WAVE EQUATION ON GENERAL NONCOMPACT SYMMETRIC SPACES

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Dedicated to the memory of Robert S. Strichartz (1943-2021)

Abstract. We establish sharp pointwise kernel estimates and dispersive properties for the wave equation on noncompact symmetric spaces of general rank. This is achieved by combining the stationary phase method and the Hadamard parametrix, and in particular, by introducing a subtle spectral decomposition, which allows us to overcome a well-known difficulty in higher rank analysis, namely the fact that the Plancherel density is not a differential symbol in general. Consequently, we deduce the Strichartz inequality for a large family of admissible pairs and prove global well-posedness results for the corresponding semi-linear equation with low regularity data as on hyperbolic spaces.

Contents

1. Introduction

This paper is devoted to prove sharp-in-time kernel estimates and dispersive properties for the wave equation on noncompact symmetric spaces of higher rank. Consequently, we prove the Strichartz inequality and study their applications to associated semi-linear Cauchy problems. Relevant theories are well established on Euclidean spaces, see for instance [23, 27, 14, 25, 11], and the references therein.

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Given the rich Euclidean results, several works have been made in other settings. We are interested in Riemannian symmetric spaces of noncompact type, where relevant questions are now well answered in rank one, see for instance [12, 21, 36, 28, 29, 5, 3] on hyperbolic spaces, and [6] on Damek-Ricci spaces. A first study of the wave equation on general symmetric spaces of higher rank was carried out in [17], where some non optimal estimates were obtained under a strong smoothness assumption. Recently, sharp-in-time kernel estimates and dispersive properties have been proven in [38] on noncompact symmetric spaces G/K, with G complex. In this case, the Harish-Chandra $\mathbf{c}$-function and the spherical function have elementary expressions, which is not the case in general.

In this paper, we establish pointwise wave kernel estimates and dispersive properties for the wave equation on general noncompact symmetric spaces, which are sharp in time and extend previous results obtained on real hyperbolic spaces [5, 3] to higher rank. The main challenge is that the Plancherel density involved in the wave kernel is not a polynomial, nor even a differential symbol in general. To bypass this problem, we consider barycentric decompositions of the Weyl chambers into subcones and differentiate in each subcone along a well chosen direction.

For suitable $\sigma \in \mathbb{C}$, we consider the wave operator $W_t^\sigma = (-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}}$ associated to the Laplace-Beltrami operator Δ on a d-dimensional noncompact symmetric space $\mathbb{X} = G/K$. To avoid possible singularities (see Sect. 3.2), we consider actually the analytic family of operators

$$\widetilde{W}_t^\sigma = \frac{e^{\sigma^2}}{\Gamma(\frac{d+1}{2} - \sigma)} (-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}} \tag{1.1}$$

in the vertical strip $0 \le \operatorname{Re}\sigma \le \frac{d+1}{2}$. Let us denote by $\widetilde{\omega}_t^\sigma$ its K-bi-invariant convolution kernel. Our first main result is the following pointwise estimate, which summarizes Theorem 3.3, Theorem 3.7 and Theorem 3.10 proved in Sect. 3.

Theorem 1.1 (Pointwise kernel estimates)**.** *Let $d \ge 3$ and $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \frac{d+1}{2}$. There exist $C > 0$ and $N \in \mathbb{N}$ such that the following estimates hold for all $t \in \mathbb{R}^*$ and $x \in \mathbb{X}$:*

$$|\widetilde{\omega}_t^\sigma(x)| \le C(1+|x^+|)^N e^{-\langle \rho, x^+\rangle} \begin{cases} |t|^{-\frac{d-1}{2}} & \text{if } 0<|t|<1, \\ |t|^{-\frac{D}{2}} & \text{if } |t| \ge 1, \end{cases}$$

where $x^+ \in \overline{\mathfrak{a}^+}$ denotes the radial component of x in the Cartan decomposition, and $D = \ell + 2|\Sigma_r^+|$ is the so-called dimension at infinity of $\mathbb{X}$.

Remark 1.2. *These kernel estimates are sharp in time and similar results hold obviously in the easier case where $\operatorname{Re}\sigma > \frac{d+1}{2}$. The value of N will be specified in Sect. 3. However, the polynomial $(1+|x^+|)^N$ is not crucial for further computations because of the exponential decay $e^{-\langle \rho, x^+\rangle}$.*

By interpolation arguments, we deduce our second main result.

Theorem 1.3 (Dispersive property)**.** *Assume that* $d \geq 3$*,* $2 < q, \widetilde{q} < +\infty$ *and* $\sigma \geq (d+1)\max(\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\widetilde{q}})$*. Then there exists a constant* $C > 0$ *such that following dispersive estimates hold:*

$$\|W_t^\sigma\|_{L^{\widetilde{q}'}(\mathbb{X})\to L^q(\mathbb{X})} \leq C \begin{cases} |t|^{-(d-1)\max(\frac{1}{2}-\frac{1}{q},\frac{1}{2}-\frac{1}{\widetilde{q}})} & \text{if } 0<|t|<1, \\ |t|^{-\frac{D}{2}} & \text{if } |t| \geq 1. \end{cases}$$

Remark 1.4. *At the endpoint* $q = \widetilde{q} = 2$*,* $t \mapsto e^{it\sqrt{-\Delta}}$ *is a one-parameter group of unitary operators on* $L^2(\mathbb{X})$*.*

Remark 1.5. *Theorem 1.1 and Theorem 1.3 generalize earlier results obtained for real hyperbolic spaces [5, 3] (which extend straightforwardly to all noncompact symmetric spaces of rank one), or for noncompact symmetric spaces* G/K *with* G *complex [38]. Notice that* $D = 3$ *in rank one and that* $D = d$ *if* G *is complex.*

Remark 1.6. *For simplicity, we omit the* 2*-dimensional case where the small time bounds in Theorem 1.1 and Theorem 1.3 involve an additional logarithmic factor, see [3, Theorem 3.2 and 4.2]. Notice that* $d \geq 4$ *in higher rank, see* (2.1)*.*

Let us sketch the proofs of our main results. We prove the dispersive properties of W_t^σ by using interpolation arguments based on pointwise estimates of $\widetilde{\omega}_t^\sigma$, which are sharp in time. By the way, let us point out that the kernel analysis carried out on hyperbolic spaces [3] can not be extend straightforwardly in higher rank, since the Plancherel density is not a differential symbol in general. Consider the Poisson operator $\mathcal{P}_\tau = e^{-\tau\sqrt{-\Delta}}$, for all $\tau \in \mathbb{C}$ with $\operatorname{Re}\tau \geq 0$. Along the lines of [31, 15, 10], we can write formally our wave operator (1.1) as

$$\widetilde{W}_t^\sigma = \frac{e^{\sigma^2}}{\Gamma(\frac{d+1}{2}-\sigma)} \frac{1}{\Gamma(\sigma)} \int_0^{+\infty} ds\, s^{\sigma-1}\mathcal{P}_{s-it}.$$

Our analysis is focused on kernel estimates of the Poisson operator $\mathcal{P}_{s-it}$ where $s \in \mathbb{R}^+$ and $t \in \mathbb{R}^*$. We adopt different methods depending whether s, $|t|$ and $\frac{|x|}{|t|}$ ($x \in \mathbb{X}$) are small or large. Specifically,

- If s is bounded from above and $\frac{|x|}{|t|}$ is sufficiently small with $|t|$ large, we develop an effective stationary phase method based on barycentric decompositions of Weyl chambers described in Sect. 2.3. In each subdivision, the Plancherel density becomes a differential symbol for a well chosen directional derivative, see Sect. 3.1.
- If s is bounded from above but $\frac{|x|}{|t|}$ is large (with $|t|$ small or large), we estimate the kernel along the lines of [9], where Cowling, Guilini and Meda have studied the Poisson operator $\mathcal{P}_\tau$ for $\tau \in \mathbb{C}$ with $\operatorname{Re}\tau \geq 0$. Unfortunately, their estimates are not sharp when τ is large and nearly imaginary, which happens in our context when s is small and $|t|$ is large. To deal with this case, we resume and improve slightly their method by writing down more explicitly the Hadamard parametrix on noncompact symmetric spaces along the lines of [7], see Sect. 3.2.

- If s is large, the kernel is estimated by using the standard stationary phase method, which is similar to the rank one analysis, see Sect. 3.3.

This paper is organized as follows. We recall spherical Fourier analysis on noncompact symmetric spaces and introduce the barycentric decomposition of Weyl chambers in Sect. 2. Next, we derive pointwise wave kernel estimates in Sect. 3. By using interpolation arguments, we prove in Sect. 4 the dispersive property for the wave operator. As consequences, we establish the Strichartz inequality for a large family of admissible pairs and obtain well-posedness results for the associated semi-linear wave equation in Sect. 5. We give further results about the Klein-Gordon equation in Sect. 6. Finally, we collect in the appendices some useful results: in Appendix A, we study by the stationary phase method an oscillatory integral occurring in the wave kernel analysis; next we describe in Appendix B the Hadamard parametrix on noncompact symmetric spaces and consider its application to the Poisson operator in Appendix C.

2. Preliminaries

In this section, we first review briefly spherical Fourier analysis on noncompact symmetric spaces. Next we introduce a barycentric decomposition for Weyl chambers, which will be crucial for the forthcoming kernel estimates.

2.1. **Notations.** We adopt the standard notation and refer to [18, 19] for more details. Let G be a semisimple Lie group, connected, noncompact, with finite center, and K be a maximal compact subgroup of G. The homogeneous space $\mathbb{X}=G/K$ is a Riemannian symmetric space of noncompact type. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of the Lie algebra of G. There is a natural identification between $\mathfrak{p}$ and the tangent space of $\mathbb{X}$ at the origin. The Killing form of $\mathfrak{g}$ induces a K-invariant inner product on $\mathfrak{p}$, hence a G-invariant Riemannian metric on $\mathbb{X}$.

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. The rank of $\mathbb{X}$ is the dimension ℓ of $\mathfrak{a}$. Let $\Sigma\subset\mathfrak{a}$ be the root system of $(\mathfrak{g},\mathfrak{a})$ and denote by W the Weyl group associated to Σ. Once a positive Weyl chamber $\mathfrak{a}^+\subset\mathfrak{a}$ has been selected, Σ^+ (resp. Σ_r^+ or Σ_s^+) denotes the corresponding set of positive roots (resp. positive reduced roots or simple roots). Let d be the dimension of $\mathbb{X}$ and D be the dimension at infinity of $\mathbb{X}$:

$$d=\ell+\textstyle\sum_{\alpha\in\Sigma^+}m_\alpha \quad\text{and}\quad D=\ell+2|\Sigma_r^+|, \tag{2.1}$$

where m_α is the dimension of the positive root subspace $\mathfrak{g}_\alpha$. Notice that one cannot compare d and D without specifying the geometric structure of $\mathbb{X}$. For example, when G is complex, we have $d=D$; but when $\mathbb{X}$ has normal real form, we have $d=\ell+|\Sigma_r^+|$ which is strictly smaller than D. Since we focus on the higher rank analysis, we may assume that $d\ge3$.

Let $\mathfrak{n}$ be the nilpotent Lie subalgebra of $\mathfrak{g}$ associated to Σ^+ and let $N=\exp\mathfrak{n}$ be the corresponding Lie subgroup of G. We have the decompositions

$$\begin{cases} G=N\,(\exp\mathfrak{a})\,K & \text{(Iwasawa)},\\ G=K(\exp\overline{\mathfrak{a}^+})K & \text{(Cartan)}.\end{cases}$$

In the Cartan decomposition, the Haar measure on G writes

$$\int_G f(x)dx = \text{const.} \int_K dk_1 \int_{\mathfrak{a}^+} dx^+\ \delta(x^+) \int_K dk_2\ f(k_1(\exp x^+)k_2),$$

with

$$\delta(x^+) = \prod_{\alpha\in\Sigma^+} \big(\sinh\alpha(x^+)\big)^{m_\alpha} \asymp \Big\{ \prod_{\alpha\in\Sigma^+} \frac{\langle\alpha,x^+\rangle}{1+\langle\alpha,x^+\rangle} \Big\}^{m_\alpha} e^{\langle 2\rho,x^+\rangle} \quad \forall\, x^+\in\overline{\mathfrak{a}^+}.$$

Here $\rho\in\mathfrak{a}^+$ denotes the half sum of all positive roots $\alpha\in\Sigma^+$ counted with their multiplicities m_α:

$$\rho = \tfrac{1}{2}\textstyle\sum_{\alpha\in\Sigma^+} m_\alpha\alpha.$$

2.2. **Spherical Fourier analysis on $\mathbb{X}$.** Let $\mathcal{S}(K\backslash G/K)$ be the Schwartz space of K-bi-invariant functions on G. The spherical Fourier transform $\mathcal{H}$ is defined by

$$\mathcal{H}f(\lambda) = \int_G dx\ \varphi_{-\lambda}(x)f(x) \quad \forall\,\lambda\in\mathfrak{a},\ \forall\, f\in\mathcal{S}(K\backslash G/K),$$

where $\varphi_\lambda \in \mathcal{C}^\infty(K\backslash G/K)$ denotes the spherical function of index $\lambda\in\mathfrak{a}_{\mathbb{C}}$, which is a smooth K-bi-invariant eigenfunction for all invariant differential operators on $\mathbb{X}$, in particular for the Laplace-Beltrami operator:

$$-\Delta\varphi_\lambda(x) = \big(|\lambda|^2+|\rho|^2\big)\,\varphi_\lambda(x).$$

In the noncompact case, spherical functions have the integral representation

$$\varphi_\lambda(x) = \int_K dk\ e^{\langle i\lambda+\rho, A(kx)\rangle} \quad \forall\,\lambda\in\mathfrak{a}_{\mathbb{C}}, \tag{2.2}$$

where $A(kx)$ denotes the $\mathfrak{a}$-component in the Iwasawa decomposition of kx. It satisfies the basic estimate

$$|\varphi_\lambda(x)| \le \varphi_0(x) \quad \forall\,\lambda\in\mathfrak{a},\ \forall\, x\in G,$$

where

$$\varphi_0(\exp x^+) \asymp \Big\{ \prod_{\alpha\in\Sigma_r^+} 1+\langle\alpha,x^+\rangle \Big\} e^{-\langle\rho,x^+\rangle} \quad \forall\, x^+\in\overline{\mathfrak{a}^+}.$$

Denote by $\mathcal{S}(\mathfrak{a})^W$ the subspace of W-invariant functions in the Schwartz space $\mathcal{S}(\mathfrak{a})$. Then $\mathcal{H}$ is an isomorphism between $\mathcal{S}(K\backslash G/K)$ and $\mathcal{S}(\mathfrak{a})^W$. The inverse spherical Fourier transform is given by

$$f(x) = C_0 \int_{\mathfrak{a}} d\lambda\ |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x)\,\mathcal{H}f(\lambda) \quad \forall\, x\in G,\ \forall\, f\in\mathcal{S}(\mathfrak{a})^W,$$

where $C_0>0$ is a constant depending only on the geometric structure of $\mathbb{X}$, and which has been computed explicitly for instance in [2, Theorem 2.2.2]. By using the Gindikin & Karpelevič formula of the Harish-Chandra $\mathbf{c}$-function (see [19] or [13]), we can write the Plancherel density as

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha\in\Sigma_r^+} |\mathbf{c}_\alpha(\langle\alpha,\lambda\rangle)|^{-2}, \tag{2.3}$$

with

$$\mathbf{c}_\alpha(v) = \frac{\Gamma(\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle}+\frac12 m_\alpha)}{\Gamma(\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle})}\frac{\Gamma(\frac12\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle}+\frac14 m_\alpha+\frac12 m_{2\alpha})}{\Gamma(\frac12\frac{\langle\alpha,\rho\rangle}{\langle\alpha,\alpha\rangle}+\frac14 m_\alpha)}\frac{\Gamma(iv)}{\Gamma(iv+\frac12 m_\alpha)}\frac{\Gamma(\frac{i}{2}v+\frac14 m_\alpha)}{\Gamma(\frac{i}{2}v+\frac14 m_\alpha+\frac12 m_{2\alpha})}.$$

Since $|\mathbf{c}_\alpha|^{-2}$ is a homogeneous symbol on $\mathbb{R}$ of order $m_\alpha + m_{2\alpha}$ for every $\alpha \in \Sigma_r^+$, then $|\mathbf{c}(\lambda)|^{-2}$ is a product of one-dimensional symbols, but not a symbol on $\mathfrak{a}$ in general. The Plancherel density satisfies

$$|\mathbf{c}(\lambda)|^{-2} \asymp \prod_{\alpha\in\Sigma_r^+} \langle\alpha,\lambda\rangle^2(1+|\langle\alpha,\lambda\rangle|)^{m_\alpha+m_{2\alpha}-2} \lesssim \begin{cases} |\lambda|^{D-\ell} & \text{if } |\lambda|\le 1,\\ |\lambda|^{d-\ell} & \text{if } |\lambda|\ge 1,\end{cases}$$

together with all its derivatives.

2.3. **Barycentric decomposition of the Weyl chamber.** Let $\Sigma_s^+ = \{\alpha_1,\dots,\alpha_\ell\}$ be the set of positive simple roots, and let $\{\Lambda_1,\dots,\Lambda_\ell\}$ be the dual basis of $\mathfrak{a}$, which is defined by

$$\langle\alpha_j,\Lambda_k\rangle = \delta_{jk} \quad \forall\, 1\le j,k\le\ell. \tag{2.4}$$

Notice that $\overline{\mathfrak{a}^+} = \mathbb{R}^+\Lambda_1 + \dots + \mathbb{R}^+\Lambda_\ell$ and recall that

$$\begin{cases} \langle\alpha_j,\alpha_k\rangle \le 0 & \forall\, 1\le j\ne k\le\ell\\ \langle\Lambda_j,\Lambda_k\rangle \ge 0 & \forall\, 1\le j,k\le\ell\end{cases} \tag{2.5}$$

(see [18, Chap.VII, Lemmas 2.18 and 2.25], see also [26, p.590]). Let $\mathfrak{B}$ be the convex hull of $W.\Lambda_1 \sqcup \dots \sqcup W.\Lambda_\ell$, and let $\mathfrak{S}$ be its polyhedral boundary. Notice that $\mathfrak{B}\cap\overline{\mathfrak{a}^+}$ is the ℓ-simplex with vertices $0,\Lambda_1,\dots,\Lambda_\ell$, and $\mathfrak{S}\cap\overline{\mathfrak{a}^+}$ is the $(\ell-1)$-simplex with vertices $\Lambda_1,\dots,\Lambda_\ell$. The following tiling is obtained by regrouping the barycentric subdivisions of the simplices $\mathfrak{S}\cap\overline{w.\mathfrak{a}^+}$:

$$\mathfrak{S} = \bigcup_{w\in W}\bigcup_{1\le j\le\ell} w.\mathfrak{S}_j \tag{2.6}$$

where

$$\mathfrak{S}_j = \{\lambda\in\mathfrak{S}\cap\overline{\mathfrak{a}^+} \mid \langle\alpha_j,\lambda\rangle = \max_{1\le j\le\ell}\langle\alpha_k,\lambda\rangle\}.$$

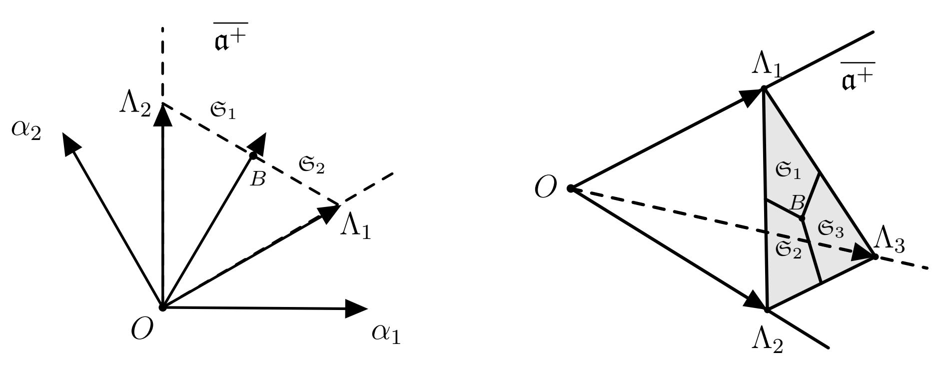

FIGURE 1. Examples of barycentric subdivisions in A_2 and in A_3.

