

LOW REGULARITY ILL-POSEDNESS FOR ELASTIC WAVES DRIVEN BY SHOCK FORMATION

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ABSTRACT. In this paper, we construct counterexamples to the local existence of low-regularity solutions to elastic wave equations in three spatial dimensions (3D). Inspired by the recent works of Christodoulou, we generalize Lindblad's classic results on the scalar wave equation by showing that the Cauchy problem for 3D elastic waves, a physical system with multiple wave-speeds, are ill-posed in $H^3(\mathbb{R}^3)$. We further prove that the ill-posedness is caused by instantaneous shock formation, which is characterized by the vanishing of the inverse foliation density. The main difficulties of the 3D case come from the multiple wave-speeds and its associated non-strict hyperbolicity. We obtain the desired results by designing and combining a geometric approach and an algebraic approach, equipped with detailed studies and calculations of the structures and coefficients of the corresponding non-strictly hyperbolic system. Moreover, the ill-posedness we depict also applies to 2D elastic waves, which corresponds to a strictly hyperbolic case.

1. INTRODUCTION

We study the low regularity ill-posedness problem for elastic waves in three spatial dimensions. For homogeneous isotropic hyperelastic materials, the motion of displacement $U = (U^1, U^2, U^3)$ satisfies a quasilinear wave system with multiple wave-speeds:

$$(1.1) \quad \partial_t^2 U - c_2^2 \Delta U - (c_1^2 - c_2^2) \nabla(\nabla \cdot U) = G(\nabla U, \nabla^2 U),$$

where c_1, c_2 are two constants satisfying $c_1 > c_2 > 0$. The precise form of $G(\nabla U, \nabla^2 U)$ will be discussed in Section 1.1.

The study of system (1.1) was pioneered by John. For the Cauchy problem of the three dimensional elastic wave equations with (smooth) small initial data, he proved that the singularities could arise in the radially symmetric case [16]. Without symmetry assumptions, the existence of almost-global solutions was later obtained in [17] by John and [21] by Klainerman-Sideris. In the case that the nonlinearities satisfy the null conditions, the solutions to the Cauchy problem of (1.1) with small initial data exist globally. See Agemi [1] and Sideris [29, 30].

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In this paper, we focus on the low regularity solutions to elastic wave equations. The main conclusion we obtain is that *The Cauchy problems of the 3D elastic wave equations are ill-posed in $H^3(\mathbb{R}^3)$* . We further show that the mechanism behind ill-posedness is the formation of shock, which is characterized by the vanishing of the inverse foliation density.

Our research is motivated by a series of classic works on the scalar wave equations. With the aid of planar symmetry, Lindblad gave sharp counterexamples to the local existence of low regularity solutions to semilinear and certain quasilinear wave equations in [22–24]. With no symmetry assumption, the first results presenting singularity formation for solutions to quasilinear wave equations in more than one spatial dimension were due to Alinhac [2–4]. He employed a Nash-Moser iteration scheme. But this approach is not capable of revealing information beyond the first blow-up point. Hence Alinhac imposed a non-degeneracy condition for initial data to close his arguments. In [7], a breakthrough was made by Christodoulou, where he offered a detailed understanding and a complete description of shock formation in 3D without imposing non-degeneracy assumptions. This remarkable work was extended to a large class of equations and data settings, see [25–27, 32–35]. Granowski made an important observation in his thesis, for a scalar wave equation, he connected Lindblad’s counterexample of local existence in [11] to the result of shock formation via the approach of Speck-Holzegel-Luk-Wong in [35]. He found that the low regularity ill-posedness of the quasilinear wave equation is driven by a shock formation. Moreover, he showed that this phenomenon is stable under a perturbation out of planar symmetry.

Compared with a single wave equation, fewer results are known for quasilinear wave systems. In [10], Ettinger-Lindblad studied Einstein’s equations and constructed a sharp counterexample for local well-posedness of Einstein vacuum equations in wave coordinates. In [8], Christodoulou-Perez studied the propagation of electromagnetic waves in nonlinear crystals. Under planar symmetry, these electromagnetic waves satisfy a first-order genuinely nonlinear and strictly hyperbolic system. By revisiting and further extending John’s work [15], Christodoulou-Perez gave a more detailed description of the behaviour of solutions at the shock-formation time. Speck [33] proved a stable shock formation result for a class of quasilinear wave systems with multiple wave-speeds under suitable assumptions on the nonlinearities.

In this paper, we study another physical system, the elastic waves. And we prove that the Cauchy problems for 3D elastic waves are ill-posed in $H^3(\mathbb{R}^3)$. We further show that the ill-posedness of elastic waves is also driven by shock formation. This $H^3(\mathbb{R}^3)$ ill-posedness is consistent with Lindblad’s sharp result for the scalar wave in [24], where he showed that the Cauchy problem for a (generic) single quasilinear wave equation is ill-posed in $H^3(\mathbb{R}^3)$. And the sharp local well-posedness was proven by Smith and Tataru for H^s with $s > 3$, and by Wang [44] via a geometrical approach. Our result extends Lindblad’s classic results on scalar wave equations to

a physical wave system and generalizes Christodoulou-Perez's work to the non-strictly hyperbolic case.

1.1. Statements of equations and main theorems. The motion of an elastic body in 3D is described by time-dependent orientation-preserving diffeomorphisms, written as $\Psi : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$, with $\Psi = \Psi(Y, t)$ satisfying $\Psi(Y, 0) = Y = (Y^1, Y^2, Y^3)$. The deformation gradient is defined as $F = \nabla \Psi$ with $F_j^i := \partial \Psi^i / \partial Y^j$. For a homogeneous isotropic hyperelastic material, the stored energy W depends only on the principal invariants of FF^T (the Cauchy-Green strain tensor). The equations of motion could be derived by employing the least action principle to \mathcal{L} :

$$\mathcal{L} := \int \int_{\mathbb{R}^2} \left(\frac{1}{2} |\partial_t \Psi|^2 - W(FF^T) \right) dY dt.$$

Then the Euler-Lagrange equations take the form:

$$(1.2) \quad \frac{\partial^2 \Psi^i}{\partial t^2} - \frac{\partial}{\partial Y^l} \frac{\partial W(FF^T)}{\partial F_l^i} = 0.$$

Let $U := \Psi - Y$ be the displacement and $G := \nabla U = F - I$ be the displacement gradient. Hence $FF^T = I + G + G^T + GG^T$. We thus have the stored energy W being a functional of $C = G + G^T + GG^T$. We further rewrite it as $W = \hat{W}(k_1, k_2, k_3)$. Here, we define k_1, k_2, k_3 to be the principal invariants of $C = G + G^T + GG^T$. In detail,

$$\begin{aligned} k_1 &= \mu_1 + \mu_2 + \mu_3 = \text{tr}C, \\ k_2 &= \sum_{\substack{i,j=1,2,3 \\ i < j}} \mu_i \mu_j = \frac{1}{2} \{ (\text{tr}C)^2 - \text{tr}C^2 \}, \\ k_3 &= \mu_1 \mu_2 \mu_3 = \frac{1}{6} \{ (\text{tr}C)^3 - 3(\text{tr}C)(\text{tr}C^2) + 2\text{tr}C^3 \}, \end{aligned}$$

where μ_1, μ_2, μ_3 are the eigenvalues of C . By Taylor expansions up to terms of higher order, we have

$$(1.3) \quad \hat{W} = \gamma_0 + \gamma_1 k_1 + \frac{1}{2} \gamma_{11} k_1^2 + \gamma_2 k_2 + \frac{1}{6} \gamma_{111} k_1^3 + \gamma_{12} k_1 k_2 + \gamma_3 k_3 + \dots$$

as a functional of ∇U . Here, $\gamma_0, \gamma_1, \gamma_{11}$, etc., are constant coefficients standing for certain partial derivatives of \hat{W} at $k_i = 0$ ($i = 1, 2, 3$). Following [29], we impose the stress-free reference state condition, i.e., $\gamma_1 = 0$. We define $4(\gamma_{11} + \gamma_2)$ and $-2\gamma_2$ to be the Lamé constants. Note that the positiveness of Lamé constants imply $-2\gamma_2 > 0$ and $4(\gamma_{11} + \gamma_2) - 2\gamma_2 = 2(2\gamma_{11} + \gamma_2) > 0$. More details are referred to [1, 29].

Remark: Tahvildar-Zadeh in [40] studied a case where $\gamma_1 \neq 0$ and proved the small data global existence under certain null condition assumptions.

Applying (1.3) into (1.2), we arrive at (1.1), where

$$c_1^2 = 4\gamma_{11}, \quad c_2^2 = -2\gamma_2 \quad \text{and} \quad c_1^2 > c_2^2 > 0.$$

Keeping only the quadratic nonlinear terms of (1.1), we derive a quasilinear wave system with multiple wave-speeds:

$$(1.4) \quad \begin{cases} \partial_t^2 U - c_2^2 \Delta U - (c_1^2 - c_2^2) \nabla(\nabla \cdot U) = N(\nabla U, \nabla^2 U), \\ U(Y, 0) = U_0(Y), \quad U_t(Y, 0) = U_1(Y) \end{cases}$$

with $N(\nabla U, \nabla^2 U)$ composed of $\nabla U \nabla^2 U$ -form. This means that neither $(\nabla U)^2$ nor $(\nabla^2 U)^2$ is included. As Agemi showed in [1], the quadratic nonlinear terms could be expressed as:

$$\begin{aligned} & N(\nabla U, \nabla^2 U) \\ &= \sigma_0 \nabla(\operatorname{div} U)^2 + \sigma_1 (\nabla |\operatorname{curl} U|^2 - 2 \operatorname{curl}(\operatorname{div} U \operatorname{curl} U)) + Q(U, \nabla U), \end{aligned}$$

where Q is a summation of null forms:

$$(1.5) \quad \begin{aligned} Q^i &= \sigma_2 \sum_{j,k=1}^3 \{2Q_{jk}(\partial_j U^i, U^k) + Q_{jk}(\partial_k U^j, U^i) + Q_{ij}(\partial_j U^k, U^k)\} \\ &+ \sigma_3 \sum_{j,k=1}^3 \{2Q_{ij}(\partial_k U^k, U^j) + Q_{ji}(\partial_k U^j, U^k)\} \\ &+ \sigma_4 \sum_{j,k=1}^3 \{Q_{ik}(U^k, \partial_j U^j) - Q_{jk}(U^j, \partial_i U^k)\}, \end{aligned}$$

with $Q_{ij}(f, g) = \partial_i f \partial_j g - \partial_j f \partial_i g$. The constants σ_i ($i = 0, 1, 2, 3, 4$) satisfy

$$\begin{aligned} \sigma_0 &= 2(2\gamma_{111} + 3\gamma_{11}), & \sigma_1 &= 2(\gamma_{11} - \gamma_{12}), & \sigma_2 &= 2(\gamma_2 - \gamma_3), \\ \sigma_3 &= 2\gamma_3, & \sigma_4 &= 4(\gamma_{11} - 2\gamma_{12}). \end{aligned}$$

In this paper, we will study elastic wave equations (1.4) for the general case

$$(1.6) \quad \sigma_0 \sigma_1 \neq 0.$$

Remark: In this paper, with the initial data we construct, the second derivatives of the displacement would tend to be infinite instantaneously, while the first derivatives of the displacement would remain small. This is because the smallness of the first derivatives can be transported along the characteristics. The cubic and higher-order terms take form of $(\nabla U)^\alpha \nabla U \nabla^2 U$ with $\alpha \geq 1$, which are negligible compared with terms in $N(\nabla U, \nabla^2 U)$ of the form $\nabla U \nabla^2 U$. So we omit the cubic and higher-order nonlinear terms in (1.1).

We are now ready to state our main result:

Theorem 1.1. *The Cauchy problems of the 3D elastic wave equations (1.4) are ill-posed in $H^3(\mathbb{R}^3)$ in the following sense: We construct a family of compactly supported smooth initial data $(U_0^{(\eta)}, U_1^{(\eta)})$ with*

$$\|U_0^{(\eta)}\|_{\dot{H}^3(\mathbb{R}^3)} + \|U_1^{(\eta)}\|_{\dot{H}^2(\mathbb{R}^3)} \rightarrow 0, \quad \text{as } \eta \rightarrow 0.$$

Let T_η^ be the largest time such that (1.4) (with a general condition (1.6)) has a solution $U_\eta \in C^\infty(\mathbb{R}^3 \times [0, T_\eta^*])$. As $\eta \rightarrow 0$, we have $T_\eta^* \rightarrow 0$.*

Moreover, in a spatial region $\Omega_{T_\eta^*}$ the H^2 norm of the solution to elastic waves (1.4) blows up at shock formation time T_η^* :

$$\|U_\eta(\cdot, T_\eta^*)\|_{H^2(\Omega_{T_\eta^*})} = +\infty.$$

We construct the solution of (1.4) by choosing modified ‘‘Lindblad-type’’ initial data. The above result can be extended to elastic waves in 2D as well. There we apply our method to a strictly hyperbolic system. Here is the 2D version counterpart.

Theorem 1.2. *The Cauchy problems of 2D elastic wave equations are ill-posed in $H^{\frac{5}{2}}(\mathbb{R}^2)$ in the following sense: We construct a family of compactly supported smooth initial data $(U_0^{(\eta)}, U_1^{(\eta)})$ with*

$$\|U_0^{(\eta)}\|_{\dot{H}^{\frac{5}{2}}(\mathbb{R}^2)} + \|U_1^{(\eta)}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)} \rightarrow 0, \quad \text{as } \eta \rightarrow 0.$$

Let T_η^* be the largest time such that (1.4) has a solution $U_\eta \in C^\infty(\mathbb{R}^2 \times [0, T_\eta^*])$. As $\eta \rightarrow 0$, we have $T_\eta^* \rightarrow 0$.

Moreover, in a spatial region $\Omega_{T_\eta^*}$ the H^2 norm of the solution to elastic waves (1.4) blows up at shock formation time T_η^* :

$$\|U_\eta(\cdot, T_\eta^*)\|_{H^2(\Omega_{T_\eta^*})} = +\infty.$$

Recently, Ohlmann [28] generalized Lindblad’s result [24] to the 2D case and proved the ill-posedness of a 2D single quasilinear wave equation in the logarithmic Sobolev space $H^{\frac{11}{4}}(\ln H)^{-\beta}$ with $\beta > \frac{1}{2}$. In 2D, we expect that $H^{\frac{11}{4}}$ is the critical space, as suggested by the local well-posedness result of Smith-Tataru [31].

1.2. Difficulties and Strategies. Though our basic strategy to construct ill-posedness is inspired by the aforementioned works, we encounter some new difficulties.

(1) *Coupling of top order terms in elastic wave equations*

Let’s start with a quasilinear wave equation with a single speed:

$$(1.7) \quad \square\varphi = \partial\varphi\partial^2\varphi.$$

By taking rectangular derivatives $\partial_t, \partial_1, \partial_2, \partial_3$ of (1.7), we deduce:

$$(1.8) \quad (g^{-1})^{\alpha\beta}(\psi)\partial_\alpha\partial_\beta\psi_l = N(\partial\psi, \partial\psi_l),$$

where

$$\psi = (\psi_0, \psi_1, \psi_2, \psi_3) := (\partial_t\varphi, \partial_1\varphi, \partial_2\varphi, \partial_3\varphi).$$

The geometric approach introduced by Christodoulou [7] could be used to prove shock formation for (1.8). However, this approach is not applicable to the elastic waves with multiple speeds. For a

slightly simplified model, Speck [33] studied the following system of two wave equations with multiple speeds:

$$(1.9) \quad \square_{g(\Upsilon)} \Upsilon \sim Q^g(\partial \Upsilon, \partial \Upsilon) + \sum_{\gamma, \nu=0,1} \left[N(\partial^\gamma q, \partial^\nu \Upsilon) + N(\partial^\gamma q, \partial^\nu q) \right],$$

$$(1.10) \quad (h^{-1})^{\alpha\beta}(\Upsilon, \partial q) \partial_\alpha \partial_\beta q \sim Q^g(\partial \Upsilon, \partial \Upsilon) + \sum_{\gamma, \nu=0,1} \left[N(\partial^\gamma q, \partial^\nu \Upsilon) + N(\partial^\gamma q, \partial^\nu q) \right],$$

where $\square_{g(\Upsilon)} := \frac{1}{\sqrt{|\det g(\Upsilon)|}} \partial_\alpha (\sqrt{|\det g(\Upsilon)|} (g^{-1})^{\alpha\beta} \partial_\beta)$ is the covariant wave operator with respect to $g(\Upsilon)$ and Q^g is the standard null form associated to g :

$$Q^g(\partial \Upsilon, \partial \Upsilon) := (g^{-1})^{\alpha\beta}(\Upsilon) \partial_\alpha \Upsilon \partial_\beta \Upsilon.$$

Note that no $\partial^2 q$ term is included on the right hand side of (1.9) (considered as the fast wave equation), and there is no $\partial^2 \Upsilon$ term coupled in equation (1.10). Employing the geometric method as in [7], Speck proved shock formation of the “fast wave Υ ”. For the more general quasilinear wave systems in 3D, as for elastic waves, the m unknowns $\bar{\varphi} = (\bar{\varphi}^1, \dots, \bar{\varphi}^m)$ satisfy:

$$\square_i \bar{\varphi}^i = \sum_{j,k} N_{jk}^i(\partial \bar{\varphi}^j, \partial^2 \bar{\varphi}^k).$$

Take a rectangular derivative, we have

$$\bar{\psi}^i = (\bar{\psi}_0^i, \bar{\psi}_1^i, \bar{\psi}_2^i, \bar{\psi}_3^i) := (\partial_t \bar{\varphi}^i, \partial_1 \bar{\varphi}^i, \partial_2 \bar{\varphi}^i, \partial_3 \bar{\varphi}^i)$$

verifying the system of

$$(1.11) \quad (g^{-1})^{\alpha\beta}(\bar{\psi}) \partial_\alpha \partial_\beta \bar{\psi}_l^i = \sum_{j,k} N_{jk}^i(\partial \bar{\psi}_l^j, \partial \bar{\psi}_l^k) + \underbrace{\sum_{\substack{j,k \\ k \neq i}} N_{jk}^i(\bar{\psi}^j, \partial^2 \bar{\psi}_l^k)}_{\text{new terms}},$$

where $1 \leq i \leq m$ and $0 \leq l \leq 3$. Compared with (1.9), there are $\partial^2 \bar{\psi}^k$ ($k \neq i$) terms coupled in every i^{th} equation. Because of the new terms in (1.11), there is a loss of derivatives and the geometric approaches in the aforementioned works are not applicable here. To study the ill-posedness problem, under the assumption of planar symmetry we use an alternative algebraic approach. The elastic waves (u^1, u^2, u^3) under planar symmetry obey

$$(1.12) \quad \begin{cases} \partial_t^2 u^1 - c_1^2 \partial_x^2 u^1 = \sigma_0 \partial_x (\partial_x u^1)^2 + \sigma_1 \partial_x (\partial_x u^2)^2 + \sigma_1 \partial_x (\partial_x u^3)^2, \\ \partial_t^2 u^2 - c_2^2 \partial_x^2 u^2 = 2\sigma_1 \partial_x (\partial_x u^1 \partial_x u^2), \\ \partial_t^2 u^3 - c_2^2 \partial_x^2 u^3 = 2\sigma_1 \partial_x (\partial_x u^1 \partial_x u^3), \end{cases}$$

where $c_1 > c_2 > 0$. Note that we have $\partial_x^2 u^2$ and $\partial_x^2 u^3$ in the first equation of (1.12), $\partial_x^2 u^1$ in the second equation of (1.12), and $\partial_x^2 u^1$ in the third equation of (1.12). We could rewrite (1.12) as a 6×6 first-order hyperbolic system:

$$(1.13) \quad \partial_t \Phi + A(\Phi) \partial_x \Phi = 0,$$

where $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)^T = (\partial_x u^1, \partial_x u^2, \partial_x u^3, \partial_t u^1, \partial_t u^2, \partial_t u^3)^T$. We then study the ill-posedness problem by exploring the algebraic structures of (1.13) and by combining the geometric method as in [8].

(2) *Invalidity of Riemann invariants for a 6×6 system*

For waves with different traveling speeds, we want to prove shock formation for the fastest one. A key ingredient is to understand the interactions of different families of characteristics. Naturally the first and foremost step is to find a proper method to trace these characteristics. For a single wave equation (or a 2×2 first-order hyperbolic system), there exist two Riemann invariants, which could help. These two Riemann invariants verify two transport equations and can be constructed explicitly with unknown functions. The evolution of other geometric quantities along the characteristics can then be described accordingly. However, for a larger system, such as our 6×6 system, to find proper Riemann invariants with explicit formula is usually impossible. Here we adopt a different approach. We appeal to John's classic method [15], i.e., decomposition of waves. We compute the eigenvalues of the coefficient matrix $(A(\Phi))_{6 \times 6}$:

$$\lambda_6(\Phi) < \lambda_5(\Phi) \leq \lambda_4(\Phi) < \lambda_3(\Phi) \leq \lambda_2(\Phi) < \lambda_1(\Phi).$$

We give its left eigenvectors $l^i(\Phi)$ and right eigenvectors $r_i(\Phi)$, $i = 1, \dots, 6$ and require:

$$l^i(\Phi)r_j(\Phi) = \delta_j^i.$$

Then (1.13) can be rewritten as a diagonalized system of Riccati-type:

$$(1.14) \quad (\partial_t + \lambda_i \partial_x)w^i = -c_{ii}^i \cdot (w^i)^2 + \left(\sum_{m \neq i} (-c_{im}^i + \gamma_{im}^i)w^m \right)w^i + \sum_{\substack{m \neq i, k \neq i \\ m \neq k}} \gamma_{km}^i w^k w^m,$$

where $i = 1, \dots, 6$ and

$$w^i := l^i(\Phi)\partial_x \Phi.$$

Here c_{im}^i, γ_{km}^i are coefficients depending on unknowns.

(3) *Non-strict hyperbolicity*

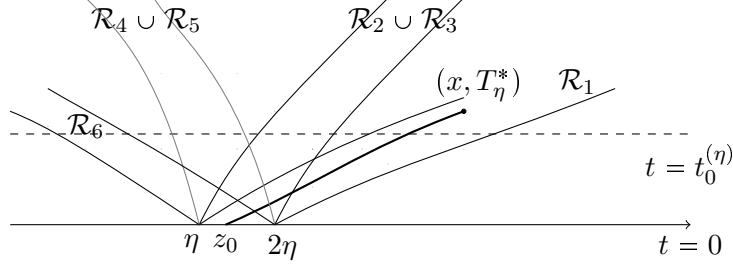
Equation (1.14) is the starting point of our further exploration. For the aforementioned system (1.13), for Φ being a small perturbation around zero, we have that two pairs of characteristic speeds are almost the same

$$\lambda_2(\Phi) \approx \lambda_3(\Phi), \quad \lambda_4(\Phi) \approx \lambda_5(\Phi).$$

This means two pairs of characteristic strips could overlap for a long time. Our strategy to overcome this non-strict hyperbolicity is to consider the wave propagations in four characteristic strips:

$$\{\mathcal{R}_1, \mathcal{R}_2 \cup \mathcal{R}_3, \mathcal{R}_4 \cup \mathcal{R}_5, \mathcal{R}_6\}.$$

These four strips will be completely separated when $t > t_0^{(\eta)}$, where $t_0^{(\eta)}$ can be precisely calculated. We then study the inverse foliation density ρ_1 corresponding to the fast wave. We show that ρ_1 vanishes (shock forms) in the strip \mathcal{R}_1 at some time T_η^* after $t = t_0^{(\eta)}$. And ρ_1 does not vanish in the overlapped characteristic strips. See the following picture:



Subtle structures of (1.14) are the key in our analysis, which makes it possible to trace ρ_i in the above four characteristic strips up to time T_η^* . We list some crucial structures here:

$$(1.15) \quad \begin{cases} (\partial_t + \lambda_1 \partial_x) w^1 \sim (w^1)^2 + \cancel{w^2 w^3} + \cancel{w^4 w^5} + \dots, \\ (\partial_t + \lambda_2 \partial_x) w^2 \sim \cancel{(w^2)^2} + (\lambda_2 - \lambda_3) w^2 w^3 + (\lambda_4 - \lambda_5) w^4 w^5 + \dots, \\ (\partial_t + \lambda_3 \partial_x) w^3 \sim (w^3)^2 + \cancel{w^2 w^3} + \cancel{w^4 w^5} + \dots, \\ (\partial_t + \lambda_4 \partial_x) w^4 \sim (w^4)^2 + \cancel{w^2 w^3} + \cancel{w^4 w^5} + \dots, \\ (\partial_t + \lambda_5 \partial_x) w^5 \sim \cancel{(w^5)^2} + (\lambda_2 - \lambda_3) w^2 w^3 + (\lambda_4 - \lambda_5) w^4 w^5 + \dots, \\ (\partial_t + \lambda_6 \partial_x) w^6 \sim (w^6)^2 + \cancel{w^2 w^3} + \cancel{w^4 w^5} + \dots. \end{cases}$$

The deleted terms with cancellation symbols in (1.15) mean their coefficients are zero. That is to say there is no interaction of the almost-repeated characteristic waves (w^2 and w^3 , w^4 and w^5) appearing in the equations of $\{w^i\}_{i=1,3,4,6}$.

Moreover, in the equations for w^2 and w^5 we also have $\lambda_2 - \lambda_3 = \lambda_4 - \lambda_5$ being small, and it shows that the related interactions are weak. These small coefficients allow us to use the bi-characteristic transformation to get desired bounds for non-strictly hyperbolic systems. For instance, with the equation $(\partial_t + \lambda_2 \partial_x) w^2$, to estimate w^2 we will need to bound the integration of $(\lambda_2 - \lambda_3) w^2 w^3$ within $\mathcal{R}_2 \cup \mathcal{R}_3$. See the exact estimates in (5.23) and (5.32). There we will use bi-characteristic transformation as in [8]

$$dt = \frac{\rho_2}{\lambda_3 - \lambda_2} dy_2 + \frac{\rho_3}{\lambda_2 - \lambda_3} dy_3.$$

This transformation could be singular owing to the non-strict hyperbolicity, since $\lambda_2(0) - \lambda_3(0) = 0$. Nevertheless, the additional small coefficient $\lambda_2 - \lambda_3$ in front of $w^2 w^3$ save us!

(4) *Meaning of blow-ups*

Notice that (1.15) are a set of Riccati-type equations. The key structures mentioned above ensure that singularities (blow-ups) will

form in finite time. A new ingredient of this paper is to study the meaning of the blow-ups and to study the ill-posedness mechanism of the first-order non-strictly hyperbolic system. Based on the decomposition of waves in an algebraic manner, we trace the evolution of inverse foliation density ρ_i ($i = 1, \dots, 6$) coming from geometry. Here ρ_i depicts the density of nearby characteristics in \mathcal{R}_i and it verifies

$$(\partial_t + \lambda_i \partial_x) \rho_i = c_{ii}^i v^i + \left(\sum_{m \neq i} c_{im}^i w^m \right) \rho_i,$$

where $v^i = \rho_i w^i$ for fixed i . By constructing suitable initial data, we get a positive lower bound for $\{\rho_i\}_{i=2, \dots, 6}$:

$$\min_{i \in \{2, \dots, 6\}} \inf_{0 \leq t \leq T_\eta^*} \rho_i \geq \frac{(1 - \varepsilon)^2}{2},$$

for some sufficiently small ε . Furthermore, ρ_1 is proved to obey

$$\begin{aligned} & (1 - \varepsilon) \left(1 - (1 + \varepsilon)^3 |c_{11}^1(0)| t W_0^{(\eta)} \right) \\ & \leq \rho_1(X_1(z_0, t), t) \leq (1 + \varepsilon) \left(1 - (1 - \varepsilon)^4 |c_{11}^1(0)| t W_0^{(\eta)} \right), \end{aligned}$$

where $W_0^{(\eta)} = \max_i \sup_z |w_{(\eta)}^i(z, 0)| = w_{(\eta)}^1(z_0, 0)$. In view of the above inequalities, we conclude that $\rho_1(z_0, t) \rightarrow 0$ as $t \rightarrow T_\eta^*$ with $T_\eta^* \sim 1/W_0^{(\eta)}$. A shock forms at time T_η^* and we can show that T_η^* is the first time when blow-up appears.

