

SCARCITY OF CONGRUENCES FOR THE PARTITION FUNCTION

SCOTT AHLGREN, OLIVIA BECKWITH, AND MARTIN RAUM

ABSTRACT. The arithmetic properties of the ordinary partition function $p(n)$ have been the topic of intensive study for the past century. Ramanujan proved that there are linear congruences of the form $p(\ell n + \beta) \equiv 0 \pmod{\ell}$ for the primes $\ell = 5, 7, 11$, and it is known that there are no others of this form. On the other hand, for every prime $\ell \geq 5$ there are infinitely many examples of congruences of the form $p(\ell Q^m n + \beta) \equiv 0 \pmod{\ell}$ where $Q \geq 5$ is prime and $m \geq 3$. This leaves open the question of the existence of such congruences when $m = 1$ or $m = 2$ (no examples in these cases are known). We prove in a precise sense that such congruences, if they exist, are exceedingly scarce. Our methods involve a careful study of modular forms of half integral weight on the full modular group which are related to the partition function. Among many other tools, we use work of Radu which describes expansions of such modular forms along square classes at cusps of the modular curve $X(\ell Q)$, Galois representations and the arithmetic large sieve.

1. INTRODUCTION

The partition function $p(n)$ counts the number of ways to represent the positive integer n as the sum of a non-increasing sequence of positive integers. By convention we agree that $p(0) := 1$ and that $p(n) := 0$ if $n \notin \{0, 1, 2, \dots\}$. This function has been long-studied in combinatorics and number theory. Ramanujan proved the famous congruences

$$p(\ell n + \beta_\ell) \equiv 0 \pmod{\ell} \quad \text{for } \ell = 5, 7, 11, \tag{1.1}$$

where $\beta_\ell := \frac{1}{24} \pmod{\ell}$ (when we speak of such a congruence we mean that the partition values vanish modulo ℓ for all integers n). Much of the interest in such congruences arises from the relationship between the partition function and Dedekind's eta function. These congruences are a prototypical example of arithmetic phenomena which occur for a wide class of weakly holomorphic modular forms.

Ramanujan conjectured extensions of these results for arbitrary powers of these primes. Proofs in the cases $\ell = 5, 7$ are attributed to Ramanujan and Watson [Ram21, Ram20, Ram19, Wat38]; the case $\ell = 11$ is much more difficult and was resolved by Atkin [Atk67]. Several decades later, examples of congruences for primes $\ell \leq 31$ were found by Newman, Atkin, and O'Brien [New57, AO67, Atk68a]. These examples are not as simple as (1.1); they take the form

$$p(\ell Q^m n + \beta) \equiv 0 \pmod{\ell}, \tag{1.2}$$

where Q is a prime distinct from ℓ . In Ono's groundbreaking work [Ono00] it was shown for each prime $\ell \geq 5$ that there are infinitely many primes Q for which (1.2) holds with $m = 4$. This result was subsequently generalized in several directions in the case of the partition

Date: November 2, 2022.

The first author was supported by a grant from the Simons Foundation (#426145 to Scott Ahlgren). The third author was partially supported by Vetenskapsrådet Grant 2019-03551.

function [Ahl00, AO01b] and for general weakly holomorphic modular forms [Tre06, Tre08]. For a more complete history, one may consult for example [BO99, AO01a, AO01c].

While there is a long literature proving that congruences of the form (1.2) exist, there are fewer results on the non-existence of congruences. Ramanujan [Ram20] speculated that the congruences (1.1) are the only examples of the form $p(\ell n + \beta) \equiv 0 \pmod{\ell}$. Kiming and Olsson [KO92] proved that any example must have $\beta \equiv \frac{1}{24} \pmod{\ell}$. Ramanujan's speculation was later confirmed by the first author and Boylan [AB03]. Radu [Rad12] confirmed an old conjecture of Subbarao by proving that there are no congruences of the form

$$p(mn + \beta) \equiv 0 \pmod{\ell} \quad \text{with } \ell = 2, 3.$$

In later work [Rad13], Radu confirmed a conjecture of the first author and Ono by proving that if there is a congruence

$$p(mn + \beta) \equiv 0 \pmod{\ell} \quad \text{with } \ell \geq 5 \text{ prime,}$$

then $\ell \mid m$ and $(\frac{1-24\beta}{\ell}) \in \{0, -1\}$; these results have been generalized to a wide class of weakly holomorphic modular forms and mock theta functions [AK15, And14].

After Ono's work, we know that for any prime $\ell \geq 5$ there are infinitely many primes Q for which there is a congruence of the form

$$p(\ell Q^4 n + \beta) \equiv 0 \pmod{\ell}.$$

Atkin [Atk68a] discovered congruences of the form

$$p(\ell Q^3 n + \beta) \equiv 0 \pmod{\ell}. \tag{1.3}$$

For $\ell = 5, 7$ and 13 , he proved that such congruences exist for infinitely many Q ; for example, there are 30 values of β which give rise to a congruence

$$p(13 \cdot 11^3 n + \beta) \equiv 0 \pmod{13}.$$

Atkin describes a method to produce such congruences which requires "accidental" values of certain Hecke eigenvalues, and he gives examples of congruences (1.3) for $\ell \leq 31$. Using this method, Weaver [Wea01] and Johansson [Joh12] found many more examples (now more than 22 billion) for these primes ℓ . In recent work [AAT21] the first author, Allen and Tang have shown that there are infinitely many congruences of the form (1.3) for every prime $\ell \geq 5$.

On the other hand, we know from [AB03] that there are no congruences

$$p(\ell n + \beta) \equiv 0 \pmod{\ell} \quad \text{with } \ell \geq 13.$$

In view of these results, it is natural to ask if there are any congruences of the form

$$p(\ell Q n + \beta_{\ell, Q}) \equiv 0 \pmod{\ell} \tag{1.4}$$

or

$$p(\ell Q^2 n + \beta_{\ell, Q}) \equiv 0 \pmod{\ell}. \tag{1.5}$$

For every Q such congruences occur for $\ell = 5, 7$, or 11 because of the original Ramanujan congruences. However, no other example of congruences (1.4), (1.5) is known (and we will show below that the first example, if it exists, would involve very large values of ℓ and Q).

It is therefore natural to speculate that there are no such congruences outside of the trivial examples arising from (1.1). The goal of this paper is to prove that such congruences, if they exist, are extremely scarce.

To state the first result, suppose that $\ell \geq 5$ and $Q \geq 5$ are primes with $Q \neq \ell$. As mentioned above, Radu [Rad13] proved that (1.4) can occur only when

$$\left(\frac{1-24\beta_{\ell,Q}}{\ell}\right) \in \{0, -1\}.$$

We begin by stating one of our main results in terms of the partition function (a stronger version is given below in Theorem 1.4). The theorem shows that either congruences modulo ℓ are scarce, or that values of $p(n)$ which are not divisible by ℓ are themselves scarce (as described in (1.6)). It is most useful to consider this in the context of (1.8), which relates the partition values in (1.6) to the coefficients of a modular form of half-integral weight. In particular, we would have (1.6) if the form $f_{\ell,\delta}$ which is defined in (1.7) below were congruent modulo ℓ to an iterated derivative of a theta series; this is explained in more detail in the remark following Theorem 1.4.

Theorem 1.1. *Suppose that $\ell \geq 5$ is prime, and fix $\delta \in \{0, -1\}$. Let S be the set of primes Q for which there exists a congruence of the form (1.4) with $\left(\frac{1-24\beta_{\ell,Q}}{\ell}\right) = \delta$. Then one of the following is true.*

- (1) S has density zero, or
- (2) we have

$$\#\left\{n \leq X : \left(\frac{-n}{\ell}\right) = \delta, \quad p\left(\frac{n+1}{24}\right) \not\equiv 0 \pmod{\ell}\right\} \ll \sqrt{X} \log X. \quad (1.6)$$

Using the statement (1.10) below, we are able to obtain the following corollary.

Corollary 1.2. *Suppose that $17 \leq \ell < 10,000$. Let S be the set of primes Q for which we have a congruence*

$$p(\ell Q n + \beta_{\ell,Q}) \equiv 0 \pmod{\ell}.$$

Then S has density zero.

Remark. In the case $\ell = 13$, statement (1.10), which we falsify to derive Corollary 1.2, holds for $\left(\frac{1-24\beta_{\ell,Q}}{\ell}\right) = \delta = -1$ by Atkin's work [Atk68b]. When $\ell = 13$ and $\delta = 0$, the set S has density zero.

We are able to show that there are no congruences (1.4) for a large range of ℓ and Q . The method of computation relies heavily on results of Radu [Rad13] and requires computing relatively few values of the partition function. In particular, we have

Theorem 1.3. *Apart from the cases arising from the Ramanujan congruences (1.1), there are no congruences (1.4) for $\ell < 1,000$ and $Q < 10^{13}$, and for $\ell < 10,000$ and $Q < 10^9$.*

Remark. If Q is any positive integer, then by Theorem 4.3 and Lemma 4.5 of [Rad13], the non-existence of a congruence with modulus ℓQ implies the non-existence of congruences with modulus $\ell Q Q'$ for many integers Q' .

Much of the interest in the arithmetic properties of the partition function arises from its connection to modular forms. The Dedekind eta function is defined for τ in the upper complex half-plane \mathbb{H} by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

with the standard notation $q := e^{2\pi i\tau}$. This is a modular form of weight $\frac{1}{2}$ on the full modular group (details are given in the next section) which is connected to $p(n)$ via Euler's formula

$$\frac{1}{\eta(\tau)} = \sum p\left(\frac{n+1}{24}\right) q^{\frac{n}{24}} = q^{-\frac{1}{24}} + q^{\frac{23}{24}} + 2q^{\frac{47}{24}} + \dots$$

Ramanujan's congruences may be more succinctly written in the form

$$\frac{1}{\eta(\tau)} \Big| U_\ell = \sum p\left(\frac{\ell n+1}{24}\right) q^{\frac{n}{24}} \equiv 0 \pmod{\ell} \quad \text{for } \ell = 5, 7, 11,$$

where U_ℓ is the standard operator defined in Lemma 2.1.

Our results are stated most naturally in terms of a family of modular forms. If $k \in \frac{1}{2}\mathbb{Z}$, N is a positive integer, and ν is a multiplier system on $\Gamma_0(N)$ in weight k , then we denote by $S_k(N, \nu)$ the space of modular forms of weight k and multiplier ν on $\Gamma_0(N)$ (details will be given in the next section). With this notation, we have

$$\eta \in S_{\frac{1}{2}}(1, \nu_\eta),$$

where ν_η is the multiplier described in (2.2).

For $\delta \in \{0, -1\}$, there is a modular form

$$f_{\ell, \delta} \in \begin{cases} S_{\frac{\ell^2-2\ell}{2}}(1, \nu_\eta^{-1}) & \text{if } \delta = 0, \\ S_{\frac{\ell^2-2}{2}}(1, \nu_\eta^{-1}) & \text{if } \delta = -1, \end{cases} \quad (1.7)$$

with

$$f_{\ell, \delta} = \sum a_{\ell, \delta}(n) q^{\frac{n}{24}} \equiv \sum_{\left(\frac{-n}{\ell}\right) = \delta} p\left(\frac{n+1}{24}\right) q^{\frac{n}{24}} \pmod{\ell}. \quad (1.8)$$

Remark. We have defined the modular forms in this manner for ease of exposition throughout the paper. When $\delta = 0$, the relevant values of the partition function are in fact captured by a modular form of lower weight. In particular, defining

$$F_\ell := f_{\ell, 0} \Big| U_\ell \in S_{\frac{\ell-2}{2}}(1, \nu_\eta^{-\ell}),$$

we have

$$F_\ell \equiv \sum p\left(\frac{\ell n+1}{24}\right) q^{\frac{n}{24}} \pmod{\ell}.$$

For each prime $Q \geq 5$ we have a Hecke operator T_{Q^2} (defined in (2.11)) on the space $S_k(1, \nu_\eta^{-1})$. The following result implies Theorem 1.1 above.

Theorem 1.4. *Suppose that $\ell \geq 5$ is prime, and fix $\delta \in \{0, -1\}$. Let S be the set of primes Q for which there exists a congruence of the form (1.4) with $\left(\frac{1-24\beta_{\ell, Q}}{\ell}\right) = \delta$. Then one of the following is true.*

- (1) S has density zero, or
- (2) we have

$$\#\{n \leq X : a_{\ell, \delta}(n) \not\equiv 0 \pmod{\ell}\} \ll \sqrt{X} \log X \quad (1.9)$$

and

$$f_{\ell, \delta} \Big| T_{Q^2} \equiv 0 \pmod{\ell} \quad \text{for all primes } Q \equiv -1 \pmod{\ell}. \quad (1.10)$$

Remark. By work of Kiming-Olsson [KO92] we know that $f_{\ell,-1} \not\equiv 0 \pmod{\ell}$, and by work of the first author with Boylan [AB03] we know that $f_{\ell,0} \not\equiv 0 \pmod{\ell}$ if $\ell \geq 13$. Apart from the three cases in which these modular forms vanish $\pmod{\ell}$, a result of Bellaïche, Green and Soundararajan [BGS18] (which improves previous results [AB08], [Ahl99] by a log-factor) implies that we have the lower bound

$$\#\{n \leq X : a_{\ell,\delta}(n) \not\equiv 0 \pmod{\ell}\} \gg \frac{\sqrt{X}}{\log \log X}.$$

This is a natural barrier in this setting due to the presence of modular forms which are congruent to theta functions and their derivatives.

Remark. In terms of the partition function, condition (1.10) is equivalent to the statement that for all n and Q with $\left(\frac{-n}{\ell}\right) = \delta$ and $Q \equiv -1 \pmod{\ell}$, we have

$$p\left(\frac{Q^2n+1}{24}\right) + Q^{-2}\left(\frac{-12n}{Q}\right)p\left(\frac{n+1}{24}\right) + Q^{-3}p\left(\frac{\frac{n}{Q^2}+1}{24}\right) \equiv 0 \pmod{\ell} \quad (1.11)$$

(this can be seen from a computation involving the weights of the modular forms $f_{\ell,\delta}$).

The next main result gives a strong necessary condition for the existence of a congruence (1.4). It is a consequence of more general results obtained for squarefree modulus Q in Section 3 below. Here V_Q is the standard operator defined in Lemma 2.1.