Remark 2.1. *$\mathfrak{S}_j$ is the convex hull of the points*

$$\frac{\Lambda_{k_1}+\dots+\Lambda_{k_r}}{r}$$

where $\{\Lambda_{k_1},\dots,\Lambda_{k_r}\}$ runs through all subsets of $\{\Lambda_1,\dots,\Lambda_\ell\}$ containing Λ_j.

Lemma 2.2. *Let $w\in W$ and $1\le j\le \ell$. Then*

(i) a root $\alpha\in\Sigma$ is orthogonal to some vectors in the tile $w.\mathfrak{S}_j$ if and only if α is orthogonal to its vertex $w.\lambda_j$.

(ii) $\langle w.\Lambda_j,\lambda\rangle\ge\frac{1}{\ell}|\Lambda_j|^2$ for every $\lambda\in w.\mathfrak{S}_j$.

Proof. (i) Let us show that $\langle\alpha,w.\Lambda_j\rangle=0$ if there exists $\lambda\in w.\mathfrak{S}_j$ such that $\langle\alpha,\lambda\rangle=0$. By symmetry, we may assume that $w=\mathrm{id}$ and that α is a positive root. On the one hand, since α is spanned by the positive simple roots $\alpha_1,\dots,\alpha_\ell$, we have

$$\alpha=\sum_{1\le k\le\ell}\langle\alpha,\Lambda_k\rangle\alpha_k$$

with $\langle\alpha,\Lambda_k\rangle\in\mathbb{N}$. On the other hand, since $\langle\alpha_1,\lambda\rangle,\dots,\langle\alpha_\ell,\lambda\rangle$ are the barycentric coordinates of $\lambda\in\mathfrak{S}\cap\overline{\mathfrak{a}^+}$, we have

$$\lambda=\sum_{1\le k\le\ell}\langle\alpha_k,\lambda\rangle\Lambda_k \tag{2.7}$$

which is a convex combination. In particular, $\langle\alpha_j,\lambda\rangle>0$ for all $\lambda\in\mathfrak{S}_j$. Hence the inner product

$$\langle\alpha,\lambda\rangle=\sum_{1\le k\le\ell}\underbrace{\langle\alpha,\Lambda_k\rangle}_{\ge0}\underbrace{\langle\alpha_k,\lambda\rangle}_{\ge0}\underbrace{\langle\alpha_k,\Lambda_k\rangle}_{=1}$$

cannot vanish unless $\langle\alpha,\Lambda_j\rangle=0$.

(ii) By symmetry, we may assume again that $w=\mathrm{id}$. By taking the inner product of Λ_j with (2.7), we obtain

$$\langle\Lambda_j,\lambda\rangle=\sum_{1\le k\le\ell}\langle\Lambda_j,\Lambda_k\rangle\langle\alpha_k,\lambda\rangle=|\Lambda_j|^2\underbrace{\langle\alpha_j,\lambda\rangle}_{\ge\frac{1}{\ell}}+\sum_{k\ne j}\underbrace{\langle\Lambda_j,\Lambda_k\rangle}_{\ge0}\underbrace{\langle\alpha_k,\lambda\rangle}_{\ge0}\ge\frac{1}{\ell}|\Lambda_j|^2,$$

according to the property (2.5), and the fact that $\langle\alpha_j,\lambda\rangle$ is the largest barycentric coordinates for $\lambda\in\mathfrak{S}_j$. □

Now, consider the tiling of the unit sphere obtained by projecting (2.6):

$$S(\mathfrak{a})=\bigcup_{w\in W}\bigcup_{1\le j\le\ell}w.S_j$$

where S_j are the projections of the barycentric subdivisions $\mathfrak{S}_j$ on the unit sphere.

We establish next a smooth version of the partition of unity

$$\sum_{w\in W}\sum_{1\le j\le\ell}\mathbf{1}_{w.S_j}\big(\tfrac{\lambda}{|\lambda|}\big)=1\quad\text{a.e..}$$

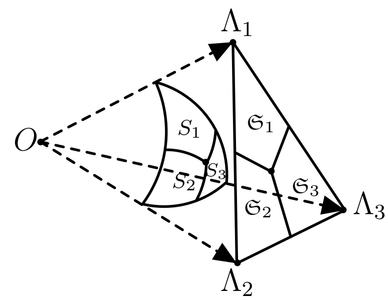

FIGURE 2. Example of the projection in A_3

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\chi(r) = 1$ when $r \geq 0$ and $\chi(r) = 0$ when $r \leq -c_1$, where $c_1 > 0$ will be specified in Remark 2.5. For every $w \in W$ and $1 \leq j \leq \ell$, we define

$$\widetilde{\chi}_{w.S_j}(\lambda) = \prod_{1\leq k\leq \ell, k\neq j} \chi\Big(\frac{\langle w.\alpha_k, \lambda\rangle}{|\lambda|}\Big)\chi\Big(\frac{\langle w.\alpha_j, \lambda\rangle - \langle w.\alpha_k, \lambda\rangle}{|\lambda|}\Big) \quad \forall\, \lambda \in \mathfrak{a} \smallsetminus \{0\},$$

and

$$\widetilde{\chi} = \sum_{w\in W} \sum_{1\leq j\leq \ell} \widetilde{\chi}_{w.S_j},$$

which satisfy the following properties.

Proposition 2.3. *Let $w \in W$ and $1 \leq j \leq \ell$. For all $\lambda \in \mathfrak{a} \smallsetminus \{0\}$, we have*

(i) $\widetilde{\chi}_{w.S_j}(w.\lambda) = \widetilde{\chi}_{S_j}(\lambda)$ *and* $\widetilde{\chi}$ *is W-invariant.*

(ii) $\widetilde{\chi}_{w.S_j} = 1$ *on* $w.S_j$ *and* $\widetilde{\chi} \geq 1$ *on* $\mathfrak{a} \smallsetminus \{0\}$.

(iii) $\widetilde{\chi}_{w.S_j}$ *and* $\widetilde{\chi}$ *are homogeneous symbols of order* 0.

Proof. *(i)* follows from immediately from the definitions. In order to prove *(ii)*, we may assume that $w = \mathrm{id}$ by symmetry. For all $\lambda \in S_j$, we have

$$\langle \alpha_k, \lambda\rangle \geq 0 \quad \text{and} \quad \langle \alpha_j, \lambda\rangle \geq \langle \alpha_k, \lambda\rangle$$

for every $1 \leq k \leq \ell$ with $k \neq j$, hence $\widetilde{\chi}_{S_j}(\lambda) = 1$. We deduce straightforwardly that $\widetilde{\chi} \geq 1$ on $\mathfrak{a} \smallsetminus \{0\}$. *(iii)* is obvious, since $\chi\big(\frac{\langle w.\alpha_k,\lambda\rangle}{|\lambda|}\big)$ and $\chi\big(\frac{\langle w.\alpha_j,\lambda\rangle - \langle w.\alpha_k,\lambda\rangle}{|\lambda|}\big)$ are homogeneous symbols of order 0 for all $\lambda \in \mathfrak{a} \smallsetminus \{0\}$ and $1 \leq k \leq \ell$. □

For every $w \in W$ and $1 \leq j \leq \ell$, we set

$$\chi_{w.S_j} = \frac{\widetilde{\chi}_{w.S_j}}{\widetilde{\chi}}$$

on $\mathfrak{a} \smallsetminus \{0\}$. It follows from Proposition 2.3 that $\chi_{w.S_j}(w.\lambda) = \chi_{S_j}(\lambda)$ and that $\chi_{w.S_j}$ is a homogeneous symbol of order 0. In particular, we have

$$\sum_{w\in W} \sum_{1\leq j\leq \ell} \chi_{w.S_j} = 1 \quad \text{on} \quad \mathfrak{a} \smallsetminus \{0\}. \tag{2.8}$$

In addition, the vectors in the support of $\chi_{w.S_j}$ satisfy further properties, which require some preliminaries.

Lemma 2.4. *There exists $c_2 > 0$ such that, if $\lambda \in \mathfrak{a}$ satisfies*

$$-c_2|\lambda| \leq \langle \alpha_k, \lambda \rangle \leq \langle \alpha_j, \lambda \rangle + c_2|\lambda| \quad \forall\, k \in \{1, \ldots, \ell\} \smallsetminus \{j\},$$

for some $1 \leq j \leq \ell$, then $\langle \alpha_j, \lambda \rangle \geq c_2|\lambda|$.

Proof. By homogeneity, we may reduce to $|\lambda| = 1$. Since all norms are equivalent on $\mathfrak{a}$, there exists $c_3 > 0$ such that

$$\sum_{1 \leq j \leq \ell} |\langle \alpha_k, \lambda \rangle| \geq c_3 \quad \forall\, \lambda \in S(\mathfrak{a}). \tag{2.9}$$

Set $c_2 = \frac{c_3}{2\ell}$. On the one hand, if

$$-c_2 \leq \langle \alpha_k, \lambda \rangle \leq 2c_2 \quad \forall\, k \in \{1, \ldots, \ell\} \smallsetminus \{j\},$$

then $\langle \alpha_j, \lambda \rangle \geq 2c_2$. Otherwise,

$$\sum_{1 \leq j \leq \ell} |\langle \alpha_k, \lambda \rangle| = \underbrace{|\langle \alpha_j, \lambda \rangle|}_{<2c_2} + \sum_{k \neq j} \underbrace{|\langle \alpha_k, \lambda \rangle|}_{\leq 2c_2} < 2\ell c_2 = c_3,$$

which contradicts (2.9). On the other hand, if

$$2c_2 \leq \langle \alpha_k, \lambda \rangle \leq \langle \alpha_j, \lambda \rangle + c_2$$

for some $k \in \{1, \ldots, \ell\} \smallsetminus \{j\}$, then $\langle \alpha_j, \lambda \rangle \geq c_2$ is obvious. □

Remark 2.5. *We clarify in this remark all constants appearing in this subsection. Denote by L_1 the highest root length and by L_2 the sum of lengths of the dual basis*

$$L_1 = \max_{\alpha \in \Sigma^+} \sum_{1 \leq k \leq \ell} \langle \alpha, \Lambda_k \rangle \quad \text{and} \quad L_2 = \sum_{1 \leq k \leq \ell} |\Lambda_k|.$$

In addition, we denote by M_1 and M_2 the shortest and the longest generators

$$M_1 = \min_{1 \leq k \leq \ell} |\Lambda_k| \quad \text{and} \quad M_2 = \max_{1 \leq k \leq \ell} |\Lambda_k|.$$

Then we choose $c_1 > 0$ such that $c_1 < c_2 \min\{\frac{1}{L_1}, \frac{M_1^2}{M_2 L_2}\}$, where $c_2 = \frac{c_3}{2\ell}$ with c_3 defined in (2.9). Let $c_4 = c_2 - L_1 c_1$ and $c_5 = M_1^2 c_2 - M_2 L_2 c_1$. Notice that $L_1 \in \mathbb{N}^*$, $c_1 < c_2$, $c_4 > 0$ and $c_5 > 0$. All these constants depend only on the geometric structure of the roots system corresponding to $\mathbb{X}$.

The following result is an analog of Lemma 2.2 for the wider regions $\operatorname{supp} \chi_{\omega.S_j}$.

Proposition 2.6. *Let $w \in W$ and $1 \leq j \leq \ell$. Then*

(i) a root $\alpha \in \Sigma$ satisfies either $\langle \alpha, w.\lambda_j \rangle = 0$ or

$$|\langle \alpha, \lambda \rangle| \geq c_4|\lambda| \quad \forall\, \lambda \in \operatorname{supp} \chi_{w.S_j}, \tag{2.10}$$

(ii) $|\langle w.\Lambda_j, \lambda \rangle| \geq c_5|\lambda|$ for every $\lambda \in \operatorname{supp} \chi_{w.S_j}$.

Proof. *(i)* By symmetry, we may assume that $w = \mathrm{id}$ and that α is a positive root. Notice that $\langle \alpha, \Lambda_j \rangle$ is a nonnegative integer, we suppose that $\langle \alpha, \Lambda_j \rangle > 0$ and let us prove (2.10). As

$$-c_1|\lambda| \le \langle \alpha_k, \lambda \rangle \le \langle \alpha_j, \lambda \rangle + c_1|\lambda| \quad \forall\, \lambda \in \operatorname{supp} \chi_{S_j},\ \forall\, k \in \{1, \dots, \ell\} \smallsetminus \{j\},$$

we have indeed

$$\begin{aligned} \langle \alpha, \lambda \rangle &= \sum_{1 \le k \le \ell} \langle \alpha, \Lambda_k \rangle \langle \alpha_k, \lambda \rangle \\ &= \underbrace{\langle \alpha, \Lambda_j \rangle}_{\ge 1} \underbrace{\langle \alpha_j, \lambda \rangle}_{\ge c_2|\lambda|} + \sum_{k \ne j} \langle \alpha, \Lambda_k \rangle \underbrace{\langle \alpha_k, \lambda \rangle}_{\ge -c_1|\lambda|} \ge (c_2 - L_1 c_1)|\lambda| = c_4|\lambda|, \end{aligned}$$

according to Lemma 2.4 since $c_1 < c_2$.

(ii) By symmetry, we assume again $w = \mathrm{id}$. By taking the inner product of Λ_j with (2.7), we obtain, for every $\lambda \in \operatorname{supp} \chi_{S_j}$,

$$\begin{aligned} \langle \Lambda_j, \lambda \rangle &= \sum_{1 \le k \le \ell} \langle \Lambda_j, \Lambda_k \rangle \langle \alpha_k, \lambda \rangle = \underbrace{|\Lambda_j|^2}_{\ge M_1^2} \underbrace{\langle \alpha_j, \lambda \rangle}_{\ge c_2|\lambda|} + \sum_{k \ne j} \underbrace{\langle \Lambda_j, \Lambda_k \rangle}_{\le |\Lambda_j||\Lambda_k|} \underbrace{\langle \alpha_k, \lambda \rangle}_{\ge -c_1|\lambda|} \\ &\ge (M_1^2 c_2 - M_2 L_2 c_1)|\lambda| = c_5|\lambda|. \end{aligned}$$

□

Remark 2.7. *The partition of unity* (2.8) *plays an important role in the kernel analysis carried out in Sect. 3. It allows us to overcome a well-known problem in spherical Fourier analysis in higher rank, namely the fact that the Plancherel density is not a symbol in general. This new tool should certainly help solving other problems.*

3. Pointwise estimates of the wave kernel

In this section, we derive pointwise estimates for the K-bi-invariant convolution kernel ω_t^σ of the operator $W_t^\sigma = (-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}}$ on the symmetric space $\mathbb{X}$:

$$W_t^\sigma f(x) = f * \omega_t^\sigma(x) = \int_G dy\ \omega_t^\sigma(y^{-1}x) f(y)$$

for suitable exponents $\sigma \in \mathbb{C}$. By using the inverse formula of the spherical Fourier transform, we have

$$\omega_t^\sigma(x) = C_0 \int_{\mathfrak{a}} d\lambda\ |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) (|\lambda|^2 + |\rho|^2)^{-\frac{\sigma}{2}} e^{it\sqrt{|\lambda|^2 + |\rho|^2}}$$

Let us point out that the analysis of this oscillatory integral carried out on hyperbolic spaces or on symmetric spaces G/K with G complex (see [3, 38]) does not hold in general since the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$ is no more a differential symbol. We write

$$\omega_t^\sigma(x) = \frac{1}{\Gamma(\sigma)} \int_0^{+\infty} ds\ s^{\sigma-1} \underbrace{C_0 \int_{\mathfrak{a}} d\lambda\ |\mathbf{c}(\lambda)|^{-2} \varphi_\lambda(x) e^{-(s-it)\sqrt{|\lambda|^2 + |\rho|^2}}}_{p_{s-it}(x)}.$$

according to the formula

$$r^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^{+\infty} \frac{ds}{s}\, s^{\sigma} e^{-sr} \quad \forall\, r > 0.$$

Here p_{s-it} is the K-bi-invariant convolution kernel of the Poisson operator $\mathcal{P}_{s-it}$. Let us split up $\omega_t^{\sigma}(x) = \omega_t^{\sigma,0}(x) + \omega_t^{\sigma,\infty}(x)$ with

$$\omega_t^{\sigma,0}(x) = \frac{1}{\Gamma(\sigma)} \int_0^1 ds\ s^{\sigma-1} p_{s-it}(x)$$

and

$$\omega_t^{\sigma,\infty}(x) = \frac{1}{\Gamma(\sigma)} \int_1^{+\infty} ds\ s^{\sigma-1} p_{s-it}(x).$$

We shall see in Sect. 3.2 that the kernel $\omega_t^{\sigma,0}(x)$ has a logarithmic singularity on the sphere $|x| = t$ when $\sigma = \frac{d+1}{2}$. To bypass this problem, we consider the analytic family of operators

$$\widetilde{W}_t^{\sigma,0} = \underbrace{\frac{e^{\sigma^2}}{\Gamma(\frac{d+1}{2} - \sigma)\Gamma(\sigma)}}_{C_{\sigma,d}} \int_0^1 ds\ s^{\sigma-1} \mathcal{P}_{s-it} \tag{3.1}$$

in the vertical strip $0 \le \operatorname{Re}\sigma \le \frac{d+1}{2}$ and the corresponding kernels

$$\widetilde{\omega}_t^{\sigma,0}(x) = C_{\sigma,d} \int_0^1 ds\ s^{\sigma-1} p_{s-it}(x) \quad \forall\, x \in \mathbb{X}.$$

Notice that the Gamma function $\Gamma(\frac{d+1}{2} - \sigma)$ allows us to deal with the boundary point $\sigma = \frac{d+1}{2}$, while the exponential function ensures boundedness at infinity in the vertical strip. More precisely, by using the inequality

$$|\Gamma(z)| \ge \Gamma(\operatorname{Re} z) \big(\cosh(\pi \operatorname{Im} z)\big)^{-\frac{1}{2}} \quad \forall\, z \in \mathbb{C} \text{ with } \operatorname{Re} z \ge \tfrac{1}{2}$$

(see for instance [30, Eq. 5.6.7]), we can estimate

$$|C_{\sigma,d}| \lesssim |\sigma|\, |\sigma - \tfrac{d+1}{2}|\, e^{\pi\, |\operatorname{Im}\sigma| - (\operatorname{Im}\sigma)^2} \tag{3.2}$$

for all $\sigma \in \mathbb{C}$ with $0 \le \operatorname{Re}\sigma \le \frac{d+1}{2}$.

