

THE TAYLOR-WILES METHOD FOR COHERENT COHOMOLOGY, II

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ABSTRACT. We show that, under certain specific hypotheses, the Taylor-Wiles method can be applied to the cohomology of a Shimura variety S of PEL type attached to a unitary similitude group G , with coefficients in the coherent sheaf attached to an automorphic vector bundle \mathcal{F} , when S has a smooth model over a p -adic integer ring. As an application, we show that, when the hypotheses are satisfied, the congruence ideal attached to a coherent cohomological realization of an automorphic Galois representation is independent of the signatures of the hermitian form to which G is attached. We also show that the Gorenstein hypothesis used to construct p -adic L -functions in [15], as elements of Hida's ordinary Hecke algebra, is valid rather generally.

The present paper generalizes the main results of the article [28], which treated the case when S is compact. As in the previous article, the starting point is a theorem of Lan and Suh that proves the vanishing of torsion in the cohomology under certain conditions on the parameters of the bundle \mathcal{F} and the prime p . Most of the additional difficulty in the non-compact case is related to showing that the contributions of boundary cohomology are all of Eisenstein type. We also need to show that the coverings giving rise to the diamond operators can be extended to étale coverings of appropriate toroidal compactifications.

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INTRODUCTION

The Taylor-Wiles method was introduced as a way of proving automorphy lifting theorems, and it has been applied in a variety of situations to show, under appropriate technical hypotheses, that all liftings to characteristic zero of an n -dimensional Galois representation $\bar{\rho}$ with coefficients in a finite field are attached to automorphic forms, provided one knows that one of the liftings is automorphic. As a by-product of the method, one finds – in the *minimal* case – that the localization of the relevant Hecke algebra at the maximal ideal attached to $\bar{\rho}$ is a local complete intersection, and that the module of automorphic forms is free over the localized Hecke algebra \mathbb{T} . These properties of the Hecke algebra and its module of automorphic forms are of independent interest. They are used, for example, in the construction of p -adic L -functions in [15].

The primary aim of the present paper, like its predecessor [28], is to prove that these properties are satisfied, in some generality, by the *coherent* cohomology of the Shimura varieties attached to unitary similitude groups. To this end we develop the Taylor-Wiles method in this situation and indicate some of the applications of the local complete intersection property to automorphic forms.

The reader should understand that, in contrast to most papers on the Taylor-Wiles method, our aim is *not* to show that certain Galois representations are automorphic (or potentially automorphic). This is already known for the kinds of Galois representations that arise in the cohomology of the PEL Shimura varieties that are the subject of these papers. In contrast, the primary purpose of the present paper, as well as its predecessor [28], is to obtain results about the structure of the localized Hecke algebra \mathbb{T} , with a view to applications to arithmetic problems. Thus the main results of the paper are those contained in §4 and §5. Thus, let F be a CM field in which every prime dividing p splits over the maximal totally real subfield F^+ of F , and let π be a cuspidal cohomological automorphic representation of $GL(n)_F$. We assume the contragredient of π is isomorphic to π^c , where c is the non-trivial element of $Gal(F/F^+)$. When V is an n -dimensional hermitian space over F , and G its unitary similitude group, we consider the L -packet $\Pi_V(\pi)$ of automorphic representations of discrete series type that base change to π , and we assume $\Pi_V(\pi)$ is non-empty; the conjugate duality hypothesis is a necessary condition for this, but there may also be local conditions at places where G is not quasi-split (see [42]). We temporarily denote by S_V the corresponding Shimura variety. We specifically prove – under the hypotheses that allow us to apply the Taylor-Wiles method – that

- The exterior power relation between the $\overline{\mathbb{Q}}_p$ -valued cohomology spaces of S_V , for V of different signatures, attached to a given cuspidal automorphic π , is in fact integral over the Hecke algebra localized at the maximal ideal corresponding to π (Proposition 44);
- The congruence ideals for the localized Hecke algebras attached to $\Pi_V(\pi)$ coincide for different V (Proposition 45);
- If π is ordinary at p and $\pi_V^0 \in \Pi_V(\pi)$ is the holomorphic member, then the Gorenstein hypothesis of [15] is valid for the Hida family through π_V^0 ; in

particular, the p -adic L -function attached to this Hida family in [15] takes values in the localized (big) Hecke algebra.

These results are all obtained as consequences of the Taylor-Wiles method for coherent cohomology.