Theorem 1.5. *Suppose that $\ell \geq 5$ and $Q \geq 5$ are primes with $Q \neq \ell$.*

- (1) *There are no congruences of the form (1.4) with $\left(\frac{24\beta_{\ell,Q}-1}{Q}\right) = 0$.*
- (2) *Fix $\delta \in \{0, -1\}$ and $\varepsilon \in \{\pm 1\}$. If there is a congruence of the form (1.4) with*

$$\left(\frac{1-24\beta_{\ell,Q}}{\ell}\right) = \delta \quad \text{and} \quad \left(\frac{24\beta_{\ell,Q}-1}{Q}\right) = \varepsilon,$$

then

$$f_{\ell,\delta}|U_Q \equiv -\varepsilon\left(\frac{-12}{Q}\right)Q^{-1}f_{\ell,\delta}|V_Q \pmod{\ell}. \quad (1.12)$$

Remark. In terms of the partition function, (1.12) can be expressed in the form

$$\sum_{\left(\frac{-n}{\ell}\right)=\delta} p\left(\frac{Qn+1}{24}\right)q^{\frac{n}{24}} \equiv -\varepsilon\left(\frac{-12}{Q}\right)Q^{-1} \sum_{\left(\frac{-n}{\ell}\right)=\delta} p\left(\frac{n+1}{24}\right)q^{\frac{Qn}{24}} \pmod{\ell}.$$

Remark. We can also deduce in Theorem 1.5 that $f_{\ell,\delta}$ is an eigenform $\pmod{\ell}$ of the Hecke operator T_{Q^2} , with eigenvalue

$$-\varepsilon\left(\frac{12}{Q}\right)(Q^{-1} + Q^{-2}) \pmod{\ell}.$$

However, this statement is much weaker than (1.12).

Our last theorem gives necessary and sufficient conditions for the existence of congruences (1.5). The statement involves the twist $f_{\ell,\delta} \otimes \chi_Q$, which is defined in (2.10).

Theorem 1.6. *Suppose that $\ell \geq 5$ and $Q \geq 5$ are primes with $Q \neq \ell$.*

- (1) *If $Q^2 \mid (24\beta_{\ell,Q} - 1)$ then the only congruences of the form (1.5) arise from the three Ramanujan congruences.*
- (2) *If $(Q, 24\beta_{\ell,Q} - 1) = 1$ then we have (1.5) if and only if (1.4).*

(3) Fix $\delta \in \{0, -1\}$. If $Q \parallel (24\beta_{\ell, Q} - 1)$ then we have a congruence (1.5) with $\left(\frac{1-24\beta_{\ell, Q}}{\ell}\right) = \delta$ if and only if both of the following are true:

$$f_{\ell, \delta}|U_{Q^2} \equiv \left(\frac{-12}{Q}\right) Q^{-1} f_{\ell, \delta} \otimes \chi_Q + Q^{-2} f_{\ell, \delta}|V_{Q^2} \pmod{\ell} \quad (1.13)$$

and

$$f_{\ell, \delta}|U_Q \equiv f_{\ell, \delta}|U_{Q^2} V_Q \pmod{\ell}. \quad (1.14)$$

Using Theorem 1.6, we are able to prove that there are no congruences (1.5) for small values of ℓ and Q . In particular, we have the following

Corollary 1.7. *There are no congruences (1.5) for $17 \leq \ell < 1,000$ and $5 \leq Q < 10,000$.*

Remark. We verified that there are no congruences (1.5) for $\ell = 13$ and $5 \leq Q < 10,000$ if $\left(\frac{1-24\beta_{\ell, Q}}{\ell}\right) = \delta = 0$. The case $\delta = -1$ evades our obstructions in analogy to the situation in Corollary 1.2.

We give a broad sketch of the arguments which we use to prove these results. The arguments are technical in many places, and they rely heavily on the theory of modular forms for powers of the eta-multiplier which is described in the next section. We remark that many of our results will extend to a wider class of weakly holomorphic modular forms; here we have focused on the prototypical case of the partition function since the technical difficulties which arise are already formidable.

In Section 3 we prove a generalization of Theorem 1.5 with squarefree modulus Q . An important result of Radu (Theorem 3.1 below) shows that congruences (1.4) are stable on square-classes of the parameter $1 - 24\beta_{\ell, Q}$. This allows us to relate the existence of a single congruence (1.4) to properties of the modular forms $f_{\ell, \delta}$. If there is a congruence (1.4), then for each β along a square-class, we use another result of Radu to compute the expansion of the weakly holomorphic modular form $g_{\beta} = q^{\frac{24\beta-1}{24\ell Q}} \sum p(\ell Q n + \beta) q^n$ at a particular cusp of $X(\ell Q)$. Theorem 1.5 follows from applying the q -expansion principle as described in the next section and assembling the contributions from each β .

Section 4, which contains the proof of Theorem 1.4, is the heart of the paper. Let S denote the set of primes for which there is a congruence. Using Theorem 1.5 we identify two possible conditions which may hold at each member of a set of auxiliary primes. If the first condition holds for infinitely many primes, we are able to conclude that S has density zero. If the second holds for all but finitely many primes then a delicate argument involving Galois representations and the arithmetic large sieve shows that (1.9) holds.

In Section 5 we prove Theorem 1.6 with methods which are similar to those in Section 3. Here we are able to give necessary and sufficient conditions for the existence of a congruence. Finally, Section 6 describes the computations which lead to Corollary 1.2, Theorem 1.3, and Corollary 1.7.

ACKNOWLEDGMENTS

We thank Nickolas Andersen, Alexander Dunn, Kevin Ford, Olav Richter and Will Sawin for their helpful comments.

2. BACKGROUND

The Dedekind eta function and the theta function are defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{and} \quad \theta(\tau) := \sum_{n=-\infty}^{\infty} q^{n^2},$$

where we use the notation

$$q := e(\tau) = e^{2\pi i \tau}, \quad \tau \in \mathbb{H}.$$

The eta function is a modular form of weight $\frac{1}{2}$ on $\mathrm{SL}_2(\mathbb{Z})$; in particular, there is a multiplier ν_η with

$$\eta(\gamma\tau) = \nu_\eta(\gamma)(c\tau + d)^{\frac{1}{2}} \eta(\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \quad (2.1)$$

Here and throughout, we choose the principal branch of the square root. For $c > 0$ we have the explicit formula [Kno70, §4.1]:

$$\nu_\eta(\gamma) = \begin{cases} \left(\frac{d}{c}\right) e\left(\frac{1}{24}((a+d)c - bd(c^2 - 1) - 3c)\right), & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right) e\left(\frac{1}{24}((a+d)c - bd(c^2 - 1) + 3d - 3 - 3cd)\right), & \text{if } c \text{ is even.} \end{cases} \quad (2.2)$$

The multiplier for the theta function is given by

$$\nu_\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (c\tau + d)^{-\frac{1}{2}} \frac{\theta(\gamma\tau)}{\theta(\tau)} = \left(\frac{c}{d}\right) \varepsilon_d^{-1},$$

where

$$\varepsilon_d = \begin{cases} 1, & d \equiv 1 \pmod{4}, \\ i, & d \equiv 3 \pmod{4}. \end{cases}$$

For odd values of d, d_1, d_2 we have the useful formulas

$$e\left(\frac{1-d}{8}\right) = \left(\frac{2}{d}\right) \varepsilon_d \quad \text{and} \quad \varepsilon_{d_1 d_2} = \varepsilon_{d_1} \varepsilon_{d_2} (-1)^{\frac{d_1-1}{2} \frac{d_2-1}{2}}. \quad (2.3)$$

Define the Gauss sum for odd d by

$$G(a, d) := \sum_{n \pmod{d}} \left(\frac{n}{d}\right) e\left(\frac{an}{d}\right).$$

Write $G(d) = G(1, d)$ for simplicity and recall that $G(a, d) = \left(\frac{a}{d}\right) G(d)$ if $(a, d) = 1$. For d odd and squarefree we have the evaluation

$$G(d) = \varepsilon_d \sqrt{d}. \quad (2.4)$$

Following Shimura [Shi73] let G be the group of pairs $[\alpha, \phi(\tau)]$ where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ and ϕ is a holomorphic function on \mathbb{H} with $\phi(\tau)^2 = t(\det \alpha)^{-\frac{1}{2}}(c\tau + d)$, where $|t| = 1$. The group operation is given by

$$[\alpha, \phi(\tau)] \cdot [\beta, \rho(\tau)] = [\alpha\beta, \phi(\beta\tau)\rho(\tau)].$$

For $k \in \frac{1}{2}\mathbb{Z}$, G acts on holomorphic functions f on \mathbb{H} by

$$(f|_k[\alpha, \phi(\tau)])(\tau) := \phi(\tau)^{-2k} f(\alpha\tau).$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we define $\gamma^* := [\gamma, (c\tau + d)^{\frac{1}{2}}] \in G$.

Throughout the paper, $\ell \geq 5$ will denote a fixed prime number. Given $k \in \frac{1}{2}\mathbb{Z}$, a positive integer N , and a multiplier system ν on $\Gamma_0(N)$, we denote by $M_k(N, \nu)$, $S_k(N, \nu)$, and $M_k^!(N, \nu)$ the spaces of modular forms, cusp forms, and weakly holomorphic modular forms of weight k and multiplier ν on $\Gamma_0(N)$ whose Fourier coefficients are algebraic numbers which are integral at all primes above ℓ . Forms in these spaces satisfy the transformation law

$$f|_k \gamma^* = \nu(\gamma)f \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

as well as the appropriate conditions at the cusps of $\Gamma_0(N)$ (weakly holomorphic forms are allowed poles at the cusps). We assume familiarity with the situation when $\nu = \chi\nu_\theta^r$, where $r \in \mathbb{Z}$ and χ is a Dirichlet character.

We will be mostly concerned with the spaces $M_k(N, \chi\nu_\eta^r)$ where $(r, 24) = 1$ and χ is a Dirichlet character modulo N , and we summarize some of their important properties. If $f \in M_k(N, \chi\nu_\eta^r)$, then $\eta^{-r}f \in M_{k-\frac{r}{2}}^!(N, \chi)$. It follows that f has a Fourier expansion of the form

$$f = \sum_{n \equiv r \pmod{24}} a(n)q^{\frac{n}{24}}. \quad (2.5)$$

We also see that

$$M_k(N, \chi\nu_\eta^r) = \{0\} \quad \text{unless } 2k - r \equiv 1 - \chi(-1) \pmod{4}. \quad (2.6)$$

In particular, the assumption that $(r, 24) = 1$ implies that $k \notin \mathbb{Z}$.

We will frequently make use of the U and V operators, whose properties are summarized in the next lemma.

Lemma 2.1. *Suppose that $(r, 24) = 1$, that $f \in M_k(N, \chi\nu_\eta^r)$ has Fourier expansion (2.5), and that m is a positive integer. Define*

$$f|U_m := \sum a(mn)q^{\frac{n}{24}} \quad \text{and} \quad f|V_m := \sum a(n)q^{\frac{mn}{24}}.$$

Then

$$f|U_m = m^{\frac{k}{2}-1} \sum_{v(m)} f|_k \left[\begin{pmatrix} 1 & 24v \\ 0 & m \end{pmatrix}, m^{\frac{1}{4}} \right] \quad \text{and} \quad f|V_m = m^{-\frac{k}{2}} f|_k \left[\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}, m^{-\frac{1}{4}} \right].$$

Proof. Letting ζ_m denote a primitive m th root of unity, the sum over $v \pmod{m}$ becomes

$$m^{-1} \sum_n a(n)q^{\frac{n}{24m}} \sum_{v(m)} \zeta_m^{vn} = \sum_n a(mn)q^{\frac{n}{24}}. \quad \square$$

From a computation involving (2.2) and (2.3) it follows that for $(r, 24) = 1$ we have

$$f \in M_k(N, \chi\nu_\eta^r) \implies f|V_{24} \in M_k(576N, \chi\left(\frac{12}{\bullet}\right)\nu_\theta^r). \quad (2.7)$$

If M is an odd positive squarefree integer, denote by $\chi_M = \left(\frac{\bullet}{M}\right)$ the quadratic character of modulus M . Using Lemma 2.1 and (2.2) it can be checked (via a somewhat tedious calculation which relies on (2.3)) that if $(r, 24) = 1$ and $Q \geq 5$ is prime, then

$$U_Q: M_k(N, \chi\nu_\eta^r) \longrightarrow M_k\left(N\frac{Q}{(N, Q)}, \chi\chi_Q\nu_\eta^{Qr}\right), \quad (2.8)$$

$$V_Q: M_k(N, \chi\nu_\eta^r) \longrightarrow M_k(NQ, \chi\chi_Q\nu_\eta^{Qr}). \quad (2.9)$$

If $Q \geq 5$ is prime and $(r, 24) = 1$, we define the twist of $f \in M_k(N, \chi\nu_\eta^r)$ with Fourier expansion (2.5) by

$$f \otimes \chi_Q := \sum \chi_Q(n) a(n) q^{\frac{n}{24}}. \quad (2.10)$$

(this normalization, which disregards the denominator of the exponents, is chosen for ease of notation in the proofs). We have

$$f \otimes \chi_Q = \frac{1}{G(Q)} \sum_{v(Q)} \chi_Q(v) f|_k \left[\begin{pmatrix} 1 & \frac{24v}{Q} \\ 0 & 1 \end{pmatrix}, 1 \right],$$

from which a computation as above gives

$$f \otimes \chi_Q \in M_k(NQ^2, \chi\nu_\eta^r).$$

For each prime $Q \geq 5$ we have the Hecke operator

$$T_{Q^2} : S_k(1, \nu_\eta^r) \rightarrow S_k(1, \nu_\eta^r).$$

If $f \in S_k(1, \nu_\eta^r)$ with $(r, 24) = 1$ has Fourier expansion (2.5) then we have (see for example [Yan14, Proposition 11])

$$\begin{aligned} f|T_{Q^2} &= \sum \left(a(Q^2n) + Q^{k-\frac{3}{2}} \left(\frac{-1}{Q} \right)^{k-\frac{1}{2}} \left(\frac{12n}{Q} \right) a(n) + Q^{2k-2} a\left(\frac{n}{Q^2} \right) \right) q^{\frac{n}{24}} \\ &= f|U_{Q^2} + Q^{k-\frac{3}{2}} \left(\frac{-1}{Q} \right)^{k-\frac{1}{2}} \left(\frac{12}{Q} \right) f \otimes \chi_Q + Q^{2k-2} f|V_{Q^2}. \end{aligned} \quad (2.11)$$

For each squarefree t with $(t, 6) = 1$ there is a Shimura lift Sh_t on $S_k(1, \nu_\eta^r)$, defined via the relationship (2.7) and the usual Shimura lift [Shi73] on $S_k(576, \left(\frac{12}{\bullet}\right)\nu_\theta^r)$. The lift Sh_t can be described by its action on Fourier expansions:

$$\text{Sh}_t \left(\sum a(n) q^{\frac{n}{24}} \right) = \sum A_t(n) q^n, \quad (2.12)$$

where the coefficients $A_t(n)$ are given by

$$A_t(n) = \sum_{d|n} \left(\frac{-1}{d} \right)^{k-\frac{1}{2}} \left(\frac{12t}{d} \right) d^{k-\frac{3}{2}} a\left(\frac{tn^2}{d^2} \right). \quad (2.13)$$

It follows that

$$f \equiv 0 \pmod{\ell} \iff \text{Sh}_t(f) \equiv 0 \pmod{\ell} \quad \text{for all squarefree } t. \quad (2.14)$$

For the non-obvious direction of this equivalence, we argue as follows: if $f \not\equiv 0 \pmod{\ell}$ then for some squarefree t there is an index n with $a(tn^2) \not\equiv 0 \pmod{\ell}$. Letting n_0 denote the minimal such n , we see from (2.13) that $A_t(n_0) \not\equiv 0 \pmod{\ell}$.