We divide the argument into three parts depending whether $|t|$ and $\frac{|x|}{|t|}$ are small or large. When $|t|$ is large but $\frac{|x|}{|t|}$ is sufficiently small, we estimate $\widetilde{\omega}_t^{\sigma,0}$ in Theorem 3.3 by combining the method of stationary phase with our barycentric decomposition of Weyl chambers; when $\frac{|x|}{|t|}$ is large, we estimate $\widetilde{\omega}_t^{\sigma,0}$ in Theorem 3.7 by using the Hadamard parametrix along the lines of [9]; $\omega_t^{\sigma,\infty}(x)$ is easily handled by a standard stationary phase argument, see Theorem 3.10.

3.1. **Estimates of $\widetilde{\omega}_t^{\sigma,0}(x)$ when $|t|$ is large and $\frac{|x|}{|t|}$ is sufficiently small.** According to the integral expression (2.2) of the spherical functions, we write

$$\widetilde{\omega}_t^{\sigma,0}(x) = C_{\sigma,d}\, C_0 \int_K dk\, e^{\langle\rho, A(kx)\rangle} \int_0^1 ds\, s^{\sigma-1} I(s,t,x),$$

where

$$I(s,t,x) = \int_{\mathfrak{a}} d\lambda\ |\mathbf{c}(\lambda)|^{-2} e^{-s\sqrt{|\lambda|^2+|\rho|^2}} e^{it\psi_t(\lambda)}$$

is an oscillatory integral with phase

$$\psi_t(\lambda) = \sqrt{|\lambda|^2+|\rho|^2} + \langle \tfrac{A(kx)}{t}, \lambda\rangle. \tag{3.3}$$

Let us split up

$$I(s,t,x) = I^-(s,t,x) + I^+(s,t,x) = \int_{\mathfrak{a}} d\lambda\ \chi_0^\rho(\lambda)\cdots\ +\ \int_{\mathfrak{a}} d\lambda\ \chi_\infty^\rho(\lambda)\cdots$$

by using smooth radial cut-off functions χ_0^ρ and $\chi_\infty^\rho = 1-\chi_0^\rho$, where $\chi_0^\rho(\lambda)$ equals 1 when $|\lambda|\le|\rho|$ and vanishes if $|\lambda|\ge 2|\rho|$. Then we have the following estimates for I^- and I^+.

Proposition 3.1. *There exists $0 < C_\Sigma \le \frac{1}{2}$ such that the following estimates hold when $0 < s < 1$, $|t|\ge 1$ and $\frac{|x|}{|t|}\le C_\Sigma$:*

$$|I^-(s,t,x)| \lesssim |t|^{-\frac{D}{2}}\, (1+|x|)^{\frac{D-\ell}{2}}, \tag{3.4}$$

and

$$|I^+(s,t,x)| \lesssim |t|^{-N}, \tag{3.5}$$

for every $N\in\mathbb{N}$.

Remark 3.2. *C_Σ is a small constant depending only on the geometric structure of the root system Σ, which will be specified later in the proof of* (3.5)*. Notice that the upper bounds* (3.4) *of I^- and* (3.5) *of I^+ hold uniformly in $s\in(0,1)$.*

Proof of the estimate (3.4). Recall that

$$I^-(s,t,x) = \int_{\mathfrak{a}} d\lambda\ a_0(s,\lambda) e^{it\psi_t(\lambda)}$$

is an oscillatory integral with amplitude

$$a_0(s,\lambda) = \chi_0^\rho(\lambda)|\mathbf{c}(\lambda)|^{-2} e^{-s\sqrt{|\lambda|^2+|\rho|^2}}$$

and phase $\psi_t(\lambda)$ which is defined by (3.3). The amplitude $a_0(s,\lambda)$ is compactly supported for $|\lambda|\le 2|\rho|$, and the phase ψ_t has, in the support of χ_0^ρ, a single nondegenerate critical point λ_0 which is given by

$$(|\lambda_0|^2+|\rho|^2)^{-\frac{1}{2}}\lambda_0 = -\tfrac{A}{t} \tag{3.6}$$

where $A = A(kx)$, and which satisfies

$$|\lambda_0| = |\rho|\tfrac{|A|}{|t|}(1-\tfrac{|A|^2}{t^2})^{-\frac{1}{2}} \le |\rho|\tfrac{|x|}{t}(1-\tfrac{|x|^2}{t^2})^{-\frac{1}{2}} < \tfrac{|\rho|}{\sqrt{3}}, \tag{3.7}$$

as $|A| \le |x|$ and $\frac{|x|}{t} \le C_\Sigma < \frac{1}{2}$. We conclude by resuming straightforwardly the computations carried out in the proof of [38, Theorem 3.1]. For the sake of completeness and for the reader's convenience, we include a detailed study of the oscillatory integral I^- in Appendix A (see Lemma A.1). □

Let us turn to the oscillatory integral

$$I^+(s,t,x) = \int_{\mathfrak{a}} d\lambda\ \chi_\infty^\rho(\lambda) |\mathbf{c}(\lambda)|^{-2} e^{-s\sqrt{|\lambda|^2+|\rho|^2}} e^{it\psi_t(\lambda)},$$

which vanishes unless $|\lambda| > |\rho|$. According to (3.7), ψ_t has no critical point in the support of χ_∞^ρ. In rank one or in higher rank with G complex, one can handle this integral by performing several integrations by parts. This approach fails in general since the Plancherel density $|\mathbf{c}(\lambda)|^{-2}$ is not a differential symbol. To get around this problem, we split up the Weyl chamber according to the barycentric decomposition carried out in Sect. 2.3, and perform integrations by parts based along a well chosen directional derivative in each component.

Proof of the estimate (3.5). According to the partition of unity (2.8), we split up

$$I^+(s,t,x) = \sum_{w\in W} \sum_{1\le j\le \ell} I_{w.S_j}(i\tau, x)$$

with $\tau = s - it$, and we estimate

$$I_{w.S_j}(i\tau, x) = \int_{\mathfrak{a}} d\lambda\, \chi_{w.S_j}(\lambda)\, \chi_\infty^\rho(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, e^{-\tau\psi_{i\tau}(\lambda)} \tag{3.8}$$

by performing integrations by parts based on

$$e^{-\tau\psi_{i\tau}(\lambda)} = -\tfrac{1}{\tau} \tfrac{1}{\partial_{w.\Lambda_j}\psi_{i\tau}(\lambda)}\, \partial_{w.\Lambda_j} e^{-\tau\psi_{i\tau}(\lambda)}. \tag{3.9}$$

Notice that

$$\partial_{w.\Lambda_j}\psi_{i\tau}(\lambda) = \langle w.\Lambda_j, \tfrac{\lambda}{\sqrt{|\lambda|^2+|\rho|^2}} - i\tfrac{A(kx)}{\tau}\rangle$$

is a symbol of order 0, which satisfies in addition

$$\begin{aligned} |\partial_{w.\Lambda_j}\psi_{i\tau}(\lambda)| &\ge \tfrac{|\langle w.\Lambda_j, \lambda\rangle|}{\sqrt{|\lambda|^2+|\rho|^2}} - \big|\langle w.\Lambda_j, \tfrac{A(kx)}{\tau}\rangle\big| \\ &\ge c_5 \tfrac{|\lambda|}{\sqrt{|\lambda|^2+|\rho|^2}} - |\Lambda_j| \tfrac{|A(kx)|}{|\tau|} \ge \tfrac{c_5}{\sqrt{2}} - M_2 \tfrac{|x|}{|t|} \end{aligned}$$

on $(\operatorname{supp}\chi_{w.S_j}) \cap (\operatorname{supp}\chi_\infty^\rho)$ according to Proposition (2.6), where the constants c_5 and M_2 are specified in Remark 2.5. By choosing $C_\Sigma = \min\{\frac{c_5}{2M_2}, \frac{1}{2}\}$, we obtain

$$|\partial_{w.\Lambda_j}\psi_{i\tau}(\lambda)| \ge \tfrac{\sqrt{2}-1}{2} c_5 > 0.$$

Let us return to (3.8), which becomes

$$\begin{aligned} I_{w.S_j}(i\tau, x) = \tau^{-N} &\int_{\mathfrak{a}} d\lambda\ e^{-\tau\psi_{i\tau}(\lambda)} \\ &\times \big\{\partial_{w.\Lambda_j} \circ \tfrac{1}{\partial_{w.\Lambda_j}\psi_{i\tau}(\lambda)}\big\}^N \big\{\chi_{w.S_j}(\lambda)\chi_\infty^\rho(\lambda)|\mathbf{c}(\lambda)|^{-2}\big\}, \end{aligned}$$

after N integrations by parts based on (3.9). If some derivatives hit $\chi_\infty^\rho(\lambda)$, the above integral is reduced to the spherical shell $|\lambda| \asymp |\rho|$ and thus converges. Assume that no derivative is applied to $\chi_\infty^\rho(\lambda)$ and that

- N_1 derivatives are applied to the factors $\frac{1}{\partial_{w.\Lambda_j}\psi_{i\tau}(\lambda)}$,
- N_2 derivatives are applied to $\chi_{w.S_j}(\lambda)$,
- N_3 derivatives are applied to $|\mathbf{c}(\lambda)|^{-2}$,

with $N = N_1 + N_2 + N_3$. The contribution of the first item is $O(|\lambda|^{-N_1})$, as $\partial_{w.\Lambda_j}\psi_{i\tau}(\lambda)$ is a symbol or order 0, which stays away from 0. Similarly, the contribution of the second item is $O(|\lambda|^{-N_2})$, as $\chi_{w.S_j}(\lambda)$ is a symbol of order 0 according to Proposition 2.3. As far as the third item is concerned, the derivatives $(\partial_{w.\Lambda_j})^{N_3}$ are applied to the various factors in (2.3). According to Proposition 2.6, for every λ in the support of $\chi_{w.S_j}$, any root $\alpha \in \Sigma$ satisfies either $\langle \alpha, w.\Lambda_j \rangle = 0$ or $|\langle \alpha, \lambda \rangle| \gtrsim |\lambda|$. On the one hand, if $\langle \alpha, w.\Lambda_j \rangle = 0$, all derivatives

$$(\partial_{w.\Lambda_j})^{N_\alpha} |\mathbf{c}_\alpha(\langle \alpha, \lambda \rangle)|^{-2} \quad \forall\, N_\alpha \in \mathbb{N}^*$$

vanish. On the other hand, if $\langle \alpha, w.\Lambda_j \rangle \neq 0$, we use the fact that $|\mathbf{c}_\alpha|^{-2}$ is a symbol on $\mathbb{R}$ of order $m_\alpha + m_{2\alpha}$, together with (2.10), in order to estimate

$$\big|(\partial_{w.\Lambda_j})^{N_\alpha} |\mathbf{c}_\alpha(\langle \alpha, \lambda \rangle)|^{-2}\big| \lesssim |\langle \alpha, \lambda \rangle|^{m_\alpha + m_{2\alpha} - N_\alpha} \asymp |\lambda|^{m_\alpha + m_{2\alpha} - N_\alpha} \quad \forall\, N_\alpha \in \mathbb{N}^*$$

for $\lambda \in (\operatorname{supp} \chi_{w.S_j}) \cap (\operatorname{supp} \chi_\infty^\rho)$. Hence

$$(\partial_{w.\Lambda_j})^{N_3} |\mathbf{c}(\lambda)|^{-2} = O(|\lambda|^{d-\ell-N_3}) \quad \forall\, \lambda \in (\operatorname{supp} \chi_{w.S_j}) \cap (\operatorname{supp} \chi_\infty^\rho).$$

In conclusion,

$$|I_{w.S_j}(i\tau, x)| \lesssim |\tau|^{-N} \int_{\mathfrak{a}} d\lambda\, |\lambda|^{d-\ell-N_1-N_2-N_3} \lesssim |t|^{-N}$$

provided that $N > d$, and consequently

$$I^+(s,t,x) = O(|t|^{-N}).$$

□

We deduce from (3.4) and (3.5) that, for all $0 < s < 1$, $|t| \geq 1$ and $\frac{|x|}{|t|} \leq C_\Sigma$,

$$|I(s,t,x)| \lesssim |t|^{-\frac{D}{2}} \, (1 + |x|)^{\frac{D-\ell}{2}} \tag{3.10}$$

uniformly in s. Notice that

$$\frac{\partial}{\partial s} I(s,t,x) = -\int_{\mathfrak{a}} d\lambda\, |\mathbf{c}(\lambda)|^{-2} \sqrt{|\lambda|^2 + |\rho|^2}\, e^{-s\sqrt{|\lambda|^2+|\rho|^2}}\, e^{it\psi_t(\lambda)}$$

has the same phase as $I(s,t,x)$. Hence the estimate (3.10) holds for $\frac{\partial}{\partial s} I(s,t,x)$ by similar computations. Since

$$\int_0^1 ds\, s^{\sigma-1} I(s,t,x) = \big[\tfrac{1}{\sigma}\, s^\sigma\, I(s,t,x)\big]_0^1 - \tfrac{1}{\sigma} \int_0^1 ds\, s^\sigma\, \tfrac{\partial}{\partial s}\, I(s,t,x),$$

we deduce that

$$\Big|C_{\sigma,d}\int_0^1 ds\, s^{\sigma-1} I(s,t,x)\Big| \lesssim |t|^{-\frac{D}{2}}\,(1+|x|)^{\frac{D-\ell}{2}}$$

according to (3.2). Then we obtain the following kernel estimate of $\widetilde{\omega}_t^{\sigma,0}$.

Theorem 3.3. *There exists* $0 < C_\Sigma \le \frac{1}{2}$ *such that the following estimate holds, when* $|t| \ge 1$ *and* $\frac{|x|}{|t|} \le C_\Sigma$*, uniformly in the vertical strip* $0 \le \operatorname{Re}\sigma \le \frac{d+1}{2}$*:*

$$|\widetilde{\omega}_t^{\sigma,0}(x)| \lesssim |t|^{-\frac{D}{2}}(1+|x|)^{\frac{D-\ell}{2}}\varphi_0(x). \tag{3.11}$$

3.2. **Estimates of $\widetilde{\omega}_t^{\sigma,0}(x)$ in the remaining range.** Recall that $\tau = s - it$ with $t \in \mathbb{R}^*$ and $s \in (0,1)$ throughout this subsection. We are looking for pointwise estimates of

$$\widetilde{\omega}_t^{\sigma,0}(x) = C_{\sigma,d}\int_0^1 ds\ s^{\sigma-1}\, \underbrace{C_0\int_{\mathfrak{a}} d\lambda\ |\mathbf{c}(\lambda)|^{-2}\varphi_\lambda(x)\,\overbrace{e^{-\tau\sqrt{|\lambda|^2+|\rho|^2}}}^{\widetilde{p}_\tau(\lambda)}}_{p_\tau(x)} \quad \forall\, x \in \mathbb{X},$$

where $p_\tau(x)$ is the Poisson kernel and $\widetilde{p}_\tau(\lambda)$ denotes its spherical Fourier transform. This subsection focuses on pointwise estimates of p_τ along the lines of [9, pp.1054-1063].

Remark 3.4. *Notice that the Gaussian factor ensures the convergence of the integral defining* p_τ*, but yields a large negative power* s^{-d}*. Then* $\widetilde{\omega}_t^{\sigma,0}$ *converges under the strong smoothness assumption* $\operatorname{Re}\sigma \ge d$*. We will sharpen it to* $\operatorname{Re}\sigma = \frac{d+1}{2}$*. Notice that the stationary phase method carried out in the previous subsection fails since the critical point can be very large when* $\frac{|x|}{|t|}$ *is not bounded from above.*

As in [9], let us denote by $p_\tau^{\mathbb{R}}(v) = \frac{\tau}{\pi(\tau^2+v^2)}$ the Poisson kernel on $\mathbb{R}$. We may write

$$\widetilde{p}_\tau(\lambda) = e^{-\tau\sqrt{|\lambda|^2+|\rho|^2}} = \int_{\mathbb{R}} dv\ p_\tau^{\mathbb{R}}(v)\cos(v\sqrt{|\lambda|^2+|\rho|^2}) \quad \forall\lambda \in \mathfrak{a}.$$

Consider a smooth even cut-off function $\chi : \mathbb{R} \to [0,1]$, which is supported in $[-\sqrt{2},\sqrt{2}]$, and equals 1 on $[-1,1]$. Denote by $\chi_T = \chi(\frac{\cdot}{2T})$ with $T = \sqrt{2}$ (uniformly in t) when $|t| \le 1$ or $T = \sqrt{2}|t|$ when $|t| \ge 1$. Then χ_T is supported in $[-2\sqrt{2}T, 2\sqrt{2}T] \subset (-3T, 3T)$. We denote by a_τ and b_τ the K-bi-invariant kernels of operators

$$A_\tau = \int_{\infty}^{+\infty} dv\,\chi_T(v)\,p_\tau^{\mathbb{R}}(v)\ \cos(v\sqrt{-\Delta})$$

and

$$B_\tau = \int_{-\infty}^{\infty} dv\,\{1-\chi_T(v)\}\,p_\tau^{\mathbb{R}}(v)\ \cos(v\sqrt{-\Delta}).$$

Then $p_\tau = a_\tau + b_\tau$ and a_τ is supported in a ball of radius $3T$ in $\mathbb{X}$ by finite propagation speed. b_τ is easily estimated by straightforward computations, see Proposition 3.6. In order to analyze a_τ, we will expand $\cos(v\sqrt{-\Delta})$ by using Hadamard parametrix.

Let $\{R_+^z \mid z \in \mathbb{C}\}$ be the analytic family of Riesz distributions on $\mathbb{R}$ defined by

$$R_+^z(r) = \begin{cases} \Gamma(z)^{-1} r^{z-1} & \text{if } r > 0, \\ 0 & \text{if } r \leq 0, \end{cases}$$

for $\operatorname{Re} z > 0$. The K-bi-invariant convolution kernel Φ_v of the operator $\cos(v\sqrt{-\Delta})$ has the asymptotic expansion

$$\Phi_v(\exp H) = J(H)^{-\frac{1}{2}} \sum_{k=0}^{[d/2]} 4^{-k}\, |v|\, U_k(H)\, R_+^{k-\frac{d-1}{2}}(v^2 - |H|^2) + E_\Phi(v, H) \tag{3.12}$$

where

$$J(H) = \prod_{\alpha \in \Sigma^+} \Big(\frac{\sinh\langle \alpha, H\rangle}{\langle \alpha, H\rangle} \Big)^{m_\alpha}$$

denotes the Jacobian of the exponential map from $\mathfrak{p}$ equipped with Lebesgue measure to $\mathbb{X}$ equipped with Riemannian measure. Moreover, the coefficients satisfy

$$\nabla_{\mathfrak{p}}^n U_k = O(1) \tag{3.13}$$

for every $k, n \in \mathbb{N}$, and the remainder is estimated as

$$|E_\Phi(v, H)| \lesssim (1+v)^{3(\frac{d}{2}+1)} e^{-\langle \rho, H\rangle}. \tag{3.14}$$

The Hadamard parametrix has been described and applied in various settings, see for instance [7, 20, 9]. For the reader's convenience, we give in Appendix B some details about this construction in the particular case of noncompact symmetric spaces. By resuming the proof of Lemma 3.3 in [9] (see Appendix C for details), we deduce the following expansion of the K-bi-invariant convolution kernel a_τ of the operator A_τ:

$$a_\tau(\exp H) = \frac{\tau}{\pi}\, J(H)^{-\frac{1}{2}} \sum_{k=0}^{[d/2]} 4^{-k}\, U_k(H)\, \Gamma\big(\tfrac{d+1}{2} - k\big)\, (|H|^2 + \tau^2)^{k-\frac{d+1}{2}} + E(\tau, H) \tag{3.15}$$

where

$$|E(\tau, H)| \lesssim |T|^{3(\frac{d}{2}+1)}\, (\log T - \log s)\, e^{-\langle \rho, H\rangle} \quad \forall\, H \in \overline{\mathfrak{a}^+}. \tag{3.16}$$

Remark 3.5. *As a consequence, we may deduce that*

$$|a_\tau(\exp H)| \lesssim s^{-\frac{d+1}{2}}\, e^{-\langle \rho, H\rangle} \begin{cases} |t|^{-\frac{d-1}{2}} & \text{if } |t| \text{ is small}, \\ |t|^{3(\frac{d}{2}+1)} \log|t| & \text{if } |t| \text{ is large}, \end{cases}$$

for all $H \in \overline{\mathfrak{a}^+}$. However, we cannot apply straightforwardly such estimates to study the kernel ω_t^σ, since it kills the imaginary part of σ and yields a logarithmic singularity on the sphere $|x| = t$ when $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \frac{d+1}{2}$.

