empty. Let X be a vector field on M so that $\operatorname{spt} X \subset U$ and for $x \in \partial M$ satisfying $X(x) \neq 0$, $X(x)/|X(x)|$ is the inward unit normal vector of ∂M. Denote by $(F_t)_{0 \leq t \leq 1}$ a family of maps generated by X. Clearly, F_t is a diffeomorphism from M to its image. Set

$$g_t = F_t^* g' \quad \text{and} \quad M_t = F_t(M).$$

Then F_t gives an isometry between $(M, \partial M, g_t)$ and $(M_t, \partial M_t, g')$ (see Figure I). Now we take $\delta > 0$ sufficiently small so that $g_t \in \mathcal{V}$ for all $t \in [0, \delta]$.

Claim 1. *Let Γ be an integral varifold in M_t whose support is a free boundary minimal hypersurface Σ (possibly disconnected) in $(M_t, \partial M_t, g')$. Assuming that $\partial\Sigma \cap F_t(V) = \emptyset$, then Σ is an almost properly embedded free boundary minimal hypersurface in $(M, \partial M, g')$. Particularly, $\mathbf{M}(\Gamma) \in \mathcal{C}$.*

Proof of Claim 1. Note that $(M_t, \partial M_t, g')$ is a domain of $(M, \partial M, g')$. Hence Σ is also embedded in $(M, \partial M, g')$. Then it suffices to show that $\partial\Sigma \subset \partial M$ and Σ meets ∂M orthogonally along $\partial\Sigma$.

Recall that X is supported on V. Therefore,

$$\partial M_t \setminus F_t(V) = \partial M \setminus V.$$

Together with the assumption of $\partial\Sigma \cap F_t(V) = \emptyset$, we have that $\partial\Sigma \subset \partial M \setminus V$. This also gives that Σ meets ∂M orthogonally along $\partial\Sigma$.

Overall, Σ is an almost properly embedded free boundary minimal hypersurface in $(M, \partial M, g')$. Thus we complete the proof of Claim 1. $\square$

Claim 2. *There exist $t_1 \in [0, \delta]$ and an almost properly embedded free boundary minimal hypersurface $(\Sigma_1, \partial\Sigma_1) \subset (M_{t_1}, \partial M_{t_1}, g')$ satisfying $\partial\Sigma_1 \cap F_{t_1}(V) \neq \emptyset$.*

Proof of Claim 2. Suppose not, then for all $t \in [0, \delta]$, all of the almost properly embedded minimal hypersurfaces in $(M_t, \partial M_t, g')$ have no boundary in $F_t(V)$. Recall that Proposition 2.4 gives that $\omega_k(M_t; g')$ is realized by the area of free boundary minimal hypersurfaces with multiplicities in $(M_t, \partial M_t, g')$. Together with Claim 1, we conclude that

$$\omega_k(M_t; g') \in \mathcal{C} \quad \text{for all } t \in [0, \delta] \text{ and } k \in \mathbb{N}.$$

Recall that $\mathcal{C}$ is a countable set. On the other hand, the Weyl law (see Theorem 2.2) implies that $\omega_k(M_\delta; g') < \omega_k(M; g')$ for k large enough. The Lemma 2.3 deduces that $\omega_k(M_t; g')$ is continuous, which leads to a contradiction that $\omega_k(M_t; g')$ lies in a countable set. The proof is finished. $\square$

Thus we have proved that for some $t_1 \in [0, \delta]$, there exists an almost properly embedded free boundary minimal hypersurface

$$(\Sigma_1, \partial\Sigma_1) \subset (M_{t_1}, \partial M_{t_1}, g')$$

satisfying $\partial\Sigma_1 \cap F_{t_1}(V) \neq \emptyset$. Denote by

$$(\Sigma', \partial\Sigma') := (F_{t_1}^{-1}(\Sigma_1), F_{t_1}^{-1}(\partial\Sigma_1)).$$

Recall that F_{t_1} is an isometry from $(M, \partial M, g_{t_1})$ to $(M_{t_1}, \partial M_{t_1}, g')$. Then $(\Sigma', \partial\Sigma') \subset (M, \partial M, g_{t_1})$ is an almost properly embedded free boundary minimal hypersurface whose boundary intersects V. By [11, Proposition 2.3] (see also [8, Proposition 7.6; 19, Lemma 4]), g_{t_1} can be perturbed to $g'' \in \mathcal{V}$ so that $(M, \partial M, g'')$ contains an almost properly embedded, non-degenerate, free boundary minimal hypersurfaces Σ'' whose boundary intersects V. Finally, [8, Proposition 7.7] would allow us to perturb g'' to $\widetilde{g} \in \mathcal{V}$ and Σ'' is a properly embedded free boundary minimal hypersurface in $(M, \partial M, \widetilde{g})$. This implies that $\widetilde{g} \in \mathcal{M}_V$ and we are done. $\square$

Now we are ready to prove Theorem 1.2. The proof is the same with that of [11, Main theorem].

Proof of Theorem 1.2. Let $\{V_i\}$ be a countable basis of ∂M. Since, by Proposition 3.1, each $\mathcal{M}_{V_i}$ is open and dense in $\mathcal{M}$, and hence the set $\bigcap_i \mathcal{M}_{V_i}$ is C^∞ Baire-generic in $\mathcal{M}$. This finishes the proof. ☐

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