

# TOTALLY UMBILIC SURFACES IN HYPERBOLIC 3-MANIFOLDS OF FINITE VOLUME

COLIN ADAMS, WILLIAM H. MEEKS III, AND ALVARO K. RAMOS

**ABSTRACT.** For every connected surface  $S$  of finite negative Euler characteristic and every  $H \in [0, 1)$ , we construct a hyperbolic 3-manifold  $N(S, H)$  of finite volume and a proper, two-sided, totally umbilic embedding  $f: S \rightarrow N(S, H)$  with mean curvature  $H$ . Conversely, we prove that a complete, totally umbilic surface with mean curvature  $H \in [0, 1)$  embedded in a hyperbolic 3-manifold of finite volume must be proper and have finite negative Euler characteristic.

*Mathematics Subject Classification:* Primary 53A10, Secondary 49Q05, 57K32, 53C42.

*Key words and phrases:* Calabi-Yau problem, hyperbolic 3-manifolds of finite volume, totally umbilic surfaces.

## 1. INTRODUCTION.

In this manuscript we develop the theory of totally umbilic surfaces in hyperbolic 3-manifolds of finite volume; all spaces considered here are assumed to be complete and connected and all surfaces in them will be assumed to be complete, connected and embedded, unless otherwise stated.

**Theorem 1.1.** *Let  $\Sigma$  be a totally umbilic surface in a hyperbolic 3-manifold  $N$  of finite volume with mean curvature  $H_\Sigma \geq 0$ . Then:*

- (1)  $\Sigma$  is proper in  $N$ .
- (2)  $\Sigma$  has positive Euler characteristic if and only if  $\Sigma$  is a geodesic sphere. In particular,  $\Sigma$  is not diffeomorphic to a plane or a projective plane.
- (3)  $\Sigma$  has zero Euler characteristic if and only if  $N$  is non-compact and  $H_\Sigma = 1$ . In this case,  $\Sigma$  is a flat torus or a flat Klein bottle that is contained in some cusp end of  $N$ .
- (4)  $\Sigma$  has negative Euler characteristic if and only if it has finite negative Euler characteristic and if and only if  $H_\Sigma \in [0, 1)$ . Furthermore, in this setting:
  - (a)  $\Sigma$  has finite area  $A(\Sigma) = \frac{2\pi}{H_\Sigma^2 - 1} \chi(\Sigma)$ , where  $\chi(\Sigma)$  is the Euler characteristic of  $\Sigma$ .
  - (b) If  $H_\Sigma > 0$ , then, for every  $H \in (0, H_\Sigma)$ , there is a totally umbilic surface with mean curvature  $H$  in the ambient isotopy class of  $\Sigma$ .

---

The second and third authors were partially supported by CNPq - Brazil, grant no. 400966/2014-0.

- (c) If  $H_\Sigma > 0$ , then there is a totally geodesic surface  $\Sigma_0$  in the isotopy class of  $\Sigma$ ; also,  $\Sigma$  is diffeomorphic to  $\Sigma_0$  if  $\Sigma_0$  is two-sided and  $\Sigma$  is diffeomorphic to the two-sided cover of  $\Sigma_0$  if  $\Sigma_0$  is one-sided.

The next theorem characterizes the admissible topological types of totally umbilic surfaces in hyperbolic 3-manifolds of finite volume with mean curvature in  $[0, 1)$ . It is a direct consequence of Theorems 1.1 and 5.1.

**Theorem 1.2** (Admissible Topology Theorem). *A surface  $S$  appears topologically as a totally umbilic surface with mean curvature  $H \in [0, 1)$  in some hyperbolic 3-manifold of finite volume if and only if  $S$  has finite negative Euler characteristic.*

Our construction of a hyperbolic 3-manifold of finite volume with a given admissible, two-sided, totally geodesic surface depends on the Switch Move Theorem [3, Theorem 4.1] and the Switch Move Gluing Operation [3, Theorem 5.1] from our previous study of modifications of hyperbolic 3-manifolds that are link complements; see Theorem 3.1 for the statement of the Switch Move Theorem. More specifically, for each admissible surface  $S$ , we apply these theorems to construct a finite volume hyperbolic 3-manifold  $N(S)$  with an order-2 isometry whose fixed point set is two-sided and contains a component  $\Sigma$  diffeomorphic to  $S$ ; see Theorem 4.1 for additional topological properties satisfied by  $N(S)$  and  $\Sigma$ . In Section 5, we apply geometric arguments to prove that for any  $T > 0$ , there is a finite cover  $N_T(S)$  of  $N(S)$  together with a lift  $\tilde{\Sigma}$  of  $\Sigma$ , so that each of the two  $T$ -parallel surfaces to  $\tilde{\Sigma}$  in  $N_T(S)$  is a properly embedded, totally umbilic surface diffeomorphic to  $S$  with mean curvature  $\tanh(T)$ ; crucial in these arguments is the property that the fundamental group of a hyperbolic 3-manifold of finite volume is LERF (see Definition 5.4).

## 2. THE PROOF OF THEOREM 1.1.

In this section, we explain why a totally umbilic surface in a hyperbolic 3-manifold of finite volume must be proper and then show how Theorem 1.1 follows from this properness property. In order to carry out these proofs, we need the following definition, which will also be used in Section 5 to construct admissible totally umbilic surfaces that are parallel to totally geodesic ones.

For a given Riemannian manifold  $N$ , we let  $\exp: TN \rightarrow N$  denote the exponential map.

**Definition 2.1.** Let  $N$  be a Riemannian 3-manifold,  $S$  a surface and  $f: S \rightarrow N$  be a two-sided embedding with image  $\Sigma$  with a unitary normal vector field  $\eta$ . We define, for  $t > 0$ , the  $t$ -parallel surface to  $\Sigma$  as the image  $\Sigma_t$  of the immersion

$$f_t: S \rightarrow N, \quad x \mapsto \exp(t\eta(f(x))).$$

Thus,

$$\Sigma_t = \{\exp(t\eta(p)) \mid p \in \Sigma\}.$$

*Proof of Theorem 1.1.* We first prove item 1 of the theorem. Since it is well-known that totally umbilic surfaces with mean curvature  $H \geq 1$  in a hyperbolic 3-manifold

of finite volume  $N$  are either flat tori or Klein bottles of constant mean curvature 1 in cusp ends of  $N$  or geodesic spheres (following, for instance, from Theorem 29 of Volume 4 of [12]), item 1 holds for  $H \geq 1$ .

Next, we consider the totally geodesic case. For a given manifold  $M$ , let  $\hat{T}(M)$  denote the bundle of unoriented tangent two-planes of  $M$ . Then, the next result follows from the work of Shah [11] (also see Ratner [10] and Payne [9]). The phrase “immersed surface  $f(S)$ ” in a 3-manifold  $N$  is used to indicate that the image surface  $f(S)$  of an immersion  $f: S \rightarrow N$  may have points of self-intersection.

**Theorem 2.2.** *Let  $f: S \rightarrow N$  be a complete, totally geodesic immersion of a surface  $S$  to a hyperbolic 3-manifold  $N$  of finite volume. Then, either  $f(S)$  is a properly immersed surface of finite area or  $f_*(\hat{T}(S))$  is dense in  $\hat{T}(N)$ .*

In fact, Theorem 2.2 can be seen to hold by the following discussion. Let  $f: S \rightarrow N$  be as stated and assume that  $S$  is endowed with a hyperbolic metric. Then, both  $S$  and  $N$  are examples of locally symmetric spaces of rank one, where  $N$  has finite volume by assumption. By the last statement of [9, Theorem 1.1], the closure of the image  $f(S)$  is a totally geodesic submanifold of  $N$ , which, in the context of Theorem 2.2, means  $f(S)$  is either proper or it is dense in  $N$ . If  $f(S)$  is not proper, then, by [11, Theorem D],  $f_*(\hat{T}(S))$  is dense in  $\hat{T}(N)$  and the theorem holds.

Let  $\Sigma$  be a totally geodesic surface in a hyperbolic 3-manifold  $N$  of finite volume. Observe that the closure of  $\Sigma$  in  $N$  is a minimal lamination of  $N$  of class  $C^\alpha$ , for all  $\alpha \in (0, 1)$ . Therefore,  $\hat{T}(\Sigma)$  is not dense in  $\hat{T}(N)$  and so Theorem 2.2 implies that  $\Sigma$  must have finite area and be proper in  $N$ , which proves item 1 of Theorem 1.1 when  $H = 0$ .

To finish the proof of item 1, let  $\Sigma$  be a totally umbilic surface in  $N$  with mean curvature  $H \in (0, 1)$ . Assume  $\Sigma$  is oriented with respect to the unit normal field  $\eta$  pointing towards its mean convex side. Let  $T = \tanh^{-1}(H) > 0$  and consider the  $T$ -parallel immersion  $f_T: \Sigma \rightarrow N$  with immersed image surface  $\Sigma_T$ . Then,  $\Sigma_T$  is a complete totally geodesic immersed surface.