In the article [28], the vanishing theorem [49] of Lan and Suh was used to show that the Taylor-Wiles method can be applied to the (coherent) cohomology of a compact Shimura variety S , attached to a reductive algebraic group G/\mathbb{Q} , with coefficients in an automorphic vector bundle \mathcal{F} . Assuming

- (i) the parameter of \mathcal{F} is in the range to which the vanishing theorem applies;
- (ii) S has a smooth model \mathbb{S}_K over a p -adic integer ring \mathcal{O} ; here the subscript K denotes a level subgroup, and
- (iii) there is a theory of Galois representations attached to cohomological automorphic forms on S ,

the cohomology $H^\bullet(\mathbb{S}_K, \mathcal{F})$, with coefficients in \mathcal{O} is concentrated in a single degree q and $H^q(\mathbb{S}_K, \mathcal{F})$, localized at a maximal ideal as above, is free over the localized Hecke algebra. The crucial input in the Taylor-Wiles method, applied to a reductive group G , is the action of a product of groups of the form

$$\Delta_{Q_N} = \left(\prod_{q \in Q_N} K_0(q)/K_1(q) \right)_p$$

where Q_N is a finite set of *Taylor-Wiles primes* of the base field F (usually totally real or a CM field), with $Nq \equiv 1 \pmod{p}$, $K_0(q)$ and $K_1(q)$ are open compact subgroups of $G(F_q)$ —This doesn't literally make sense unless $F = \mathbb{Q}$, but we ignore this in the present exposition — that generalize the classical congruence subgroups $\Gamma_0(q)$ and $\Gamma_1(q)$ respectively, so that $K_0(q)/K_1(q)$ is isomorphic to the multiplicative group of the residue field $k(q)$, and the subscript p denotes the maximal quotient of exponent p^N . Let C^{Q_N} denote the kernel of the map $\prod_{q \in Q_N} K_0(q)/K_1(q) \rightarrow \Delta_{Q_N}$, and let $\mathbb{S}_K(Q_N)$ denote the quotient by C^{Q_N} of the Shimura variety at level $K(Q)$, which is the subgroup obtained by replacing the local component of K at a prime $q \in Q$ by $K_1(q)$. Provided the original level K is sufficiently deep, the action of Δ_{Q_N} on $\mathbb{S}_K(Q_N)$ is *free*. A theorem of S. Nakajima [51], combined with the Lan-Suh vanishing theorem, then implies that $H^q(\mathbb{S}_K(Q_N), \mathcal{F})$ is a free $\mathcal{O}[\Delta_{Q_N}]$ -module. This, together with item (iii) above, allows us to prove (under favorable hypotheses on $\bar{\rho}$) that the original $H^q(\mathbb{S}_K, \mathcal{F})$ is free over the localized Hecke algebra \mathbb{T} .

The present paper is a sequel to [28]; its purpose is to extend these results to non-compact Shimura varieties, specifically those attached to unitary similitude groups of hermitian vector spaces over CM fields. To this end we apply the vanishing theorem of a different Lan-Suh paper [50], which extends their previous results to non-compact Shimura varieties.

The project faces two immediate difficulties. In the first place, coherent cohomology in the non-compact case is computed on toroidal compactifications, not on the open Shimura variety. In order to apply Nakajima's result, we need to

show that the action of the group denoted Δ_{Q_N} above on some toroidal compactification $\mathbb{S}_K(Q_N)^{tor}$ of $\mathbb{S}_K(Q_N)$ is still free; in other words, the $\mathbb{S}_K(Q_N)^{tor}$ is an étale cover of its quotient by Δ_{Q_N} . This is certainly not true for an arbitrary toroidal compactification, and one of the main objectives of the present paper is to construct toroidal compactifications of the integral models constructed in [44] on which Δ_{Q_N} acts freely. In the classical case of elliptic modular curves, the fact that the complete modular curve $X_1(q)$ is (usually) étale over $X_0(q)$ is a well known special feature of this particular pair of congruence subgroups. The constructions in the present paper make extensive use of the results of [44] and [45].

The second difficulty is that the Lan-Suh vanishing theorem in the non-compact case applies only to *interior coherent cohomology*, in other words the image $H_!^\bullet(\mathbb{S}_K, \mathcal{F})$ of the cohomology of \mathbb{S}_K^{tor} with coefficients in the subcanonical extension \mathcal{F}^{sub} in the cohomology of the canonical extension \mathcal{F}^{can} . Thus, although Nakajima's result applies to the canonical extension, one cannot apply it directly to the problem at hand. The freeness of the action allows us to represent $H^\bullet(\mathbb{S}_K^{tor}, \mathcal{F}^{can})$ as the cohomology of a perfect complex $R\Gamma^\bullet$ of $\mathcal{O}[\Delta_{Q_N}]$ -modules with an action of the (non-localized) Hecke algebra T , which we define, following Khare and Thorne [39], as an algebra of endomorphisms of the object $R\Gamma^\bullet$ in an appropriate derived category. Thus, even though our final results concern a Hecke algebra module that occurs in a single degree of cohomology, we find ourselves obliged to work, at least in the first stages, with the extension of the Taylor-Wiles method to cohomological complexes introduced by Calegari and Geraghty [10], and pursued by Hansen [20] as well as Khare-Thorne.