The work of Shimura and Niwa [Niw75] shows that if $f \in S_k(1, \nu_\eta^r)$, then $\text{Sh}_t(f) \in S_{2k-1}(288)$. Moreover, for all primes $Q \geq 5$ we have

$$\text{Sh}_t(f|T_{Q^2}) = (\text{Sh}_t f)|T_Q, \quad (2.15)$$

where T_Q is the Hecke operator of index Q on the integral weight space.

From recent work of Yang [Yan14] it follows that

$$\text{Sh}_t : S_k(1, \nu_\eta^r) \longrightarrow S_{2k-1}^{\text{new}}(6) \otimes \left(\frac{12}{\bullet} \right) := \left\{ f \otimes \left(\frac{12}{\bullet} \right) : f \in S_{2k-1}^{\text{new}}(6) \right\}. \quad (2.16)$$

To establish this, it suffices to prove that if $f \in S_k(1, \nu_\eta^r)$ is an eigenform of T_{Q^2} for primes $Q \geq 5$, then $\text{Sh}_t(f)$ is in the space described in (2.16). Suppose that f is such an eigenform and that t is a squarefree positive integer with $(t, 6) = 1$ and $\text{Sh}_t(f) \neq 0$. By [Yan14, Thm 1], there is a newform $F \in S_{2k-1}^{\text{new}}(6)$ with the same Hecke eigenvalues as $\text{Sh}_t(f) \otimes \left(\frac{12}{\bullet}\right)$ at all primes $Q \geq 5$. By strong multiplicity one [JS81b, JS81a], it follows that $\text{Sh}_t(f) \otimes \left(\frac{12}{\bullet}\right)$ is a constant multiple of F . The claim follows since $\text{Sh}_t(f)$ is supported on exponents coprime to 6.

If M is a positive integer, we define

$$W_M := \left[\begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}, M^{\frac{1}{4}} \tau^{\frac{1}{2}} \right].$$

We require the following version of the q -expansion principle (see [DR73, VII, Corollary 3.12] or [Rad13, Theorem 4.8]).

Proposition 2.2. *Suppose that k and N are positive integers, that ℓ is prime, and that π is prime ideal above ℓ in a number field \mathbb{F} which contains all N th roots of unity. Write $\mathcal{O}_\pi \subseteq \mathbb{F}$ for the ring of elements which are integral at π . Suppose that $f \in M_k(\Gamma(N)) \cap \mathcal{O}_\pi[[q^{\frac{1}{N}}]]$ and that $\gamma \in \Gamma_0(\ell^m)$, where ℓ^m is the highest power of ℓ dividing N . Then $f|_k \gamma^* \in \mathcal{O}_\pi[[q^{\frac{1}{N}}]]$, and for $n \geq 0$ we have*

$$f \equiv 0 \pmod{\pi^n} \iff f|_k \gamma^* \equiv 0 \pmod{\pi^n}.$$

For convenience, we record a lemma which allows us to apply Proposition 2.2 in a straightforward way to weakly holomorphic modular forms of half-integral weight.

Lemma 2.3. *Suppose that $k \in \frac{1}{2}\mathbb{Z}$, that $r \in \mathbb{Z}$, and that ℓ is a prime number. Write $\mathcal{O}_\ell \subseteq \overline{\mathbb{Q}}$ for the ring of algebraic numbers which are integral at all primes dividing ℓ . Let h be a positive integer. Suppose that $f \in \overline{\mathbb{Q}}[[q^{\frac{1}{hN}}]][q^{-1}]$ converges absolutely and locally uniformly if $0 < |q| < 1$, that $f^h \in M_{hk}^1(\Gamma(N), \nu_\eta^r) \cap \mathcal{O}_\ell[[q^{\frac{1}{N}}]][q^{-1}]$ or $f^h \in M_{hk}^1(\Gamma(N), \nu_\theta^r) \cap \mathcal{O}_\ell[[q^{\frac{1}{N}}]][q^{-1}]$, and that $\gamma \in \Gamma_0(\ell^m)$, where ℓ^m is the highest power of ℓ dividing N . Then $f|_k \gamma^* \in \mathcal{O}_\ell[[q^{\frac{1}{N}}]][q^{-1}]$, and for $n \geq 0$ we have*

$$f \equiv 0 \pmod{\ell^n} \iff f|_k \gamma^* \equiv 0 \pmod{\ell^n}. \quad (2.17)$$

If ℓ and N are coprime, then for any positive integer M with $\ell \nmid M$ we have

$$f \equiv 0 \pmod{\ell^n} \iff f|_k W_M \equiv 0 \pmod{\ell^n}. \quad (2.18)$$

Proof. Since $W_M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^* \left[\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}, M^{-\frac{1}{4}} \right]$, the second assertion follows from (2.17).

Assume that the lemma holds for $h = 1$. Given f and an arbitrary h as in the statement, apply the lemma with n replaced by hn . This yields $f^h|_{hk} \gamma^* \in \mathcal{O}_\ell[[q^{\frac{1}{N}}]][q^{-1}]$ and the equivalence

$$f^h \equiv 0 \pmod{\ell^{hn}} \iff f^h|_{hk} \gamma^* \equiv 0 \pmod{\ell^{hn}}.$$

Observe that $f^h|_{hk} \gamma^* = (f|_k \gamma^*)^h$, and hence $f|_k \gamma^* \in \mathcal{O}_\ell[[q^{\frac{1}{hN}}]][q^{-1}]$. Since we further have the equivalences

$$\begin{aligned} f^h \equiv 0 \pmod{\ell^{hn}} &\iff f \equiv 0 \pmod{\ell^n}, \\ f^h|_{hk} \gamma^* \equiv 0 \pmod{\ell^{hn}} &\iff f|_k \gamma^* \equiv 0 \pmod{\ell^n}, \end{aligned}$$

we can and will assume that $h = 1$ in the remainder of the proof.

To prove (2.17) in the case of $h = 1$, we employ an argument of Jochnowitz [Joc]; a slightly different argument was given by Radu [Rad13]. We restrict to the case of the eta multiplier, since the theta multiplier can be handled in an analogous way. Further, we can and will replace \mathcal{O}_ℓ by its intersection with a suitable number field that contains the Fourier coefficients of f and all N th roots of unity. Let $\ell^n = \pi_1^{n_1} \cdots \pi_m^{n_m}$ be the prime ideal factorization of ℓ^n in \mathcal{O}_ℓ . It suffices to show that

$$f \equiv 0 \pmod{\pi_i^{n_i}} \iff f|_k \gamma^* \equiv 0 \pmod{\pi_i^{n_i}} \text{ for all } i.$$

Observe that $f \equiv 0 \pmod{\pi_i^{n_i}}$ is equivalent to $\eta^j f \equiv 0 \pmod{\pi_i^{n_i}}$ for any $j \in \mathbb{Z}$. Choosing a sufficiently large j with $j \equiv -r \pmod{24}$, we have $\eta^j f \in M_{k+\frac{j}{2}}(\Gamma(N))$. The lemma follows by applying Proposition 2.2 to $\eta^j f$. \square

Finally, we justify the statements (1.7) and (1.8) from the introduction. Let $\Delta = \eta^{24}$ be the unique normalized cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$, and let U_ℓ and V_ℓ denote the usual operators on spaces of integral weight modular forms. By [AB03, (3.2)] we have

$$\sum_{\left(\frac{-n}{\ell}\right)=0} p\left(\frac{n+1}{24}\right) q^{\frac{n}{24}} \equiv \left(\frac{\Delta^{\frac{\ell^2-1}{24}}|U_\ell}{\eta^\ell}\right)|V_\ell \pmod{\ell}.$$

By [Ser73, Lemma 2], $\Delta^{\frac{\ell^2-1}{24}}|U_\ell$ is congruent modulo ℓ to a modular form G_ℓ of weight $\ell - 1$ on $\mathrm{SL}_2(\mathbb{Z})$. The form G_ℓ vanishes modulo ℓ to order $> \frac{\ell}{24}$. Note that $M_{\ell-1}(1) \cap \mathbb{Z}[[q]]$ has a basis $\{h_1, \dots, h_d\}$ with $h_j = q^j + \dots$. After subtracting a suitable integral linear combination of these basis elements, we may assume that G_ℓ vanishes to order $> \frac{\ell}{24}$. Therefore we can take

$$F_\ell = \frac{G_\ell}{\eta^\ell} \in S_{\frac{\ell-2}{2}}(1, \nu_\eta^{-\ell})$$

and $f_{\ell,0} = F_\ell \in S_{\frac{\ell^2-2\ell}{2}}(1, \nu_\eta^{-1})$ as claimed. After dividing the modular form which is described in [AO01c, (3.4)] by η^{ℓ^2} , we deduce that there is a modular form $g \in S_{\frac{\ell^2-2}{2}}(1, \nu_\eta^{-1})$ such that

$$g \equiv \sum_{\left(\frac{-n}{\ell}\right)=0} p\left(\frac{n+1}{24}\right) q^{\frac{n}{24}} + 2 \sum_{\left(\frac{-n}{\ell}\right)=-1} p\left(\frac{n+1}{24}\right) q^{\frac{n}{24}} \pmod{\ell}.$$

Subtracting $f_{\ell,0}E_{\ell-1}$, where $E_{\ell-1} \equiv 1 \pmod{\ell}$ is the Eisenstein series of weight $\ell - 1$, it follows that we can also take $f_{\ell,-1}$ as stated.

3. PROOF OF THEOREM 1.5

In this section we address the issue of congruences $p(\ell Qn + \beta) \equiv 0 \pmod{\ell}$ for squarefree integers Q . Theorem 1.5 will follow from Proposition 3.2 and Theorem 3.4 below.

Given a positive integer m and an integer β , define the set

$$S_{m,\beta} := \{\beta' \pmod{m} : 24\beta' - 1 \equiv a^2(24\beta - 1) \pmod{m} \text{ for some } a \text{ with } (a, 6m) = 1\}. \quad (3.1)$$

Crucial to our arguments is the following result of Radu.

Theorem 3.1. [Rad12, Theorem 5.4] *Suppose that m and ℓ are positive integers and that for some integer β we have a congruence $p(mn + \beta) \equiv 0 \pmod{\ell}$. Then for all $\beta' \in S_{m,\beta}$ we have a congruence $p(mn + \beta') \equiv 0 \pmod{\ell}$.*

We begin by making a reduction. From [AB03] we know that for $\ell \geq 13$ there are no congruences of the form

$$p(\ell n + \beta) \equiv 0 \pmod{\ell}.$$

It follows from the next result that for such ℓ there are no congruences of the form

$$p(\ell Q n + \beta) \equiv 0 \pmod{\ell},$$

where Q is a squarefree positive integer with $Q \mid 24\beta - 1$. In particular, this result implies the first assertion in Theorem 1.5.

Proposition 3.2. *Suppose that $\ell \geq 5$ is prime, that Q is a squarefree positive integer with $(Q, 6\ell) = 1$, and that $\beta \in \mathbb{Z}$. Write*

$$Q = Q'Q'', \quad \text{where } (Q', 24\beta - 1) = 1 \quad \text{and } Q'' \mid 24\beta - 1. \quad (3.2)$$

Then there is a congruence

$$p(\ell Q n + \beta) \equiv 0 \pmod{\ell} \quad (3.3)$$

if and only if there is a congruence

$$p(\ell Q' n + \beta) \equiv 0 \pmod{\ell}. \quad (3.4)$$

Proof. For the non-obvious direction, suppose that there is a congruence (3.3). Set

$$\delta := \left(\frac{1 - 24\beta}{\ell} \right) \in \{0, -1\}$$

and let $Q' = Q_1 \cdots Q_t$ be the prime factorization of Q' . For $i = 1, \dots, t$ set

$$\delta_i := \left(\frac{24\beta - 1}{Q_i} \right) \in \{\pm 1\}.$$

Suppose that $f = \sum a(n)q^{\frac{n}{24}} \in S_k(N, \nu_\eta^{-1})$. For each i , define the operator

$$f|B_i := \frac{1}{2} (f - f|U_{Q_i}V_{Q_i} + \delta_i f \otimes \chi_{Q_i}).$$

Then we have

$$f|B_i = \sum_{\left(\frac{n}{Q_i}\right) = \delta_i} a(n)q^{\frac{n}{24}} \in S_k(NQ_i^2, \nu_\eta^{-1}).$$

Recall the definition (1.8), and define

$$g := f_{\ell, \delta} | B_{Q_1} \cdots B_{Q_t} \in S_k((Q')^2, \nu_\eta^{-1})$$

(where the particular value of k depends on δ and is unimportant in what follows). We have

$$g \equiv \sum_{\substack{\left(\frac{-n}{\ell}\right) = \delta \\ \left(\frac{n}{Q_i}\right) = \delta_i \forall i}} p\left(\frac{n+1}{24}\right) q^{\frac{n}{24}} \pmod{\ell}.$$

After changing variables we find that

$$g \equiv \sum_{\beta' \in \mathcal{S}_{\ell Q', \beta}} \sum_n p(\ell Q' n + \beta') q^{\ell Q' n + \beta' - \frac{1}{24}} \pmod{\ell}.$$

So by Theorem 3.1, we have the congruence (3.4) if and only if $g \equiv 0 \pmod{\ell}$.

Let $Q'' = P_1 \dots P_s$ be the prime factorization of Q'' , and define

$$h := g|U_{P_1} \dots U_{P_s} V_{P_1} \dots V_{P_s} \equiv \sum_{\substack{\left(\frac{-n}{\ell}\right) = \delta \\ \left(\frac{n}{Q_i}\right) = \delta_i \forall i \\ P_j | n \forall j}} p\left(\frac{n+1}{24}\right) q^{\frac{n}{24}} \pmod{\ell}.$$

Then

$$h \equiv \sum_{\beta' \in \mathcal{S}_{\ell Q, \beta}} \sum_n p(\ell Q n + \beta') q^{\ell Q n + \beta' - \frac{1}{24}} \pmod{\ell},$$

so we have the congruence (3.3) if and only if $h \equiv 0 \pmod{\ell}$.