We claim that  $\Sigma_T$  is proper. Otherwise,  $\Sigma_T$  has infinite area and Theorem 2.2 implies that  $\hat{T}(\Sigma_T)$  is dense in  $\hat{T}(N)$ . Since  $\Sigma_T$  has bounded norm of its second fundamental form,  $\Sigma_T$  intersects itself transversely in a dense set of points in  $N$ . Let  $\Pi: \mathbb{H}^3 \rightarrow N$  denote the universal Riemannian covering map and let  $\tilde{\Sigma}_T^1, \tilde{\Sigma}_T^2 \subset \mathbb{H}^3$  be two components of  $\Pi^{-1}(\Sigma_T)$  corresponding to two lifts of  $\Sigma_T$  that intersect transversely in  $\mathbb{H}^3$ . Let  $C_1, C_2$  be the respective boundary circles of  $\tilde{\Sigma}_T^1, \tilde{\Sigma}_T^2$ , intersecting transversely in the boundary sphere at infinity of  $\mathbb{H}^3$ . Consider two respective lifts  $\tilde{\Sigma}_1, \tilde{\Sigma}_2$  of  $\Sigma$  in  $\mathbb{H}^3$  with the same circles  $C_1, C_2$  at infinity. Then,  $\tilde{\Sigma}_1$  intersects  $\tilde{\Sigma}_2$  transversely along a proper arc, which implies that  $\Sigma$  is not embedded, a contradiction. It also follows from this argument that  $\Sigma_T$  must be a proper, totally geodesic (embedded) surface in  $N$ .

To show that  $\Sigma$  is proper, there are two cases to consider. First, assume that  $\Sigma_T$  is two-sided and oriented with respect to the unitary normal field  $\hat{\eta}$  corresponding to the opposite orientation from the induced immersion  $f_T$ . Then, as  $\Sigma$  is connected,

it is the image of the  $T$ -parallel immersion  $\widehat{f}_T: \Sigma_T \rightarrow N$ , which must be a proper map, since the inclusion map of  $\Sigma_T$  in  $N$  is proper and the distance between any two corresponding points  $x \in \Sigma_T$ ,  $\widehat{f}_T(x) \in \Sigma$  is bounded by  $T$ . On the other hand, if  $\Sigma_T$  is one-sided, we may pass to the (proper) two-sheeted, two-sided cover of  $\Sigma_T$  and repeat the same argument, finishing the proof of item 1 of Theorem 1.1.

As already observed, a surface  $\Sigma$  appears as a totally umbilic surface with mean curvature  $H_\Sigma \geq 1$  in a hyperbolic 3-manifold  $N$  of finite volume if and only if  $\Sigma$  is a geodesic sphere with  $H_\Sigma > 1$  or it is a flat torus or a flat Klein bottle in a cusp end of  $N$  when  $H_\Sigma = 1$ . On the other hand, if  $\Sigma$  is a totally umbilic surface with  $H_\Sigma \in [0, 1)$ , which must be proper by item 1, then Corollary 4.7 of [7] implies that the Euler characteristic of  $\Sigma$  is negative, completing the proof of items 2 and 3.

Note that the main statement of item 4 follows from the above discussion and the fact that any properly immersed, infinite topology surface with constant mean curvature  $H \in [0, 1)$  in a hyperbolic 3-manifold of finite volume has unbounded norm of its second fundamental form; see item 4 of [7, Theorem 1.3] for this unboundedness property.

The other statements of item 4 will be explained next. Suppose that  $\Sigma$  is a totally umbilic surface in a hyperbolic 3-manifold  $N$  of finite volume, with finite negative Euler characteristic and  $H_\Sigma \in [0, 1)$ . Then, item 4a follows immediately from [7, Corollary 4.7].

Next, we prove items 4b and 4c by showing that for any  $t \in (0, T)$ , the  $t$ -parallel immersion  $f_t: \Sigma \rightarrow N$  is injective, where  $T = \tanh^{-1}(H_\Sigma)$ . Recall, from the proof of item 1, that the image surface  $\Sigma_T = f_T(\Sigma)$  is a totally geodesic (embedded) surface in  $N$ . Let  $\widetilde{\Sigma}_T \subset \mathbb{H}^3$  be a component of  $\Pi^{-1}(\Sigma_T)$  and let  $\widetilde{\Sigma}$  be a component of  $\Pi^{-1}(\Sigma)$  with the same boundary circle at infinity as  $\widetilde{\Sigma}_T$ . Let  $W \subset \mathbb{H}^3$  be the closed region with boundary  $\widetilde{\Sigma} \cup \widetilde{\Sigma}_T$ . Since both  $\Sigma$  and  $\Sigma_T$  are embedded, if  $\sigma$  is a covering transformation of  $\Pi$ , then either  $\sigma$  maps  $W$  to itself, in which case  $\sigma$  leaves invariant each surface in  $W$  parallel to  $\widetilde{\Sigma}$ , or  $\sigma(W) \cap W = \widetilde{\Sigma}_T$ , in which case  $\sigma$  is a glide reflection along  $\widetilde{\Sigma}_T$  or a loxodromic transformation of  $\mathbb{H}^3$  with respect to a geodesic  $\gamma$  in  $\widetilde{\Sigma}_T$  that has order-two rotational part about  $\gamma$ , or  $\sigma(W) \cap W = \emptyset$ . It follows that  $f_t$  is injective for all  $t \in (0, T)$  and  $f_T$  is injective if and only if  $\Sigma_T$  is two-sided. In the case where  $\Sigma_T$  is one-sided, then the induced immersion  $f_T: \Sigma \rightarrow \Sigma_T$  is a double covering of  $\Sigma_T$ , and items 4b and 4c follow.  $\square$

### 3. RESULTS ON HYPERBOLIC LINK COMPLEMENTS.

In this section, we present some results concerning the hyperbolicity of link complements in 3-manifolds that will be used to construct the totally geodesic examples described in Theorem 4.1.

**Theorem 3.1** (Switch Move Theorem [3, Theorem 4.1]). *Let  $L$  be a link in a 3-manifold  $M$  such that  $M \setminus L$  admits a complete hyperbolic metric of finite volume. Let  $\alpha \subset M$  be a compact arc which intersects  $L$  transversely in its two distinct*

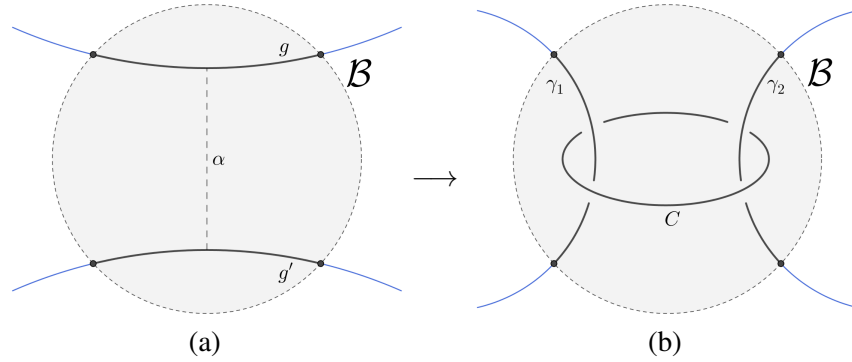


FIGURE 1. The Switch Move replaces the arcs  $g$  and  $g'$  by the tangle  $\gamma_1 \cup \gamma_2 \cup C$ .

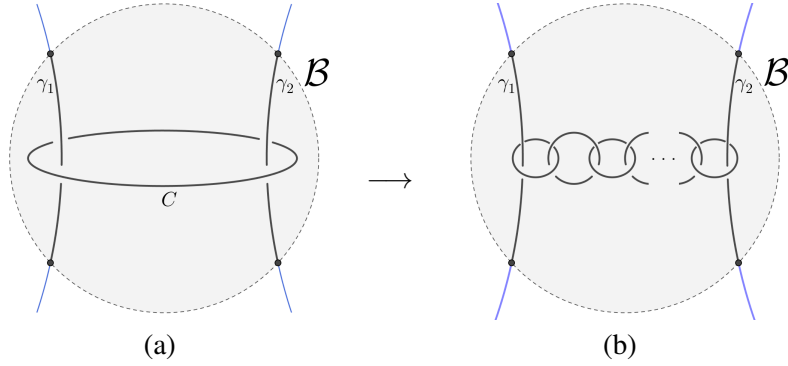


FIGURE 2. For any positive integer  $k$ , replacing the trivial component  $C$  in (a) with the untwisted chain with  $2k + 1$  components as in (b) preserves hyperbolicity of the complement.

endpoints, and such that  $\text{int}(\alpha)$  is a properly embedded geodesic in the hyperbolic metric of  $M \setminus L$ . Let  $\mathcal{B}$  be a closed ball in  $M$  containing  $\alpha$  in its interior and such that  $\mathcal{B} \cap L$  is composed of two arcs in  $L$ , as in Figure 1(a). Let  $L'$  be the resulting link in  $M$  obtained by replacing  $L \cap \mathcal{B}$  by the components as appearing in Figure 1(b). Then  $M \setminus L'$  admits a complete hyperbolic metric of finite volume.

A consequence of the Switch Move Theorem is the following Untwisted Chain Theorem.