We then check the $H^2(\mathbb{R}^3)$ norm of solutions to (1.4) at time T_η^* . In a suitable constructed spatial region $\Omega_{T_\eta^*}$, we have

$$\begin{aligned} \|U(\cdot, T_\eta^*)\|_{H^2(\Omega_{T_\eta^*})}^2 & \gtrsim C_\eta \int_{z_0}^{z_0^*} \frac{1}{\rho_1(z, T_\eta^*)} dz \\ & \gtrsim C_\eta \int_{z_0}^{z_0^*} \frac{1}{(\sup_{z_1 \in (z_0, z_0^*]} |\partial_{z_1} \rho_1|)(z - z_0)} dz \\ & = +\infty. \end{aligned}$$

In the above inequalities, we utilize

$$\rho_1(z, T_\eta^*) = \rho_1(z, T_\eta^*) - \rho_1(z_0, T_\eta^*) = \partial_{z_1} \rho_1(z', T_\eta^*)(z - z_0)$$

for some $z' \in (z_0, z_0^*]$ and employ the uniform bound of $\partial_{z_1} \rho_1$ obtained in Section 9. This shows that the ill-posedness is driven by the shock formation: $\rho_1(z_0, t) \rightarrow 0$ as $t \rightarrow T_\eta^*$. Moreover, as $\eta \rightarrow 0$, we have $T_\eta^* \rightarrow 0$.

1.3. New ingredients. The method we develop in this paper might be useful for studying other problems. We list some of the key points:

- (1) For elastic waves (1.4), both our lower regularity ill-posedness result and our exploration of the ill-posedness mechanism in this paper are

new. We extend the previous result on a single quasilinear wave equation to a physical wave system.

- (2) For elastic waves with multiple speeds, under plane symmetry we give a complete description of the wave dynamics up to the time T_η^* , when the first (shock) singularity happens. For $t \leq T_\eta^*$, there is no other singular point in the spacetime region. And the solution is smooth before time T_η^* . We summarize it in Proposition 8.1.
- (3) Our algebraic approach of rewriting (1.12) into a 6×6 system and using r_j in (2.13) and l^i in (2.15) as right and left eigenvectors for wave decomposition is new. By algebraic calculations, we find the system (1.12) is non-strictly hyperbolic, with eigenvalues of the coefficient matrix $(A(\Phi))_{6 \times 6}$ satisfying

$$\lambda_6(\Phi) < \lambda_5(\Phi) \leq \lambda_4(\Phi) < \lambda_3(\Phi) \leq \lambda_2(\Phi) < \lambda_1(\Phi).$$

For $i, j = 1, \dots, 6$, for the left eigenvectors $l^i(\Phi)$ and right eigenvectors $r_j(\Phi)$ we construct, it holds

$$l^i(\Phi)r_j(\Phi) = \delta_j^i,$$

but there is not requirement of $l^i(\Phi)$ and $r_j(\Phi)$ being unit. This flexibility enables us to use bi-characteristic coordinates for our non-strictly hyperbolic system.

- (4) It is the first time the subtle structures of (1.15) are explored. See also the structures in (3.23) and (3.24). As aforementioned, these subtle structures are really the key to our proof. And utilizing them may lead to other applications.
- (5) Tracing dynamics of the 6×6 non-strictly hyperbolic system with four characteristic strips $\{\mathcal{R}_1, \mathcal{R}_2 \cup \mathcal{R}_3, \mathcal{R}_4 \cup \mathcal{R}_5, \mathcal{R}_6\}$ is new. Here \mathcal{R}_2 and \mathcal{R}_3 (or \mathcal{R}_4 and \mathcal{R}_5) could overlap with each other for a long time. This treatment is applied together with using the subtle structures of (1.12) and (1.15).

For instance, \mathcal{R}_2 and \mathcal{R}_3 could overlap for a long time. But for the equation $(\partial_t + \lambda_2 \partial_x)w^2$ in (1.15), there is a small coefficient $\lambda_2 - \lambda_3$ in front of $w^2 w^3$. This means that the worrisome nonlinear interaction of w^2 and w^3 inside $\mathcal{R}_2 \cup \mathcal{R}_3$ is small. In addition, the corresponding coefficient of $w^2 w^3$ in the equation of $(\partial_t + \lambda_3 \partial_x)w^3$ is zero. We incorporate the use of these subtle structures in our proof.

- (6) Because of the non-strict hyperbolicity, we need to include \underline{S} , i.e., estimates of the lower bound of $\{\rho_i\}_{i=2, \dots, 6}$ as part of the bootstrap argument. This is an extension of Christodoulou-Perez [8]. To get the desired results, we also employ the modified Lindblad-type initial data by introducing a small parameter η . This allows us to prove that our constructed initial data are in $H^3(\mathbb{R}^3)$ for 3D and are in $H^{\frac{5}{2}}(\mathbb{R}^2)$ for 2D and for T_η^* being the first blow-up time, it holds $T_\eta^* \rightarrow 0$ as $\eta \rightarrow 0$.
- (7) For elastic waves (1.4), not only we prove H^3 ill-posedness in 3D and $H^{\frac{5}{2}}$ ill-posedness in 2D, but also we demonstrate the ill-posedness mechanism, and it is driven by shock formation. In addition, at the

shock formation time T_η^* , the H^2 norms of the solutions in 3D and 2D are infinity.

For elastic waves (1.4) in 3D, its critical norm is $H^{\frac{5}{2}}(\mathbb{R}^3)$. And for the 2D case, its critical norm is $H^2(\mathbb{R}^2)$. Our ill-posedness results are with Sobolev norms $\frac{1}{2}$ -derivative higher than the critical norms. Let's take a look at another physical quasilinear wave system *Einstein's equations*. In $3 + 1$ dimensions, its critical norm is $H^{\frac{3}{2}}(\mathbb{R}^3)$, but the sharp local well-posedness and ill-posedness results are all at the level of $H^2(\mathbb{R}^3)$, i.e., a $\frac{1}{2}$ -derivative higher than the critical norm. As an analogue, our $H^3(\mathbb{R}^3)$ ill-posedness of elastic waves (1.4) in 3D is a desired result.

1.4. A single wave model. In this section, we demonstrate the basic geometric and algebraic approaches with a single wave model:

$$(1.16) \quad \partial_t^2 \varphi - \partial_x^2 \varphi = \partial_x (\partial_x \varphi)^2.$$

For a simpler scenario, if we set the waves u^2 and u^3 to vanish, the first equation of (1.12) takes the form of (1.16). The constants c_1 and σ_0 have no essential influence on these two approaches. This is a simplified case of the 3D quasilinear scalar wave equation:

$$\partial_t^2 \varphi - \Delta \varphi = \partial (\partial \varphi)^2, \quad \partial \text{ is a rectangular spatial derivative.}$$

To study the ill-posedness of a scalar wave equation, Lindblad considered in [24] the following quasilinear wave model

$$\partial_t^2 \varphi - \Delta \varphi = D(D\varphi)^2,$$

where $D = \partial_x - \partial_t$. In this case, the explicit formula of solutions can be obtained by solving the initial data problem along the characteristics. However, this is not applicable for (1.16) to establish explicit formula along characteristics. In order to gain some insights about solutions to (1.16), we take a glance at two different methods. One is the geometric method. The shock formation can be described in a more explicit manner by studying the corresponding geometric wave equations of (1.16). The other is an algebraic approach, with which we rewrite (1.16) as a 2×2 system and we algebraically diagonalize this system by introducing and finding Riemann invariants. This leads to two Riccati-type equations, which may have singularities formed in finite time.

1.4.1. Geometric approach. We first rewrite the model equation (1.16) into the following form,

$$(g^{-1}(\partial_x \varphi))^{\alpha\beta} \partial_\alpha \partial_\beta \varphi = 0.$$

The ill-posedness and shock formation of the above equation have been studied in Speck [33] [35] and Granowski [11]. For demonstration purpose, we review the ideas and methods therein.

Take $\psi = \partial_x \varphi$. We have ψ satisfies a geometrically covariant wave equation:

$$(1.17) \quad \square_{g(\psi)} \psi = \mathcal{Q}(\partial \psi, \partial \psi).$$

According to (1.16), the nonlinear terms are given by

$$\mathcal{Q}(\partial\psi, \partial\psi) = -(1 + 2\psi)^{-1} \mathcal{Q}_0(\partial\psi, \partial\psi),$$

where $\mathcal{Q}_0(\partial\psi, \partial\psi)$ is the standard null form with respect to the geometric metric $g(\psi)$.

To understand the causal structure with respect to $g(\psi)$, one can use the geometric coordinates (t, u) , with u an eikonal function satisfying the eikonal equation

$$g^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0.$$

The intersection of characteristics is then described by the vanishing of a geometric quantity called *inverse foliation density* μ ,

$$\mu := -\frac{1}{(g^{-1})^{\alpha\beta}(\psi) \partial_\alpha t \partial_\beta u} = \frac{1}{\partial_t u}.$$

We then define geometric frame (L, \check{L}) , where \check{L} and L represent incoming and outgoing null directions respectively. In particular, we have $L = \partial_t$ in (t, u) coordinates. The equation (1.17) for ψ can be reformulated in this frame. And it also holds that μ satisfies a transport equation along the outgoing characteristic. In detail, equation (1.16) can be written as a system with the following form:

$$\begin{cases} L\check{L}\psi = -(1 + 2\psi)^{-1} L\psi \cdot \check{L}\psi, \\ \check{L}L\psi = -\frac{\mu}{2}(1 + 2\psi)^{-1} (L\psi)^2 - \frac{1}{2}(1 + 2\psi)^{-1} L\psi \cdot \check{L}\psi, \\ L\mu = -\frac{\mu}{2}(1 + 2\psi)^{-1} L\psi - \frac{1}{2}(1 + 2\psi)^{-1} \check{L}\psi. \end{cases}$$

By properly choosing initial data, and by setting and proving a bootstrap argument for ψ , $L\psi$, $\check{L}\psi$ and μ , one can arrive at a description of the inverse foliation density

$$\mu \sim 1 - \left(\sup_{0 \leq u \leq 1} [f(u)]_- \right) t,$$

where f is the initial data given at $t = 0$ along $\Sigma_0^1 := \{(x, 0) | 0 \leq u(x, 0) \leq 1\}$. This leads to shock formation at time $T_{(shock)} \sim 1 / \left(\sup_{0 \leq u \leq 1} [f(u)]_- \right)$. We refer to [33, 35] for more details.

Granowski [11] studied ill-posedness theory of the above geometric wave equation (1.17). By prescribing Lindblad-type initial data, he showed the instantaneous blow-up of the $H^2(\mathbb{R}^3)$ norm. In his proof, based on the setup of Speck [33], the inverse foliation density also satisfies a transport equation:

$$L\mu = \frac{1}{2} G_{LL} \check{X} \Phi,$$

in geometric coordinates (t, u) . We hence obtain

$$\mu(t, u) = \mu(0, u) + \frac{t}{2} G_{LL}(u) \check{X} \Phi(u).$$

From the calculation of $H^2(\mathbb{R}^3)$ norm, we can read its instantaneous blow-up is driven by the vanishing of μ , which demonstrates the shock formation. See [11] for more details.

1.4.2. *Algebraic approach.* We also outline another approach to study (1.16). Taking $V^{(1)} = \varphi_t$ and $V^{(2)} = \varphi_x$, equation (1.16) can be transformed into a first-order genuinely nonlinear strictly hyperbolic system:

$$(1.18) \quad \begin{cases} \partial_t V^{(1)} - (1 + 2V^{(2)})\partial_x V^{(2)} = 0, \\ \partial_t V^{(2)} - \partial_x V^{(1)} = 0. \end{cases}$$

The definitions of genuinely nonlinear and strictly hyperbolic are given in Lemma 3.1 and Section 2. By direct calculation, the eigenvalues of its coefficient matrix are obtained:

$$\lambda_1(V) = -\sqrt{1 + 2V^{(2)}}, \quad \lambda_2(V) = \sqrt{1 + 2V^{(2)}}.$$

Now we can introduce $W = (W^{(1)}, W^{(2)})$ and they are defined through

$$\begin{aligned} W^{(1)} &= V^{(1)} - \frac{1}{3}(1 + 2V^{(2)})^{\frac{3}{2}} + \frac{1}{3}, \\ W^{(2)} &= V^{(2)} + \frac{1}{3}(1 + 2V^{(2)})^{\frac{3}{2}} - \frac{1}{3}. \end{aligned}$$

Then the eigenvalues λ_1, λ_2 can be viewed as functions depending on W . It is a straight forward check that $W^{(1)}$ and $W^{(2)}$ diagonalize (1.18), i.e., they satisfy the following transport equations:

$$(1.19) \quad \begin{cases} \partial_t W^{(1)} + \lambda_2(W)\partial_x W^{(1)} = 0, \\ \partial_t W^{(2)} + \lambda_1(W)\partial_x W^{(2)} = 0. \end{cases}$$

Then one can define two families of characteristics, which are the solutions to the following ODEs:

$$\begin{cases} \frac{dx^{(1)}(t)}{dt} = \lambda_1(W(x^{(1)}(t), t)), & \frac{dx^{(2)}(t)}{dt} = \lambda_2(W(x^{(2)}(t), t)), \\ x^{(1)}(0) = y_1, & x^{(2)}(0) = y_2. \end{cases}$$

By (1.19), $W^{(1)}$ and $W^{(2)}$ are invariant along the characteristics $(x^{(2)}(t), t)$ and $(x^{(1)}(t), t)$, respectively. We hence call $W^{(1)}$ and $W^{(2)}$ Riemann invariants for equation (1.18). Note that for the Cauchy problem of (1.19), $W(x, t)$ remains bounded for any given bounded initial data $W(x, 0)$. But $\partial_x W^{(1)}$ or $\partial_x W^{(2)}$ could blow up. It can be deduced that $\partial_x W^{(1)}$ and $\partial_x W^{(2)}$ also satisfy Riccati-type equations. Along the second characteristic $(x^{(2)}(t), t)$, we have

$$(1.20) \quad \begin{aligned} \frac{dW_x^{(1)}(x^{(2)}(t), t)}{dt} + \frac{\partial \lambda_2}{\partial W^{(2)}} \frac{1}{\lambda_2 - \lambda_1} \frac{dW_x^{(2)}(x^{(2)}(t), t)}{dt} W_x^{(1)}(x^{(2)}(t), t) \\ = -\frac{\partial \lambda_2}{\partial W^{(1)}} [W_x^{(1)}(x^{(2)}(t), t)]^2. \end{aligned}$$

And along the first characteristic $(x^{(1)}(t), t)$, $\partial_x W^{(1)}$ verifies:

$$(1.21) \quad \begin{aligned} \frac{dW_x^{(2)}(x^{(1)}(t), t)}{dt} + \frac{\partial \lambda_1}{\partial W^{(1)}} \frac{1}{\lambda_1 - \lambda_2} \frac{dW_x^{(1)}(x^{(1)}(t), t)}{dt} W_x^{(2)}(x^{(1)}(t), t) \\ = -\frac{\partial \lambda_1}{\partial W^{(2)}} [W_x^{(2)}(x^{(1)}(t), t)]^2. \end{aligned}$$

The solution of (1.20) can be expressed as:

(1.22)

$$\partial_x W^{(1)}(x^{(2)}(t), t) = \frac{e^{-I_t} \partial_x W_0^{(1)}(y_2)}{1 + \partial_x W_0^{(1)}(y_2) e^{-I_t} \int_0^t e^{-I_\tau} \frac{\partial \lambda_2}{\partial W^{(1)}}(W(x^{(2)}(\tau), \tau)) d\tau},$$

where $\partial_x W_0^{(1)}(y_2) = \partial_x W^{(1)}(x^{(2)}(0), 0) = \partial_x W^{(1)}(y_2, 0)$ and

$$I_t = \int_{W^{(2)}(x^{(2)}(0), 0)}^{W^{(2)}(x^{(2)}(t), t)} \frac{1}{\lambda_2 - \lambda_1} \frac{\partial \lambda_2(W_0^{(1)}, W^{(2)})}{\partial W^{(2)}} dW^{(2)}.$$

Moreover, e^{I_t} has positive lower bound and upper bound. If further assuming that there exist a uniform positive constant C and a point x_0 such that

$$\frac{\partial \lambda_2}{\partial W^{(1)}} > C, \quad \partial_x W_0^{(1)}(x_0) < 0,$$

then from (1.22), we have $\partial_x W^{(1)}$ blows up in finite time.

Picking Lindblad-type initial data as in [24], by further analysis we can show that at the blow-up point, the corresponding inverse foliation density $\left(\frac{\partial x^{(2)}(y_2, t)}{\partial y_2}\right)$ also vanishes, and it renders $H^2(\mathbb{R}^3)$ norm to blow up, which is driven by shock formation.

Remark: The classic algebraic approach on constructing explicit Riemann invariants and using them for a genuinely nonlinear strictly hyperbolic system, relies heavily on its being a 2×2 first-order system. However, for more complicated cases, such as our setting of a 6×6 system for elastic waves, it is almost impossible to construct proper Riemann invariants with explicit forms. John [15] extended this idea of constructing Riemann invariants to calculating and analyzing the decomposition of waves, which could be useful for larger hyperbolic systems. In [46], with John's formula, Zhou studied the local well-posedness of low regularity solutions for linearly degenerate hyperbolic systems in one spatial dimension. In this paper, we will also adopt John's approach.

1.5. Main steps in the proof. Here we outline our proof of Theorem 1.1. We take an algebraic approach first and we decompose the waves as [8, 15]. The components of decomposed waves satisfy a quasilinear hyperbolic system. It is worthwhile to mention that our system is not strictly hyperbolic: the coefficient matrix has almost-the-same eigenvalues for small perturbation of the unknowns around zero. In the end, we will show that (1.4) is ill-posed in $H^3(\mathbb{R}^3)$ and it is caused by shock formation of the fast characteristic wave.

Step 1: Reduction to a first-order quasilinear hyperbolic system

We first transform the equations of elastic plane waves to a first-order

$$(1.23) \quad \partial_t \Phi + A(\Phi) \partial_x \Phi = 0,$$

for $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)^T = (\partial_x u^1, \partial_x u^2, \partial_x u^3, \partial_t u^1, \partial_t u^2, \partial_t u^3)^T$. This system is *not* uniformly strictly hyperbolic in a small ball $B_{2\delta}^6(0)$ where

the amplitude of Φ is bounded by a small parameter δ . In particular, the eigenvalues of the coefficients matrix $(A(\Phi))_{6 \times 6}$ satisfy

$$\lambda_6(\Phi) < \lambda_5(\Phi) \leq \lambda_4(\Phi) < \lambda_3(\Phi) \leq \lambda_2(\Phi) < \lambda_1(\Phi),$$

where $\lambda_2(\Phi)$ and $\lambda_3(\Phi)$ or $\lambda_4(\Phi)$ and $\lambda_5(\Phi)$ could be the same for some $\Phi \in B_{2\delta}^6(0)$. Based on the eigenvalues, the left eigenvectors $\{l^i(\Phi)\}_{i=1, \dots, 6}$ and the right eigenvectors $\{r_i(\Phi)\}_{i=1, \dots, 6}$ can also be calculated, and we require their verifying the property

$$l^i(\Phi)r_j(\Phi) = \delta_j^i.$$

Step 2: Decomposition of waves

The next is to decompose the waves and to derive the equations for their components. As in [8], we use characteristic coordinates and bi-characteristic coordinates. For any $(x, t) \in \mathbb{R} \times [0, T]$, there is a unique $(z_i, s_i) \in \mathbb{R} \times [0, T]$, called characteristic coordinates, such that

$$(x, t) = (X_i(z_i, s_i), s_i),$$

where the flow map $X_i(z_i, s_i)$ is defined by

$$\begin{cases} \frac{\partial}{\partial t} X_i(z, t) = \lambda_i(\Phi(X_i(z_i, t), t)), & t \in [0, T], \\ X_i(z, 0) = z_i. \end{cases}$$

Given $(x, t) \in \mathbb{R} \times [0, T]$, along different characteristics, there is a unique $(y_i, y_j) \in \mathbb{R}^2$, called bi-characteristic coordinates, such that $t = t'(y_i, y_j)$ and

$$(x, t) = (X_i(y_i, t'(y_i, y_j)), t'(y_i, y_j)) = (X_j(y_j, t'(y_i, y_j)), t'(y_i, y_j)).$$

Let $\mathcal{C}_i(z_i)$ be the i^{th} characteristic with propagation speed λ_i starting at z_i . Denote the corresponding i^{th} characteristic strip to be $\mathcal{R}_i := \cup_{z_i \in I_0} \mathcal{C}_i(z_i)$, where I_0 is the support of initial data. Define

$$\rho_i := \partial_{z_i} X_i$$

and use it to describe the inverse foliation density of the i^{th} characteristics. For fixed i , let

$$w^i := l^i \partial_x \Phi, \quad \text{and} \quad v^i := \rho_i w^i.$$

It can be checked that these geometric quantities satisfy

$$(1.24) \quad \partial_{s_i} \rho_i = c_{ii}^i v^i + \left(\sum_{m \neq i} c_{im}^i w^m \right) \rho_i,$$

$$(1.25) \quad \partial_{s_i} w^i = -c_{ii}^i (w^i)^2 + \left(\sum_{m \neq i} (-c_{im}^i + \gamma_{im}^i) w^m \right) w^i + \sum_{\substack{m \neq i, k \neq i \\ m \neq k}} \gamma_{km}^i w^k w^m,$$

$$(1.26) \quad \partial_{s_i} v^i = \left(\sum_{m \neq i} \gamma_{im}^i w^m \right) v^i + \sum_{\substack{m \neq i, k \neq i \\ m \neq k}} \gamma_{km}^i w^k w^m \rho_i,$$

where the characteristic vectorfields $\{\partial_{s_i}\}$ are given by

$$\partial_{s_i} = \partial_t + \lambda_i \partial_x.$$

We note that the non-zero of c_{ii}^i means the genuine non-linearity in (1.25), while the vanishing of c_{ii}^i denotes degeneration. Since λ_1 is the largest

eigenvalue, we have w^1 being the wave of the fastest speed. Being genuinely nonlinear means $c_{11}^1 \neq 0$ and without loss of generality we assume $c_{11}^1 < 0$, and clearly (1.25) gives a Riccati-type equation. In [8], Christodoulou-Perez studied the shock formation of the uniformly strictly hyperbolic systems. While for elasticity a lot of attention should be paid to the structures of elastic wave equations related to the 2nd, 3rd, 4th, and 5th characteristics, which make the system not uniformly strictly hyperbolic. We explore the structures and calculate the coefficients $\{c_{im}^i\}$ and γ_{km}^i very carefully. We observe that some crucial coefficients are zero and some key terms have almost-zero factor in front. These structures are vital in the proof of the main theorem. See Section 3 for the details.

Step 3: Construction of initial data

We employ a family of Lindblad-type smooth initial data $\{w_{(\eta)}^i(z_i, 0)\}$ and modify them a bit such that they are supported in $[\eta, 2\eta]$ for a given small η and satisfy

$$\begin{cases} W_0^{(\eta)} := \max_{i=1, \dots, 6} \sup_{z_i} |w_{(\eta)}^i(z_i, 0)| = w_{(\eta)}^1(z_0, 0), \\ \max_{i=3, 4} \sup_{z_i} |w_{(\eta)}^i(z_i, 0)| \leq \min \left\{ \frac{(1-\varepsilon)^4 |c_{11}^1(0)|}{(1+\varepsilon)^3} W_0^{(\eta)}, W_0^{(\eta)} \right\}, \\ \sup_{z_6} |w^6(z_6, 0)| \leq \frac{(1-\varepsilon)^4}{2(1+\varepsilon)^3} W_0^{(\eta)}, \end{cases}$$

with

$$w_{(\eta)}^1(z, 0) := \theta \int_{\mathbb{R}} \zeta_{\frac{\eta}{10}}(y) |\ln(z-y)|^\alpha \chi(z-y) dy, \quad 0 < \alpha < \frac{1}{2},$$

where χ is a characteristic function, $\zeta_{\frac{\eta}{10}}$ is a test function and θ is a small constant to be chosen. Note that we can choose $\hat{w}(x, Y^2, Y^3) = w_{(\eta)}^1(x, 0)$ such that $\hat{w} \in H^1(\mathbb{R}^3)$ by Lemma A.1 in Appendix. This implies that for (1.4) the corresponding initial data $(U_0, U_1) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$.

As mentioned before, two pairs of characteristic strips (\mathcal{R}_2 and \mathcal{R}_3 , \mathcal{R}_4 and \mathcal{R}_5) may overlap for a long time. Our strategy is to consider the following four separated strips:

$$\{\mathcal{R}_1, \mathcal{R}_2 \cup \mathcal{R}_3, \mathcal{R}_4 \cup \mathcal{R}_5, \mathcal{R}_6\},$$

and to derive desired estimates in \mathcal{R}_1 . By calculation, we find these four strips are separated when $t > t_0^{(\eta)} := \frac{\eta}{\sigma}$. Here, σ is a constant that describes the amplitude of the eigenvalues, a uniform nonzero constant. The main quantities to be estimated are as follows:

$$(1.27) \quad S_i(t) := \sup_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} \rho_i(z'_i, s'_i), \quad S(t) := \max_{i \in \{1, 2, 3, 4, 5, 6\}} S_i(t),$$

$$(1.28) \quad J_i(t) := \sup_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} |v^i(z'_i, s'_i)|, \quad J(t) := \max_{i \in \{1, 2, 3, 4, 5, 6\}} J_i(t),$$

$$(1.29) \quad \bar{U}(t) := \sup_{\substack{(x',t') \\ 0 \leq t' \leq t}} |\Phi(x', t')|, \quad W(t) := \max_{i \in \{1,2,3,4,5,6\}} \sup_{\substack{(x',t') \\ 0 \leq t' \leq t}} |w^i(x', t')|,$$

$$(1.30) \quad V_1(t) := \sup_{\substack{(x',t') \notin \mathcal{R}_1, \\ 0 \leq t' \leq t}} |w^1(x', t')|,$$

$$(1.31) \quad V_2(t) := \sup_{\substack{(x',t') \notin \mathcal{R}_2 \cup \mathcal{R}_3, \\ 0 \leq t' \leq t}} \{|w^2(x', t')|, |w^3(x', t')|\},$$

$$(1.32) \quad V_6(t) := \sup_{\substack{(x',t') \notin \mathcal{R}_6, \\ 0 \leq t' \leq t}} |w^6(x', t')|,$$

$$(1.33) \quad V_5(t) := \sup_{\substack{(x',t') \notin \mathcal{R}_4 \cup \mathcal{R}_5, \\ 0 \leq t' \leq t}} \{|w^4(x', t')|, |w^5(x', t')|\},$$

$$(1.34) \quad V(t) := \max_{i=1,2,5,6} V_i(t).$$

See Section 4 for the precise expressions and the details.

