HYPERBOLICITY AND BIFURCATIONS IN HOLOMORPHIC FAMILIES OF POLYNOMIAL SKEW PRODUCTS

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We study holomorphic families of polynomial skew products, i.e., polynomial endomorphisms of $\mathbb{C}^2$ of the form $F(z,w) = (p(z), q(z,w))$ that extend to holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$. We prove that stability in the sense of [BBD18] preserves hyperbolicity within such families, and give a complete classification of the hyperbolic components that are the analogue, in this setting, of the complement of the Mandelbrot set for the family $z^2 + c$. We also precisely describe the geometry of the bifurcation locus and current near the boundary of the parameter space. One of our tools is an asymptotic equidistribution property for the bifurcation current. This is established in the general setting of families of endomorphisms of $\mathbb{P}^k$, and is the first equidistribution result of this kind for holomorphic dynamical systems in dimension larger than one.

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1. Introduction and results

A polynomial skew product is a polynomial endomorphism of $\mathbb{C}^2$ of the form $F(z,w) = (p(z), q(z,w))$ that extends to an endomorphism of $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$. The dynamics of these maps was studied in detail in [Jon99]. Despite (and actually because of) their specific form, they have already provided examples of dynamical phenomena not displayed by one-dimensional polynomials, see for instance [Ast+16; Duj16; Duj17; Taf21]. Their understanding has already proved to be a necessary step in the study of endomorphisms of $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$ in any dimension. In this paper we address the question of understanding the dynamical stability of such maps. In order to do this, let us first introduce the framework for our work.

A *holomorphic family* of endomorphisms of $\mathbb{P}^k$ is a holomorphic map $f\colon M \times \mathbb{P}^k \to M \times \mathbb{P}^k$ of the form $f(\lambda, z) = (\lambda, f_\lambda(z))$. The complex manifold M is the *parameter space* and we require that all the endomorphisms f_λ of $\mathbb{P}^k$ have the same degree. In dimension $k = 1$, the study of *stability* and *bifurcation* within such families was initiated

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by Mané-Sad-Sullivan [MSS83] and Lyubich [Lyu83] in the 80's. They proved that many natural definitions of stability are equivalent, allowing one to decompose the parameter space M of any holomorphic family of rational maps f_λ into a *stability locus* and a *bifurcation locus*. Moreover, their notion of stability preserves hyperbolicity, in the sense that if at least one parameter in a given component of the stability locus in M is hyperbolic, then all parameters in this component are hyperbolic. This fact is of crucial importance for the theory. In 2000, by means of the *Lyapunov function* $L(f_\lambda)$ on M, DeMarco [DeM01] constructed a natural *bifurcation current* $T_{\text{bif}} := dd^c_\lambda L(f_\lambda)$ precisely supported on the bifurcation locus. This allowed for the start of a pluripotential study of the bifurcations of rational maps.

The theory by Mané-Sad-Sullivan, Lyubich and DeMarco was recently extended to family of endomorphisms of $\mathbb{P}^k$ in any dimension by Berteloot, Dupont, and the second author [BBD18; Bia19] (see also [DL15; BD17] for a parallel theory in the setting of polynomial diffeomorphisms of $\mathbb{C}^2$). Despite the quite precise understanding of the relation between the various phenomena related to stability and bifurcation (motion of the repelling cycles, Lyapunov function, Misiurewicz parameters), apart from specific examples ([BT17]) or near special parameters ([BB18a; Duj17; Taf21; Bie19]), we still miss a concrete and somehow general family whose bifurcations can be explicitly exhibited and studied, which may possibly play the role of the quadratic family z^2+c for the higher dimensional theory. Moreover, it is an open question whether stability preserves hyperbolicity in this context (observe that this is the case for polynomial diffeomorphisms of $\mathbb{C}^2$ [BD17]).

This paper aims at the precise understanding of the phenomena above, hyperbolicity *in primis*, within families of polynomial skew products.

1.1. **Main results.** While many of the results apply to more general families, we mainly focus here on the family of *quadratic skew products*, i.e., polynomial skew products of (algebraic) degree 2, that are in this context the analogue of the family z^2+c. By means of an affine change of coordinates, the dynamical study of this family can be reduced to that of the family

$$(*) \qquad f_\lambda : (z,w) \mapsto (z^2+d, w^2+az^2+bz+c)$$

with d and $\lambda := (a,b,c)$ as (complex) parameters. Since bifurcations due to the parameter d are of one-dimensional nature, we fix here $p(z) := z^2+d$ and consider the parameter space $\mathbb{C}^3$ of the family $\mathbf{Sk}(p,2) := \{f_\lambda : (a,b,c) \in \mathbb{C}^3\}$.

We are especially interested in parameters near the boundary of this space, i.e., near the hyperplane at infinity, that we denote by $\mathbb{P}^2_\infty$. The following is our first main result, giving a complete description of the bifurcation locus near $\mathbb{P}^2_\infty$ from both a topological and measure-theoretical point of view. We denote by J_p the Julia set of p. Given $z \in \mathbb{C}$, we set $E_z := \{\,[a,b,c] : az^2+bz+c=0\,\} \subset \mathbb{P}^2_\infty$ and $E := \cup_{z\in J_p} E_z$. An analogous result for quadratic rational maps is proved in [BG15a].

Theorem A. *The accumulation on $\mathbb{P}^2_\infty$ of the bifurcation locus of the family* (*) *coincides with E. Moreover, the bifurcation current T_{bif} on $\mathbb{C}^3$ extends as a positive closed current $\hat{T}_{\text{bif}}$ to $\mathbb{P}^3 = \mathbb{C}^3 \cup \mathbb{P}^2_\infty$ and*

$$\hat{T}_{\text{bif}} \wedge [\mathbb{P}^2_\infty] = \int_{J_p} [E_z]\mu_p(z).$$

The proof of this result relies on several ingredients. The first is a decomposition for the bifurcation current (and locus), see Theorem 3.3. We then prove that special dynamically defined hypersurfaces $\mathrm{Per}_n^v(\eta)$ in $\mathbb{C}^3$ *equidistribute* towards the bifurcation current T_{bif} (and $\hat{T}_{\mathrm{bif}}$), see the next Section 1.2 for more details. Moreover, we can precisely control the intersections of these hypersurfaces with $\mathbb{P}^2_\infty$. We thus obtain the convergences

$$\frac{1}{d^{2n}}[\mathrm{Per}_n^v(\eta)] \to \hat{T}_{\mathrm{bif}} \quad \text{and} \quad \frac{1}{d^{2n}}[\mathrm{Per}_n^v(\eta)] \wedge [\mathbb{P}^2_\infty] \to \int_{z \in J_p} [E_z]\mu_p.$$

Theorem A then reduces to proving that the convergences above imply that

$$\frac{1}{d^{2n}}[\mathrm{Per}_n^v(\eta)] \wedge [\mathbb{P}^2_\infty] \to \hat{T}_{\mathrm{bif}} \wedge [\mathbb{P}^2_\infty],$$

which is a problem of intersection of currents. To do this, we exploit the theory of *horizontal positive closed currents* as developed by Dujardin [Duj04], see also [DS06]. This requires proving some uniform estimates on the directions at which the bifurcation locus approaches $\mathbb{P}^2_\infty$.

Once the bifurcation locus near the hyperplane at infinity is understood, we turn our attention to its complement, and in particular to the characterization of the *hyperbolic components*. Notice that, in order for those to exist, p must be hyperbolic.

The stability of a polynomial skew product as in (*) is determined by the behaviour of the critical points of the form $(z, 0)$ with $z \in J_p$. As is the case for polynomials, when all these points escape to infinity by iteration, the map is hyperbolic. It is however not clear a priori that the presence of a hyperbolic map in a component forces all the other maps in the same stability component to be hyperbolic.

In our next result not only do we solve this general problem in the setting of polynomial skew products (thus giving meaning to the expression *hyperbolic components* here), but we also give a complete classification of hyperbolic components that are analoguous to the so-called *shift locus* from dimension 1.

More precisely, let $\mathcal{D}$ be the set of parameters for which all critical points in $J_p \times \mathbb{C}$ escape, and let $\mathcal{D}' \subset \mathcal{D}$ be the subset of parameters λ for which there is an arc joining λ to $\mathbb{P}^2_\infty \backslash E$ inside $\mathcal{D}$. Set

$$\mathcal{S}_p := \Big\{ s : \pi_0(\mathring{K}_p) \to \{0, 1, 2\} : \sum_{U \in \pi_0(\mathring{K}_p)} s(U) \leq 2 \Big\},$$

where $\pi_0(\mathring{K}_p)$ denotes the set of bounded Fatou components of p.

Theorem B. *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of polynomial skew products.*

(1) Any f_λ in a stable component containing a hyperbolic parameter is hyperbolic.

(2) Assume that $M = \mathbf{Sk}(p, 2)$. All connected components of $\mathcal{D}'$ are hyperbolic components, and there is a natural bijection between $\mathcal{S}_p$ and the connected components of $\mathcal{D}'$.

The condition of the base polynomial p of being hyperbolic is actually not necessary, if we replace hyperbolicity with *vertical expansion*, see [Jon99] and Section 2.2. Our Theorem holds in this case too (see Section 5), and proves that stability preserves vertical expansion (Theorem 5.1), and gives a classification of vertical expanding component (Theorem 5.7).

The proof of the first item of Theorem B is based on a characterization of hyperbolicity (and vertical expansion) due to Jonsson (see Theorem 2.3) based on the (non) accumulation of the postcritical set on the Julia set. Our task is to prove that stability preserves this equivalent notion. The proof of the second item is topological in nature. Our main task is to exclude that a given hyperbolic component can accumulate two distinct components of $\mathbb{P}^2_\infty \backslash E$. To prove this, we show that the combinatorial invariants $s \in \mathcal{S}_p$ encode the isotopy class of the Julia set in $J_p \times \mathbb{C}$.

1.2. **Equidistribution towards the bifurcation current.** As mentioned above, one of our main tools in the proof of Theorem A is an approximation result for the bifurcation current by means of dynamically defined hypersurfaces in the parameter space. In dimension 1, the idea of seeing T_{bif} as a limit of currents detecting dynamically interesting parameters goes back to Levin [Lev82] (see also [Lev90]), who proved that the centres of the hyperbolic components of the Mandelbrot set equidistribute the bifurcation current, which is supported on its boundary. This result was later generalized in order to cover any family of polynomials (and actually rational maps) [BB11; Oku14], the distribution of maps with a cycle of any given multiplier [BB11; BG15b; Gau16; GOV19] or with preperiodic critical points [DF08; FG15].

In our situation, in the proof of Theorem A we need an equidistribution property towards T_{bif} of the parameters admitting a periodic point with *vertical* multiplier η. Since the same techniques allow to prove a general result valid for any family of endomorphisms of $\mathbb{P}^k$, in any dimensions k, we give a full proof of this in the Section 6 The following is also one of our main results: it is the first equidistribution result in the parameter space for holomorphic dynamical systems in dimension larger than one.

Theorem C. *Let $(f_\lambda)_{\lambda \in M}$ be the family of all holomorphic endomorphisms of $\mathbb{P}^k$ of a given degree $d \geq 2$. For all $\eta \in \mathbb{C}$ outside of a polar subset, we have*

$$\frac{1}{d^{2n}}[\text{Per}_n(\eta)] \to T_{\text{bif}},$$

where $\text{Per}_n(\eta) := \{\lambda \colon \exists z \in J_{f_\lambda}$ *of exact period n for f_λ and such that* $\text{Jac}_z f_\lambda = \eta\}$.

The general strategy of the proof of Theorem C follows the main line of the one dimensional case and is based of techniques and tools from pluripotential theory. However, one of the difficulties we have to face here is the possible presence of infinitely many non-repelling cycles for an endomorphism of $\mathbb{P}^k$ – something which is excluded for $k = 1$ by a Theorem due to Fatou. We thus need more quantitative estimates on the number of repelling cycles with small multiplier, which are related to the approximation formula for the Lyapunov exponent valid in any dimension established in [BDM08].

1.3. **Organization of the paper.** After recalling the notions of vertical expansion, stability and bifurcation and fixing the notations in Section 2, in Section 3 we prove our approximation formulas for the vertical Lyapunov exponent. This motivates the study of *vertical bifurcations*. Theorems A and B are proved in Sections 4 and 5. Theorem C is proved in Section 6, together with its adapted version for families of polynomial skew product needed in the proof of Theorem A.

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2. Preliminaries and notations

2.1. **Polynomial skew products.** A polynomial skew product is an endomorphism of $\mathbb{P}^2$ of the form $f(z,w) = (p(z), q(z,w))$, for polynomials p, q. The second coordinate will be also written as $q_z(w)$. We shall denote by $z_j := p^j(z), j \in \mathbb{N}$ the points of the orbit of $z \in \mathbb{C}$ under the base polynomial p. In this way, for each $n \in \mathbb{N}$, we can write

$$f^n(z,w) = (p^n(z), q_{z_{n-1}} \cdots q_{z_1} \circ q_z(w)) =: (z_n, Q^n_z(w)). \qquad (1)$$

The dynamics of polynomial skew products has been studied in detail by Jonsson [Jon99]. In particular, he proved that it is possible to associate to each $z \in J_p$ a *vertical Julia set* J_z, defined as the boundary of the set of points K_z that have bounded orbit under the sequence Q^n_z. The map $z \mapsto J_z$ is lower semicontinuous. The following result describes the structure of the *Julia set* J_f of f, i.e., the support of its measure of maximal entropy μ_f.

Theorem 2.1 (Jonsson [Jon99]). *Let f be a polynomial skew product. Then $J_f = \overline{\bigcup_{z \in J_p} (\{z\} \times J_z)}$. Moreover, J_f is the closure of the repelling periodic points for f.*

2.2. **Vertical expansion.** Recall that an endomorphism f of $\mathbb{P}^k$ is *hyperbolic* (or *uniformly expanding*) *on the Julia set* if there exist constants $c > 0, K > 1$ such that, for every $x \in J_f$ and $v \in T_x\mathbb{P}^k$, we have $\|Df^n_x(v)\|_{\mathbb{P}^k} \geq cK^n$ (with respect for instance to the standard norm on $\mathbb{P}^k$). In the case of polynomial skew products, this condition in particular forces the base polynomial p to be hyperbolic. Jonsson thus introduced an adapted notion of hyperbolicity valid for any base polynomial p. Given an invariant set Z for p, set

(1) $C_Z := \bigcup_{z \in Z} (\{z\} \times C_z)$ for the *critical set over* Z, where $C_z := \{w \in \mathbb{C} : q'_z(w) = 0\}$
(2) $D_Z := \overline{\bigcup_{\geq 1} f^n C_Z}$ for the *postcritical set over* Z.

Definition 2.2 (Jonsson, [Jon99]). *Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product and $Z \subset \mathbb{C}$ be invariant for p. We say that f is* vertically expanding over Z *if there exist constants $c > 0$ and $K > 1$ such that $\left|(Q^n_z)'(w)\right| \geq cK^n$ for every $z \in Z$, $w \in J_z$ and $n \geq 1$.*

For polynomials on $\mathbb{C}$, hyperbolicity is equivalent to the fact that the closure of the postcritical set is disjoint from the Julia set. In our situation, we have the following analogous characterization.

Theorem 2.3 (Jonsson [Jon99]). *Let $f(z,w) = (p(z), q(z,w))$ be a polynomial skew product. Then f is vertically expanding over Z if and only if $D_Z \cap J_Z = \emptyset$, and the following conditions are equivalent:*

(1) f is hyperbolic on its Julia set;
(2) $D_{J_p} \cap J = \emptyset$;
(3) p is hyperbolic, and f is vertically expanding over J_p.

2.3. **Stability and bifurcations.** The definition and study of the notions of stability and bifurcations for endomorphisms of projective spaces of any dimension is given in [BBD18; Bia19], see also [BB18b]. Since we will be mainly concerned with families of polynomial skew products in dimension 2, we cite an adapted version in our setting.