**Corollary 3.2** (Untwisted Chain Theorem). *Let  $L$  be a link in a 3-manifold  $M$  such that the link complement  $M \setminus L$  admits a complete hyperbolic metric of finite volume. Suppose that there is a ball  $\mathcal{B}$  in  $M$  that intersects  $L$  as in Figure 2 (a). For any positive integer  $k$ , let  $L'$  be the resulting link in  $M$  obtained by replacing  $L \cap \mathcal{B}$  by the untwisted chain with  $2k + 1$  components as in Figure 2 (b). Then  $M \setminus L'$  admits a complete hyperbolic metric of finite volume.*

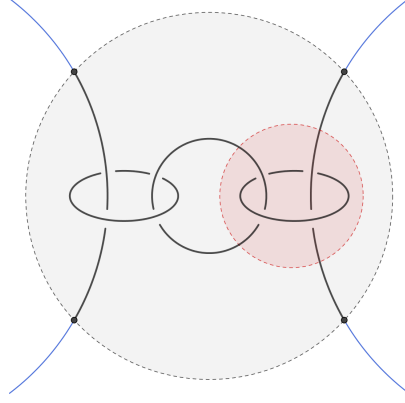


FIGURE 3. The Untwisted Chain Theorem with  $k = 1$  can be repeated in the highlighted subball to obtain any odd number of components.

*Proof.* We first prove Corollary 3.2 when  $k = 1$ . Let  $L$ ,  $\mathcal{B}$  and  $M$  be as stated and let  $D \subset \mathcal{B}$  be a two-punctured disk in  $\mathcal{B} \setminus L$  with  $\partial D = C$ , where  $C$  is the circle component of  $\mathcal{B} \cap L$ . Since  $D$  is an incompressible thrice-punctured sphere in the hyperbolic 3-manifold  $M \setminus L$ , then [1, Theorem 3.1] gives that, up to isotopy,  $D$  is totally geodesic. Hence, there exists a compact arc  $\alpha \subset \overline{D}$  in the closure of  $D$ , with two distinct endpoints in  $C$ , separating the two punctures of  $D$  and such that its interior  $\tilde{\alpha}$  is a proper geodesic in the hyperbolic metric of  $M \setminus L$ . Moreover,  $\alpha$  can be chosen to be transverse to  $L$ . After applying the Switch Move Theorem in a neighborhood of  $\alpha$ , we obtain a link  $L'$  as in Figure 3, such that  $M \setminus L'$  admits a complete, hyperbolic metric of finite volume. The general case follows by induction on  $k$ , as indicated in Figure 3.  $\square$

**Theorem 3.3** (Theorem 1.1 of [2]). *Let  $S$  be a closed (possibly non-orientable) surface with  $\chi(S) \leq 0$ . Then, there exists a link  $L$  in  $S \times (0, 1)$  such that:*

- a. *If  $\chi(S) < 0$ ,  $(S \times [0, 1]) \setminus L$  admits a complete hyperbolic metric of finite volume with totally geodesic boundary.*
- b. *If  $\chi(S) = 0$ ,  $(S \times (0, 1)) \setminus L$  admits a complete hyperbolic metric of finite volume.*

**Remark 3.4.** In fact, the link  $L$  given by Theorem 3.3 can be any *prime fully alternating link*, see [2]. A fully alternating link  $L$  in  $S \times (0, 1)$  is a link that admits a projection to  $S$  that is *alternating* in the sense that the link can be oriented so that two consecutive crossings have distinct over/under signs and is *full* in the sense that every component of the complement of the projected image of  $L$  on  $S$  is a disk.

We also note that Theorem 3.3 is proved in [6] in the case where  $S$  is orientable, without obtaining a totally geodesic boundary when  $\chi(S) < 0$ . The proof in [6] could be extended to show these additional facts and it uses different techniques than those applied in [2].

#### 4. CONSTRUCTION OF HYPERBOLIC 3-MANIFOLDS WITH TOTALLY GEODESIC SURFACES.

Next, we construct, in Theorem 4.1 below, examples of totally geodesic surfaces in hyperbolic 3-manifolds that will be used in Section 5 to prove Theorem 1.2. We remark that Theorem 4.1 contrasts with the rarity of totally geodesic surfaces in certain categories of hyperbolic 3-manifolds, as exemplified by the thirty-year old conjecture of Menasco and Reid from [8] that no hyperbolic knot complement in  $S^3$  contains a closed totally geodesic surface.

**Theorem 4.1.** *Let  $S$  be a surface with finite negative Euler characteristic. There exists a finite volume hyperbolic 3-manifold  $N$  and a proper, two-sided embedding  $f: S \rightarrow N$  with totally geodesic image  $\Sigma$ . Moreover:*

- (1) *If  $S$  is closed (resp. orientable),  $N$  is closed (resp. orientable).*
- (2) *If  $e_1$  and  $e_2$  are distinct ends of  $\Sigma$ , then  $N$  contains disjoint cusp ends  $C_1, C_2$  such that, for  $i = 1, 2$ ,  $\Sigma \cap C_i$  is an annular representative of  $e_i$ .*
- (3)  *$\Sigma$  is a two-sided component of the fixed point set of an order-two isometry of  $N$ .*

*Proof.* Let  $S$  be as stated. To prove Theorem 4.1, we construct a complete hyperbolic 3-manifold of finite volume  $N$  together with an order-two diffeomorphism  $\varphi$  that has a two-sided fixed point set containing a component  $\Sigma$  diffeomorphic to  $S$ . By the Mostow-Prasad Rigidity Theorem, after changing coordinates by a diffeomorphism isotopic to the identity, we may assume that  $\varphi$  is an isometry of  $N$ , from where it follows that  $\Sigma$  is totally geodesic and item 3 holds.

The proof breaks up into cases which are treated separately.

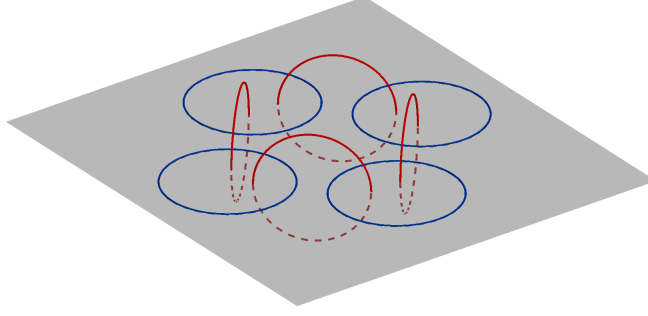
**Case 4.2.**  *$S$  is an  $n$ -punctured sphere with  $n \geq 3$ .*

*Proof.* For  $n \geq 3$ , let  $L_n \subset \mathbb{R}^3$  be the  $2n$ -component untwisted chain link; the  $L_4$  version appears in Figure 4. Then, by adding to  $\mathbb{R}^3$  the point at infinity and considering  $L_n \subset \mathbb{S}^3$ ,  $\mathbb{S}^3 \setminus L_n$  has an explicit hyperbolic metric of finite volume, as described in Example 6.8.7 of [13].

Thinking of the compactified  $xy$ -plane as a sphere  $\mathcal{S} \subset \mathbb{S}^3$ , we can view every other component of  $L_n$  as being contained in  $\mathcal{S}$  with the remaining components perpendicular to  $\mathcal{S}$  and symmetric with respect to reflection  $R$  through  $\mathcal{S}$ . Then, the restriction of  $R$  to the link complement  $\mathbb{S}^3 \setminus L_n$  is an order-two isometry of the hyperbolic metric described above, with two-sided fixed point set  $\mathcal{S} \setminus L_n$ . This fixed point set contains an  $n$ -punctured sphere  $S_n$ , where the  $n$  punctures come from the  $n$  components of  $L_n$  in  $\mathcal{S}$ . It follows directly that  $S_n$  is totally geodesic and satisfies the statements of the theorem.  $\square$

**Case 4.3.**  *$S$  is an  $n$ -punctured projective plane with  $n \geq 2$ .*

*Proof.* Fix  $n \geq 2$  and let  $k = n - 1$ . Let  $L_{2k}$  be the  $4k$ -component untwisted chain link in  $\mathbb{S}^3$ . As in Case 4.2,  $\mathcal{S}$  denotes the compactification of the  $xy$ -plane and we assume that every other component of  $L_{2k}$  is contained in  $\mathcal{S}$ . Furthermore, we assume that the  $2k$  components of  $L_{2k}$  that are not contained in  $\mathcal{S}$  are perpendicular

FIGURE 4. The untwisted chain link  $L_4$ .

to  $S$  and lie on the unit sphere  $\mathbb{S}^2$  centered at the origin of  $\mathbb{R}^3$  and that  $L_{2k}$  is invariant under the inversion  $\phi(\mathbf{x}) = -\mathbf{x}$  through the origin and the reflection  $\Phi$  through  $\mathbb{S}^2$ .

Let  $Z \subset \mathbb{S}^3$  denote the  $z$ -axis with the point at infinity. Define  $J_k = L_{2k} \cup Z$  and let  $M = \mathbb{S}^3 \setminus J_k$ . Then,  $M$  is a  $2k$ -cover of  $\mathbb{S}^3 \setminus B_3$ , where  $B_3$  is the Borromean ring with three components. Since  $\mathbb{S}^3 \setminus B_3$  admits a complete hyperbolic metric of finite volume (see [13, Section 3.4]), there exists a complete hyperbolic metric  $g$  on  $M$ . Moreover,  $\phi$  and  $\Phi$  restrict to isometries of  $(M, g)$ .

Note that  $\mathbb{S}^2 \setminus J_k$  contains a connected component  $\tilde{S}$  which is a  $(2k+2)$ -punctured sphere, where  $2k$  punctures come from  $L_{2k}$  and the other two from  $\mathbb{S}^2 \cap Z$ . By construction,  $\Phi|_{\tilde{S}} = \text{Id}_{\tilde{S}}$ , and so  $\tilde{S}$  is totally geodesic.