Step 4: A priori estimates

We give estimates of the quantities in (1.27)-(1.34). Assume, for some $T > 0$, the Cauchy problem of system (1.23) has a solution $\Phi \in C^\infty(\mathbb{R} \times [0, T]; B_{2\delta}^6(0))$. We then derive estimates of these quantities in the non-separated characteristic region $t \in [0, t_0^{(\eta)}]$ and in the separated characteristic regions $t \in [t_0^{(\eta)}, T]$. We will show that the formation of shock happens in the separated regions.

- **Estimates for $t \in [0, t_0^{(\eta)}]$**

In the non-separated region $t \in [0, t_0^{(\eta)}]$, all the characteristic strips $\{\mathcal{R}_i\}$ overlap. All the quantities defined in (1.27)-(1.34) are bounded in such a way

$$\begin{aligned} W(t) &= O(W_0^{(\eta)}), \quad V(t) = O(\eta[W_0^{(\eta)}]^2), \quad S(t) = O(1), \\ J(t) &= O(W_0^{(\eta)}), \quad \bar{U}(t) = O(\eta W_0^{(\eta)}). \end{aligned}$$

In particular, for the inverse foliation densities $\{\rho_i\}$, we have

$$\rho_i(z_i, t) \geq 1 - \varepsilon, \quad i = 1, \dots, 6, \quad \forall t \in [0, t_0^{(\eta)}],$$

for some given small parameter ε . See Section 5 for the details.

- **Estimates for $t \in [t_0^{(\eta)}, T]$**

For $t \in [t_0^{(\eta)}, T]$, though \mathcal{R}_2 and \mathcal{R}_3 or \mathcal{R}_4 and \mathcal{R}_5 are not completely separated due to the non-strict hyperbolicity, the aforementioned four characteristic strips $\mathcal{R}_1, \mathcal{R}_2 \cup \mathcal{R}_3, \mathcal{R}_4 \cup \mathcal{R}_5, \mathcal{R}_6$ are well separated. We give estimates in different strips, and show that the

first singularity (a shock) happens in \mathcal{R}_1 along \mathcal{C}_1 . Subtle cancellations are explored. When $t \in [t_0^{(\eta)}, T]$, a priori estimates are obtained in these strip regions:

$$\begin{aligned} S &= O(1 + tVS + tJ + \eta SJ), \\ J &= O\left(W_0^{(\eta)} + tVJ + tV^2S + \eta J^2 + \frac{tVSJ}{\underline{S}}\right), \\ V &= O\left(\eta[W_0^{(\eta)}]^2 + tV^2 + \eta VJ + \eta \frac{\varepsilon}{\underline{S}} J^2\right), \\ \bar{U} &= O(\eta J + \eta V + \eta tV). \end{aligned}$$

Here $\underline{S}(t)$ is the infimum of inverse foliation densities except for ρ_1 :

$$\underline{S}(t) := \min_{i \in \{2, \dots, 6\}} \inf_{\substack{z' \in [\eta, 2\eta] \\ 0 \leq t' \leq t}} \rho_i(z', t').$$

Step 5: Bounds of the norms and a positive lower bound for \underline{S}

Based on the above estimates, we obtain a desired positive lower bound of \underline{S} via a bootstrap argument. We prove:

$$\begin{aligned} J &= O(W_0^{(\eta)}), \quad S = O(1), \quad tV = O(\eta W_0^{(\eta)} + \eta \varepsilon \theta^{-\frac{1}{3}} W_0^{(\eta)}), \\ V &= O(\eta [W_0^{(\eta)}]^2 + \eta \varepsilon \theta^{-\frac{1}{3}} [W_0^{(\eta)}]^2) \end{aligned}$$

and

$$\underline{S}(s) \geq \frac{(1 - \varepsilon)^2}{2}.$$

See Section 6 and Section 8 for the details.

Step 6: Bound for $\partial_{z_1} \rho_1$

With characteristic coordinates and bi-characteristic coordinates, we prove that $\tau_1^{(6)} := \partial_{y_1} \rho_1$ and $\pi_1^{(6)} := \partial_{y_1} v^1$ satisfy a linear system:

$$\begin{cases} \partial_{s_1} \tau_1^{(6)} = B_{11}^\eta \tau_1^{(6)} + B_{12}^\eta \pi_1^{(6)} + B_{13}^\eta \\ \partial_{s_1} \pi_1^{(6)} = B_{21}^\eta \tau_1^{(6)} + B_{22}^\eta \pi_1^{(6)} + B_{23}^\eta, \end{cases}$$

where $\{B_{ij}^\eta\}_{i=1,2; j=1,2,3}$ are uniformly bounded constants depending on η . Hence $\tau_1^{(6)} := \partial_{y_1} \rho_1$ is bounded. With expression

$$\partial_{z_1} \rho_1 = \partial_{y_1} \rho_1 + \frac{\rho_1}{2\lambda_1} \partial_{s_1} \rho_1 = \partial_{y_1} \rho_1 + \frac{\rho_1}{2\lambda_1} (c_{11}^1 v^1 + \sum_{k \neq 1} c_{1k}^1 w^k \rho_1),$$

and estimate for v^1 and estimates for w^k where $k \neq 1$ in Step 4, we then have a bound for $\partial_{z_1} \rho_1$ by a uniform constant depending on η .

Step 7: Ill-posedness mechanism

The formation of shock happens in the first characteristic strip \mathcal{R}_1 . With the obtained estimates, we control ρ_1 by a sharp description:

$$\rho_1 \sim 1 - tW_0^{(\eta)}.$$

This implies that the shock forms at a time T_η^* with

$$T_\eta^* \sim \frac{1}{W_0^{(\eta)}}.$$

With our constructed initial data, which are finite in $H^3(\mathbb{R}^3)$, we calculate the Sobolev norms at later moments. We observe that as $t \rightarrow T_\eta^*$, the $H^2(\mathbb{R}^3)$ norm of solutions to (1.4) at time t approaches to infinite, and it is driven by shock formation. That is

$$\|U_\eta(\cdot, T_\eta^*)\|_{H^2(\Omega_{T_\eta^*})} \gtrsim \|w^1(\cdot, T_\eta^*)\|_{L^2(\Omega_{T_\eta^*})} = +\infty.$$

Moreover, as $\eta \rightarrow 0$, we have $T_\eta^* \rightarrow 0$. This is the desired ill-posedness result for 3D elastic waves in $H^3(\mathbb{R}^3)$. See Section 10 for the details.

1.6. Other related works. In this section, we refer to some related works on the local well-posedness of low regularity solutions.

For quasilinear wave equations, Bahouri-Chemin [5, 6] and Tataru [41] showed the local well-posedness in $H^s(\mathbb{R}^n)$ with $s > \frac{n}{2} + \frac{3}{4}$ (later enhanced to $s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6}$ in [43]) with Strichartz estimates. In [18] Klainerman-Rodnianski improved their results to $H^{s_*}(\mathbb{R}^3)$ with $s_* > s_0 = 2 + (2 - \sqrt{3})/2$ by combing a geometric method with a paradifferential calculation. Tataru-Smith obtained in [31] sharper results where they used a parametrix construction to improve the estimates. They showed that for n dimensional quasilinear wave equations, the Cauchy problems are locally well-posed in $H^s(\mathbb{R}^n)$ with $s > n/2 + 3/4$ for $n = 2$ and $s > (n + 1)/2$ for $n = 3, 4, 5$. The corresponding low regularity result for Einstein vacuum equations was obtained by Klainerman-Rodnianski [19] via a vectorfield method. In [44], Wang adopted the geometric vectorfield approach and show the $H^s(\mathbb{R}^3)$ local well-posedness of three dimensional quasilinear wave equations for any $s > 2$. This result is generalized to the 3D compressible Euler equations in [9] by Disconzi-Luo-Mazzone-Speck, via a decomposition of Euler flow into the wave part and the transport part. Recently, Wang [45] revisited this topic and enhanced the regularity of the transport part in [9] by 1/2-derivative.

For Einstein vacuum equations, it was shown in [19] that the local existence holds for initial data g_0 being in $H^s(\mathbb{R}^3)$ for $s > 2$. And it can be seen from the counterexamples Ettinger-Lindblad constructed in [10] that the above result is sharp in wave coordinates. Through a series of remarkable works by Klainerman, Rodnianski, Szeftel [20, 36–39], the celebrated L^2 bounded curvature theorem with Yang-Mills frames was achieved. They proved that Einstein vacuum equations are well-posedness in $H^2(\mathbb{R}^3)$.

For elastic wave equations, local well-posedness in $H^s(\mathbb{R}^3)$ with $s > 3/2 + 2$ was obtained by Hughes-Kato-Marsden [14]. For the radially symmetric case, Hidano-Zha proved almost global existence [12] and global existence under null condition [13] for small initial data in $H_{\text{rad}}^3(\mathbb{R}^3)$.

2. REDUCTION OF EQUATIONS

For $Y = (Y^1, Y^2, Y^3) \in \mathbb{R}^3$, $U(Y, t) = (U^1(Y, t), U^2(Y, t), U^3(Y, t))$ being a solution to (1.4). Under planar symmetry (with respect to Y^2 and Y^3), we have

$$U^1(Y, t) = u^1(Y^1, t), \quad U^2(Y, t) = u^2(Y^1, t), \quad U^3(Y, t) = u^3(Y^1, t).$$

Denote Y^1 to be x . Then $u(x, t) = (u^1(x, t), u^2(x, t), u^3(x, t))$ is a solution to (1.4). In the following proposition, we will show that $u(x, t)$ satisfies a quasilinear wave system with multiple speeds:

Proposition 2.1. *Assume that $u(x, t)$ is a solution of (1.4) under a planar symmetry. Then $u = (u^1, u^2, u^3)$ verifies the following quasilinear wave system:*

$$(2.1) \quad \begin{cases} \partial_t^2 u^1 - c_1^2 \partial_x^2 u^1 = \sigma_0 \partial_x (\partial_x u^1)^2 + \sigma_1 \partial_x (\partial_x u^2)^2 + \sigma_1 \partial_x (\partial_x u^3)^2, \\ \partial_t^2 u^2 - c_2^2 \partial_x^2 u^2 = 2\sigma_1 \partial_x (\partial_x u^1 \partial_x u^2), \\ \partial_t^2 u^3 - c_2^2 \partial_x^2 u^3 = 2\sigma_1 \partial_x (\partial_x u^1 \partial_x u^3), \end{cases}$$

with initial data $u(x, 0) = U_0(x)$, $u_t(x, 0) = U_1(x)$.

Proof. For $U = u(x, t) = (u^1(x, t), u^2(x, t), u^3(x, t))$, we revisit (1.4). With definition of $Q_{jk}(f, g) = \partial_j f \partial_k g - \partial_k f \partial_j g$ ($j \neq k$) for $f, g \in C^1(\mathbb{R})$, it is a straight forward check that Q^1, Q^2, Q^3 given in (1.5) are all zero. Hence

$$Q(U, \nabla U) = 0.$$

We then calculate the remaining terms, since

$$\text{curl } U = (0, -\partial_x u^3, \partial_x u^2),$$

it holds that

$$\nabla |\text{curl } U|^2 = (\partial_x (\partial_x u^2)^2 + \partial_x (\partial_x u^3)^2, 0, 0)$$

and

$$\text{curl} (\text{div } U \text{curl } U) = (0, -\partial_x (\partial_x u^1 \partial_x u^2), -\partial_x (\partial_x u^1 \partial_x u^3)).$$

Back to (1.4)-(1.5), Proposition 2.1 is proved. \square

We then change (2.1) into a first-order hyperbolic system. Let us recall some important definitions. Assume unknowns $\Phi \in \mathbb{R}^n$ and a coefficient matrix $A(\Phi) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfy a first-order system:

$$(2.2) \quad \partial_t \Phi + A(\Phi) \partial_x \Phi = 0.$$

The system (2.2) is called *hyperbolic* if $A(\Phi)$ has n real eigenvalues noted as:

$$\lambda_1(\Phi), \dots, \lambda_n(\Phi)$$

and $A(\Phi)$ is diagonalizable. If furthermore all of the eigenvalues are distinct, then (2.2) is called *strictly hyperbolic*. Otherwise, system (2.2) is called *non-strictly hyperbolic*.

The following Lemma shows that the elastic plane waves in 3D verify a non-strictly hyperbolic system. We will use a notation $B_{2\delta}^6(0)$ to mean an open ball of radius 2δ around $0 \in \mathbb{R}^6$. Let

$$\phi_1 := \partial_x u^1, \quad \phi_2 := \partial_x u^2, \quad \phi_3 := \partial_x u^3, \quad \phi_4 := \partial_t u^1, \quad \phi_5 := \partial_t u^2, \quad \phi_6 := \partial_t u^3.$$

Lemma 2.1. *Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)^T$. Then system (2.1) is equivalent to*

$$(2.3) \quad \partial_t \Phi + A(\Phi) \partial_x \Phi = 0,$$

where

$$A(\Phi) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ -(c_1^2 + 2\sigma_0\phi_1) & -2\sigma_1\phi_2 & -2\sigma_1\phi_3 & 0 & 0 & 0 \\ -2\sigma_1\phi_2 & -(c_2^2 + 2\sigma_1\phi_1) & 0 & 0 & 0 & 0 \\ -2\sigma_1\phi_3 & 0 & -(c_2^2 + 2\sigma_1\phi_1) & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, system (2.3) is not uniformly strictly hyperbolic for $\Phi \in B_{2\delta}^6(0)$.

Proof. For notational simplicity, we set

$$(2.4) \quad a = c_1^2 + 2\sigma_0\phi_1, \quad b = c_2^2 + 2\sigma_1\phi_1, \quad c = 2\sigma_1\phi_2, \quad d = 2\sigma_1\phi_3.$$

We then calculate the eigenvalues of $A(\Phi)$. By direct computing, we have

$$\begin{aligned} \det(\lambda I - A) &= \lambda^6 - (a + 2b)\lambda^4 + (2ab + b^2 - c^2 - d^2)\lambda^2 - ab^2 + b(c^2 + d^2) \\ &= (\lambda^2 - b)[(\lambda^2 - a)(\lambda^2 - b) - (c^2 + d^2)]. \end{aligned}$$

And its six roots are:

$$(2.5) \quad \lambda_1 = \sqrt{\frac{1}{2}(a + b) + \frac{1}{2}\sqrt{(a - b)^2 + 4(c^2 + d^2)}},$$

$$(2.6) \quad \lambda_2 = \sqrt{b},$$

$$(2.7) \quad \lambda_3 = \sqrt{\frac{1}{2}(a + b) - \frac{1}{2}\sqrt{(a - b)^2 + 4(c^2 + d^2)}},$$

$$(2.8) \quad \lambda_4 = -\sqrt{\frac{1}{2}(a + b) - \frac{1}{2}\sqrt{(a - b)^2 + 4(c^2 + d^2)}},$$

$$(2.9) \quad \lambda_5 = -\sqrt{b},$$

$$(2.10) \quad \lambda_6 = -\sqrt{\frac{1}{2}(a + b) + \frac{1}{2}\sqrt{(a - b)^2 + 4(c^2 + d^2)}}.$$

Note that for $|\Phi| < 2\delta$ being small, we have

$$(2.11) \quad \lambda_6(\Phi) < \lambda_5(\Phi) \leq \lambda_4(\Phi) < \lambda_3(\Phi) \leq \lambda_2(\Phi) < \lambda_1(\Phi).$$

Equalities in (2.11) hold when $\phi_2 = \phi_3 = 0$, i.e., $c = d = 0$. In this case $a - b = c_1^2 + 2(\sigma_0 - \sigma_1)\phi_1 - c_2^2 > 0$ and

$$\lambda_3 = \sqrt{\frac{1}{2}(a + b) - \frac{1}{2}\sqrt{(a - b)^2}} = \sqrt{b} = \lambda_2,$$

$$\lambda_4 = -\sqrt{\frac{1}{2}(a + b) - \frac{1}{2}\sqrt{(a - b)^2}} = -\sqrt{b} = \lambda_5.$$

For $\Phi = 0 \in \mathbb{R}^6$, we further have

$$(2.12) \quad \lambda_6 = -c_1, \quad \lambda_5(0) = \lambda_4(0) = -c_2, \quad \lambda_3(0) = \lambda_2(0) = c_2, \quad \lambda_1 = c_1.$$

Though $A(0)$ has repeated eigenvalues, we still can find six linearly independent right eigenvectors. By calculation, we have

$$e_{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -c_1 \\ 0 \\ 0 \end{pmatrix}, \quad e_{02} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -c_2 \\ 0 \end{pmatrix}, \quad e_{03} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -c_2 \end{pmatrix},$$

$$e_{04} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ c_2 \end{pmatrix}, \quad e_{05} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ c_2 \\ 0 \end{pmatrix}, \quad e_{06} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ c_1 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly, for eigenvalues λ_1 to λ_6 as in (2.5)-(2.10) we compute their corresponding right eigenvectors with $A(\Phi)$:

$$(2.13) \quad r_1 = \begin{pmatrix} \frac{\lambda_1^2 - b}{2\sigma_1} \\ \phi_2 \\ \phi_3 \\ -\frac{\lambda_1(\lambda_1^2 - b)}{2\sigma_1} \\ -\lambda_1\phi_2 \\ -\lambda_1\phi_3 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 0 \\ \phi_3 \\ -\phi_2 \\ 0 \\ -\lambda_2\phi_3 \\ \lambda_2\phi_2 \end{pmatrix}, \quad r_3 = \begin{pmatrix} \frac{\lambda_3^2 - b}{2\sigma_1} \\ \phi_2 \\ \phi_3 \\ -\frac{\lambda_3(\lambda_3^2 - b)}{2\sigma_1} \\ -\lambda_3\phi_2 \\ -\lambda_3\phi_3 \end{pmatrix},$$

$$r_4 = \begin{pmatrix} \frac{\lambda_3^2 - b}{2\sigma_1} \\ \phi_2 \\ \phi_3 \\ \frac{\lambda_3(\lambda_3^2 - b)}{2\sigma_1} \\ \lambda_3\phi_2 \\ \lambda_3\phi_3 \end{pmatrix}, \quad r_5 = \begin{pmatrix} 0 \\ \phi_3 \\ -\phi_2 \\ 0 \\ \lambda_2\phi_3 \\ -\lambda_2\phi_2 \end{pmatrix}, \quad r_6 = \begin{pmatrix} \frac{\lambda_1^2 - b}{2\sigma_1} \\ \phi_2 \\ \phi_3 \\ \frac{\lambda_1(\lambda_1^2 - b)}{2\sigma_1} \\ \lambda_1\phi_2 \\ \lambda_1\phi_3 \end{pmatrix}.$$

□

In order to use John's approach in [15], we also compute the left eigenvectors $\{l^i(\Phi)\}_{i=1, \dots, 6}$ and require

$$(2.14) \quad l^i(\Phi) \cdot r_j(\Phi) = \delta_j^i, \quad \text{for } i, j \in \{1, \dots, 6\}.$$

Requirement (2.14) is important for deriving the decomposition of waves in the next Section. Via calculations, we have

$$\begin{aligned}
(2.15) \quad l^1 &= \frac{1}{K} \left(\frac{\lambda_1^2 - b}{2\sigma_1}, \phi_2, \phi_3, -\frac{\lambda_1^2 - b}{2\sigma_1\lambda_1}, -\frac{\phi_2}{\lambda_1}, -\frac{\phi_3}{\lambda_1} \right), \\
l^2 &= \frac{1}{M} \left(0, \phi_3, -\phi_2, 0, -\frac{\phi_3}{\lambda_2}, \frac{\phi_2}{\lambda_2} \right), \\
l^3 &= \frac{1}{N} \left(\frac{\lambda_3^2 - b}{2\sigma_1}, \phi_2, \phi_3, -\frac{\lambda_3^2 - b}{2\sigma_1\lambda_3}, -\frac{\phi_2}{\lambda_3}, -\frac{\phi_3}{\lambda_3} \right), \\
l^4 &= \frac{1}{N} \left(\frac{\lambda_3^2 - b}{2\sigma_1}, \phi_2, \phi_3, \frac{\lambda_3^2 - b}{2\sigma_1\lambda_3}, \frac{\phi_2}{\lambda_3}, \frac{\phi_3}{\lambda_3} \right), \\
l^5 &= \frac{1}{M} \left(0, \phi_3, -\phi_2, 0, \frac{\phi_3}{\lambda_2}, -\frac{\phi_2}{\lambda_2} \right), \\
l^6 &= \frac{1}{K} \left(\frac{\lambda_1^2 - b}{2\sigma_1}, \phi_2, \phi_3, \frac{\lambda_1^2 - b}{2\sigma_1\lambda_1}, \frac{\phi_2}{\lambda_1}, \frac{\phi_3}{\lambda_1} \right),
\end{aligned}$$

where

$$\begin{aligned}
K &= \frac{(a-b)^2 + 4(c^2 + d^2)}{4\sigma_1^2} + \frac{(a-b)\sqrt{(a-b)^2 + 4(c^2 + d^2)}}{4\sigma_1^2}, \\
M &= 2(\phi_2^2 + \phi_3^2), \\
N &= \frac{(a-b)^2 + 4(c^2 + d^2)}{4\sigma_1^2} - \frac{(a-b)\sqrt{(a-b)^2 + 4(c^2 + d^2)}}{4\sigma_1^2} \\
&= \frac{(\lambda_3^2 - b)^2}{2\sigma_1^2} + 2(\phi_2^2 + \phi_3^2).
\end{aligned}$$

3. DECOMPOSITION OF WAVES

In this section, we introduce useful characteristic coordinates and bi-characteristic coordinates. We assume that $\Phi \in C^2(\mathbb{R} \times [0, T], B_{2\delta}^6(0))$ is a solution to (2.3) for some $T > 0$. We define the i^{th} characteristic $\mathcal{C}_i(z)$ to be the image $(X_i(z, t), t)$ of solutions of the following ODE:

$$(3.1) \quad \begin{cases} \frac{\partial}{\partial t} X_i(z, t) = \lambda_i(\Phi(X_i(z, t), t)), & t \in [0, T], \\ X_i(z, 0) = z. \end{cases}$$

Define \mathcal{R}_i to be the i^{th} characteristic strip evolving from initial data supported in I_0 . That is,

$$\mathcal{R}_i = \bigcup_{z \in I_0} \mathcal{C}_i(z).$$

By the uniqueness of the solution to (3.1), for any given $(x, t) \in \mathbb{R} \times [0, T]$, there is a unique $(z_i, s_i) \in \mathbb{R} \times [0, T]$ such that along the i^{th} characteristic $\mathcal{C}_i(z)$ with $X_i(z_i, 0) = z_i$ we have

$$(3.2) \quad (x, t) = (X_i(z_i, s_i), s_i).$$

We call (z_i, s_i) the *characteristic coordinates*. We then define inverse foliation density

$$(3.3) \quad \rho_i := \frac{\partial}{\partial z_i} X_i(z_i, s_i),$$

and it implies

$$(3.4) \quad \partial_{z_i} = \rho_i \partial_x, \quad \partial_{s_i} = \lambda_i \partial_x + \partial_t$$

and

$$(3.5) \quad dx = \rho_i dz_i + \lambda_i ds_i, \quad dt = ds_i.$$

The bi-characteristic coordinates are introduced to study the intersection of the i^{th} and j^{th} characteristics $C_i(z_i)$ and $C_j(z_j)$ when $i \neq j$. Given (z_i, z_j) for $t > 0$, the characteristics $C_i(z_i)$ and $C_j(z_j)$ defined by (3.1) share the unique point of intersection. One can thus locate this point with coordinates (z_i, z_j) . To avoid ambiguity, we denote (z_i, z_j) by (y_i, y_j) as a new coordinate system, i.e., the *bi-characteristic coordinates*. Under the new coordinates, we denote $t = t'(y_i, y_j)$, and we have

$$(3.6) \quad (x, t) = (X_i(y_i, t'(y_i, y_j)), t'(y_i, y_j)) = (X_j(y_j, t'(y_i, y_j)), t'(y_i, y_j))$$

with

$$(3.7) \quad \partial_{y_i} t' = \frac{\rho_i}{\lambda_j - \lambda_i}, \quad \partial_{y_j} t' = \frac{\rho_j}{\lambda_i - \lambda_j}.$$

Note that (3.7) implies $[\partial_{y_i}, \partial_{y_j}] = 0$ for $1 \leq i \neq j \leq 6$. We then have

$$(3.8) \quad dx = \frac{\rho_i \lambda_j}{\lambda_j - \lambda_i} dy_i + \frac{\rho_j \lambda_i}{\lambda_i - \lambda_j} dy_j, \quad dt = \frac{\rho_i}{\lambda_j - \lambda_i} dy_i + \frac{\rho_j}{\lambda_i - \lambda_j} dy_j$$

and

$$(3.9) \quad dz_i = dy_i, \quad dz_j = dy_j.$$

Now we are ready to study the decomposition of waves. For fixed $i \in \{1, \dots, 6\}$, let

$$(3.10) \quad w^i := l^i \partial_x \Phi, \quad \text{and} \quad v^i := l^i \partial_{z_i} \Phi = \rho_i w^i.$$

The transformation $w^1 = l^1 \partial_x \Phi$ is well-defined for $|\Phi| < 2\delta$, since

$$l^1(0) = \left(\frac{\sigma_1}{c_1^2 - c_2^2}, 0, 0, -\frac{\sigma_1}{c_1(c_1^2 - c_2^2)}, 0, 0 \right)$$

being a non-zero vector. Using (2.14) and (3.10), we have

$$(3.11) \quad \partial_x \Phi = \sum_k w^k r_k.$$

By John's formula [15], we diagonalize the system (2.3) as:

$$(3.12) \quad \partial_{s_i} w^i = -c_{ii}^i (w^i)^2 + \left(\sum_{m \neq i} (-c_{im}^i + \gamma_{im}^i) w^m \right) w^i + \sum_{\substack{m \neq i, k \neq i \\ m \neq k}} \gamma_{km}^i w^k w^m,$$

where

$$(3.13) \quad c_{im}^i = \nabla_{\Phi} \lambda_i \cdot r_m,$$

and

$$(3.14) \quad \begin{aligned} \gamma_{im}^i &= -(\lambda_i - \lambda_m)l^i \cdot (\nabla_{\Phi} r_i \cdot r_m - \nabla_{\Phi} r_m \cdot r_i), & m \neq i, \\ \gamma_{km}^i &= -(\lambda_k - \lambda_m)l^i \cdot (\nabla_{\Phi} r_k \cdot r_m), & k \neq i, m \neq i. \end{aligned}$$

Here the notation ∇_{Φ} denotes taking the gradient with respect to Φ . Moreover, computed as in Christoudoulou-Perez [8], the inverse foliation density of characteristics ρ_i and quantity v^i satisfy:

$$(3.15) \quad \partial_{s_i} \rho_i = c_{ii}^i v^i + \left(\sum_{m \neq i} c_{im}^i w^m \right) \rho_i,$$

$$(3.16) \quad \partial_{s_i} v^i = \left(\sum_{m \neq i} \gamma_{im}^i w^m \right) v^i + \sum_{\substack{m \neq i, k \neq i \\ m \neq k}} \gamma_{km}^i w^k w^m \rho_i.$$

We next analyze the detailed algebraic structures of (3.12), (3.15) and (3.16) when applying to elastic waves. We have

Lemma 3.1. *The 1st and 6th characteristics of system (2.3) are genuinely non-linear in the sense of Lax:*

$$\nabla_{\Phi} \lambda_i(\Phi) \cdot r_i(\Phi) \neq 0, \quad i = 1, 6, \quad \forall \Phi \in B_{2\delta}^6(0).$$

The 2nd and 5th characteristics of system (2.3) are linearly degenerate in the sense of Lax:

$$\nabla_{\Phi} \lambda_j \cdot r_j = 0, \quad j = 2, 5, \quad \forall \Phi \in B_{2\delta}^6(0).$$

Proof. For notational simplicity, we define

$$\Delta := (a - b)^2 + 4(c^2 + d^2).$$

By direct calculation, we get

$$(3.17) \quad \begin{aligned} \nabla_{\Phi} \lambda_1 &= -\nabla_{\Phi} \lambda_6 = \left(\frac{(\sigma_0 + \sigma_1)\sqrt{\Delta} + (a-b)(\sigma_0 - \sigma_1)}{2\lambda_1\sqrt{\Delta}}, \frac{4\sigma_1^2\phi_2}{\lambda_1\sqrt{\Delta}}, \frac{4\sigma_1^2\phi_3}{\lambda_1\sqrt{\Delta}}, 0, 0, 0 \right), \\ \nabla_{\Phi} \lambda_2 &= -\nabla_{\Phi} \lambda_5 = \left(\frac{\sigma_1}{\lambda_2}, 0, 0, 0, 0, 0 \right), \\ \nabla_{\Phi} \lambda_3 &= -\nabla_{\Phi} \lambda_4 = \left(\frac{(\sigma_0 + \sigma_1)\sqrt{\Delta} - (a-b)(\sigma_0 - \sigma_1)}{2\lambda_3\sqrt{\Delta}}, -\frac{4\sigma_1^2\phi_2}{\lambda_3\sqrt{\Delta}}, -\frac{4\sigma_1^2\phi_3}{\lambda_3\sqrt{\Delta}}, 0, 0, 0 \right). \end{aligned}$$

With (2.13) and (3.17), in the expression of (3.13) we have

$$c_{22}^2(\Phi) = c_{55}^5(\Phi) = 0.$$

And it also holds

$$c_{11}^1(\Phi) = -c_{66}^6(\Phi) = \frac{2\sigma_0(a-b)(\lambda_1^2 - b) + (2\sigma_0 + 6\sigma_1)(c^2 + d^2)}{4\sigma_1\lambda_1\sqrt{\Delta}}.$$

By a direct check of

$$c_{11}^1(0) = -c_{66}^6(0) = \frac{\sigma_0(c_1^2 - c_2^2)}{2\sigma_1 c_1} \neq 0,$$

we have

$$c_{11}^1(\Phi) = -c_{66}^6(\Phi) \neq 0$$

for any $\Phi \in B_{2\delta}^6(0)$ with sufficiently small δ . \square

Without loss of generality, we assume that

$$c_{11}^1(0) < 0 \quad \text{and} \quad c_{11}^1(\Phi) < 0, \quad \forall \Phi \in B_{2\delta}^6(0).$$

Correspondingly, we require the initial data of $w^1(z, 0)$ to be positive at some $z_0 \in I_0$. This condition ensures the positivity of v^1 in a bootstrap argument.