The following proposition concerning the estimate of b_τ will be proved by straightforward computations.

Proposition 3.6. *Let $N > d$ be an even integer. Then*

$$|b_\tau(x)| \lesssim (1+|t|)^{-N}\,\varphi_0(x) \tag{3.17}$$

for every $x \in \mathbb{X}$ and for every $\tau = s - it$ with $s \in (0,1]$ and $t \in \mathbb{R}^*$.

Proof. Let is study

$$B_\tau(\lambda) = \frac{2\tau}{\pi} \int_0^{+\infty} dv\,\{1-\chi_T(v)\}\,\frac{1}{\tau^2+v^2}\,\cos(v\sqrt{|\lambda|^2+|\rho|^2}),$$

which vanishes unless $v > 2T$. By performing N integrations by parts based on

$$\cos(v\sqrt{|\lambda|^2+|\rho|^2}) = -\frac{1}{|\lambda|^2+|\rho|^2}\,\frac{\partial^2}{\partial v^2}\,\cos(v\sqrt{|\lambda|^2+|\rho|^2}),$$

we obtain

$$\begin{aligned} B_\tau(\lambda) = {} & \frac{2\tau}{\pi}\,(-1)^{-\frac{N}{2}}\,(|\lambda|^2+|\rho|^2)^{-\frac{N}{2}} \\ & \times \int_0^{+\infty} dv\,\cos(v\sqrt{|\lambda|^2+|\rho|^2})\,\big(\tfrac{\partial}{\partial v}\big)^N\,\big(\tfrac{1-\chi_T(v)}{\tau^2+v^2}\big). \end{aligned}$$

Since $v > 2T$, we have $|\tau^2+v^2| \gtrsim v^2$ uniformly in $\tau = s - it$. Hence

$$B_\tau(\lambda) \lesssim |\tau|\,(|\lambda|^2+|\rho|^2)^{-\frac{N}{2}} \int_{2T}^{+\infty} dv\,v^{-2-N} \lesssim |T|^{-N}\,(|\lambda|^2+|\rho|^2)^{-\frac{N}{2}}.$$

By the inverse formula of the spherical Fourier transform, we deduce

$$\begin{aligned} |b_\tau(x)| &= \Big|\int_{\mathfrak{a}} d\lambda\,|\mathbf{c}(\lambda)|^{-2}\,\varphi_\lambda(x)\,B_\tau(\lambda)\Big| \\ &\lesssim |T|^{-N}\,\varphi_0(x) \int_{\mathfrak{a}} d\lambda\,|\mathbf{c}(\lambda)|^{-2}\,(|\lambda|^2+|\rho|^2)^{-\frac{N}{2}} \end{aligned}$$

where the last integral converges provided that $N > d$. ☐

According to the asymptotic expansion (3.15) of a_τ and to the estimate (3.17) of b_τ, we establish the pointwise estimates of $\widetilde{\omega}_t^{\sigma,0}$ in the case where $\frac{|x|}{|t|}$ is bounded from below.

Theorem 3.7. *Let $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \frac{d+1}{2}$. The following estimates hold for all $t \in \mathbb{R}^*$ and $x \in \mathbb{X}$.*

(i) If $0 < |t| < 1$, then

$$|\widetilde{\omega}_t^{\sigma,0}(x)| \lesssim |t|^{-\frac{d-1}{2}}\,(1+|x^+|)^{\frac{\max\{d,D\}-\ell}{2}}\,e^{-\langle\rho,x^+\rangle}.$$

(ii) If $|t| \ge 1$ and $\frac{|x|}{|t|} > C_\Sigma$, then

$$|\widetilde{\omega}_t^{\sigma,0}(x)| \lesssim |t|^{-N_1}\,(1+|x^+|)^{N_2}\,e^{-\langle\rho,x^+\rangle},$$

for every $N_1 \in \mathbb{N}$ and $N_2 \ge N_1 + 2(d+1) + \frac{\max\{d,D\}-\ell}{2}$.

Proof. Recall that we are looking for a pointwise estimate of the kernel

$$\widetilde{\omega}_t^{\sigma,0}(x) = C_{\sigma,d} \int_0^1 ds\, s^{\sigma-1}\, p_\tau(x),$$

where $\tau = s - it$ with $s \in (0,1)$ and $t \in \mathbb{R}^*$. According to the Cartan decomposition, for every $x \in \mathbb{X}$, there exist $k_1, k_2 \in K$ and $x^+ \in \overline{\mathfrak{a}^+}$ such that $x = k_1(\exp x^+)k_2$. Then

$$p_\tau(x) = a_\tau(\exp x^+) + b_\tau(\exp x^+)$$

by the K-bi-invariance. According to the expansion (3.15), we split up

$$\begin{aligned}
\widetilde{\omega}_t^{\sigma,0}(x) &= I_1(t,x^+) + I_2(t,x^+) + I_3(t,x^+)\\
&= \frac{1}{\pi}\, J(x^+)^{-1/2} \sum_{k=0}^{[d/2]} 4^{-k}\, U_k(x^+)\, \Gamma\big(\tfrac{d+1}{2} - k\big)\, I_{1,k}(t,x^+)\\
&+ C_{\sigma,d} \int_0^1 ds\, s^{\sigma-1} E(\tau,x^+) + C_{\sigma,d} \int_0^1 ds\, s^{\sigma-1} b_\tau(\exp x^+)
\end{aligned}$$

where

$$I_{1,k}(t,x^+) = C_{\sigma,d} \int_0^1 ds\, s^{\sigma-1}\, \tau(|x^+|^2 + \tau^2)^{k-\frac{d+1}{2}}$$

satisfies

$$|I_{1,k}(t,x^+)| \lesssim 1 + |t|^{-\frac{d-1}{2}}$$

according to next lemma. Hence

$$I_1(t,x^+) \lesssim (\sqrt{|t|} + |t|^{-\frac{d-1}{2}})\, J(x^+)^{-1/2} \quad \forall\, t \in \mathbb{R}^*. \tag{3.18}$$

The last two terms $I_2(t,x^+)$ and $I_3(t,x^+)$ are easily handled: on the one hand, we have

$$\begin{aligned}
|I_2(t,x^+)| &\lesssim \int_0^1 ds\, s^{\operatorname{Re}\sigma-1}\, |E(\tau,x^+)|\\
&\lesssim (1+|t|)^{3(\frac{d}{2}+1)} \log(2+|t|)\, e^{-\langle\rho,x^+\rangle},
\end{aligned} \tag{3.19}$$

according to (3.16); on the other hand, (3.17) yields

$$|I_3(t,x^+)| \lesssim \int_0^1 ds\, s^{\operatorname{Re}\sigma-1}\, |b_\tau(\exp x^+)| \lesssim (1+|t|)^{-N}\, \varphi_0(\exp x^+) \tag{3.20}$$

for all $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \frac{d+1}{2}$. By summing up the estimates (3.18), (3.19) and (3.20), we deduce, on the one hand,

$$|\widetilde{\omega}_t^{\sigma,0}(x)| \lesssim |t|^{-\frac{d-1}{2}}\, (1+|x^+|)^{\frac{\max\{d,D\}-\ell}{2}}\, e^{-\langle\rho,x^+\rangle}$$

if $|t| < 1$, and on the other hand,

$$|\widetilde{\omega}_t^{\sigma,0}(x)| \lesssim |t|^{3(\frac{d}{2}+1)} \log(2+|t|)\, (1+|x^+|)^{\frac{\max\{d,D\}-\ell}{2}}\, e^{-\langle\rho,x^+\rangle}$$

if $|t| \ge 1$. Since $\frac{|x|}{|t|}$ is bounded from below, we obtain finally

$$|\widetilde{\omega}_t^{\sigma,0}(x)| \lesssim |t|^{-N_1}\, (1+|x^+|)^{N_2}\, e^{-\langle\rho,x^+\rangle} \quad \forall\, |t| \ge 1$$

for every $N_1 \in \mathbb{N}$ and $N_2 \geq N_1 + 2(d+1) + \frac{\max\{d,D\}-\ell}{2}$. □

Remark 3.8. *Notice that the above method works only in small time, or in large time under the assumption that $\frac{|x|}{|t|}$ is bounded from below. The large polynomial growth in $|x^+|$ appearing in the estimate is not crucial for further computations because of the exponential decay $e^{-\langle \rho, x^+ \rangle}$.*

Lemma 3.9. *For every integer $0 \leq k < \frac{d+1}{2}$, the integral*

$$I_{1,k}(t,x^+) = C_{\sigma,d} \int_0^1 ds\, s^{\sigma-1}\, \tau(|x^+|^2 + \tau^2)^{k-\frac{d+1}{2}}$$

satisfies

$$|I_{1,k}(t,x^+)| \lesssim 1 + |t|^{k-\frac{d-1}{2}} \quad \forall\, t \in \mathbb{R}^*,\ \forall\, x \in \overline{\mathfrak{a}^+}$$

uniformly in $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \frac{d+1}{2}$.

Proof. Since $\tau = s - it$, we write $I_{1,k}(t,x^+) = P_1 + P_2$ with

$$P_1 = C_{\sigma,d} \int_0^1 ds\, s^{\sigma}\, (s^2 + |x^+|^2 - t^2 - 2sti)^{k-\frac{d+1}{2}}$$

and

$$P_2 = C_{\sigma,d}\, (-it) \int_0^1 ds\, s^{\sigma-1}\, (s^2 + |x^+|^2 - t^2 - 2sti)^{k-\frac{d+1}{2}}.$$

As

$$\big|s^2 + |x^+|^2 - t^2 - 2sti\big| = \sqrt{s^4 + 2s^2(|x^+|^2 + t^2) + (|x^+|^2 - t^2)^2},$$

notice that

$$\big|s^2 + |x^+|^2 - t^2 - 2sti\big| \geq \begin{cases} s^2 & (3.21) \\ s|t| & (3.22) \\ \big||x^+|^2 - t^2\big| & (3.23) \end{cases}$$

P_1 is easily estimated. By using (3.22), we obtain first

$$|P_1| \lesssim |t|^{k-\frac{d+1}{2}} \int_0^1 ds\, s^k \leq |t|^{k-\frac{d+1}{2}} \quad \forall\, t \in \mathbb{R}^*.$$

By using in addition (3.21), we obtain next, for $|t| < 1$,

$$|P_1| \lesssim |t|^{k-\frac{d+1}{2}} \int_0^{|t|} ds\, s^k + \int_{|t|}^1 ds\, s^{2k-\frac{d+1}{2}} \lesssim 1 + |t|^{2k-\frac{d-1}{2}}$$

We deduce that

$$|P_1| \lesssim \begin{cases} 1 + |t|^{2k-\frac{d-1}{2}} & \text{if } |t| < 1, \\ |t|^{k-\frac{d+1}{2}} & \text{if } |t| \geq 1. \end{cases} \tag{3.24}$$

Let us turn to P_2. Consider first the easy case where $1 \le k < \frac{d+1}{2}$. By using (3.22) again, we get

$$|P_2| \lesssim |t| \cdot |t|^{k-\frac{d+1}{2}} \int_0^1 ds\, s^{\operatorname{Re}\sigma-1+k-\frac{d+1}{2}} \lesssim |t|^{k-\frac{d-1}{2}} \tag{3.25}$$

for all $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \frac{d+1}{2}$. In order to estimate P_2 in the remaining case where $k = 0$, we write

$$P_2 = C_{\sigma,d}\,(-it) \int_0^1 ds\, s^{i\operatorname{Im}\sigma-1} \big(\tfrac{s}{s^2+|x^+|^2-t^2-2sti}\big)^{\frac{d+1}{2}}.$$

By performing an integration by parts, P_2 becomes the sum of P_2^- and P_2^+ where

$$P_2^- = \big[\tfrac{C_{\sigma,d}}{\operatorname{Im}\sigma}\,(-it)\, s^{\operatorname{Im}\sigma} \big(\tfrac{s}{s^2+|x^+|^2-t^2-2sti}\big)^{\frac{d+1}{2}}\big]_0^1$$

and

$$P_2^+ = \tfrac{C_{\sigma,d}}{\operatorname{Im}\sigma}\,(it) \int_0^1 ds\, s^{\operatorname{Im}\sigma}\, \tfrac{\partial}{\partial s}\big\{\big(\tfrac{s}{s^2+|x^+|^2-t^2-2sti}\big)^{\frac{d+1}{2}}\big\}.$$

By using (3.2) together with (3.21) in small time and (3.22) in large time, we obtain

$$|P_2^-| \lesssim \begin{cases} 1 & \text{if } |t| < 1, \\ |t|^{-\frac{d-1}{2}} & \text{if } |t| \ge 1. \end{cases} \tag{3.26}$$

Since

$$\tfrac{\partial}{\partial s}\big\{\big(\tfrac{s}{s^2+|x^+|^2-t^2-2sti}\big)^{\frac{d+1}{2}}\big\} = \underbrace{\tfrac{d+1}{2}\Big(\tfrac{s}{s^2+|x^+|^2-t^2-2sti}\Big)^{\frac{d-1}{2}}}_{O(|t|^{-\frac{d-1}{2}})}\, \tfrac{|x^+|^2-t^2-s^2}{(s^2+|x^+|^2-t^2-2sti)^2},$$

we have

$$|P_2^+| \lesssim |t|^{-\frac{d-1}{2}}\, \underbrace{|t| \int_0^1 ds\, \tfrac{\big||x^+|^2-t^2-s^2\big|}{\big|s^2+|x^+|^2-t^2-2sti\big|^2}}_{Q}. \tag{3.27}$$

It remains for us to estimate Q, which is bounded by the sum of

$$Q_1 = |t| \int_0^1 ds\, \tfrac{s^2}{\big|s^2+|x^+|^2-t^2-2sti\big|^2} \quad \text{and} \quad Q_2 = |t| \int_0^1 ds\, \tfrac{\big||x^+|^2-t^2\big|}{\big|s^2+|x^+|^2-t^2-2sti\big|^2}.$$

Q_1 is estimated as P_1. According to (3.22) and (3.21), we have

$$Q_1 \lesssim \begin{cases} |t| \int_0^1 ds\, |t|^{-2} = |t|^{-1} \le 1 & \text{if } |t| \ge 1, \\ |t| \int_0^{|t|} ds\, |t|^{-2} + |t| \int_{|t|}^1 ds\, s^{-2} \le 2 & \text{if } |t| < 1. \end{cases}$$

Let us finally estimate Q_2. On the one hand, if $\big||x^+|^2 - t^2\big| \ge |t|$, by using (3.23), we get

$$|Q_2| \lesssim |t| \int_0^1 ds\, \big||x^+|^2 - t^2\big|^{-1} \lesssim 1.$$

On the other hand, if $\big||x^+|^2-t^2\big|\le|t|$, we have $Q_2=O(1)$ since

$$\Big||t|\int_{0\le s\le\frac{||x^+|^2-t^2|}{|t|}}ds\,\frac{\big||x^+|^2-t^2\big|}{\big|s^2+|x^+|^2-t^2-2sti\big|^2}\Big|\lesssim|t|\int_{0\le s\le\frac{||x^+|^2-t^2|}{|t|}}ds\,\big||x^+|^2-t^2\big|^{-1}$$
$$\le 1$$

according to (3.23), and

$$\Big||t|\int_{\frac{||x^+|^2-t^2|}{|t|}\le s\le 1}ds\,\frac{\big||x^+|^2-t^2\big|}{\big|s^2+|x^+|^2-t^2-2sti\big|^2}\Big|\lesssim\frac{||x^+|^2-t^2|}{|t|}\int_{\frac{||x^+|^2-t^2|}{|t|}\le s\le 1}ds\,s^{-2}$$
$$\le 2$$

according to (3.22). Hence $Q=O(1)$ and we deduce from (3.27) that $|P_2^+|\lesssim|t|^{-\frac{d-1}{2}}$ for all $t\in\mathbb{R}^*$. By combining with (3.26) and (3.25), we obtain

$$|P_2|\lesssim|t|^{k-\frac{d-1}{2}}\quad\forall\,t\in\mathbb{R}^*.$$

Together with (3.24), this concludes the proof. □

3.3. **Estimates of $\omega_t^{\sigma,\infty}$.** We establish in this last subsection the pointwise estimates of $\omega_t^{\sigma,\infty}$. Recall that

$$\omega_t^{\sigma,\infty}(x)=\frac{1}{\Gamma(\sigma)}\int_1^{+\infty}ds\,s^{\sigma-1}\,p_{s-it}(x)\quad\forall\,x\in\mathbb{X},\ \forall\,t\in\mathbb{R}^*.$$

According to the integral expression (2.2) of the spherical function, we may write

$$\omega_t^{\sigma,\infty}(x)=\frac{C_0}{\Gamma(\sigma)}\int_K dk\,e^{\langle\rho,A(kx)\rangle}\int_1^{+\infty}ds\,s^{\sigma-1}\,I(s,t,x),$$

where, let us recall,

$$I(s,t,x)=\int_{\mathfrak{a}}d\lambda\,|\mathbf{c}(\lambda)|^{-2}\,e^{-s\sqrt{|\lambda|^2+|\rho|^2}}\,e^{it\psi_t(\lambda)}.$$

We have considered this oscillatory integral in the case where $s\in(0,1)$.For $s\ge1$, the factor $e^{-s\sqrt{|\lambda|^2+|\rho|^2}}$ plays an important role. On the one hand, for λ close to the critical point of $\psi_t(\lambda)$, this Gaussian produces an exponential decay in s, which ensures the convergence of the integral over $s\in(1,+\infty)$. For λ away from the critical point, it produces an exponential decay in $|\lambda|$, which ensures the convergence of the integral over $\lambda\in\mathfrak{a}$. Let us elaborate.