Theorem 2.4 ([BBD18])**.** *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew products of degree $d \geq 2$. Then the following are equivalent:*

(1) the repelling cycles move holomorphically;
(2) $dd^c_\lambda L(\lambda) \equiv 0$ on M;
(3) there are no Misiurewicz parameters *in M.*

We say that a family is *stable* if any of the conditions above is satisfied. The holomorphic motion of the repelling cycles is defined as in dimension 1 (see e.g. [Ber13; Duj11], or [BBD18, Definition 1.2] in this context). Notice that, in our setting, all repelling points are contained in J_{f_λ} see Theorem 2.1. We denote by $L(\lambda)$ the sum of the Lyapunov exponents of f_λ, with respect to $\mu_\lambda := \mu_{f_\lambda}$ which is a plurisubharmonic (psh for short) function on M. Thus, $dd^c_\lambda L$ is a positive closed (1,1) current on M. We call it the *bifurcation current* for the family. Its support is the *bifurcation locus*. A *Misiurewicz parameter* is a generalization to any dimension of a map with a critical point that is (non-persistently) preperiodic to a repelling cycle. More precisely, they are defined as follows:

Definition 2.5. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ and let C_F be the critical set of the map $F(\lambda, z) := (\lambda, f_\lambda(z))$. A point λ_0 of the parameter space M is called a* Misiurewicz parameter *if there exist a neighbourhood $N_{\lambda_0} \subset M$ of λ_0 and a holomorphic map $\sigma\colon N_{\lambda_0} \to \mathbb{P}^k$ such that:*

(1) for every $\lambda \in N_{\lambda_0}$, $\sigma(\lambda)$ is a repelling periodic point;
(2) $\sigma(\lambda_0)$ is in the Julia set J_{λ_0} of f_{λ_0};
(3) there exists an n_0 such that $(\lambda_0, \sigma(\lambda_0))$ belongs to some irreducible component of $F^{n_0}(C_F)$;
(4) $\sigma(N_{\lambda_0})$ is not contained in some irreducible component of $f^{n_0}(C_F)$ satisfying (3)*.*

2.4. **Non autonomous bifurcations.** Given a family of polynomial skew products of the form $f_\lambda(z,w) = (p(z), q_{\lambda,z}(w))$, $\lambda \in M$, for every $z \in J_p$ we can consider the non-autonomous iteration of the sequence $q_{\lambda,p^j(z)}$ associated to the fibre over z. The corresponding *vertical Green function* is given by $G_{\lambda,z}(w) = \lim_{n\to\infty} \frac{1}{n}\log^+ \left|Q^n_{\lambda,z}(w)\right|$ (where $Q^n_{\lambda,z} := q_{\lambda,z_{n-1}} \circ \cdots \circ q_{\lambda,z}$) and is psh on $M \times \mathbb{C}$. The proof of the following results are completely analogous to the autonomous case.

Proposition 2.6. *Take $z \in J_p$ and let $c(\lambda)$ be a (marked) critical point of $q_{\lambda,z}$. The sequence of maps $\lambda \mapsto Q^n_{\lambda,z}(c(\lambda))$ is normal if and only if $dd^c_\lambda G_\lambda(z, c(\lambda)) \equiv 0$.*

Definition 2.7. *We denote by $\mathcal{B}_{z,c} := \{\, \lambda\colon G(\lambda, z, c(\lambda)) = 0 \,\}$, $T_{\mathrm{bif},z,c} := dd^c_\lambda G(\lambda, z, c(\lambda))$, and* $\mathrm{Bif}_{z,c} := \mathrm{Supp}\, T_{\mathrm{bif},z,c}$ *the* boundedness locus, *the* bifurcation current *and the* bifurcation locus *associated to a marked critical point c in the fibre z. $\mathcal{B}_z, \mathrm{Bif}_z$ and $T_{\mathrm{bif},z}$ are the unions (or the sum) of the sets (currents) above, for c critical point for q_z.*

Lemma 2.8 (cp. [Jon99], Prop. 2.1)**.** *Take $z \in J_p$ and let $c(\lambda)$ be a (marked) critical point of $q_{\lambda,z}$. We have $\mathrm{Bif}_{z,c} = \partial\mathcal{B}_{z,c}$ and $\mathrm{Bif}_z = \partial\mathcal{B}_z$. For every compact $M' \Subset M$ the set $M' \cap \mathcal{B}_z$ (resp., $M' \cap \mathrm{Bif}_z$) varies upper (resp. lower) semicontinuously with z (in the Hausdorff distance).*

2.5. **Quadratic skew products.** We now specialize to quadratic polynomial skew products. The general form is

$$(p(z), Az^2 + Bzw + Cw^2 + Dz + Ew + F),$$

where p is a quadratic polynomial. Notice that we necessarily have $C \neq 0$ in order to extend the map above to an endomorphism to $\mathbb{P}^2(\mathbb{C})$.

Lemma 2.9. *Every quadratic skew product is affinely conjugated to a map of the form $(z, w) \mapsto p(z), w^2 + az^2 + bz + c)$, where p is a monic centered quadratic polynomial.*

We can thus consider the space $\mathbf{Sk}(p, 2)$ of quadratic skew products over the base p as identified with $\mathbb{C}^3$. We will also work with the compactification of $\mathbf{Sk}(p, 2)$ as $\mathbb{P}^3$, and denote by $\mathbb{P}^2_\infty$ the hyperplane at infinity.

Notice that, for all maps in $\mathbf{Sk}(p, 2)$, the fiber at any z contains a unique critical point for q_z, $w = 0$. By the results of the previous section, in order to understand the stability of the family we then need to study the Green function at the points of the form $(z, 0)$, with $z \in J_p$. This leads to the following definition.

Definition 2.10. *We partition the parameter space $\mathbb{C}^3$ of $\mathbf{Sk}(p, 2)$ as follows:*

(1) $\mathcal{C} := \{\lambda \in \mathbb{C}^3 : \forall z \in J_p, G(z, 0) = 0\} = \bigcap_{z \in J_p} \mathcal{B}_z$;
(2) $\mathcal{D} := \{\lambda \in \mathbb{C}^3 : \forall z \in J_p, G(z, 0) > 0\} = \bigcap_{z \in J_p} (\mathbb{C}^3 \setminus \mathcal{B}_z)$;
(3) $\mathcal{M} := \mathbb{C}^3 \backslash (\mathcal{C} \cup \mathcal{D})$.

In the case where J_p is connected, $\mathcal{C}$ is (in restriction to our family) what in [Jon99] is called the *connectedness locus*, meaning the set of parameters $\lambda \in \mathbb{C}^3$ such that J_p is connected, and $J_{\lambda,z}$ is connected for all $z \in J_p$. The set $\mathcal{C}$ is closed (as an intersection of closed sets) and $\mathcal{D}$ is open (by the continuity of G and the compactness of J_p). It follows from [Jon99] that $\mathcal{D}$ is in fact a union of vertically expanding components (hyperbolic, if p is hyperbolic). As we will see below, $\mathcal{C}$ is bounded (Corollary 4.3) but $\mathcal{M}$ and $\mathcal{D}$ are not (and actually contain unbounded hyperbolic components, see Sections 5.2 and 5.3).

We further define $\mathcal{D}'$ as the subset of $\mathcal{D}$ with access to infinity:

$$\mathcal{D}' := \{\lambda \in \mathcal{D} : \text{ there exists a path joining } \lambda \text{ to } \mathbb{P}^2_\infty \backslash E \text{ in } \mathcal{D}\}. \tag{2}$$

Note that connected components of $\mathcal{D}'$ are also connected components of $\mathcal{D}$.

Remark 2.11. *Dujardin [Duj17] and Taflin [Taf21] recently proved that some polynomial skew products are in the interior of the bifurcation locus, a phenomenon that contrasts with the one-variable situation. Such behaviour can only occur in $\mathcal{M}$: indeed, parameters in $\mathcal{D}$ are vertically expanding hence in the stability locus. As for parameters in $\mathcal{C}$, any connected component of $\mathring{\mathcal{C}}$ is a stable component.*

For technical reasons, we will also need to consider some 1-codimension subfamilies of $\mathbf{Sk}(p, 2)$ defined as follows. Given any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$, we denote by $\mathbf{Sk}(p, 2, \alpha)$ the family parametrized by the hyperplane $\alpha_1 a + \alpha_2 b + \alpha_3 c = 0$ in the parameter space $\mathbb{C}^3$. We denote by Bif^α and T^α_{bif} the bifurcation locus and current of the sub-family parametrized by this hyperplane. For every $z \in J_p$, the definition of $\mathcal{B}_z, \mathrm{Bif}_z, T_{\mathrm{bif},z}$ can also easily be adapted for $\mathbf{Sk}(p, 2, \alpha)$, and we can define $\mathcal{B}^\alpha_z, \mathrm{Bif}^\alpha_z, T^\alpha_{\mathrm{bif},z}$ in an analogous way as in Definition 2.7. We denote by E^α the hyperplane $\{\, [a, b, c] \colon \alpha_1 a + \alpha_2 b + \alpha_3 c = 0 \,\} \subset \mathbb{P}^2_\infty$.

3. Lyapunov exponents and fiber-wise bifurcations

In this Section we establish decomposition formulas for the bifurcation current and locus. Both will be used in the proof of Theorem A, and they motivate the classification in Theorem B.

3.1. **The vertical bifurcation.** Consider a family of polynomial skew products of $\mathbb{C}^2$ of the form $f_\lambda(z,w) = (p_\lambda(z), q_{\lambda,z}(w))$, $\lambda \in M$. By [Jon99], the two Lyapunov exponents of f_λ with respect to its maximal entropy measure are equal to

$$L_p(\lambda) = \log d + \sum_{z \in C_{p_\lambda}} G_{p_\lambda}(z) \quad \text{and} \quad L_v(\lambda) = \log d + \int \Big(\sum_{w \in C_{\lambda,z}} G_\lambda(z,w) \Big) \mu_{p_\lambda}, \tag{3}$$

where C_{p_λ} and μ_{p_λ} are the critical set and the equilibrium measure of p_λ and $C_{\lambda,z}$ is the critical set of $q_{\lambda,z}$.

By [Pha05; DS10] the sum $L(\lambda) = L_p(\lambda) + L_v(\lambda)$ is a psh function. In our situation, we are interested in the two functions L_p and L_v separately. The first is psh, since it is the Lyapunov function of a polynomial family on $\mathbb{C}$. The following result ensures that L_v enjoys the same property.

Proposition 3.1. *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of polynomial skew product. The function $\lambda \mapsto L_v(\lambda)$ is psh. In particular, the current $T^v_{\rm bif} := dd^c L_v = T_{\rm bif} - dd^c_\lambda L(p_\lambda)$ is positive and closed.*

The Proposition above is a direct consequence of the (pointwise and L^1_{loc}) convergence of the first sequence in the following Lemma. We denote by $\mathcal{R}_N(\lambda) \subset \mathcal{P}_N(\lambda)$ the sets

$$\mathcal{P}_N(\lambda) := \{\, z \in \mathbb{C} \colon p^n_\lambda(z) = z \,\} \quad \text{and} \quad \mathcal{R}_N(\lambda) := \{\, z \in \mathcal{P}_N \colon \,\} \left|(p^n_\lambda)'(z)\right| > 1.$$

Lemma 3.2. *We have*

$$\begin{aligned} L_v(\lambda) &= \log d + \lim_{N \to \infty} \frac{1}{d^N} \sum_{z \in \mathcal{P}_N(\lambda)} \sum_{w \in C_{\lambda,z}} G_\lambda(z,w) \\ &= \log d + \lim_{N \to \infty} \frac{1}{d^N} \sum_{z \in \mathcal{R}_N(\lambda)} \sum_{w \in C_{\lambda,z}} G_\lambda(z,w), \end{aligned}$$

where the convergence is pointwise on M and in $L^1_{loc}(M)$.

Proof. The pointwise convergence of both sequences follows from the equidistribution of the periodic (or repelling periodic) points towards the equilibrium measure μ_{p_λ} for the polynomial p_λ, and the continuity of the Green function, and (3).

The continuity of G and the fact that $\operatorname{card} \mathcal{P}_N(\lambda), \operatorname{card} \mathcal{R}_N(\lambda) \leq d^N$ for every $N \in \mathbb{N}$ also imply that both sequences are locally uniformly bounded. It thus suffices to prove that the first sequence consists of psh functions to get that there exist L^1_{loc} limits. By the previous part the only possible limit will then be $L_v(\lambda)$, proving the statement. In order to do so, since G is psh, it suffices to notice that the set C_N given by $C_N := \{\, (\lambda, z, w) \colon z \in \mathcal{P}_N(\lambda), w \in C_{\lambda,z} \,\}$ is an analytic subset of $M \times \mathbb{C}^2$. The assertion follows. □

In view of Proposition 3.1, and since $\operatorname{Bif}(p) \equiv \operatorname{Supp} dd^c L_p$, we can decompose the bifurcation locus $\operatorname{Bif}((f_\lambda)_{\lambda \in M})$ as a union (non necessarily disjoint)

$$\operatorname{Bif}((f_\lambda)_{\lambda \in M}) = \operatorname{Bif}(p) \cup \operatorname{Bif}(q)$$

where we denoted $\mathrm{Bif}(q) := \mathrm{Supp}\, T^v_{\mathrm{bif}} = \mathrm{Supp}\, dd^c L_v$. We will call the bifurcations occurring in $\mathrm{Bif}(q)$ *vertical*. Our next goal consists in getting a better understanding of the set $\mathrm{Bif}(q) \setminus \mathrm{Bif}(p)$.

3.2. **A decomposition for the bifurcation current and locus.**

Theorem 3.3. *Let $f_\lambda(z,w) = (p(z), q_\lambda(z,w))$, $\lambda \in M$, be a holomorphic family of polynomial skew products of degree d over a fixed base. Then*

$$T_{\mathrm{bif}} = \int_{z \in J_p} T_{\mathrm{bif},z} \mu_p \quad \textit{and} \quad \mathrm{Bif}\left((f_\lambda)_{\lambda \in M}\right) = \overline{\bigcup_{z \in J_p} \mathrm{Bif}_z}.$$

Proof. The first formula follows from the expression for L_v in (3) The inclusion $\subseteq$ in the second formula is an immediate consequence of the first formula.

By the lower semicontinuity of $z \mapsto \mathrm{Bif}_z$, in order to prove the reversed inclusion it is enough to show that, for every N and $z \in \mathcal{R}_N$, we have $\mathrm{Bif}\left((f_\lambda)_{\lambda \in M}\right) \supseteq \mathrm{Bif}(Q^N_{\lambda,z})$, where $Q^N_{\lambda,z}$ denotes the 1-dimensional family $(\lambda, w) \mapsto (\lambda, Q^N_{\lambda,z}(w))$. In order to prove this, given any such N and z, let us consider a parameter $\lambda_0 \in \mathrm{Bif}(Q^N_{\lambda,z})$. There exists a parameter λ_1 close to λ_0 which is Misiurewicz for the family $Q^N_{\lambda,z}$, i.e., there exist a critical point c for $Q^N_{\lambda_1,z}$, a number $N_0 \geq 1$ and a N_1-periodic repelling point w for $Q^N_{\lambda_1,z}$ such that $Q^{NN_0}_{\lambda,z}(c) = w$, and the relation $Q^{NN_0}_{\lambda,z}(c(\lambda)) = w(\lambda)$ does not persistently hold for every λ near λ_1. Here $c(\lambda)$ and $w(\lambda)$ are the local holomorphic motions of c and w as a critical point and as a N_1-periodic repelling point, respectively (we assume for simplicity that we can mark the critical point c, the argument being similar otherwise). The point (z,c) is in particular critical also for f_{λ_1}, and the point (z,w) is NN_1-periodic and repelling for f_{λ_1}. So, it is enough to check that there does not exist any holomorphic map $\lambda \mapsto (z(\lambda), \widetilde{c}(\lambda)) \in C(f_\lambda)$ such that $(z(\lambda_1), \widetilde{c}(\lambda_1)) = (z,c)$ and the relation $f_\lambda^{NN_0}(z(\lambda), \widetilde{c}(\lambda)) = (z, w(\lambda))$ holds persistently in a neighbourhood of λ_1. First of all, by the finiteness of the p^{N_0}-preimages of z, up to restricting ourselves to a small neighbourhood of this point, we can assume that every $f_\lambda^{N_0}$-preimage of $(z, w(\lambda))$ belongs to the fiber of z, too. In this way, any persistent critical relation must happen in the fibers of z. This in excluded, since the parameter is Misiurewicz for the restricted family. □

3.3. **Approximations for the bifurcation current: periodic fibres and preimages.**

We characterize here the Lyapunov exponents of a skew product map by means of the Green functions of the return maps of the periodic vertical fibers. This allows us to approximate the bifurcation current by means of the bifurcation currents of these return maps. We fix the base polynomial p for simplicity, but the results are generalizable to families where also p is allowed to depend from a parameter, see also [DT21].