In [8], Christodoulou-Perez studied mechanism for shock formation of strictly hyperbolic systems. However, here elastic waves are reduced to a non-strictly hyperbolic system below. We explore some subtle structures in (3.12). These structures are critical for the proof of the main theorem.

Proposition 3.1. *For the formula (3.12) of the non-strictly hyperbolic system (2.3), with $\Phi \in B_{2\delta}^6(0)$ we have the following properties:*

- For $i = 1, 3, 4, 6$, terms of w^2w^3 and w^4w^5 disappear;
- For $i = 2, 5$, the coefficients of w^2w^3 and w^4w^5 always have a small factor $\lambda_2(\Phi) - \lambda_3(\Phi)$ in front.

Proof. We may worry about the interaction of waves with almost-the-same speeds. The corresponding terms are w^2w^3 and w^4w^5 . To get the coefficient of w^2w^3 , with (2.13), (2.15), (3.12) and (3.14), we first calculate

$$\nabla_{\Phi} r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\phi_3 \partial_{\phi_1} \lambda_2 & 0 & -\lambda_2 & 0 & 0 & 0 \\ \phi_2 \partial_{\phi_1} \lambda_2 & \lambda_2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\nabla_{\Phi} r_3 =$$

$$\begin{pmatrix} \frac{\partial_{\phi_1}(\lambda_3^2 - b)}{2\sigma_1} & \frac{\partial_{\phi_2}(\lambda_3^2 - b)}{2\sigma_1} & \frac{\partial_{\phi_3}(\lambda_3^2 - b)}{2\sigma_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\lambda_3 \partial_{\phi_1}(\lambda_3^2 - b) + (\lambda_3^2 - b) \partial_{\phi_1} \lambda_3}{2\sigma_1} & -\frac{\lambda_3 \partial_{\phi_2}(\lambda_3^2 - b) + (\lambda_3^2 - b) \partial_{\phi_2} \lambda_3}{2\sigma_1} & -\frac{\lambda_3 \partial_{\phi_3}(\lambda_3^2 - b) + (\lambda_3^2 - b) \partial_{\phi_3} \lambda_3}{2\sigma_1} & 0 & 0 & 0 \\ -\phi_2 \partial_{\phi_1} \lambda_3 & -\lambda_3 - \phi_2 \partial_{\phi_2} \lambda_3 & -\phi_2 \partial_{\phi_3} \lambda_3 & 0 & 0 & 0 \\ -\phi_3 \partial_{\phi_1} \lambda_3 & -\phi_3 \partial_{\phi_2} \lambda_3 & -\lambda_3 - \phi_3 \partial_{\phi_3} \lambda_3 & 0 & 0 & 0 \end{pmatrix}.$$

These imply

$$(3.18) \quad \nabla_{\Phi} r_2 \cdot r_3 = \begin{pmatrix} 0 \\ \phi_3 \\ -\phi_2 \\ 0 \\ -\lambda_2 \phi_3 - \frac{\phi_3(\lambda_3^2 - b)}{2\lambda_2} \\ \lambda_2 \phi_2 + \frac{\phi_2(\lambda_3^2 - b)}{2\lambda_2} \end{pmatrix}, \quad \nabla_{\Phi} r_3 \cdot r_2 = \begin{pmatrix} 0 \\ \phi_3 \\ -\phi_2 \\ 0 \\ -\lambda_3 \phi_3 \\ \lambda_3 \phi_2 \end{pmatrix}.$$

For w^4w^5 , by similar calculations we get

$$\nabla_{\Phi} r_4 = \begin{pmatrix} \frac{\partial_{\phi_1}(\lambda_3^2-b)}{2\sigma_1} & \frac{\partial_{\phi_2}(\lambda_3^2-b)}{2\sigma_1} & \frac{\partial_{\phi_3}(\lambda_3^2-b)}{2\sigma_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{\lambda_3\partial_{\phi_1}(\lambda_3^2-b)+(\lambda_3^2-b)\partial_{\phi_1}\lambda_3}{2\sigma_1} & \frac{\lambda_3\partial_{\phi_2}(\lambda_3^2-b)+(\lambda_3^2-b)\partial_{\phi_2}\lambda_3}{2\sigma_1} & \frac{\lambda_3\partial_{\phi_3}(\lambda_3^2-b)+(\lambda_3^2-b)\partial_{\phi_3}\lambda_3}{2\sigma_1} & 0 & 0 & 0 \\ \phi_2\partial_{\phi_1}\lambda_3 & \lambda_3 + \phi_2\partial_{\phi_2}\lambda_3 & \phi_2\partial_{\phi_3}\lambda_3 & 0 & 0 & 0 \\ \phi_3\partial_{\phi_1}\lambda_3 & \phi_3\partial_{\phi_2}\lambda_3 & \lambda_3 + \phi_3\partial_{\phi_3}\lambda_3 & 0 & 0 & 0 \end{pmatrix},$$

$$\nabla_{\Phi} r_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \phi_3\partial_{\phi_1}\lambda_2 & 0 & \lambda_2 & 0 & 0 & 0 \\ -\phi_2\partial_{\phi_1}\lambda_2 & -\lambda_2 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and then

$$(3.19) \quad \nabla_{\Phi} r_4 \cdot r_5 = \begin{pmatrix} 0 \\ \phi_3 \\ -\phi_2 \\ 0 \\ \lambda_3\phi_3 \\ -\lambda_3\phi_2 \end{pmatrix}, \quad \nabla_{\Phi} r_5 \cdot r_4 = \begin{pmatrix} 0 \\ \phi_3 \\ -\phi_2 \\ 0 \\ \lambda_2\phi_3 + \frac{\phi_3(\lambda_3^2-b)}{2\lambda_2} \\ -\lambda_2\phi_2 - \frac{\phi_2(\lambda_3^2-b)}{2\lambda_2} \end{pmatrix}.$$

Now we derive the coefficients of w^2w^3 and w^4w^5 in the formulas of $\partial_{s_i}w^i (i = 1, \dots, 6)$.

Take $\partial_{s_1}w^1$ and $\partial_{s_2}w^2$ for examples. For $\partial_{s_1}w^1$, the coefficient of w^2w^3 is $\gamma_{23}^1 + \gamma_{32}^1$, with γ_{km}^i given in (3.14). Since

$$l^1 \cdot (\nabla_{\Phi} r_2 \cdot r_3) = \frac{1}{K} [\phi_2\phi_3 - \phi_2\phi_3 - (\lambda_2 + \frac{\lambda_3^2-b}{2\lambda_2})(\phi_2\phi_3 - \phi_2\phi_3)] = 0,$$

we have

$$(3.20) \quad \gamma_{23}^1 = -(\lambda_2 - \lambda_3)l^1 \cdot (\nabla_{\Phi} r_2 \cdot r_3) = 0.$$

We proceed to calculate γ_{32}^1 . With the expression for $\nabla_{\Phi}\lambda_3$ in (3.17), it holds that

$$(\phi_3\partial_{\phi_2} - \phi_2\partial_{\phi_3})\lambda_3^2 = -\frac{8\sigma_1^2}{\sqrt{\Delta}}\phi_2\phi_3 + \frac{8\sigma_1^2}{\sqrt{\Delta}}\phi_2\phi_3 = 0.$$

Hence we have

$$\begin{aligned}
(3.21) \quad & \gamma_{32}^1 = -(\lambda_3 - \lambda_2)l^1 \cdot (\nabla_{\Phi} r_3 \cdot r_2) \\
& = \frac{\lambda_2 - \lambda_3}{K} \left(\begin{array}{c} \diamond \\ \phi_2 + \frac{\lambda_3}{\lambda_1} \phi_2 + \left[\frac{\lambda_1^2 - b}{4\sigma_1^2} + \frac{\lambda_3(\lambda_1^2 - b)}{4\sigma_1^2 \lambda_1} \right] \partial_{\phi_2} (\lambda_3^2 - b) + \left[\frac{(\lambda_1^2 - b)(\lambda_3^2 - b)}{4\sigma_1^2 \lambda_1} + \frac{\phi_2^2 + \phi_3^2}{\lambda_1} \right] \partial_{\phi_2} \lambda_3 \\ \phi_3 + \frac{\lambda_3}{\lambda_1} \phi_3 + \left[\frac{\lambda_1^2 - b}{4\sigma_1^2} + \frac{\lambda_3(\lambda_1^2 - b)}{4\sigma_1^2 \lambda_1} \right] \partial_{\phi_3} (\lambda_3^2 - b) + \left[\frac{(\lambda_1^2 - b)(\lambda_3^2 - b)}{4\sigma_1^2 \lambda_1} + \frac{\phi_2^2 + \phi_3^2}{\lambda_1} \right] \partial_{\phi_3} \lambda_3 \\ 0 \\ 0 \\ 0 \end{array} \right)^{\top} \\
& \quad \cdot \left(\begin{array}{c} 0 \\ \phi_3 \\ -\phi_2 \\ 0 \\ -\lambda_2 \phi_3 \\ \lambda_2 \phi_2 \end{array} \right) \\
& = \frac{\lambda_2 - \lambda_3}{K} \left[\frac{\lambda_1^2 - b}{4\sigma_1^2} + \frac{\lambda_3(\lambda_1^2 - b)}{4\sigma_1^2 \lambda_1} + \frac{(\lambda_1^2 - b)(\lambda_3^2 - b)}{8\sigma_1^2 \lambda_1 \lambda_3} + \frac{\phi_2^2 + \phi_3^2}{2\lambda_1 \lambda_3} \right] (\phi_3 \partial_{\phi_2} - \phi_2 \partial_{\phi_3}) \lambda_3^2 \\
& = 0.
\end{aligned}$$

Notation “ \diamond ” in (3.21) represents certain functions depending on the unknown Φ . The detailed expression is not used in the above calculations, so we omit it. From (3.20) and (3.21), we conclude the coefficient of $w^2 w^3$ in $\partial_{s_1} w^1$ is

$$\gamma_{23}^1(\Phi) + \gamma_{32}^1(\Phi) = 0.$$

Similarly, with (3.19), we have the coefficient of $w^4 w^5$ in $\partial_{s_1} w^1$ is

$$\gamma_{45}^1(\Phi) + \gamma_{54}^1(\Phi) = 0.$$

Hence, there is no $w^2 w^3$ and $w^4 w^5$ terms in $\partial_{s_1} w^1$, i.e.,

$$\partial_{s_1} w^1 \sim (w^1)^2 + w^2 w^3 + w^4 w^5 + \dots$$

For the formula of $\partial_{s_2} w^2$ in (3.12), by definition of c_{23}^2 , γ_{23}^2 , γ_{45}^2 , γ_{54}^2 in (3.14), we get the coefficient of $w^2 w^3$ is

$$\begin{aligned}
-c_{23}^2(\Phi) + \gamma_{23}^2(\Phi) &= -\frac{\lambda_3^2 - \lambda_2^2}{2\lambda_2} - (\lambda_2 - \lambda_3) [l^2 \cdot (\nabla_{\Phi} r_2 \cdot r_3) - l^2 \cdot (\nabla_{\Phi} r_3 \cdot r_2)] \\
&= (\lambda_2 - \lambda_3) \left(1 + \frac{\lambda_3^2 - \lambda_2^2}{4\lambda_2^2} \right),
\end{aligned}$$

and the coefficient of $w^4 w^5$ is

$$\gamma_{45}^2(\Phi) + \gamma_{54}^2(\Phi) = -(\lambda_4 - \lambda_5) [l^2 \cdot (\nabla_{\Phi} r_4 \cdot r_5) - l^2 \cdot (\nabla_{\Phi} r_5 \cdot r_4)] = \frac{(\lambda_3 - \lambda_2)^3}{4\lambda_2^2}.$$

Since $\lambda_2(0) = \lambda_3(0) = c_2$, it holds that $\lambda_2(\Phi) - \lambda_3(\Phi)$ is small for $\Phi \in B_{2\delta}^6(0)$ with small δ . Combing with Lemma 3.1, we have

$$\partial_{s_2} w^2 \sim (w^2)^2 + (\lambda_2 - \lambda_3) w^2 w^3 + (\lambda_2 - \lambda_3) w^4 w^5 + \dots$$

Similarly, with (2.13), (2.15), (3.17), (3.18) and (3.19), we obtain

$$\begin{aligned}
-c_{32}^3(\Phi) &= \gamma_{32}^3(\Phi) = 0, & \gamma_{45}^3(\Phi) &= \gamma_{54}^3(\Phi) = 0, \\
-c_{45}^4(\Phi) &= \gamma_{45}^4(\Phi) = 0, & \gamma_{23}^4(\Phi) &= \gamma_{32}^4(\Phi) = 0, \\
-c_{54}^5(\Phi) + \gamma_{54}^5(\Phi) &= (\lambda_2 - \lambda_3) \left(-\frac{\lambda_3}{\lambda_2} + \frac{\lambda_3^2 - \lambda_2^2}{4\lambda_2^2} \right), \\
\gamma_{23}^5(\Phi) + \gamma_{32}^5(\Phi) &= \frac{(\lambda_2 - \lambda_3)^3}{4\lambda_2^2}, \\
\gamma_{23}^6(\Phi) &= \gamma_{32}^6(\Phi) = 0, & \gamma_{45}^6(\Phi) &= \gamma_{54}^6(\Phi) = 0.
\end{aligned}$$

This means that expressions of $\partial_{s_i} w^i$ admit the following property: when $i = 1, 3, 4, 6$, terms of $w^2 w^3$ and $w^4 w^5$ in $\partial_{s_i} w^i$ disappear; when $i = 2, 5$, the coefficients in front of $w^2 w^3$ and $w^4 w^5$ always have a factor $\lambda_2(\Phi) - \lambda_3(\Phi)$. By (2.12), for small $\varepsilon \in (0, \frac{1}{100}]$, we can choose sufficiently small δ such that

$$(3.22) \quad |\lambda_2(\Phi) - \lambda_3(\Phi)| \leq \varepsilon, \quad \text{for } \Phi \in B_{2\delta}^6(0).$$

□

In fact, all the coefficients c_{im}^i, γ_{km}^i are uniformly bounded. This is obvious for c_{im}^i and γ_{km}^j with $j = 1, 6$ since all the related eigenvalues, eigenvectors and their derivatives are regular. The only potentially problematic terms might come from the factors M, N in $\{l^i\}_{i=2,3,4,5}$ for γ_{km}^i . Via a direct calculation, one can verify that these coefficients are all of $O(1)$. For example, in the proof of the above proposition, we use the fact γ_{45}^2 is bounded and it follows from

$$\gamma_{45}^2 = -(\lambda_4 - \lambda_5) l^2 \cdot (\nabla_{\Phi} r_4 \cdot r_5) = -\frac{(\lambda_2 - \lambda_3)^2}{M\lambda_2} (\phi_2^2 + \phi_3^2) = -\frac{(\lambda_2 - \lambda_3)^2}{2\lambda_2}.$$

Similarly, we can obtain boundedness of other terms in the estimates of other γ_{km}^2 and γ_{km}^5 . For coefficients containing N , we take γ_{25}^3 as an example. It holds

$$\gamma_{25}^3 = -(\lambda_2 - \lambda_5) l^3 \cdot (\nabla_{\Phi} r_2 \cdot r_5) = \frac{(\lambda_2 - \lambda_5)(\lambda_2 + \lambda_3)(\phi_2^2 + \phi_3^2)}{\lambda_3 N}.$$

Noting that $N = \frac{(\lambda_3^2 - b)^2}{2\sigma_1^2} + 2(\phi_2^2 + \phi_3^2)$, hence we have $\gamma_{25}^3 \lesssim 1$. One can verify the boundedness of other γ_{km}^3 and γ_{km}^4 in the same fashion.

In summary, we list the important structures of $\partial_{s_i} w^i$ here:

$$(3.23) \quad \begin{cases} \partial_{s_1} w^1 \sim (w^1)^2 + w^2 w^3 + w^4 w^5 + \dots, \\ \partial_{s_2} w^2 \sim (w^2)^2 + (\lambda_2 - \lambda_3) w^2 w^3 + (\lambda_2 - \lambda_3) w^4 w^5 + \dots, \\ \partial_{s_3} w^3 \sim (w^3)^2 + w^2 w^3 + w^4 w^5 + \dots, \\ \partial_{s_4} w^4 \sim (w^4)^2 + w^2 w^3 + w^4 w^5 + \dots, \\ \partial_{s_5} w^5 \sim (w^5)^2 + (\lambda_2 - \lambda_3) w^2 w^3 + (\lambda_2 - \lambda_3) w^4 w^5 + \dots, \\ \partial_{s_6} w^6 \sim (w^6)^2 + w^2 w^3 + w^4 w^5 + \dots. \end{cases}$$

The deleted terms mean their coefficients are zero. That is to say that no interactions of the almost-repeated characteristic waves (w^2 and w^3 , w^4 and w^5) appear in the equations of $\{\partial_{s_i} w^i\}_{i=1,3,4,6}$. Moreover, we have $\lambda_2 - \lambda_3 = \lambda_4 - \lambda_5$ being small, and it shows that the interactions $w^2 w^3$, $w^4 w^5$ in $\partial_{s_2} w^2$ and $\partial_{s_5} w^5$ are also weak. This allows us to use the bi-characteristic transformation to get desired bounds.

Back to (3.15)-(3.16) with the above coefficients, we also derive the equations for $\{\rho_i\}$ and $\{v^i\}$:

$$(3.24) \quad \begin{cases} \partial_{s_1} \rho_1 \sim v^1 + \rho_1(w^i)_{i \neq 1}, \\ \partial_{s_2} \rho_2 \sim \cancel{\rho_2^2} + (\lambda_2 - \lambda_3) \rho_2 w^3 + \rho_2(w^i)_{i \neq 2,3}, \\ \partial_{s_3} \rho_3 \sim v^3 + \cancel{\rho_3 w^2} + \rho_3(w^i)_{i \neq 2,3}, \\ \partial_{s_4} \rho_4 \sim v^4 + \cancel{\rho_4 w^5} + \rho_4(w^i)_{i \neq 4,5}, \\ \partial_{s_5} \rho_5 \sim \cancel{\rho_5^2} + (\lambda_2 - \lambda_3) \rho_5 w^4 + \rho_5(w^i)_{i \neq 4,5}, \\ \partial_{s_6} \rho_6 \sim v^6 + \rho_6(w^i)_{i \neq 6}, \end{cases}$$

and

$$(3.25) \quad \begin{cases} \partial_{s_1} v^1 \sim v^1(w^i)_{i \neq 1} + \rho_1(\cancel{w^2 w^3} + \cancel{w^4 w^5} + \dots), \\ \partial_{s_2} v^2 \sim v^2[(\lambda_2 - \lambda_3)w^3 + \dots] + \rho_2[(\lambda_2 - \lambda_3)w^4 w^5 + \dots], \\ \partial_{s_3} v^3 \sim v^3(\cancel{w^2} + \dots) + \rho_3(\cancel{w^4 w^5} + \dots), \\ \partial_{s_4} v^4 \sim v^4(\cancel{w^5} + \dots) + \rho_4(\cancel{w^2 w^3} + \dots), \\ \partial_{s_5} v^5 \sim v^5[(\lambda_2 - \lambda_3)w^4 + \dots] + \rho_5[(\lambda_2 - \lambda_3)w^2 w^3 + \dots], \\ \partial_{s_6} v^6 \sim v^6(w^i)_{i \neq 6} + \rho_6(\cancel{w^2 w^3} + \cancel{w^4 w^5} + \dots). \end{cases}$$

The deleted terms mean their coefficients are zero.

Remark: Here we only list the terms, whose nonlinear interactions might be large, because they are from the same or almost-the-same eigenvalues.

4. CONSTRUCTION OF INITIAL DATA

Let $\varepsilon \in (0, \frac{1}{100}]$ be a small parameter. Given a small fixed parameter η ($0 < \eta \ll 1$), we choose initial data $w_{(\eta)}^i(z_i, 0)$ such that $\text{supp } w_{(\eta)}^i(z_i, 0) \in [\eta, 2\eta]$, and

$$(4.1) \quad W_0^{(\eta)} := \max_i \sup_{z_i} |w_{(\eta)}^i(z_i, 0)| = w_{(\eta)}^1(z_0, 0) > 0.$$

Furthermore, we require

$$(4.2) \quad \sup_{z_6} |w^6(z_6, 0)| \leq \frac{(1 - \varepsilon)^4}{2(1 + \varepsilon)^3} W_0^{(\eta)}$$

and

$$(4.3) \quad \max_{i=3,4} \sup_{z_i} |w_{(\eta)}^i(z_i, 0)| \leq \min \left\{ \frac{(1 - \varepsilon)^4 |c_{11}^1(0)|}{(1 + \varepsilon)^3} W_0^{(\eta)}, W_0^{(\eta)} \right\}.$$

For $w_{(\eta)}^1(z, 0)$, we choose

$$(4.4) \quad w_{(\eta)}^1(z, 0) = \theta \int_{\mathbb{R}} \zeta_{\frac{\eta}{10}}(y) |\ln(z-y)|^\alpha \chi(z-y) dy, \quad 0 < \alpha < \frac{1}{2},$$

where θ is a small parameter to be determined later. Here χ is the characteristic function

$$\chi(z) = \begin{cases} 1, & z \in [\frac{6}{5}\eta, \frac{9}{5}\eta], \\ 0, & z \notin (\frac{6}{5}\eta, \frac{9}{5}\eta), \end{cases}$$

and $\zeta_{\frac{\eta}{10}}(z)$ is a test function in $C_0^\infty(\mathbb{R})$ and satisfying

$$\text{supp } \zeta_{\frac{\eta}{10}}(z) = \{z : |z| \leq \frac{\eta}{10}\}, \quad \text{and} \quad \int_{\mathbb{R}} \zeta_{\frac{\eta}{10}}(z) dz = 1.$$

We then define the following norms for $i = 1, 2, 3, 4, 5, 6$,

$$(4.5) \quad S_i(t) := \sup_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} \rho_i(z'_i, s'_i), \quad S(t) := \max_i S_i(t),$$

$$(4.6) \quad J_i(t) := \sup_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} |v^i(z'_i, s'_i)|, \quad J(t) := \max_i J_i(t),$$

$$(4.7) \quad W(t) := \max_i \sup_{\substack{(x', t') \\ 0 \leq t' \leq t}} |w^i(x', t')|, \quad \bar{U}(t) := \sup_{\substack{(x', t') \\ 0 \leq t' \leq t}} |\Phi(x', t')|.$$

We also denote

$$V_1(t) := \sup_{\substack{(x', t') \notin \mathcal{R}_1, \\ 0 \leq t' \leq t}} |w^1(x', t')|,$$

$$V_2(t) := \max_{\substack{(x', t') \notin \mathcal{R}_2 \cup \mathcal{R}_3, \\ 0 \leq t' \leq t}} \{|w^2(x', t')|, |w^3(x', t')|\},$$

$$V_5(t) := \max_{\substack{(x', t') \notin \mathcal{R}_4 \cup \mathcal{R}_5, \\ 0 \leq t' \leq t}} \{|w^4(x', t')|, |w^5(x', t')|\},$$

$$V_6(t) := \sup_{\substack{(x', t') \notin \mathcal{R}_6, \\ 0 \leq t' \leq t}} |w^6(x', t')|$$

and

$$(4.8) \quad V(t) := \max_i V_i(t), \quad \text{for } i = 1, \bar{2}, \bar{5}, 6.$$

Finally, we set

$$(4.9) \quad \underline{S}(t) := \min_{i \in \{2, \dots, 6\}} \inf_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} \rho_i(z'_i, s'_i).$$

5. ESTIMATES OF NORMS

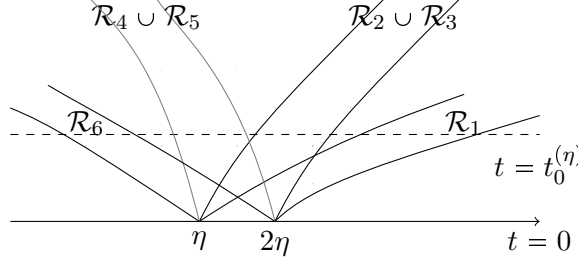
Recall that for the aforementioned system (2.2), two pairs of characteristic wave speeds are almost the same,

$$\lambda_2(\Phi) \approx \lambda_3(\Phi), \quad \lambda_4(\Phi) \approx \lambda_5(\Phi),$$

for $\Phi \in B_{2\delta}^6(0)$. This means two pairs of characteristic strips \mathcal{R}_2 and \mathcal{R}_3 or \mathcal{R}_4 and \mathcal{R}_5 could overlap for a long time. Our strategy is to consider the wave propagations in four characteristic strips:

$$(5.1) \quad \{\mathcal{R}_1, \mathcal{R}_2 \cup \mathcal{R}_3, \mathcal{R}_4 \cup \mathcal{R}_5, \mathcal{R}_6\}.$$

These four strips will be completely separated when $t > t_0^{(\eta)}$, where $t_0^{(\eta)}$ can be precisely calculated as in (5.2) below.