Theorem 3.10. *The following estimate holds, uniformly in the strip $0\le\operatorname{Re}\sigma\le\frac{d+1}{2}$, for all $t\in\mathbb{R}^*$ and $x\in\mathbb{X}$:*

$$|\omega_t^{\sigma,\infty}(x)|\lesssim\varphi_0(x).\tag{3.28}$$

Moreover, if $|t|\ge1$,

$$|\omega_t^{\sigma,\infty}(x)|\lesssim|t|^{-\frac{D}{2}}\,(1+|x|)^{\frac{D}{2}}\,\varphi_0(x).\tag{3.29}$$

Proof. The global estimate (3.28) is obtained by a straightforward computation. On the one hand,

$$\int_{|\lambda|\le 1} d\lambda\, |\mathbf{c}(\lambda)|^{-2}\, e^{-s\sqrt{|\lambda|^2+|\rho|^2}} \le e^{-s|\rho|} \underbrace{\int_{|\lambda|\le 1} d\lambda\, |\lambda|^{D-\ell}}_{<+\infty}.$$

On the other hand,

$$\int_{|\lambda|\ge 1} d\lambda\, |\mathbf{c}(\lambda)|^{-2}\, e^{-s\sqrt{|\lambda|^2+|\rho|^2}} \le e^{-\frac{s}{2}|\rho|} \underbrace{\int_{|\lambda|\ge 1} d\lambda\, |\lambda|^{d-\ell}\, e^{-\frac{s}{2}|\lambda|}}_{<+\infty}.$$

Hence

$$|\omega_t^{\sigma,\infty}(x)| \lesssim \varphi_0(x) \underbrace{\int_1^{+\infty} ds\, s^{\operatorname{Re}\sigma-1}\, e^{-\frac{s}{2}|\rho|}}_{<+\infty}. \tag{3.30}$$

The estimate (3.29) follows from (3.28) if $\frac{|x|}{|t|}$ is bounded from below. Let us prove it if $\frac{|x|}{|t|}$ is bounded from above, let say by $\frac{1}{2}$. We study the oscillatory integral I along the lines of Sect. 3.1. Let split up again

$$I(s,t,x) = I^-(s,t,x) + I^+(s,t,x) = \int_{\mathfrak{a}} d\lambda\, \chi_0^\rho(\lambda) \cdots + \int_{\mathfrak{a}} d\lambda\, \chi_\infty^\rho(\lambda) \cdots$$

according to cut-off functions χ_0^ρ and $\chi_\infty^\rho = 1 - \chi_0^\rho$, which have been defined in Sect. 3.1. Recall that $\chi_0^\rho(\lambda) = 1$ when $|\lambda| \le |\rho|$ and vanishes if $|\lambda| \ge 2|\rho|$.

On the one hand, I^- is estimated by studying the oscillatory integral

$$I^-(s,t,x) = \int_{\mathfrak{a}} d\lambda\, \underbrace{\chi_0^\rho(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, e^{-s\sqrt{|\lambda|^2+|\rho|^2}}}_{a_0(s,\lambda)}\, e^{it\psi_t(\lambda)}$$

where the amplitude a_0 is compactly supported for $|\lambda| \le 2|\rho|$, and in this range, the phase ψ_t, defined by (3.3), has a single critical point, which is nondegenerate and small if $\frac{|x|}{|t|} \le \frac{1}{2}$. According to Lemma A.1, we obtain

$$|I^-(s,t,x)| \lesssim |t|^{-\frac{D}{2}}\, (1+|x|)^{\frac{D-\ell}{2}}\, e^{-\frac{|\rho|}{2}s}. \tag{3.31}$$

On the other hand,

$$I^+(s,t,x) = \int_{\mathfrak{a}} d\lambda\, \chi_\infty^\rho(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, e^{-s\sqrt{|\lambda|^2+|\rho|^2}}\, e^{it\psi_t(\lambda)}$$

is easily estimated with no barycentric decomposition. Let

$$\widetilde{\psi}_\infty(\lambda) = \frac{|\lambda|^2}{\sqrt{|\lambda|^2+|\rho|^2}} + \langle \tfrac{A(kx)}{t}, \lambda \rangle \quad \forall\, \lambda \in \operatorname{supp} \chi_\infty^\rho.$$

Then $\widetilde{\psi}_\infty$ is a symbol of order 1, and satisfies

$$|\widetilde{\psi}_\infty(\lambda)| = \underbrace{\frac{|\lambda|^2}{\sqrt{|\lambda|^2+|\rho|^2}}}_{\geq \frac{|\lambda|}{\sqrt{2}}} - \underbrace{\frac{\langle \lambda_0, \lambda\rangle}{\sqrt{|\lambda_0|^2+|\rho|^2}}}_{\leq \frac{|\lambda|}{\sqrt{3}}} \geq (\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{3}})|\rho| > 0$$

on $\operatorname{supp}\chi_\infty^\rho$ according to (3.6) and (3.7). By performing N integrations by parts based on

$$e^{it\psi_t(\lambda)} = \tfrac{1}{it}\,\widetilde{\psi}_\infty(\lambda)^{-1}\,\textstyle\sum_{j=1}^{\ell}\lambda_j\,\frac{\partial}{\partial\lambda_j}\,e^{it\psi_t(\lambda)},$$

we write

$$I_\infty^+(s,t,x) = (it)^{-N}\int_{\mathfrak{a}} d\lambda\, e^{it\psi_t(\lambda)} \times \big\{-\textstyle\sum_{j=1}^{\ell}\frac{\partial}{\partial\lambda_j}\circ\frac{\lambda_j}{\widetilde{\psi}_\infty(\lambda)}\big\}^N \big\{\chi_\infty^\rho(\lambda)\,|\mathbf{c}(\lambda)|^{-2}\,e^{-s\sqrt{|\lambda|^2+|\rho|^2}}\big\}.$$

If some derivatives hit $\chi_\infty^\rho(\lambda)$, the range of the above integral is reduced to a spherical shell where $|\lambda|\asymp|\rho|$, and

$$I^+(s,t,x)\asymp|t|^{-N}e^{-s|\rho|}.$$

Assume next that no derivative is applied to χ_∞^ρ and

- N_1 derivatives are applied to the factors $\lambda_j/\widetilde{\psi}_\infty(\lambda)$, which are inhomogeneous symbols of order 0, producing contributions which are $O(|\lambda|^{-N_1})$,
- N_2 derivatives are applied to the factor $|\mathbf{c}(\lambda)|^{-2}$ which is not a symbol in general, producing a contribution which is $O(|\lambda|^{d-\ell})$,
- N_3 derivatives are applied to the factor $e^{-s\sqrt{|\lambda|^2+|\rho|^2}}$, producing a contribution which is $O(s^{N_3}e^{-s\sqrt{|\lambda|^2+|\rho|^2}})$,

with $N_1+N_2+N_3=N$. Then we get the upper bound

$$|t|^{-N}\,s^{N_3}\int_{|\lambda|>|\rho|} d\lambda\,|\lambda|^{d-\ell-N_1}\,e^{-s\sqrt{|\lambda|^2+|\rho|^2}},$$

which yields

$$|I^+(s,t,x)|\lesssim|t|^{-N}\,s^N\,e^{-\frac{s}{2}|\rho|}\underbrace{\int_{|\lambda|>|\rho|}d\lambda\,|\lambda|^{d-\ell}\,e^{-\frac{s}{2}|\lambda|}}_{<+\infty}.$$

Together with (3.31), we obtain

$$|I(s,t,x)|\lesssim|t|^{-\frac{D}{2}}\,(1+|x|)^{\frac{D-\ell}{2}}\,s^Ne^{-\frac{s}{2}|\rho|}.$$

for all $s\geq1$ and for $N\geq\frac{D}{2}$. We deduce

$$|\omega_t^{\sigma,\infty}(x)|\lesssim|t|^{-\frac{D}{2}}\,(1+|x|)^{\frac{D-\ell}{2}}\,\varphi_0(x)\underbrace{\int_1^{+\infty}ds\,s^{\operatorname{Re}\sigma-1+N}\,e^{-\frac{|\rho|}{2}s}}_{<+\infty}$$

for all $x\in\mathbb{X}$ and $|t|\geq1$. □

4. Dispersive estimates

In this section, we prove our second main result about the $L^{q'} \to L^q$ estimates for the operator $W_t^\sigma = (-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}}$. We introduce the following criterion based on the Kunze-Stein phenomenon, which is crucial for the proof of dispersive estimates.

Lemma 4.1. *Let κ be a reasonable K-bi-invariant function on G. Then*

$$\| \cdot * \kappa \|_{L^{q'}(\mathbb{X}) \to L^q(\mathbb{X})} \le \Big\{ \int_G dx\, \varphi_0(x)\, |\kappa(x)|^{\frac{q}{2}} \Big\}^{\frac{2}{q}}$$

for every $q \in [2, +\infty)$. In the limit case $q = \infty$,

$$\| \cdot * \kappa \|_{L^1(\mathbb{X}) \to L^\infty(\mathbb{X})} = \sup_{x \in G} |\kappa(x)|.$$

Remark 4.2. *This lemma has been proved in several contexts. For $q = 2$, it is the so-called Herz's criterion, see for instance [8]. For $q > 2$, the proof carried out on Damek-Ricci spaces [4, Theorem 4.2] is adapted straightforwardly in the higher rank case.*

Theorem 4.3 (Small time dispersive estimate)**.** *Let $d \ge 3$ and $0 < |t| < 1$. Then*

$$\| (-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}} \|_{L^{q'}(\mathbb{X}) \to L^q(\mathbb{X})} \lesssim |t|^{-(d-1)(\frac{1}{2} - \frac{1}{q})}$$

for all $2 < q < +\infty$ and $\sigma \ge (d+1)(\frac{1}{2} - \frac{1}{q})$.

Proof. We divide the proof into two parts, corresponding to the kernel decomposition $\omega_t^\sigma = \omega_t^{\sigma,0} + \omega_t^{\sigma,\infty}$. According to Lemma 4.1 and to the pointwise estimate Theorem 3.10, we obtain on one hand

$$\begin{aligned} \| \cdot * \omega_t^{\sigma,\infty} \|_{L^{q'}(\mathbb{X}) \to L^q(\mathbb{X})} &\le \Big\{ \int_G dx\, \varphi_0(x)\, |\omega_t^{\sigma,\infty}(x)|^{\frac{q}{2}} \Big\}^{\frac{2}{q}} \\ &\lesssim \Big\{ \int_{\mathfrak{a}^+} dx^+\, |\varphi_0(x^+)|^{\frac{q}{2}+1}\, \delta(x^+) \Big\}^{\frac{2}{q}} \\ &\lesssim \Big\{ \int_{\mathfrak{a}^+} dx^+\, (1 + |x^+|)^{\frac{D-\ell}{2}(\frac{q}{2}+1)}\, e^{-(\frac{q}{2}-1)\langle \rho, x^+ \rangle} \Big\}^{\frac{2}{q}} \\ &< +\infty \end{aligned}$$

for all $q > 2$. On the other hand, we use an analytic interpolation between $L^2 \to L^2$ and $L^1 \to L^\infty$ estimates for the family of operators $\widetilde{W}_t^{\sigma,0}$ defined by (3.1) in the vertical strip $0 \le \operatorname{Re}\sigma \le \frac{d+1}{2}$. When $\operatorname{Re}\sigma = 0$, the spectral theorem yields

$$\| \widetilde{W}_t^{\sigma,0} \|_{L^2(\mathbb{X}) \to L^2(\mathbb{X})} = \| e^{it\sqrt{-\Delta}} \|_{L^2(\mathbb{X}) \to L^2(\mathbb{X})} = 1$$

for all $t \in \mathbb{R}^*$. According to Theorem 3.7, when $\operatorname{Re}\sigma = \frac{d+1}{2}$,

$$\| \widetilde{W}_t^{\sigma,0} \|_{L^1(\mathbb{X}) \to L^\infty(\mathbb{X})} \lesssim \| \widetilde{\omega}_t^{\sigma,0} \|_{L^\infty(\mathbb{X})} \lesssim |t|^{-\frac{d-1}{2}}.$$

By Stein's interpolation theorem applied to the analytic family of operators $\widetilde{W}_t^{\sigma,0}$, we conclude for $\sigma = (d+1)(\frac{1}{2}-\frac{1}{q})$ that

$$\|W_t^{\sigma}\|_{L^{q'}(\mathbb{X})\to L^q(\mathbb{X})} \lesssim |t|^{-(d-1)(\frac{1}{2}-\frac{1}{q})},$$

for all $0<|t|<1$ and $2<q<+\infty$. □

Theorem 4.4 (Large time dispersive estimate)**.** *Assume that* $|t|\geq 1$, $2<q<+\infty$ *and* $\sigma \geq (d+1)(\frac{1}{2}-\frac{1}{q})$. *Then*

$$\|(-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}}\|_{L^{q'}(\mathbb{X})\to L^q(\mathbb{X})} \lesssim |t|^{-\frac{D}{2}}.$$

Proof. We divide the proof into three parts, corresponding to the kernel decomposition

$$\omega_t^{\sigma} = \mathbb{1}_{B(0,C_\Sigma|t|)}\,\omega_t^{\sigma,0} + \mathbb{1}_{\mathbb{X}\setminus B(0,C_\Sigma|t|)}\,\omega_t^{\sigma,0} + \omega_t^{\sigma,\infty}$$

where the constant C_Σ has been specified in the proof of Theorem 3.3. The first and the last terms are estimated by straightforward computations. By combining Lemma 4.1 with the pointwise kernel estimates in Theorem 3.3 and Theorem 3.10, we obtain

$$\begin{aligned}
&\|\cdot * \{\mathbb{1}_{B(0,C_\Sigma|t|)}\,\omega_t^{\sigma,0}\}\|_{L^{q'}(\mathbb{X})\to L^q(\mathbb{X})}\\
&\lesssim \Big\{\int_G dx\,\varphi_0(x)\,|\mathbb{1}_{B(0,C_\Sigma|t|)}(x)\,\omega_t^{\sigma,0}(x)|^{\frac{q}{2}}\Big\}^{\frac{2}{q}}\\
&\lesssim |t|^{-\frac{D}{2}} \underbrace{\Big\{\int_{|x^+|<C_\Sigma|t|} dx^+\,(1+|x^+|)^{\frac{D-\ell}{2}(q+1)}\,e^{-(\frac{q}{2}-1)\langle\rho,x^+\rangle}\Big\}^{\frac{2}{q}}}_{<+\infty}
\end{aligned}$$

and

$$\begin{aligned}
&\|\cdot * \omega_t^{\sigma,\infty}\|_{L^{q'}(\mathbb{X})\to L^q(\mathbb{X})}\\
&\lesssim \Big\{\int_G dx\,\varphi_0(x)\,|\omega_t^{\sigma,\infty}(x)|^{\frac{q}{2}}\Big\}^{\frac{2}{q}}\\
&\lesssim |t|^{-\frac{D}{2}} \underbrace{\Big\{\int_{\mathfrak{a}^+} dx^+\,(1+|x^+|)^{\frac{D-\ell}{2}+(D-\frac{\ell}{2})\frac{q}{2}}\,e^{-(\frac{q}{2}-1)\langle\rho,x^+\rangle}\Big\}^{\frac{2}{q}}}_{<+\infty}.
\end{aligned}$$

Here $q\leq 2<\infty$ and the above estimates are uniform in the strip $0\leq \operatorname{Re}\sigma\leq \frac{d+1}{2}$. As far as the middle term is concerned, we use again the analytic interpolation for the family of operators associated with the convolution kernel $\mathbb{1}_{\mathbb{X}\setminus B(0,C_\Sigma|t|)}\,\widetilde{\omega}_t^{\sigma,0}$. On the one hand, if $\operatorname{Re}\sigma = 0$, then

$$\begin{aligned}
&\|\cdot * \mathbb{1}_{\mathbb{X}\setminus B(0,C_\Sigma|t|)}\,\widetilde{\omega}_t^{\sigma,0}\|_{L^2(\mathbb{X})\to L^2(\mathbb{X})}\\
&\leq \|\cdot * \widetilde{\omega}_t^{\sigma,0}\|_{L^2(\mathbb{X})\to L^2(\mathbb{X})} + \|\cdot * \mathbb{1}_{B(0,C_\Sigma|t|)}\,\widetilde{\omega}_t^{\sigma,0}\|_{L^2(\mathbb{X})\to L^2(\mathbb{X})} \lesssim 1.
\end{aligned}$$

On the other hand, if $\operatorname{Re}\sigma = \frac{d+1}{2}$, we deduce from Theorem 3.7 that

$$\|\cdot * \mathbb{1}_{\mathbb{X}\setminus B(0,C_\Sigma|t|)}\,\widetilde{\omega}_t^{\sigma,0}\|_{L^1(\mathbb{X})\to L^\infty(\mathbb{X})} = \sup_{x\in\mathbb{X}} |\mathbb{1}_{\mathbb{X}\setminus B(0,C_\Sigma|t|)}(x)\,\widetilde{\omega}_t^{\sigma,0}(x)| \lesssim |t|^{-N}$$

for any $N \in \mathbb{N}$. By using Stein's interpolation theorem between above $L^2 \to L^2$ and $L^1 \to L^\infty$ estimates, we obtain

$$\| \cdot * \mathbb{1}_{\mathbb{X}\setminus B(0,C_\Sigma|t|)}\, \omega_t^{\sigma,0} \|_{L^{q'}(\mathbb{X}) \to L^q(\mathbb{X})} \lesssim |t|^{-N},$$

for all $|t| \geq 1$, $2 < q < +\infty$ and for any $N \in \mathbb{N}$. This concludes the proof. $\square$

Remark 4.5. *The standard TT^* method used to prove the Strichartz inequality breaks down in the critical case. In order to take care of these endpoints, we need the dyadic decomposition method carried out in [25] and the following stronger dispersive property, which is obtained by interpolation arguments.*

Corollary 4.6. *Let $d \geq 3$, $2 < q, \widetilde{q} < +\infty$ and $\sigma \geq (d+1) \max(\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\widetilde{q}})$. Then there exists a constant $C > 0$ such that the following dispersive estimates hold:*

$$\|(-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}}\|_{L^{\widetilde{q}'}(\mathbb{X}) \to L^q(\mathbb{X})} \leq C \begin{cases} |t|^{-(d-1)\max(\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\widetilde{q}})} & \text{if } \ 0 < |t| < 1, \\ |t|^{-\frac{D}{2}} & \text{if } \ |t| \geq 1. \end{cases}$$

5. Strichartz inequality and applications

In this section, we use the dispersive properties proved in the previous section to establish the Strichartz inequality. This inequality serves as a tool for finding minimal regularity conditions on the initial data ensuring well-posedness of related semi-linear wave equations. Such results were previously known to hold for real hyperbolic spaces [3] (actually for all noncompact symmetric spaces of rank one) and for noncompact symmetric spaces G/K with G complex [38]. For simplicity, we may assume that $\ell \geq 2$ throughout this section, thus $d \geq 4$.

Let $\sigma \in \mathbb{R}$ and $1 < q < \infty$. Recall that the Sobolev space $H^{\sigma,q}(\mathbb{X})$ is the image of $L^q(\mathbb{X})$ under the operator $(-\Delta)^{-\frac{\sigma}{2}}$, equipped with the norm

$$\|f\|_{H^{\sigma,q}(\mathbb{X})} = \|(-\Delta)^{\frac{\sigma}{2}} f\|_{L^q(\mathbb{X})}.$$

If $\sigma = N$ is a nonnegative integer, then $H^{\sigma,q}(\mathbb{X})$ coincides with the classical Sobolev space

$$W^{N,q}(\mathbb{X}) = \{ f \in L^q(\mathbb{X}) \,|\, \nabla^j f \in L^q(\mathbb{X}) \,\forall\, 1 \leq j \leq N \},$$

defined by means of covariant derivatives. We refer to [37] for more details about function spaces on Riemannian manifolds. Let us state the Strichartz inequality and some applications. The proofs are adapted straightforwardly from [5, 3] and are therefore omitted.