Proposition 3.4. *Let $f_\lambda(z,w) = (p(z), q_\lambda(z,w)), \lambda \in M$ be a family of polynomial skew products. Then*

$$L_v(\lambda) = \lim_{N \to \infty} \frac{1}{Nd^N} \sum_{z \in \mathcal{R}_N} \sum_{w \in C\left(Q^N_{\lambda,z}\right)} G_{Q^N_{\lambda,z}}(w) \tag{4}$$

where $\mathcal{R}_N$ is the set of repelling periodic points of p, and the convergence holds pointwise and in $L^1_{loc}(M)$. In particular,

$$T^v_{\text{bif}} = \lim_{N\to\infty} \frac{1}{Nd^N} \sum_{z\in\mathcal{R}_N} T_{\text{bif}}(Q^N_{\lambda,z}).$$

Proof. By Lemma 3.2 we only have to prove that, for any λ, N and $z \in \mathcal{R}_N$,

$$\sum_{w\in C_{\lambda,z}} G_\lambda(z,w) = \frac{1}{N} \sum_{w\in C\left(Q^N_{\lambda,z}\right)} G_{Q^N_{\lambda,z}}(w). \tag{5}$$

First notice that, for every $\lambda, N, z \in \mathcal{R}_N$ and $w \in \mathbb{C}$, we have $G_\lambda(z,w) = G_{Q^N_{\lambda,z}}(w)$. So, the left hand side of (5) is equal to $\sum_{w\in C_{\lambda,z}} G_{Q^N_{\lambda,z}}(w)$. We are thus left with checking that, for a given skew product $f(z,w) = (p(z), q_z(w))$, for every N-periodic point z of p, we have

$$\sum_{j=0}^{N-1} \sum_{w\in C_{p^j(z)}} G_{Q^N_{p^j(z)}}(w) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{w\in C\left(Q^N_{p^j(z)}\right)} G_{Q^N_{p^j(z)}}(w).$$

Let us first describe the critical set of $Q^N_{p^j(z)}$, that we denote by C^j. Since $Q^N_{p^j(z)}$ is by definition equal to $q_{p^{j-1}(z)} \circ \cdots \circ q_z \circ q_{p^{N-1}(z)} \circ \cdots \circ q_{p^{j+1}(z)} \circ q_{p^j(z)}$, we have $C^j = C\left(Q^N_{p^j(z)}\right) = \bigcup_{i=0}^{N-1} C^j_i$, where

$$\begin{aligned}
C^j_0 &= C(q_{p^j(z)})\\
C^j_1 &= q^{-1}_{p^j(z)} C(q_{p^{j+1}(z)})\\
C^j_2 &= q^{-1}_{p^j(z)} q^{-1}_{p^{j+1}(z)} C(q_{p^{j+2}(z)}) = \left[Q^2_{p^j(z)}\right]^{-1} C(q_{p^{j+2}(z)})\\
&\vdots\\
C^j_{N-1} &= q^{-1}_{p^j(z)} q^{-1}_{p^{j+1}(z)} \dots q_{p^{j-2}(z)} C(q_{p^{j-1}(z)}) = \left[Q^{N-1}_{p^j(z)}\right]^{-1} C\left(q_{p^{j-1}(z)}\right)
\end{aligned}$$

(each term C^k_l is to be thought of as a subset of the fibre over $p^k(z)$ which is the preimage of the critical set of $q_{p^{k+l(\mod N)}(z)}$ by $Q^l_{p^k(z)}$). So, it suffices to prove that, for any $0 \le j, i \le N-1$, we have

$$\sum_{w\in C^j_0 = C\left(q_{p^j(z)}\right)} G_{Q^N_{p^j(z)}}(w) = \sum_{w\in C^i_{j-i} = \left[Q^{j-i}_{p^i(z)}\right]^{-1}\left(C\left(q_{p^j(z)}\right)\right)} G_{Q^N_{p^i(z)}}(w),$$

where $j-i$ has to be taken modulo N. But this follows from the identity $G(f(\cdot)) = dG(\cdot)$. Indeed, for points $(p^l(z), w)$ in the fibre $\{ p^l(z) \} \times \mathbb{C}$, we have $G_f(p^l(z), w) = G_{Q^N_{p^l(z)}}(w)$. Moreover $\{ p^i(z) \} \times C^i_{j-1}$ contains exactly d^{j-i} preimages (counting multiplicities) by F^{j-i} of any point in $\{ p^j(z) \} \times C^j_0$ (out of the total $d^{2(j-i)}$, since we do not consider preimages other than the ones contained in the fiber over $p^j(z)$). So, for each $w \in C^j_0$ in the left sum, with value $G_{Q^N_{p^j(z)}}(w)$, there are d^{j-i} preimages $w_1, \dots, w_{d^{j-i}}$ in the right sum, each one with value $G_{Q^N_{p^i(z)}}(w_l) = G_{Q^N_{p^j(z)}}(w)/d^{j-i}$. The assertion follows. □

By similar arguments, by exploiting the equidistribution of preimages of generic points, we can establish the following further approximation of the bifurcation current and locus.

Proposition 3.5. *Let $f_\lambda(z,w) = (p(z), q_\lambda(z,w)), \lambda \in M$, be a holomorphic family of polynomial skew products of degree $d \geq 2$. Let $z \in J_p$.*

$$T_{\mathrm{bif}} = \lim_{N\to\infty} \frac{1}{d^N} \sum_{y\colon p^N(y)=z} T_{\mathrm{bif},y} \text{ and } \mathrm{Bif}(M) = \overline{\bigcup_{N\in\mathbb{N}} \bigcup_{y\colon p^N(y)=z} \mathrm{Bif}\, y}.$$

4. Bifurcations near $\mathbb{P}^2_\infty$ and Theorem A

4.1. **Accumulation at $\mathbb{P}^2_\infty$.** In this Section we prove the first part of Theorem A. Since we will need it in the next Section 4.2, we actually characterize the accumulation of the bifurcation locus also for generic subfamilies of the form $\mathbf{Sk}(p,2,\alpha)$, see Section 2.5. Namely, we will consider in the following families $\mathbf{Sk}(p,2,\alpha)$ corresponding to $\alpha \in \mathbb{C}^3$ satisfying

$$(6) \qquad [\alpha_1,\alpha_2,\alpha_3] \neq [z^2,z,1] \text{ for all } z \in J_p.$$

Condition (6) means that the line at infinity E^α of the parameter plane in $\mathbb{C}^3$ of the family $\mathbf{Sk}(p,2,\alpha)$ is different from any line E_z corresponding to any $z \in J_p$. Notice in particular that, for every $[a,b,c] \in \mathbb{P}^2_\infty$, among all the families $\mathbf{Sk}(p,2,\alpha)$ such that $[a,b,c] \in E^\alpha$, at most two do not satisfy condition (6).

Theorem 4.1. *In the family $\mathbf{Sk}(p,2)$ (resp., $\mathbf{Sk}(p,2,\alpha)$ for any α satisfying in (6)), the following hold.*

(1) For every $z \in J_p$, the cluster set at infinity of $\mathcal{B}_z$ and Bif_z (resp., $\mathcal{B}_z^\alpha$ and Bif_z^α) is exactly E_z (resp., $E_z \cap E^\alpha$).
(2) The cluster set at infinity of Bif (resp., Bif^α) is exactly E (resp., $E \cap E^\alpha$).

Recall that, for $\lambda := (a,b,c) \in \mathbb{C}^3$ and p a monic quadratic polynomial, we set

$$f_\lambda(z,w) = (p(z), w^2 + az^2 + bz + c).$$

For $\lambda \in \mathbb{C}^3$, recall that $q_{\lambda,z}(0) = az^2 + bz + c$ and set $R(f_\lambda) := \sup_{z\in J_p} |az^2 + bz + c||q_{\lambda,z}(0)|$. Let us also fix a norm $\|\cdot\|$ on $\mathbb{C}^3$. We then have the following elementary lemma:

Lemma 4.2. *For all λ with $\|\lambda\|$ sufficiently large, for all $z_0 \in J_p$, if $G_\lambda(z_0,0) = 0$ then $|q_{\lambda,z_0}(0)| \leq 2\sqrt{R(f_\lambda)}$. In particular, the cluster set of $\mathcal{B}_{z_0}$ in $\mathbb{P}^2_\infty$ is contained in E_{z_0}.*

Proof. For $n \in \mathbb{N}$, set $z_n := p^n(z_0)$ and $\rho_n := az_n^2 + bz_n + c$. Let $w_n := Q_{z_0}^{n+1}(0)$: then $w_0 = \rho_0$ and $w_{n+1} = w_n^2 + \rho_n$. Therefore we have $|w_{n+1}| \geq |w_n|^2 - R(f_\lambda)$, and since by assumption $(w_n)_{n\in\mathbb{N}}$ is bounded, if $R(f_\lambda) \geq 1$ (which is true for $\|\lambda\|$ sufficiently large) we must have $|w_n| \leq 2\sqrt{R(f_\lambda)}$ for all $n \in \mathbb{N}$. The first assertion follows by taking $n = 0$.

Take now $\lambda_0 \notin E_{z_0}$. In particular, we have $|q_{\lambda_0,z_0}(0)| > 2\varepsilon_0$ for some positive ε_0, which implies that $|q_{\lambda_0,z_0}(0)| > \varepsilon_0$ for all λ sufficiently close to λ_0. It follows that for all such λ, both $q_{z_0,t\cdot\lambda}(0)$ and $R(f_{t\cdot\lambda})$ grow linearly in $|t|$ as $|t| \to \infty$. By the first part of the statement, this implies that $G_{t\cdot\lambda}(z_0) > 0$ for all λ sufficiently close to λ_0 and t with $|t|$ large enough. The assertion follows. □

Proof of Theorem 4.1. By Lemma 4.2, the cluster set of $\mathcal{B}_z$ is included in E_z. We thus prove the opposite inclusion. We first consider z such that $z = p^n(z)$. Since E_z is an irreducible curve (more precisely, a projective line), it is enough to note that there is a component C of $\mathrm{Per}_n(0)$ such that for all $\lambda \in C$, $f_\lambda^n(z,0) = (z,0)$. Indeed, that component C intersects the plane at infinity in some (1 dimensional) hypersurface that is contained in E_z and is therefore equal to E_z. Moreover, it is clear that $C \subset \mathcal{B}_z$.

Let us now pick any (non necessarily periodic) $z \in J_p$. It is enough to prove the statement for the family $\mathbf{Sk}(p,2,\alpha)$ for all α satisfying (6). Since $E_z \cap E^\alpha$ is a single point, because of the inclusion already proved it is enough to show that $\mathcal{B}_z^\alpha$ is not compact. This follows by the continuity of the Green function and the fact that for the dense subset of periodic points the corresponding $\mathcal{B}_z^\alpha$ is not compact, as proved above. The assertion for Bif_z also easily follows.

Let us now prove the second assertion. Observe that $\mathrm{Bif} \subset \bigcup_{z\in J_p} \mathcal{B}_z$. Therefore, the cluster set at infinity of Bif is contained in the cluster set of $\bigcup_{z\in J_p} \mathcal{B}_z$, which a priori might be larger than the union of cluster sets of $\mathcal{B}_z$; but the estimate from Lemma 4.2 implies that this is not the case. Indeed, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of points in $\bigcup_{z\in J_p} \mathcal{B}_z$ going to infinity, and such that $[\lambda_n] \to [\lambda] \in \mathbb{P}^2_\infty$. For each n there is at least one $z_n \in J_p$ such that $\lambda_n \in \mathcal{B}_{z_n}$, and thus, by Lemma 4.2,

$$|q_{\lambda_n,z_n}(0)| \leq 2\sqrt{R(f_{\lambda_n})}.$$

Since $\lambda \mapsto R(f_\lambda)$ is a vector space norm on $\mathbb{C}^3$, there is some constant $C_p > 0$ such that for all $\lambda \in \mathbb{C}^3$,

$$\frac{1}{C_p}\|\lambda\| \leq R(f_\lambda) \leq C_p\|\lambda\|$$

and therefore, setting $M_n := \|\lambda_n\|$, we have

$$\left|\frac{q_{\lambda_n,z_n}(0)}{M_n}\right| \leq 2C_p\sqrt{\frac{1}{M_n}}.$$

Passing to the limit, we conclude that $z \mapsto q_\lambda(0)$ must vanish at least once on J_p.

This takes care of one inclusion. Now let us prove that the cluster of the bifurcation locus on the hyperplane at infinity contains the set E. Take $[\lambda] \in E$, so that $q_{\lambda,z}(0) = 0$ for some $z \in J_p$. By Theorem 3.3, we know that $\partial\mathcal{B}_z \subset \mathrm{Bif}$ (here, the boundary is taken in $\mathbb{C}^3$). By the first item, we know that $\partial\mathcal{B}_z$ accumulates on $\mathbb{P}^2_\infty$ to the set $\{[\lambda] : q_{\lambda,z}(0) = 0\}$. This concludes the proof. □

Corollary 4.3. *Let $z_1, z_2, z_3 \in J_p$ be three distinct points. Then $\mathcal{B}_{z_1} \cap \mathcal{B}_{z_2} \cap \mathcal{B}_{z_3}$ is compact. In particular, $\mathcal{C}$ is compact.*

Proof. If $[a,b,c] \in \mathbb{P}^2_\infty$ were accumulated by $\mathcal{B}_{z_1} \cap \mathcal{B}_{z_2} \cap \mathcal{B}_{z_3}$, then $aX^2 + bX + c$ would have z_1, z_2, z_3 as roots, and we would have $a = b = c = 0$, which is impossible. So $\mathcal{B}_{z_1} \cap \mathcal{B}_{z_2} \cap \mathcal{B}_{z_3}$ is closed and bounded in $\mathbb{C}^3$. In particular, $\mathcal{C} = \bigcap_{z\in J_p} \mathcal{B}_z$ is compact. □

4.2. **The bifurcation current at infinity.** We prove here the second part of Theorem A. Recall that we are considering the family $\mathbf{Sk}(p,2)$ given by maps of the form

$$f_\lambda = \big(p(z), w^2 + az^2 + bz + c\big)$$

where p is a fixed polynomial of degree 2, and $\lambda = (a,b,c) \in \mathbb{C}^3$.

First of all, we prove that we can extend the bifurcation current of the family $\mathbf{Sk}(p,2)$ (resp $\mathbf{Sk}(p,2,\alpha)$, see Section 2.5) to the compactification $\mathbb{P}^3$ (resp. $\mathbb{P}^2$) of the parameter space (see also [BG15a] for an analogous result for quadratic rational maps).

Lemma 4.4. *There exists a positive closed* $(1,1)-$ *current* $\hat{T}_{\rm bif}$ *on* $\mathbb{P}^3$ *(resp., for every* $\alpha \in \mathbb{C}^3$ *satisfying* (6) *a positive closed* $(1,1)$ *current* $\hat{T}^{\alpha}_{\rm bif}$ *on* $\mathbb{P}^2$*) of mass 1 and such that*

(1) $\hat{T}_{\rm bif}|_{\mathbb{C}^3} = T_{\rm bif}$ *(resp.,* $\hat{T}^{\alpha}_{\rm bif}|_{\mathbb{C}^2} = T^{\alpha}_{\rm bif}$*);*
(2) for a generic $\eta \in \mathbb{C}$*, the sequence* $4^{-n}[{\rm Per}^J_n(\eta)]$ *(resp.,* $4^{-n}[{\rm Per}^v_n(\eta)]$*) converges to* $\hat{T}_{\rm bif}$ *(resp.,* $\hat{T}^{\alpha}_{\rm bif}$*) in the sense of currents of* $\mathbb{P}^3$ *(resp.,* $\mathbb{P}^2$*).*

Proof. We prove the statement for the family $\mathbf{Sk}(p,2)$, the proof for $\mathbf{Sk}(p,2,\alpha)$ being completely analogous.

The existence of $\hat{T}_{\rm bif}$ follows by an application of Skoda-El Mir Theorem. Indeed, by the equidistribution results in Appendix 6 (which we can apply since the vertically expanding map $(z,w) \mapsto (p(z), w^2)$ belongs to the family), the mass of $T_{\rm bif}$ with respect to the Fubini-Study form on $\mathbb{P}^3$ is equal to 1. We thus can trivially extend $T_{\rm bif}$ to $\mathbb{P}^3$, and the mass of the extension is still equal to 1.