For notational simplicity, we denote

$$\mathcal{R}_{\bar{2}} := \mathcal{R}_2 \cup \mathcal{R}_3, \quad \mathcal{R}_{\bar{5}} := \mathcal{R}_4 \cup \mathcal{R}_5.$$

Let

$$\bar{\lambda}_i := \sup_{\Phi \in B_{2\delta}^6(0)} \lambda_i(\Phi), \quad \underline{\lambda}_i := \inf_{\Phi \in B_{2\delta}^6(0)} \lambda_i(\Phi), \quad \text{for } i = 1, 6,$$

$$\bar{\lambda}_{\bar{2}} := \sup_{\Phi \in B_{2\delta}^6(0)} \{\lambda_2(\Phi), \lambda_3(\Phi)\}, \quad \underline{\lambda}_{\bar{2}} := \inf_{\Phi \in B_{2\delta}^6(0)} \{\lambda_2(\Phi), \lambda_3(\Phi)\}$$

$$\bar{\lambda}_{\bar{5}} := \sup_{\Phi \in B_{2\delta}^6(0)} \{\lambda_4(\Phi), \lambda_5(\Phi)\}, \quad \underline{\lambda}_{\bar{5}} := \inf_{\Phi \in B_{2\delta}^6(0)} \{\lambda_4(\Phi), \lambda_5(\Phi)\}$$

and

$$\sigma := \min_{\substack{\alpha < \beta \\ \alpha, \beta \in \{1, \bar{2}, \bar{5}, 6\}}} (\underline{\lambda}_\alpha - \bar{\lambda}_\beta).$$

According to (2.11), with δ sufficiently small we have that σ has a uniform positive lower bound. By the defining equation of characteristics (3.1), for $\alpha \in \{1, \bar{2}, \bar{5}, 6\}$, $z \in [\eta, 2\eta]$, we have

$$z + \underline{\lambda}_\alpha t \leq X_\alpha(z, t) \leq z + \bar{\lambda}_\alpha t.$$

And for all $\alpha < \beta$, with $\alpha, \beta \in \{1, \bar{2}, \bar{5}, 6\}$, it holds

$$\begin{aligned} & X_\alpha(\eta, t) - X_\beta(2\eta, t) \\ & \geq (\eta + \underline{\lambda}_\alpha t) - (2\eta + \bar{\lambda}_\beta t) = -\eta + (\underline{\lambda}_\alpha - \bar{\lambda}_\beta)t \geq -\eta + \sigma t. \end{aligned}$$

Note that the above difference is strictly positive when

$$(5.2) \quad t > t_0^{(\eta)} := \frac{\eta}{\sigma}.$$

This implies that the four characteristic strips in (5.1) are well separated when $t > t_0^{(\eta)}$.

- **Estimates for $t \in [0, t_0^{(\eta)}]$**

In the non-separated region before $t = t_0^{(\eta)}$, all the characteristic strips \mathcal{R}_i are overlapped. But even then, the inverse foliation density of characteristics still obey positive lower bounds in this time region.

Let Γ be the maximum of all the coefficients $\{c_{im}^i\}$ and $\{\gamma_{km}^i\}$ in (3.15)-(3.16). According to the calculations in Section 3, we have

$$(5.3) \quad \Gamma = O(1).$$

In the following part, we will bound $W(t)$, $V(t)$, $S(t)$, $J(t)$ and $\bar{U}(t)$ defined in (4.5)-(4.8). We estimate $W(t)$ first. For $t \in [0, t_0^{(\eta)}]$, by (3.12), we have

$$\frac{\partial}{\partial s_i} |w^i| \leq \Gamma W^2.$$

Comparing with solutions to

$$\begin{cases} \frac{d}{dt} Y = \Gamma Y^2, \\ Y(0) = W_0^{(\eta)}, \end{cases}$$

we have

$$(5.4) \quad |w^i| \leq Y(t) = \frac{W_0^{(\eta)}}{1 - \Gamma W_0^{(\eta)} t}, \quad \text{for } t < \min \left\{ \frac{1}{\Gamma W_0^{(\eta)}}, T \right\}.$$

With (5.2) and (5.3), we obtain

$$(5.5) \quad \Gamma W_0^{(\eta)} t_0 = O(\eta W_0^{(\eta)}).$$

Applying (5.5) to (5.4), it holds

$$|w^i(x, t)| \leq (1 + \varepsilon) W_0^{(\eta)}, \quad \forall x \in \mathbb{R}, t \in [0, t_0^{(\eta)}],$$

for some $\varepsilon > 0$ small. This could be achieved by requiring θ in (4.4) sufficiently small. Back to the definition in (4.7), this implies that

$$(5.6) \quad |W(t)| \leq (1 + \varepsilon) W_0^{(\eta)}, \quad \forall t \in [0, t_0^{(\eta)}].$$

We proceed to bound $V(t)$. Any $(x', t') \notin \mathcal{R}_i$ can be characterised by the corresponding characteristic coordinates (z'_i, s'_i) satisfying $z'_i \notin [\eta, 2\eta]$. Since our constructed initial data are supported in $[\eta, 2\eta]$, for $z'_i \notin [\eta, 2\eta]$ we have $w_{(\eta)}^i(z_i, 0) = 0$. Integrating (3.12) along the characteristic \mathcal{C}_i , we obtain

$$V(t) = O\left(\int_0^{t_0^{(\eta)}} w^i w^j ds_i\right) = O(\eta [W(t)]^2) = O(\eta [W_0^{(\eta)}]^2).$$

We then estimate $S(t)$. From (3.15), we have

$$\frac{\partial \rho_i}{\partial s_i} = O(\rho_i W).$$

Integrating the above equation along \mathcal{C}_i , we get

$$(5.7) \quad \rho_i(z_i, t) = \rho_i(z_i, 0) \exp(O(tW(t))).$$

Note that by definitions of (3.1) and (3.3) we have

$$(5.8) \quad \rho_i(z_i, 0) = 1,$$

then via (5.7) it holds $\rho_i(z_i, t) > 0$. Moreover, by (5.6), we obtain

$$(5.9) \quad \rho_i(z_i, t) = \exp(O(\eta W_0^{(\eta)})), \quad \forall t \in [0, t_0^{(\eta)}].$$

For η being small, we can choose sufficiently small θ such that

$$(5.10) \quad 1 - \varepsilon \leq \exp(O(\eta W_0^{(\eta)})) \leq 1 + \varepsilon.$$

Inserting (5.10) into (5.9), we get

$$1 - \varepsilon \leq \rho_i(z_i, t) \leq 1 + \varepsilon, \quad \forall t \in [0, t_0^{(\eta)}].$$

So we have

$$(5.11) \quad S(t) = O(1), \quad \forall t \in [0, t_0^{(\eta)}].$$

For $J(t)$, from (3.16), we have

$$\frac{\partial v^i}{\partial s_i} = O(S(t)[W(t)]^2).$$

Using (5.6) and (5.11), for $t \in [0, t_0^{(\eta)}]$, we get

$$(5.12) \quad J(t) = O(W_0^{(\eta)} + t[W(t)]^2) = O(W_0^{(\eta)} + \eta[W_0^{(\eta)}]^2) = O(W_0^{(\eta)}).$$

Next, we give an estimate for \bar{U} . By (3.11), we have

$$(5.13) \quad \Phi(x, t) = \int_{X_6(\eta, t)}^x \frac{\partial \Phi(x', t)}{\partial x} dx' = \int_{X_6(\eta, t)}^x \sum_k w^k r_k(x', t) dx'.$$

since

$$r_k(\Phi) = O(1),$$

equality (5.13) implies

$$(5.14) \quad |\Phi(x, t)| = O\left(\sum_k \int_{X_6(\eta, t)}^{X_1(2\eta, t)} |w^k(x', t)| dx'\right).$$

By (5.6) and definition (4.7), we get

$$\bar{U}(t) = O(W(t)(\eta + (\bar{\lambda}_1 - \underline{\lambda}_6)t)) = O(\eta W_0^{(\eta)}), \quad \forall t \in [0, t_0^{(\eta)}].$$

In summary, we have proved

$$(5.15) \quad W(t) = O(W_0^{(\eta)}), \quad \forall t \in [0, t_0^{(\eta)}],$$

$$(5.16) \quad V(t) = O(\eta[W_0^{(\eta)}]^2), \quad \forall t \in [0, t_0^{(\eta)}],$$

$$(5.17) \quad S(t) = O(1), \quad \forall t \in [0, t_0^{(\eta)}],$$

$$(5.18) \quad J(t) = O(W_0^{(\eta)}), \quad \forall t \in [0, t_0^{(\eta)}],$$

$$(5.19) \quad \bar{U}(t) = O(\eta W_0^{(\eta)}), \quad \forall t \in [0, t_0^{(\eta)}].$$

• **Estimates for $t \in [t_0^{(\eta)}, T]$**

Though six characteristic strips only separate partially because of the non-strict hyperbolicity, the aforementioned four characteristic strips (5.1) are well separated after $t > t_0^{(\eta)}$. In the following part we estimate geometric

quantities in different strips, and we will show that a shock forms in \mathcal{R}_1 along \mathcal{C}_1 . Subtle structures of (3.23)-(3.25) yield cancellations of some potentially dangerous terms.

Based on the bounds for $t \in [0, t_0^{(\eta)}]$, we carry on estimates in the region $t \in [t_0^{(\eta)}, T]$. We first estimate $S(t)$, i.e., the supremum of inverse foliation densities. For $\alpha = 1$ or 6, if $(x, t) \in \mathcal{R}_\alpha$, we have

$$(5.20) \quad \frac{\partial \rho_\alpha}{\partial s_\alpha} = O(J_\alpha + VS_\alpha), \quad \alpha = 1 \text{ or } 6.$$

Thus, by integrating (5.20) along the characteristic \mathcal{C}_α , we have

$$(5.21) \quad \rho_\alpha(z_\alpha, t) = \rho_0(z_\alpha, 0) + \int_0^t O(J_\alpha + VS_\alpha) dt'.$$

We then conclude

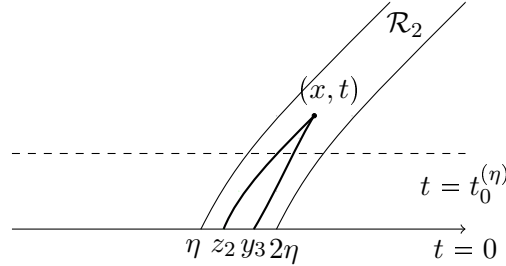
$$S_\alpha(t) = O(1 + tJ_\alpha + tVS_\alpha), \quad \alpha = 1 \text{ or } 6.$$

For ρ_2 , if $(x, t) \in \mathcal{R}_2$, the characteristic \mathcal{C}_2 crossing (x, t) may also intersect \mathcal{R}_3 . By a crucial structure displayed in (3.15)

$$c_{23}^2 = (\lambda_2 - \lambda_3)O(1),$$

we have

$$(5.22) \quad \frac{\partial \rho_2}{\partial s_2} = O(VS_2 + (\lambda_2 - \lambda_3)w^3\rho_2).$$



Along the characteristic \mathcal{C}_2 , by definitions of the characteristic coordinates and bi-characteristic coordinates, we have $dy_2 = dz_2 = 0$. Thus, integrating (5.22) along \mathcal{C}_2 and applying the bi-characteristic coordinate transformation as in (3.8), we have

$$(5.23) \quad \begin{aligned} & \int_0^t (\lambda_2 - \lambda_3)w^3(X_2(z_2, t'), t')\rho_2(X_2(z_2, t'), t') dt' \\ &= O\left(\int_{[\eta, 2\eta]} (\lambda_2 - \lambda_3)\frac{\rho_3}{\lambda_2 - \lambda_3}w^3\rho_2 dy_3\right) = O(\eta\rho_2v^3) = O(\eta S_2 J_3). \end{aligned}$$

Hence, by integrating (5.22) along the characteristic \mathcal{C}_2 , we get

$$(5.24) \quad \rho_2(z_2, t) = \rho_2(z_2, 0) + \int_0^t O(VS_2 + (\lambda_2 - \lambda_3)w^3\rho_2) dt'.$$

Hence, we have

$$(5.25) \quad S_2 = O(1 + tVS_2 + \eta S_2 J_3).$$

For ρ_3 , since we have

$$c_{32}^3 = 0,$$

it follows

$$(5.26) \quad S_3 = O(1 + tJ_3 + tVS_3).$$

For ρ_4 and ρ_5 , if $(x, t) \in \mathcal{R}_5$, we proceed as for ρ_3 and ρ_2 , respectively. Note

$$c_{45}^4 = 0, \quad \text{and} \quad c_{54}^5 = (\lambda_4 - \lambda_5)O(1).$$

As estimates obtained in (5.26), we have

$$S_4 = O(1 + tJ_4 + tVS_4).$$

And proceed as in (5.22)-(5.25), we also obtain

$$(5.27) \quad S_5 = O(1 + tVS_5 + \eta S_5 J_4).$$

So we have

$$(5.28) \quad S = O(1 + tVS + tJ + \eta SJ).$$

We then bound $J(t)$, the supremum of $\{v^i\}_{i=1, \dots, 6}$. For $\alpha = 1, 6$, and $(x, t) \in \mathcal{R}_\alpha$, with the cancellations of the deleted terms in (3.25), we have

$$\frac{\partial v^\alpha}{\partial s_\alpha} = O(VJ_\alpha + V^2S_\alpha).$$

Thus,

$$(5.29) \quad J_\alpha(t) = O(W_0^{(\eta)} + tVJ_\alpha + tV^2S_\alpha), \quad \text{for } \alpha = 1, 6.$$

For v^3 , $(x, t) \in \mathcal{R}_3$, we have

$$\frac{\partial v^3}{\partial s_3} = O\left(VJ_3 + S_3V^2 + VS_3w^2\right).$$

Since

$$\int_0^t VS_3w^2 dt' = \int_0^t VS_3 \frac{v^2}{\rho_2} dt' = O\left(\frac{tVS_3J}{\underline{S}}\right),$$

we obtain

$$(5.30) \quad J_3(t) = O\left(W_0^{(\eta)} + tVJ_3 + tV^2S_3 + \frac{tVS_3J}{\underline{S}}\right).$$

For v^4 , in the same fashion we get

$$(5.31) \quad J_4(t) = O\left(W_0^{(\eta)} + tVJ_4 + tV^2S_4 + \frac{tVS_4J}{\underline{S}}\right).$$

For v^2 , $(x, t) \in \mathcal{R}_2$, we have

$$\frac{\partial v^2}{\partial s_2} = O((\lambda_2 - \lambda_3)w^3J_2 + V^2S_2).$$

By a similar estimate to (5.23), it holds

$$\begin{aligned}
 & \int_0^t (\lambda_2 - \lambda_3) w^3(X_2(z_2, t'), t') J_2(X_2(z_2, t'), t') dt' \\
 (5.32) \quad & = O\left(\int_{[\eta, 2\eta]} (\lambda_2 - \lambda_3) \frac{\rho_3}{\lambda_2 - \lambda_3} w^3 J_2 dy_3\right) \\
 & = O\left(J_2 \int_{[\eta, 2\eta]} v^3 dy_3\right) = O(\eta J_2 J_3).
 \end{aligned}$$

Integrating along characteristic \mathcal{C}_2 , we arrive at

$$(5.33) \quad J_2 = O(W_0^{(\eta)} + \eta J_2 J_3 + tV^2 S_2).$$

Analogously, for v^5 we obtain

$$(5.34) \quad J_5 = O(W_0^{(\eta)} + \eta J_4 J_5 + tV^2 S_5).$$

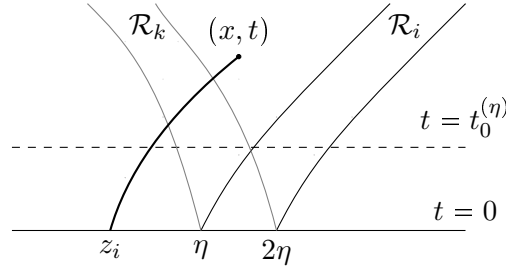
In conclusion, from (5.29), (5.30)-(5.31), and (5.33)-(5.34), we have

$$(5.35) \quad J = O\left(W_0^{(\eta)} + tVJ + tV^2S + \eta J^2 + \frac{tV S J}{\underline{S}}\right).$$

We next bound $V(t)$, i.e., the supremum of w^i outside the corresponding characteristic strip \mathcal{R}_i . We first estimate $w^1(x, t)$ with $(x, t) \notin \mathcal{R}_1$ and $w^6(x, t)$ with $(x, t) \notin \mathcal{R}_6$. Let $i = \{1, 6\}$. From (3.23), we have

$$(5.36) \quad \frac{\partial w^i}{\partial s_i} = O(V^2) + O\left(\sum_{k \neq i} w^k\right)V + O\left(\sum_{\substack{m \neq i, k \neq i \\ m \neq k}} w^m w^k\right).$$

Note that \mathcal{C}_i starts from $z_i \notin [\eta, 2\eta]$ and ends at $(x, t) \notin \mathcal{R}_i$. When $t' \geq t_0^{(\eta)}$, for any point $(X_i(z_i, t'), t')$ on \mathcal{C}_i , it holds either $(X_i(z_i, t'), t') \in (\mathbb{R} \times [t_0^{(\eta)}, t]) \setminus \bigcup_k \mathcal{R}_k$ or $(X_i(z_i, t'), t') \in \mathcal{R}_k$ for some $k \neq i$.

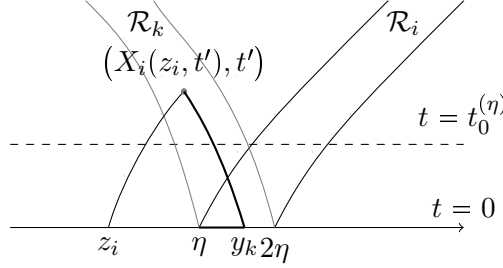


For the term $O\left(\sum_{\substack{m \neq i, k \neq i \\ m \neq k}} w^m w^k\right)$ in (5.36), we recall that both $w^2 w^3$ and $w^4 w^5$ vanish in equation (5.36). If $(x, t) \notin \mathcal{R}_i$, then there exist only three cases: $(x, t) \in \mathcal{R}_m$, $(x, t) \in \mathcal{R}_k$, (x, t) stays out of all the characteristics. In all these three cases, the term $O\left(\sum_{\substack{m \neq i, k \neq i \\ m \neq k}} w^m w^k\right)$ can be absorbed by the second term $O\left(\sum_{k \neq i} w^k\right)V$. Let $I_k^i = \{t' \in [t_0^{(\eta)}, t] : (x, t') \in \mathcal{C}_i \cap \mathcal{R}_k\}$ for

$k \neq i$. Integrating (5.36) along \mathcal{C}_i and using $w_{(\eta)}^i(z_i, 0) = 0$, we have

$$\begin{aligned}
w^i(x, t) &= O\left(tV^2 + V \sum_{k \neq i} \int_0^t w^k(X_i(z_i, t'), t') dt'\right) \\
&= O\left(tV^2 + V \sum_{k \neq i} \int_0^{t_0^{(\eta)}} w^k(X_i(z_i, t'), t') dt'\right) \\
(5.37) \quad &+ O\left(V \sum_{k \neq i} \int_{t_0^{(\eta)}}^t w^k(X_i(z_i, t'), t') dt'\right) \\
&= O\left(tV^2 + \eta[W_0^{(\eta)}]^2 + V \underbrace{\sum_{k \neq i} \int_{I_k^i} w^k(X_i(z_i, t'), t') dt'}_M\right).
\end{aligned}$$

Here we use the fact $V(t) \leq W(t) = O(W_0^{(\eta)})$ for $t \leq t_0^{(\eta)}$. We proceed to bound M . When $(X_i(z_i, t'), t') \in I_k^i$ for some $k \neq i$, the picture is as below



we then employ the bi-characteristic coordinates and get

$$\begin{aligned}
&\int_{I_k^i} w^k(X_i(z_i, t'), t') dt' \\
(5.38) \quad &= O\left(\int_{y_k \in [\eta, 2\eta]} \left| \frac{\rho_k(y_k, t'(y_i, y_k))}{\lambda_i - \lambda_k} w^k(y_k, t'(y_i, y_k)) \right| dy_k\right) \\
&= O(\eta J_k).
\end{aligned}$$

Together with (5.37) and (5.38), for $i = 1, 6$, we hence obtain

$$(5.39) \quad w^i(x, t) = O(tV^2 + \eta[W_0^{(\eta)}]^2 + \eta V J), \quad \text{for } \forall (x, t) \notin \mathcal{R}_i.$$

We next estimate $w^3(x, t)$ with $(x, t) \notin \mathcal{R}_2 \cup \mathcal{R}_3$ and $w^4(x, t)$ with $(x, t) \notin \mathcal{R}_4 \cup \mathcal{R}_5$. From (3.23), we have that $w^4 w^5$ and $w^2 w^3$ vanish in $\partial_{s_3} w^3$ and $\partial_{s_4} w^4$. In the same fashion as for (5.37), (5.38) and (5.39), we hence obtain

$$(5.40) \quad w^i(x, t) = O(tV^2 + \eta[W_0^{(\eta)}]^2 + \eta V J), \quad \text{for } (x, t) \notin \mathcal{R}_i, \quad i = 3, 4.$$

We then deal with $w^2(x, t)$ for $(x, t) \notin \mathcal{R}_2 \cup \mathcal{R}_3$. We discuss the following case first: \mathcal{R}_2 and \mathcal{R}_3 do not separate before time t . Note that $\lambda_2 \geq \lambda_3$ by (2.6)-(2.7) and $a > b$. For \mathcal{C}_2 being a characteristic curve ending at (x, t) and starting from $z_2 \notin [\eta, 2\eta]$, for any point $(X_2(z_2, t'), t')$ on \mathcal{C}_2 , the fact $\lambda_2 \geq \lambda_3$ implies $(X_2(z_2, t'), t') \notin \mathcal{R}_3$. Thus when $t' \in [t_0^{(\eta)}, t]$, it holds that either $(X_2(z_2, t'), t')$ stays in $(\mathbb{R} \times [t_0^{(\eta)}, t]) \setminus \bigcup_k \mathcal{R}_k$ or it lies in one of

the characteristic strips $\{\mathcal{R}_1, \mathcal{R}_6, \mathcal{R}_4 \cup \mathcal{R}_5\}$. Integrating $\partial_{s_2} w^2$ along \mathcal{C}_2 , we hence get

$$\begin{aligned}
& w^2(x, t) \\
&= O\left(tV^2 + V \sum_{k=1,6} \int_0^t w^k(X_2(z_2, t'), t') dt'\right) \\
(5.41) \quad & + O\left(\int_0^t |(\lambda_2 - \lambda_3)w^4(X_2(z_2, t'), t')w^5(X_2(z_2, t'), t')| dt'\right) \\
&= O\left(tV^2 + \eta[W_0^{(\eta)}]^2 + V \sum_{k=1,6} \int_{I_k^2} w^k(X_2(z_2, t'), t') dt'\right) \\
& + O\left(\int_{I_4^2 \cup I_5^2} |(\lambda_2 - \lambda_3)w^4(X_2(z_2, t'), t')w^5(X_2(z_2, t'), t')| dt'\right).
\end{aligned}$$

Similarly to (5.38), it holds that

$$\begin{aligned}
& \sum_{k=1,6} \int_{I_k^2} w^k(X_2(z_2, t'), t') dt' \\
(5.42) \quad &= O\left(\sum_{k=1,6} \int_{y_k \in [\eta, 2\eta]} \left| \frac{\rho_k(y_k, t'(y_2, y_k))}{\lambda_2 - \lambda_k} w^k(y_k, t'(y_2, y_k)) \right| dy_k\right) \\
&= O(\eta J).
\end{aligned}$$

Using bi-characteristic coordinates (y_2, y_5) and together with (3.22), we have

$$\begin{aligned}
(5.43) \quad & \int_{I_4^2 \cup I_5^2} |(\lambda_2 - \lambda_3)w^4(X_2(z_2, t'), t')w^5(X_2(z_2, t'), t')| dt' \\
&= O\left(\varepsilon \int_{y_5 \in [\eta, 2\eta]} \left| w^4(y_5, t'(y_2, y_5))w^5(y_5, t'(y_2, y_5)) \frac{\rho_5(y_5, t'(y_2, y_5))}{\lambda_2 - \lambda_5} dy_5 \right|\right) \\
&= O\left(\varepsilon \int_{y_5 \in [\eta, 2\eta]} \left| \frac{v^4 v^5}{\rho_4} dy_5 \right|\right) = O\left(\eta \frac{\varepsilon}{\underline{S}} J^2\right).
\end{aligned}$$

If \mathcal{R}_2 and \mathcal{R}_3 separate before t , for $(x, t) \notin \mathcal{R}_2 \cup \mathcal{R}_3$, the characteristic curve \mathcal{C}_2 ending at (x, t) may overlap \mathcal{R}_3 but does not intersect \mathcal{R}_2 . In this case, we calculate the integration along \mathcal{C}_2 inside \mathcal{R}_3

$$\begin{aligned}
& \int_{I_3^2} |(\lambda_2 - \lambda_3)w^2(X_2(z_2, t'), t')w^3(X_2(z_2, t'), t')| dt' \\
(5.44) \quad & \leq \int_{y_3 \in [\eta, 2\eta]} \left| (\lambda_2 - \lambda_3)w^2w^3(y_3, t'(y_2, y_3)) \frac{\rho_3(y_3, t'(y_2, y_3))}{\lambda_2 - \lambda_3} \right| dy_3 \\
& \leq \int_{y_3 \in [\eta, 2\eta]} \left| \frac{v^2}{\rho_2} v^3(y_3, t'(y_2, y_3)) \right| dy_3 \\
& = O\left(\eta \frac{\varepsilon}{\underline{S}} J^2\right).
\end{aligned}$$

In sum, by (5.41), (5.42), (5.43) and (5.44), we consequently obtain

$$(5.45) \quad w^2(x, t) = O\left(\eta[W_0^{(\eta)}]^2 + tV^2 + \eta VJ + \eta \frac{\varepsilon}{\underline{S}} J^2\right), \quad \forall (x, t) \notin \mathcal{R}_2 \cup \mathcal{R}_3.$$

Analogously, it also holds

$$(5.46) \quad w^5(x, t) = O\left(\eta[W_0^{(\eta)}]^2 + tV^2 + \eta VJ + \eta \frac{\varepsilon}{\underline{S}} J^2\right), \quad \forall (x, t) \notin \mathcal{R}_4 \cup \mathcal{R}_5.$$

Hence, from (5.39)-(5.40), (5.45) and (5.46), we get

$$(5.47) \quad V = O\left(\eta[W_0^{(\eta)}]^2 + tV^2 + \eta VJ + \eta \frac{\varepsilon}{\underline{S}} J^2\right).$$

Finally, we bound Φ . If (x, t) does not belong to any characteristic strip, we go back to (5.14) and then obtain

$$(5.48) \quad \bar{U}(t) = O((\eta + (\bar{\lambda}_1 - \underline{\lambda}_6)t)V).$$

If $(x, t) \in \mathcal{R}_k$ for some k , with characteristic coordinates we have

$$(5.49) \quad \begin{aligned} |\Phi(x, t)| &= \left| \int_{X_k(\eta, t)}^x \frac{\partial \Phi(x', t)}{\partial x} dx' \right| = \left| \int_{X_k(\eta, t)}^x \sum_k w^k r_k(x', t) dx' \right| \\ &\leq \int_{X_k(\eta, t)}^{X_k(2\eta, t)} |w^k(x', t)| dx' = O\left(\int_{\eta}^{2\eta} |w^k(x', t)| \rho_k dz_k\right) \\ &= O(\eta J). \end{aligned}$$

Together with (5.48) and (5.49), we get

$$(5.50) \quad \bar{U}(t) = O(\eta J + \eta V + \eta tV), \quad \forall t \in [t_0^{(\eta)}, T].$$

In summary, for all $t \in [0, T]$, we have

$$(5.51) \quad S = O(1 + tVS + tJ + \eta SJ),$$

$$(5.52) \quad J = O\left(W_0^{(\eta)} + tVJ + tV^2S + \eta J^2 + \frac{tVSJ}{\underline{S}}\right),$$

$$(5.53) \quad V = O\left(\eta[W_0^{(\eta)}]^2 + tV^2 + \eta VJ + \eta \frac{\varepsilon}{\underline{S}} J^2\right),$$

$$(5.54) \quad \bar{U} = O(\eta J + \eta V + \eta tV).$$

Remark: The terms $\{\eta SJ, \eta J^2, \eta VJ, \frac{tVSJ}{\underline{S}}, \eta \frac{\varepsilon}{\underline{S}} J^2\}$ in (5.51)-(5.54) are coming from system (2.3) being non-strictly hyperbolic.