Let  $N = M/\phi$ . Since  $\phi$  is a fixed-point free, orientation reversing order-two isometry of  $(M, g)$ ,  $N$  is a non-orientable hyperbolic 3-manifold that is double covered by  $M$ . Since  $\phi|_{\mathbb{S}^2}$  is the antipodal map and  $\phi(\tilde{S}) = \tilde{S}$ , the surface  $S = \tilde{S}/\phi$  is a  $(k+1)$ -punctured projective plane in  $N$ . Also, since  $\Phi$  and  $\phi$  commute, the map  $\Phi$  descends to  $N$  as an order-two isometry of  $N$  which contains  $S$  in its fixed point set. Since  $k+1 = n$  and  $S$  satisfies the properties stated by Theorem 4.1, this proves Case 4.3.  $\square$

**Case 4.4.**  $S$  is closed.

*Proof.* For the following construction, see Figure 5 (a). Consider  $\mathbb{S}^1$  to be the unit circle in the  $yz$ -plane and let  $P = S \times \mathbb{S}^1$ . Let  $\mathbb{S}_+^1 = \mathbb{S}^1 \cap \{z \geq 0\}$ ,  $\mathbb{S}_-^1 = \mathbb{S}^1 \cap \{z \leq 0\}$ ,  $M_1 = S \times \mathbb{S}_+^1$  and  $M_2 = S \times \mathbb{S}_-^1$ . Then,  $M_1, M_2$  are subsets of  $P$  glued along their boundary surfaces  $S_1 = S \times \{(-1, 0)\}$  and  $S_2 = S \times \{(1, 0)\}$ . Let  $R: P \rightarrow P$  be the reflective symmetry interchanging  $M_1$  with  $M_2$ ; the fixed point set of  $R$  is  $S_1 \cup S_2$ .

By Theorem 3.3, there exists a link  $L$  in  $\text{int}(M_1)$  such that  $M_1 \setminus L$  admits a finite volume hyperbolic metric with totally geodesic boundary  $S_1 \cup S_2$ . Let  $L' = R(L) \subset \text{int}(M_2)$  and  $\Gamma = L \cup L'$ . Then,  $P \setminus \Gamma$  admits a complete hyperbolic metric  $g$  for which  $R$  is an isometry and the surfaces  $S_1, S_2$  are totally geodesic surfaces forming the fixed point set of  $R$ .

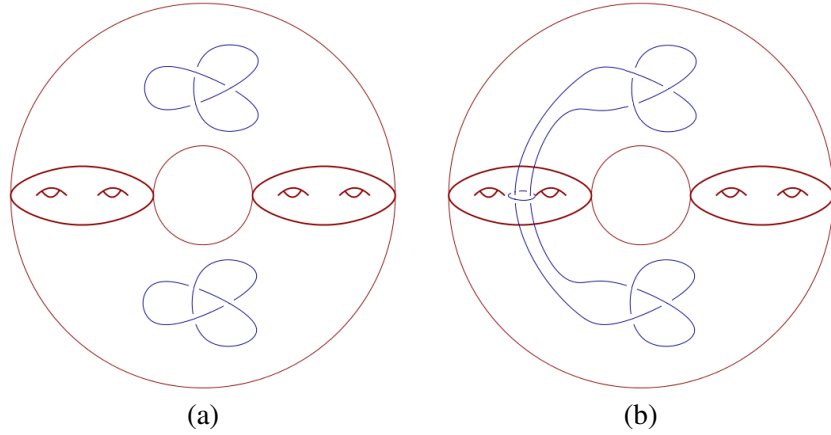


FIGURE 5. (a) is the manifold  $P = S \times \mathbb{S}^1$  with the link  $L$  and its reflection  $R(L)$ . (b) is the manifold  $P = S' \times \mathbb{S}^1$  with the link  $L_1$  that creates a puncture to  $S_1$ .

Note that  $P \setminus \Gamma$  is orientable if and only if  $S$  is orientable. Moreover, after performing an appropriate Dehn filling in the ends of  $P \setminus \Gamma$  in a symmetric manner with respect to  $R$ , we obtain a closed hyperbolic 3-manifold  $N$ , where  $S_1$  and  $S_2$  are each as stated in Theorem 4.1.  $\square$

**Case 4.5.**  $S$  is  $S'$  punctured one time, where  $S'$  is a closed surface with  $\chi(S') < 0$ .

*Proof.* The starting point to this case is the closed manifold  $P = S' \times \mathbb{S}^1$ . As in Case 4.4, there exists a link  $\Gamma = L \cup R(L)$ , where  $L$  is a link in the interior of  $M_1 = S' \times \mathbb{S}_+^1$ , such that  $P \setminus \Gamma$  is hyperbolic, the reflection  $R$  restricts to an isometry and the fixed point set of  $R$  consists of the two totally geodesic surfaces  $S_1 = S' \times \{(-1, 0)\}$  and  $S_2 = S' \times \{(1, 0)\}$ .

In the hyperbolic metric of  $M_1 \setminus L$ , let  $\alpha_1$  be a minimizing geodesic ray from  $S_1$  to  $L$ . Then,  $\alpha_1$  is proper, perpendicular to  $S_1$  and  $\alpha_1$  does not intersect  $S_2$ . Let  $\alpha = \alpha_1 \cup R(\alpha_1)$ . Then,  $\alpha$  is a complete geodesic of  $P \setminus \Gamma$  and the closure of  $\alpha$  in  $P$  admits a neighborhood  $\mathcal{B} \subset P$  with the following properties:  $R(\mathcal{B}) = \mathcal{B}$ ,  $\mathcal{B} \cap S_1 = E$  is a disk,  $\mathcal{B} \cap S_2 = \emptyset$  and  $\mathcal{B}$  intersects  $\Gamma$  in two arcs  $g \subset L$  and  $g' = R(g) \subset R(L)$ . Then,  $\mathcal{B}$  satisfies the hypothesis of Theorem 3.1. Then, we can replace the arcs  $g \cup g'$  in  $\mathcal{B} \cap \Gamma$  by a tangle  $\gamma_1 \cup \gamma_2 \cup C$  as in Figure 1 (b) to form a new link  $L_1 \subset P$ , where we may choose  $C$  as a circle in  $\text{int}(E)$  that bounds a disk in  $E$ , punctured once by  $\alpha$  (see Figure 5 (b)). Then, Theorem 3.1 gives that  $P \setminus L_1$  admits a complete, hyperbolic metric of finite volume, and since we may choose the arcs  $\gamma_1, \gamma_2$  to be invariant under  $R$ , it follows that  $R$  restricts to an isometry  $\varphi$ . Note that the fixed point set of  $\varphi$  contains three connected components, one being  $S_2$  and the other two being the connected components of  $S_1 \setminus (C \cup \gamma_1 \cup \gamma_2)$ , one a thrice-punctured sphere and the other diffeomorphic to  $S$  and satisfying the conclusions required by Theorem 4.1.  $\square$

**Case 4.6.**  $S$  is a torus or a Klein bottle punctured once.

*Proof.* Let  $T$  be either a torus or a Klein bottle and let  $P_1$  be the product manifold  $T \times [0, 1]$ . Let  $\Gamma_1$  be a link in  $\text{int}(P_1)$  such that  $\text{int}(P_1 \setminus \Gamma_1)$  is hyperbolic, as given by item b of Theorem 3.3. Let  $P = T \times [-1, 1]$  and  $R: P \rightarrow P$  be the reflection  $R(x, t) = (x, -t)$ ; take  $P_2$  as the reflected image of  $P_1$  in  $P$ , with respective hyperbolic link  $\Gamma_2 = R(\Gamma_1) \subset P_2$ . Also, let  $\alpha_1$  be a complete geodesic in the hyperbolic metric of  $\text{int}(P_1 \setminus \Gamma_1)$  with one endpoint in  $T \times \{0\}$  and another endpoint in a component  $L_1$  of  $\Gamma_1$  and let  $\alpha_2 = R(\alpha_1)$ .

Let  $\alpha \subset P$  be the concatenation of  $\alpha_1$  and  $\alpha_2^{-1}$ . Then,  $\alpha$  is an arc with one endpoint in  $L_1$  and another endpoint in  $L_2 = R(L_1)$ . Let  $\mathcal{B}$  be a regular neighborhood of  $\alpha$  in  $P$ , invariant under  $R$  and that intersects  $\Gamma = \Gamma_1 \cup \Gamma_2$  in two arcs  $g \subset L_1$  and  $g' = R(g) \subset L_2$  and intersects  $T \times \{0\}$  in a disk  $\Delta$ . Let  $\Gamma'$  be the link in  $\text{int}(P)$  obtained from  $\Gamma$  by replacing  $g \cup g'$  in  $\mathcal{B}$  by the tangle  $\gamma_1 \cup \gamma_2 \cup C$  as in Figure 1 (b), where  $C \subset \Delta$ . Then, Theorem 5.1 in [3] implies that the interior of the link complement  $P \setminus \Gamma'$  admits a complete hyperbolic metric of finite volume. Furthermore, we may choose  $\gamma_1 \cup \gamma_2 \cup C$  so that it is invariant under  $R$ . Hence,  $R$  restricts to an isometry  $\varphi$  of the hyperbolic metric of  $P \setminus \Gamma'$ . This proves Case 4.6, since the fixed point set of  $\varphi$  contains a thrice-punctured sphere  $D$  bounded by  $C$  in  $T$  and a component  $T \setminus \overline{D}$  diffeomorphic to  $S$  that satisfies the properties required by Theorem 4.1.  $\square$