We then localize T at a non-Eisenstein maximal ideal \mathfrak{m} , and write $\mathbb{T} = T_{\mathfrak{m}}$. Using the results of [30] and the fact, due to Scholze, Boxer, Pilloni-Stroh, and Goldring-Koskivirta [57, 6, 54, 18], that there is a theory of Galois representations attached to *torsion* cohomology classes as well, we conclude that the localization at \mathfrak{m} of $H^\bullet(\mathbb{S}_K^{tor}, \mathcal{F}^{can})$ coincides with the localization of $H_!^\bullet(\mathbb{S}_K, \mathcal{F})$. Thus it is concentrated in a single degree q , and is p -torsion free, when the Lan-Suh results apply. It then follows easily that the localized cohomology is free over $\mathcal{O}[\Delta_{Q_N}]$, and we conclude the argument as in [28].

This material occupies the first three sections of the paper. Sections 5 and 6 develop applications of these results. In Section 5, we apply the results for coherent cohomology in order to prove that the interior de Rham and p -adic étale cohomology are also free over the Hecke algebra, after localization at a non-Eisenstein prime, and again in the range to which the results of [50] apply. As V varies among hermitian spaces of a fixed dimension n , the Galois representations on the corresponding p -adic étale cohomology groups are related by a formula conjectured by Langlands that has essentially been proved [58] using the methods of Kottwitz and the stable trace formula. Roughly speaking, up to twist by an explicit character, all of the cohomology groups can be obtained from a few of them by tensor operations. We show that the analogous relations hold over the localized Hecke algebra. We also observe that the congruence ideal for the automorphic representations realized on the unitary Shimura varieties attached to varying V depends only on the associated Galois representation and not on V . These results are only proved, of

course, when the Taylor-Wiles hypotheses are valid and assuming the parameters are in the range of the Lan-Suh vanishing theorem.

In section 6 we specialize to degree 0 but apply the results to holomorphic modular forms varying in ordinary Hida families. We prove that, when the Taylor-Wiles hypotheses apply to residual representation of a Hida family, then the family satisfies the Gorenstein hypotheses used in [15] to construct p -adic L -functions as elements of Hecke algebras.

The last two sections are devoted to the proof that the contribution of the boundary cohomology is purely Eisenstein. This roughly follows the pattern of the analysis of boundary coherent cohomology in [32, 33, 34]. However, some of the arguments in those papers are analytic in nature, or are only valid for cohomology with characteristic zero coefficients, and there is no reason to suppose that the results are as clean as in the analytic setting. Fortunately, for our purposes we can be satisfied with qualitative results. To prove these we make extensive use of Lan's study of the geometry of integral models of Kuga varieties in [44, 45, 46]. We also borrow some ideas of Newton and Thorne from [52].

An appendix explains the translation between the boundary cohomology of connected Shimura varieties, as developed in [32, 33], which is more convenient for the local study of the boundary, and the adelic theory of [34], which simplifies level-raising arguments.

Conventions. Let F be a CM field, quadratic over a totally real field F^+ and contained in a fixed algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} , $d = [F^+ : \mathbb{Q}]$. We let $\Gamma_F = \text{Gal}(\overline{\mathbb{Q}}/F)$, $\Gamma_{F^+} = \text{Gal}(\overline{\mathbb{Q}}/F^+)$. Let S_∞ , resp. S_∞^+ denote the set of complex embeddings of F , respectively real places of F^+ , and let \tilde{S}_∞ denote a CM type for F , i.e. a choice for each $v \in S_\infty^+$ of a complex place $\tilde{v} \in S_\infty$. Let $S(F^+)$ denote the set of primes of F^+ that ramify in F .

Our modular forms and deformation problems will be defined over a p -adic integer ring \mathcal{O} , with maximal ideal $\mathfrak{m}_{\mathcal{O}}$. It will always be taken big enough to contain the rings of definition of all local deformation problems.

1. SHIMURA VARIETIES ATTACHED TO UNITARY GROUPS

1.1. Notation for automorphic vector bundles. Let V be an n -dimensional space over F with nondegenerate hermitian form (\bullet, \bullet) , let $U = U(V)$ be its unitary group, and define the reductive group G over \mathbb{Q} by its values on \mathbb{Q} -algebras R :

$$(1) \quad G(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid (g(u_1), g(u_2)) = \nu(g)(u_1, u_2)\}$$

for all $u_1, u_2 \in V \otimes_{\mathbb{Q}} R$ for some $\nu(g) \in R^\times$. As in [29] we extend G to a Shimura datum (G, X) so that

$$S(G, X)(\mathbb{C}) := \varprojlim_{K \subset G(\mathbf{A}_f)} G(\mathbb{Q}) \backslash X \times G(\mathbf{A}_f)/K = \varprojlim_{K \subset G(\mathbf{A}_f)} S_K(G, X)(\mathbb{C}),$$

where K varies over open compact subgroups of $G(\mathbf{A}_f)$, is the set of complex points of a Shimura variety $S(G, X) = \varprojlim S_K(G, X)$ of PEL type, with reflex field $E = E(G, X)$. We let $d_V = \dim S_K(G, X)$. The CM type \tilde{S}_∞ is part of the

data defining X . For each $v \in S_\infty^+$, there is a partition $r_v + s_v = n$ such that the hermitian space $V \otimes_{F, \tilde{v}} \mathbb{C}$ has signature (r_v, s_v) ; then $U(F_v^+) \xrightarrow{\sim} U(r_v, s_v)$, with the ambiguity between $U(r_v, s_v)$ and $U(s_v, r_v)$ resolved by the choice of \tilde{v} .