6. BOOTSTRAP ARGUMENT

We now design a bootstrap argument to bound S , J , V and to improve assumption $|\Phi| \leq 2\delta$.

When $t = 0$, by (5.8) and definitions (4.6)-(4.7) we have

$$S(0) = 1, \quad J(0) = W_0^{(\eta)}, \quad V(0) = 0.$$

Let κ be a fixed small constant satisfying $0 < \kappa < \frac{(1-\varepsilon)^2}{2}$. This κ measures the lower bound of ρ_j with $j \neq 1$. Based on estimates (5.51)-(5.53), our goal is to prove:

$$(6.1) \quad S(t) = O(1),$$

$$(6.2) \quad J(t) = O(W_0^{(\eta)}),$$

$$(6.3) \quad V(t) = O(\eta[W_0^{(\eta)}]^2 + \eta \frac{\varepsilon}{\kappa} \theta^{-\frac{1}{3}} [W_0^{(\eta)}]^2),$$

for $t \in [0, T_\eta^*]$, where

$$(6.4) \quad T_\eta^* \leq \frac{C}{W_0^{(\eta)}}.$$

And C is a uniform constant. Once these estimates are achieved, we can improve bound for Φ and obtain

$$(6.5) \quad \begin{aligned} |\Phi| &\leq \bar{U}(t) \\ &= O(\eta W_0^{(\eta)} + \eta^2 W_0^{(\eta)} + \eta^2 [W_0^{(\eta)}]^2 + \eta^2 \frac{\varepsilon}{\kappa} \theta^{-\frac{1}{3}} W_0^{(\eta)} + \eta^2 \frac{\varepsilon}{\kappa} \theta^{-\frac{1}{3}} [W_0^{(\eta)}]^2) \\ &= O(\eta W_0^{(\eta)} + \eta^2 \frac{\varepsilon}{\kappa} \theta^{-\frac{1}{3}} W_0^{(\eta)}), \quad t \in [0, T_\eta^*]. \end{aligned}$$

Choosing θ being sufficiently small, we hence prove $\Phi \in B_\delta^6(0)$.

To obtain (6.1)-(6.3), with θ small, we use bootstrap assumptions:

$$(6.6) \quad tV \leq \theta^{\frac{1}{2}},$$

$$(6.7) \quad J \leq \theta^{-\frac{1}{3}} W_0^{(\eta)}$$

$$(6.8) \quad \underline{S} := \min_{i=2, \dots, 6} \inf_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} \rho_i(z'_i, s'_i) \geq \frac{\kappa}{2}.$$

In the argument below, we first improve bootstrap assumptions in (6.6) and (6.7). An improvement of (6.8) for \underline{S} will be given in Section 8.

Applying (6.6) and (6.7) to (5.51), we have

$$(6.9) \quad S = O(1 + tJ + \theta^{\frac{1}{2}} S + \eta \theta^{-\frac{1}{3}} W_0^{(\eta)} S) \quad \Rightarrow \quad S = O(1 + tJ).$$

For $J(t)$, we go back to (5.52). With (6.6)-(6.7) we obtain

$$(6.10) \quad \begin{aligned} J &= O(W_0^{(\eta)} + \theta^{\frac{1}{2}} J + \theta^{\frac{1}{2}} VS + \eta \theta^{-\frac{1}{3}} W_0^{(\eta)} J + \frac{\theta^{\frac{1}{2}}}{\kappa} SJ) \\ &\Rightarrow J = O(W_0^{(\eta)} + \theta^{\frac{1}{2}} VS + \theta^{\frac{1}{2}} SJ). \end{aligned}$$

Together with (6.9), (6.7) and (6.6), equality (6.10) implies

$$(6.11) \quad \begin{aligned} J &= O(W_0^{(\eta)} + \theta^{\frac{1}{2}} V + \theta^{\frac{1}{2}} tVJ + \theta^{\frac{1}{2}} J + \theta^{\frac{1}{2}} tJ^2) \\ &= O(W_0^{(\eta)} + \theta^{\frac{1}{2}} V + \theta^{\frac{1}{4}} J + \theta^{\frac{1}{2}} J + \theta^{\frac{1}{6}} J) \\ &\Rightarrow J = O(W_0^{(\eta)} + \theta^{\frac{1}{2}} V). \end{aligned}$$

For $V(t)$, employing (6.6)-(6.7) to (5.53), we get

$$V = O\left(\eta[W_0^{(\eta)}]^2 + \theta^{\frac{1}{2}}V + \eta\theta^{-\frac{1}{3}}W_0^{(\eta)}V + \eta\frac{\varepsilon}{\kappa}\theta^{-\frac{1}{3}}W_0^{(\eta)}J\right).$$

Using (6.11), we hence have

$$V = O\left(\eta[W_0^{(\eta)}]^2 + \eta\frac{\varepsilon}{\kappa}\theta^{-\frac{1}{3}}[W_0^{(\eta)}]^2 + \eta\frac{\varepsilon}{\kappa}\theta^{\frac{1}{6}}W_0^{(\eta)}V\right).$$

This yields the desired bound of (6.3)

$$(6.12) \quad V = O\left(\eta[W_0^{(\eta)}]^2 + \eta\frac{\varepsilon}{\kappa}\theta^{-\frac{1}{3}}[W_0^{(\eta)}]^2\right).$$

Furthermore, by (6.4), we have

$$(6.13) \quad tV = O\left(\eta W_0^{(\eta)} + \eta\frac{\varepsilon}{\kappa}\theta^{-\frac{1}{3}}W_0^{(\eta)}\right) = O\left(\theta\eta(\ln\eta)^\alpha + \frac{\varepsilon}{\kappa}\theta^{\frac{2}{3}}\eta(\ln\eta)^\alpha\right) < O(\theta^{\frac{1}{2}}),$$

with $0 < \alpha < \frac{1}{2}$. Estimate (6.13) improves bootstrap assumption (6.6). Back to (6.11) and (6.9) with (6.12), we obtain the desired bounds of $J(t)$ and $S(t)$:

$$(6.14) \quad J = O(W_0^{(\eta)} + \theta^{\frac{1}{2}}\eta[W_0^{(\eta)}]^2 + \eta\frac{\varepsilon}{\kappa}\theta^{\frac{1}{6}}[W_0^{(\eta)}]^2) = O(W_0^{(\eta)}),$$

$$(6.15) \quad S = O(1 + tW_0^{(\eta)}) = O(1).$$

And (6.14) improves bootstrap assumption in (6.7).

7. SHOCK FORMATION

In this section, we show that in \mathcal{R}_1 the inverse foliation density ρ_1 goes to zero as time goes to a certain T_η^* .

With the following transport equation for ρ_1

$$\frac{\partial\rho_1}{\partial s_1} = c_{11}^1(\Phi)v^1 + O\left(\sum_{k \neq 1} w^k\right)\rho_1,$$

and the fact $c_{11}^1(\Phi) < 0$, we have

$$(7.1) \quad -|c_{11}^1||v^1| - \left|O\left(\sum_{k \neq 1} w^k\right)\right|\rho_1 \leq \frac{\partial\rho_1}{\partial s_1} \leq -|c_{11}^1||v^1| + \left|O\left(\sum_{k \neq 1} w^k\right)\right|\rho_1.$$

By (6.5), $|\Phi| = O(\eta W_0^{(\eta)} + \eta^2\frac{\varepsilon}{\kappa}\theta^{-\frac{1}{3}}W_0^{(\eta)}) \leq \delta$, choosing θ sufficiently small, it holds

$$(7.2) \quad (1 - \varepsilon)|c_{11}^1(0)| \leq |c_{11}^1(\Phi)| \leq (1 + \varepsilon)|c_{11}^1(0)|.$$

With bi-characteristic coordinates, we have

$$\int_0^t \sum_{k \neq 1} w^k(X_1(z_1, t'), t') dt' = O(\eta W_0^{(\eta)} + \eta J) = O(\eta W_0^{(\eta)}).$$

For $0 < \eta \ll 1$, this implies

$$(7.3) \quad 1 - \varepsilon \leq \exp\left(\int_0^t O\left(\sum_{k \neq 1} w^k(X_1(z_1, t'), t')\right) dt'\right) \leq 1 + \varepsilon,$$

and

$$(7.4) \quad 1 - \varepsilon \leq \exp \left(- \int_0^t O \left(\sum_{k \neq 1} w^k(X_1(z_1, t'), t') \right) dt' \right) \leq 1 + \varepsilon.$$

Employing Grönwall inequality to (7.1) and combining with (7.2), (7.3)-(7.4), we get

$$(7.5) \quad \begin{aligned} & (1 - \varepsilon) \left(1 - (1 + \varepsilon)^2 |c_{11}^1(0)| \int_0^t |v^1(z_1, t')| dt' \right) \\ & \leq \rho_1(z_1, t) \\ & \leq (1 + \varepsilon) \left(1 - (1 - \varepsilon)^2 |c_{11}^1(0)| \int_0^t |v^1(z_1, t')| dt' \right) \\ & \leq 1 + \varepsilon. \end{aligned}$$

For v^1 , we integrate

$$(7.6) \quad \frac{\partial v^1}{\partial s_1} = O \left(\sum_{m \neq 1} w^m \right) v^1 + O \left(\sum_{\substack{m \neq 1, k \neq 1 \\ \{m, k\} \neq \{2, 3\}, \{m, k\} \neq \{4, 5\}}} w^m w^k \right) \rho_1$$

along \mathcal{C}_1 and obtain

$$(7.7) \quad \begin{aligned} v^1(z_1, t) & \leq w_{(\eta)}^1(z_1, 0) + O(tVJ + tV^2S) \\ & = w_{(\eta)}^1(z_1, 0) + O(\eta[W_0^{(\eta)}]^2 + \eta \frac{\varepsilon}{\kappa} \theta^{-\frac{1}{3}} [W_0^{(\eta)}]^2). \end{aligned}$$

For sufficiently small θ and η , the above inequality yields

$$v^1(z_0, t) \leq (1 + \varepsilon) W_0^{(\eta)}.$$

By using the first inequality of (7.5), we arrive at

$$(7.8) \quad \rho_1(z_0, t) \geq (1 - \varepsilon) \left(1 - (1 + \varepsilon)^3 |c_{11}^1(0)| t W_0^{(\eta)} \right),$$

which shows that $\rho_1(z_0, t) > 0$ when

$$t < \frac{1}{(1 + \varepsilon)^3 |c_{11}^1(0)| W_0^{(\eta)}}.$$

Meanwhile, applying $\rho_1 \leq 1 + \varepsilon$ to (7.6), we have

$$\frac{\partial v^1}{\partial s_1} \geq - \left| O \left(\sum_{m \neq 1} w^m \right) v^1 - (1 + \varepsilon) \right| O \left(\sum_{\substack{m \neq 1, k \neq 1 \\ \{m, k\} \neq \{2, 3\}, \{m, k\} \neq \{4, 5\}}} w^m w^k \right),$$

provided $v^1 > 0$. Since

$$O \left(\sum_{m \neq 1} \int_0^t w^m dt' \right) = O(\eta W_0^{(\eta)} + \eta J) = O(\eta W_0^{(\eta)})$$

and

$$\begin{aligned} & O\left(\sum_{\substack{m \neq 1, k \neq 1 \\ \{m,k\} \neq \{2,3\}, \{m,k\} \neq \{4,5\}}} \int_0^t w^m w^k dt'\right) \\ &= O(tV^2) = O(\eta^2 [W_0^{(\eta)}]^3 + \eta^2 \frac{\varepsilon^2}{\kappa^2} \theta^{-\frac{2}{3}} [W_0^{(\eta)}]^3), \end{aligned}$$

via Grönwall inequality, we have

$$(7.9) \quad v^1(z_1, t) \geq (1-\varepsilon) \left[w_{(\eta)}^1(z_1, 0) - (1+\varepsilon)^2 (\eta^2 [W_0^{(\eta)}]^3 + \eta^2 \frac{\varepsilon^2}{\kappa^2} \theta^{-\frac{2}{3}} [W_0^{(\eta)}]^3) \right].$$

Choosing θ sufficiently small, we have

$$(7.10) \quad (1+\varepsilon)^2 O(\eta^2 [W_0^{(\eta)}]^3 + \eta^2 \frac{\varepsilon^2}{\kappa^2} \theta^{-\frac{2}{3}} [W_0^{(\eta)}]^3) \leq \varepsilon W_0^{(\eta)},$$

and setting $z_1 = z_0$, it holds

$$v^1(z_0, t) \geq (1-\varepsilon) [W_0^{(\eta)} - \varepsilon W_0^{(\eta)}] = (1-\varepsilon)^2 W_0^{(\eta)}.$$

By the second inequality of (7.5), we get

$$\rho_1(z_0, t) \leq (1+\varepsilon) \left(1 - (1-\varepsilon)^4 |c_{11}^1(0)| t W_0^{(\eta)} \right).$$

Together with (7.8), we conclude that there exists T_η^* (shock formation time) such that

$$\lim_{t \rightarrow T_\eta^*} \rho_1(z_0, t) = 0.$$

And T_η^* obeys

$$(7.11) \quad \frac{1}{(1+\varepsilon)^3 |c_{11}^1(0)| W_0^{(\eta)}} \leq T_\eta^* \leq \frac{1}{(1-\varepsilon)^4 |c_{11}^1(0)| W_0^{(\eta)}},$$

which is consistent with the requirement (6.4)

$$T_\eta^* \leq \frac{C}{W_0^{(\eta)}}.$$

8. LOWER BOUND FOR \underline{S}

We next improve bootstrap assumption (6.8) via obtaining a uniformly positive lower bound for $\{\rho_i\}_{i=2,3,4,5,6}$.

We start from ρ_6 . Similarly to (7.5), we have

$$\rho_6(z_6, t) \geq (1-\varepsilon) \left(1 - (1+\varepsilon)^2 |c_{66}^6(0)| \int_0^t |v^6(z_6, t')| dt' \right).$$

Taking $i = 6$ in (3.16) and integrating it along \mathcal{R}_6 , proceeding as in (7.7) we get

$$\begin{aligned} v^6(z, t) &\leq w_{(\eta)}^6(z, 0) + O(tVJ + tV^2S) \\ &= w_{(\eta)}^6(z, 0) + O(\eta [W_0^{(\eta)}]^2 + \eta \frac{\varepsilon}{\kappa} \theta^{-\frac{1}{3}} [W_0^{(\eta)}]^2). \end{aligned}$$

With initial data condition (4.2) and the fact $0 < \theta, \eta \ll 1$, the above inequality implies

$$v^6(z, t) \leq \frac{(1 - \varepsilon)^4}{2(1 + \varepsilon)^2} W_0^{(\eta)}.$$

Noting that $c_{66}^6(0) = -c_{11}^1(0)$, we obtain a lower bound for ρ_6 :

$$\begin{aligned} \rho_6(z_6, t) &\geq (1 - \varepsilon) \left(1 - (1 + \varepsilon)^2 |c_{11}^1(0)| \frac{(1 - \varepsilon)^4}{2(1 + \varepsilon)^2} W_0^{(\eta)} t \right) \\ &> (1 - \varepsilon) \left(1 - |c_{11}^1(0)| \frac{(1 - \varepsilon)^4 W_0^{(\eta)}}{2} \frac{1}{(1 - \varepsilon)^4 |c_{11}^1(0)| W_0^{(\eta)}} \right) \\ &= \frac{(1 - \varepsilon)}{2} > 0, \quad \text{for any } t < T_\eta^*. \end{aligned}$$

For ρ_3 and ρ_4 , we use the structure of equations in (3.24), i.e., $\rho_3 w^2$ and $\rho_4 w^5$ vanish in $\partial_{s_3} \rho_3$ and $\partial_{s_4} \rho_4$, respectively. Using similar arguments as above, with initial data condition (4.3) we have

$$\begin{aligned} \min_{i=3,4} \rho_i(z_i, t) &\geq (1 - \varepsilon) \left(1 - (1 + \varepsilon)^2 \varepsilon \frac{(1 - \varepsilon)^4 |c_{11}^1(0)|}{(1 + \varepsilon)^2} W_0^{(\eta)} t \right) \\ &> (1 - \varepsilon) \left(1 - \varepsilon (1 - \varepsilon)^4 |c_{11}^1(0)| W_0^{(\eta)} \frac{1}{(1 - \varepsilon)^4 |c_{11}^1(0)| W_0^{(\eta)}} \right) \\ &= (1 - \varepsilon)^2 > 0, \quad \text{for } t \leq T_\eta^*. \end{aligned}$$

For ρ_2 and ρ_5 , since $c_{22}^2 = c_{55}^5 = 0$, a direct calculation yields

$$\rho_2(z, t) \geq 1 - \varepsilon > 0, \quad \text{and} \quad \rho_5(z, t) \geq 1 - \varepsilon > 0.$$

In conclusion, we obtain the following lower bound for \underline{S}

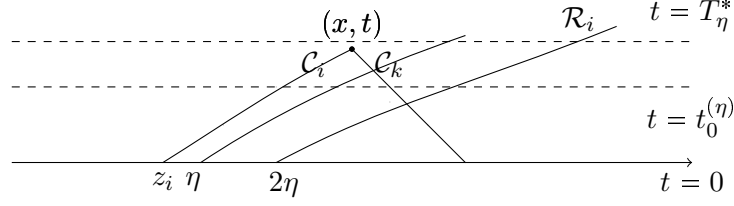
$$(8.1) \quad \underline{S}(s) := \min_{i=2, \dots, 6} \inf_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} \rho_i(z'_i, s'_i) \geq \frac{(1 - \varepsilon)^2}{2}.$$

With $\varepsilon \in (0, \frac{1}{100}]$ and $\kappa \in (0, \frac{1}{100}]$, the inequality (8.1) improves (6.8).

We then move to prove that $\{\rho_i\}_{i=1, \dots, 6}$ obey a positive lower bound at points (x, t) outside \mathcal{R}_i . For $t < T_\eta^*$ we have

$$\min_{i=1, \dots, 6} \inf_{\substack{(z'_i, s'_i) \\ z'_i \notin [\eta, 2\eta], 0 \leq s'_i \leq t}} \rho_i(z'_i, s'_i) \geq \frac{(1 - \varepsilon)^2}{2} \text{ for some } \varepsilon \in (0, \frac{1}{100}].$$

Proof: For (x, t) outside of \mathcal{R}_i , lying at the intersection of \mathcal{C}_i and $\{\mathcal{C}_k\}_{k \neq i}$, we estimate ρ_i along \mathcal{C}_i .



From (3.15), we have

$$(8.2) \quad \frac{\partial \rho_i}{\partial s_i} \geq - \left| O\left(\sum_k w^k\right) \right| \rho_i.$$

For $z_i \notin [\eta, 2\eta]$, $t \in [t_0^{(\eta)}, T_\eta^*)$, utilizing (5.6) and (5.38) we obtain

$$(8.3) \quad \begin{aligned} & \int_0^t \left| O\left(\sum_k w^k(z_i, s)\right) \right| ds \\ &= \int_0^{t_0^{(\eta)}} \left| O\left(\sum_k w^k(z_i, s)\right) \right| ds + \int_{t_0^{(\eta)}}^t \left| O\left(\sum_k w^k(z_i, s)\right) \right| ds \\ &= O\left(\eta W_0^{(\eta)} + tV + \eta J\right). \end{aligned}$$

By the bound (6.13) for tV and bound (6.2) for J , we have

$$(8.4) \quad \int_0^t \left| O\left(\sum_k w^k(z_i, s)\right) \right| ds = O\left(\eta W_0^{(\eta)} + \eta \varepsilon \theta^{-\frac{1}{3}} W_0^{(\eta)}\right).$$

Employing Grönwall inequality for (8.2), by choosing θ small, we conclude

$$(8.5) \quad \rho_i(z_i, t) \geq \exp\left(-\int_0^t \left| O\left(\sum_k w^k(z_i, s)\right) \right| ds\right) \geq 1 - \varepsilon,$$

for $z_i \notin [\eta, 2\eta]$, $t \in [t_0^{(\eta)}, T_\eta^*)$. This estimate illustrates that there is no shock formed outside $\mathcal{R}_1, \dots, \mathcal{R}_6$. And by the lower bound of $\underline{S}(t)$, we also conclude that no shock emerges in $\mathcal{R}_2, \dots, \mathcal{R}_6$. For the whole spacetime region before $t = T_\eta^*$, the only shock happens in \mathcal{R}_1 , i.e., $\rho_1(t) \rightarrow 0$ as $t \rightarrow T_\eta^*$.

For $i \in \{1, \dots, 6\}$, recall that w^i is uniformly bounded outside \mathcal{R}_i . Utilizing the fact that $v^i = \rho_i w^i$ is uniformly bounded inside \mathcal{R}_i , together with the lower bound of $\{\rho_i\}_{i \in \{2, \dots, 6\}}$, we conclude that $\{w^i\}_{i \in \{2, \dots, 6\}}$ are also uniformly bounded inside $\{\mathcal{R}_i\}_{i \in \{2, \dots, 6\}}$. For the whole spacetime region before $t = T_\eta^*$, the only singularity happens in \mathcal{R}_1 , i.e., $w^1(t) \rightarrow +\infty$ as $t \rightarrow T_\eta^*$.

We summarize the above conclusions into

Proposition 8.1. *For $t \leq T_\eta^*$, we have*

- $\{\rho_i\}_{i=2, \dots, 6}$ are bounded away from zero in the whole (x, t) -plane. ρ_1 obeys a positive lower bound outside \mathcal{R}_1 and within \mathcal{R}_1 it holds $\rho_1 \rightarrow 0$ as $t \rightarrow T_\eta^*$. And the first singularity (shock) forms.

- $\{w^i\}_{i=2,\dots,6}$ are uniformly bounded in the whole (x, t) -plane. w^1 obeys a uniform bound outside \mathcal{R}_1 . Within \mathcal{R}_1 it holds $w^1 \rightarrow +\infty$ as $t \rightarrow T_\eta^*$ and w^1 being finite for $t < T_\eta^*$.

These further imply that the solutions to the Cauchy problem of system (1.4) are smooth before time T_η^* .