5.1. Strichartz inequality.

We study the linear inhomogeneous wave equation on $\mathbb{X}$

$$\begin{cases} \partial_t^2\, u(t,x) - \Delta\, u(t,x) = F(t,x), \\ u(0,x) = f(x),\ \partial_t|_{t=0} u(t,x) = g(x) \end{cases} \tag{5.1}$$

whose solution is given by Duhamel's formula:

$$u(t,x) = (\cos t\sqrt{-\Delta}) f(x) + \tfrac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} g(x) + \int_0^t ds\, \tfrac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} F(s,x).$$

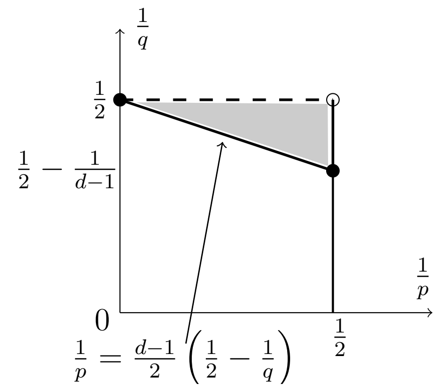

FIGURE 3. Admissibility in dimension $d \geq 4$.

Recall that a couple (p, q) is called admissible if $(\frac{1}{p}, \frac{1}{q})$ belongs to the triangle

$$\Big\{\Big(\tfrac{1}{p}, \tfrac{1}{q}\Big) \in \Big(0, \tfrac{1}{2}\Big] \times \Big(0, \tfrac{1}{2}\Big) \,\Big|\, \tfrac{1}{p} \geq \tfrac{d-1}{2}\Big(\tfrac{1}{2}-\tfrac{1}{q}\Big)\Big\} \bigcup \Big\{\Big(0, \tfrac{1}{2}\Big)\Big\}.$$

Theorem 5.1. *Let (p, q) and $(\tilde{p}, \tilde{q})$ be two admissible couples, and let*

$$\sigma \geq \tfrac{d+1}{2}\left(\tfrac{1}{2}-\tfrac{1}{q}\right) \quad \text{and} \quad \widetilde{\sigma} \geq \tfrac{d+1}{2}\left(\tfrac{1}{2}-\tfrac{1}{\tilde{q}}\right).$$

Then all solutions u to the Cauchy problem (5.1) *satisfy the following Strichartz inequality:*

$$\|\nabla_{\mathbb{R}\times\mathbb{X}} u\|_{L^p(I;H^{-\sigma,q}(\mathbb{X}))} \lesssim \|f\|_{H^1(\mathbb{X})} + \|g\|_{L^2(\mathbb{X})} + \|F\|_{L^{\tilde{p}'}(I;H^{\tilde{\sigma},\tilde{q}'}(\mathbb{X}))}. \tag{5.2}$$

Remark 5.2. *As have already been observed on hyperbolic spaces, the admissible set for $\mathbb{X}$ is much larger than the admissible set for $\mathbb{R}^d$, which corresponds only to the lower edge of the triangle. This is due to large scale dispersive effects in negative curvature.*

The admissible range in (5.2) can be widened by using the Sobolev embedding theorem.

Corollary 5.3. *Assume that (p, q) and $(\tilde{p}, \tilde{q})$ are two couples corresponding to the square*

$$\Big[0, \tfrac{1}{2}\Big] \times \Big(0, \tfrac{1}{2}\Big) \bigcup \Big\{\Big(0, \tfrac{1}{2}\Big)\Big\},$$

Let $\sigma, \tilde{\sigma} \in \mathbb{R}$ such that $\sigma \geq \sigma(p, q)$, where

$$\sigma(p, q) = \tfrac{d+1}{2}\Big(\tfrac{1}{2}-\tfrac{1}{q}\Big) + \max\Big\{0, \tfrac{d-1}{2}\Big(\tfrac{1}{2}-\tfrac{1}{q}\Big) - \tfrac{1}{p}\Big\},$$

and similarly $\tilde{\sigma} \geq \sigma(\tilde{p}, \tilde{q})$. Then the Strichartz inequality (5.2) *holds for all solutions to the Cauchy problem* (5.1).

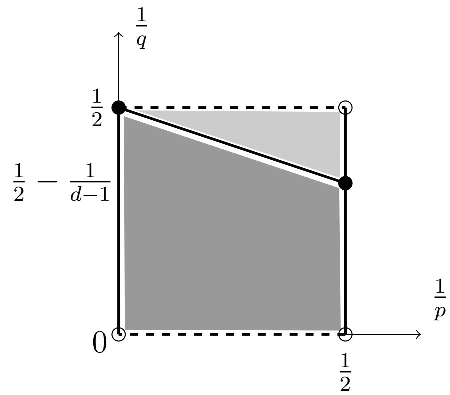

FIGURE 4. Extended admissibility in dimension $d \geq 4$.

5.2. **Global well-posedness in $L^p(\mathbb{R}, L^q(\mathbb{X}))$.** By combining the classical fixed point scheme with the previous Strichartz inequalities, one obtains the global well-posedness for the semi-linear equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) = F(u(t,x)), \\ u(0,x) = f(x),\, \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{5.3}$$

on $\mathbb{X}$ with small initial data f and g and power-like nonlinearities F satisfying

$$|F(u)| \lesssim |u|^\gamma \quad \text{and} \quad |F(u) - F(v)| \lesssim (|u|^{\gamma-1} + |v|^{\gamma-1})|u - v|$$

where $\gamma > 1$. Let $\gamma_c = 1 + \frac{4}{d-1}$ be the conformal power. The global existence of solutions to the semi-linear wave equation (5.3) on $\mathbb{R}^d$ is related to the Strauss conjecture: the critical power γ_0, i.e., the infimum of all $\gamma \in (1, \gamma_c]$ such that (5.3) has global solutions for small initial data, is the positive root of the quadratic equation

$$(d-1)\gamma_0^2 - (d+1)\gamma_0 - 2 = 0 \quad (d \geq 2).$$

In other words,

$$\gamma_0 = \tfrac{1}{2} + \tfrac{1}{d-1} + \sqrt{(\tfrac{1}{2} + \tfrac{1}{d-1})^2 + \tfrac{2}{d-1}} > 1.$$

We refer to [22, 24, 35, 14, 36] and the references therein for more details about the Strauss conjecture in the Euclidean setting. In negative curvature, the global existence for small initial data has been proved, for any $\gamma \in (1, \gamma_c]$, on real hyperbolic spaces of dimension $d = 3$ [28, 29] and then of any dimension $d \geq 2$ [3]. In other words, there is no phenomenon analogous to Strauss conjecture on such spaces. Similar results were extended later to Damek-Ricci spaces, which contain all noncompact symmetric spaces of rank one [6], and have been established recently on simply connected complete Riemannian manifolds with strictly negative sectional curvature [32], and on non-trapping asymptotically hyperbolic manifolds [33]. Next theorem shows that the same phenomenon holds on general noncompact symmetric spaces. More precisely, we prove that the semi-linear wave equation (5.3) on $\mathbb{X}$ is globally well-posed.

To state the following theorem, we need to introduce some notation. Consider the following powers

$$\gamma_1 = 1 + \frac{3}{d} \qquad \gamma_2 = 1 + \frac{2}{\frac{d-1}{2} + \frac{2}{d-1}}$$

$$\gamma_3 = \begin{cases} 1 + \frac{4}{d-2} & \text{if } d \leq 5, \\ \frac{d-1}{2} + \frac{3}{d+1} - \sqrt{(\frac{d-3}{2} + \frac{3}{d+1})^2 - 4\frac{d-1}{d+1}} & \text{if } d \geq 6, \end{cases}$$

and the following curves

$$\sigma_1(\gamma) = \frac{d+1}{4} - \frac{(d+1)(d+5)}{8d} \frac{1}{\gamma - \frac{d+1}{2d}},$$

$$\sigma_2(\gamma) = \frac{d+1}{4} - \frac{1}{\gamma - 1}, \quad \sigma_3(\gamma) = \frac{d}{2} - \frac{2}{\gamma - 1}.$$

Theorem 5.4. *The semi-linear Cauchy problem* (5.3) *is globally well-posed for small initial data in* $H^{\sigma,2}(\mathbb{X}) \times H^{\sigma-1,2}(\mathbb{X})$ *provided that*

$$\begin{cases} \sigma > 0 & \text{if } 1 < \gamma \leq \gamma_1, \\ \sigma \geq \sigma_1(\gamma) & \text{if } \gamma_1 < \gamma \leq \gamma_2, \\ \sigma \geq \sigma_2(\gamma) & \text{if } \gamma_2 \leq \gamma \leq \gamma_c, \\ \sigma \geq \sigma_3(\gamma) & \text{if } \gamma_c \leq \gamma \leq \gamma_3. \end{cases}$$

More precisely, in each case, there exists $2 \leq p, q < \infty$ *such that for any small initial data* (f, g) *in* $H^{\sigma,2}(\mathbb{X}) \times H^{\sigma-1,2}(\mathbb{X})$, *the Cauchy problem* (5.3) *has a unique solution in the Banach space*

$$\mathcal{C}(\mathbb{R}; H^{\sigma,2}(\mathbb{X})) \cap \mathcal{C}^1(\mathbb{R}; H^{\sigma-1,2}(\mathbb{X})) \cap L^p(\mathbb{R}; L^q(\mathbb{X})).$$

6. Further results for Klein-Gordon equations

The kernel estimates and dispersive estimates proved above for the wave equation still hold if we replace the operator $(-\Delta)^{-\frac{\sigma}{2}} e^{it\sqrt{-\Delta}}$ by $\mathbf{D}^{-\sigma} e^{it\mathbf{D}}$, where

$$\mathbf{D} = \sqrt{-\Delta - |\rho|^2 + \kappa^2} \quad \text{with} \quad \kappa > 0.$$

Then, for every admissible couple (p, q), the operator

$$\mathbf{T}f(t, x) = \mathbf{D}_x^{-\frac{\sigma}{2}} e^{it\mathbf{D}_x} f(x)$$

is again bounded from $L^2(\mathbb{X})$ to $L^p(\mathbb{R}; L^q(\mathbb{X}))$, and its adjoint

$$\mathbf{T}^* F(x) = \int_{-\infty}^{+\infty} ds \, \mathbf{D}_x^{-\frac{\sigma}{2}} e^{-is\mathbf{D}_x} F(s, x)$$

from $L^{p'}(\mathbb{R}; L^{q'}(\mathbb{X}))$ to $L^2(\mathbb{X})$. While L^2 Sobolev spaces may be defined in terms of $\mathbf{D}$, we need the operator

$$\widetilde{\mathbf{D}} = \sqrt{-\Delta - |\rho|^2 + \widetilde{\kappa}^2} \quad \text{with} \quad \widetilde{\kappa} \geq |\rho|,$$

in order to define L^q Sobolev spaces when q gets large. As $\widetilde{\mathbf{D}}^{-\frac{\sigma}{2}} \circ \mathbf{D}^{\frac{\sigma}{2}}$ is a topological automorphism of $L^2(\mathbb{X})$, the operator

$$\widetilde{\mathbf{T}} f(t,x) = \widetilde{\mathbf{D}}_x^{-\frac{\sigma}{2}}\, e^{it\mathbf{D}_x} f(x)$$

is also bounded from $L^2(\mathbb{X})$ to $L^p(\mathbb{R}; L^q(\mathbb{X}))$, and its adjoint

$$\widetilde{\mathbf{T}}^* F(x) = \int_{-\infty}^{+\infty} ds\, \widetilde{\mathbf{D}}_x^{-\frac{\sigma}{2}}\, e^{-is\mathbf{D}_x}\, F(s,x)$$

from $L^{p'}(\mathbb{R}; L^{q'}(\mathbb{X}))$ to $L^2(\mathbb{X})$, hence

$$\widetilde{\mathbf{T}}\widetilde{\mathbf{T}}^* F(t,x) = \int_{-\infty}^{+\infty} ds\, \widetilde{\mathbf{D}}_x^{-\sigma}\, e^{i(t-s)\mathbf{D}_x}\, F(s,x)$$

from $L^{p'}(\mathbb{R}; L^{q'}(\mathbb{X}))$ to $L^{\widetilde{p}}(\mathbb{R}; L^{\widetilde{q}}(\mathbb{X}))$ for all admissible couples (p,q) and $(\widetilde{p},\widetilde{q})$. We deduce that Theorem 5.1 and Corollary 5.3 still hold for solutions to the inhomogeneous Klein-Gordon equation

$$\begin{cases} \partial_t^2\, u(t,x) + \mathbf{D}^2\, u(t,x) = F(t,x), \\ u(0,x) = f(x),\ \partial_t|_{t=0} u(t,x) = g(x), \end{cases}$$

and that the corresponding semi-linear equation is globally well-posed with low regularity data, see Theorem 5.4.

Appendix A. Oscillatory integral on $\mathfrak{a}$

In this appendix, we prove the following lemma which is used in the proofs of Theorem 3.3 and Theorem 3.10. Recall that $A = A(kx)$ is the $\mathfrak{a}$-component of $kx \in \mathbb{X}$ in the Iwasawa decomposition, and that $C_\Sigma \in (0, \frac{1}{2}]$ is a fixed constant.

Lemma A.1. *Let $s \in \mathbb{R}_+$ and $|t| \geq 1$. Consider the oscillatory integral*

$$I^-(s,t,x) = \int_{\mathfrak{a}} d\lambda\, a_0(s,\lambda)\, e^{it\psi_t(\lambda)}$$

where the phase is given by

$$\psi_t(\lambda) = \sqrt{|\lambda|^2 + |\rho|^2} + \langle \tfrac{A}{t}, \lambda \rangle$$

and the amplitude

$$a_0(s,\lambda) = \chi_0^\rho(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, e^{-s\sqrt{|\lambda|^2+|\rho|^2}}$$

vanishes unless $|\lambda| \leq 2|\rho|$. Then, for all $x \in \mathbb{X}$ such that $\frac{|x|}{|t|} \leq C_\Sigma$,

$$|I^-(s,t,x)| \lesssim |t|^{-\frac{D}{2}}\, (1+|x|)^{\frac{D-\ell}{2}}\, e^{-\frac{|\rho|}{2}s}.$$

Remark A.2. *The proof of this lemma is similar to the proof of [38, Theorem 3.1.(ii)], except that our amplitude involves the general Plancherel density and in addition a Gaussian factor depending on s.*

Proof. By symmetry, we may assume that $t \geq 1$. Recall that the critical point λ_0 of the phase ψ is given by

$$(|\lambda_0|^2 + |\rho|^2)^{-\frac{1}{2}} \lambda_0 = -\tfrac{A}{t} \tag{A.1}$$

and satisfies

$$|\lambda_0| = |\rho| \tfrac{|A|}{|t|} (1 - \tfrac{|A|^2}{t^2})^{-\frac{1}{2}} \leq |\rho| \tfrac{|x|}{t} (1 - \tfrac{|x|^2}{t^2})^{-\frac{1}{2}} < \tfrac{|\rho|}{\sqrt{3}}, \tag{A.2}$$

as $|A| \leq |x|$ and $\frac{|x|}{t} \leq \frac{1}{2}$. Denote by

$$B(\lambda_0, \eta) = \{\lambda \in \mathfrak{a} \,|\, |\lambda - \lambda_0| \leq \eta\}$$

the ball in $\mathfrak{a}$ centered at λ_0, where the radius η will be specified later. Notice that $|\lambda| < |\rho| + \eta$ for all $\lambda \in B(\lambda_0, \eta)$. Let P_λ be the projection onto the vector spanned by $\frac{\lambda}{|\lambda|}$. Then $|\lambda|^2 P_\lambda = \lambda \otimes \lambda$ and the Hessian matrix of ψ_t is given by

$$\begin{aligned}
\operatorname{Hess} \psi_t(\lambda) &= (|\lambda|^2 + |\rho|^2)^{-\frac{1}{2}} I_\ell - (|\lambda|^2 + |\rho|^2)^{-\frac{3}{2}} \lambda \otimes \lambda \\
&= (|\lambda|^2 + |\rho|^2)^{-\frac{3}{2}} \{|\rho|^2 P_\lambda + (|\lambda|^2 + |\rho|^2)(I_\ell - P_\lambda)\} \\
&= (|\lambda|^2 + |\rho|^2)^{-\frac{3}{2}} \left(\begin{array}{c|c} |\rho|^2 & 0 \\ \hline 0 & (|\lambda|^2 + |\rho|^2) I_{\ell-1} \end{array} \right)
\end{aligned}$$

which is a positive definite symmetric matrix. Hence λ_0 is a nondegenerate critical point. Since $\nabla_{\mathfrak{a}} \psi_t(\lambda_0) = 0$, we write

$$\begin{aligned}
&\psi_t(\lambda) - \psi_t(\lambda_0) \\
&= (\lambda - \lambda_0)^T \underbrace{\Big\{ \int_0^1 ds\, (1 - s) \operatorname{Hess} \psi_t\big(\lambda_0 + s(\lambda - \lambda_0)\big) \Big\}}_{\mathcal{M}(\lambda)} (\lambda - \lambda_0),
\end{aligned}$$

where $\mathcal{M}(\lambda)$ belongs, for every $\lambda \in B(\lambda_0, \eta)$, to a compact subset of the set of positive definite symmetric matrices. We introduce a new variable $\mu = \mathcal{M}(\lambda)^{\frac{1}{2}}(\lambda - \lambda_0)$, then $|\mu|^2 = \psi_t(\lambda) - \psi_t(\lambda_0)$ and $\mu = 0$ if and only if $\lambda = \lambda_0$. There exist $0 < \widetilde{\eta}_1 \leq \widetilde{\eta}_2$ such that $\mu \in B(0, \widetilde{\eta}_1)$ implies $\lambda \in B(\lambda_0, \eta)$, and $\lambda \in B(\lambda_0, \eta)$ implies $\mu \in B(0, \widetilde{\eta}_2)$. Notice that for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$|\nabla_{\mathfrak{a}}^k \mathcal{M}(\lambda)^{\frac{1}{2}}| \leq C_k \quad \forall\, \lambda \in B(\lambda_0, \eta). \tag{A.3}$$

Denote by $j(\lambda)$ the Jacobian matrix such that $d\mu = \det[\, j(\lambda)\,] d\lambda$, then we can choose $\eta > 0$ small enough such that

$$\det[\, j(\lambda)\,] > \frac{1}{2} \det[\, \mathcal{M}(\lambda)^{\frac{1}{2}}\,] \quad \forall\, \lambda \in B(\lambda_0, \eta). \tag{A.4}$$

Now, we split up

$$\begin{aligned} I^-(s,t,x) &= I_0^-(s,t,x) + I_\infty^-(s,t,x) \\ &= \int_{\mathfrak{a}} d\lambda\, \chi_0^\eta(\lambda)\, a_0(s,\lambda)\, e^{it\psi_t(\lambda)} + \int_{\mathfrak{a}} d\lambda\, \chi_\infty^\eta(\lambda)\, a_0(s,\lambda)\, e^{it\psi_t(\lambda)} \end{aligned}$$

where $\chi_0^\eta : \mathfrak{a} \to [0,1]$ is a smooth cut-off function which vanishes unless $|\lambda - \lambda_0| \le \frac{\eta}{2}$, $\chi_0^\eta(\lambda) = 1$ if $|\lambda - \lambda_0| \le \frac{\eta}{4}$, and $\chi_\infty^\eta = 1 - \chi_0^\eta$.