Proposition 3.2 therefore follows from the next lemma, applied successively with the primes P_1, \dots, P_s . \square

Lemma 3.3. *Suppose that $\ell \geq 5$ is prime, that $(r, 24) = 1$, that $(N, 6\ell) = 1$, that Q is a prime with $(Q, 6N\ell) = 1$, and that χ is a Dirichlet character modulo N . Suppose that $f \in M_k(N, \chi \nu_\eta^r)$. Then*

$$f|U_Q \not\equiv 0 \pmod{\ell} \iff f \not\equiv 0 \pmod{\ell}.$$

Proof of Lemma 3.3. It suffices to prove the assertion for the modular form $f|V_{24}$. Using (2.7) and setting $N_1 = 576N$, we will assume for simplicity in the proof that

$$f \in M_k\left(N_1, \left(\frac{12}{\bullet}\right) \chi \nu_\theta^r\right).$$

Only one direction requires proof. Since f has a Fourier expansion in integral powers of q , we have

$$f|U_Q|_k W_{Q N_1} = Q^{\frac{k}{2}-1} \sum_{v(Q)} f|_k \left[\begin{pmatrix} v Q N_1 & -1 \\ Q^2 N_1 & 0 \end{pmatrix}, Q^{\frac{1}{2}} N_1^{\frac{1}{4}} \tau^{\frac{1}{2}} \right]. \quad (3.5)$$

Write

$$g := f|_k W_{N_1} = \sum b(n) q^n$$

for the image under the Fricke involution W_{N_1} . The term with $v = 0$ in (3.5) gives

$$Q^{\frac{k}{2}-1} f|_k W_{N_1} \left[\begin{pmatrix} Q^2 & 0 \\ 0 & 1 \end{pmatrix}, Q^{-\frac{1}{2}} \right] = Q^{\frac{3k}{2}-1} g|V_{Q^2}. \quad (3.6)$$

For each $v \not\equiv 0 \pmod{Q}$ in (3.5) we choose α with

$$\alpha v N_1 \equiv 1 \pmod{Q}.$$

Then the terms in (3.5) with $v \not\equiv 0 \pmod{Q}$ contribute

$$Q^{\frac{k}{2}-1} \sum_{v(Q)^*} f|_k W_{N_1} \left(\begin{matrix} Q & \alpha \\ -v N_1 & 1 - \frac{\alpha v N_1}{Q} \end{matrix} \right)^* \left[\begin{pmatrix} Q & -\alpha \\ 0 & Q \end{pmatrix}, 1 \right].$$

To compute the automorphy factors here and below it is helpful to use the facts that

$$\left(\frac{z}{w}\right)^{\frac{1}{2}} = \frac{z^{\frac{1}{2}}}{w^{\frac{1}{2}}}, \quad (-z)^{\frac{1}{2}} = -iz^{\frac{1}{2}} \quad \text{for } z, w \in \mathbb{H}.$$

Since

$$W_{N_1} \begin{pmatrix} Q & \alpha \\ -vN_1 & \frac{1-\alpha vN_1}{Q} \end{pmatrix}^* = \begin{pmatrix} \frac{1-\alpha vN_1}{Q} & v \\ -N_1\alpha & Q \end{pmatrix}^* W_{N_1}$$

and r is odd, this simplifies to

$$\begin{aligned} & Q^{\frac{k}{2}-1} \chi(Q) \left(\frac{12N_1}{Q}\right) \varepsilon_Q^{-r} \sum_{v(Q)^*} \left(\frac{-\alpha}{Q}\right) g\left(\tau - \frac{\alpha}{Q}\right) \\ &= Q^{\frac{k}{2}-1} \chi(Q) \left(\frac{12N_1}{Q}\right) \varepsilon_Q^{-r} \sum_n b(n) q^n \sum_{v(Q)^*} \left(\frac{-\alpha}{Q}\right) \zeta_Q^{-n\alpha} = Q^{\frac{k}{2}-1} \chi(Q) \left(\frac{12N_1}{Q}\right) \varepsilon_Q^{-r} G(Q) g \otimes \chi_Q. \end{aligned}$$

Summarizing, we have

$$f|U_Q|_k W_{Q_{N_1}} = Q^{\frac{k}{2}-1} \chi(Q) \left(\frac{12N_1}{Q}\right) \varepsilon_Q^{-r} G(Q) g \otimes \chi_Q + Q^{\frac{3k}{2}-1} g|V_{Q^2}.$$

By (2.18), we have $f|U_Q \equiv 0 \pmod{\ell}$ only if this expression is zero modulo ℓ , which can happen only if $g \equiv 0 \pmod{\ell}$. By another application of (2.18), this can occur only if $f \equiv 0 \pmod{\ell}$. \square

The second assertion of Theorem 1.5 follows from the next result.

Theorem 3.4. *Suppose that $\ell \geq 5$ is prime and that Q is a squarefree positive integer with $(Q, 6\ell) = 1$ and prime factorization $Q = Q_1 \dots Q_t$. Suppose that $\beta_0 \in \mathbb{Z}$ has*

$$(Q, 24\beta_0 - 1) = 1$$

and that there is a congruence

$$p(\ell Qn + \beta_0) \equiv 0 \pmod{\ell}.$$

Define

$$\delta := \left(\frac{1-24\beta_0}{\ell}\right) \in \{0, -1\},$$

and for $d \mid Q$ define

$$\lambda_d := d^{-1} \left(\frac{-12}{d}\right) \left(\frac{24\beta_0 - 1}{d}\right). \quad (3.7)$$

Then we have

$$f_{\ell, \delta} | (U_{Q_1} + \lambda_{Q_1} V_{Q_1}) \cdots (U_{Q_t} + \lambda_{Q_t} V_{Q_t}) \equiv 0 \pmod{\ell}. \quad (3.8)$$

To prove Theorem 3.4, we require another result of Radu [Rad13]. Suppose that m is a positive integer with $(m, 6) = 1$, and that $\beta \in \mathbb{Z}$. As in [Rad13, Lemma 4.11], define

$$g(m, \beta, \tau) := q^{\frac{24\beta-1}{24m}} \sum p(mn + \beta) q^n. \quad (3.9)$$

If $\ell \geq 5$ is prime and Q is a positive integer with $(Q, 6\ell) = 1$, we choose integers X and Y with

$$576\ell^2 X + QY = 1 \quad (3.10)$$

and define

$$\gamma_{\ell, Q} := \begin{pmatrix} 1 & -24^2 \ell X \\ \ell & QY \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \quad (3.11)$$

The next proposition follows from Lemma 5.1 of [Rad13]. Note that we have corrected a typographical error in that lemma which arises from dropping the term $\left(\frac{24\ell}{Q/d}\right)$ in equation (44). The correct version, which we quote below, can also be found in equation (12) of the preprint version available at the author's homepage. Note also that for the values of Q and d below we have $(-1)^{\frac{Q+1}{2} \frac{d-1}{2}} = (-1)^{\frac{Qd-1}{2} \frac{d-1}{2}}$.

Proposition 3.5. *Suppose that $\ell \geq 5$ is prime, that Q is a positive integer with $(Q, 6\ell) = 1$, and that $\beta \in \mathbb{Z}$. Let $\gamma_{\ell, Q}$ be defined as in (3.11). Then*

$$\begin{aligned} & Qe\left(-\frac{\pi i Q}{12}\right) e\left(-\frac{48\pi i X(24\beta - 1)}{Q}\right) (\ell\tau + QY)^{\frac{1}{2}} g(\ell Q, \beta, \gamma_{\ell, Q}\tau) \\ &= \sum_{d|Q} d^{-\frac{1}{2}} e\left(\frac{1-d}{8}\right) \left(\frac{24\ell}{Q/d}\right) (-1)^{\frac{Qd-1}{2} \frac{d-1}{2}} q^{\frac{(24t_{d,\beta}-1)d^2}{24\ell Q}} \sum_{n=0}^{\infty} q^{\frac{nd^2}{Q}} p(\ell n + t_{d,\beta}) T(n, d), \end{aligned} \quad (3.12)$$

where $t_{d,\beta}$ is the integer satisfying $0 \leq t_{d,\beta} \leq \ell - 1$ and

$$d^2(24t_{d,\beta} - 1) \equiv 24\beta - 1 \pmod{\ell}, \quad (3.13)$$

\bar{s} is any integer such that $s\bar{s} \equiv 1 \pmod{Q/d}$, and

$$T(n, d) := \sum_{s(Q/d)} \left(\frac{24\ell s}{Q/d}\right) e\left(\frac{-48\pi i X}{Q/d} (\bar{s}(24(\ell n + t_{d,\beta}) - 1) + s(24\beta - 1))\right).$$

Note that the definition of $T(n, d)$ depends implicitly on Q and that we have $T(n, Q) = 1$ for all n (since $\binom{0}{1} = 1$). Define the Salie sum by

$$S(a, b, c) := \sum_{n(c)} \left(\frac{n}{c}\right) e\left(\frac{an + b\bar{n}}{c}\right). \quad (3.14)$$

Replacing $-\overline{24\ell^2}$ by s and using (3.10), we see that

$$\begin{aligned} T(n, d) &= \left(\frac{-\ell}{Q/d}\right) \sum_{s(Q/d)} \left(\frac{s}{Q/d}\right) e\left(\frac{2\pi i}{Q/d} (\overline{24^2 \ell^4} \bar{s}(24(\ell n + t_{d,\beta}) - 1) + s(24\beta - 1))\right) \\ &= \left(\frac{-\ell}{Q/d}\right) S\left(24\beta - 1, \overline{24^2 \ell^4} (24(\ell n + t_{d,\beta}) - 1), Q/d\right). \end{aligned} \quad (3.15)$$

Proof of Theorem 3.4. Let ℓ and $Q = Q_1 \dots Q_t$ be as in the statement of the theorem, and suppose that there is a congruence

$$p(\ell Qn + \beta_0) \equiv 0 \pmod{\ell} \quad \text{with} \quad \left(\frac{1 - 24\beta_0}{\ell}\right) = \delta \in \{0, -1\}.$$

By Theorem 3.1 we have congruences

$$p(\ell Qn + \beta) \equiv 0 \pmod{\ell} \quad \text{for all } \beta \in S_{\ell Q, \beta_0}. \quad (3.16)$$

Suppose that $\beta \in S_{\ell Q, \beta_0}$. By equations (17) and (18) of [Rad13], there exists a positive integer h such that $g(\ell Q, \beta, \tau)^h \in M_{\frac{h}{2}}^1(\Gamma(\ell Q))$. Applying Lemma 2.3 to $g(\ell Q, \beta, \tau) \equiv 0 \pmod{\ell}$, we conclude that

$$(\ell\tau + YQ)^{\frac{1}{2}} g(\ell Q, \beta, \gamma_{\ell, Q}\tau) \equiv 0 \pmod{\ell}. \quad (3.17)$$

Therefore, the expression on the right side of (3.12) is 0 (mod ℓ). We write this expression in the form

$$\sum_{r \in \frac{1}{24\ell}\mathbb{Z}} a(r)q^r + \sum_{r \in \mathbb{Q} \setminus \frac{1}{24\ell}\mathbb{Z}} a(r)q^r,$$

and note that each of these summands is 0 (mod ℓ). Define

$$F_{\ell, Q, \beta} := \sum_{r \in \frac{1}{24\ell}\mathbb{Z}} a(r)q^r \equiv 0 \pmod{\ell}.$$

The theorem will follow from computing each $F_{\ell, Q, \beta}$ explicitly. In the term arising from the divisor d in (3.12), the exponents have the form

$$r = \frac{nd^2}{Q} + \frac{(24t_{d, \beta} - 1)d^2}{24\ell Q} = \frac{d(24(\ell n + t_{d, \beta}) - 1)}{24\ell Q/d}. \quad (3.18)$$

Since $(d, Q/d) = 1$, it follows that $F_{\ell, Q, \beta}$ is the sum of those terms in (3.12) with

$$24(\ell n + t_{d, \beta}) - 1 \equiv 0 \pmod{Q/d}. \quad (3.19)$$

To compute $F_{\ell, Q, \beta}$, we may therefore assume that (3.19) holds. Since $(24\beta - 1, Q) = 1$, (3.15) gives

$$T(n, d) = \left(\frac{-\ell}{Q/d}\right) \left(\frac{24\beta - 1}{Q/d}\right) G(Q/d).$$

We write

$$F_{\ell, Q, \beta} = \sum_{d|Q} F_{\ell, Q, \beta, d}, \quad (3.20)$$

where for each divisor d , $F_{\ell, Q, \beta, d}$ is the contribution from those terms in (3.12) satisfying (3.19). In other words,

$$F_{\ell, Q, \beta, d} = d^{-\frac{1}{2}} e\left(\frac{1-d}{8}\right) (-1)^{\frac{Qd-1}{2} \frac{d-1}{2}} G(Q/d) \left(\frac{-24(24\beta - 1)}{Q/d}\right) \cdot \sum_{24(\ell n + t_{d, \beta}) - 1 \equiv 0 \pmod{Q/d}} p(\ell n + t_{d, \beta}) q^{\frac{nd^2}{Q} + \frac{(24t_{d, \beta} - 1)d^2}{24\ell Q}}.$$

From (2.3) and (2.4), we have the formula

$$e\left(\frac{1-d}{8}\right) (-1)^{\frac{Qd-1}{2} \frac{d-1}{2}} G(Q/d) = d^{-\frac{1}{2}} \left(\frac{2}{d}\right) G(Q),$$

and from (3.1) we have

$$\left(\frac{24\beta - 1}{d}\right) = \left(\frac{24\beta_0 - 1}{d}\right) \text{ for } d | Q.$$

Using these facts, replacing $\frac{24(\ell n + t_{d, \beta}) - 1}{Q/d}$ by n , and recalling the definition (3.7), we find that

$$F_{\ell, Q, \beta, d} = G(Q) \left(\frac{-24(24\beta_0 - 1)}{Q}\right) \lambda_d \sum_{\frac{Q}{d}n \equiv 24t_{d, \beta} - 1 \pmod{\ell}} p\left(\frac{\frac{Q}{d}n + 1}{24}\right) q^{\frac{nd}{24\ell}},$$

which may be written in the form

$$F_{\ell, Q, \beta, d} | V_\ell = G(Q) \left(\frac{-24(24\beta_0 - 1)}{Q} \right) \lambda_d \left(\sum_{n \equiv 24t_{d, \beta} - 1 (\ell)} p \left(\frac{n+1}{24} \right) q^{\frac{n}{24}} \right) | U_{Q/d} V_d. \quad (3.21)$$

We are now in a position to prove Theorem 3.4. Suppose first that $\delta = \left(\frac{1-24\beta_0}{\ell} \right) = 0$. Then for every $\beta \in S_{\ell Q, \beta_0}$ we have $\ell \mid 24\beta - 1$. It follows from (3.13) that for each d and β we have $t_{d, \beta} \equiv \beta_0 \pmod{\ell}$, from which

$$\sum_{n \equiv 24t_{d, \beta} - 1 (\ell)} p \left(\frac{n+1}{24} \right) q^{\frac{n}{24}} = \sum_{\ell \mid n} p \left(\frac{n+1}{24} \right) q^{\frac{n}{24}} \equiv f_{\ell, 0} \pmod{\ell}.$$

Using (3.20) and (3.21) together with the fact that $F_{\ell, Q, \beta} \equiv 0 \pmod{\ell}$, we find that

$$\sum_{d \mid Q} \lambda_d f_{\ell, 0} | U_{Q/d} V_d \equiv 0 \pmod{\ell}.$$

Factoring gives

$$f_{\ell, 0} | (U_{Q_1} + \lambda_{Q_1} V_{Q_1}) \cdots (U_{Q_t} + \lambda_{Q_t} V_{Q_t}) \equiv 0 \pmod{\ell}.$$

So the theorem follows in this case.