In what follows we always assume $K = \prod_q K_q$, where q runs over rational prime numbers and $K_q \subset G(\mathbb{Q}_q)$ is open compact. We fix an odd prime p for the remainder of this paper and assume p is not in the set $S = S(K)$ of bad primes for K , defined in §1.1.1 below; this can be weakened slightly, but we do not bother to treat the more general situation.

For any q , we break up the set of primes $D(q)$ of F^+ dividing q into the subsets $D_+(q) \amalg D_-(q)$, where $v \in D_+(q)$ if and only if v is split in F/F^+ . For each $v \in D(q)$ choose a prime $w(v)$ of F dividing v . Then there is a natural isomorphism

$$(2) \quad G_q := G(\mathbb{Q}_q) \xrightarrow{\sim} \prod_{v \in D_+(q)} GL(n, F_{w(v)}) \times G_{-,q}$$

where we set $V_- = \prod_{v \in D_-(q)} V \otimes_{\mathbb{Q}} F_v^+$ and

$$G_{-,q} = \{g \in GL(V_-) \mid (g(u_1), g(u_2)) = \nu(g)(u_1, u_2) \text{ for some } \nu(g) \in \mathbb{Q}_q^\times\},$$

where u_1, u_2 vary over vectors in V_- . For all q we assume K_q admits a factorization

$$(3) \quad K_q = \prod_{v \in D_+(q)} K_v \times K_{-,q}$$

where $K_v \subset GL(n, F_{w(v)})$ and $K_{-,q} \subset G_{-,q}$ are open compact subgroups.

1.1.1. Type data. It was pointed out in [28, Remark 6.9] that the hypotheses of that paper restricted ramification of the residual representation $\bar{\rho}$ at places of F^+ not dividing p . In this paper those restrictions are somewhat relaxed, although we only consider deformation problems that involve minimal ramification. Let $S = S(K)$ be the set of primes q where either K_v , for some $v \in D_+(q)$, or $K_{-,q}$, is not hyperspecial maximal; we call S the set of *ramified primes* for K . The set S contains the set $S(G)$ of primes q at which the group G is ramified, and in particular contains the rational primes divisible by (the set) $S(F^+)$ of primes of F^+ that ramify in F/F^+ . At primes $q \in S(G)$ we take $K_{-,q}$ to be special maximal. If $q \in S \setminus S(G)$ we assume $K_{-,q}$ is a hyperspecial maximal subgroup of $G_{-,q}$. For $v \in D_+(q)$, whether or not $q \in S(G)$, we will choose a set of *type data*, namely quadruples $(K_v^+, K_v, \iota_v, \Lambda_{\iota_v})$, with $K_v \subset K_v^+$ a pair of open compact subgroups, Λ_{ι_v} a finite free \mathcal{O} -module and

$$(4) \quad \iota_v : K_v^+ / K_v \rightarrow \text{Aut}(\Lambda_{\iota_v})$$

a representation that determines the inertial type of the corresponding representation of $GL(n, F_v^+)$, as in [BK99].

We will also use S to denote the set of v dividing primes $q \in S(K)$ where the factor K_v of K_q is not hyperspecial maximal; this in particular allows us to use the same notation even when q is ramified in F^+ .

Our level subgroups K will always be assumed to be *neat*, as in [25, 55], so that the Shimura variety $S_K(G, X)$ is smooth and we can choose toroidal compactifications that are smooth and projective. The set S will be assumed to contain a prime \mathfrak{r} all of whose divisors in F^+ split in F/F^+ . We will choose one such divisor \mathfrak{r}_0 so that $K_{\mathfrak{r}_0}$ is the subgroup of a standard maximal compact subgroup of $GL(n, F_{\mathfrak{r}_0}^+)$ defined by

$$(5) \quad K_{\mathfrak{r}_0} = \{k \in GL(n, \mathcal{O}_{\mathfrak{r}_0}) \mid k \equiv u \pmod{\mathfrak{m}_{\mathfrak{r}_0}}\},$$

where $\mathfrak{m}_{\mathfrak{r}_0}$ is the maximal ideal in the integer ring and u is upper triangular unipotent. This suffices to guarantee that K is neat, and \mathfrak{r}_0 will be chosen so that the deformation problems to be studied below are minimal and unrestricted at \mathfrak{r} . In the statement of [28, Theorem 6.8] this set is called S_a .