Estimate of I_0^-. We estimate I_0^- by using the stationary phase analysis described in [34, Chap.VIII 2.3]. Notice that $\operatorname{supp} \chi_0^\eta \subset B(\lambda_0, \eta)$. By substituting $\psi_t(\lambda) = |\mu|^2 + \psi_t(\lambda_0)$, we get

$$I_0^-(s,t,x) = e^{it\psi_t(\lambda_0)} \int_{\mathfrak{a}} d\mu\, \widetilde{a}(s,\lambda(\mu))\, e^{it|\mu|^2}$$

where the amplitude

$$\begin{aligned} \widetilde{a}(s,\lambda(\mu)) &= \chi_0^\eta(\lambda(\mu))\, \chi_0^\rho(\lambda(\mu)) \\ &\quad \times |\mathbf{c}(\lambda(\mu))|^{-2}\, e^{-s\sqrt{|\lambda(\mu)|^2+|\rho|^2}}\, \det[\, j(\lambda(\mu))\,]^{-1} \end{aligned} \tag{A.5}$$

is smooth and compactly supported in $B(0,\widetilde{\eta}_2)$. We deduce, from (A.3) and (A.4) that $\widetilde{a}(s,\lambda(\mu))$ is bounded, together with all its derivatives. Let $\chi_{\widetilde{\eta}_2} \in \mathcal{C}_c^\infty(\mathfrak{a})$ be a bump function such that $\chi_{\widetilde{\eta}_2} = 1$ on $B(0,\widetilde{\eta}_2)$. Then

$$I_0^-(s,t,x) = e^{it\psi_t(\lambda_0)} \int_{\mathfrak{a}} d\mu\, \chi_{\widetilde{\eta}_2}(\mu)\, e^{it|\mu|^2}\, e^{-|\mu|^2} \{e^{|\mu|^2}\, \widetilde{a}(s,\lambda(\mu))\}.$$

Let $M = [\frac{D}{2}] + 1$ be the smallest integer $> \frac{D}{2}$, the coefficients of the Taylor expansion

$$e^{|\mu|^2}\, \tilde{a}(s,\lambda(\mu)) = \sum_{|k|\le 2M} c_k\, \mu^k + R_{2M}(\mu)$$

at the origin satisfy

$$|c_k| \lesssim |\mathbf{c}(\lambda_0)|^{-2}\, (1+s)^k\, e^{-s\sqrt{|\lambda_0|^2+|\rho|^2}} \lesssim (\tfrac{|x|}{t})^{D-\ell}\, (1+s)^k\, e^{-|\rho| s}, \tag{A.6}$$

according to (A.2), and the remainder satisfies

$$|\nabla_{\mathfrak{a}}^n\, R_{2M}(\mu)| \lesssim |\mu|^{2M+1-n}\, (1+s)^{2M+1+n}\, e^{-|\rho| s} \quad \forall\, 0 \le n \le 2M+1. \tag{A.7}$$

By substituting this expansion in the above integral, $I_0^-(s,t,x)$ is the sum of following three terms:

$$I_1 = \sum_{|k|\le 2M} c_k \int_{\mathfrak{a}} d\mu\, \mu^k\, e^{it|\mu|^2}\, e^{-|\mu|^2}$$

$$I_2 = \int_{\mathfrak{a}} d\mu\, \chi_{\widetilde{\eta}_2}(\mu)\, R_{2M}(\mu)\, e^{it|\mu|^2}\, e^{-|\mu|^2},$$

and

$$I_3 = \sum_{|k|\le 2M} c_k \int_{\mathfrak{a}} d\mu\, \{\chi_{\widetilde{\eta}_2}(\mu) - 1\}\, \mu^k\, e^{it|\mu|^2}\, e^{-|\mu|^2}.$$

To estimate I_1, we write

$$I_1 = \sum_{|k|\le 2M} c_k \prod_{j=1}^{\ell} \int_{-\infty}^{+\infty} d\mu_j \, e^{it\mu_j^2} \, e^{-\mu_j^2} \, \mu_j^{k_j}$$

where

$$\int_{-\infty}^{+\infty} d\mu_j \, e^{-(1-it)\mu_j^2} \, \mu_j^{k_j} = 0$$

if k_j is odd, while

$$\int_{-\infty}^{+\infty} d\mu_j \, e^{-(1-it)\mu_j^2} \, \mu_j^{k_j} = 2\,(1-it)^{-\frac{k_j+1}{2}} \int_0^{+\infty} dz_j \, e^{-z_j^2} \, z_j^{k_j}$$

by a change of contour if k_j is even. We deduce from (A.6)

$$|I_1| \lesssim t^{-\frac{\ell}{2}} \, (\tfrac{|x|}{t})^{D-\ell} \, (1+s)^{2M} \, e^{-|\rho| s} \lesssim t^{-\frac{D}{2}} \, |x|^{-\frac{D-\ell}{2}} \, e^{-\frac{|\rho|}{2} s}$$

since $\frac{|x|}{t} \le C_\Sigma$. Next, we perform M integrations by parts based on

$$e^{it|\mu|^2} = -\tfrac{i}{2t} \, \textstyle\sum_{j=1}^{\ell} \tfrac{\mu_j}{|\mu|^2} \, \tfrac{\partial}{\partial \mu_j} \, e^{it|\mu|^2} \tag{A.8}$$

and obtain

$$|I_2| \lesssim t^{-M} \, (1+s)^{3M+1} \, e^{-|\rho| s} \lesssim t^{-M} \, e^{-\frac{|\rho|}{2} s}$$

according to (A.7). Finally, as $\mu \mapsto \mu^k \, e^{-|\mu|^2} \, (\tilde{\chi}(\mu) - 1)$ is exponentially decreasing and vanishes near the origin, we perform $N \ge \frac{D}{2}$ integrations by parts based on (A.8) again and obtain

$$|I_3| \lesssim t^{-N} \, (1+s)^{2M} \, e^{-\frac{|\rho|}{2} s}.$$

By summing up the estimates of I_1, I_2 and I_3, we deduce that

$$\begin{aligned} |I_0^-(s,t,x)| &\lesssim t^{-\frac{D}{2}} \, (1+|x|)^{\frac{D-\ell}{2}} \, (1+s)^{2d+3} \, e^{-|\rho| s} \\ &\lesssim t^{-\frac{D}{2}} \, (1+|x|)^{\frac{D-\ell}{2}} \, e^{-\frac{|\rho|}{2} s}. \end{aligned} \tag{A.9}$$

Estimate of I_∞^-. Since the phase ψ_t has a unique critical point λ_0 which is defined by (A.1) and satisfies (A.2), then for all $\lambda \in \operatorname{supp} \chi_\infty^\eta$, we have $\nabla_{\mathfrak{a}} \psi_t(\lambda) \neq 0$. In order to get large time decay, we estimate

$$I_\infty^-(s,t,x) = \int_{\mathfrak{a}} d\lambda \, \chi_\infty^\eta(\lambda) \, a_0(s,\lambda) \, e^{it\psi_t(\lambda)}$$

by using several integrations by parts based on

$$e^{it\psi_t(\lambda)} = \tfrac{1}{it} \, \widetilde{\psi}_0(\lambda)^{-1} \, \textstyle\sum_{j=1}^{\ell} \big(\tfrac{\lambda_j}{\sqrt{|\lambda|^2+|\rho|^2}} + \tfrac{A_j}{t} \big) \tfrac{\partial}{\partial \lambda_j} \, e^{it\psi_t(\lambda)},$$

where

$$\widetilde{\psi}_0(\lambda) = \big| \tfrac{\lambda}{\sqrt{|\lambda|^2+|\rho|^2}} + \tfrac{A}{t} \big|^2$$

is a smooth function, which is bounded from below on the compact set $(\operatorname{supp}\chi_\infty^\eta)\cap(\operatorname{supp}\chi_0^\rho)$, uniformly in $\frac{A}{t}$. After performing N such integrations by parts, $I_\infty^-(s,t,x)$ becomes

$$\text{const.}\,(it)^{-N}\int_{\mathfrak{a}}d\lambda\,e^{it\psi_t(\lambda)}$$
$$\times\Big\{-\sum_{j=1}^{\ell}\frac{\partial}{\partial\lambda_j}\circ\big[\widetilde{\psi}_0(\lambda)^{-1}\big(\tfrac{\lambda_j}{\sqrt{|\lambda|^2+|\rho|^2}}+\tfrac{A_j}{t}\big)\big]\Big\}^N\Big\{\chi_\infty^\eta(\lambda)\,a_0(s,\lambda)\Big\}$$

where the last integral is bounded from above by $(1+s)^N\,e^{-|\rho|s}\lesssim e^{-\frac{|\rho|}{2}s}$. Hence

$$|I_\infty^-(s,t,x)|\lesssim t^{-N}\,e^{-\frac{|\rho|}{2}s} \tag{A.10}$$

for every $N\in\mathbb{N}$. By combining (A.9) and (A.10), we conclude that

$$|I^-(s,t,x)|\lesssim t^{-\frac{D}{2}}\,(1+|x|)^{\frac{D-\ell}{2}}\,e^{-\frac{|\rho|}{2}s}.$$

□

Appendix B. Hadamard parametrix on symmetric spaces

Let Φ_v be the K-bi-invariant convolution kernel of the operator $\cos(v\sqrt{-\Delta})$ whose spherical Fourier transform is given by $\widetilde{\Phi}_v(\lambda)=\cos(v\sqrt{|\lambda|^2+|\rho|^2})$. Then $\Phi_v(x)$ solves the following Cauchy problem

$$\begin{cases}\partial_v^2\,U(v,x)-\Delta_x\,U(v,x)=0,\\ U(0,x)=\delta_0(x),\ \ \partial_v|_{v=0}U(v,x)=0.\end{cases}$$

We are looking for the asymptotic expansion of the kernel Φ_v. Recall that J denotes the Jacobian of the exponential map from $\mathfrak{p}$ equipped with Lebesgue measure to $\mathbb{X}$ equipped with Riemannian measure. It satisfies

$$\begin{aligned}J(H)^{-\frac{1}{2}}&=\prod_{\alpha\in\Sigma^+}\Big(\frac{\langle\alpha,H\rangle}{\sinh\langle\alpha,H\rangle}\Big)^{\frac{m_\alpha}{2}}\\&\asymp\Big\{\prod_{\alpha\in\Sigma^+}(1+\langle\alpha,H\rangle)^{\frac{m_\alpha}{2}}\Big\}e^{-\langle\rho,H\rangle}\quad\forall H\in\overline{\mathfrak{a}^+}.\end{aligned}$$

Let f be a K-bi-invariant function on G, then f is also $\operatorname{Ad}K$-invariant on $\mathfrak{p}$ and W-invariant on $\mathfrak{a}$. Recall that $\Delta_{\mathfrak{p}}$ and $\Delta_{\mathfrak{a}}$ denote the usual Laplacian on the Euclidean spaces $\mathfrak{p}$ and $\mathfrak{a}\subset\mathfrak{p}$. The radial part of the Laplacian Δ on $\mathbb{X}$ is defined by

$$\Delta^{\text{rad}}f(H)=\Delta_{\mathfrak{a}}f(H)+\sum_{\alpha\in\Sigma^+}m_\alpha\coth\langle\alpha,H\rangle\partial_\alpha f(H)\quad\forall H\in\mathfrak{a}^+,$$

and that of $\Delta_{\mathfrak{p}}$ is given by

$$\Delta_{\mathfrak{p}}^{\text{rad}}f(H)=\Delta_{\mathfrak{a}}f(H)+\sum_{\alpha\in\Sigma^+}m_\alpha\langle\alpha,H\rangle^{-1}\partial_\alpha f(H)\quad\forall H\in\mathfrak{a}^+,$$

see [19, Propositions 3.9 and 3.11]. The following proposition provides a relation between Δ^{rad} and $\Delta_{\mathfrak{p}}^{\mathrm{rad}}$, it allows us to simplify the computations about the parametrix.

Proposition B.1. *Let $f \in \mathcal{C}^{\infty}(\mathfrak{a})$ be a W-invariant function. Then*

$$\big[J(H)^{\frac{1}{2}} \circ \Delta^{\mathrm{rad}} \circ J(H)^{-\frac{1}{2}}\big] f(H) = \big[\Delta_{\mathfrak{p}}^{\mathrm{rad}} + \omega(H)\big] f(H) \quad \forall H \in \mathfrak{a},$$

where

$$\omega(H) = \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{2}\Big(\frac{m_\alpha}{2} - 1\Big)|\alpha|^2 \Big\{\frac{1}{\langle \alpha, H\rangle} - \frac{1}{\sinh^2\langle \alpha, H\rangle}\Big\}$$
$$+ \sum_{\substack{\alpha \in \Sigma^+ \\ \text{s.t. } 2\alpha \in \Sigma^+}} \frac{m_\alpha m_{2\alpha}}{2}|\alpha|^2\Big(\frac{m_\alpha}{2} - 1\Big)|\alpha|^2 \Big\{\frac{1}{\langle \alpha, H\rangle} - \frac{1}{\sinh^2\langle \alpha, H\rangle}\Big\} - |\rho|^2$$

is a smooth W-invariant function, which is uniformly bounded together with all its derivatives.

Proof. Notice that

$$\big[J(H)^{\frac{1}{2}} \circ \Delta^{\mathrm{rad}} \circ J(H)^{-\frac{1}{2}}\big] f(H) = J(H)^{\frac{1}{2}}\big(\Delta^{\mathrm{rad}} J^{-\frac{1}{2}}\big)(H) f(H)$$
$$+ \underbrace{\Delta^{\mathrm{rad}} f(H) + 2J(H)^{\frac{1}{2}}\big(\nabla_{\mathfrak{a}} J^{-\frac{1}{2}}\big)(H) \cdot \nabla_{\mathfrak{a}} f(H)}_{\Delta_{\mathfrak{p}}^{\mathrm{rad}} f(H)},$$

since

$$J(H)^{\frac{1}{2}}\big(\nabla_{\mathfrak{a}} J^{-\frac{1}{2}}\big)(H) = \sum_{\alpha \in \Sigma^+} \frac{m_\alpha}{2}\Big\{\frac{1}{\langle \alpha, H\rangle} - \coth\langle \alpha, H\rangle\Big\}\alpha.$$

We deduce from the next lemma that

$$J(H)^{\frac{1}{2}}\big(\Delta^{\mathrm{rad}} J^{-\frac{1}{2}}\big)(H) = \omega(H) \quad \forall H \in \mathfrak{a},$$

and this conclude the proof. □

Lemma B.2 (Cancellations)**.** *The following equations hold for all $H \in \mathfrak{a}$:*

$$\sum_{\alpha,\beta \in \Sigma^+,\ \mathbb{R}\alpha \neq \mathbb{R}\beta} m_\alpha m_\beta \frac{\langle \alpha, \beta\rangle}{\langle \alpha, H\rangle\langle \beta, H\rangle} = 0$$

$$\sum_{\alpha,\beta \in \Sigma^+,\ \mathbb{R}\alpha \neq \mathbb{R}\beta} m_\alpha m_\beta \langle \alpha, \beta\rangle\big(\coth\langle \alpha, H\rangle \coth\langle \beta, H\rangle - 1\big) = 0$$

Proof. See [16, Appendix] for a detailed proof of this "*folklore*" result. □

Recall that $\{R_+^z \mid z \in \mathbb{C}\}$ denotes the analytic family of Riesz distributions on $\mathbb{R}$ defined by

$$R_+^z(r) = \begin{cases} \Gamma(z)^{-1} r^{z-1} & \text{if } r > 0, \\ 0 & \text{if } r \leq 0. \end{cases}$$

Consider the asymptotic expansion

$$\Phi_v(\exp H) = J(H)^{-\frac{1}{2}} \sum_{k=0}^{+\infty} 4^{-k}\,|v|\,U_k(H)\,R_+^{k-\frac{d-1}{2}}(v^2-|H|^2) \tag{B.1}$$

where U_0 is a constant such that $U_0\,J(H)^{-\frac{1}{2}}\,|v|\,R_+^{-\frac{d-1}{2}}(v^2-|H|^2) \to \delta_0(H)$ as $v \to 0$ and $U_k \in \mathcal{C}^\infty(\mathfrak{p})$ are smooth $\operatorname{Ad} K$-invariant functions. By expanding

$$\begin{aligned} 0 &= J(H)^{\frac{1}{2}}\big[\partial_v^2 - \Delta^{\mathrm{rad}}\big]\Phi_v(\exp H) \\ &= \sum_{k=0}^{+\infty} 4^{-k}\,\big[\partial_v^2 - \Delta_{\mathfrak{p}}^{\mathrm{rad}} - \omega(H)\big]\Big\{|v|\,U_k(H)\,R_+^{k-\frac{d-1}{2}}(v^2-|H|^2)\Big\}, \end{aligned}$$

we deduce

$$\big[(k+1)+\partial_H\big]U_{k+1}(H) = \big[\Delta_{\mathfrak{p}}^{\mathrm{rad}} + \omega(H)\big]U_k(H), \tag{B.2}$$

for every $k \in \mathbb{N}$. In other words,

$$U_{k+1}(H) = \int_0^1 ds\, s^k\,\big[\Delta_{\mathfrak{p}}^{\mathrm{rad}} + \omega(sH)\big]U_k(sH). \tag{B.3}$$

As ω and all its derivatives are uniformly bounded, we obtain

$$\nabla_{\mathfrak{p}}^n U_k = O(1) \tag{B.4}$$

for any $k,n \in \mathbb{N}$.