The remaining cases to treat in Theorem 4.1 are those where  $S$  is diffeomorphic to  $S'$  punctured  $n \geq 2$  times, where  $S'$  is a closed surface with  $\chi(S') \leq 0$ . If  $\chi(S') < 0$ , let  $P = S' \times \mathbb{S}^1$  with respective hyperbolic link  $L_1$  as given by Case 4.5. Let  $\mathcal{B} \subset P$  be the ball where the Switch Move Theorem was applied. Then, in  $P \setminus L_1$ ,  $\mathcal{B}$  satisfies the hypothesis of the Untwisted Chain Theorem, Corollary 3.2. Then, we can replace the circle component  $C$  of  $\mathcal{B} \cap L_1$  by an untwisted chain with  $2n - 1$  components as in Figure 2 (b), where every other component is in the interior of  $D = \mathcal{B} \cap S_1$  and the remaining components lie in  $\mathcal{B}$  and are symmetric with respect to  $R$  to create a hyperbolic link  $L_n \subset P$ . Again, the reflection  $R$  restricts to an isometry  $\varphi$  of the hyperbolic metric of  $P \setminus L_n$ , and the fixed point set of  $\varphi$  contains  $S_2$ ,  $n - 1$  thrice-punctured spheres and a surface diffeomorphic to  $S$ , which finishes the proof of Theorem 4.1 when  $\chi(S') < 0$ . The proof for the case when  $S'$  is a torus or a Klein bottle is analogous.  $\square$

## 5. THE PROOF OF THEOREM 1.2.

To finish the proof of Theorem 1.2, it suffices to prove Theorem 5.1 below.

**Theorem 5.1.** *Let  $H \in [0, 1)$  and suppose that  $S$  is a connected surface of finite topology and negative Euler characteristic. Then, there exists a hyperbolic 3-manifold  $N$  of finite volume and a proper, two-sided embedding  $f: S \rightarrow N$ , with totally umbilic image  $\Sigma$  having mean curvature  $H$  and satisfying:*

- (1) *If  $S$  is closed (resp. orientable), then  $N$  is closed (resp. orientable).*
- (2) *Each end  $e$  of  $\Sigma$  admits an annular end representative  $E$  which is embedded in a cusp end  $\mathcal{C}_e$  of  $N$ . In addition, if  $e$  and  $f$  are two distinct ends of  $\Sigma$ , the respective cusp ends  $\mathcal{C}_e, \mathcal{C}_f$  are distinct.*

To prove Theorem 5.1, we make use of the totally geodesic examples provided by Theorem 4.1 to construct hyperbolic 3-manifolds of finite volume with two-sided, totally umbilic surfaces which are properly embedded with any admissible topology and mean curvature  $H \in (0, 1)$ . In order to define these associated totally umbilic examples, we recall, from Definition 2.1, that if  $N$  is a Riemannian 3-manifold and  $f: S \rightarrow N$  is a two-sided embedding with image  $\Sigma$  and unitary normal vector field  $\eta$ , then the  $t$ -parallel surface to  $\Sigma$  is the image  $\Sigma_t$  of the immersion  $f_t: S \rightarrow N$  defined by  $f_t(x) = \exp(t\eta(f(x)))$ . Note that the  $t$ -parallel surface  $\Sigma_t$  to a totally geodesic surface  $\Sigma$  in a hyperbolic 3-manifold  $N$  is totally umbilic and has mean curvature  $\tanh(t) \in (0, 1)$ . For convenience, we will often assume that the domain of the immersion  $f_t$  is  $\Sigma$  instead of the abstract surface  $S$ . Note that the ambient distance of two respective points  $x \in \Sigma$  and  $f_t(x) \in \Sigma_t$  is uniformly bounded by  $t$ .

**Lemma 5.2.** *Let  $f: S \rightarrow N$  satisfy the properties given by Theorem 4.1, and assume that  $N$  has  $m \geq 0$  ends. Then, for any  $T > 0$ :*

- (1) *There exists a pairwise disjoint collection of cusp end representatives of the ends of  $N$ ,  $\{\mathcal{C}_1(T), \dots, \mathcal{C}_m(T)\}$ , such that for any  $t \in (0, T]$ ,  $f_t$  is injective on  $f^{-1}(\cup_{i=1}^m \mathcal{C}_i(T))$ .*
- (2) *For  $t \in (0, T]$  sufficiently small, the immersions  $f_t: S \rightarrow N$  are injective.*
- (3) *If  $\Lambda = \{t > 0 \mid f_t \text{ is not injective}\} \neq \emptyset$ , then  $t_0 = \inf \Lambda > 0$  and there exists a prime closed geodesic in  $N$  with length  $4t_0$ . Furthermore, this geodesic meets  $\Sigma$  orthogonally in exactly two (distinct) points.*

*Proof.* First, we notice that if  $S$  is compact, the proofs of items 1 and 2 are immediate.

Assume now that  $S$  is noncompact and let  $e_1, \dots, e_n$  be the ends of  $\Sigma = f(S)$ . A straightforward consequence of item 2 of Theorem 4.1 is that there exist pairwise disjoint annular end representatives  $E_1, \dots, E_n \subset \Sigma$  and a collection  $\{\mathcal{C}_1, \dots, \mathcal{C}_m\}$  of pairwise disjoint cusp ends of  $N$  such that, after possibly passing to subends, it holds, for  $1 \leq i \leq n$ , that  $\Sigma \cap \mathcal{C}_i = E_i$  with  $\partial \mathcal{C}_i$  intersecting  $\Sigma$  orthogonally and, for  $n+1 \leq j \leq m$ ,  $\mathcal{C}_j \cap \Sigma = \emptyset$ .

For a given  $i \in \{1, \dots, n\}$ , consider the cusp end  $\mathcal{C}_i$ . There exists a family of compact surfaces  $\{\mathcal{T}_i(s)\}_{s \geq 0}$  (if  $\mathcal{C}_i$  is orientable, each  $\mathcal{T}_i(s)$  is a torus, otherwise each  $\mathcal{T}_i(s)$  is a Klein bottle) arising from the descent of parallel horospheres of  $\mathbb{H}^3$  via the universal covering projection, parameterized by  $s = \text{dist}(\mathcal{T}_i(s), \mathcal{T}_i(0) = \partial \mathcal{C}_i)$  such that  $\cup_{s \geq 0} \mathcal{T}_i(s) = \mathcal{C}_i$ . For any  $\lambda > 0$ , let

$$(1) \quad \mathcal{C}_i(\lambda) = \cup_{s \geq \lambda} \mathcal{T}_i(s), \quad N(\lambda) = N \setminus (\cup_{i=1}^n \mathcal{C}_i(\lambda)),$$

and

$$(2) \quad E_i(\lambda) = E_i \cap \mathcal{C}_i(\lambda), \quad \Sigma(\lambda) = \Sigma \cap N(\lambda).$$

The assumption that  $\Sigma \cap \mathcal{C}_i = E_i$ , for  $1 \leq i \leq n$ , implies that  $E_i(\lambda) = \Sigma \cap \mathcal{C}_i(\lambda)$ . In particular, for every  $\lambda > 0$ , we can express  $\Sigma$  as a disjoint union  $\Sigma = (\cup_{i=1}^n E_i(\lambda)) \cup \Sigma(\lambda)$ .

Fix  $T > 0$ ,  $t \in (0, T]$  and  $i \in \{1, 2, \dots, n\}$ . Using the universal covering of  $\mathcal{C}_i$  by a horoball of  $\mathbb{H}^3$ , it is easy to see that, for any  $s > T$ ,  $E_i$  is orthogonal to  $\mathcal{T}_i(s)$  and  $E_i \cap \mathcal{T}_i(s)$  is a geodesic of  $\mathcal{T}_i(s)$ , which is injectively mapped by  $f_t$  to a geodesic of  $\mathcal{T}_i(h_t(s))$ , where  $h_t(s) \in (s - t, s)$  and the function  $h_t$  is increasing; hence  $f_t|_{E_i(T)}$  is injective. In order to finish the proof of item 1, just note that  $f_t(E_i(T)) \subset \mathcal{C}_i$  and, if  $i \neq j$ ,  $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ .

Next, we prove the second statement of the lemma. Since  $\overline{\Sigma(3T)}$  is compact, then there exists some  $\varepsilon(T) \in (0, T)$  such that  $f_t|_{\Sigma(3T)}$  is injective for all  $t \in (0, \varepsilon(T))$ . We claim that  $f_t$  is injective for all  $t \in (0, \varepsilon(T))$ .

Assume that  $f_t(x) = f_t(y) = p$ , for some  $x, y \in \Sigma$  and  $t \in (0, \varepsilon(T))$ . If  $x \neq y$ , since  $f_t|_{\Sigma(3T)}$  is injective and  $\{E_1(3T), E_2(3T), \dots, E_n(3T), \Sigma(3T)\}$  is a partition of  $\Sigma$ , there exists some  $i \in \{1, 2, \dots, n\}$  such that either  $x \in E_i(3T)$  or  $y \in E_i(3T)$ . In particular,  $p \in \mathcal{C}_i(2T)$ . Without loss of generality, assume that  $x \in E_i(3T)$ . Item 1 and the fact that  $t \in (0, T)$  gives that  $f_t|_{E_i(T)}$  is injective. Moreover,  $E_i(3T) \subset E_i(T)$ , and therefore  $y \notin E_i(T)$ . There are two possibilities: either  $y \in E_j(T)$  for  $j \neq i$  or  $y \in \Sigma(T)$ . If  $y \in E_j(T)$ , then  $p \in \mathcal{C}_j$ , which is impossible since  $\mathcal{C}_j \cap \mathcal{C}_i = \emptyset$ . On the other hand, if  $y \in \Sigma(T)$ , then  $p \in N(2T)$ . However,  $N(2T) \cap \mathcal{C}_i(2T) = \emptyset$ , and this contradiction proves item 2.