9. ESTIMATE FOR $\partial_{z_1}\rho_1$

In this section, we fix η and employ characteristic coordinates and bi-characteristic coordinates introduced in (3.2), (3.6) and (3.7) for $i \neq j$:

$$(x, t) = (X_i(z_i, s_i), s_i) = (X_i(y_i, t'(y_i, y_j)), t'(y_i, y_j)) = (X_j(y_j, t'(y_i, y_j)), t'(y_i, y_j)).$$

For any smooth function $f(x, t) = f(X_i(y_i, t'(y_i, y_j)), t'(y_i, y_j))$, with (3.4), (3.7), we calculate

$$\begin{aligned} \partial_{y_i} f &= (\partial_{z_i} X_i + \partial_{s_i} X_i \partial_{y_i} t') \partial_x f + \partial_{y_i} t' \partial_t f \\ &= \left(\rho_i + \lambda_i \frac{\rho_i}{\lambda_j - \lambda_i} \right) \partial_x f + \frac{\rho_i}{\lambda_j - \lambda_i} \partial_t f \\ (9.1) \quad &= \rho_i \partial_x f + \frac{\rho_i}{\lambda_j - \lambda_i} (\partial_t + \lambda_i \partial_x) f \\ &= \partial_{z_i} f + \frac{\rho_i}{\lambda_j - \lambda_i} \partial_{s_i} f. \end{aligned}$$

Bi-characteristic coordinates also means $f(x, t) = f(X_j(y_j, t'(y_i, y_j)), t'(y_i, y_j))$, thus we deduce

$$\begin{aligned} \partial_{y_i} f &= \partial_{s_j} X_j \partial_{y_i} t' \partial_x f + \partial_{y_i} t' \partial_t f \\ &= \lambda_j \frac{\rho_i}{\lambda_j - \lambda_i} \partial_x f + \frac{\rho_i}{\lambda_j - \lambda_i} \partial_t f \\ (9.2) \quad &= \frac{\rho_i}{\lambda_j - \lambda_i} (\partial_t + \lambda_j \partial_x) f \\ &= \frac{\rho_i}{\lambda_j - \lambda_i} \partial_{s_j} f. \end{aligned}$$

From (9.1) and (9.2), we hence have that transformations between these coordinates satisfy

$$(9.3) \quad \partial_{y_i} = \frac{\rho_i}{\lambda_j - \lambda_i} \partial_{s_j} = \partial_{z_i} + \frac{\rho_i}{\lambda_j - \lambda_i} \partial_{s_i}.$$

We fix y_1 along \mathcal{C}_1 and choose y_6 acting as a parameter. From (9.3), we have

$$\partial_{y_1} = \partial_{z_1} + \frac{\rho_1}{\lambda_6 - \lambda_1} \partial_{s_1} = \partial_{z_1} - \frac{\rho_1}{2\lambda_1} \partial_{s_1},$$

where we use $\lambda_6 = -\lambda_1$. Hence,

$$(9.4) \quad \partial_{z_1} \rho_1 = \partial_{y_1} \rho_1 + \frac{\rho_1}{2\lambda_1} \partial_{s_1} \rho_1.$$

To bound $\partial_{z_1} \rho_1$, we start with controlling $\partial_{y_1} \rho_1$ by studying its evolution equation. Let

$$\tau_1^{(6)} := \partial_{y_1} \rho_1, \quad \pi_1^{(6)} := \partial_{y_1} v^1.$$

With the help of $[\partial_1, \partial_6] = 0$, we have

$$(9.5) \quad \partial_{y_6} \tau_1^{(6)} = \partial_{y_6} \partial_{y_1} \rho_1 = \partial_{y_1} \partial_{y_6} \rho_1.$$

Since

$$(9.6) \quad \partial_{y_6} = \frac{\rho_6}{\lambda_1 - \lambda_6} \partial_{s_1} = \frac{\rho_6}{2\lambda_1} \partial_{s_1},$$

with (9.5) and (9.6), by calculation we obtain

$$(9.7) \quad \begin{aligned} \partial_{y_6} \tau_1^{(6)} &= \partial_{y_1} \left(\frac{\rho_6}{2\lambda_1} \partial_{s_1} \rho_1 \right) \stackrel{(3.15)}{=} \partial_{y_1} \left(\frac{\rho_1 \rho_6}{2\lambda_1} \sum_m c_{1m}^1 w^m \right) \\ &= \frac{\rho_6}{2\lambda_1} \left(c_{11}^1 \partial_{y_1} v^1 + \sum_{m \neq 1} c_{1m}^1 w^m \partial_{y_1} \rho_1 \right) \\ &\quad + \frac{\rho_1}{2\lambda_1} \left(\sum_{m \neq 6} c_{1m}^1 w^m \partial_{y_1} \rho_6 + c_{16}^1 \partial_{y_1} v^6 \right) \\ &\quad + \frac{\rho_1 \rho_6}{2\lambda_1} \left(\sum_{m \neq 1, m \neq 6} c_{1m}^1 \frac{1}{\rho_m} \partial_{y_1} v^m - \sum_{m \neq 1, m \neq 6} c_{1m}^1 \frac{w^m}{\rho_m} \partial_{y_1} \rho_m \right) \\ &\quad - \frac{\partial_{y_1} \lambda_1}{2\lambda_1^2} \rho_1 \rho_6 \sum_m c_{1m}^1 w^m + \frac{\rho_1 \rho_6}{2\lambda_1} \sum_m \partial_{y_1} c_{1m}^1 w^m. \end{aligned}$$

With bi-characteristic coordinates (y_1, y_m) ($m \neq 1$), we have

$$(9.8) \quad \begin{aligned} \partial_{y_1} \lambda_1 &= \nabla_{\Phi} \lambda_1 \cdot \partial_{y_1} \Phi = \nabla_{\Phi} \lambda_1 \cdot [\partial_{s_m} X_m \partial_{y_1} t' \partial_x \Phi + \partial_{y_1} t' \partial_t \Phi] \\ &= \nabla_{\Phi} \lambda_1 \cdot \left[\lambda_m \frac{\rho_1}{\lambda_m - \lambda_1} \sum_k w^k r_k + \frac{\rho_1}{\lambda_m - \lambda_1} (-A(\Phi) \sum_k w^k r_k) \right] \\ &= O(v^1 + \rho_1 \sum_{k \neq 1} w^k) \end{aligned}$$

and

$$(9.9) \quad \begin{aligned} \partial_{y_1} c_{1m}^1 &= \nabla_{\Phi} c_{1m}^1 \cdot \partial_{y_1} \Phi = \nabla_{\Phi} c_{1m}^1 \cdot [\partial_{s_m} X_m \partial_{y_1} t' \partial_x \Phi + \partial_{y_1} t' \partial_t \Phi] \\ &= \nabla_{\Phi} c_{1m}^1 \cdot \left[\lambda_m \frac{\rho_1}{\lambda_m - \lambda_1} \sum_k w^k r_k + \frac{\rho_1}{\lambda_m - \lambda_1} (-A(\Phi) \sum_k w^k r_k) \right] \\ &= O(v^1 + \rho_1 \sum_{k \neq 1} w^k). \end{aligned}$$

With (9.3), it also holds

$$\partial_{y_1} = \frac{\rho_1}{\lambda_m - \lambda_1} \partial_{s_m}, \quad \text{for } m \neq 1.$$

By (3.15) and (3.16), we obtain

$$(9.10) \quad \partial_{y_1} \rho_m = \frac{\rho_1}{\lambda_m - \lambda_1} \partial_{s_m} \rho_m = O\left(\rho_m v^1 + \rho_1 \rho_m \sum_{k \neq 1} w^k\right),$$

and

$$(9.11) \quad \partial_{y_1} v^m = \frac{\rho_1}{\lambda_m - \lambda_1} \partial_{s_m} v^m = O\left(\rho_m v^1 \sum_{k \neq 1} w^k + \rho_1 \rho_m \sum_{\substack{j \neq 1, k \neq 1 \\ j \neq k}} w^j w^k\right).$$

Together with estimates in Section 5, (9.8)-(9.9), (9.10)-(9.11) imply that the right hand side of (9.7) are bounded as below

$$(9.12) \quad \partial_{s_1} \tau_1^{(6)} = B_{11}^\eta \tau_1^{(6)} + B_{12}^\eta \pi_1^{(6)} + B_{13}^\eta,$$

where $B_{11}^\eta, B_{12}^\eta, B_{13}^\eta$ are uniformly bounded constants depending on η .

Similarly, we have the evolution equation for $\pi_1^{(6)}$ with bounded coefficients:

$$(9.13) \quad \begin{aligned} & \partial_{y_6} \pi_1^{(6)} \\ &= \frac{\rho_6}{2\lambda_1} \left(\sum_{\substack{p \neq 1, q \neq 1 \\ p \neq q}} \gamma_{pq}^1 w^p w^q \tau_1^{(6)} + \sum_{p \neq 1} \gamma_{1p}^1 w^p \pi_1^{(6)} \right) \\ & \quad - \frac{\partial_{y_1}(\lambda_1)}{4\lambda_1^2} \left(\sum_{p \neq 1} \gamma_{1p}^1 w^p v^1 \rho_6 + \sum_{\substack{p \neq 1, q \neq 1 \\ p \neq q}} \gamma_{pq}^1 w^p w^q \rho_6 \rho_1 \right) \\ & \quad + \frac{\rho_6 \rho_1}{2\lambda_1} \left(\sum_{p \neq 1} \partial_{y_1} \gamma_{1p}^1 w^p w^m + \sum_{\substack{p \neq 1, q \neq 1 \\ p \neq q}} \partial_{y_1} \gamma_{pq}^1 w^p w^q \right) \\ & \quad + \frac{\rho_1}{2\lambda_1} \left(\sum_{p \neq 1} \gamma_{1p}^1 w^p w^1 + \sum_{\substack{p \neq 1, q \neq 1 \\ p \neq q}} \gamma_{pq}^1 w^p w^q \right) \partial_{y_1} \rho_6 \\ & \quad + \frac{\rho_6 \rho_1}{2\lambda_1} \left(\sum_{p \neq 1, p \neq 6} \gamma_{1p}^1 \frac{w^1}{\rho_p} + \sum_{\substack{p \neq 1, q \neq 1 \\ p \neq q}} \gamma_{pq}^1 \frac{w^q}{\rho_p} \right) (\partial_{y_1} v^p - w^p \partial_{y_1} \rho_p) \\ & \quad + \frac{\rho_6 \rho_1}{2\lambda_1} \sum_{\substack{p \neq 1, q \neq 1 \\ p \neq q}} \gamma_{pq}^1 \frac{w^p}{\rho_q} (\partial_{y_1} v^q - w^q \partial_{y_1} \rho_q) + \frac{\rho_1}{2\lambda_1} \gamma_{16}^1 w^1 \partial_{y_1} v^6 \\ &= B_{21}^\eta \tau_1^{(6)} + B_{22}^\eta \pi_1^{(6)} + B_{23}^\eta, \end{aligned}$$

where $B_{21}^\eta, B_{22}^\eta, B_{23}^\eta$ are uniformly bounded constants depending on η .

We next check that the initial data of $\tau_1^{(6)}$ and $\pi_m^{(6)}$ are bounded. Since

$$\rho_1(z_1, 0) = 1, \quad v^1(z_1, 0) = w^1(z_1, 0).$$

using (9.3), for fixed η we have

$$\begin{aligned} \tau_1^{(6)}(z_1, 0) &:= \partial_{y_1} \rho_1(z_1, 0) \\ &= \partial_{z_1} \rho_1(z_1, 0) - \frac{\rho_1(z_1, 0)}{2\lambda_1} \partial_{s_1} \rho_1(z_1, 0) \\ &= -\frac{1}{2\lambda_1} \sum_k c_{1k}^1 w^k(z_1, 0) \rho_1(z_1, 0) \\ &= -\frac{1}{2\lambda_1} \sum_k c_{1k}^1 w^k(z_1, 0) \\ &= O(W_0^{(\eta)}) < +\infty, \end{aligned}$$

and

$$\begin{aligned}
\pi_1^{(6)}(z_1, 0) &:= \partial_{y_1} v^1(z_1, 0) \\
&= \partial_{z_1} v^1(z_1, 0) - \frac{\rho_1(z_1, 0)}{2\lambda_1} \partial_{s_1} v^1(z_1, 0) \\
&= \partial_{z_1} w^1(z_1, 0) - \frac{1}{2\lambda_1} \sum_{q \neq 1, q \neq p} \gamma_{pq}^1 w^p(z_1, 0) w^q(z_1, 0) \rho_1(z_1, 0) \\
&= O(\partial_{z_1} w^1(z_1, 0) + [W_0^{(\eta)}]^2) < +\infty.
\end{aligned}$$

Applying Gröwall inequality to (9.12) and (9.13), for $t \leq T_\eta^*$ we have that $\tau_1^{(6)} := \partial_{y_1} \rho_1$ is bounded by a constant depending on η .

Back to (9.4) and (3.15), we have

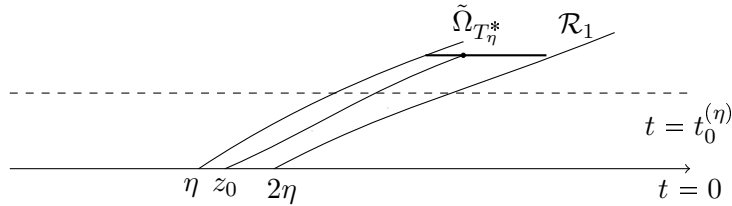
$$\partial_{z_1} \rho_1 = \partial_{y_1} \rho_1 + O(v^1 + \sum_{m \neq 1} w^m \rho_1).$$

With the bounds for $J(t)$, $S(t)$ and $V(t)$ in Section 5, consequently, we conclude $\partial_{z_1} \rho_1$ is bounded by C_η , a constant depending on η .

10. ILL-POSEDNESS MECHANISM

From our construction of modified Lindblad-type initial data (4.1)-(4.4) and the bounds (7.11) obtained for the shock formation time T_η^* , we immediately conclude that $T_\eta^* \rightarrow 0$ as $\eta \rightarrow 0$. This gives the ill-posedness in Theorem 1.1.

In the following part of this section, we further show that the H^2 norm of the solution to elastic waves (1.4) is infinite at the shock formation time T_η^* when restricted to a spatial region $\tilde{\Omega}_{T_\eta^*}$. In the picture below, $\tilde{\Omega}_{T_\eta^*} := \{(x, T_\eta^*) : x = X_1(z, T_\eta^*) \text{ and } \eta \leq z \leq 2\eta\}$.



Considering acoustic metric g_{c_1} according to the wave equation for U^1 in (1.4), we have

$$(g_{c_1}^{-1})^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(c_1^2 + 2\sigma_0 \partial_x u^1) & \bar{\sigma} \partial_x u^2 & \bar{\sigma} \partial_x u^3 \\ 0 & \bar{\sigma} \partial_x u^2 & -(c_2^2 + 2\sigma_1 \partial_x u^1) & 0 \\ 0 & \bar{\sigma} \partial_x u^3 & 0 & -(c_2^2 + 2\sigma_1 \partial_x u^1) \end{pmatrix},$$

where $\bar{\sigma} = 2\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4$. For $\Phi \in B_\delta^6(0)$, g_{c_1} is a small perturbation of $g_{c_1}^{(Flat)}$

$$g_{c_1}^{(Flat)} = dt^2 - c_1^{-2} dx^2 - c_2^{-2} (dY^2)^2 - c_2^{-2} (dY^3)^2.$$

This indicates that domain of future dependence with respect to our metric g_{c_1} is uniformly close to the domain with respect to the flat metric $g_{c_1}^{(Flat)}$. Denote the T_η^* -slice of the domain of future dependence as $\Omega_{T_\eta^*}$, then we have that: in 3D, $\Omega_{T_\eta^*}$ is uniformly close to an ellipsoidal ball, centered at

$$P = \left(\frac{X_1(2\eta, T_\eta^*) + X_1(\eta, T_\eta^*)}{2}, 0, 0 \right) = \left(c_1 T_\eta^* + \frac{3}{2}\eta, 0, 0 \right)$$

with major axis $X_1(2\eta, T_\eta^*) - X_1(\eta, T_\eta^*) = \eta$ and two minor axes $\frac{c_2}{c_1} [X_1(2\eta, T_\eta^*) - X_1(\eta, T_\eta^*)] = \frac{c_2\eta}{c_1}$.

To calculate $\int_{\Omega_{T_\eta^*}} |w^1(Y, T_\eta^*)|^2 dx dY^2 dY^3$, we first compute:

Proposition 10.1. *Let $\Omega_{T_\eta^*}^r = \{(Y^2, Y^3) : (x, Y^2, Y^3) \in \Omega_{T_\eta^*}\}$. For $(x, Y^2, Y^3) \in \Omega_{T_\eta^*}$ with x along \mathcal{C}_1 starting from z , we have*

$$(10.1) \quad \int_{\Omega_{T_\eta^*}^r} dY^2 dY^3 \approx (z - \eta + O(\varepsilon)\eta)(2\eta - z + O(\varepsilon)\eta),$$

and

$$(10.2) \quad |\Omega_{T_\eta^*}| = \int_{\Omega_{T_\eta^*}^r} dx dY^2 dY^3 \sim \eta^3.$$

Proof. In fact, $\Omega_{T_\eta^*}$ is given by

$$(10.3) \quad \frac{c_2^2(x - c_1 T_\eta^* - \frac{3\eta}{2})^2}{c_1^2} + (Y^2)^2 + (Y^3)^2 \leq \frac{c_2^2}{c_1^2} \left(\frac{\eta}{2} \right)^2.$$

It follows,

$$(10.4) \quad (Y^2)^2 + (Y^3)^2 \leq \frac{c_2^2}{c_1^2} \left[\left(\frac{\eta}{2} \right)^2 - \left(x - c_1 T_\eta^* - \frac{3\eta}{2} \right)^2 \right]$$

Since $\lambda_1(0) = c_1$, it holds that

$$\lambda_1(\Phi) = c_1(1 + O(\varepsilon)), \quad \text{for } \Phi \in B_\delta^6(0)$$

with sufficiently small δ . Hence, along \mathcal{C}_1 starting from z , we have

$$(10.5) \quad x = z + (1 + O(\varepsilon))c_1 T_\eta^*.$$

Employing (10.5) to (10.4), we obtain

$$\begin{aligned} \int_{\Omega_{T_\eta^*}^r} dY^2 dY^3 &\approx \frac{c_2^2}{c_1^2} \left[\left(\frac{\eta}{2} \right)^2 - \left(x - c_1 T_\eta^* - \frac{3\eta}{2} \right)^2 \right] \\ &\approx \frac{c_2^2}{c_1^2} \left[\left(\frac{\eta}{2} \right)^2 - \left(z - \frac{3\eta}{2} + O(\varepsilon)T_\eta^* \right)^2 \right] \\ &\approx (z - \eta + O(\varepsilon)\eta)(2\eta - z + O(\varepsilon)\eta). \end{aligned}$$

So we complete proof of (10.1). Estimate (10.2) is obvious by the expression of $\Omega_{T_\eta^*}$ in (10.3). \square

With bi-characteristic coordinates and Proposition (10.1), we have

$$\begin{aligned} & \int_{\Omega_{T_\eta^*}} |w^1|^2 dx dY^2 dY^3 \\ & \geq C \int_\eta^{2\eta} \left| \frac{v^1}{\rho_1}(z, T_\eta^*) \right|^2 \rho_1(z, T_\eta^*) (z - \eta + O(\varepsilon)\eta) (2\eta - z + O(\varepsilon)\eta) dz. \end{aligned}$$

By (7.9), (7.10), we have

$$(10.6) \quad v^1(z, t) \geq (1 - \varepsilon)[w_{(\eta)}^1(z, 0) - \varepsilon W_0^{(\eta)}].$$

Restrict the spatial integration region to a subinterval $(z_0, z_0^*] \subseteq [\eta, 2\eta]$, where $w_{(\eta)}^1(z, 0) > \frac{1}{2}W_0^{(\eta)}$ for $z \in (z_0, z_0^*]$. Then by (10.6) and fact that $\rho_1(z_0, T_\eta^*) = 0$, we have

$$\begin{aligned} & \|w^1(\cdot, T_\eta^*)\|_{L^2(\Omega_{T_\eta^*})}^2 \\ & \geq C \int_{z_0}^{z_0^*} \frac{(v^1)^2}{\rho_1(z, T_\eta^*)} (z - \eta + O(\varepsilon)\eta) (2\eta - z + O(\varepsilon)\eta) dz \\ & \geq C(1 - \varepsilon)^2(1 - 2\varepsilon)^2 [W_0^{(\eta)}]^2 \int_{z_0}^{z_0^*} \frac{1}{\rho_1(z, T_\eta^*)} (z - \eta + O(\varepsilon)\eta) (2\eta - z + O(\varepsilon)\eta) dz \\ & \geq C(z_0 - \eta)(2\eta - z_0^*) [W_0^{(\eta)}]^2 \int_{z_0}^{z_0^*} \frac{1}{\rho_1(z, T_\eta^*) - \rho_1(z_0, T_\eta^*)} dz \\ & \geq C(z_0 - \eta)(2\eta - z_0^*) [W_0^{(\eta)}]^2 \int_{z_0}^{z_0^*} \frac{1}{(\sup_{z \in (z_0, z_0^*]} |\partial_z \rho_1|)(z - z_0)} dz. \end{aligned}$$

With the crucial boundedness in Section 9 for $\partial_z \rho_1$, we conclude that

$$\|w^1(\cdot, T_\eta^*)\|_{L^2(\Omega_{T_\eta^*})} \geq C_\eta \int_{z_0}^{z_0^*} \frac{1}{z - z_0} dz = +\infty.$$

For the component U^1 of elastic waves, we derive from (3.11), (3.10) and (2.13)

$$\begin{aligned} \|\partial_x^2 U^1\|_{L^2(\Omega_{T_\eta^*})} &= \left\| \sum_{k=1}^6 w^k r_{k1} \right\|_{L^2(\Omega_{T_\eta^*})} \geq \|w^1 r_{11}\|_{L^2(\Omega_{T_\eta^*})} - \sum_{k=2}^6 \left\| \frac{v^k}{\rho_k} r_{k1} \right\|_{L^2(\Omega_{T_\eta^*})} \\ &\geq C \left[\|w^1\|_{L^2(\Omega_{T_\eta^*})} - \left\| \frac{v^3}{\rho_3} \right\|_{L^2(\Omega_{T_\eta^*})} - \left\| \frac{v^4}{\rho_4} \right\|_{L^2(\Omega_{T_\eta^*})} - \left\| \frac{v^6}{\rho_6} \right\|_{L^2(\Omega_{T_\eta^*})} \right] \\ &\geq C \left[\|w^1\|_{L^2(\Omega_{T_\eta^*})} - 3\varepsilon\eta^{\frac{3}{2}} \frac{W_0^{(\eta)}}{(1 - \varepsilon)^2} \right]. \end{aligned}$$

Finally, since $\|w^1\|_{L^2(\Omega_{T_\eta^*})} = +\infty$, we obtain

$$\|\partial_x^2 U^1\|_{L^2(\Omega_{T_\eta^*})} = +\infty.$$

In summary, with initial data requirements:

$$\begin{cases} W_0^{(\eta)} = \max_{i=1,\dots,6} \sup_z |w_{(\eta)}^i(z_i, 0)| = w_{(\eta)}^1(z_0, 0), \\ \max_{i=3,4} \sup_z |w_{(\eta)}^i(z_i, 0)| \leq \min \left\{ \frac{(1-\varepsilon)^4 |c_{11}^1(0)|}{(1+\varepsilon)^3} W_0^{(\eta)}, W_0^{(\eta)} \right\}, \\ \sup_z |w_{(\eta)}^6(z_6, 0)| \leq \frac{(1-\varepsilon)^4}{2(1+\varepsilon)^3} W_0^{(\eta)}, \end{cases}$$

and a general condition

$$c_{11}^1(\Phi) < 0, \quad \forall \Phi \in B_{2\delta}^6(0),$$

we prove

Theorem 10.1. *The Cauchy problems of the 3D elastic wave equations (1.4) are ill-posed in $H^3(\mathbb{R}^3)$ in the following sense: We construct a family of compactly supported smooth initial data $(U_0^{(\eta)}, U_1^{(\eta)})$ with*

$$\|U_0^{(\eta)}\|_{\dot{H}^3(\mathbb{R}^3)} + \|U_1^{(\eta)}\|_{\dot{H}^2(\mathbb{R}^3)} \rightarrow 0, \quad \text{as } \eta \rightarrow 0.$$

Let T_η^* be the largest time such that (1.4) (with a general condition (1.6)) has a solution $U_\eta \in C^\infty(\mathbb{R}^3 \times [0, T_\eta^*))$. As $\eta \rightarrow 0$, we have $T_\eta^* \rightarrow 0$.

Moreover, in a spatial region $\Omega_{T_\eta^*}$ the H^2 norm of the solution to elastic waves (1.4) blows up at shock formation time T_η^* :

$$\|U_\eta(\cdot, T_\eta^*)\|_{H^2(\Omega_{T_\eta^*})} = +\infty.$$

11. 2D CASE

The aim of this section is to prove low regularity ill-posedness for Cauchy problems of two dimensional elastic waves.

For a 2D elastic waves model with plane symmetry, equations in (1.4)

$$(11.1) \quad \partial_t^2 U - c_2^2 \Delta U - (c_1^2 - c_2^2) \nabla(\nabla \cdot U) = N(\nabla U, \nabla^2 U)$$

can be reduced to

$$(11.2) \quad \begin{cases} \partial_t^2 u^1 - c_1^2 \partial_x^2 u^1 = \sigma_0 \partial_x (\partial_x u^1)^2 + \sigma_1 \partial_x (\partial_x u^2)^2, \\ \partial_t^2 u^2 - c_2^2 \partial_x^2 u^2 = 2\sigma_1 \partial_x (\partial_x u^1 \partial_x u^2). \end{cases}$$

Let

$$\phi_1 := \partial_x u^1, \quad \phi_2 := \partial_x u^2, \quad \phi_3 := \partial_t u^1, \quad \phi_4 := \partial_t u^2.$$

And set $\Phi := (\phi_1, \phi_2, \phi_3, \phi_4)^T$. Then system (11.2) is equivalent to

$$(11.3) \quad \partial_t \Phi + A(\Phi) \partial_x \Phi = 0,$$

where

$$A(\Phi) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -(c_1^2 + 2\sigma_0 \phi_1) & -2\sigma_1 \phi_2 & 0 & 0 \\ -2\sigma_1 \phi_2 & -(c_2^2 + 2\sigma_1 \phi_1) & 0 & 0 \end{pmatrix}.$$

Lemma 11.1. *There exists a small $\delta > 0$. For $\Phi \in B_{2\delta}^4(0)$, we have that the reduced system (11.3) of the two dimensional elastic wave equations under planar symmetry is uniformly strictly hyperbolic. Moreover, the 1st and 4th characteristics are genuinely non-linear:*

$$\nabla_{\Phi} \lambda_i \cdot r_i \neq 0, \quad i = 1, 4, \quad \forall \Phi \in B_{2\delta}^4.$$

Proof. Let a, b, c be given by (2.4). We compute

$$\begin{aligned} \det(\lambda I - A) &= \lambda^4 - (a + b)\lambda^2 + (ab - c^2) \\ &= (\lambda^2 - a)(\lambda^2 - b) - c^2, \end{aligned}$$

and obtain the eigenvalues of $(A(\Phi))_{4 \times 4}$:

$$(11.4) \quad \begin{aligned} \lambda_1 &= \sqrt{\frac{1}{2}(a + b) + \frac{1}{2}\sqrt{(a - b)^2 + 4c^2}}, \\ \lambda_2 &= \sqrt{\frac{1}{2}(a + b) - \frac{1}{2}\sqrt{(a - b)^2 + 4c^2}}, \\ \lambda_3 &= -\sqrt{\frac{1}{2}(a + b) - \frac{1}{2}\sqrt{(a - b)^2 + 4c^2}}, \\ \lambda_4 &= -\sqrt{\frac{1}{2}(a + b) + \frac{1}{2}\sqrt{(a - b)^2 + 4c^2}}. \end{aligned}$$

These four eigenvalues are completely distinct,

$$\lambda_4(\Phi) < \lambda_3(\Phi) < \lambda_2(\Phi) < \lambda_1(\Phi).$$

Then we calculate their corresponding right eigenvectors and get:

$$\begin{aligned} r_1 &= \begin{pmatrix} \frac{\lambda_1^2 - b}{2\sigma_1} \\ \phi_2 \\ -\frac{\lambda_1(\lambda_1^2 - b)}{2\sigma_1} \\ -\lambda_1 \phi_2 \end{pmatrix}, \quad r_2 = \begin{pmatrix} \frac{\lambda_2^2 - b}{2\sigma_1} \\ \phi_2 \\ -\frac{\lambda_2(\lambda_2^2 - b)}{2\sigma_1} \\ -\lambda_2 \phi_2 \end{pmatrix}, \\ r_3 &= \begin{pmatrix} \frac{\lambda_2^2 - b}{2\sigma_1} \\ \phi_2 \\ \frac{\lambda_2(\lambda_2^2 - b)}{2\sigma_1} \\ \lambda_2 \phi_2 \end{pmatrix}, \quad r_4 = \begin{pmatrix} \frac{\lambda_1^2 - b}{2\sigma_1} \\ \phi_2 \\ \frac{\lambda_1(\lambda_1^2 - b)}{2\sigma_1} \\ \lambda_1 \phi_2 \end{pmatrix}. \end{aligned}$$

Since system (11.3) is strictly hyperbolic for $|\Phi| < 2\delta$ with small δ , Christodoulou-Perez's formulas and discussion of shock formation in [8] are applicable. Here we follow our argument in Section 3- Section 10 to study the ill-posedness theory of system (11.3).