Remark 6. Erratum for [28]: *In the published version of [28] the set S_a was mentioned appropriately in the statement of its main theorem but its definition was inadvertently omitted. As already mentioned, this condition is required in order to guarantee smoothness. Theorem 37 requires as well that the deformation condition at such an \mathfrak{r} be unrestricted and minimal. We choose \mathfrak{r} as on [11, p. 111] (where it is called S_a) or as on [62, p. 893] (where it is called v_1). The hypothesis on \mathfrak{r} – that $[F(\zeta_{\mathfrak{r}}) : F] > n$ – guarantees that \mathfrak{r} satisfies condition (1) of Theorem 37.*

We recall the standard constructions of automorphic vector bundles. The points $x \in X$ are identified with homomorphisms

$$h_x : \mathbb{S} = R_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{R}}$$

satisfying Deligne's list of axioms [De, 2.1.1]. The centralizer in $G(\mathbb{R})$ of h_x is a reductive group K_x whose intersection with the derived subgroup $G^{der}(\mathbb{R})$ of $G(\mathbb{R})$ is a maximal compact connected subgroup. Deligne's axiom (2.1.1.1) concerns the adjoint action

$$Ad \circ h_x : \mathbb{S} \rightarrow GL(\mathrm{Lie}(G)_{\mathbb{C}})$$

that yields an eigenspace decomposition (the *Harish-Chandra decomposition*)

$$(7) \quad \mathrm{Lie}(G)_{\mathbb{C}} = \mathfrak{p}_x^- \oplus \mathrm{Lie}(K_x)_{\mathbb{C}} \oplus \mathfrak{p}_x^+.$$

Here $z \in \mathbb{S}(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^\times$ acts trivially on $\mathrm{Lie}(K_x)$ and as (z/\bar{z}) (resp. \bar{z}/z) on \mathfrak{p}_x^+ (resp. \mathfrak{p}_x^-). The Lie subalgebras \mathfrak{p}_x^- and \mathfrak{p}_x^+ are naturally identified, respectively, with the anti-holomorphic and holomorphic tangent spaces of X at x .

Let \check{X} denote the compact dual of X , and $X \hookrightarrow \check{X}$ the Borel embedding. Concretely, \check{X} is a flag variety of maximal parabolic subgroups of $G_{\mathbb{C}}$ and the image in \check{X} of $x \in X$ is a maximal parabolic P_x with Levi subgroup K_x . In [H85] it is explained how to define a canonical $E = E(G, X)$ -rational structure on the flag variety \check{X} , following Deligne, and how to define a functor $\mathcal{V} \mapsto [\mathcal{V}]$ from G -equivariant vector bundles on X to $G(\mathbf{A}_f^p)$ -equivariant vector bundles on $S_K(G, X)$. The latter are called *automorphic vector bundles*. The functor is compatible with the E -structure in the sense that, if \mathcal{V} is defined as G -equivariant vector

bundle over a field $E(\mathcal{V})$ (which can always be taken to be a number field), then for any $\sigma \in \text{Gal}(\bar{E}/E)$, we have

$$(8) \quad \sigma[\mathcal{V}] = [\sigma(\mathcal{V})].$$

The bracket notation in the previous paragraph was introduced in order to make reference to the functor. Automorphic vector bundles will in general be denoted \mathcal{F} , and we let $E(\mathcal{F})$ denote a field of definition for the corresponding equivariant vector bundle over \tilde{X} .

Fix a point $x \in X$ with stabilizer $P_x \subset G_{\mathbb{C}}$; thus

$$(9) \quad \text{Lie}(P_x) = \text{Lie}(K_x)_{\mathbb{C}} \oplus \mathfrak{p}_x^+$$

in the Harish-Chandra decomposition. There is a natural equivalence of categories between G -equivariant vector bundles \mathcal{V} on X and finite-dimensional representations (τ, W_{τ}) of P_x ; W_{τ} is the fiber of \mathcal{V} at x , and τ is the isotropy representation. In particular, the natural representation of P_x

$$ad^+ : P_x \rightarrow K_x \rightarrow \text{Aut}(\mathfrak{p}_x^+),$$

where the second arrow is the adjoint representation, defines an automorphic vector bundle canonically isomorphic to the tangent bundle $\mathcal{T}_{S(G,X)}$. Likewise $\wedge^{\text{top}}(ad^+)^{\vee}$, the dual of the top exterior power of the adjoint action on \mathfrak{p}_x^+ , defines the canonical bundle $\Omega_{S(G,X)}^{\text{top}}$ as an automorphic vector bundle.