Finally, suppose that $\delta = \left(\frac{1-24\beta_0}{\ell} \right) = -1$. In this case, the situation is complicated by the fact that the values of $t_{d, \beta}$ in (3.21) vary with d and β . To proceed, define

$$S' := \{ \beta \in S_{\ell Q, \beta_0} : \beta \equiv \beta_0 \pmod{Q} \}.$$

Then S' contains one representative for each residue class $\beta \pmod{\ell}$ with

$$\left(\frac{1-24\beta}{\ell} \right) = \left(\frac{1-24\beta_0}{\ell} \right) = -1.$$

For each d , we see by (3.13) that $t_{d, \beta}$ ranges over those residue classes $t \pmod{\ell}$ with $\left(\frac{1-24t}{\ell} \right) = -1$ as β ranges over S' . We conclude that

$$f_{\ell, -1} \equiv \sum_{\left(\frac{-n}{\ell} \right) = -1} p \left(\frac{n+1}{24} \right) q^{\frac{n}{24}} \equiv \sum_{\beta \in S'} \sum_{n \equiv 24t_{d, \beta} - 1 (\ell)} p \left(\frac{n+1}{24} \right) q^{\frac{n}{24}} \pmod{\ell}.$$

Combining this with (3.20) and (3.21), we obtain

$$\sum_{\beta \in S'} F_{\ell, Q, \beta} | V_\ell = \sum_{d \mid Q} \sum_{\beta \in S'} F_{\ell, Q, \beta, d} | V_\ell \equiv G(Q) \left(\frac{-24(24\beta_0 - 1)}{Q} \right) \sum_{d \mid Q} \lambda_d f_{\ell, -1} | U_{Q/d} V_d \pmod{\ell}.$$

Since $F_{\ell, Q, \beta} \equiv 0 \pmod{\ell}$ for each β in the sum, we obtain

$$\sum_{d \mid Q} \lambda_d f_{\ell, -1} | U_{Q/d} V_d \equiv 0 \pmod{\ell}.$$

Factoring gives

$$f_{\ell, -1} | (U_{Q_1} + \lambda_{Q_1} V_{Q_1}) \cdots (U_{Q_t} + \lambda_{Q_t} V_{Q_t}) \equiv 0 \pmod{\ell}.$$

This proves Theorem 3.4. □

4. PROOF OF THEOREM 1.4

Theorem 1.4 will follow from the next result together with Theorem 1.5.

Theorem 4.1. *Suppose that $\ell \geq 5$ is prime, that $(r, 24) = 1$ and that $f = \sum a(n)q^{\frac{n}{24}} \in S_k(1, \nu_\eta^r)$ has $f \not\equiv 0 \pmod{\ell}$. Fix $\varepsilon \in \{\pm 1\}$. Let S be the set of primes p such that*

$$f|U_p \equiv c_p f|V_p \pmod{\ell} \quad \text{for some } c_p \not\equiv 0 \pmod{\ell} \quad (4.1)$$

and

$$f \equiv \sum_{\left(\frac{n}{p}\right)=\varepsilon} a(n)q^{\frac{n}{24}} + \sum a(p^2 n)q^{\frac{p^2 n}{24}} \pmod{\ell}. \quad (4.2)$$

Then one of the following is true.

- (1) S has density zero, or
- (2) we have

$$\#\{n \leq X : a(n) \not\equiv 0 \pmod{\ell}\} \ll \sqrt{X} \log X$$

and

$$f|T_{Q^2} \equiv 0 \pmod{\ell} \quad \text{for all primes } Q \equiv -1 \pmod{\ell}.$$

Assuming this result for the moment, we prove Theorem 1.4.

Proof of Theorem 1.4. Recall the modular forms $f_{\ell, \delta}$ from (1.8). If $\ell \in \{5, 7, 11\}$ and $\delta = 0$, we have $f_{\ell, \delta} \equiv 0 \pmod{\ell}$ and both (1.9) and (1.10) hold trivially. We exclude these cases from further consideration. In all other cases, we have $f_{\ell, \delta} \not\equiv 0 \pmod{\ell}$ by the work cited after the statement of Theorem 1.4. Fix $\delta \in \{0, -1\}$. For $\varepsilon \in \{\pm 1\}$, let S_ε be the set of primes p for which there is a congruence

$$p(\ell p n + \beta) \equiv 0 \pmod{\ell} \quad \text{with} \quad \left(\frac{1 - 24\beta}{\ell}\right) = \delta, \quad \left(\frac{24\beta - 1}{p}\right) = \varepsilon.$$

In view of the first assertion of Theorem 1.5, we see that $S = S_1 \cup S_{-1}$ is the set described in the statement of Theorem 1.4.

Suppose that $p \in S_\varepsilon$. Then

$$\sum_{\left(\frac{n}{p}\right)=\varepsilon} a_{\ell, \delta}(n)q^{\frac{n}{24}} \equiv 0 \pmod{\ell},$$

and by Theorem 1.5 we have

$$f_{\ell, \delta}|U_p \equiv -\varepsilon \left(\frac{-12}{p}\right) p^{-1} f_{\ell, \delta}|V_p \pmod{\ell}.$$

Using this fact together with Theorem 3.1 we see that

$$f_{\ell, \delta} = \sum a_{\ell, \delta}(n)q^{\frac{n}{24}} \equiv \sum_{\left(\frac{n}{p}\right)=-\varepsilon} a_{\ell, \delta}(n)q^{\frac{n}{24}} + \sum a_{\ell, \delta}(p^2 n)q^{\frac{p^2 n}{24}} \pmod{\ell}.$$

Suppose that S does not have density zero. Then the same is true of S_ε for some choice of ε , and Theorem 1.4 follows from applying Theorem 4.1 to $f_{\ell, \delta}$. \square

We turn to the proof of Theorem 4.1. Let f and S be as in the hypotheses. If Q is a prime with $Q \geq 5$ and $Q \neq \ell$ then one of two mutually exclusive things must occur. Either

$$\text{there exists an integer } n_Q \text{ with } \text{ord}_Q(n_Q) \text{ odd and } a(n_Q) \not\equiv 0 \pmod{\ell}, \quad (4.3)$$

or

$$f|U_{Q^{2j+1}} \equiv f|U_{Q^{2j+2}}V_Q \pmod{\ell} \quad \text{for all } j \geq 0. \quad (4.4)$$

Note that if $Q \in S$, then (4.4) holds by virtue of (4.1) and induction on $\text{ord}_Q(n_Q)$.

Suppose that (4.3) holds for infinitely many primes Q . For each such prime, write

$$n_Q = Q^{1+2e_Q} n'_Q \quad \text{with } Q \nmid n'_Q.$$

Letting Q_1 denote the smallest such prime, we may then choose an infinite sequence of such primes Q_2, Q_3, \dots successively with

$$Q_j \nmid n_{Q_1} \cdots n_{Q_{j-1}}. \quad (4.5)$$

For each j we have

$$a(Q_j^{1+2e_{Q_j}} n'_{Q_j}) \not\equiv 0 \pmod{\ell}. \quad (4.6)$$

For each $p \in S$, it follows from (4.4) that $\text{ord}_p(n'_{Q_j})$ is even, and from (4.1) that

$$a(p^2 n) \equiv c_p a(n) \pmod{\ell} \quad \text{for all } n.$$

Using these facts, we may remove even powers of each $p \in S$ from each n'_{Q_j} , and we may therefore assume in (4.6) that for all j and all $p \in S$ we have

$$p \nmid n'_{Q_j}. \quad (4.7)$$

Let $t \geq 1$. From (4.6), (4.7) and (4.2), we see that

$$\left(\frac{Q_j n'_{Q_j}}{p} \right) = \varepsilon \quad \text{for } p \in S \text{ and } 1 \leq j \leq t.$$

Each of these quadratic relations imposes a residue class restriction on the primes p that may belong to S . More specifically, the quadratic relation corresponding to Q_j prohibits from belonging to S a proportion of $1/2$ of the primes not already prohibited by the quadratic relations corresponding to Q_1, Q_2, \dots, Q_{j-1} . Therefore

$$\limsup_{X \rightarrow \infty} \frac{\#\{p \in S : p \leq X\}}{X/\log X} \leq 2^{-t}.$$

Since t is arbitrary we conclude that S has density zero if (4.3) holds for infinitely many primes Q .

From (4.3) and (4.4) we conclude that to prove Theorem 4.1 it suffices to establish the following proposition, whose proof occupies the remainder of this section.

Proposition 4.2. *Suppose that $\ell \geq 5$ is prime, that $(r, 24) = 1$ and that $f = \sum a(n)q^{\frac{n}{24}} \in S_k(1, \nu_\eta^r)$ has $f \not\equiv 0 \pmod{\ell}$. Suppose that for all but finitely many primes $Q \geq 5$ we have*

$$f|U_{Q^{2j+1}} \equiv f|U_{Q^{2j+2}}V_Q \pmod{\ell} \quad \text{for all } j \geq 0. \quad (4.8)$$

Then

$$\#\{n \leq X : a(n) \not\equiv 0 \pmod{\ell}\} \ll \sqrt{X} \log X,$$

and

$$f|T_{Q^2} \equiv 0 \pmod{\ell} \text{ for all primes } Q \equiv -1 \pmod{\ell}.$$

In the proof of Proposition 4.2 we will need the following result.

Proposition 4.3. *Let $\ell \geq 5$ and $Q \geq 5$ be primes with $Q \neq \ell$ and $Q \not\equiv 1 \pmod{\ell}$. Suppose that $(r, 24) = 1$ and that $f \in M_k(1, \nu_\eta^r)$ has*

$$f|U_Q \equiv f|U_{Q^2}V_Q \pmod{\ell}. \quad (4.9)$$

Then we have

$$f|U_{Q^2} \equiv Q^{k-\frac{1}{2}} \left(\frac{-1}{Q}\right)^{k-\frac{1}{2}} \left(\frac{12}{Q}\right) f \otimes \chi_Q + Q^{2k-1} f|V_{Q^2} \pmod{\ell}, \quad (4.10)$$

and

$$f|T_{Q^2} \equiv (1 + Q^{-1}) f|U_{Q^2} \pmod{\ell}. \quad (4.11)$$

Proof of Proposition 4.3. Let ℓ , Q and f be as in the hypothesis. By (2.8) and (2.9) we have

$$f|U_Q \in M_k(Q, \chi_Q \nu_\eta^{Qr}), \quad f|U_{Q^2}V_Q \in M_k(Q^2, \chi_Q \nu_\eta^{Qr}).$$

Then (4.9) and (2.18) give

$$f|U_Q|_k W_Q \equiv f|U_{Q^2}V_Q|_k W_Q \pmod{\ell}. \quad (4.12)$$

We first claim that

$$f|U_Q|_k W_Q = Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \left(\frac{-24}{Q}\right) G(Q) f \otimes \chi_Q + Q^{\frac{3k}{2}-1} e\left(\frac{-r}{8}\right) f|V_{Q^2}. \quad (4.13)$$

To establish (4.13) we compute using Lemma 2.1. We have

$$f|U_Q|_k W_Q = Q^{\frac{k}{2}-1} \sum_{v(Q)} f|_k \left[\begin{pmatrix} 24vQ & -1 \\ Q^2 & 0 \end{pmatrix}, Q^{\frac{1}{2}} z^{\frac{1}{2}} \right]. \quad (4.14)$$

Using (2.2), the term with $v = 0$ in (4.14) gives

$$\begin{aligned} Q^{\frac{k}{2}-1} f|_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^* \left[\begin{pmatrix} Q^2 & 0 \\ 0 & 1 \end{pmatrix}, Q^{-\frac{1}{2}} \right] &= Q^{\frac{k}{2}-1} e\left(\frac{-r}{8}\right) f|_k \left[\begin{pmatrix} Q^2 & 0 \\ 0 & 1 \end{pmatrix}, Q^{-\frac{1}{2}} \right] \\ &= Q^{\frac{3k}{2}-1} e\left(\frac{-r}{8}\right) f|V_{Q^2}. \end{aligned} \quad (4.15)$$

For each $v \not\equiv 0 \pmod{Q}$ in (4.14) we choose α with

$$24\alpha v \equiv 1 \pmod{Q} \quad \text{and} \quad \alpha = 24\alpha' \text{ with } \alpha' \in \mathbb{Z}.$$

Then the terms in (4.14) with $v \not\equiv 0 \pmod{Q}$ contribute

$$Q^{\frac{k}{2}-1} \sum_{v(Q)^*} f|_k \left(\begin{matrix} 24v & \frac{24\alpha v-1}{\alpha} \\ Q & \alpha \end{matrix} \right)^* \left[\begin{pmatrix} Q & -\alpha \\ 0 & Q \end{pmatrix}, 1 \right].$$

Using (2.2) this becomes

$$Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \sum_{v(Q)^*} \left(\frac{\alpha}{Q}\right) f\left(\tau - \frac{\alpha}{Q}\right) = Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \sum_n a(n) q^{\frac{n}{24}} \sum_{v(Q)^*} \left(\frac{24\alpha'}{Q}\right) \zeta_Q^{-n\alpha'}.$$

Since the inner sum evaluates to $\left(\frac{-24n}{Q}\right) G(Q)$, the claim (4.13) follows from this equation together with (4.15).