By calculation, we obtain

$$\begin{aligned} l^1 &= \frac{1}{K} \left(\frac{\lambda_1^2 - b}{2\sigma_1}, \phi_2, -\frac{\lambda_1^2 - b}{2\sigma_1 \lambda_1}, -\frac{\phi_2}{\lambda_1} \right), \quad l^2 = \frac{1}{N} \left(\frac{\lambda_2^2 - b}{2\sigma_1}, \phi_2, -\frac{\lambda_2^2 - b}{2\sigma_1 \lambda_2}, -\frac{\phi_2}{\lambda_2} \right), \\ l^3 &= \frac{1}{N} \left(\frac{\lambda_2^2 - b}{2\sigma_1}, \phi_2, \frac{\lambda_2^2 - b}{2\sigma_1 \lambda_2}, \frac{\phi_2}{\lambda_2} \right), \quad l^4 = \frac{1}{K} \left(\frac{\lambda_1^2 - b}{2\sigma_1}, \phi_2, \frac{\lambda_1^2 - b}{2\sigma_1 \lambda_1}, \frac{\phi_2}{\lambda_1} \right), \end{aligned}$$

where

$$\begin{aligned} K &= \frac{(a - b)^2 + 4c^2}{4\sigma_1^2} + \frac{(a - b)\sqrt{(a - b)^2 + 4c^2}}{4\sigma_1^2}, \\ N &= \frac{(a - b)^2 + 4c^2}{4\sigma_1^2} - \frac{(a - b)\sqrt{(a - b)^2 + 4c^2}}{4\sigma_1^2}. \end{aligned}$$

Moreover, since

$$(\lambda_1^2 - b)(\lambda_2^2 - b) + c^2 = 0,$$

we verify that

$$l^i(\Phi) \cdot r_j(\Phi) = \delta_j^i.$$

As discussed in Section 3, we employ wave decomposition and explore the coefficients of equations (3.15)-(3.16). Denote

$$\Delta = (a - b)^2 + 4c^2.$$

Then, by calculation it follows

$$\begin{aligned} \nabla_{\Phi} \lambda_1 &= -\nabla_{\Phi} \lambda_4 = \left(\frac{(\sigma_0 + \sigma_1)\sqrt{\Delta} + (a - b)(\sigma_0 - \sigma_1)}{2\lambda_1\sqrt{\Delta}}, \frac{4\sigma_1^2\phi_2}{\lambda_1\sqrt{\Delta}}, 0, 0 \right), \\ \nabla_{\Phi} \lambda_2 &= -\nabla_{\Phi} \lambda_3 = \left(\frac{(\sigma_0 + \sigma_1)\sqrt{\Delta} - (a - b)(\sigma_0 - \sigma_1)}{2\lambda_2\sqrt{\Delta}}, -\frac{4\sigma_1^2\phi_2}{\lambda_2\sqrt{\Delta}}, 0, 0 \right). \end{aligned}$$

And from (3.13) and (3.14), we have

$$\begin{aligned} c_{11}^1(\Phi) &= -c_{44}^4(\Phi) = \nabla_{\Phi} \lambda_1 \cdot r_1 = \frac{2\sigma_0(a - b)(\lambda_1^2 - b) + (2\sigma_0 + 6\sigma_1)c^2}{4\sigma_1\lambda_1\sqrt{\Delta}}, \\ c_{22}^2(\Phi) &= -c_{33}^3(\Phi) = \nabla_{\Phi} \lambda_2 \cdot r_2 = -\frac{2\sigma_0(a - b)(\lambda_2^2 - b) - (2\sigma_0 + 6\sigma_1)c^2}{4\sigma_1\lambda_2\sqrt{\Delta}}. \end{aligned}$$

By (11.4) and definitions of a, b, c in (2.4), for $\Phi \in B_{2\delta}^4(0)$, we have

$$a - b \sim c_1^2 - c_2^2, \quad \lambda_1^2 - b \sim c_1^2 - c_2^2, \quad c \sim 0.$$

This implies

$$c_{11}^1(\Phi) = -c_{44}^4(\Phi) \neq 0.$$

This concludes the proof of Lemma 11.1. \square

By the same definitions for ρ_i, w^i, v^i as in (3.3) and (3.10) for $i = 1, 2, 3, 4$, we have decomposition of waves corresponding to this strictly hyperbolic system (11.3):

$$\begin{aligned} \partial_{s_i} w^i &= -c_{ii}^i (w^i)^2 + \left(\sum_{m \neq i} (-c_{im}^i + \gamma_{im}^i) w^m \right) w^i + \sum_{\substack{m \neq i, k \neq i \\ m \neq k}} \gamma_{km}^i w^k w^m, \\ \partial_{s_i} \rho_i &= c_{ii}^i v^i + \left(\sum_{m \neq i} c_{im}^i w^m \right) \rho_i, \\ \partial_{s_i} v^i &= \left(\sum_{m \neq i} \gamma_{im}^i w^m \right) v^i + \sum_{\substack{m \neq i, k \neq i \\ m \neq k}} \gamma_{km}^i w^k w^m \rho_i. \end{aligned}$$

We construct the initial data satisfying

$$W_0^{(\eta)} := \max_i \sup_{z_i} |w_{(\eta)}^i(z_i, 0)| = w_{(\eta)}^1(z_0, 0).$$

We further choose

$$w_{(\eta)}^1(z, 0) = \theta \int_{\mathbb{R}} \zeta_{\frac{\eta}{10}}(y) |\ln(x - y)|^\alpha \chi(x - y) dy, \quad \text{for } 0 < \alpha < 1,$$

and require

$$\max_{i=2,3,4} \sup_{z_i} |w_{(\eta)}^i(z_i, 0)| \leq \frac{(1-\varepsilon)^4}{2(1+\varepsilon)^3} W_0^{(\eta)}.$$

Let

$$\begin{aligned} S(t) &:= \max_i \sup_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} \rho_i(z'_i, s'_i), & J(t) &:= \max_i \sup_{\substack{(z'_i, s'_i) \\ z'_i \in [\eta, 2\eta], 0 \leq s'_i \leq t}} |v^i(z'_i, s'_i)|, \\ V(t) &:= \max_i \sup_{\substack{(x', t') \notin \mathcal{R}_i, \\ 0 \leq t' \leq t}} |w^i(x', t')|, & \bar{U}(t) &:= \sup_{\substack{(x', t') \\ 0 \leq t' \leq t}} |\Phi(x', t')|. \end{aligned}$$

For $\Phi \in C^2(\mathbb{R} \times [0, T], B_{2\delta}^4(0))$ being a solution to (11.1) for some $T > 0$, with a general assumption $c_{11}^1(0) < 0$, by analogous arguments as in Section 5, we obtain estimates:

$$\begin{aligned} S &= O(1 + tJ + tVS), & J &= O(W_0^{(\eta)} + tVJ + tV^2S), \\ V &= O(\eta[W_0^{(\eta)}]^2 + tV^2 + \eta VJ), & \bar{U} &= O(\eta J + \eta V + \eta tV), \end{aligned}$$

where $t < T$. For $tW_0^{(\eta)} \leq C$, by a bootstrap argument we hence have the following bounds:

$$\begin{aligned} S &= O(1), & J &= O(W_0^{(\eta)}), \\ V &= O(\eta[W_0^{(\eta)}]^2), & \bar{U} &= O(\eta W_0^{(\eta)}). \end{aligned}$$

These imply there exists T_η^* satisfying

$$(11.5) \quad \frac{1}{(1+\varepsilon)^3 |c_{11}^1(0)| W_0^{(\eta)}} \leq T_\eta^* \leq \frac{1}{(1-\varepsilon)^4 |c_{11}^1(0)| W_0^{(\eta)}}.$$

And a shock forms at time T_η^* ,

$$\lim_{t \rightarrow T_\eta^*} \rho_1(z_0, t) = 0.$$

Moreover, deriving estimates as in Section 9, we can show $\partial_{z_1} \rho_1$ is bounded.

We construct $\Omega_{T_\eta^*} \subseteq \{(x, Y^2, t) : t = T_\eta^*\}$ to be the region

$$\frac{c_2^2(x - c_1 T_\eta^* - \frac{3\eta}{2})^2}{c_1^2} + (Y^2)^2 \leq \frac{c_2^2}{c_1^2} \left(\frac{\eta}{2}\right)^2,$$

hence we obtain

$$\int_{\Omega_{T_\eta^*}^r} dY^2 \sim (1 + O(\varepsilon)) \sqrt{(z - \eta + O(\varepsilon)\eta)(2\eta - z + O(\varepsilon)\eta)},$$

where $\Omega_{T_\eta^*}^r = \{Y^2 : (x, Y^2) \in \Omega_{T_\eta^*}\}$. This yields

$$\begin{aligned}
& \|w^1(\cdot, T_\eta^*)\|_{L^2(\Omega_{T_\eta^*}^*)}^2 \\
& \geq C \int_\eta^{2\eta} \left| \frac{v^1}{\rho_1}(z, T_\eta^*) \right|^2 \rho_1(z, T_\eta^*) \sqrt{(z - \eta + O(\varepsilon)\eta)(2\eta - z + O(\varepsilon)\eta)} dz \\
& \geq C(1 - \varepsilon)^2(1 - 2\varepsilon)^2 [W_0^{(\eta)}]^2 \int_{z_0}^{z_0^*} \frac{1}{\rho_1(z, T_\eta^*)} \sqrt{(z - \eta + O(\varepsilon)\eta)(2\eta - z + O(\varepsilon)\eta)} dz \\
& \geq C \sqrt{(z_0 - \eta)(2\eta - z_0^*)} [W_0^{(\eta)}]^2 \int_{z_0}^{z_0^*} \frac{1}{\rho_1(z, T_\eta^*) - \rho_1(z_0, T_\eta^*)} dz \\
& \geq C \sqrt{(z_0 - \eta)(2\eta - z_0^*)} [W_0^{(\eta)}]^2 \int_{z_0}^{z_0^*} \frac{1}{(\sup_{z \in (z_0, z_0^*]} |\partial_z \rho_1|)(z - z_0)} dz.
\end{aligned}$$

With the boundedness of $\partial_{z_1} \rho_1$, we arrive at

$$\|U_\eta(\cdot, T_\eta^*)\|_{H^2(\Omega_{T_\eta^*}^*)} \gtrsim \|w^1(\cdot, T_\eta^*)\|_{L^2(\Omega_{T_\eta^*}^*)} = +\infty.$$

In summary, we prove the following result.

Theorem 11.1. *The Cauchy problems of 2D elastic wave equations are ill-posed in $H^{\frac{5}{2}}(\mathbb{R}^2)$ in the following sense: We construct a family of compactly supported smooth initial data $(U_0^{(\eta)}, U_1^{(\eta)})$ with*

$$\|U_0^{(\eta)}\|_{\dot{H}^{\frac{5}{2}}(\mathbb{R}^2)} + \|U_1^{(\eta)}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^2)} \rightarrow 0, \quad \text{as } \eta \rightarrow 0.$$

Let T_η^* be the largest time such that (1.4) has a solution $U_\eta \in C^\infty(\mathbb{R}^2 \times [0, T_\eta^*])$. As $\eta \rightarrow 0$, we have $T_\eta^* \rightarrow 0$.

Moreover, in a spatial region $\Omega_{T_\eta^*}$ the H^2 norm of the solution to elastic waves (1.4) blows up at shock formation time T_η^* :

$$\|U_\eta(\cdot, T_\eta^*)\|_{H^2(\Omega_{T_\eta^*}^*)} = +\infty.$$

APPENDIX A. NORMS FOR INITIAL DATA

In this appendix, we construct examples of initial data in 3D and 2D satisfying requirements in Section 4 and in Section 11.

Assume $\eta > 0$ and $\theta > 0$. Let $\chi(x)$ be the characteristic function

$$(A.1) \quad \chi(x) = \begin{cases} 1, & x \in [\frac{6}{5}\eta, \frac{9}{5}\eta], \\ 0, & x \notin [\frac{6}{5}\eta, \frac{9}{5}\eta]. \end{cases}$$

Set $\zeta_{\frac{\eta}{10}}(x)$ to be a test function in $C_c^\infty(\mathbb{R})$ satisfying:

$$\text{supp } \zeta_{\frac{\eta}{10}}(x) \subseteq \{x : |x| \leq \frac{\eta}{10}\}, \quad 0 \leq \zeta_{\frac{\eta}{10}}(x) \leq \frac{20}{\eta}, \quad \text{and} \quad \int_{\mathbb{R}} \zeta_{\frac{\eta}{10}}(x) dx = 1.$$

We start with the 3D case. Denote $B_{\frac{\eta}{2}}^3 := \{(x, Y^2, Y^3) : (x - \frac{3\eta}{2})^2 + (Y^2)^2 + (Y^3)^2 \leq (\frac{\eta}{2})^2\}$. We have:

Lemma A.1. *Let $0 < \alpha < \frac{1}{2}$. We can construct a function $\hat{w} \in H^1(\mathbb{R}^3)$ satisfying*

$$(A.2) \quad \hat{w}(x, Y^2, Y^3) = \theta \int_{\mathbb{R}} \zeta_{\frac{\eta}{10}}(y) |\ln(x-y)|^\alpha \chi(x-y) dy, \text{ in } B_{\frac{\eta}{2}}^3,$$

and

$$(A.3) \quad \|\hat{w}\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \theta \sqrt{\eta} (1 + |\ln \frac{6\eta}{5}|^\alpha + |\ln \frac{9\eta}{5}|^\alpha).$$

Proof. By construction, we have $\text{supp } \hat{w} \subset \{(x, Y^2, Y^3) : \eta \leq x \leq 2\eta\}$. For $\Omega_0^r = \{(Y^2, Y^3) : (x, Y^2, Y^3) \in B_{\frac{\eta}{2}}^3\}$, we have

$$(A.4) \quad \begin{aligned} \int_{\Omega_0^r} dY^2 dY^3 &= \left(\frac{\eta}{2}\right)^2 - \left(x - \frac{3\eta}{2}\right)^2 = (x - \eta)(2\eta - x) \\ &= \left[x - \frac{\eta}{10}\right] - \frac{9\eta}{10} \left[\frac{19\eta}{10} - \left(x - \frac{\eta}{10}\right)\right] \\ &\leq \left[x - \frac{\eta}{10}\right] - \frac{9\eta}{10} \frac{19\eta}{10} \\ &\leq \frac{19}{10} \eta \left(x - \frac{\eta}{10}\right). \end{aligned}$$

It follows that

$$\begin{aligned} &\int_{B_{\frac{\eta}{2}}^3} |\partial_x \hat{w}|^2 dx dY^2 dY^3 \\ &\leq \underbrace{\int_{B_{\frac{\eta}{2}}^3} \left| \int_{|y| \leq \frac{\eta}{10}} \zeta_{\frac{\eta}{10}}(y) \frac{\alpha \theta}{(x-y) |\ln(x-y)|^{1-\alpha}} \chi(x-y) dy \right|^2 dx dY^2 dY^3}_{L_1} \\ &\quad + \underbrace{\int_{B_{\frac{\eta}{2}}^3} \left| \int_{|y| \leq \frac{\eta}{10}} \zeta_{\frac{\eta}{10}}(y) \theta |\ln(x-y)|^\alpha \delta(x-y - \frac{6\eta}{5}) dy \right|^2 dx dY^2 dY^3}_{L_2} \\ &\quad + \underbrace{\int_{B_{\frac{\eta}{2}}^3} \left| \int_{|y| \leq \frac{\eta}{10}} \zeta_{\frac{\eta}{10}}(y) \theta |\ln(x-y)|^\alpha \delta(x-y - \frac{9\eta}{5}) dy \right|^2 dx dY^2 dY^3}_{L_3} \end{aligned}$$

For L_1 , by (A.4), we have

$$(A.5) \quad \begin{aligned} L_1 &\lesssim \int_{\eta}^{2\eta} \frac{\alpha^2 \theta^2 \cdot (x - \eta)(2\eta - x)}{\left(x - \frac{\eta}{10}\right)^2 |\ln(x - \frac{\eta}{10})|^{2-2\alpha}} dx \\ &\lesssim \int_{\eta}^{2\eta} \frac{\alpha^2 \theta^2 \eta}{\left(x - \frac{\eta}{10}\right) |\ln(x - \frac{\eta}{10})|^{2-2\alpha}} dx \\ &\lesssim \theta^2 \eta. \end{aligned}$$

For $L_2 + L_3$, we proceed and get

$$\begin{aligned}
(A.6) \quad L_2 + L_3 &\lesssim \int_{B_{\frac{\eta}{2}}^3} \left| \theta \sup_z |\zeta(z)| \int_{\mathbb{R}} |\ln(x-y)|^\alpha \delta(x-y - \frac{6\eta}{5}) dy \right|^2 dx dY^2 dY^3 \\
&\quad + \int_{B_{\frac{\eta}{2}}^3} \left| \theta \sup_z |\zeta(z)| \int_{\mathbb{R}} |\ln(x-y)|^\alpha \delta(x-y - \frac{9\eta}{5}) dy \right|^2 dx dY^2 dY^3 \\
&\lesssim \theta^2 \frac{1}{\eta^2} \cdot (|\ln \frac{6\eta}{5}|^{2\alpha} + |\ln \frac{9\eta}{5}|^{2\alpha}) \cdot \eta^3 \lesssim \theta^2 \eta \cdot (|\ln \frac{6\eta}{5}|^{2\alpha} + |\ln \frac{9\eta}{5}|^{2\alpha}).
\end{aligned}$$

Together with (A.5) and (A.6), we conclude

$$\int_{B_{\frac{\eta}{2}}^3} |\partial_x \hat{w}|^2 dx dY^2 dY^3 \lesssim \theta^2 \eta (1 + |\ln \frac{6\eta}{5}|^{2\alpha} + |\ln \frac{9\eta}{5}|^{2\alpha}).$$

We hence have $\|\hat{w}\|_{\dot{H}^1(B_{\frac{\eta}{2}}^3)} \lesssim \theta \sqrt{\eta} (1 + |\ln \frac{6\eta}{5}|^\alpha + |\ln \frac{9\eta}{5}|^\alpha)$. We then extend \hat{w} to a compactly supported H^1 function in the whole region satisfying (A.3). This finishes the proof. \square

For 2D case, we will use the following identity: for $0 < s < 1$, it holds that

$$(A.7) \quad \|f\|_{\dot{H}^s(\mathbb{R}^d)}^2 = C_s \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{[f(x+y) - f(x)]^2}{|y|^{2s+d}} dx dy.$$

Denote $B_{\frac{\eta}{2}}^2 := \{(x, Y^2) : (x - \frac{3\eta}{2})^2 + (Y^2)^2 \leq (\frac{\eta}{2})^2\}$. We have:

Lemma A.2. *Let $0 < \alpha < 1$. We can construct a function $\hat{w} \in H^{\frac{1}{2}}(\mathbb{R}^2)$ satisfying*

$$(A.8) \quad \hat{w}(x, Y^2) = \theta \int_{\mathbb{R}} \zeta_{\frac{\eta}{10}}(y) |\ln(x-y)|^\alpha \chi(x-y) dy, \text{ in } B_{\frac{\eta}{2}}^2,$$

and

$$(A.9) \quad \|\hat{w}\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} \lesssim \theta \eta^{\frac{1}{4}} (1 + |\ln \frac{6\eta}{5}|^\alpha + |\ln \frac{9\eta}{5}|^\alpha).$$

Proof. By construction, we have $\text{supp } \hat{w} \subset \{(x, Y^2) : \eta \leq x \leq 2\eta\}$. For $\Omega_1^r = \{Y^2 : (x, Y^2) \in B_{\frac{\eta}{2}}^2\}$, it holds that

$$\int_{\Omega_1^r} dY^2 = \sqrt{(x-\eta)(2\eta-x)} \leq \frac{1}{2}\eta.$$

Let $w(x) = \theta(\zeta_{\frac{\eta}{10}} * (|\ln \cdot|^\alpha \chi))(x)$. We have $\text{supp } w \subseteq [\eta, 2\eta]$ and

$$\int_{B_{\frac{\eta}{2}}^2} \|\nabla_x |^{\frac{1}{2}} \hat{w}\|^2 dx dY^2 \leq \frac{1}{2}\eta \int_{\eta}^{2\eta} \|\nabla_x |^{\frac{1}{2}} w\|^2 dx.$$

Applying (A.7) for $d = 1$, we get

$$(A.10) \quad \int_{B_{\frac{\eta}{2}}^2} \left| |\nabla_x|^{\frac{1}{2}} \hat{w} \right|^2 dx dY^2 \lesssim \eta \iint_{\mathbb{R} \times \mathbb{R}} \frac{|w(x+y) - w(x)|^2}{|y|^2} dx dy.$$

Now we split the domain of integration on the right hand side of (A.10) into six parts:

$$\begin{aligned} I_1 &= \{(x, y) : x \in [\eta, 2\eta], y \in [-\sqrt{\eta}, \sqrt{\eta}]\}, \\ I_2 &= \{(x, y) : x \in [\eta, 2\eta], y \in (\sqrt{\eta}, +\infty)\}, \\ I_3 &= \{(x, y) : x \in [\eta, 2\eta], y \in (-\infty, -\sqrt{\eta}]\}, \\ I_4 &= \{(x, y) : x \in (2\eta, +\infty), x + y \in [\eta, 2\eta]\}, \\ I_5 &= \{(x, y) : x \in (-\infty, \eta), x + y \in [\eta, 2\eta]\}, \\ I_6 &= \mathbb{R}^2 \setminus \cup_{k \in \{1, \dots, 5\}} I_k. \end{aligned}$$

In this way, $w(x+y) - w(x)$ is supported in $I_j, j = 1, \dots, 5$. P

In I_1 , we have

$$(A.11) \quad \begin{aligned} \eta \iint_{I_1} \frac{|w(x+y) - w(x)|^2}{|y|^2} dx dy &\leq \eta \int_{-\sqrt{\eta}}^{\sqrt{\eta}} \int_{\eta}^{2\eta} \frac{|\int_0^y \partial_x w(x+z) dz|^2}{|y|^2} dx dy \\ &\leq \eta \int_{-\sqrt{\eta}}^{\sqrt{\eta}} \int_{\eta}^{2\eta} \sup_{z \in [\eta, 2\eta]} |\partial_z w(z)|^2 dx dy. \end{aligned}$$

Since

$$\begin{aligned} &\sup_{z \in [\eta, 2\eta]} |\partial_z w(z)| \\ &\leq \sup_{z \in [\eta, 2\eta]} \int_{|y| \leq \frac{\eta}{10}} \zeta_{\frac{\eta}{10}}(y) \frac{\alpha \theta}{(z-y) |\ln(z-y)|^{1-\alpha}} dy \\ &\quad + \sup_{z \in [\eta, 2\eta]} \int_{|y| \leq \frac{\eta}{10}} \zeta_{\frac{\eta}{10}}(y) \alpha \theta |\ln(z-y)|^\alpha \delta(z-y - \frac{6\eta}{5}) dy \\ &\quad + \sup_{z \in [\eta, 2\eta]} \int_{|y| \leq \frac{\eta}{10}} \zeta_{\frac{\eta}{10}}(y) \alpha \theta |\ln(z-y)|^\alpha \delta(z-y - \frac{9\eta}{5}) dy \\ &\lesssim \sup_{z \in [\eta, 2\eta]} \frac{\theta}{(z - \frac{\eta}{10}) |\ln(z - \frac{\eta}{10})|^{1-\alpha}} + \frac{1}{\eta} \theta \cdot (|\ln \frac{6\eta}{5}|^\alpha + |\ln \frac{9\eta}{5}|^\alpha) \\ &\lesssim \frac{\theta}{\eta} (1 + |\ln \frac{6\eta}{5}|^\alpha + |\ln \frac{9\eta}{5}|^\alpha), \end{aligned}$$

from (A.11), we get

$$(A.12) \quad \eta \iint_{I_1} \frac{|w(x+y) - w(x)|^2}{|y|^2} dx dy \lesssim \theta^2 \sqrt{\eta} (1 + |\ln \frac{6\eta}{5}|^{2\alpha} + |\ln \frac{9\eta}{5}|^{2\alpha}).$$

In I_2 , we have $w(x+y) = 0$. Thus, it holds that

$$\begin{aligned}
& \eta \iint_{I_2} \frac{|w(x+y) - w(x)|^2}{|y|^2} dx dy = \eta \int_{\sqrt{\eta}}^{+\infty} \int_{\eta}^{2\eta} \frac{|w(x)|^2}{|y|^2} dx dy \\
\text{(A.13)} \quad & \lesssim \eta \sup_{x \in [\eta, 2\eta]} |w(x)|^2 \int_{\sqrt{\eta}}^{+\infty} \int_{\eta}^{2\eta} \frac{1}{|y|^2} dx dy \lesssim \eta \theta^2 |\ln \eta|^{2\alpha} \frac{\eta}{\sqrt{\eta}} \\
& \lesssim \eta^{\frac{3}{2}} \theta^2 |\ln \eta|^{2\alpha}.
\end{aligned}$$

By the same calculation as in (A.13), we obtain

$$\text{(A.14)} \quad \eta \iint_{I_3} \frac{|w(x+y) - w(x)|^2}{|y|^2} dx dy \lesssim \eta^{\frac{3}{2}} \theta^2 |\ln \eta|^{2\alpha}.$$

In I_4 , it holds that $w(x) = 0$. Note $\text{supp } w \subseteq [\eta, 2\eta]$. We have

$$\begin{aligned}
\text{(A.15)} \quad & \eta \iint_{I_4} \frac{|w(x+y) - w(x)|^2}{|y|^2} dx dy \\
& = \eta \int_{2\eta}^{+\infty} \int_{-\infty}^{-\eta} \frac{|w(x+y)|^2}{|y|^2} dy dx + \eta \int_{2\eta}^{+\infty} \int_{-\eta}^0 \frac{|w(x+y) - w(x)|^2}{|y|^2} dy dx \\
& \leq \eta \int_{\eta}^{2\eta} \int_{-\infty}^{-\eta} \frac{|w(z)|^2}{|y|^2} dy dz + \eta \int_{2\eta}^{+\infty} \int_{-\eta}^0 \frac{|\int_0^y \partial_x w(x+z) dz|^2}{|y|^2} dy dx \\
& \lesssim \eta \sup_{z \in [\eta, 2\eta]} |w(z)|^2 \int_{\eta}^{2\eta} \int_{-\infty}^{-\eta} \frac{1}{|y|^2} dy dz + \eta \sup_{z \in [\eta, 2\eta]} |\partial_z w(z)|^2 \eta \int_{-\eta}^0 dy \\
& \lesssim \eta \theta^2 |\ln \eta|^{2\alpha} + \eta^3 \frac{\theta^2}{\eta^2} (1 + |\ln \frac{6\eta}{5}|^{2\alpha} + |\ln \frac{9\eta}{5}|^{2\alpha}) \\
& \lesssim \eta \theta^2 (1 + |\ln \frac{6\eta}{5}|^{2\alpha} + |\ln \frac{9\eta}{5}|^{2\alpha}).
\end{aligned}$$

For I_5 , it holds that $y \in [0, +\infty)$. We divide $[0, +\infty)$ into $[\eta, +\infty)$ and $[0, \eta)$. With a similar calculation as in (A.15), we also get

$$\text{(A.16)} \quad \eta \iint_{I_5} \frac{|w(x+y) - w(x)|^2}{|y|^2} dx dy \lesssim \eta \theta^2 (1 + |\ln \frac{6\eta}{5}|^{2\alpha} + |\ln \frac{9\eta}{5}|^{2\alpha}).$$

Finally, in I_6 , since $w(x+y) = w(x) = 0$, we have

$$\text{(A.17)} \quad \eta \iint_{I_6} \frac{|w(x+y) - w(x)|^2}{|y|^2} dx dy = 0.$$

Consequently, together with (A.10), (A.12), (A.13), (A.14), (A.15), (A.16) and (A.17), we obtain

$$\int_{B_{\frac{\eta}{2}}^2} \|\nabla_x |\frac{1}{2} \hat{w}|^2\|^2 dx dY^2 \lesssim \theta^2 \sqrt{\eta} (1 + |\ln \frac{6\eta}{5}|^{2\alpha} + |\ln \frac{9\eta}{5}|^{2\alpha}).$$

We hence have $\|\hat{w}\|_{\dot{H}^{\frac{1}{2}}(B_{\eta/2}^2)} \lesssim \theta\eta^{\frac{1}{4}}(1 + |\ln \frac{6\eta}{5}|^\alpha + |\ln \frac{9\eta}{5}|^\alpha)$. We then extend \hat{w} to a compactly supported $H^{\frac{1}{2}}$ function in the whole region satisfying (A.9). This completes the proof. \square

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