We now promote the equidistribution of $[{\rm Per}_n(\eta)]$ to $T_{\rm bif}$ on $\mathbb{C}^3$ to an equidistribution to $\hat{T}_{\rm bif}$ on $\mathbb{P}^3$ (we denote by ${\rm Per}_n(\eta)$ both ${\rm Per}^J_n$ and ${\rm Per}^v_n$, the proof is the same). First recall (see Appendix 6) that the ${\rm Per}_n(\eta)$ are actually algebraic surfaces on $\mathbb{P}^3$, of mass $\sim 4^n$. Thus, the sequence $4^{-n}[{\rm Per}_n(\eta)]$ gives a sequence of uniformly bounded (in mass) positive closed currents. We have to prove that any limit of this sequence coincides with $\hat{T}_{\rm bif}$. Let us denote by T a cluster of the sequence. By Siu's decomposition Theorem, we have $T = S + \beta[\mathbb{P}^2_\infty]$, where S has no mass on $\mathbb{P}^2_\infty$. It follows from the description of the accumulation of the bifurcation locus given in Section 2.5 that $\beta = 0$. Moreover, we have $S = T_{\rm bif}$ on $\mathbb{C}^3$. This completes the proof. □

In order to study the trace of $\hat{T}_{\rm bif} \wedge [\mathbb{P}^2_\infty]$, we will first need to obtain an analogous statement for the families $\mathbf{Sk}(p,2,\alpha)$.

Theorem 4.5. *For any* $\alpha \in \mathbb{C}^3$ *satisfying* (6) *we have* $\hat{T}_{\rm bif} \wedge [\mathbb{P}^1_\infty] = \int [E_z \cap E_\alpha]\mu_p$.

Notice that the support of the measure on $\mathbb{P}^1_\infty$ in the right hand side can be seen as the image of J_p by a polynomial map π_∞ of degree at most 2 (and equal to 2 for generic α). This polynomial can be explicitly computed from the polynomial az^2+bz+c after substituting the relation on a,b,c defining the family $\mathbf{Sk}(p,2,\alpha)$. We denote by $J_{p,\infty}$ this support, and by $\mu_{p,\infty}$ the measure $\int [E_z \cap E_\alpha]\mu_p = (\pi_\infty)_* \mu_p$.

Lemma 4.6. *For a generic* $\eta \in \mathbb{D}$ *we have* $4^{-n}[{\rm Per}^v_n(\eta)] \wedge [\mathbb{P}^1_\infty] \to \mu_{p,\infty}$.

Proof. By the equidistribution of the periodic points of p towards μ_p, it is enough to prove that

$$[{\rm Per}^v_n(\eta)] \wedge [\mathbb{P}^1_\infty] \sim 2^n (\pi_\infty)_* \sum_{p^n(y)=y} \delta_y.$$

The sum in the right hand side can be taken with or without multiplicity.

First, notice that the support of $[{\rm Per}^v_n(\eta)] \wedge [\mathbb{P}^1_\infty]$ is contained in the image by π_∞ of the union of the solutions of $p^n(y) = y$. Indeed, every ${\rm Per}^v_n(\eta)$ is contained in the boundedness locus $\mathcal{B}_z$ of some fibre z of period (dividing) n (since a periodic cycle of vertical multiplier $\eta \in \mathbb{D}$ attracts a critical point). By Theorem 4.1, $\mathcal{B}_z$ precisely clusters at $\pi_\infty(z)$.

To conclude, it is enough to prove that for every point y of period n for p the Lelong number of $[\mathrm{Per}_n^v(\eta)] \wedge [\mathbb{P}^1_\infty]$ at $(\pi_\infty)_*(y)$ is at least $\sim 2^n$. Since the return map of the fibre corresponding to y is of degree 2^n, the above follows since the mass of $\mathrm{Per}_1(\eta)$ in this one-dimensional family is $\sim 2^n$. □

Proof of Theorem 4.5. The good definition of the intersection follows the same argument as in [BG15a, Lemma 4.3]. We give it for completeness, also to highlight that a different approach will be needed when considering the complete family. We take any complex line L intersecting $\mathbb{P}^1_\infty$ in a point disjoint from $J_{p,\infty}$. The complement of this line is a copy of $\mathbb{C}^2$. Since the set $J_{p,\infty}$ is compact in this copy of $\mathbb{C}^2$, we can define the intersection here by means of [Dem97, Proposition 4.1]. We then trivially extend this intersection as zero on the line L.

Remark 4.7. *When considering the full family, with the three-dimensional parameter space, we cannot find a line in $\mathbb{P}^2_\infty$ disjoint from E (and thus decompose $\mathbb{P}^3$ as the union of $\mathbb{C}^3$ and a hyperplane disjoint from E) and, hence, cannot apply the argument above.*

We now prove that $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{P}^1_\infty] = \mu_{p,\infty}$ By Lemma 4.6, it is enough to prove that

$$4^{-n}[\mathrm{Per}_n^v(\eta)] \wedge [\mathbb{P}^1_\infty] \to \hat{T}_{\mathrm{bif}} \wedge [\mathbb{P}^1_\infty].$$

The idea is the following: the main obstacle in getting the convergence above would be that some components of $\mathrm{Per}_n^v(\eta)$ become more and more tangent to $\mathbb{P}^1_\infty$ as $n \to \infty$ (possibly with some multiple of the plane at infinity in their cluster set). But this cannot happen, because of Lemma 4.2.

To make the above precise we use the theory of *horizontal positive closed currents*, introduced by Dujardin [Duj04], see also [DS06; Pha05; Duj07]. Recall that a closed positive $(1,1)$-current in the product $\mathbb{D} \times \mathbb{D}$ is *horizontal* if its support is contained in $\mathbb{D} \times K$, for some K compact in $\mathbb{D}$. We will use the following result.

Theorem 4.8 (Dinh-Sibony [DS06])**.** *Let $\mathcal{R}$ be a closed positive horizontal $(1,1)$-current on $\mathbb{D} \times \mathbb{D}$, with support contained in $\mathbb{D} \times K$. Then the* slice *$\mathcal{R}_z$ of $\mathcal{R}$ by $\{z\} \times \mathbb{D}$ is well defined for every $z \in \mathbb{D}$. The slices are measures on $\mathbb{D}$, supported in K, of constant mass. If φ is a smooth psh function on $\mathbb{D} \times \mathbb{D}$ then the function $z \mapsto \langle \mathcal{R}_z, \varphi(z, \cdot) \rangle$ is psh.*

By the description of the cluster set of the $\mathcal{B}_z$'s given in Theorem 4.1, we can find a biholomorphic image of a polydisc $\Delta \subset \mathbb{P}^2$ such that the following hold (by abuse of notation, we also denote by Δ its image):

1. $\{0\} \times \mathbb{D} \subset \mathbb{P}^1_\infty$;
2. there exists $K \Subset \mathbb{D}$ such that $\mathrm{supp}\hat{T}_{\mathrm{bif}} \cap \Delta \subset \mathbb{D} \times K$ and $\mathrm{supp}[\mathrm{Per}_n^v(0)] \cap \Delta \subset \mathbb{D} \times K$ for every n.

Indeed, suppose this is not true. We then find points in $\mathrm{Per}_n^v(0)$ accumulating some point in $\mathbb{P}^1_\infty \setminus J_{p,\infty}$. Since all the $\mathrm{Per}_n^v(0)$ cluster on $J_{p,\infty}$, this contradicts Lemma 4.2

With this setting, we see that all the $[\mathrm{Per}_n^v(\eta)]$ and $\hat{T}_{\mathrm{bif}}$ are (uniformly) horizontal currents on Δ. The convergence above can thus be rephrased as a convergence for the slices at 0:

$$4^{-n}[\mathrm{Per}_n^v(\eta)]_0 \to \left(\hat{T}_{\mathrm{bif}}\right)_0.$$

By standard arguments, the convergence can be tested against smooth psh test functions on $\mathbb{D}$. By Theorem 4.8 above we know that, for every φ smooth and psh in Δ, the functions $u_n(z) := 4^{-n}[\mathrm{Per}_n^v(\eta)]_z(\varphi(z,\cdot))$ and $u(z) := \left(\hat{T}_{\mathrm{bif}}\right)_z(\varphi(z,\cdot))$ are psh. We claim that $u_n \to u$ in L^1_{loc}. Indeed, the convergence of $4^{-n}[\mathrm{Per}_n^v(0)]$ to $\hat{T}_{\mathrm{bif}}$ implies that of $\varphi 4^{-n}[\mathrm{Per}_n^v(\eta)]$ to $\varphi\hat{T}_{\mathrm{bif}}$ in the product space Δ. Since the projection on the first coordinate of Δ is continuous, we have $u_n \to u$ as distributions. Thus, by [Hör07, Theorem 3.2.12], we have $u_n \to u$ in L^1_{loc}. This also implies that $u_n \to u$ almost everywhere.

Now, by Hartogs' Lemma the L^1_{loc} limit of a sequence of psh function is greater than or equal to the pointwise limit. In our case, the pointwise limit of the u_n is given by $\widetilde{u}(z) = \langle \lim_{n\to\infty}[\mathrm{Per}_n^v(\eta)]_z, \varphi\rangle$. Since $\widetilde{u}(0) = \langle \mu_{p,\infty}, \varphi\rangle$ (by Lemma 4.6), we just need to prove that $u'(0) \geq u(0)$. Since u is psh and $u = u'$ almost everywhere, we have a sequence of $z_m \in \mathbb{D}$ converging to 0 and such that $u(z_m) = u'(z_m) \to u(0)$. It is then enough to prove that the limit of the $u'(z_m)$ is equal to $u'(0)$, i.e., that

$$\langle \left(\hat{T}_{\mathrm{bif}}\right)_{z_m}, \varphi\rangle \to \langle \mu_{p,\infty}, \varphi\rangle.$$

Since $u(z_m) = u'(z_m)$, every limit ν of the slice measures on the left hand side is a measure on $J_{p,\infty}$ of the form $(\pi_\infty)_*\nu'$, for some ν' probability measure on J_p. It is enough to prove that $\nu = \mu_{p,\infty}$. Suppose this is not the case. Lemma 4.9 below gives a contradiction with the fact that $\langle \mu_{p,\infty}, \psi\rangle \leq \langle \nu, \psi\rangle$ for every psh function ψ, as proved in the previous part. This completes the proof. □

Lemma 4.9. *Let p be any polynomial on $\mathbb{C}$, μ its equilibrium measure and μ' a probability measure supported on the Julia set of p. If $\mu \neq \mu'$ there exists a subharmonic function ψ on $\mathbb{C}$ such that $\langle \mu, \psi\rangle > \langle \mu', \psi\rangle$.*

Proof. Let $p_{\mu'}$ and p_μ be the respective logarithmic potentials of μ' and μ, that is, $p_\mu(z) = \int_{\mathbb{C}} \log|z-w|d\mu(w)$ and similarly for μ'. Recall that the energy of a compactly supported Radon probability measure m is defined by $I(m) = \int_{\mathbb{C}} p_m(z)dm(z)$. Since μ is the equilibrium measure of the Julia set of p, we have that $I(\mu) > I(\mu')$ for every $\mu' \neq \mu$, see for instance [Ran95]. Therefore there must exist z_0 such that $p_\mu(z_0) > p_{\mu'}(z_0)$. Setting $\psi(z) = \log|z - z_0|$, by definition of p_μ and $p_{\mu'}$ we have $\langle \mu, \psi\rangle > \langle \mu', \psi\rangle$. Thus, ψ has the required property. □

We can now describe the intersection of the bifurcation current $\hat{T}_{\mathrm{bif}}$ with the hyperplane at infinity $\mathbb{P}^2_\infty$ in the full family. This completes the proof of Theorem A

Theorem 4.10. *The intersection $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{P}^2_\infty]$ is well defined and equal to $\int_z [E_z]\mu_p(z)$*

Proof. We start proving that the intersection is well defined. Since the support of T_{bif} only clusters on $E = \cup_{z\in J_p} E_z$, we need only prove the statement in a neighbourhood of E. Take a point $[a_0, b_0, c_0] \in E$. There exist z_0 and z_1 (not necessarily distinct) such that $[a_0, b_0, c_0] \in E_{z_0} \cap E_{z_1}$ but $[a_0, b_0, c_0] \notin E_z$ for every $z \neq z_0, z_1$. To prove that the intersection is well defined, we prove that $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{P}^2_\infty]$ has locally bounded mass near $[a_0, b_0, c_0]$. We fix local coordinates x, y, centred at $[a_0, b_0, c_0]$ and such that the coordinate axis are transversal to both E_{z_0} and E_{z_1} at the origin. Theorem 4.5 implies that the intersection $\hat{T}_{\mathrm{bif}} \wedge [\mathbb{P}^2_\infty] \wedge [L]$ is well defined for lines L parallel (or almost parallel) to the x and y axis. Since all these intersections are measures with uniformly bounded mass, the intersection between $\hat{T}_{\rm bif} \wedge [\mathbb{P}^2_\infty]$ and the currents $\int_{x\in I}[L_x]$ and $\int_{y\in I}[L_y]$ are well defined, where I is a small open neighbourhood of 0, L_x the line $\{x = \text{constant}\}$, L_y the line $\{y = \text{constant}\}$ and the integrations are against the standard Lebesgue measure. Since (locally and up to positive constants) we have $dx \wedge id\overline{x} = \int_{y\in I}[L_y]$ and $dy \wedge id\overline{y} = \int_{x\in I}[L_x]$, this implies that the intersections between $\hat{T}_{\rm bif} \wedge [\mathbb{P}^2_\infty]$ and respectively $dx \wedge id\overline{x}$ and $dy \wedge id\overline{y}$ are of locally bounded mass, thus well defined.

We can now prove the formula in the statement. For every η the intersection at infinity of the current $[{\rm Per}^v_n(\eta)]$ is given by an average of currents of the form $[E_z]$, with z such that $p^n(z) = z$. This implies that $\hat{T}_{\rm bif} \wedge [\mathbb{P}^2_\infty] = \int [E_z]\nu$ for some measure ν on J_p. We can thus find ν by considering a slice of the current above by a complex line corresponding to a family $\mathbf{Sk}(p, 2, \alpha)$ with α satisfying (6). The assertion then follows from Theorem 4.5. □

5. Vertical expansion, hyperbolicity, and Theorem B

In this section we prove Theorem B. The first point is proved in Section 5.1, the second in Section 5.2. In Section 5.3 we give an example of a different kind of unbounded hyperbolic component.

5.1. **Stability preserves hyperbolicity.**

As mentioned in the introduction, a crucial point of the one-dimensional theory of stability and bifurcations is that stability preserves hyperbolicity. This result crucially relies on a characterization of stability not available in higher dimension, and it is then an open problem whether the same would hold in this generality. The following answers this question for any family of polynomial skew products, and gives a parallel in this setting to [BD17, Theorem C].

Theorem 5.1. *Let (f_λ) be a stable family of polynomial skew products, and let λ_0 be a parameter.*

(1) If (f_λ) has constant base p and f_{λ_0} is vertically expanding over J_p, then for all λ, f_λ is vertically expanding over J_p.
(2) If f_{λ_0} is hyperbolic, then for all λ, f_λ is hyperbolic.

Lemma 5.2. *Let f be a polynomial skew product with base p. Assume $(z, w) \in J_p \times \mathbb{C}$ is accumulated by D_{J_p}. Then there exists a sequence $(z_m, w_m) \in J_p \times \mathbb{C}$ of iterates of critical points such that $(z_m, w_m) \to (z, w)$ and z_m is a repelling periodic point for p.*

Proof. By assumption, there is a sequence $(y_m, c_m) \in J_p\times\mathbb{C}$, such that $q'_{y_m}(c_m) = 0$ and $f^{n_m}(y_m, c_m) \to (z, w)$. Given any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $\|f^{n_m}(y_m, c_m) - (z, w)\| \le \varepsilon$ for all $m \ge M$. Since f^{n_m} is continuous, there exists $\delta_m > 0$ such that if $\|(z_m, c'_m) - (y_m, c_m)\| \le \delta_m$, then $\|f^{n_m}(z_m, c'_m) - f^{n_m}(y_m, c_m)\| \le \varepsilon$. This implies that $\|f^{n_m}(z_m, c'_m) - (z, w)\| \le 2\varepsilon$. Since repelling periodic points are dense in J_p, we can find z'_m periodic and repelling arbitrarily close to y_m. We can then take c'_m such that $(z'_m, c'_m) \in C_{J_p}$ is δ_m-close to (y_m, c_m). The point $(z_m, w_m) := f^{n_m}(z_m, c'_m)$ is then 2ε-close to (z, w). Since z'_m is periodic and repelling for p, the same holds for $z_m = p^{n_m}(z'_m)$. Since $\varepsilon > 0$ was arbitrary, the lemma is proved. □

Proof of Theorem 5.1. Assume by contradiction that there exists λ_1 such that f_{λ_1} is not vertically expanding. We can replace our parameter space with any relatively compact connected open subset containing λ_0 and λ_1. By Theorem 2.3, there exists $(z, w) \in J_{f(\lambda_1)}$ such that (z, w) is accumulated by the post-critical set of f_{λ_1} over J_p. By Lemma 5.2, there is a sequence (z_m, w_m) of iterates of critical points such that z_m is periodic for p and $(z_m, w_m) \to (z, w)$.