Next, by resuming the proof of [7, Proposition 27] with our asymptotic expansion (B.1), we deduce that the remainder of the truncated expansion

$$\begin{aligned} \Phi_v(\exp H) = J(H)^{-\frac{1}{2}} &\sum_{k=0}^{N} 4^{-k}\,|v|\,U_k(H)\,R_+^{k-\frac{d-1}{2}}(v^2-|H|^2) \\ &+ E_N(v,\exp H) \end{aligned} \tag{B.5}$$

is a solution to the inhomogeneous Cauchy problem

$$\begin{cases} \big[\partial_v^2 - \Delta^{\mathrm{rad}}\big]E_N(v,\exp H) = J(H)^{-\frac{1}{2}}\widetilde{U}_N(v,H), \\ \lim_{v\to 0} E_N(v,\exp H) = 0,\ \lim_{v\to 0}\frac{\partial E_N}{\partial v}(v,\exp H) = 0, \end{cases}$$

where $\widetilde{U}_N(v,H) = -4^{-N}\,|v|\,U_N(H)\,R_+^{N-\frac{d-1}{2}}(v^2-|H|^2)$. Hence, by Duhamel's formula

$$E_N(v,\exp H) = \int_0^v du\,\tfrac{\sin(v-u)\sqrt{-\Delta^{\mathrm{rad}}}}{\sqrt{-\Delta^{\mathrm{rad}}}}\{J(H)^{-\frac{1}{2}}\,\widetilde{U}_N(u,H)\}.$$

According to next lemma and by L^2 conservation, we have

$$\begin{aligned} |E_N(v,\exp H)| &\lesssim e^{-\langle\rho,H\rangle}\,\|E_N(v,\cdot)\|_{H^{2\sigma+1}(\mathbb{X})} \\ &\lesssim e^{-\langle\rho,H\rangle}\int_0^v du\,\|\widetilde{U}_N(u,\cdot)J^{-\frac{1}{2}}\|_{H^{2\sigma}(\mathbb{X})} \end{aligned}$$

provided that $2\sigma+1>\frac{d}{2}$, and

$$\begin{aligned}
\|\widetilde{U}_N(u,\cdot)J^{-\frac12}\|^2_{H^{2\sigma}(\mathbb{X})} &= \|\Delta^{\sigma}\{\widetilde{U}_N(u,\cdot)J^{-\frac12}\}\|^2_{L^2(\mathbb{X})}\\
&= \text{const.}\int_{\mathfrak{p}} dX\,\big|J(X)^{\frac12}(\Delta^{\mathrm{rad}})^{\sigma}\{J(X)^{-\frac12}\widetilde{U}_N(u,X)\}\big|^2\\
&= \text{const.}\; u^2\int_{\mathfrak{p}} dX\,\big|[\Delta^{\mathrm{rad}}_{\mathfrak{p}}+\omega(X)]^{\sigma}(\widetilde{U}_N(u,X))\big|^2\\
&\lesssim u^2\sum_{j=0}^{2\sigma}\int_{\{X\in\mathfrak{p}\,|\,|X|<u\}} dX\,\big|\nabla^j_{\mathfrak{p}}(u^2-|X|^2)^{N-\frac{d+1}{2}}\big|^2\\
&\lesssim (1+u)^{4N-d},
\end{aligned}$$

since ω and U_N, together with all their derivatives are uniformly bounded. Here we assume $N>\frac{d+1}{2}+2\sigma$ to avoid possible singularities on the sphere $|X|=u$. We may set $2\sigma=\left[\frac{d}{2}\right]$ and $N>d$. Finally, we obtain

$$|E_N(v,\exp H)|\lesssim e^{-\langle\rho,H\rangle}\int_0^v du\,(1+u)^{2N-\frac{d}{2}}\lesssim(1+v)^{2N-\frac{d}{2}+1}e^{-\langle\rho,H\rangle}.$$

Lemma B.3 (Sobolev embedding theorem for K-bi-invariant functions on $\mathbb{X}$)**.** *Let $\sigma>\frac{d}{2}$ be an integer. Then*

$$|f(\exp H)|\lesssim e^{-\langle\rho,H\rangle}\|f\|_{H^{\sigma}(\mathbb{X})}\quad\forall\,H\in\overline{\mathfrak{a}^+}$$

for all K-bi-invariant functions $f\in H^{\sigma}(\mathbb{X})$.

Proof. See [1, Lemma 2.3]. □

Notice that, for all $N>\frac{d}{2}$, we have

$$\Big|J(H)^{-\frac12}\sum_{k=[d/2]+1}^{N}4^{-k}\,|v|\,U_k(H)R_+^{k-\frac{d-1}{2}}(v^2-|H|^2)\Big|\lesssim(1+v)^{2N-\frac{d+\ell}{2}}\,e^{-\langle\rho,H\rangle}.$$

Then we deduce the following corollary.

Corollary B.4. *The K-bi-invariant convolution kernel Φ_v has the asymptotic expansion*

$$\begin{aligned}
\Phi_v(\exp H) = J(H)^{-\frac12}&\sum_{k=0}^{[d/2]}4^{-k}\,|v|\,U_k(H)\,R_+^{k-\frac{d-1}{2}}(v^2-|H|^2)\\
&+E_{\Phi}(v,H),
\end{aligned}\tag{B.6}$$

where the remainder satisfies

$$|E_{\Phi}(v,H)|\lesssim(1+v)^{3(\frac{d}{2}+1)}e^{-\langle\rho,H\rangle}\quad\forall\,H\in\overline{\mathfrak{a}^+}.\tag{B.7}$$

Appendix C. Asymptotic expansion of the Poisson kernel

The Hadamard parametrix described above provides an asymptotic development of the kernel of the truncated Poisson operator

$$A_\tau = \int_{-\infty}^{+\infty} dv\ \chi_T(v) p_\tau^{\mathbb{R}}(v) \cos(v\sqrt{-\Delta}).$$

Here $\tau = s - it$ with $s \in (0,1]$ and $t \in \mathbb{R}^*$,

$$T = \begin{cases} \sqrt{2} & \text{if } 0 < |t| \le 1, \\ \sqrt{2}|t| & \text{if } |t| \ge 1, \end{cases}$$

$\chi : \mathbb{R} \to [0,1]$ is a smooth even cut-off function such that $\chi = 1$ on $[-1,1]$ and $\operatorname{supp} \chi \subset [-2\sqrt{2}, 2\sqrt{2}]$, $\chi_T(v) = \chi(\frac{v}{2T})$ is supported in $[-2\sqrt{2}T, 2\sqrt{2}T] \subset (-3T, 3T)$, and $p_\tau^{\mathbb{R}}(v) = \frac{1}{\pi}\frac{\tau}{\tau^2+v^2}$ is the Poisson kernel on $\mathbb{R}$ (with complex time τ). Notice that $|\tau| \le T$. By resuming and improving slightly [9, Lemma 3.3], we deduce the following parametrix for the kernel of A_τ.

Proposition C.1. *The kernel a_τ of the operator A_τ is a smooth K-bi-invariant function on G, which is supported in the ball of radius $3T$ in $\mathbb{X}$. Moreover*

$$a_\tau(\exp H) = \frac{\tau}{\pi} J(H)^{-\frac{1}{2}} \sum_{k=0}^{[d/2]} 4^{-k} U_k(H) \Gamma\Big(\frac{d+1}{2} - k\Big) (|H|^2 + \tau^2)^{\frac{d+1}{2}-k} + E(\tau, H) \tag{C.1}$$

where the remainder satisfies

$$|E(\tau,H)| \lesssim |T|^{3(\frac{d}{2}+1)} (\log T - \log s)\, e^{-\langle \rho, H\rangle} \quad \forall\, H \in \overline{\mathfrak{a}^+}. \tag{C.2}$$

Here the coefficients U_k are the same as in Corollary B.4 and are uniformly bounded.

Remark C.2. *The proof of Proposition C.1 is similar to the proof of Lemma 3.3 in [9]. Notice that the latter statement contains a minor error in the Gamma factor and that our estimates contain an additional exponential decay, which is crucial for the dispersive estimates.*

Let us state and reprove some technical results borrowed from [9].

Lemma C.3. *Let $n \ge 1$ and $\gamma \in \mathbb{R}_+$. Then*

$$|z|^{2\gamma - n} \int_0^{3T} dr\ r^{n-1} |r^2 + z^2|^{-\gamma} \asymp$$

$$\asymp \begin{cases} (\frac{|z|}{\operatorname{Re} z})^{\gamma-1} & \text{if } \gamma > 1 \text{ and } n < 2\gamma, \\ (\frac{T}{|z|})^{n-2} + \log(\frac{|z|}{\operatorname{Re} z}) & \text{if } \gamma = 1 \text{ and } n > 2, \\ 1 + \log(\frac{T}{\operatorname{Re} z}) & \text{if } \gamma = 1 \text{ and } n = 2, \\ 1 + \log(\frac{|z|}{\operatorname{Re} z}) & \text{if } \gamma = 1 \text{ and } n < 2. \end{cases} \tag{C.3}$$

for every $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$ and $|z| \le T$.

Proof. Write $z = |z|e^{i\theta}$ in polar coordinates, with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. By performing the change of variables $r = |z|w$, the left hand side of (C.3) becomes

$$I = \int_0^{\frac{3T}{|z|}} dw\ w^{n-1} |w^2 + e^{i2\theta}|^{-\gamma}.$$

Notice that $\frac{3T}{|z|} > 2$ and that

$$|w^2 - 1| \leq |w^2 + e^{i2\theta}| \leq |w^2 + 1|. \tag{C.4}$$

Let us split up $I = I_0 + I_1 + T_\infty$ according to

$$\int_0^{\frac{3T}{|z|}} dw\ = \int_0^{\frac{1}{2}} dw\ + \int_{\frac{1}{2}}^{2} dw\ + \int_2^{\frac{3T}{|z|}} dw.$$

The first and the last integrals are easily estimated. According (C.4),

$$\begin{cases} \frac{3}{4} \leq |w^2 + e^{i2\theta}| \leq \frac{5}{4} & \text{if } 0 < w \leq \frac{1}{2}, \\ \frac{3}{4} w^2 \leq |w^2 + e^{i2\theta}| \leq \frac{5}{4} w^2 & \text{if } w \geq 2, \end{cases}$$

we deduce

$$I_0 = \int_0^{\frac{1}{2}} dw\ w^{n-1} \asymp 1 \tag{C.5}$$

and

$$I_\infty = \int_2^{\frac{3T}{|z|}} dw\ w^{n-2\gamma-1} \asymp \begin{cases} 1 & \text{if } n < 2\gamma, \\ 1 + \log \frac{T}{|z|} & \text{if } n = 2\gamma, \\ (\frac{T}{|z|})^{n-2\gamma} & \text{if } n > 2\gamma. \end{cases} \tag{C.6}$$

Let us turn to the remaining integral, where $\frac{1}{2} \leq w \leq 2$. In this case we use the following improvement of (C.4):

$$\begin{aligned} |w^2 + e^{i2\theta}|^2 &= w^2 + 1 + 2w^2 \cos 2\theta = (w^2 - 1)^2 + 4w^2 \cos^2 \theta \\ &\asymp \big(w - \frac{1}{w}\big)^2 + \cos^2 \theta. \end{aligned}$$

By performing the change of variables $u = w - \frac{1}{w}$ and noticing that $\frac{du}{dw} = 1 + \frac{1}{w^2} \asymp 1$, we get

$$\begin{aligned} I_1 &\asymp \int_{-\frac{3}{2}}^{\frac{3}{2}} du\ (u^2 + \cos^2 \theta)^{-\frac{\gamma}{2}} \asymp \int_0^{\frac{3}{2}} du\ (u + \cos \theta)^{-\gamma} \\ &\asymp \begin{cases} (\cos \theta)^{-\gamma-1} & \text{if } \gamma > 1, \\ 1 - \log(\cos \theta) & \text{if } \gamma = 1, \\ 1 & \text{if } \gamma < 1. \end{cases} \end{aligned} \tag{C.7}$$

In conclusion, (C.3) is obtained by combining (C.5), (C.6) and (C.7). $\square$

Lemma C.4. *Let* $z \in \mathbb{C}$ *with* $\operatorname{Re} z > 0$ *and* $u \in \mathbb{R}$. *Then*

$$\int_0^{+\infty} d(w^2)\ R_+^{1-\varepsilon}(w^2-u^2)\frac{1}{\pi}\frac{\tau}{w^2+z^2} = \begin{cases} \frac{1}{\pi}\frac{z}{u^2+z^2} & \text{if } \varepsilon = 1, \\ \frac{1}{\sqrt{\pi}}\frac{z}{\sqrt{u^2+z^2}} & \text{if } \varepsilon = \frac{1}{2}. \end{cases} \tag{C.8}$$

Proof. The case $\varepsilon = 1$ follows immediately from the fact the distribution R_+^0 is equal to the Dirac measure at the origin. In the case $\varepsilon = \frac{1}{2}$, the formula is proved first for $z > 0$ and then extended straightforwardly by analytic continuation to all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Specifically, the left hand side of (C.8) becomes

$$\pi^{-\frac{3}{2}} \int_0^{+\infty} \frac{d(w^2)}{w} \frac{z}{w^2+u^2+z^2} = \underbrace{2\pi^{-\frac{3}{2}} \int_0^{+\infty} \frac{dr}{r^2+1}}_{\frac{1}{\sqrt{\pi}}} \frac{z}{\sqrt{u^2+z^2}}$$

after performing the change of variables $w = \sqrt{u^2+z^2}\,r$. ☐

Proof of Proposition C.1. According to the asymptotic expansion (B.6), we write

$$a_\tau(\exp H) = J(H)^{-\frac{1}{2}} \sum_{k=0}^{[d/2]} 4^{-k}\, U_k(H)\, I_k(\tau,H) + E(\tau,H)$$

with

$$I_k(\tau,H) = \int_0^{+\infty} d(v^2)\, p_\tau^{\mathbb{R}}(v)\, R_+^{k-\frac{d-1}{2}}(v^2-|H|^2)$$

and

$$\begin{aligned} E(\tau,H) = {} & J(H)^{-\frac{1}{2}} \sum_{k=0}^{[d/2]} 4^{-k}\, U_k(H) \\ & \times \int_0^{+\infty} d(v^2)\, \{\chi_T(v)-1\}\, p_\tau^{\mathbb{R}}(v)\, R_+^{k-\frac{d-1}{2}}(v^2-|H|^2) \\ & +\ 2\int_0^{+\infty} dv\, \chi_T(v)\, p_\tau^{\mathbb{R}}(v)\, E_\Phi(v,H) \end{aligned}$$

Let $\varepsilon = 1$ if d is even and $\varepsilon = \frac{1}{2}$ if d odd. Then

$$I_k(\tau,H) = \big(-\tfrac{\partial}{\partial(|H|^2)}\big)^{[\frac{d}{2}]-k} \textstyle\int_0^{+\infty} d(v^2)\, p_\tau^{\mathbb{R}}(v)\, R_+^{1-\varepsilon}(v^2-|H|^2)$$

where

$$\int_0^{+\infty} d(v^2)\, p_\tau^{\mathbb{R}}(v)\, R_+^{1-\varepsilon}(v^2-|H|^2) = \begin{cases} \frac{1}{\pi}\frac{\tau}{|H|^2+\tau^2} & \text{if } \varepsilon = 1, \\ \frac{1}{\sqrt{\pi}}\frac{\tau}{\sqrt{|H|^2+\tau^2}} & \text{if } \varepsilon = \frac{1}{2}, \end{cases}$$

according to (C.8). Then we obtain

$$I_k(\tau,H) = \frac{\tau}{\pi}\, \frac{\Gamma(\frac{d+1}{2}-k)}{(|H|^2+\tau^2)^{\frac{d+1}{2}-k}}.$$

Next, we estimate the remainder $E(\tau,H)$ whose second part is easily handled. By using (B.7), we have

$$|E_2(\tau,H)| \le 2\int_0^{+\infty} dv\, \chi_T(v)\, |p_\tau^{\mathbb{R}}(v)|\, |E_\Phi(v,H)|$$
$$\lesssim e^{-\langle\rho,H\rangle} \int_0^{3T} dv\, \frac{|\tau|}{|v^2+\tau^2|}\, (1+v^{3(\frac{d}{2}+1)}).$$

where

$$|\tau| \int_0^{3T} dv\, |v^2+\tau^2|^{-1} \lesssim 1+\log\tfrac{|\tau|}{\operatorname{Re}\tau}$$

and

$$|\tau| \int_0^{3T} dv\, |v^2+\tau^2|^{-1}\, v^{3(\frac{d}{2}+1)} \lesssim |\tau|^{3(\frac{d}{2}+1)} \big\{ \big(\tfrac{T}{|\tau|}\big)^{\frac{3d+1}{2}} + \log\tfrac{|\tau|}{\operatorname{Re}\tau} \big\}$$

according to the formulas in (C.3). We deduce

$$|E_2(\tau,H)| \lesssim T^{3(\frac{d}{2}+1)} (\log T - \log s) e^{-\langle\rho,H\rangle}. \tag{C.9}$$

It remains to estimate

$$E_1(\tau,H) = J(H)^{-\frac12} \sum_{k=0}^{[d/2]} 4^{-k}\, U_k(H)$$
$$\times \underbrace{\int_0^{+\infty} d(v^2)\, \{\chi_T(v)-1\} p_\tau^{\mathbb{R}}(v)\, R_+^{k-\frac{d-1}{2}}(v^2-|H|^2)}_{\widetilde{I}_k(\tau,H)}.$$

By repeating the previous calculations for I_k,

$$\widetilde{I}(\tau,H) = \frac{\tau}{\pi} \big(-\tfrac{\partial}{\partial(|H|^2)}\big)^{[\frac{d}{2}]-k}$$
$$\int_0^{+\infty} d(v^2)\, \{\chi_T(\sqrt{v^2+|H|^2})-1\}\, \frac{1}{v^2+|H|^2+\tau^2}\, R_+^{1-\varepsilon}(v^2).$$

Let's first consider the case where $\varepsilon=1$, i.e., d is odd. Then

$$\widetilde{I}_k(\tau,H) = \frac{\tau}{\pi} \big(-\tfrac{\partial}{\partial(|H|^2)}\big)^{\frac{d-1}{2}-k} \Big\{ \big(\chi_T(|H|)-1\big) \tfrac{1}{|H|^2+\tau^2} \Big\}$$
$$= \frac{\tau}{\pi} \sum_{j+j'=\frac{d-1}{2}-k} \tfrac{(\frac{d-1}{2}-k)!}{j!j'!} \big(-\tfrac{\partial}{\partial(|H|^2)}\big)^{j} \big(\chi_T(|H|)-1\big) \big(-\tfrac{\partial}{\partial(|H|^2)}\big)^{j'} \tfrac{1}{|H|^2+\tau^2}.$$

On the one hand, the expression $\big(-\tfrac{\partial}{\partial(|H|^2)}\big)^{j}\big(\chi_T(|H|)-1\big)$ vanishes when $|H|\le 2T$. In addition, it is $O(T^{-2j})$. On the other hand,

$$\big(-\tfrac{\partial}{\partial(|H|^2)}\big)^{j'} \tfrac{1}{|H|^2+\tau^2} = \tfrac{j'!}{(|H|^2+\tau^2)^{j'+1}} = O(T^{-2j'-2})$$

when $|H|\ge 2T$. We deduce that

$$|\widetilde{I}_k(\tau,H)| = O(T^{2k-d}).$$

Let's then consider the case where $\varepsilon=\frac{1}{2}$, i.e., d is even. Then

$$\widetilde{I}_k(\tau,H)=\frac{2\tau}{\pi}\sum_{j+j'=\frac{d}{2}-k}\frac{(\frac{d}{2}-k)!}{j!}$$

$$\times\int_0^{+\infty}dv\,\big(-\tfrac{\partial}{\partial(|H|^2)}\big)^j\{\chi_T(\sqrt{v^2+|H|^2})-1\}\,(v^2+|H|^2+\tau^2)^{-j'-1}.$$

Again, the expression $\big(-\tfrac{\partial}{\partial(|H|^2)}\big)^j\{\chi_T(\sqrt{v^2+|H|^2})-1\}$ is $O(T^{-2j})$, which vanishes when $v^2+|H|^2\le 4T^2$, as well as $v^2+|H|^2\ge 9T^2$ if $j>0$. It follows that the integral above is $O(T^{2k-d-1})$ if $j>0$, and that it is estimated by

$$\int_{v^2+|H|^2\ge 4T^2}dv\,\big|v^2+|H|^2+\tau^2\big|^{k-\frac{d}{2}-1}\lesssim\int_{v+|H|\ge 2T}dv\,(v+|H|)^{2k-d-2}$$

$$\asymp T^{2k-d-1}$$

if $j=0$. In any case, we obtain

$$|\widetilde{I}_k(\tau,H)|\lesssim T^{2k-d}\lesssim 1$$

and therefore

$$|E_1(\tau,H)|\lesssim J(H)^{-\frac{1}{2}}$$

since the coefficients U_k are bounded. By combining with (C.9), we conclude that

$$|E(\tau,H)|\lesssim|E_1(\tau,H)|+|E_1(\tau,H)|\lesssim T^{3(\frac{d}{2}+1)}\,(\log T-\log s)\,e^{-\langle\rho,H\rangle}$$

for all $H\in\overline{\mathfrak{a}^+}$. □

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