A representation (τ, W_{τ}) of K_x extends trivially to a representation of P_x and thus defines an automorphic vector bundle $\mathcal{F} = \mathcal{F}_{\tau}$ on $S_{K_p}(G, X)$ whose fiber at a point beneath $x \times g$ for any $g \in G(\mathbf{A}_f)$ can be identified with W_{τ} . The automorphic vector bundle \mathcal{F} can also be identified with the family of bundles \mathcal{F}_K on $S_K(G, X)$. If the family \mathcal{F}_K extends to a $G(\mathbf{A}_f^p)$ -equivariant family of vector bundles on a family of \mathcal{O} -integral models \mathbb{S}_K of $S_K(G, X)$, where \mathcal{O} is some p -adic integer ring, we denote the extension $\mathcal{F}_{K,\mathbb{S}}$. In the applications \mathbb{S}_K is a moduli space for abelian varieties with PEL structure and natural extensions will be specified in terms of this identification.

1.1.2. Notation for highest weights. Notation is as in [28, §2]. We choose a maximal torus $T \subset K_x$; then T is also a maximal torus in G . Let $\Phi = \Phi(G, T)$ denote the set of roots of G relative to T , and let Φ^+ be a system of positive roots compatible with P_x , i.e. containing the roots of \mathfrak{p}_x^- ; let $X^+(T)$ (resp. $X_x^+(T)$) denote the set of dominant weights for G (resp. for K_x) relative to this choice. Let $W = W(G, T)$ be the absolute Weyl group and let $W^x \subset W$ be the set of Kostant representatives.

Definition 10. (cf. [49, 2.32]) Let $\mu \in X^+(T)$. Say μ is *p-small*, resp. *p-small relative to x*, if

$$\langle \mu + \rho, \alpha^{\vee} \rangle \leq p, \forall \alpha \in X^+(T) \text{ (resp. } \forall \alpha \in X_x^+(T))$$

and if $p > |\mu|_L$, where if we write $\mu = (\mu_{\sigma}, \sigma \in \tilde{S}_{\infty})$,

$$|\mu|_L = \sum_{\sigma \in \tilde{S}_{\infty}} |\mu_{\sigma}|$$

in the notation of [49, Definition 3.2]. (In [49] this is called “ p -small for the geometric realization of Weyl’s construction.”)

1.2. Integral models. The Shimura datum (G, X) is of PEL type and G is of type A , so by [41] we may consider integral models of the Shimura variety $S_K(G, X)$ over p -adic integer rings, at least when p is unramified for G and K . Let $g = n \cdot [F^+ : \mathbb{Q}]$.

Proposition 11. *Let p be a prime at which G is unramified, and fix a hyperspecial maximal compact $K_p \subseteq G(\mathbb{Q}_p)$. Then the Shimura variety $S_K(G, X)$ admits a smooth model \mathbb{S}_K over a p -adic integer ring \mathcal{O} for all neat $K \supset K_p$. More precisely, if K is neat and contains K_p , then up to replacing K by a normal subgroup K' of finite index, for any prime v in $E(G, X)$ above p there exists a smooth moduli scheme $\mathbb{S}_{K'}$ over $\text{Spec}(\mathcal{O}_v)$ of abelian varieties of dimension g , with PEL structure defined in terms of the hermitian space V , whose generic fiber is isomorphic to $S_{K'}(G, X)$. The quotient $\mathbb{S}_{K'}$ by K/K' supplies the integral model \mathbb{S}_K .*

Remark 12. *It is well known that $S_K(G, X)$ is a subvariety of the moduli space parametrizing all quadruples (A, i, λ, κ) where A is an abelian variety of dimension g with endomorphisms by (a subring of) F determined by $i : F \hookrightarrow \text{End}(A)$, polarization λ , and level structure κ , all satisfying the usual hypotheses adapted to the signatures of V at archimedean places of F . We denote this moduli space $M_K(G, X)$; it is a union of a finite number of copies of $S_K(G, X)$, corresponding to the number of global forms of G that are locally isomorphic everywhere. The moduli space $M_K(G, X)$ plays no separate role in the theory developed in this paper.*

1.3. Terminology for toroidal compactifications. For any level subgroup K , the Shimura variety $S_K(G, X)$ has a family of toroidal compactifications, each one attached to a collection Σ of combinatorial data adapted to K . The precise definition of Σ is recalled below. The corresponding toroidal compactification is denoted $S_K(G, X)_\Sigma$, or $S_K(G, X)^{\text{tor}}$ when we don’t need to specify Σ . When K is allowed to vary through a collection \mathcal{B} of open compact subgroups of $G(\mathbb{A}_f)$, we choose $\Sigma(K)$ adapted to $K \in \mathcal{B}$ in such a way that, if $K' \subset K$ with both K' and K in \mathcal{B} , the natural covering map $S_{K'}(G, X) \rightarrow S_K(G, X)$ extends to a map

$$S_{K'}(G, X)_{\Sigma(K')} \rightarrow S_K(G, X)_{\Sigma(K)}$$

of toroidal compactifications.