We first treat the case where it is possible to follow holomorphically all critical points over J_p as holomorphic functions of the parameter λ. Notice that this is the case in particular for the polynomial skew products of degree 2, whose critical points are of the form $(z, 0)$ (and so independent from λ). Set

$$h_m(\lambda) := f_\lambda^{n_m}(y_m, c_m(\lambda))$$

where $f_{\lambda_1}^{n_m}(y_m, c_m(\lambda_1)) = (z_m, w_m)$, and $(y_m, c_m(\lambda))$ is a critical point of f_λ. By definition we have $(\lambda, h_m(\lambda))$ is in the postcritical set. We write $h_m(\lambda) =: (z_m, w_m(\lambda))$.

Since $(z, w) \in J_{f(\lambda_1)}$, there exists a sequence of repelling cycles of the form $(z_m, \gamma_m(\lambda_1)$ converging to (z, w) (by the lower semi-continuity of $z \mapsto J_z$ and the density of repelling cycles). Since f_λ is stable, repelling cycles can be followed holomorphically. We denote by $(z_m, \gamma_m(\lambda))$ the motion of $(z_m, \gamma(\lambda_1))$. Again by the stability of the family, since there are no Misiurewicz parameters, we must have $\gamma_m(\lambda) \neq w_m(\lambda)$ for all m and for all λ. Since the sequence γ_m is uniformly bounded, it is normal and we can assume that γ_m converges to some holomorphic map γ with $\gamma(\lambda_1) = w$.

Claim 5.3. *The sequence $(w_m(\lambda))$ is also normal.*

Assuming this claim, we can get the desired contradiction by taking a limit $w(\lambda)$ for the sequence $w_m(\lambda)$. Indeed, recall that $\gamma(\lambda) \neq w_m(\lambda)$ for all λ and m. Since $\gamma(\lambda_1) = w(\lambda_1)$, by Hurwitz's Theorem the only possibility if that $\gamma(\lambda) \equiv w(\lambda)$ for all λ. Since $\gamma(\lambda_0) \neq w(\lambda_0)$ by assumption, this gives the desired contradiction.

Proof of Claim 5.3. The argument is classical, see for instance [MSS83]. Since the family is stable, $w_m(\lambda)$ avoids the repelling cycles for all m and λ. Let $a_m(\lambda), b_m(\lambda)$ be two sequences of (holomorphic motions of) repelling periodic points in the fibre z_m. Up to passing to subsequences, we can assume that $a_m(\lambda) \to a(\lambda)$ and $b_m(\lambda) \to b(\lambda)$ (as holomorphic functions in λ). We can also assume that $|a_m(\lambda) - b_m(\lambda)| \geq \varepsilon_0 > 0$ for all m and λ. Then, for all λ, we have $w_m(\lambda) \notin \{a_m(\lambda), b_m(\lambda), \infty\}$. It follows that the family $g_m(\lambda) := \frac{w_m(\lambda) - a_m(\lambda)}{b_m(\lambda) - a_m(\lambda)}$ avoids $0, 1, \infty$, hence is normal by Montel Theorem and converges, up to extraction, to some $g(\lambda)$. Since $|a_m(\lambda) - b_m(\lambda)| \geq \varepsilon_0$, the sequence $w_m(\lambda) = a_m(\lambda) + g_m(\lambda) \cdot (b_m(\lambda) - a_m(\lambda))$ converges to $w(\lambda) := a(\lambda) + g(\lambda) \cdot (b(\lambda) - a(\lambda))$. The claim is proved. $\square$

We now explain how to adapt the above arguments in the case where it is not possible to follow all critical points as holomorphic functions of λ. As before, we start with sequences of integers n_m and points $y_m \in J_p$ such that $f_{\lambda_1}^{n_m}(y_m, c_m)$ accumulates to some point in $J(f_{\lambda_1})$, and c_m is a critical point of q_{y_m, λ_1}. The accumulation point in $J(f_{\lambda_1})$ can also be accumulated by repelling periodic points $(z_m, \gamma_m(\lambda_1))$.

We now define the function

$$h_m(\lambda) := \prod_{c_i} (f^{n_m}(y_m, c_i) - \gamma_m(\lambda)),$$

where the product is taken over the set of critical points c_i of $q_{y_m, \lambda}$ whose orbits are bounded. Observe now that the function h_m is holomorphic. Indeed, *for a fixed m*, it is always possible to mark the critical points of $q_{y_m,t}$ as holomorphic functions $c_i(t)$, up to passing to a re-parametrization $\varphi(t) = \lambda$, where φ is a finite branched cover.

Since the family is stable, each critical point $c_i(t)$ either has bounded orbit for all t or unbounded orbit for all t. Therefore, $t \mapsto h_m \circ \varphi(t)$ is holomorphic, and since $\lambda \mapsto h_m(\lambda)$ is continuous and holomorphic outside the branch locus of φ, it is also holomorphic on the whole family. Moreover, the sequence (h_m) is locally uniformly bounded in λ, hence normal; and for all m and λ we must have $h_m(\lambda) \neq 0$ since otherwise this would create a Misiurewicz parameter, contradicting the stability of the family. From there, the proof works as in the previous case. □

For families of polynomial skew products, it thus makes sense to talk about *hyperbolic components* (respectively *vertically expanding components*), i.e., stable components whose elements are (all) hyperbolic (respectively, vertically expanding). We will characterize and classify some components of this kind in the next sections. An ingredient in our classification is given by the following result.

Lemma 5.4. *Let $f_\lambda(z,w) = (p(z), q_\lambda(z,w))$ be a family of polynomial skew products parametrized by M. Assume that f_{λ_0} is uniformly vertically expanding above J_p. Then for a small enough neighbourhood U of λ_0 in M, there exists a unique continuous map $h : U \times J(f_{\lambda_0}) \to \mathbb{C}^2$, such that:*

1. *for all $\lambda \in U$, $h_\lambda := h(\lambda, \cdot) : J(f_{\lambda_0}) \to J(f_\lambda)$ is a homeomorphism conjugating the dynamics, and*
2. *h_λ is of the form $h_\lambda(z,w) = (z, g_\lambda(z,w))$.*

Proof. We follow the classical one dimensional construction of the conjugation valid for hyperbolic polynomial maps, see e.g., [BH]. For ease of notation, we write f_0 for f_{λ_0} and assume that $\lambda \in \mathbb{D}$.

By uniform expansion and continuity, there exist ε and $C > 1$ such that, for every λ sufficiently small and every $(z,w) \in J_2(f_0)$ we have $\left|q'_{\lambda,z}(w')\right| > C > 1$ for every $w' \in B(w,\varepsilon)$. This implies that, denoting by (z_n, w_n) the orbit of (z,w) under f_0, we have $q_{\lambda,z}(B(w_n,\varepsilon)) \supset B(w_{n+1}, C'\varepsilon)$ for some $1 < C' < C$. It follows that the diameter of $B(w_{n+1},\varepsilon)$ inside $q_{\lambda,z}(B(w_n,\varepsilon))$ is uniformly bounded from above and that, if $x, y \in B(w_n,\varepsilon)$ and $q_{\lambda,z}(x), q_{\lambda,z}(y) \in B(w_{n+1},\varepsilon)$, then

$$d_{B(w_{n+1},\varepsilon)}(q_{\lambda,z}(x), q_{\lambda,z}(y)) > C'' d_{B(w_n,\varepsilon)}(x,y).$$

for some uniform constant $C'' > 1$. Thus, the intersection

$$B(w,\varepsilon) \cap q_{\lambda,z}^{-1}(B(w_1,\varepsilon)) \cap \cdots \cap q_{\lambda,z_{n-1}}^{-1} \circ \cdots \circ q_{\lambda,z}^{-1}(B(w_n,\varepsilon))$$

consists of a single point. Denote it by $g_\lambda(z,w)$. Then, $q_{z,\lambda} \circ g_\lambda(z,w) = g_\lambda(z_1, q_{z,0}(w))$.

Let us prove that the map g constructed above is continuous at (λ_0, z_0, w_0). As proved above, for $\varepsilon > 0$ small enough a basis of open neighbourhoods of $g_{\lambda_0}(z_0, w_0)$ is given by the intersections $V_n(\varepsilon) := \bigcap_{i=0}^n \left(Q^n_{z_0,\lambda_0}\right)^{-1}(B(w_n,\varepsilon))$. Let

$$U_n(\varepsilon) := \{(\lambda, z, w) \in \mathbb{D} \times \mathbb{C}^2 : \forall k \leq n, \left|Q^n_{z,\lambda}(w) - Q^n_{z_0,\lambda_0}(w_0)\right| < \varepsilon\}.$$

Then $U_n(\varepsilon)$ is an open neighbourhood of (λ_0, z_0, w_0), and for all $(\lambda, z, w) \in U_n(\varepsilon)$, $g_\lambda(z,w) \in V_n(2\varepsilon)$. This proves the continuity of g.

Now we set $h_\lambda(z,w) := (z, g_\lambda(z,w))$. Since we can start the construction at a different λ near 0, the map h_λ is invertible and thus a homeomorphism. Finally, to prove the uniqueness of h, just note that for any $\lambda \in U$, h_λ must map periodic points of f_0 to periodic points of f_λ of same period; since periodic points of a given period are discrete, the values of h_λ are uniquely defined on periodic points, and so uniquely defined by density. □

Corollary 5.5. *Let f_0, f_1 be two polynomial skew products in the same vertically expanding component. Then there exists an isotopy $(h_t)_{t\in[0,1]}$ in $J_p \times \mathbb{C}$ between $J(f_0)$ and $J(f_1)$, fixing each vertical fiber.*

Proof. Since f_0 and f_1 are in the same vertically expanding component, there is a continuous path $(f_t)_{t\in[0,1]}$ joining them, such that f_t is vertically expanding for all $t \in [0,1]$. Covering the path (f_t) with finitely many small enough balls, we can apply Lemma 5.4 to find the required isotopy h_t (the uniqueness in Lemma 5.4 makes it possible to glue each piece of the isotopy in a coherent manner). □

5.2. **Unbounded hyperbolic components in $\mathcal{D}$.** We establish here the second part of Theorem B. We work with the family $\mathbf{Sk}(p,2)$ defined as in Section 2.5. Recall that this means working with the family $f_{\lambda=(a,b,c)}(z,w) = (p(z), w^2 + az^2 + bz + c)$ for some fixed polynomial p of degree 2.

Proposition 5.6. *Let $[\lambda_i] := [a_i, b_i, c_i] \in \mathbb{P}^2_\infty \backslash E$, $i \in \{0,1\}$, and for every bounded Fatou component U of p, let $s_i(U)$ denote the number of roots of $a_iX^2 + b_iX + c_i$ in U, counting multiplicity. If for every U we have $s_0(U) = s_1(U)$ then both $[\lambda_1]$ and $[\lambda_2]$ are accumulated by the same connected component of $\mathcal{D}$.*

Proof. Theorem 4.1 implies that if $[\lambda_0]$ and $[\lambda_1]$ are in the same connected component of $\mathbb{P}^2_\infty \backslash E$, then they are accumulated by the same connected component of $\mathcal{D}$. This can be seen by picking a path $t \mapsto [\lambda(t)]$ in $\mathbb{P}^2_\infty \backslash E$ joining $[\lambda_0]$ and $[\lambda_1]$, and lifting it to a path $t \mapsto \lambda(t) \in \mathbb{C}^3$. Then, for all $n \in \mathbb{N}$ large enough, $n\lambda(t) \in \mathcal{D}$ for all $t \in [0,1]$, so $n\lambda(0)$ and $n\lambda(1)$ are in the same component of $\mathcal{D}$. Moreover, as $n \to +\infty$, $n\lambda(t) \to [\lambda(t)] \in \mathbb{P}^2_\infty$.

Therefore, it remains to see that if $s_0(U) = s_1(U)$ for all bounded Fatou component U of p, then $[\lambda_0]$ and $[\lambda_1]$ belong to the same connected component of $\mathbb{P}^2_\infty \backslash E$. Since E is closed, we may slightly perturb $[\lambda_0]$ and $[\lambda_1]$ if necessary to assume that $a_i \neq 0$, and so choose representatives of the form $(1, b_i, c_i)$, so that both polynomials $X^2 + b_iX + c_i$ have two roots x_i, y_i counted with multiplicity. By assumption, we may assume that x_0 and x_1 (respectively y_0 and y_1) belong to the same Fatou components of p. Choosing paths $x(t)$ (respectively $y(t)$) joining x_0 and x_1 (respectively y_0 and y_1) inside those Fatou components, the path $t \mapsto [1, -x(t) - y(t), x(t) \cdot y(t)]$ joins $[\lambda_0]$ and $[\lambda_1]$ inside $\mathbb{P}^2_\infty \backslash E$. This concludes the proof. □

Let $\pi_0(\mathring{K}_p)$ denote the set of all bounded Fatou components of p, and set

$$\mathcal{S}_p = \Big\{ s : \pi_0(\mathring{K}_p) \to \{0,1,2\} : \sum_{U \in \pi_0(\mathring{K}_p)} s(U) \leq 2 \Big\}. \tag{7}$$

To any $[\lambda] = [a,b,c] \in \mathbb{P}^2_\infty \backslash E$, we can associate an element $s \in \mathcal{S}_p$ defined as follows: $s(U)$ is the number of roots of $aX^2 + bX + c$ in U. It is easy to check that all $s \in \mathcal{S}_p$ can be realized in that way. Proposition 5.6 asserts that to any such s is associated a unique hyperbolic component of $\mathcal{D}'$: in other words, we have defined a map $\omega : \mathcal{S}_p \to \pi_0(\mathcal{D}')$, where $\pi_0(\mathcal{D}')$ is the set of hyperbolic components of $\mathcal{D}'$.

Our result below completes the classification of these components, for p with locally connected Julia set. It also completes the proof of Theorem B. Notice that the assumption that J_p is locally connected implies that the boundary of every bounded Fatou component of p is a Jordan curve, see [DH84]. This assumption is automatically satisfied if p is hyperbolic. Recall further that all bounded Fatou components of p are simply connected.

Theorem 5.7. *Assume that J_p is locally connected. Then $\omega : \mathcal{S}_p \to \pi_0(\mathcal{D}')$ is bijective.*

Since ω is surjective, all it remains is to prove that it is injective. The rest of the section is devoted to that task. We will need the following definition.

Definition 5.8. *Given $\lambda \in \mathcal{D}$ set $r(f_\lambda) := \inf_{z\in J_p} |az^2 + bz + c|$. Let $\gamma\colon [0,1] \to \mathbb{C}^2$ be a simple closed curve, given by $\gamma(t) = (\gamma_z(t), \gamma_w(t))$. We say that γ is admissible (for f_λ) if for all $t \in [0,1]$, $|\gamma_w(t)| < r(f_\lambda)$.*

Lemma 5.9. *Let $[a_0, b_0, c_0] \in \mathbb{P}^2_\infty$ be such that the roots of $a_0X^2 + b_0X + c_0$ are in the Fatou set of p. Let $\lambda = (a, b, c) \in \mathbb{C}^3$ be such that $[a, b, c] = [a_0, b_0, c_0]$. If $|\lambda|$ is large enough, the map f_λ satisfies the following properties:*

1. *if C is a curve such that $C \subset K(f_\lambda)$, then C is admissible;*
2. *If C is an admissible curve, then so is every component of $f_\lambda^{-1}(C)$;*
3. *There exists $0 < r^*(f_\lambda) < r(f_\lambda)$ such that for all $z \in J_p$, $K_z \subset \mathbb{D}(0, r^*(f_\lambda))$.*

Proof. Set $R(f_\lambda) := \sup_{z\in J_p} |az^2 + bz + c|$. Then there exists a positive constant $\alpha = \alpha(a_0, b_0, c_0, p)$ such that $\frac{1}{\alpha}|\lambda| \leq r(f_\lambda) \leq R(f_\lambda) \leq \alpha|\lambda|$. The first item then follows from Lemma 4.2. Moreover $\lambda \in \mathcal{D}$ for $|\lambda|$ large enough and for all $z \in J_p$, we have $K_z \subset \mathbb{D}(0, 2\sqrt{R(f_\lambda)})$, again by Lemma 4.2. Therefore we may take $r^* := 2\sqrt{R(f_\lambda)}$ for item (3). For item (2), observe that if $(z, w) \in F^{-1}(C)$ and C is admissible, then $|w| = O(\sqrt{|\lambda|})$ and therefore any component of $F^{-1}(C)$ is also admissible. □

In the following, we fix a pair U, V of bounded Fatou components of p with $p(U) = V$. Our assumption implies that ∂U and ∂V are Jordan curves. We denote by s the number of roots of $aX^2 + bX + c$ lying in U, counted with multiplicity. Given λ and C a simple closed curve in $\partial V \times \mathbb{C}$, we will set $\hat{C} := f_\lambda^{-1}(C) \cap (\partial U \times \mathbb{C})$.