We next claim that if (4.9) holds then

$$f|U_{Q^2}V_Q|_k W_Q \equiv Q^{-1}f|U_Q|_k W_Q + Q^{-\frac{k}{2}}(1 - Q^{-1})e\left(\frac{-r}{8}\right)f|U_{Q^2} \pmod{\ell}. \quad (4.16)$$

To prove this, we use Lemma 2.1 to compute

$$f|U_{Q^2}V_Q|_k W_Q = Q^{\frac{k}{2}-2} \sum_{v(Q^2)} f|_k \left[\begin{pmatrix} 24v & -1 \\ Q^2 & 0 \end{pmatrix}, Q^{\frac{1}{2}}z^{\frac{1}{2}} \right]. \quad (4.17)$$

Using (4.14), we see that the terms in (4.17) with $v \equiv 0 \pmod{Q}$ give

$$Q^{\frac{k}{2}-2} \sum_{v'(Q)} f|_k \left[\begin{pmatrix} 24v'Q & -1 \\ Q^2 & 0 \end{pmatrix}, Q^{\frac{1}{2}}z^{\frac{1}{2}} \right] = Q^{-1}f|U_Q|_k W_Q. \quad (4.18)$$

For $v \not\equiv 0 \pmod{Q}$, choose α with

$$24\alpha v \equiv 1 \pmod{Q^2} \quad \text{and} \quad \alpha = 24\alpha' \quad \text{with} \quad \alpha' \in \mathbb{Z}.$$

Using (2.2), the terms in (4.17) with $v \not\equiv 0 \pmod{Q}$ give

$$\begin{aligned} Q^{\frac{k}{2}-2} \sum_{\substack{v(Q^2) \\ v \not\equiv 0(Q)}} f|_k \left(\frac{24v}{Q^2} \frac{24\alpha v - 1}{\alpha} \right)^* \left[\begin{pmatrix} 1 & -\alpha \\ 0 & Q^2 \end{pmatrix}, Q^{\frac{1}{2}} \right] &= Q^{\frac{k}{2}-2} e\left(\frac{-r}{8}\right) \sum_{\substack{v(Q^2) \\ v \not\equiv 0(Q)}} f|_k \left[\begin{pmatrix} 1 & -\alpha \\ 0 & Q^2 \end{pmatrix}, Q^{\frac{1}{2}} \right] \\ &= Q^{-\frac{k}{2}-2} e\left(\frac{-r}{8}\right) \sum_n a(n) q^{\frac{n}{24Q^2}} \sum_{\substack{v(Q^2) \\ v \not\equiv 0(Q)}} \zeta_{Q^2}^{-n\alpha'}. \end{aligned} \quad (4.19)$$

The inner sum is

$$\sum_{\substack{v(Q^2) \\ v \not\equiv 0(Q)}} \zeta_{Q^2}^{nv} = \sum_{v(Q^2)} \zeta_{Q^2}^{nv} - \sum_{v(Q)} \zeta_{Q^2}^{nv} = \begin{cases} Q^2 - Q & \text{if } Q^2 \mid n, \\ -Q & \text{if } Q \parallel n, \\ 0 & \text{if } Q \nmid n. \end{cases}$$

From (4.9) we have $a(n) \equiv 0 \pmod{\ell}$ if $Q \parallel n$. Therefore the expression in (4.19) is congruent to

$$Q^{-\frac{k}{2}-2}(Q^2 - Q)e\left(\frac{-r}{8}\right) \sum a(Q^2 n) q^{\frac{n}{24}} \equiv Q^{-\frac{k}{2}}(1 - Q^{-1})e\left(\frac{-r}{8}\right)f|U_{Q^2} \pmod{\ell}.$$

The claim (4.16) follows from this together with (4.18).

If $Q \not\equiv 1 \pmod{\ell}$ it follows from (4.12) and (4.16) that

$$f|U_Q|_k W_Q \equiv Q^{-\frac{k}{2}}e\left(\frac{-r}{8}\right)f|U_{Q^2} \pmod{\ell}.$$

Combining this with (4.13) gives

$$f|U_{Q^2} \equiv Q^{k-1}e\left(\frac{r(1-Q)}{8}\right)\left(\frac{-24}{Q}\right)G(Q)f \otimes \chi_Q + Q^{2k-1}f|V_{Q^2} \pmod{\ell}.$$

From (2.3) and (2.4), we obtain

$$f|U_{Q^2} \equiv Q^{k-\frac{1}{2}}\varepsilon_Q^{r+1}\left(\frac{-12}{Q}\right)f \otimes \chi_Q + Q^{2k-1}f|V_{Q^2} \pmod{\ell}.$$

From (2.6) we have $2k \equiv r \pmod{4}$. This gives (4.10), and (4.11) then follows from the definition (2.11) of the Hecke operator. \square

We require a lemma before turning to the proof of Proposition 4.2.

Lemma 4.4. *Under the hypotheses of Proposition 4.2, suppose that $Q_0 \geq 5$, $Q_0 \neq \ell$ is a prime with $Q_0 \not\equiv \pm 1 \pmod{\ell}$ for which (4.8) holds. Define*

$$\mathcal{Q} := \{Q \text{ prime} : (4.8) \text{ holds for } Q, \quad Q \equiv Q_0 \pmod{12\ell} \quad \text{and} \\ f|T_{Q^2} \equiv f|T_{Q_0^2} \pmod{\ell}\}.$$

Then

$$\#\{Q \in \mathcal{Q} : Q \leq X\} \gg \frac{X}{\log X}. \quad (4.20)$$

Proof of Lemma 4.4. Suppose that $f \in S_k(1, \nu_\eta^r)$ and that $f \not\equiv 0 \pmod{\ell}$. For each square-free t let

$$F_t \in S_{2k-1}^{\text{new}}(6)$$

be the form with

$$\text{Sh}_t f = F_t \otimes \left(\frac{12}{\bullet}\right).$$

By (2.14) and (2.15), for all primes $Q \geq 5$ we have

$$f|T_{Q^2} \equiv f|T_{Q_0^2} \pmod{\ell} \iff F_t|T_Q \equiv F_t|T_{Q_0} \pmod{\ell} \quad \text{for all } t. \quad (4.21)$$

As t ranges over all squarefree integers, there are only finitely many non-zero possibilities for $F_t \pmod{\ell}$. Let $\{F_{t_1}, \dots, F_{t_k}\}$ be a collection which represents all of these possibilities; from the definition (2.13) we see that this collection is not empty.

The space $S_{2k-1}^{\text{new}}(6)$ is spanned by newforms g_1, \dots, g_d . Write

$$F_{t_j} = \sum_{i=1}^d c_{i,j} g_i.$$

Let L be the number field generated by the coefficients of g_1, \dots, g_d as well as the collection $\{c_{i,j}\}$ and let \mathcal{O}_L be the ring of integers. Let π be a prime ideal above ℓ in \mathcal{O}_L . Define

$$m := \max(1, 1 - \min(\text{ord}_\pi(c_i))) > 0.$$

For each i , it follows from the work of Deligne (see, for example, [DS74, Thm. 6.7]) that there is a Galois representation

$$\rho_i : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathcal{O}_L/\pi^m),$$

unramified outside of 6ℓ , such that if

$$g_i = \sum_{n=1}^{\infty} a_i(n) q^n,$$

then for all primes $Q \nmid 6\ell$, we have

$$\text{Tr}(\rho_i(\text{Frob}_Q)) \equiv a_i(Q) \pmod{\pi^m}.$$

Let ψ denote the mod 12ℓ cyclotomic character.

All of the ρ_i and ψ have finite images. By the Chebotarev density theorem, there is a set of primes \mathcal{P} with positive lower density such that for $Q \in \mathcal{P}$ we have

$$a_i(Q) \equiv a_i(Q_0) \pmod{\pi^m} \quad \text{for all } i$$

and $\psi(Q) = \psi(Q_0)$. For such Q we have

$$F_{t_j}|T_Q = \sum_{i=1}^d c_{i,j} a_i(Q) g_i \equiv \sum_{i=1}^d c_{i,j} a_i(Q_0) g_i \equiv F_{t_j}|T_{Q_0} \pmod{\pi}.$$

By assumption, all but finitely many primes Q satisfy (4.8). The lemma follows from (4.21). \square

Proof of Proposition 4.2. Fix a prime $Q_0 \geq 5$ with $Q_0 \neq \ell$ and $Q_0 \not\equiv \pm 1 \pmod{\ell}$ for which (4.8) holds, and let \mathcal{Q} be the set provided by Lemma 4.4 (recall that $Q_0 \in \mathcal{Q}$). Define

$$c := Q_0^{k-\frac{1}{2}} \left(\frac{-1}{Q_0}\right)^{k-\frac{1}{2}} \left(\frac{12}{Q_0}\right).$$

Then for $Q \in \mathcal{Q}$, (4.10) and (4.11) give

$$\sum \binom{n}{Q} a(n) q^{\frac{n}{24}} + c \sum a(n) q^{\frac{Q^2 n}{24}} \equiv \sum \binom{n}{Q_0} a(n) q^{\frac{n}{24}} + c \sum a(n) q^{\frac{Q_0^2 n}{24}} \pmod{\ell}. \quad (4.22)$$

It follows that

$$\sum_{(n, Q Q_0)=1} \left[\binom{n}{Q} - \binom{n}{Q_0} \right] a(n) q^{\frac{n}{24}} \equiv 0 \pmod{\ell}. \quad (4.23)$$

From (4.8), we see that

$$a(n) \not\equiv 0 \pmod{\ell} \implies \text{ord}_Q(n) \text{ is even for all } Q \in \mathcal{Q},$$

and from (4.10) we see that for all $Q \in \mathcal{Q}$ we have

$$a(Q^2 n) \not\equiv 0 \pmod{\ell} \implies a(n) \not\equiv 0 \pmod{\ell}.$$

We conclude that if $a(n) \not\equiv 0 \pmod{\ell}$ then

$$n = M^2 n' \text{ where } M \text{ is divisible only by primes in } \mathcal{Q}, \text{ and } Q \nmid n' \text{ for all } Q \in \mathcal{Q}. \quad (4.24)$$

Moreover, for such values of n , (4.23) gives

$$\binom{n'}{Q} = \binom{n'}{Q_0} \text{ for all } Q \in \mathcal{Q}. \quad (4.25)$$

We now apply the arithmetic large sieve [Mon68]. Let N be a parameter, and let \mathcal{N} be the set of integers consisting of all $n \leq N$ with $a(n) \not\equiv 0 \pmod{\ell}$. Write $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_{-1}$, where

$$\mathcal{N}_\varepsilon := \left\{ n \in \mathcal{N} : \binom{n'}{Q_0} = \varepsilon \right\}, \quad (4.26)$$

and let $Z_\varepsilon := |\mathcal{N}_\varepsilon|$.

Fix $\varepsilon \in \{\pm 1\}$ and write $\mathcal{N}_\varepsilon = \{n_1, \dots, n_{Z_\varepsilon}\}$. Using (4.24) and (4.25) we see that for each $i \in \{1, \dots, Z_\varepsilon\}$, we have

$$\binom{n_i}{Q} = 0 \text{ or } \binom{n_i}{Q} = \varepsilon \text{ for all } Q \in \mathcal{Q}.$$

For each prime Q , define

$$w(Q) := \begin{cases} \frac{Q-1}{2} & \text{if } Q \in \mathcal{Q}, \\ 0 & \text{else.} \end{cases}$$

Let X be another parameter. Then for each $Q \leq X$, there are $w(Q)$ residue classes modulo Q which contain no element of \mathcal{N}_ε . By [Mon68] we have

$$Z_\varepsilon \leq \frac{(N^{\frac{1}{2}} + X)^2}{M},$$

with

$$M := \sum_{\substack{Q \leq X \\ Q \text{ prime}}} \frac{w(Q)}{Q - w(Q)} = \sum_{\substack{Q \leq X \\ Q \in \mathcal{Q}}} \frac{Q - 1}{Q + 1} \gg \frac{X}{\log X},$$

where in the last estimate we have used (4.20). Setting $X = \sqrt{N}$ gives $Z_\varepsilon \ll \sqrt{N} \log N$. Thus

$$|\mathcal{N}| = Z_1 + Z_{-1} \ll \sqrt{N} \log N.$$

This proves the first assertion in Proposition 4.2.

We turn to the second assertion of Proposition 4.2. By the hypotheses, for all but finitely many primes $Q \equiv -1 \pmod{\ell}$, (4.11) gives

$$f|T_{Q^2} \equiv 0 \pmod{\ell}. \quad (4.27)$$

Suppose that there is a prime $Q_0 \equiv -1 \pmod{\ell}$ for which $f|T_{Q_0^2} \not\equiv 0 \pmod{\ell}$. Arguing as in Lemma 4.4, we see that a positive proportion of primes $Q \equiv -1 \pmod{\ell}$ have

$$f|T_{Q^2} \equiv f|T_{Q_0^2} \not\equiv 0 \pmod{\ell}.$$

It follows that such a prime Q_0 does not exist. In other words, (4.27) holds for all $Q \equiv -1 \pmod{\ell}$. The last assertion of Proposition 4.2 follows. \square

5. PROOF OF THEOREM 1.6

We start by proving a lemma which we will employ to handle the first case of Theorem 1.6. The proof is similar in spirit to the proof of Lemma 3.3. Since there are a number of technical differences we present a self-contained proof for the reader's benefit.

Lemma 5.1. *Suppose that $\ell \geq 5$ and $Q \geq 5$ are primes with $Q \neq \ell$. Suppose that $(r, 24) = 1$, that $a \in \{0, 1\}$, and that $f \in M_k(Q, \chi_Q^a \nu_\eta^r)$. Then*

$$f|U_Q \equiv 0 \pmod{\ell} \iff f \equiv 0 \pmod{\ell}.$$

Proof. Only one direction requires proof. By Lemma 2.1, we have

$$f|U_Q|_k W_Q = Q^{\frac{k}{2}-1} \sum_{v(Q)} f|_k \left[\begin{pmatrix} 24vQ & -1 \\ Q^2 & 0 \end{pmatrix}, Q^{\frac{1}{2}} z^{\frac{1}{2}} \right]. \quad (5.1)$$

The term with $v = 0$ in (5.1) gives

$$Q^{\frac{k}{2}-1} f|_k \left[\begin{pmatrix} 0 & -1 \\ Q & 0 \end{pmatrix}, Q^{\frac{1}{4}} z^{\frac{1}{2}} \right] \left[\begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}, Q^{-\frac{1}{4}} \right] = Q^{k-1} f|_k W_Q|V_Q. \quad (5.2)$$

For each $v \not\equiv 0 \pmod{Q}$ in (5.1) we choose α with

$$24\alpha v \equiv 1 \pmod{Q} \quad \text{and} \quad \alpha = 24\alpha' \quad \text{with} \quad \alpha' \in \mathbb{Z}.$$

Write $f = \sum a(n)q^{\frac{n}{24}}$. From (2.2), the terms in (5.1) with $v \not\equiv 0 \pmod{Q}$ contribute

$$\begin{aligned} Q^{\frac{k}{2}-1} \sum_{v(Q)^*} f|_k \left(\frac{24v}{Q} \frac{24\alpha v-1}{\alpha} \right)^* \left[\begin{pmatrix} Q & -\alpha \\ 0 & Q \end{pmatrix}, 1 \right] &= Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \sum_{v(Q)^*} \left(\frac{\alpha}{Q}\right)^{a+r} f|_k \left[\begin{pmatrix} Q & -\alpha \\ 0 & Q \end{pmatrix}, 1 \right] \\ &= Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \sum_n a(n)q^{\frac{n}{24}} \sum_{v(Q)^*} \left(\frac{\alpha}{Q}\right)^{a+r} \zeta_Q^{-n\alpha'}. \end{aligned} \quad (5.3)$$