We next prove item 3. First, item 2 gives that  $t_0 > 0$ , and our next argument shows that  $f_{t_0}$  is not injective.

Choose  $\lambda > t_0$ . By the definition of  $t_0$ , there exist a sequence  $\{t_k\}_{k \in \mathbb{N}} \subset [t_0, \lambda)$ ,  $t_k \rightarrow t_0$  and points  $x_k, y_k \in \Sigma$ ,  $x_k \neq y_k$ , such that  $f_{t_k}(x_k) = f_{t_k}(y_k) = p_k$ . From the definition of  $f_{t_k}$  and from the assumption that  $t_k < \lambda$  for all  $k \in \mathbb{N}$ , the triangle inequality implies that  $d_N(x_k, y_k) < 2\lambda$ , for all  $k \in \mathbb{N}$ .

By item 1, it follows that  $f_t|_{E_i(\lambda)}$  is injective for all  $i \in \{1, \dots, n\}$  and all  $t \in (0, \lambda)$ . Therefore,  $x_k, y_k \subset \Sigma(3\lambda)$  for all  $k \in \mathbb{N}$ , since  $x_k \in E_i(3\lambda) \subset E_i(\lambda)$  gives that  $y_k \in E_i(\lambda)$  and vice versa. The compactness of  $\overline{\Sigma(3\lambda)}$  implies that, up to subsequences, there are  $x, y \in \overline{\Sigma(3\lambda)}$  such that  $x_k \rightarrow x$  and  $y_k \rightarrow y$ . By the continuity of  $(t, z) \mapsto f_t(z)$ , it follows that  $f_{t_0}(x) = f_{t_0}(y) = p$ . Moreover,  $\{f_t|_{\overline{\Sigma(3\lambda)}}\}_{t \in [0, \lambda]}$  is a smooth, compact family of immersions of the compact surface  $\overline{\Sigma(3\lambda)}$  into  $N$ . Hence, there exists an  $\varepsilon > 0$  such that for any  $t \in [0, \lambda]$  and any  $z \in \overline{\Sigma(3\lambda)}$ ,  $f_t|_{B_{\Sigma}(z, \varepsilon)}$  is injective, which implies that  $d_{\Sigma}(x_k, y_k) \geq \varepsilon$ , thus  $x \neq y$ . This proves that  $f_{t_0}$  is not injective.

Let  $U \ni x$  and  $V \ni y$  be two disjoint open disks of  $\Sigma$  such that the restrictions  $f_{t_0}|_U$  and  $f_{t_0}|_V$  are injective. It follows that  $f_{t_0}(U)$  and  $f_{t_0}(V)$  are embedded disks intersecting at some point  $p = f_{t_0}(x) = f_{t_0}(y)$ . Note that the fact that  $f_t$  is injective for all  $t \in (0, t_0)$  gives that the intersection of  $f_t(U)$  and  $f_t(V)$  is tangential at  $p$ . Consider the two oriented geodesic rays  $\gamma_x = \{f_t(x)\}_{t \in [0, t_0]}$  and  $\gamma_y = \{f_t(y)\}_{t \in [0, t_0]}$ . Then the concatenation  $\gamma$  of  $\gamma_x$  with  $\gamma_y^{-1}$  is a smooth geodesic arc in  $N$  with length  $2t_0$  that meets  $\Sigma$  orthogonally at the points  $x$  and  $y$ , and the natural parameterization of  $\gamma$  is  $\exp_x(t\eta(x))$ ,  $t \in [0, 2t_0]$ . Let  $\Gamma: [0, 4t_0] \rightarrow N$  be the unit speed parameterization of the simple closed geodesic  $\gamma \cup \varphi(\gamma)$ , where  $\Gamma(0) = x$  and  $\Gamma|_{[0, 2t_0]}$  parameterizes  $\gamma$ ; here  $\varphi: N \rightarrow N$  is the order-two isometry containing  $\Sigma$  in its fixed point set  $\text{Fix}(\varphi)$ .

Note that each component of  $\text{Fix}(\varphi)$  is either a point, a proper arc or a proper surface in  $N$ . We will say that  $\Gamma$  is *orthogonal* to  $\text{Fix}(\varphi)$  at a time  $t_1 \in [0, 4t_0)$ , if  $\Gamma(t_1) \in \text{Fix}(\varphi)$  and  $\Gamma'(t_1)$  is a normal vector to the component of  $\text{Fix}(\varphi)$  containing  $\Gamma(t_1)$ ; we let  $\text{Fix}^\perp(\varphi) \subset [0, 4t_0)$  denote the set of all such times. By construction,  $\{x, y\} \subset \Gamma(\text{Fix}^\perp(\varphi))$ .

**Claim 5.3.**  $\text{Fix}^\perp(\varphi) = \{0, 2t_0\}$ . *In particular,  $\Gamma$  is a prime geodesic and it intersects  $\Sigma$  orthogonally only at the points  $x, y$ .*

*Proof.* Let  $[0, t_1]$  be the closure of the component of  $[0, 4t_0) \setminus \text{Fix}^\perp(\varphi)$  that contains 0. Then,  $\Gamma(t_1) = y$ , since otherwise the concatenation  $\tilde{\Gamma} = \Gamma|_{[0, t_1]}((\varphi \circ \Gamma)|_{[0, t_1]})^{-1}$  is a closed geodesic that does not intersect  $\Sigma$  orthogonally at  $y$ , which is a contradiction since the image of  $\Gamma$  equals the image of  $\tilde{\Gamma}$ . Moreover, the fact that  $t_1 \in \text{Fix}^\perp(\varphi)$  with  $\Gamma(t_1) = y \in \Sigma$  implies that  $\Gamma'(t_1) = \eta(y)$  or  $\Gamma'(t_1) = -\eta(y)$ . But, since  $[0, 4t_0) \mapsto \exp_x(t\eta(x))$  is a positive parameterization of the closed geodesic  $\Gamma$ , which satisfies  $\Gamma'(2t_0) = -\eta(y)$ , then  $\Gamma'(t_1) = -\eta(y)$ .

To finish the proof of the claim, just note that the above construction implies

$$f_{t_1/2}(x) = \exp_x((t_1/2)\eta(x)) = \exp_y((t_1/2)\eta(y)) = f_{t_1/2}(y);$$

thus,  $t_1/2 \geq t_0$ . But, on the other hand,  $t_1 \leq 2t_0$ , from where it follows  $t_1 = 2t_0$ .  $\square$

By Claim 5.3,  $\Gamma$  is a prime, closed geodesic in  $N$ , with length  $4t_0$  that intersects  $\Sigma$  orthogonally in exactly the two points  $x, y$ , proving item 3 of Lemma 5.2.  $\square$

Lemma 5.2 gives properly embedded, totally umbilic surfaces for small values of  $H$ . In order to finish the proof of Theorem 5.1, we apply a technical result stating that fundamental groups of hyperbolic 3-manifolds of finite volume satisfy the following definition.

**Definition 5.4** (Locally Extendable Residually Finite group). *A group  $G$  is called LERF if for every finitely generated subgroup  $K$  of  $G$  and any  $g \notin K$ , there exists a representation  $\sigma: G \rightarrow F$  from  $G$  to a finite group  $F$  such that  $\sigma(g) \notin \sigma(K)$ .*

The above definition can be extended as follows. *A group  $G$  is LERF if and only if for every finitely generated subgroup  $K$  of  $G$  and any finite subset  $\mathcal{F} = \{g_1, \dots, g_n\} \subset G$  such that  $\mathcal{F} \cap K = \emptyset$ , there exists a representation  $\sigma: G \rightarrow F$  from  $G$  to a finite group  $F$  such that  $\sigma(\mathcal{F}) \cap \sigma(K) = \emptyset$ . Indeed, if  $G$  is LERF and  $\mathcal{F}$  and  $K$  are as above, for each  $i \in \{1, \dots, n\}$ , there exists a representation  $\sigma_i: G \rightarrow F_i$  from  $G$  to a finite group  $F_i$  such that  $\sigma_i(g_i) \notin \sigma_i(K)$ . Let  $\sigma = \sigma_1 \times \dots \times \sigma_n: G \rightarrow F = F_1 \times \dots \times F_n$ , then  $\sigma$  is a representation as claimed. This equivalent definition will be used in the proof of Theorem 5.1.*

By a series of recent works (see [4] for a complete list of appropriate references) on group theoretical properties of fundamental groups of hyperbolic 3-manifolds, one has the following result.