Definition 5.10. *Let $C_0 \subset \mathbb{C}$ and $\widetilde{C}_0 \subset C_0 \times \mathbb{C}$ be two topological circles. We will say that $\widetilde{C}_0$* winds n times above C_0 *if the projection $\pi_1 : \widetilde{C}_0 \to C_0$ is an unbranched covering of degree n.*

Lemma 5.11. *Assume that $\lambda = (a, b, c) \in \mathcal{D}$, the roots of $aX^2 + bX + c$ are in the Fatou set of p and that $|\lambda|$ is large enough so that Lemma 5.9 holds. Assume that C winds once above ∂V and is admissible. Then*

1. *if $s = 0$ or $s = 2$, $\hat{C}$ has two connected components C_1 and C_2, and their linking number in $\partial U \times \mathbb{C}$ is equal to $s/2$. Both components wind once above ∂U;*
2. *if $s = 1$, then $\hat{C}$ is connected and winds twice above ∂U.*

Proof. Let $\delta \in \{1, 2\}$ be the degree of $p : U \to V$ ($\delta = 1$ if U contains no critical point of p, and $\delta = 2$ otherwise). Let $\gamma : \mathbb{R}/\mathbb{Z} \to C$ defined by $\gamma(t) := (\gamma_V(t), \gamma_w(t))$ be a parametrization of C. Let $\gamma_1 : \mathbb{R} \to \mathbb{C}^2$ be a lift by F of $t \mapsto \gamma(\delta t)$. We can define a parametrization γ_U of ∂U by $p \circ \gamma_U(t) = \gamma_V(\delta t)$ for all $t \in \mathbb{R}/\mathbb{Z}$. So, the map γ_1 is of the form

$$\gamma_1(t) = (\gamma_U(t), w_t)$$

and w_t satisfies the equation

$$w_t^2 = \gamma_w(t) - (a\gamma_U(t)^2 + b\gamma_U(t) + c).$$

Observe that the curve $t \mapsto \gamma_w(t) - (a\gamma_U(t)^2 + b\gamma_U(t) + c)$ turns s times around $w = 0$. We now distinguish between the cases $s \in \{0, 2\}$ or $s = 1$.

(1) If $s \in \{0, 2\}$, since the curve $t \mapsto \gamma_w(t) - (a\gamma_U(t)^2 + b\gamma_U(t) + c)$ turns an even number of times around $w = 0$ as t goes from 0 to 1, we have $w_1 = w_0$ by monodromy. Therefore $\gamma_1(1) = \gamma_1(0)$, and $\gamma_1(\mathbb{R})$ is a closed loop winding once above ∂U. Since $F : \hat{C} \to C$ has degree 2δ, and $F : \gamma_1(\mathbb{R}) \to C$ has degree δ, there is a second lift $\gamma_2 : \mathbb{R} \to C_2$ parametrizing a second connected component of $\hat{C}$. Moreover, γ_2 has the form

$$\gamma_2(t) = (\gamma_U(t), -w_t)$$

and therefore the linking number in $\partial U \times \mathbb{C}$ of C_1 and C_2 is given by the number of turns around $w = 0$ of $t \mapsto w_t$ as t varies from 0 to 1, namely $s/2$.

(2) If $s = 1$: now the curve $t \mapsto \gamma_w(t) - (a\gamma_U(t)^2 + b\gamma_U(t) + c)$ turns exactly once around $w = 0$ as t goes from 0 to 1. Therefore, by monodromy we have $w_1 = -w_0$ and $w_2 = w_0$. This means that the support of $\gamma_1(\mathbb{R})$ is a curve that winds twice above ∂U. Moreover, as $\gamma_1(\mathbb{R}) \cap \{z = \gamma_U(0)\} = \{(\gamma_U(0), \pm w_0)\}$, the degree of $F : \gamma_1(\mathbb{R}) \to C$ is 2δ and therefore $\hat{C} = \gamma_1(\mathbb{R})$. □

Lemma 5.12. *Assume that $\lambda = (a, b, c) \in \mathcal{D}$, the roots of $aX^2 + bX + c$ are in the Fatou set of p and that $|\lambda|$ is large enough so that Lemma 5.9 holds. Assume that C winds twice above ∂V. Then $\hat{C}$ has two connected components C_1 and C_2. Both are curves that wind twice above ∂U and their linking number in $\partial U \times \mathbb{C}$ is equal to s.*

Proof. The proof is similar to that of the previous Lemma. Let $\delta \in \{1, 2\}$ be the degree of $p : U \to V$. Since C winds twice above ∂V, it has a parametrization $\gamma : \mathbb{R}/\mathbb{Z} \to C$ of the form $\gamma(t) = (\gamma_V(2t), \gamma_w(t))$, where for all $t \in \mathbb{R}$, $\gamma_w(t + \frac{1}{2}) \neq \gamma_w(t)$. As before, let $\gamma_1 : \mathbb{R} \to \hat{C}$ be a lift by F of $t \mapsto \gamma(\delta t)$. Then γ_1 has the form

$$\gamma_1(t) = (\gamma_U(2t), w_t),$$

and $t \mapsto w_t$ satisfies the equation

$$w_t^2 = \gamma_w(t) - (a\gamma_U(2t)^2 + b\gamma_U(2t) + c).$$

Note that $a\gamma_U(1)^2 + b\gamma_U(1) + c = a\gamma_U(0)^2 + b\gamma_U(0) + c$ but $\gamma_w(\frac{1}{2}) \neq \gamma_w(0)$, hence $w_{1/2} \neq w_0$. Also note that as t varies from 0 to 1, the loop $t \mapsto \gamma_w(t) - (a\gamma_U(2t)^2 + b\gamma_U(2t) + c)$ turns $2s$ times around $w = 0$. Therefore by monodromy, we have $w_1 = w_0$, so that $\gamma_1(1) = \gamma_1(0)$ and $\gamma_1(\mathbb{R})$ is a closed loop that winds twice above ∂U.

Again, the degree of $F : \hat{C} \to C$ is 2δ, and the degree of $F : \gamma_1(\mathbb{R}) \to C$ is only δ. Moreover, $w_{1/2} \neq -w_0$ (since $w_{1/2}^2 \neq w_0^2$), and therefore $\gamma_2 : \mathbb{R} \to C$ defined by $\gamma_2(t) = (\gamma_U(2t), -w_t)$ parametrizes a second and different component of $\hat{C}$. For degree reasons, $\hat{C}$ is exactly equal to $C_1 \cup C_2$, where C_i is the support of $\gamma_i(\mathbb{R})$. Each C_i is a loop winding twice above ∂U, and $C_1 \cap C_2 = \emptyset$ since for all $t \in \mathbb{R}$, $w_t \neq 0$. Therefore the C_i are the connected components of $\hat{C}$. Moreover, since $\gamma_1(t) = (\gamma_U(2t), w_t)$ and $\gamma_2(t) = (\gamma_U(2t), -w_t)$, the linking number of C_1 and C_2 is given by the number of times that $t \mapsto w_t$ turns around $w = 0$ as t varies from 0 to 1, namely s. □

On our way to prove Theorem 5.7, we will need the following topological description of the Julia sets of maps in $\mathcal{D}$, which has independent interest.

Definition 5.13. *Let $\Sigma \subset \mathbb{C}$ be a Cantor set that is invariant under $w \mapsto -w$. The* suspension *S of Σ is given by $S := ([0,1] \times \Sigma)/\sim$, where $(0, w) \sim (1, -w)$.*

Theorem 5.14. *Assume that J_p is locally connected, and let U be a bounded Fatou component of p. Let $[a, b, c] \in \mathbb{P}^2_\infty \backslash E$. For all representative $\lambda = (a, b, c)$ of norm large enough, the following holds:*

(1) If there exists $n \in \mathbb{N}^*$ such that $p^n(U)$ contains exactly one root of $aX^2 + bX + c$, then $J_{\partial U} := \cup_{z \in \partial U} \{z\} \times J_z$ is homeomorphic to S.
(2) Otherwise, $J_{\partial U}$ is homeomorphic to $S^1 \times \Sigma$.

Moreover, if p has no bounded Fatou components then $\mathcal{D}'$ has only one component, in which the Julia set is homeomorphic to $J_p \times \Sigma$.

Proof. If p has no periodic bounded Fatou component, then by Sullivan's Theorem p only has the basin of infinity as a Fatou component. In this case, there is only one component in $\mathcal{D}'$. Indeed, in this case $\mathcal{S}_p$ is a singleton and $\mathcal{D}' = \omega(\mathcal{S}_p)$. This component must necessarily contain product maps; therefore $J(F)$ is homeomorphic to $J_p \times \Sigma$. From now on, we assume that p has a cycle of bounded Fatou components.

Let $r^* = r^*(f_\lambda)$ be given by Lemma 5.9. Let U_0 be a bounded periodic Fatou component for p of period $m \in \mathbb{N}^*$, and let $W_0 := \partial U_0 \times \mathbb{D}(0, r^*)$. Let $U_i := p^{m-i}(U_0)$ be a cyclic numbering of the cycle of components containing U_0, with $i = 0, \dots, m-1$, so that $p(U_{i+1}) = U_i$.

(1) Assume first that each component in the cycle $U_0, \dots, U_{m-1}$ contains either zero or two roots of $aX^2 + bX + c$. Since W_0 is homotopic to a curve winding once above ∂U_0, by Lemma 5.11, $W_1 := F^{-1}(W_0) \cap (\partial U_1 \times \mathbb{C})$ is homotopic to two disjoint curves, each winding once above ∂U_1. Therefore, W_1 is a disjoint union of the interior of two solid tori, each winding once above ∂U_1. Letting $W_n := F^{-n}(W_0) \cap (\partial U_n \times \mathbb{C})$, we therefore get by induction that W_n is a disjoint union of the interior of 2^n solid tori, each winding once above ∂U_n. Since $W_m \Subset W_0$, we get that $\bigcap_{n \in m\mathbb{N}} W_n$ is homeomorphic to $S^1 \times \Sigma$.
(2) Assume now that there exists a component in the cycle containing U_0 (we may assume without loss of generality that it is U_0 itself) that contains exactly one root of $aX^2 + bX + c$. We proceed as before, letting $W_0 := \partial U_0 \times \mathbb{D}(0, r^*)$ and $W_n := F^{-n}(W_0) \cap (\partial U_n \times \mathbb{C})$. This time, Lemma 5.11 implies that W_1 is homotopic to a double winding curve above ∂U_1. Therefore W_1 is the interior of a double winding solid torus. Moreover, by Lemma 5.12, for all $n \geq 1$ the set W_n is the disjoint union of the interior of 2^{n-1} solid tori, each winding twice above ∂U_n. Therefore, for $0 \leq j \leq m-1$, $J_{\partial U_j} = \bigcap_{n \in m\mathbb{N}} W_n$ is homeomorphic to the suspension S.

To conclude the proof of Theorem 5.14, notice that if U, V are two Fatou components of p such that $p(U) = V$, and $J_{\partial V}$ is homeomorphic to either $S^1 \times \Sigma$ or S, then Lemmas 5.11 and 5.12 allow us to determine the topology of $J_{\partial U}$. More precisely, letting

$s \in \{0,1,2\}$ be the number of roots of $aX^2 + bX + c$ contained in U, we have the following:

(1) if $s = 0$ or $s = 2$, then $J_{\partial U}$ is homeomorphic to $J_{\partial V}$;
(2) if $s = 1$, then $J_{\partial U}$ is homeomorphic to S.

Since every Fatou component of p is preperiodic to U_0, the rest of the proof follows. □

We are now ready to prove the injectivity of ω.

Proof of Theorem 5.7. Let $s_0, s_1 \in \mathcal{S}_p$ such that $\omega(s_0) = \omega(s_1)$: we need to prove that $s_0 = s_1$. In other terms, let $f_0, f_1 \in \mathbf{Sk}(p,2)$ be in a small enough neighbourhood of $\mathbb{P}^2_\infty \backslash E$ in $\mathbb{P}^3$, and belonging to the same component of $\mathcal{D}'$; we will prove that for every $U \in \pi_0(\mathring{K}_p)$, $s_0(U) = s_1(U)$.

Recall that by Corollary 5.5, if $(f_t)_{t\in[0,1]}$ is an arc in $\mathcal{D}$, there is an isotopy $h_t : J(f_0) \to J(f_t)$ of the form $h_t(z,w) = (z, g_t(z,w))$. Since f_0, f_1 are in the same connected component of $\mathcal{D}'$, they can be joined by such an arc and therefore their Julia sets are isotopic in $J_p \times \mathbb{C}$.

Let U be a bounded Fatou component of p and let $z \in \partial U$, $w \in J_z(f_0)$. By Theorem 5.14, there exists a unique closed simple curve C_0 passing through $f_0(z,w)$ and contained in $J(f_0) \cap (\partial V \times \mathbb{C})$, where $V := p(U)$. That curve winds either once or twice above ∂V. Let $C_1 := h_1(C_0)$ and $\hat{C}_i := F_i^{-1}(C_i) \cap (\partial U \times \mathbb{C})$, where $i \in \{0,1\}$. Since the number of connected components of $\hat{C}_i$ and their linking number in $\partial U \times \mathbb{C}$ are invariant under isotopy in $\partial U \times \mathbb{C}$, Lemmas 5.11 and 5.12 imply that $s_0(U) = s_1(U)$. Since this is true for any bounded Fatou component U of p, $s_0 = s_1$ and the proof is finished. □

5.3. **Unbounded hyperbolic components in $\mathcal{M}$.** We have provided in the previous section a complete classification of unbounded components of $\mathcal{D}$ accumulating on $\mathbb{P}^2_\infty \setminus E$. In this section we provide a basic classification of unbounded hyperbolic components in $\mathcal{M}$ and adapt an interesting example by Jonsson ([Jon99, Example 9.6]) to show that every type of components in such classification indeed appears. For the sake of notation, we start setting the following definition, motivated by Corollary 4.3.

Definition 5.15. *Let p be a quadratic polynomial. Let $z_1, z_2 \in J_p$ (possibly with $z_1 = z_2$). We say that a hyperbolic component $U \subset \mathbf{Sk}(p,2)$ is* of type $\{z_1, z_2\}$ *if for all $z \in J_p$, $G(z,0) = 0$ if and only if $z = z_1$ or $z = z_2$. We may write $\{z_1\}$ instead of $\{z_1, z_1\}$.*

While for components of $\mathcal{D}$ we looked for a correspondence with (pairs of) points in the Fatou set of p, for unbounded components of $\mathcal{M}$ we see that a natural correspondence exists with (pairs of) points in the Julia set of p.

Theorem 5.16. *Let p be a quadratic polynomial and $U \subset \mathbf{Sk}(p,2)$ be an unbounded hyperbolic component in $\mathcal{M}$. Then there are $z_1, z_2 \in J_p$ such that U is either of type $\{z_1\}$ or of type $\{z_1, z_2\}$. Moreover, if U is of type $\{z_1\}$ then z_1 must be periodic for p, and if it is of type $\{z_1, z_2\}$ then either both z_1 and z_2 are periodic or one is preperiodic to the other.*

Proof. By Theorem 3.3, for any $f_1, f_2 \in U$ and $z \in J_p$, we have that $(z,0)$ has a bounded orbit for f_1 if and only if it has a bounded orbit for f_2. Since U is unbounded, Corollary 4.3 implies that there are at most two points $z_1, z_2 \in J_p$ such that $(z_i, 0)$ has bounded orbit, and since U is a component in $\mathcal{M}$ there is at least one $z \in J_p$ such that $(z,0)$ has bounded orbit. Therefore there are $z_1, z_2 \in J_p$ (possibly with $z_1 = z_2$) such that U is of type $\{z_1, z_2\}$. In order to prove the remaining claims of the theorem, we will use the following lemma.