Suppose first that $a+r$ is even. Then the inner sum is $Q-1$ if $Q \mid n$ and -1 if $Q \nmid n$. So the last expression becomes

$$Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) (Q-1) f|U_Q V_Q - Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \sum_{Q \nmid n} a(n)q^{\frac{n}{24}}.$$

From this and (5.2) we obtain

$$f|U_Q|_k W_Q = Q^{k-1} f|_k W_Q|V_Q + Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) (Q-1) f|U_Q V_Q - Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \sum_{Q \nmid n} a(n)q^{\frac{n}{24}}.$$

If $f|U_Q \equiv 0 \pmod{\ell}$ then $f|U_Q V_Q \equiv 0 \pmod{\ell}$, and $f|U_Q|_k W_Q \equiv 0 \pmod{\ell}$ by (2.18). Therefore

$$Q^{k-1} f|_k W_Q|V_Q \equiv Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \sum_{Q \nmid n} a(n)q^{\frac{n}{24}} \pmod{\ell}. \quad (5.4)$$

We have

$$W_Q \begin{pmatrix} 1 & 24 \\ 0 & 1 \end{pmatrix}^* = \begin{pmatrix} -1 & 0 \\ 24Q & 1 \end{pmatrix}^* W_Q = \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, -i \right] \begin{pmatrix} -1 & 0 \\ 24Q & -1 \end{pmatrix}^* W_Q.$$

From (2.2) we see that $(f|_k W_Q)(\tau + 24) = (-i)^{-2k} e\left(\frac{-r}{4}\right) \chi_Q^\alpha(-1) (f|_k W_Q)(\tau)$. The compatibility condition (2.6) can be expressed as $\chi_Q^\alpha(-1) = \left(\frac{-1}{r}\right) (-1)^{k-\frac{1}{2}}$; it follows that $(f|_k W_Q)(\tau + 24) = (f|_k W_Q)(\tau)$. Therefore the Fourier expansion of the left side of (5.4) is supported on exponents in $\frac{Q}{24}\mathbb{Z}$. Since the right side is supported on exponents prime to Q , we see that $f|_k W_Q \equiv 0 \pmod{\ell}$. Using (2.18) again, we obtain $f \equiv 0 \pmod{\ell}$.

If on the other hand $a+r$ is odd, the inner sum in (5.3) reduces to $\left(\frac{-24n}{Q}\right) G(Q)$, and we obtain

$$f|U_Q|_k W_Q = Q^{k-1} f|_k W_Q|V_Q + Q^{\frac{k}{2}-1} e\left(\frac{-rQ}{8}\right) \left(\frac{-24}{Q}\right) G(Q) f \otimes \chi_Q.$$

If $f|U_Q \equiv 0 \pmod{\ell}$ then we conclude that $f \equiv 0 \pmod{\ell}$ as before. \square

We will need the following statement about Kloosterman sums

$$K(a, b, c) := \sum_{n(c)^*} e\left(\frac{an + b\bar{n}}{c}\right),$$

where the summation is over residue classes prime to c and $n\bar{n} \equiv 1 \pmod{c}$.

Lemma 5.2. *Let a, b be integers and p, ℓ distinct primes, ℓ odd. Then we have*

$$K(a', b', p) \not\equiv 0 \pmod{\ell}.$$

Remark. The authors are grateful to Will Sawin, who pointed out a proof of this lemma.

Proof. The Kloosterman sum takes values in algebraic integers of the p -th cyclotomic field:

$$\mathbb{Z}(\zeta_p) \cong \mathbb{Z}[X]/(X^{p-1} + X^{p-2} + \cdots + 1) \cong (\mathbb{Z}[X]/(X^p - 1))/(X^{p-1} + X^{p-2} + \cdots + 1).$$

Reduction modulo ℓ intertwines with the polynomial quotient, yielding a ring over the finite field with ℓ elements. It therefore suffices to examine the sum

$$\sum_{0 < s < p} X^{as+b\bar{s}} = \sum_{i=0}^{p-1} c_i X^i \in \mathbb{Z}[X]/(X^p - 1).$$

The lemma follows if we show that the coefficients c_i of X^i , $0 \leq i < p$, are not all equal modulo ℓ . We suppose the opposite and derive a contradiction.

Notice that $as + b\bar{s} \equiv h \pmod{p}$ for any $h \pmod{p}$ yields a quadratic equation in s , which has at most two solutions. In particular, we have $c_i \in \{0, 1, 2\}$. Since ℓ is odd, 0, 1, and 2 are distinct modulo ℓ . We are assuming that all c_i are equal modulo ℓ , and hence they coincide as integers. Comparing the number of terms in the sum with the possible values of the c_i , we find that $p - 1 = \sum_i c_i = pc_0 \in \{0, p, 2p\}$, a contradiction. \square

Proof of Theorem 1.6. We begin with the first assertion. Suppose that there is a congruence (1.5) with $Q^2 \mid 24\beta_{\ell, Q} - 1$, and set $\delta = \left(\frac{1 - 24\beta_{\ell, Q}}{\ell}\right)$. Then Theorem 3.1 implies that $f_{\ell, \delta} \mid U_{Q^2} \equiv 0 \pmod{\ell}$, while the first assertion of Theorem 1.5 implies that $f_{\ell, \delta} \mid U_Q \not\equiv 0 \pmod{\ell}$. In view of (2.8), this contradicts Lemma 5.1.

The second assertion follows directly from Lemma 4.5 of [Rad13]. The proof of the third assertion is more difficult. To begin, we proceed as in the proof of Theorem 1.5. Suppose that there is a congruence

$$p(\ell Q^2 n + \beta_0) \equiv 0 \pmod{\ell} \quad \text{with} \quad Q \parallel 24\beta_0 - 1,$$

and define

$$\delta := \left(\frac{1 - 24\beta_0}{\ell}\right) \in \{0, -1\}.$$

Recalling the definitions (3.1), (3.11), suppose that

$$\beta \in S_{\ell Q^2, \beta_0},$$

and let

$$\gamma_{\ell, Q^2} = \begin{pmatrix} 1 & -24\ell X \\ \ell & Q^2 Y \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).$$

Recall the definition (3.9). Arguing as in (3.17) with Q replaced by Q^2 , we conclude that

$$(\ell\tau + Q^2 Y)^{\frac{1}{2}} g(\ell Q^2, \beta, \gamma_{\ell, Q^2} \tau) \equiv 0 \pmod{\ell}.$$

Let $H_{\ell, Q^2, \beta}$ denote the resulting right side of (3.12) with Q replaced by Q^2 ; we have

$$H_{\ell, Q^2, \beta} \equiv 0 \pmod{\ell}. \tag{5.5}$$

There are three terms in the sum defining $H_{\ell, Q^2, \beta}$. Recalling that $T(n, Q^2) = 1$, the $d = Q^2$ term becomes

$$Q^{-1} q^{\frac{(24t_{Q^2, \beta} - 1)Q^2}{24\ell}} \sum_{n=0}^{\infty} p(\ell n + t_{Q^2, \beta}) q^{Q^2 n} = Q^{-1} \sum_{n \equiv 24t_{Q^2, \beta} - 1 \pmod{\ell}} p\left(\frac{n+1}{24}\right) q^{\frac{Q^2 n}{24\ell}}. \tag{5.6}$$

For the $d = Q$ term, (3.15) gives

$$T(n, Q) = \left(\frac{-\ell}{Q}\right) S(24\beta - 1, \overline{24}^2 \ell^4 (24(\ell n + t_{Q,\beta}) - 1), Q).$$

Since $Q \mid 24\beta - 1$, this reduces to

$$T(n, Q) = \left(\frac{-\ell}{Q}\right) \left(\frac{24(\ell n + t_{Q,\beta}) - 1}{Q}\right) G(Q).$$

Substituting n for $24(\ell n + t_{Q,\beta}) - 1$ and using (2.3) and (2.4), the $d = Q$ term becomes

$$\begin{aligned} Q^{-\frac{1}{2}} e\left(\frac{1-Q}{8}\right) G(Q) (-1)^{\frac{Q-1}{2}} \left(\frac{-24}{Q}\right) \sum_{n \equiv 24t_{Q,\beta}-1 \pmod{\ell}} \left(\frac{n}{Q}\right) p\left(\frac{n+1}{24}\right) q^{\frac{n}{24\ell}} \\ = \left(\frac{-12}{Q}\right) \sum_{n \equiv 24t_{Q,\beta}-1 \pmod{\ell}} \left(\frac{n}{Q}\right) p\left(\frac{n+1}{24}\right) q^{\frac{n}{24\ell}}. \end{aligned} \quad (5.7)$$

For the $d = 1$ term, (3.15) gives

$$T(n, 1) = S(24\beta - 1, \overline{24}^2 \ell^4 (24(\ell n + t_{1,\beta}) - 1), Q^2).$$

This Salie sum vanishes whenever $Q \nmid 24(\ell n + t_{1,\beta}) - 1$ (this follows from the explicit evaluation [IK04, Lemma 12.4]). For the other terms, the Salie sum reduces to a Kloosterman sum; we have

$$T(n, 1) = Q K\left(\frac{24\beta - 1}{Q}, \overline{24}^2 \ell^4 \frac{24(\ell n + t_{1,\beta}) - 1}{Q}, Q\right) \quad \text{if } Q \mid 24(\ell n + t_{1,\beta}) - 1.$$

When $Q^2 \mid 24(\ell n + t_{1,\beta}) - 1$ the Kloosterman sum reduces to a Ramanujan sum which evaluates to -1 . Changing variables as above, the $d = 1$ term of the sum defining $H_{\ell, Q^2, \beta}$ becomes

$$\begin{aligned} -Q \sum_{Q^2 n \equiv 24t_{1,\beta}-1 \pmod{\ell}} p\left(\frac{Q^2 n + 1}{24}\right) q^{\frac{n}{24\ell}} \\ + Q \sum_{\substack{Qn \equiv 24t_{1,\beta}-1 \pmod{\ell} \\ Q \nmid n}} K\left(\frac{24\beta - 1}{Q}, \overline{24}^2 \ell^4 n, Q\right) p\left(\frac{Qn + 1}{24}\right) q^{\frac{n}{24\ell Q}}. \end{aligned} \quad (5.8)$$

Using (5.6), (5.7) and (5.8), we collect the terms of $H_{\ell, Q^2, \beta}$ whose exponents are in $\frac{1}{24\ell}\mathbb{Z}$ to find that the following expression vanishes modulo ℓ :

$$\begin{aligned} Q^{-1} \left(\sum_{n \equiv 24t_{Q^2, \beta}-1 \pmod{\ell}} p\left(\frac{n+1}{24}\right) q^{\frac{n}{24\ell}} \right) \Big|_{V_{Q^2}} + \left(\frac{-12}{Q}\right) \sum_{n \equiv 24t_{Q, \beta}-1 \pmod{\ell}} \left(\frac{n}{Q}\right) p\left(\frac{n+1}{24}\right) q^{\frac{n}{24\ell}} \\ - Q \left(\sum_{n \equiv 24t_{1, \beta}-1 \pmod{\ell}} p\left(\frac{n+1}{24}\right) q^{\frac{n}{24\ell}} \right) \Big|_{U_{Q^2}}. \end{aligned} \quad (5.9)$$

On the other hand, the sum of those terms of $H_{\ell, Q^2, \beta}$ whose exponents are not in $\frac{1}{24\ell}\mathbb{Z}$ must also vanish modulo ℓ , giving

$$\sum_{\substack{Qn \equiv 24t_{1, \beta} - 1 \pmod{\ell} \\ Q \nmid n}} K\left(\frac{24\beta - 1}{Q}, \overline{24^2 \ell^4 n}, Q\right) p\left(\frac{Qn + 1}{24\ell}\right) q^{\frac{n}{24\ell Q}} \equiv 0 \pmod{\ell}. \quad (5.10)$$

Define

$$S' := \{\beta \in S_{\ell Q^2, \beta_0} : \beta \equiv \beta_0 \pmod{Q^2}\}.$$

Then S' contains one representative for each residue class $\beta \pmod{\ell}$ with

$$\left(\frac{1 - 24\beta}{\ell}\right) = \left(\frac{1 - 24\beta_0}{\ell}\right) = \delta.$$

For each d , we see by (3.13) that $t_{d, \beta}$ ranges over those residue classes $t \pmod{\ell}$ with $\left(\frac{1 - 24t}{\ell}\right) = \delta$ as β ranges over S' . So for each d we have

$$f_{\ell, \delta} \equiv \sum_{\beta \in S'} \sum_{n \equiv 24t_{d, \beta} - 1 \pmod{\ell}} p\left(\frac{n + 1}{24}\right) q^{\frac{n}{24}} \pmod{\ell}. \quad (5.11)$$

Replacing q by q^ℓ in (5.9) and using (5.11) gives

$$Q^{-1} f_{\ell, \delta} |V_{Q^2} + \left(\frac{-12}{Q}\right) f_{\ell, \delta} \otimes \chi_Q - Q f_{\ell, \delta} |U_{Q^2} \equiv 0 \pmod{\ell}, \quad (5.12)$$

which is equivalent to (1.13). Replacing q by $q^{\ell Q}$ in (5.10), summing over $\beta \in S'$, and noting that for fixed n the Kloosterman sums which appear are independent of the choice of β , we obtain

$$\sum_{\substack{\left(\frac{-Qn}{\ell}\right) = \delta \\ Q \nmid n}} K\left(\frac{24\beta_0 - 1}{Q}, \overline{24^2 \ell^4 n}, Q\right) p\left(\frac{Qn + 1}{24}\right) q^{\frac{n}{24}} \equiv 0 \pmod{\ell}. \quad (5.13)$$

By Lemma 5.2, the Kloosterman sums in (5.13) are not divisible by ℓ . It follows that $a_{\ell, \delta}(Qn) \equiv 0 \pmod{\ell}$ for all n with $Q \nmid n$, which is equivalent to (1.14). This establishes one direction of the third assertion of Theorem 1.6.

For the other direction, suppose that (5.12) and (5.13) hold for some β_0 with $\left(\frac{1 - 24\beta_0}{\ell}\right) = \delta$ and $Q \parallel 24\beta_0 - 1$. By (5.11), we see that (5.9) and (5.10) hold with β replaced by β_0 . In other words, we have $H_{\ell, Q^2, \beta_0} \equiv 0 \pmod{\ell}$, from which

$$(\ell\tau + Q^2 Y)^{\frac{1}{2}} g(\ell Q^2, \beta_0, \gamma_{\ell, Q^2} \tau) \equiv 0 \pmod{\ell}.$$

By Lemma 2.3 we conclude that $g(\ell Q^2, \beta_0, \tau) \equiv 0 \pmod{\ell}$. □

6. COMPUTATIONS

Throughout this section, we assume that $\ell, Q \geq 5$ are distinct primes. The source code for each computation is available on the last named author's homepage (<https://www.raum-brothers.eu/martin>).