The combinatorial data $\Sigma = \cup \Sigma_F$ adapted to the neat level K is a collection of fans, Σ_F , one for each rational boundary component F . (We will make an effort to use the letter F for boundary components and for number fields in different paragraphs, so there should be no confusion.) Each F corresponds to its stabilizer P_F , which is a maximal rational parabolic inside G . The fan Σ_F gives a polyhedral cone decomposition of a partial compactification \bar{C}_F of a certain cone C_F inside $U_F(\mathbb{R})$ that is open, convex, and self-adjoint with respect to a \mathbb{Q} -rational positive definite quadratic form, where U_F is the center of the unipotent radical of P_F . We examine the Σ_F and the inclusion $C_F \subset \bar{C}_F$ more closely below.

If Σ is $K \cap G(\mathbb{Q})$ -admissible in the sense of [2, Definition 5.1], the compactification $S_K(G, X)_\Sigma$ is smooth, and if Σ is furthermore defined by cocores then $S_K(G, X)_\Sigma$ is projective by Tai's theorem [2, IV, §2]. Lastly, we assume that Σ is such that

$$\partial S_K(G, X)_\Sigma := S_K(G, X)_\Sigma - S_K(G, X)$$

is a divisor with normal crossings. This is equivalent to the hypothesis that for all $\sigma \in \Sigma$, the semigroup $\sigma \cap (U_F(\mathbb{Q}) \cap K)$ is generated by a subset of a basis for the free abelian group $U_F(\mathbb{Q}) \cap K$. We will always choose Σ that satisfy all of the above conditions; such Σ exist and are constructed for instance in [25].

Unlike the anisotropic case treated in [28], to apply the vanishing theorem of Lan-Suh for non-compact Shimura varieties, we need work with a smooth proper integral model \mathbb{S}_K^{tor} , over a p -adic integer ring \mathcal{O} , of the toroidal compactification, for which $\mathbb{S}_K(G, X)$ of §1.2 embeds as an open dense subscheme. In the case when G is a symplectic similitude group this has been constructed in the book [16] by Faltings-Chai and subsequently extended to all PEL type Shimura varieties by K.-W. Lan in his thesis [44]. The comparison between the algebraic (moduli) construction of these compactification and the analytical one presented above is established in [43]. Below is an abridged summary of the relevant results.

Theorem 13. (Lan) *Let p be a prime at which G is unramified, and fix a hyperspecial maximal compact $K_p \subseteq G(\mathbb{Q}_p)$. Then for all neat $K \supset K_p$, there is a compatible choice of admissible smooth rational polyhedral data Σ (see [44, Def. 6.3.3.2]) such that the $\mathbb{S}_K(G, X)_\Sigma$ is a smooth proper scheme over a p -adic integer ring \mathcal{O} which contains $\mathbb{S}_K(G, X)$ of Proposition 11 as an open dense subscheme. Furthermore, Σ may be chosen so that $\mathbb{S}_K(G, X)_\Sigma - \mathbb{S}_K(G, X)$, viewed as a closed reduced subscheme, is a divisor with normal crossings. There is a canonical strata-preserving isomorphism between the basechange to $\text{Spec}(\mathbb{C})$ of this integral model and the classical analytical construction as in [2].*

1.3.1. Parabolic strata. The boundary divisor $\partial S_K(G, X)_\Sigma$ has a closed covering indexed by maximal standard rational parabolic subgroups of G , defined as follows. Let \mathbb{S}_K^{min} denote the minimal compactification of \mathbb{S}_K over $\text{Spec}(\mathcal{O})$, as in [44, Theorem 7.4.2.1]. The boundary $\mathbb{S}_K^{min} \setminus \mathbb{S}_K$ decomposes as a disjoint union of locally closed strata $\partial^P \mathbb{S}_K^{min}$ indexed by standard maximal rational parabolic subgroups $P \subset G$. Each $\partial^P \mathbb{S}_K^{min}$ is a union of certain strata denoted $\mathbf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ in Lan, where P is the stabilizer of the filtration on the hermitian vector space V concealed in the notation $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ (see [44, §5.4] for an explanation).

There is a canonical morphism [44, Theorem 7.4.2.1, 3] $\mathcal{J} : \mathbb{S}_{K, \Sigma} \rightarrow \mathbb{S}_K^{min}$. For each standard maximal rational parabolic $P \subset G$, we let the P -stratum $\partial^P \mathbb{S}_{K, \Sigma}$ of the toroidal boundary of $\mathbb{S}_{K, \Sigma}$ be the closure of $\mathcal{J}^{-1}(\partial^P \mathbb{S}_K^{min})$. Let $R \subset G$ be a standard rational parabolic, and write $R = \cap Q_j$ for a (unique) set of standard maximal parabolics Q_j . We define the R -stratum:

$$(14) \quad \partial^R \mathbb{S}_{K, \Sigma} = \cap_j \partial^{Q_j} \mathbb{S}_{K, \Sigma}.$$

Under our running hypotheses on Σ , the intersections are componentwise transversal and each $\partial^R \mathbb{S}_{K,\Sigma}$ is a union of smooth subvarieties of codimension equal to the parabolic rank $r(R)$ of R , intersecting transversally.