Lemma 5.17. *Let f be a polynomial skew product that is vertically expanding above J_p. Let $z \in J_p$ and V be a connected component of $\mathring{K}_z$. There exists $n \in \mathbb{N}^*$ such that $f^n(\{z\} \times V)$ contains a critical point for f.*

We refer to [DH08, Proposition 3.8] for a proof of this fact. It is stated there in the case of an Axiom A polynomial skew product but the proof only uses vertical expansion over J_p.

Assume first that U is of type $\{z\}$, and let V be the connected component of $\mathring{K}_z$ containing 0. By Lemma 5.17, there is $n \in \mathbb{N}^*$ such that $f^n(\{z\} \times V)$ contains a critical point for f. But since all critical points $(y, 0)$, $y \in J_p$ escape if $y \neq z$, this means that $f^n(\{0\} \times V) = \{0\} \times V$ and $(z, 0) \in V$. In particular, we must have $p^n(z) = z$. Similarly, if U is of type $\{z_1, z_2\}$, let V_i denote the component of $\mathring{K}_{z_i}$ containing 0 ($1 \leq i \leq 2$). By Lemma 5.17, there are $n_1, n_2 \in \mathbb{N}^*$ such that $f^{n_i}(\{0\} \times V_i)$ is either $\{0\} \times V_1$ or $\{0\} \times V_2$. The result follows. □

We now give examples of all three possibilities of unbounded hyperbolic components in $\mathcal{M}$. We need the following elementary lemma, following from Section 2.5.

Lemma 5.18. *Let $z_1, z_2 \in J_p$ with $z_1 \neq z_2$ and assume that U is a hyperbolic unbounded component of type $\{z_1, z_2\}$. Then the cluster of U on $\mathbb{P}^2_\infty$ is exactly $\{[1, -z_1 - z_2, z_1 z_2]\}$.*

The following is an adaptation of [Jon99, Example 9.6].

Proposition 5.19. *Take $p(z) := z^2 - 2$, and set $g_t(z, w) := (p(z), w^2 + t(z + 1)(2 - z))$. Then for all $t > 0$ large enough,*

(1) g_t is hyperbolic;
(2) for all $z \in J_p \backslash \{-1, 2\}$, the critical point $(z, 0)$ escapes to infinity;
(3) the critical points $(-1, 0)$ and $(2, 0)$ are fixed.

Proof. Observe that for all $z \in J_p$ and t large enough, $R := 3\sqrt{t}$ is an escape radius (i.e., $K_z \subset \mathbb{D}(0, 3\sqrt{t})$ and $|w| \geq 3\sqrt{t}$ implies that $|Q_z(w)| > |w| \geq 3\sqrt{t}$). Set

$$A_t := \{z \in [-2, 2] : t(z + 1)(2 - z) \geq 3\sqrt{t}\} \subset (-1, 2).$$

Claim 5.20. *For $t > 0$ large enough, for any $z \in J_p \backslash \{-2, -1, 2\}$ there exists $n \geq 0$ such that $p^n(z) \in A_t$.*

Proof of Claim 5.20. Notice that p is semi-conjugated on J_p to the doubling map on $\mathbb{R}/\mathbb{Z}$ via the map $\varphi : \mathbb{R}/\mathbb{Z} \to J_p$ given by $\varphi(x) = 2\cos(2\pi x)$. Note that $\varphi([0]) = 2$ and $\varphi([\frac{1}{3}]) = \varphi([\frac{2}{3}]) = -1$. Given $0 < \varepsilon < 1/8$, let us set

$$\widetilde{A}_\varepsilon := \left(\varepsilon, \frac{1}{3} - \varepsilon\right) \cup \left(\frac{2}{3} + \varepsilon, 1 - \varepsilon\right) \subset \mathbb{R}/\mathbb{Z}.$$

We claim that for any $\theta \in \mathbb{R}/\mathbb{Z} \backslash \{[0], [\frac{1}{3}], [\frac{1}{2}], [\frac{2}{3}]\}$, there exists $n \in \mathbb{N}$ such that $2^n\theta \in \widetilde{A}_\varepsilon$.

(1) If $\theta \in [-\varepsilon, \varepsilon]$ and $\theta \neq 0$ then $2^n\theta \in \widetilde{A}_\varepsilon$ for some n sufficiently large;

(2) If $\theta \in I_\varepsilon := [\frac{1}{3} - \varepsilon, \frac{2}{3} + \varepsilon]$, because of (1), we can assume by contradiction that $2^n\theta \in I_\varepsilon \cup \{[0]\}$ for all n. This would imply that $2^{n+1}\theta$ belongs to I for all n, and so necessarily to a small neighbourhood of $\{\frac{1}{3}, \frac{2}{3}\}$. The only possibility is that $\theta \in \{\frac{1}{3}, \frac{2}{3}\}$, which gives the desired contradiction.

By conjugating with φ, it follows from the above that, for any $\delta > 0$ small enough, for any $z \in J_p \backslash \{-2, -1, 2\}$ there exists $n \in \mathbb{N}$ such that $p^n(z) \in (-1 + \delta, 2 - \delta)$. Since for $t > 0$ large enough we have $(-1 + \delta, 2 - \delta) \subset A_t$, the assertion follows. □

Claim 5.21. *Set*

$$U_\delta := \left\{|\mathrm{Re}(w)| \leq \frac{1}{3}, |\mathrm{Im}(w)| \leq \delta\right\} \quad \textit{and} \quad U'_\delta := \left\{|\mathrm{Re}(w)| \leq \frac{1}{4}, |\mathrm{Im}(w)| \leq \delta\right\}.$$

There exists $t_0 > 0$ such that, for all $t > t_0$ and all $0 < \delta_1 < \frac{1}{40t}$, there exists $\delta_2 > 0$ small enough such that the following holds:

(1) $U_{\delta_2} \cap K_z = \emptyset$ for all $z \in J_p$ such that $\min(|z + 1|, |z - 2|) > \delta_1$;
(2) for all $z \in J_p$ such that $\min(|z + 1|, |z - 2|) \leq \delta_1$ we have $q_z(U_{\delta_2}) \subset U'_{\delta_2}$;
(3) for all $z \in J_p \backslash \{-1, 2\}$, we have $U_{\delta_2} \cap K_z = \emptyset$.

Proof of Claim 5.21. Let us prove each item separately. First, we take $t_0 > 0$ large enough so that Claim 5.20 holds for all $t > t_0$ and we have $4t - \frac{1}{9} > 3\sqrt{t}$ for all $t > t_0$. We then fix $t > t_0$. We denote by $K \subset \mathbb{C}^2$ the set of points with bounded orbit for g_t.

(1) We claim that for all $z \in J_p \backslash \{-1, 2\}$ we have $[-\frac{1}{3}, \frac{1}{3}] \cap K_z = \emptyset$.

Fix first $z \in J_p \backslash \{-2, -1, 2\}$ and $w \in \mathbb{R}$. By Claim 5.20 there is some $n \geq 0$ such that $p^n(z) \in A_t$. Set $w_n := Q_z^n(w) \in \mathbb{R}$. Then $Q_z^{n+1}(w) = w_n^2 + t(z + 1)(2 - z) \geq t(2 - z)(z + 1) \geq 3\sqrt{t}$ and therefore $g_t^n(z, w) \notin K$, hence $(z, w) \notin K$, as desired.

Then, take $z = -2$ and $w \in [-\frac{1}{3}, \frac{1}{3}]$, so that $|w^2| < 1/9$. In this case, we have $|f(-2, w)| = |w^2 - 4t| \geq 4|t| - 1/9$, which is larger than the escape radius by our choice of t_0. Together with the previous paragraph, this completes the proof of the claim above.

Fix now $\delta_1 > 0$ and set $I(\delta_1) = \{z \in J_p : |z + 1| \geq \delta_1 \text{ and } |z + 2| \geq \delta_1\}$; the existence of δ_2 as in item (1) now follows from the fact that $(I(\delta_1) \times \mathbb{C}) \cap K$ and $I(\delta_1) \times [-\frac{1}{3}, \frac{1}{3}]$ are disjoint compact subsets of $\mathbb{C}^2$.

(Note that, for (1), the condition that $\delta_1 < \frac{1}{40t}$ is not necessary; however, it will be required in the proof of (2)).

(2) Take $0 < \delta_1 < \frac{1}{40t}$, and let $\delta_2 > 0$ be given by (1). Fix $z \in J_p$ such that $\min(|z + 1|, |z - 2|) \leq \delta_1$: then $|t(z + 1)(z - 2)| \leq 4t\delta_1$. Taking $w \in U_{\delta_2}$ and setting $w_1 := q_z(w)$, we have

$$\begin{cases} \mathrm{Re}(w_1) = \mathrm{Re}(w)^2 - \mathrm{Im}(w)^2 + t(2 - z)(z + 1) \\ \mathrm{Im}(w_1) = 2\mathrm{Im}(w)\mathrm{Re}(w) \end{cases}$$

and therefore

$$\begin{cases} |\mathrm{Re}(w_1)| \leq \frac{1}{9} + \delta_2^2 + 4t\delta_1 \leq \frac{1}{9} + \frac{1}{10} + \delta_2^2 \\ |\mathrm{Im}(w_1)| \leq 2\delta_2\frac{1}{3} \leq \delta_2, \end{cases}$$

using $\delta_1 < \frac{1}{40t}$. Up to taking a smaller δ_2 if necessary (for which (1) stays true), we may assume that $\frac{1}{9} + \frac{1}{10} + \delta_2^2 \leq \frac{1}{4}$, which concludes the proof of (2).

(3) Because of item (1), we only need to consider z such that $0 < \min(|z-2|, |z+1|) \leq \delta_1$. For any such z and $w \in U_{\delta_2}$, by means of Claim 5.20 and iterating (2) we find a smallest $n \geq 1$ such that $p^n(z) \in A_t$ and $Q_z^n(w) \in U_{\delta_2}$. By (1), we have $f^n(z, w) \notin K$, hence $(z, w) \notin K$. The proof is complete.

□

Let us now return to the proof of Proposition 5.19. Item (3) is trivial. Take $t > t_0$, where t_0 is given by Claim 5.21, take $0 < \delta_1 < \frac{1}{40t}$, and let δ_2 be given by the same Claim 5.21. We can assume that $\delta_2 < \frac{1}{12}$ and that the distance between J and $J_p \times \{|w| \geq 3\sqrt{t}\}$ is also larger than δ_2.

Item (2) follows immediately from the last item of Claim 5.21. In order to prove that g_t is indeed hyperbolic, we apply Theorem 2.3 and prove that the post-critical set does not accumulate on the Julia set J. Since the critical set over J_p is given by $[-2, 2] \times \{0\}$, it is enough to prove that

$$\text{for every } z \in [-2, 2], \text{ we have } d(g_t^n(z, 0), J) > \delta_2 \text{ for every } n \geq 0.$$

Item (3) of Claim 5.21 and the lower semicontinuity of $z \mapsto J_z$ imply that $J \cap ([-2, 2] \times U_{\delta_2}) = \emptyset$. Thus, the claim is true for $n = 0$. Since $(2, 0)$ and $(-1, 0)$ are fixed, the claim is true for these two points. Moreover, the claim holds for every $z \in A_t$, since by definition $q_z(0) = t(z + 1)(z - 2) \geq 3\sqrt{t}$.

Fix any other $z \in J_p$, $z \neq -2$, and set $(z_n, w_n) := (p^n(z), Q_z^n(0))$. Notice that $w_n \in \mathbb{R}$. By Claim 5.20, there exists n such that $z_n \in A_t$. By the first item of Claim 5.21, it is then enough to prove that $d((z_j, w_j), J) \geq \delta_2$ for $1 \leq j < n$. But the second item of Claim 5.21 implies that $w_j \in \mathbb{R} \cap U'_{\delta_2}$. Since $J \cap ([-2, 2] \times U_{\delta_2}) = \emptyset$, the assertion follows.

To conclude the proof, we need to consider the orbit of $(-2, 0)$. But $|f^n(-2, 0)| > 3\sqrt{|t|}$ for all $n \geq 1$, as proved in the first item of Claim 5.20. The proof is complete. □

Proposition 5.22. *Take $p(z) = z^2 - 2$. There are unbounded hyperbolic components in $\mathbf{Sk}(p, 2)$ of type $\{-1, 2\}$, $\{2\}$, and $\{-2, 2\}$.*

Notice the the component of type $\{-1, 2\}$ corresponds to the case with two periodic points for p, while for the component of type $\{-2, 2\}$ the point 2 is periodic and -2 is preperiodic to 2.

Proof. According to Proposition 5.19, the maps g_t are all hyperbolic for t large enough, and since $t \mapsto g_t$ is a continuous, unbounded path in $\mathbf{Sk}(p, 2)$, they all belong to the same hyperbolic component which is unbounded and of type $\{-1, 2\}$. The existence of components of type $\{2\}$ and $\{-2, 2\}$ can be proved considering skew products of respective forms $(z, w) \mapsto (z^2 - 2, w^2 + t(2 - z))$ and $(z, w) \mapsto (z^2 - 2, w^2 + t(z + 2)(2 - z))$, and adapting Proposition 5.19 to those cases. □

6. Equidistribution results in parameter spaces (Theorem C)

We obtain here a general parametric equidistribution result for families of endomorphisms of $\mathbb{P}^k$, in any dimension k, see Theorem 6.3. We then describe the adapted version for families of polynomial skew products that was used in Section 4.2, see Theorem 6.9.

6.1. **Equidistributions for endomorphisms of $\mathbb{P}^k$.** Let M be a connected complex manifold, and let $f\colon M \times \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map, defining a holomorphic family $f(\lambda, z) = (\lambda, f_\lambda(z))$ of endomorphisms of $\mathbb{P}^k$. We assume here the following:

$$\forall n \in \mathbb{N}^* \exists \lambda \in M \text{ such that for all periodic points of exact period } n \text{ for } f_\lambda :$$

$$\det(Df^n_\lambda(z) - \mathrm{Id}) \neq 0. \tag{8}$$

Observe that the above condition is for instance satisfied if the family contains the map $[z_0, z_1, \dots z_k] \mapsto [z_0^d, z_1^d, \dots, z_k^d]$. Denote by Jac the determinant of the Jacobian matrix and set

$$\widetilde{\mathrm{Per}^J_n} = \{(\lambda, \eta) \in M \times \mathbb{C} : \exists z \in \mathbb{P}^k \text{ of exact period } n \text{ for } f_\lambda \text{ and such that } \mathrm{Jac} f^n_\lambda(z) = \eta\}.$$

Let Per^J_n be the closure of $\widetilde{\mathrm{Per}^J_n}$ in $M \times \mathbb{C}$. The following result in particular implies that Per^J_n is an analytic hypersurface in $M \times \mathbb{C}$.

Proposition 6.1. *Let $(f_\lambda)_{\lambda \in M}$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ satisfying* (8)*. There exists a sequence of holomorphic maps $P_n : M \times \mathbb{C} \to \mathbb{C}$ such that*

1. *for all $\lambda \in M$, $P_n(\lambda, \cdot)$ is a monic polynomial of degree $\delta_n \sim \frac{d^{nk}}{n}$;*
2. *$P_n(\lambda, \eta) = 0$ if and only $(\lambda, \eta) \in \mathrm{Per}^J_n$.*

Moreover, if $(\lambda, \eta) \in \mathrm{Per}^J_n \setminus \widetilde{\mathrm{Per}^J_n}$, there exists $z \in \mathbb{P}^k$ and $m < n$ dividing n such that $f^m_\lambda(z) = z$, $\mathrm{Jac}(f^n_\lambda)(z) = \eta$, and 1 is an eigenvalue of $Df^n_\lambda(z)$.

Proof. Let Ω_n denote the set of $\lambda \in M$ such that periodic cycles of period less than or equal to n do not have 1 as an eigenvalue. By Assumption (8), Ω_n is open and dense in M. Let $p_n\colon \Omega_n \times \mathbb{C} \to \mathbb{C}$ be defined by

$$p_n(\lambda, \eta) := \prod_{z \in E_n(\lambda)} (\eta - \mathrm{Jac}\, f^n_\lambda(z))$$

where $E_n(\lambda)$ denotes the set of periodic points of exact period n for f_λ.

By the implicit function theorem and the definition of Ω_n, p_n is holomorphic on $\Omega_n \times \mathbb{C}$. Since it is locally bounded, Riemann's extension theorem implies that it extends holomorphically to all of $M \times \mathbb{C}$.