6.1. **Proof of Theorem 1.3.** Suppose that there is a congruence

$$p(\ell Qn + \beta_{\ell, Q}) \equiv 0 \pmod{\ell}$$

as in (1.4). Define $\delta = \left(\frac{1-24\beta_{\ell, Q}}{\ell}\right) \in \{0, -1\}$, and set

$$\mathcal{B} := \left\{ \frac{24\beta - 1}{\ell} : \beta \in \mathbb{Z}, \left(\frac{24\beta - 1}{\ell}\right) = \delta, p(\beta) \not\equiv 0 \pmod{\ell} \right\}, \quad \text{if } \delta = 0,$$

$$\mathcal{B} := \left\{ 24\beta - 1 : \beta \in \mathbb{Z}, \left(\frac{24\beta - 1}{\ell}\right) = \delta, p(\beta) \not\equiv 0 \pmod{\ell} \right\}, \quad \text{if } \delta = -1.$$

Define $\varepsilon = \left(\frac{24\beta_{\ell, Q} - 1}{Q}\right) \in \{\pm 1\}$. By Theorem 3.1, it follows that there are congruences

$$p(\ell Qn + \beta) \equiv 0 \pmod{\ell}$$

for all $\beta \in S_{\ell Q, \beta_{\ell, Q}}$, where $S_{\ell Q, \beta_{\ell, Q}}$ is as in (3.1). In particular, we have

$$\left(\frac{n}{Q}\right) \neq \varepsilon \text{ for all } n \in \mathcal{B}. \quad (6.1)$$

Since the set \mathcal{B} does not depend on Q , this opens the door to rule out possible Q via a sieve-like computation.

Given a finite subset $\mathcal{N} \subset \mathcal{B}$ and a bound Q_{\max} , Algorithm 1 computes the set

$$\mathcal{Q} := \left\{ Q \leq Q_{\max} : Q \text{ prime, there is } \varepsilon \in \{\pm 1\} \text{ such that for all } n \in \mathcal{N}, \left(\frac{n}{Q}\right) \neq \varepsilon \right\}. \quad (6.2)$$

If $\mathcal{Q} \subseteq \{2, 3, \ell\}$ for $\delta = -1$ or $\mathcal{Q} \subseteq \{2, 3, 5, 7, 11, \ell\}$ for $\delta = 0$, we conclude that there are no congruences (1.4) for $Q \leq Q_{\max}$ that do not arise from the Ramanujan congruences. If there are a few excess primes in \mathcal{Q} , we can check each individually to conclude that no corresponding congruence holds.

Data: A finite list of positive integers \mathcal{N} , positive integers Q_{\max} and Q_{stride} .

Result: The set \mathcal{Q} defined in (6.2).

Choose a subset \mathcal{P} of primes in \mathcal{N} such that $\prod_{p \in \mathcal{P}} p \approx Q_{\max}/Q_{\text{stride}}$;

$\mathcal{N} \leftarrow \mathcal{N} \setminus \mathcal{P}$;

$\mathcal{Q} \leftarrow \emptyset$;

for $\varepsilon \in \{\pm 1\}$:

for $p \in \mathcal{P}$:

$\mathcal{Q}_p \leftarrow \{n \pmod{p} : \left(\frac{n}{p}\right) \neq \varepsilon\}$;

for $(Q_p) \in \prod \mathcal{Q}_p$:

for $Q \leq Q_{\max}$ a probable prime with $Q \equiv Q_p \pmod{p}$ for all $p \in \mathcal{P}$:

if $\left(\frac{n}{Q}\right) \neq \varepsilon$ for all $n \in \mathcal{N}$:

if Q is a prime:

$\mathcal{Q} \leftarrow \mathcal{Q} \cup \{Q\}$;

Algorithm 1: Search for congruences (1.4)

We conclude with two remarks on Algorithm 1. In practice, the most time consuming part of computing (6.2) is to iterate through primes, while it is very cheap to evaluate the square-class conditions. For this reason, Algorithm 1 iterates through probable primes and only filters out non-primes after ensuring the square-class conditions. Splitting a set of

primes \mathcal{P} off the original \mathcal{N} reduces the number of transitioned primes Q by a factor of $2^{\#\mathcal{P}}$. The role of Q_{stride} in this context is to avoid extra computations that in practice arise from imposing the congruences $Q \equiv Q_p \pmod{p}$. The choice of Q_{stride} does not impact the result of Algorithm 1, but its performance. We have observed that changing Q_{stride} by no more than a factor of 10 can deteriorate performance by as much as a factor 20.

We provide an implementation that is based on the computer algebra package Nemo [FHHJ20, FHHJ17]. To prove Theorem 1.3 we employed it to verify the absence of congruences (1.4) for $\ell \leq 1,000$ and $Q \leq 10^{13}$, and $\ell \leq 10,000$ and $Q \leq 10^9$. Using a set \mathcal{N} of size at most 100, Algorithm 1 in the former computation excludes all but 3019 pairs (ℓ, Q) and in the latter computation all but 2 pairs. The exceptional pairs can all be ruled out by slightly enlarging \mathcal{N} .

6.2. Proof of Corollary 1.2. In Theorem 1.1 and its refinement 1.4, we state that the set S of primes Q for which there exists a congruence (1.2) for fixed $\left(\frac{1-24\beta_{\ell,Q}}{\ell}\right) = \delta$ has density zero or the estimate (1.9) and the Hecke congruences (1.10) hold. We have ruled out the latter for all primes $17 \leq \ell \leq 10^4$, for $\ell = 11$ if $\delta = -1$, and for $\ell = 13$ if $\delta = 0$ by a computer calculation. Observe that the missing cases $\ell \in \{5, 7, 11\}$ and $\delta = 0$ correspond to the Ramanujan congruences, and that the cases $\ell \in \{5, 7, 13\}$ and $\delta = -1$ correspond to Atkin's congruences for the partition function [Atk68b, Theorem 1].

Our code relies heavily on Johansson's performant implementation of the partition function [Joh12] available in ARB [Joh20] through Nemo [FHHJ20, FHHJ17]. The computation proceeds by enumerating primes $Q \equiv -1 \pmod{\ell}$ until we have found n with $(n, Q) = 1$ and $\left(\frac{-n}{\ell}\right) = \delta$ that falsifies the congruence (1.11).

6.3. Proof of Corollary 1.7. We proceed in a similar way to rule out congruences (1.5). Theorem 1.6 lists a number of necessary conditions that can be effectively checked, which we have falsified for all primes $Q \leq 10^4$ and the following primes ℓ : for $\ell = \{5, 7, 11\}$ if $\delta = -1$, for $\ell = 13$ if $\delta = 0$ and for $17 \leq \ell \leq 10^4$ if $\delta \in \{0, -1\}$.

The implementation is more involved than the previous one, but its effectiveness relies equally on Johansson's implementation of the partition function. Specifically, for each ℓ and Q , we try to falsify (1.13) by computing

$$p\left(\frac{Q^2n+1}{24}\right) - \left(\frac{-12\ell}{Q}\right) Q^{-1}p\left(\frac{n+1}{24}\right) \pmod{\ell}$$

for a few n with $(n, Q) = 1$ and $\left(\frac{-n}{\ell}\right) = \delta$. This suffices to handle many Q . Only if this falsification fails, we employ (1.14) by finding n with $(n, Q) = 1$ and $\left(\frac{-n}{\ell}\right) = \delta$ such that

$$p\left(\frac{Qn+1}{24}\right) \not\equiv 0 \pmod{\ell}.$$

REFERENCES

- [AAT21] Scott Ahlgren, Patrick Allen, and Shiang Tang, *Congruences like Atkin's for the partition function*, <https://arxiv.org/abs/2112.09481> (2021).
- [AB03] Scott Ahlgren and Matthew Boylan, *Arithmetic properties of the partition function*, *Invent. Math.* **153** (2003), no. 3, 487–502.
- [AB08] ———, *Odd coefficients of weakly holomorphic modular forms*, *Math. Res. Lett.* **15** (2008), no. 3, 409–418.
- [Ahl99] Scott Ahlgren, *Non-vanishing of the partition function modulo odd primes l* , *Mathematika* **46** (1999), no. 1, 185–192.

- [Ahl00] ———, *Distribution of the partition function modulo composite integers M* , Math. Ann. **318** (2000), no. 4, 795–803.
- [AK15] Scott Ahlgren and Byungchan Kim, *Mock theta functions and weakly holomorphic modular forms modulo 2 and 3*, Math. Proc. Cambridge Philos. Soc. **158** (2015), no. 1, 111–129.
- [And14] Nickolas Andersen, *Classification of congruences for mock theta functions and weakly holomorphic modular forms*, Q. J. Math. **65** (2014), no. 3, 781–805.
- [AO67] A. O. L. Atkin and J. N. O’Brien, *Some properties of $p(n)$ and $c(n)$ modulo powers of 13*, Trans. Amer. Math. Soc. **126** (1967), 442–459.
- [AO01a] Scott Ahlgren and Ken Ono, *Addition and counting: the arithmetic of partitions*, Notices Amer. Math. Soc. **48** (2001), no. 9, 978–984.
- [AO01b] ———, *Congruence properties for the partition function*, Proc. Natl. Acad. Sci. USA **98** (2001), no. 23, 12882–12884.
- [AO01c] ———, *Congruences and conjectures for the partition function*, q -series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), Contemp. Math., vol. 291, Amer. Math. Soc., Providence, RI, 2001, pp. 1–10.
- [Atk67] A. O. L. Atkin, *Proof of a conjecture of Ramanujan*, Glasgow Math. J. **8** (1967), 14–32.
- [Atk68a] ———, *Multiplicative congruence properties and density problems for $p(n)$* , Proc. London Math. Soc. (3) **18** (1968), 563–576.
- [Atk68b] ———, *Multiplicative congruence properties and density problems for $p(n)$* , Proc. London Math. Soc. (3) **18** (1968), 563–576.
- [BGS18] Joël Bellaïche, Ben Green, and Kannan Soundararajan, *Nonzero coefficients of half-integral weight modular forms mod ℓ* , Res. Math. Sci. **5** (2018), no. 1, Paper No. 6, 10.
- [BO99] Bruce C. Berndt and Ken Ono, *Ramanujan’s unpublished manuscript on the partition and tau functions with proofs and commentary*, Sémin. Lothar. Combin. **42** (1999), The Andrews Festschrift (Maratea, 1998).
- [DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 143–316. Lecture Notes in Math., Vol. 349.
- [DS74] Pierre Deligne and Jean-Pierre Serre, *Formes modulaires de poids 1*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 507–530 (1975).
- [FHHJ17] Claus Fieker, William Hart, Tommy Hofmann, and Fredrik Johansson, *Nemo/Hecke: computer algebra and number theory packages for the Julia programming language*, ISSAC’17—Proceedings of the 2017 ACM International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2017, pp. 157–164.
- [FHHJ20] ———, *Nemo*, 2020, Version 0.17.2, <https://nemocas.org>.
- [IK04] Henryk Iwaniec and Emmanuel Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004.
- [Joc] Naomi Jochnowitz, *Congruences between modular forms of half integral weights and implications for class numbers and elliptic curves*, preprint.
- [Joh12] Fredrik Johansson, *Efficient implementation of the Hardy-Ramanujan-Rademacher formula*, LMS J. Comput. Math. **15** (2012), 341–359.
- [Joh20] ———, *Arb*, 2020, Version 2.17, <https://github.com/fredrik-johansson/arb>.
- [JS81a] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. **103** (1981), no. 4, 777–815.
- [JS81b] ———, *On Euler products and the classification of automorphic representations. I*, Amer. J. Math. **103** (1981), no. 3, 499–558.
- [Kno70] Marvin I. Knopp, *Modular functions in analytic number theory*, Markham Publishing Co., Chicago, Ill., 1970.
- [KO92] Ian Kiming and Jørn B. Olsson, *Congruences like Ramanujan’s for powers of the partition function*, Arch. Math. (Basel) **59** (1992), no. 4, 348–360.
- [Mon68] H. L. Montgomery, *A note on the large sieve*, J. London Math. Soc. **43** (1968), 93–98.
- [New57] Morris Newman, *Congruences for the coefficients of modular forms and some new congruences for the partition function*, Canadian J. Math. **9** (1957), 549–552.

- [Niw75] Shinji Niwa, *Modular forms of half integral weight and the integral of certain theta-functions*, Nagoya Math. J. **56** (1975), 147–161.
- [Ono00] Ken Ono, *Distribution of the partition function modulo m* , Ann. of Math. (2) **151** (2000), no. 1, 293–307.
- [Rad12] Cristian-Silviu Radu, *A proof of Subbarao’s conjecture*, J. Reine Angew. Math. **672** (2012), 161–175.
- [Rad13] ———, *Proof of a conjecture by Ahlgren and Ono on the non-existence of certain partition congruences*, Trans. Amer. Math. Soc. **365** (2013), no. 9, 4881–4894.
- [Ram19] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n* , Proc. Cambridge Philos. Soc. **19** (1919), 207–210.
- [Ram20] ———, *Congruence properties of partitions*, Proc. London Math. Soc (Records for 13 March 1919) (1920).
- [Ram21] ———, *Congruence properties of partitions*, Math. Z. **9** (1921), no. 1-2, 147–153.
- [Ser73] Jean-Pierre Serre, *Formes modulaires et fonctions zêta p -adiques*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), 1973, pp. 191–268. Lecture Notes in Math., Vol. 350.
- [Shi73] Goro Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481.
- [Tre06] Stephanie Treener, *Congruences for the coefficients of weakly holomorphic modular forms*, Proc. London Math. Soc. (3) **93** (2006), no. 2, 304–324.
- [Tre08] ———, *Quadratic twists and the coefficients of weakly holomorphic modular forms*, J. Ramanujan Math. Soc. **23** (2008), no. 3, 283–309.
- [Wat38] G. N. Watson, *Ramanujans Vermutung über Zerfallungszahlen*, J. Reine Angew. Math. **179** (1938), 97–128.
- [Wea01] Rhiannon L. Weaver, *New congruences for the partition function*, Ramanujan J. **5** (2001), no. 1, 53–63.
- [Yan14] Yifan Yang, *Modular forms of half-integral weights on $SL(2, \mathbb{Z})$* , Nagoya Math. J. **215** (2014), 1–66.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801

Email address: sahlgren@illinois.edu

DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LA 70118

Email address: obeckwith@tulane.edu

CHALMERS TEKNISKA HÖGSKOLA OCH GÖTEBORGS UNIVERSITET, INSTITUTIONEN FÖR MATEMATISKA VETENSKAPER, SE-412 96 GÖTEBORG, SWEDEN

Email address: martin@raum-brothers.eu