**Theorem 5.5.** *If  $N$  is a hyperbolic 3-manifold of finite volume, then  $\pi_1(N)$  is LERF.*

The case where  $N$  is orientable is treated in [4, Corollary 4.2.3]. However, from the discussion in the book [4], it was not clear to us if Theorem 5.5 applies to the non-orientable case. For this reason, we next explain how this property follows from the orientable case. Recall, from [4], that a group  $G$  is LERF if and only if any finitely generated subgroup  $K$  of  $G$  is closed in the profinite topology of  $G$ . Let  $N$  be a non-orientable hyperbolic 3-manifold of finite volume and let  $\tilde{N}$  be its oriented 2-sheeted cover. Then  $\pi_1(\tilde{N})$  is LERF. Since  $\pi_1(\tilde{N})$  can be viewed as a finitely generated, index-2 subgroup of  $\pi_1(N)$ ,  $\pi_1(\tilde{N})$  is closed in the profinite topology of  $\pi_1(N)$ . Let  $K \subset \pi_1(N)$  be a finitely generated subgroup. There are two cases to consider: either  $K \subset \pi_1(\tilde{N})$ , in which case  $K$  is closed in  $\pi_1(N)$  since  $\pi_1(\tilde{N})$  is LERF and closed in  $\pi_1(N)$ , or  $K \not\subset \pi_1(\tilde{N})$ , and there exists some  $a \in K$ ,  $a \notin \pi_1(\tilde{N})$ ; it then follows that  $K = (K \cap \pi_1(\tilde{N})) \cup a(K \cap \pi_1(\tilde{N}))$  is the union of two closed sets, thus  $K$  is closed in  $\pi_1(N)$ . Therefore,  $\pi_1(N)$  is LERF.

With the above discussion in mind, we now continue with the proof of Theorem 5.1. Fix a surface  $S$  of finite topology, with negative Euler characteristic and  $T > 0$ . Our goal (precisely stated in Lemma 5.6 below) is to produce a hyperbolic 3-manifold  $N_T$ , together with a two-sided, proper embedding  $\tilde{f}: S \rightarrow N_T$  with totally geodesic image  $\tilde{\Sigma}$  such that, for each  $t \in (0, T]$ , the related parallel immersion  $\tilde{f}_t: S \rightarrow N_T$  is injective. Since the image surface  $\tilde{\Sigma}_t = \tilde{f}_t(S)$  is totally umbilic with mean curvature  $H = \tanh(t)$ ,  $\lim_{T \rightarrow \infty} \tanh(T) = 1$  and  $T$  is arbitrary, Lemma 5.6 proves Theorem 5.1.

By Theorem 4.1, there exists a hyperbolic 3-manifold of finite volume  $N$ , together with a two-sided proper embedding  $f: S \rightarrow N$ , with totally geodesic image  $\Sigma$  and an order-two isometry  $\varphi: N \rightarrow N$  such that  $\varphi|_{\Sigma} = \text{Id}_{\Sigma}$ . Using this particular example, we prove next lemma.

**Lemma 5.6.** *For each  $T > 0$ , there exists a finite Riemannian covering space  $\Pi_T: N_T \rightarrow N$  satisfying:*

- (1) *The embedding  $f: S \rightarrow N$  lifts to an embedding  $\tilde{f}: S \rightarrow N_T$  with image  $\tilde{\Sigma} = \tilde{f}(S)$ .*
- (2) *If  $e_1$  and  $e_2$  are distinct ends of  $\tilde{\Sigma}$ , then  $N_T$  contains disjoint cusp ends  $\mathcal{C}_1, \mathcal{C}_2$  such that, for  $i = 1, 2$ ,  $\tilde{\Sigma} \cap \mathcal{C}_i$  is an annular representative of  $e_i$ .*
- (3) *The order-two isometry  $\varphi: N \rightarrow N$  lifts to an order-two isometry  $\tilde{\varphi}: N_T \rightarrow N_T$  such that  $\tilde{\varphi}|_{\tilde{\Sigma}} = \text{Id}_{\tilde{\Sigma}}$ .*
- (4) *The  $t$ -parallel surfaces to  $\tilde{\Sigma}$  are embedded for all  $t \in (0, T]$ .*

*Proof.* Fix  $T > 0$ . Since  $N$  is a hyperbolic 3-manifold of finite volume, there are only a finite number of smooth, closed, prime geodesics in  $N$  with length less than  $L = 5T$ . In particular, the (possibly empty) collection  $\mathcal{G}_L$  of smooth, prime, closed geodesics in  $N$  that are orthogonal to  $\Sigma$  at two points and with length less than  $L$  is finite.

Note that if  $\mathcal{G}_L$  were empty, then Lemma 5.6 follows directly from item 3 of Lemma 5.2, by letting  $N_T = N$ ,  $\Pi_T = \text{Id}$ ,  $\tilde{f} = f$  and  $\tilde{\varphi} = \varphi$ . Thus, we next

assume that

$$\mathcal{G}_L = \{\gamma_1, \gamma_2, \dots, \gamma_{n(L)}\} \neq \emptyset.$$

For each  $j \in \{1, \dots, n(L)\}$ , let  $\{p_j, q_j\} \subset \Sigma \cap \gamma_j$  be the two points where  $\gamma_j$  meets  $\Sigma$  orthogonally. The geodesic  $\gamma_j \in \mathcal{G}_L$  is invariant under the isometry  $\varphi$ ; furthermore,  $\varphi|_{\gamma_j}$  reverses the orientation of  $\gamma_j$  and fixes the points  $p_j, q_j$ . Moreover, each geodesic  $\gamma_j$  is uniquely defined by either of the two points of orthogonal intersection, so if  $i \neq j$  it holds that  $\{p_i, q_i\} \cap \{p_j, q_j\} = \emptyset$ .

Let  $i: \Sigma \rightarrow N$  be the inclusion map. Choosing  $p_1$  as a base point, we let  $i_*: \pi_1(\Sigma, p_1) \rightarrow \pi_1(N, p_1)$  be the induced homomorphism on fundamental groups. We let

$$K_1 = i_*(\pi_1(\Sigma, p_1)) \subset \pi_1(N, p_1)$$

be the image of the finitely generated group  $\pi_1(\Sigma, p_1)$ . Since  $p_1 \in \gamma_1$ ,  $\gamma_1$  can be considered to represent a nontrivial element  $[\gamma_1] \in \pi_1(N, p_1)$ . Moreover, since  $\Sigma$  is totally geodesic in a hyperbolic 3-manifold,  $i_*$  is injective and, for any positive integer  $l$ ,  $[\gamma_1]^l \notin K_1$ .

Fix a positive integer  $k$  sufficiently large so that

$$(3) \quad k \cdot \text{Length}(\gamma_1) \geq L$$

and let  $\mathcal{F} = \{[\gamma_1], [\gamma_1]^2, \dots, [\gamma_1]^k\}$ . Then,  $\mathcal{F} \cap K_1 = \emptyset$  and, since  $\pi_1(N, p_1)$  is LERF, there exists a representation  $\sigma: \pi_1(N, p_1) \rightarrow F_1$ , from  $\pi_1(N, p_1)$  to a finite group  $F_1$ , such that  $\sigma(\mathcal{F}) \cap \sigma(K_1) = \emptyset$ . Let  $\widehat{K}_1, \widetilde{K}_1$  be the subgroups of  $\pi_1(N, p_1)$  defined by

$$\widehat{K}_1 = \sigma^{-1}(\sigma(K_1)), \quad \widetilde{K}_1 = \widehat{K}_1 \cap \varphi_*(\widehat{K}_1).$$

Note that  $K_1 \subset \widetilde{K}_1$ , since  $K_1 \subset \widehat{K}_1$  and  $\varphi_*$  fixes all elements of  $K_1$ . Also, for  $l \in \{1, 2, \dots, k\}$ ,  $[\gamma_1]^l \notin \widehat{K}_1$ , hence  $[\gamma_1]^l \notin \widetilde{K}_1$ .

Next we show that  $\widetilde{K}_1$  has finite index in  $\pi_1(N, p_1)$ . Since  $\widehat{K}_1 \supset \ker(\sigma)$  and  $F_1$  is finite, the index of  $\widehat{K}_1$  in  $\pi_1(N, p_1)$  is finite. Moreover,  $\varphi_*: \pi_1(N, p_1) \rightarrow \pi_1(N, p_1)$  is a group isomorphism; hence the index of  $\varphi_*(\widehat{K}_1)$  in  $\pi_1(N, p_1)$  is also finite. Then, as the intersection of two subgroups of finite index also has finite index, the claim follows.

Let  $\Pi_1: (N_1, q_1) \rightarrow (N, p_1)$  be the Riemannian covering space of  $(N, p_1)$  with image subgroup  $(\Pi_1)_*(\pi_1(N_1, q_1)) = \widetilde{K}_1$ . Note that  $\Pi_1$  is a finite covering, since the index of  $\widetilde{K}_1$  in  $\pi_1(N, p_1)$  is finite; in particular,  $N_1$  is a hyperbolic 3-manifold of finite volume and, if  $N$  is closed (resp. orientable),  $N_1$  is also closed (resp. orientable).

Since  $\varphi_*(\widetilde{K}_1) = \widetilde{K}_1$ , then, by the lifting criterion, the maps  $i: \Sigma \rightarrow N$ ,  $\varphi: N \rightarrow N$  have respective lifts

$$i_1: (\Sigma, p_1) \rightarrow (N_1, q_1), \quad \varphi_1: (N_1, q_1) \rightarrow (N_1, q_1).$$

Let  $\Sigma_1$  denote the embedded, totally geodesic image surface of the injective immersion  $i_1$ , and note that  $\Sigma_1$  is two-sided and contained in the fixed point set of the order-two isometry  $\varphi_1$ , which proves item 3 of Lemma 5.6. It is also straightforward to check that item 2 holds for  $\Sigma_1$ .