Now notice that for all $\lambda \in \Omega_n$, n divides the multiplicity of every root w of the polynomial $p_n(\lambda, \cdot)$. Indeed, if $z \in E_n(\lambda)$ is such that $w = \mathrm{Jac}\, f^n_\lambda(z)$, then it is also the case for the other points of the cycle, namely the $f^m(z)$, $0 \leq m \leq n-1$. So, for every $\lambda \in \Omega_n$, there is a unique monic polynomial map $P_n(\lambda, \cdot)$ such that $P_n(\lambda, \cdot)^n = p_n(\lambda, \cdot)$. Its degree δ_n satisfies $\delta_n \sim \frac{\mathrm{card}\, E_n(\lambda)}{n}$, and by classical computations $\mathrm{card}\, E_n(\lambda) \sim d^{nk}$. The map $\lambda \mapsto P_n(\lambda, \cdot)$ is holomorphic on Ω_n and locally bounded, hence extends holomorphically to M.

Finally, for all $(\lambda, \eta) \in \Omega_n \times \mathbb{C}$, $P_n(\lambda, \eta) = 0$ if and only if $\lambda \in \widetilde{\mathrm{Per}^J_n}$. If $\lambda \notin \Omega_n$, by considering a sequence $(\lambda_i, \eta_i) \in \Omega_n \times \mathbb{C}$ converging to (λ, η), we find that $P_n(\lambda, \eta) = 0$ if and only if f_λ has a cycle with Jacobian η whose period divides n. The drop in period may occur if two points of the cycle collide, creating an eigenvalue 1. □

Definition 6.2. *For $\eta \in \mathbb{C}$, we denote by $\mathrm{Per}^J_n(\eta)$ the analytic hypersurface of M defined by $\mathrm{Per}^J_n(\eta) := \{\lambda \in M : (\lambda, \eta) \in \mathrm{Per}^J_n\}$ and by $L_n\colon M \times \mathbb{C} \to \mathbb{C}$ the function $L_n(\lambda, \eta) := d^{-nk} \log |P_n(\lambda, \eta)|$.*

By the Lelong-Poincaré equation, we have that $dd^c_{\lambda,\eta} L_n = d^{-nk}[\mathrm{Per}_n^J]$, where $[\mathrm{Per}_n^J]$ is the current of integration on Per_n^J. Likewise, we have $dd^c_\lambda L_n(\cdot,\eta) = d^{-nk}[\mathrm{Per}_n^J(\eta)]$.

Theorem 6.3. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of endomorphisms of $\mathbb{P}^k$ containing the map $[z_0, z_1, \ldots z_k] \mapsto [z_0^d, z_1^d, \ldots, z_k^d]$. Then $L_n \to L$, the convergence taking place in $L^1_{\mathrm{loc}}(M\times\mathbb{C})$. In particular, for any $\eta \in \mathbb{C}$ outside of a polar set, we have $d^{-nk}[\mathrm{Per}_n^J(\eta)] \to T_{\mathrm{bif}}$.*

Recall that we denote by $L : M \to \mathbb{R}^+$ the sum of the Lyapunov exponents of f_λ with respect to its equilibrium measure μ_λ. The assumptions of Theorem 6.3 are clearly satisfied when we consider the family of all endomorphisms of $\mathbb{P}^k$ of a given algebraic degree. Thus Theorem 6.3 implies Theorem C. In order to prove the convergence in Theorem 6.3, in the spirit of [BB09] we first study the convergence of the following modifications of L_n:

(1) $L_n^+(\lambda,\eta) = (nd^{nk})^{-1}\sum_{z\in E_n(\lambda)} \log^+|\eta - \eta_n(z,\lambda)|$ where $\eta_n(z,\lambda) := \mathrm{Jac} f_\lambda^n(z)$
(2) $L_n^r(\lambda) = (2\pi d^{nk})^{-1}\int_0^{2\pi} \log|P_n(\lambda, re^{it})|dt$.

We will need the following quantitative approximation of L by Berteloot-Dupont-Molino.

Lemma 6.4 ([BDM08], Lemma 4.5)**.** *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Let $\varepsilon > 0$ and let $R_n^\varepsilon(f)$ be the set of repelling periodic points z of exact period n for f, such that $\left|\frac{1}{n}\log|\mathrm{Jac} f^n(z)| - L(f)\right| \leq 2\varepsilon$. Then for n large enough, $\mathrm{card}\, R_n^\varepsilon(f) \geq d^{nk}(1-\varepsilon)^3$.*

Lemma 6.5. *For all $\eta \in \mathbb{C}$, $L_n^+(\cdot,\eta) \to L$ pointwise and in $L^1_{\mathrm{loc}}(M)$.*

Proof. In what follows, the notation $O(\cdot)$ denotes quantities that are bounded by constants depending only on f_λ and η, and not on n or ε. Fix $\eta \in \mathbb{C}$ and $\varepsilon > 0$. We have, for all $n \in \mathbb{N}^*$:

$$|L_n^+(\lambda,\eta)| \leq \frac{\mathrm{card}\, E_n(\lambda)}{d^{nk}} \sup_{z\in\mathbb{P}^k} \|Df_\lambda(z)\|$$

which is locally bounded from above. Moreover,

$$\begin{aligned} L_n^+(\lambda,\eta) =& \frac{1}{nd^{nk}}\left(\sum_{z\in R_n^\varepsilon(\lambda)} \log|\eta - \eta_n(z,\lambda)| + \sum_{z\in E_n(\lambda)\setminus R_n^\varepsilon(\lambda)} \log^+|\eta - \eta_n(z,\lambda)|\right) \\ =& \frac{1}{nd^{nk}}\left(\sum_{z\in R_n^\varepsilon(\lambda)} \log|\eta_n(z,\lambda)| + O\left((L(\lambda)-2\varepsilon)^{-n}\right)\right) \\ &+ O\left(\frac{\mathrm{card}\, E_n(\lambda)\setminus R_n^\varepsilon(\lambda)}{nd^{nk}} \log(|\eta| + (L(\lambda)+2\varepsilon)^n)\right) \end{aligned}$$

For any $\varepsilon > 0$ small enough, $\lim_{n\to\infty}(L(\lambda)-2\varepsilon)^{-n} = 0$. By Lemma 6.4, for n large enough, $\frac{\mathrm{card}\, E_n(\lambda)\setminus R_n^\varepsilon(\lambda)}{d^{nk}} = O(\varepsilon)$. Hence, for n large enough,

$$L_n^+(\lambda,\eta) = \frac{1}{nd^{nk}} \sum_{z\in R_n^\varepsilon(\lambda)} \log|\eta_n(z,\lambda)| + O(\varepsilon) = L(\lambda) + O(\varepsilon).$$

Therefore the sequence of maps L_n^+ converges pointwise to $(\lambda, \eta) \mapsto L(\lambda)$ on $M \times \mathbb{C}$. Since the L_n's are psh and locally uniformly bounded from above, by Hartogs lemma, the convergence also happens in L^1_{loc}. □

Lemma 6.6. *For all $r > 0$, $L_n^r \to L$ pointwise and in $L^1_{\mathrm{loc}}(M)$.*

The proof is a straightforward adaptation of that of [BB09, Theorem 3.4 (2)].

Proof of Theorem 6.3. First, note that the sequence L_n does not converge to $-\infty$. Indeed, denote by λ_0 the parameter corresponding to the map $[z_0, z_1, \dots z_k] \mapsto [z_0^d, z_1^d, \dots, z_k^d]$. For all n-periodic cycles at f_{λ_0}, the eigenvalues at those cycles are either d or 0; in particular, the modulus of the Jacobian takes values in the set $\{0\} \cup \{d^{kn}, n \in \mathbb{N}\}$. Let us choose $\eta_0 := i$: we claim that $(\lambda_0, \eta_0) \notin \overline{\bigcup_{n \in \mathbb{N}^*} \mathrm{Per}_n}$. Indeed, for any $\varepsilon > 0$, there exists a neighborhood V of the map $[z_0, z_1, \dots z_k] \mapsto [z_0^d, z_1^d, \dots, z_k^d]$ such that for all $\lambda \in V$, any eigenvalue ρ of any m-periodic cycle of f_λ satisfies either $|\rho| < \varepsilon^m$ or $(d - \varepsilon)^m < |\rho| < (d + \varepsilon)^m$. Therefore, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that the modulus of the Jacobian at any cycle of f_λ of period larger than N avoids the annulus $\{\frac{3}{4} < |z| < \frac{4}{3}\}$. Moreover, as mentioned above, the map $[z_0, z_1, \dots z_k] \mapsto [z_0^d, z_1^d, \dots, z_k^d]$ has no cycles with Jacobian in the larger annulus $\{\frac{1}{2} < |z| < 2\}$. Since there are only finitely many cycles of period at most N, we conclude by continuity that up to restricting V, no f_λ with $\lambda \in V$ has a cycle with Jacobian in $\{\frac{3}{4} < |z| < \frac{4}{3}\}$. In particular, the sequence $L_n(\lambda_0, \eta_0)$ does not converge to $-\infty$, as desired.

Let now $\varphi : M \times \mathbb{C} \to \mathbb{R}$ be a psh function such that a subsequence L_{n_j} converges L^1_{loc} to φ. Let $(\lambda_0, \eta_0) \in M \times \mathbb{C}$. We have to prove that $\varphi(\lambda_0, \eta_0) = L(\lambda_0)$.

First, let us prove that $\varphi(\lambda_0, \eta_0) \leq L(\lambda_0)$. Take $\varepsilon > 0$ and let B_ε be the ball of radius ε centered at (λ_0, η_0) in $M \times \mathbb{C}$. Using the submean inequality and the L^1_{loc} convergence of L_n^+, we have

$$\varphi(\lambda_0, \eta_0) \leq \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} \varphi \leq \frac{1}{|B_\varepsilon|} \lim_j \int_{B_\varepsilon} L_{n_j} \leq \frac{1}{|B_\varepsilon|} \lim_j \int_{B_\varepsilon} L_{n_j}^+ \leq \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} L.$$

Then letting $\varepsilon \to 0$, we have that $\varphi(\lambda_0, \eta_0) \leq L(\lambda_0)$, which gives the desired inequality.

Now let us prove the opposite inequality. Assume for now that $\eta_0 \neq 0$. Let $r_0 = |\eta_0|$, and let us first notice that

$$\text{for almost every } t \in S^1, \quad \limsup_j L_{n_j}(\lambda_0, r_0 e^{it}) = L(\lambda_0). \tag{9}$$

Indeed, for any $t \in S^1$ we have

$$\lim_j L_{n_j}(\lambda_0, r_0 e^{it}) \leq \limsup_j L_{n_j}^+(\lambda_0, r_0 e^{it}) = L(\lambda_0) \tag{10}$$

and by Fatou's lemma (applied to the functions $t \mapsto -L_{n_j}(\lambda_0, r_0 e^{it})$, which are bounded from below by a constant) and the pointwise convergence of $L_n^{r_0}$ we get:

$$\begin{aligned} L(\lambda_0) = \lim_n L_n^{r_0}(\lambda_0) &= \limsup_j \frac{1}{2\pi} \int_0^{2\pi} L_{n_j}(\lambda_0, r_0 e^{it}) dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \limsup_j L_{n_j}(\lambda_0, r_0 e^{it}) dt \end{aligned}$$

which, together with (10), concludes the proof of (9).

Suppose now to obtain a contradiction that $\varphi(\lambda_0,\eta_0) < L(\lambda_0)$. Since L is continuous and φ is upper semi-continuous, there is $\varepsilon > 0$ and a neighbourhood V_0 of (λ_0,η_0) such that $\varphi(\lambda,\eta) - L(\lambda) < -\varepsilon$ for all $(\lambda,\eta) \in V_0$, We may assume without loss of generality that $V_0 = B_0 \times \mathbb{D}(\eta_0,\gamma)$, where B_0 is a ball containing λ_0. Hartogs' Lemma then gives

$$\limsup_j \sup_{V_0} L_{n_j} - L \leq \sup_{V_0} \varphi - L \leq -\varepsilon.$$

But this contradicts (9).

Therefore, we have proved that any convergent subsequence of L_n in the L^1_{loc} topology of $M \times \mathbb{C}$ must agree with L on $M \times \mathbb{C}^*$. Since $M \times \{0\}$ is negligible, this proves that L_n converges in L^1_{loc} to L on $M \times \mathbb{C}$. The proof is complete. □

6.2. **Equidistribution results for polynomial skew products.** We now explain how to adapt (the proof of) the general Theorem 6.3 to get a natural equidistribution statement for a family $(f_\lambda)_{\lambda\in M}$ of polynomial skew products of $\mathbb{P}^2$. Since the construction is very similar to the one above, we will omit part of the proofs. Recall that $Q^n_{z,\lambda}$ is defined by (1).

Proposition 6.7. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew products of $\mathbb{P}^2$ over a fixed base p. There exists a sequence of holomorphic maps $P^v_n : M \times \mathbb{C} \to \mathbb{C}$ such that:*

(1) For all $\lambda \in M$, $P^v_n(\lambda,\cdot)$ is a monic polynomial
(2) If $\eta \neq 1$, then $P^v_n(\lambda,\eta) = 0$ if and only if there exists $(z,w) \in \mathbb{C}^2$ that is periodic of exact period n, and $(Q^n_z)'(w) = \eta$;
(3) If $\eta = 1$, then $P^v_n(\lambda,\eta) = 0$ if and only if there exists $(z,w) \in \mathbb{C}^2$ such that (z,w) is periodic of exact period m dividing n for f_λ, and $(Q^m_{z,\lambda})'(w)$ is a primitive $\frac{n}{m}$-th root of unity.

Proof. For any $n \in \mathbb{N}$, let $E_n(p)$ denote the set of periodic points for p of exact period n. Let

$$P^v_n(\lambda,\eta) := \prod_{m|n} \prod_{z\in E_m(p)} P_{z,\frac{n}{m}}(\lambda,\eta)$$

where $P_{z,\frac{n}{m}} : M \times \mathbb{C} \to \mathbb{C}$ is the map given by [BB11, Theorem 2.1] for the family of degree d^m polynomials $\{Q^m_{z,\lambda} : \lambda \in M\}$ with $k := \frac{n}{m}$. It is straightforward to check that P_n satisfies the required properties. □

Definition 6.8. *For any $\eta \in \mathbb{C}$, we set $\text{Per}^v_n(\eta) := \{\lambda \in M : P^v_n(\lambda,\eta) = 0\}$.*

Theorem 6.9. *Let $(f_\lambda)_{\lambda\in M}$ be a holomorphic family of polynomial skew products of $\mathbb{P}^2$ of degree $d \geq 2$ over a fixed base p. Assume that the family contains a vertical expanding map. For all $\eta \in \mathbb{C}$ outside of a polar subset, we have $d^{-kn}[\text{Per}^v_n(\eta)] \to T_{\text{bif}}$.*

Proof. Set $L^v_n(\lambda,\eta) := (d^{nk})^{-1} \log |P^v_n(\lambda,\eta)|$. Similarly to the proof of Theorem 6.3, in order to prove Theorem 6.9, it is enough to prove that $L^v_n \to L_v$ in $L^1_{\text{loc}}(M\times\mathbb{C})$. Indeed, in the family $(f_\lambda)_{\lambda\in M}$, the exponent L_p is constant so $T_{\text{bif}} = dd^c L_v$ (see (3)). Set

(1) $L^{v,+}_n(\lambda,\eta) = (nd^{nk})^{-1} \sum_{(z,w)\in E_n(\lambda)} \log^+ |\eta - (Q_{z,\lambda})'(w)|$;
(2) $L^{v,r}_n(\lambda) = (2\pi d^{nk})^{-1} \int_0^{2\pi} \log |P^v_n(\lambda, re^{it})| dt$.

The desired convergence of L_n^v follows from the convergences of $L_n^{v,+}$ and $L_n^{v,r}$ to L^v. The proof of these last two, as well as the deduction of the convergence of L_n^v, is an adaptation of the methods of the previous section, see Lemmas 6.5, 6.6 and Theorem 6.3 respectively. Note that if f_{λ_0} is vertically expanding skew product, then vertical eigenvalues must converge either to 0 or ∞ exponentially fast with respect to the period; therefore in the annulus $\{\frac{1}{2} < |\eta| < 2\}$, there can be only finitely many vertical eigenvalues for f_{λ_0}, hence the argument of the proof of Theorem 6.3 is even simpler in this case, since no compensation between eigenvalues is possible. □

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