Consider the (possibly empty) collection  $\mathcal{G}_L^1 = \{\gamma_1^1, \gamma_2^1, \dots, \gamma_{n_1(L)}^1\}$  of smooth, prime closed geodesics in  $N_1$  that have length less than  $L$  and are orthogonal to  $\Sigma_1$  at two points. Then,  $\Pi_1$  induces an injective map from  $\mathcal{G}_L^1$  to  $\mathcal{G}_L$ , since  $\Pi_1|_{\Sigma_1}$  is a diffeomorphism from  $\Sigma_1$  to  $\Sigma$  and either of the two points for which a geodesic  $\gamma_i^1$  in  $\mathcal{G}_L^1$  intersects  $\Sigma_1$  orthogonally defines  $\gamma_i^1$ . Hence  $n_1(L) \leq n(L)$ . We next prove that this construction yields  $n_1(L) < n(L)$ .

**Claim 5.7.** *The image set of geodesics  $\Pi_1(\mathcal{G}_L^1) = \{\Pi_1(\gamma_j^1) \mid j = 1, \dots, n_1(L)\}$  forms a subset of  $\{\gamma_1, \gamma_2, \dots, \gamma_{n(L)}\}$ , which does not include  $\gamma_1$ . In particular,  $n_1(L) < n(L)$ .*

*Proof.* Arguing by contradiction, suppose, after possibly reordering, that  $\gamma_1 = \Pi_1(\gamma_1^1)$  as a set. Then,  $\gamma_1^1$  is the lift of a certain smallest power  $J$  of  $\gamma_1$ , which implies that the length of  $\gamma_1^1$  is equal to  $J \cdot \text{Length}(\gamma_1)$ . However, for  $l \in \{1, \dots, k\}$ ,  $[\gamma_1]^l \notin \tilde{K}_1$ ; hence, none of the powers of  $\gamma_1$  less than or equal to  $k$  lift. Then  $J > k$ , and (3) implies that the length of  $\gamma_1^1$  is larger than  $L$ , which is a contradiction.  $\square$

By induction, Claim 5.7 allows us to produce a finite Riemannian cover  $\Pi_T: N_T \rightarrow N$  satisfying:

- (1) The embedding  $f: S \rightarrow N$  lifts to an embedding  $\tilde{f}: S \rightarrow N_T$ , with totally geodesic image surface  $\tilde{\Sigma}$ .
- (2) There are no prime closed geodesics in  $N_T$  with length less than  $L$  and intersecting  $\tilde{\Sigma}$  orthogonally at exactly two points.
- (3) Item 2 of Lemma 5.6 holds for  $\tilde{\Sigma}$ .
- (4)  $\varphi: N \rightarrow N$  lifts to an order-two isometry  $\tilde{\varphi}: N_T \rightarrow N_T$  and  $\tilde{\varphi}|_{\tilde{\Sigma}} = \text{Id}_{\tilde{\Sigma}}$ .

Note that  $\tilde{f}$  and  $\tilde{\varphi}$  satisfy the hypothesis of Lemma 5.2. In particular, since  $\tilde{\Sigma}$  was constructed in such a way that there are no prime closed geodesics in  $N_T$  with length less than  $L = 5T$  intersecting  $\tilde{\Sigma}$  orthogonally at two points, item 3 of Lemma 5.2 implies that the  $t$ -parallel surfaces to  $\tilde{\Sigma}$  are embedded for all  $t \in (0, T]$ , which completes the proof of Lemma 5.6.  $\square$

In the above construction, if  $t \in (0, T]$ , the  $t$ -parallel surface  $\tilde{\Sigma}_t = \tilde{f}_t(S)$  is a properly embedded, totally umbilic surface in  $N_T$  satisfying the conditions 1 and 2 of Theorem 5.1 for  $H = \tanh(t) \in (0, \tanh(T)]$ . Hence, Theorem 5.1 follows from Theorem 4.1 and Lemma 5.6, since  $T$  is arbitrary and  $\lim_{T \rightarrow \infty} \tanh(T) \nearrow 1$ .

**Remark 5.8.** If  $f: S \rightarrow N$  satisfies the hypothesis of Theorem 4.1, then, for each  $t > 0$ , the construction of the  $t$ -parallel surfaces to  $\Sigma = f(S)$ , as presented in the proof of Theorem 5.1, yields a pair of totally umbilic surfaces  $\Sigma_t, \Sigma_{-t}$ , corresponding to the two possible orientations of the normal bundle of  $\Sigma$ . Then, since there is an order-two isometry  $\varphi: N \rightarrow N$  such that  $\Sigma$  is a component of  $\text{Fix}(\varphi)$  and  $\Sigma_t = \varphi(\Sigma_{-t})$ ,  $\Sigma_t$  is embedded if and only if  $\Sigma_{-t}$  is embedded. Furthermore, for all  $t > 0$ , it holds that  $\Sigma_t$  and  $\Sigma_{-t}$  are two distinct totally umbilic surfaces in  $N$ , since an elementary argument shows that if  $\Sigma_{t_1} = \Sigma_{-t_1}$  for some  $t_1 > 0$ , then  $\Sigma_t = \Sigma_{-t}$  for all  $t > 0$ , contradicting the fact that  $\Sigma$  is two-sided. Hence, for any  $H \in (0, 1)$ , the proof of Theorem 5.1 actually gives a pair of properly embedded,

totally umbilic surfaces in a hyperbolic 3-manifold of finite volume mean curvature  $H$ .

## REFERENCES

- [1] C. Adams. Thrice-punctured spheres in hyperbolic 3-manifolds. *Trans. Amer. Math. Soc.*, 287(2):645–656, 1985. DOI 10.2307/1999666, <https://doi.org/10.2307/1999666>, MR768730.
- [2] C. Adams, C. Albors-Riera, B. Haddock, Z. Li, D. Nishida, B. Reinoso, and L. Wang. Hyperbolicity of links in thickened surfaces. *Topology Appl.*, 256:262–278, 2019. DOI 10.1016/j.topol.2019.01.022, <https://doi.org/10.1016/j.topol.2019.01.022>, MR3916014.
- [3] C. Adams, W. H. Meeks III, and A. Ramos. Modifications preserving hyperbolicity of link complements. Preprint at <https://arxiv.org/abs/2007.01739>, to appear in *Journal of Algebraic and Geometric Topology*.
- [4] M. Aschenbrenner, S. Friedl, and H. Wilton. *3-manifold Groups*. EMS series of lectures in mathematics. European Mathematical Society, 2015. ISBN 9783037191545.
- [5] P. Collin, L. Hauswirth, and H. Rosenberg. Minimal surfaces in finite volume hyperbolic 3-manifolds and in  $M \times S$ ,  $M$  a finite area hyperbolic surface. *Amer. J. Math.*, 140(4):1075–1112, 2018. DOI 10.1353/ajm.2018.0024, <https://doi.org/10.1353/ajm.2018.0024>, MR3828041.
- [6] J. Howie and J. Purcell. Geometry of alternating links on surfaces. *Trans. Amer. Math. Soc.*, 373(4):2349–2397, 2020. DOI 10.1090/tran/7929, <https://doi.org/10.1090/tran/7929>, MR4069222.
- [7] W. H. Meeks III and A. K. Ramos. Properly immersed surfaces in hyperbolic 3-manifolds. *J. Differential Geom.*, 112(2):233–261, 2019. DOI 10.4310/jdg/1559786424, <https://doi.org/10.4310/jdg/1559786424>, MR3960267.
- [8] W. Menasco and A. W. Reid. Totally geodesic surfaces in hyperbolic link complements. In *Topology '90 (Columbus, OH, 1990)*, volume 1 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 215–226. de Gruyter, Berlin, 1992.
- [9] T. Payne. Closures of totally geodesic immersions into locally symmetric spaces of noncompact type. *Proc. Amer. Math. Soc.*, 127(3):829–833, 1999. MR1468202, Zbl 0936.53034, DOI 10.1090/S0002-9939-99-04552-9, <https://doi.org/10.1090/S0002-9939-99-04552-9>.
- [10] M. Ratner. Raghunathan’s topological conjecture and distributions of unipotent flows. *Duke Math. J.*, 63(1):235–280, 1991. MR1106945, Zbl 0733.22007, DOI 10.1215/S0012-7094-91-06311-8, <https://doi.org/10.1215/S0012-7094-91-06311-8>.
- [11] N. Shah. Closures of totally geodesic immersions in manifolds of constant negative curvature. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 718–732. World Sci. Publ., River Edge, NJ, 1991. MR1170382, Zbl 0846.53041.
- [12] M. Spivak. *A comprehensive introduction to differential geometry*. Publish or Perish Inc., second edition, 1979. MR532834.
- [13] W. P. Thurston. *The geometry and topology of three manifolds*. Princeton Lecture Notes.

Colin Adams at [cadams@williams.edu](mailto:cadams@williams.edu)  
 Department of Mathematics and Statistics  
 Williams College, Williamstown, MA 01267

William H. Meeks, III at [profmeeks@gmail.com](mailto:profmeeks@gmail.com)  
 Mathematics Department, University of Massachusetts  
 Amherst, MA 01003

Álvaro K. Ramos at [alvaro.ramos@ufrgs.br](mailto:alvaro.ramos@ufrgs.br)  
 Departamento de Matemática Pura e Aplicada  
 Universidade Federal do Rio Grande do Sul, Brazil