Suppose R has parabolic rank r . We define the nerve $\mathfrak{N}_\Sigma(R)$ of the closed covering of $\partial^R \mathbb{S}_{K,\Sigma}$ by irreducible components of codimension r , as in [34, §3.1] (where the R -stratum was denoted $Z_\Sigma(R)$). We recall the relation between the homotopy type of $\mathfrak{N}_\Sigma(R)$ and Borel-Serre compactifications in §8.2.1.

Let R be a standard rational proper parabolic subgroup of G , and define the maximal parabolic $P(R)$ as in §A.1; we suppose $P(R) = P = P_F$, as above, and define $F(R) = F$. We can write $R = \cap_{j=1}^{r(R)} Q_j$ uniquely as the intersection of standard rational maximal parabolics, with $Q_j < Q_{j'}$, in the total order defined above, if and only if $j < j'$. Thus $Q_1 = P(R)$; $P(R)$ is the smallest maximal parabolic containing R , in the order by size of G_h , but it has the largest G_ℓ . The cone C_F is homogeneous under the action of $G_{\ell,P(R)}(\mathbb{R}) = GL(m(R), \mathbb{C})^d$, and can be identified with the symmetric space attached to the Lie group $GL(m(R), \mathbb{C})^d$. The partial compactification \bar{C}_F is a subset of the topological closure of C_F in $U_F(\mathbb{R})$ that is set-theoretically the union of cones of the form $C_{F'}$, where F' runs over boundary components of X such that the stabilizer $P_{F'}$ contains $G_{h,R}$; alternatively, such that F is a boundary component of F' . Here it is necessary to allow $P_{F'}$ to be a non-standard maximal parabolic; the construction of the toroidal compactification involves canonical rational embeddings $U_{F'} \subset U_F$ for all such F' . Alternatively, we can take the union of $C_{F'}$ for which $P_{F'} > P(R)$, together with their translates under $G_{\ell,P(R)}(\mathbb{Q}) \simeq GL(m(R), F)$ (here F designates the CM field!).

Recall that Σ_F is a fan in \bar{C}_F , and in particular can be written as a disjoint union

$$(15) \quad \Sigma_F = \coprod_{F'} \Sigma_F(F')$$

where $\Sigma_F(F')$ is the set of polyhedral cones in Σ_F contained in $C_{F'}$. To fix ideas we assume for the moment that $\Gamma_{\ell,P} \subset G_{\ell,P}(\mathbb{Q})$ is a congruence subgroup and that Σ_F is invariant under $\Gamma_{\ell,P}$; this is one of the properties of Σ that is used to construct classical (non-adelic) toroidal compactifications in [2]. We assume $\Gamma_{\ell,P}$ is neat. Then the locally symmetric space $X(\Gamma_{\ell,P}) := \Gamma_{\ell,P} \backslash C_F$ has a Borel-Serre compactification $\bar{X}(\Gamma_{\ell,P})$ and the inclusion $X(\Gamma_{\ell,P}) \subset \bar{X}(\Gamma_{\ell,P})$ is a homotopy equivalence.

The relation between toroidal and Borel-Serre compactifications will be needed for the computations in §8.2; see §8.2.1.

1.4. Level subgroups. We will be working in the following situation. Start with K as above, and let $S = S(K)$ be its set of ramified primes, as in §1.1. We recall that S contains a prime \mathfrak{r} with a divisor \mathfrak{r}_0 and that $K_{\mathfrak{r}_0}$ is chosen to guarantee that K is neat. Choose a finite set Q of primes of F^+ that split in F , with the property that, if $v \in Q$ divides the rational prime q , then $q \notin S$ and $q \equiv 1 \pmod{p}$. Let $S(Q)$ be the set of rational primes divided by primes in Q ; we assume that each $q \in S(Q)$ is divided by a unique $v \in Q$. For each $v \in Q$ let $\mathfrak{m}_v \in \mathcal{O}_v$ be the

maximal ideal, $k(v) = \mathcal{O}_v/\mathfrak{m}_v$ the residue field, and let

$$K_{0,v} = \{k \in GL(n, \mathcal{O}_v) \mid k \equiv \begin{pmatrix} a(k) & b \\ 0 & d \end{pmatrix} \pmod{\mathfrak{m}_v}\}$$

with $a(k) \in k(v)^\times$, b a $1 \times (n-1)$ row matrix over $k(v)$, and $d \in GL(n-1, k(v))$. Let $K_{1,v} \subset K_{0,v}$ denote the kernel of the map $k \mapsto a(k)$, and let $K_{\Delta,v}$ be the smallest subgroup of $K_{0,v}$ containing $K_{1,v}$ such that $\Delta_v = K_{0,v}/K_{\Delta,v}^\Delta$ is a p -group. Let $\Delta_Q = \prod_{v \in Q} \Delta_v$. For $q \in S(Q)$ let $K_{i,q} = K_{i,v} \times [\prod_{v' \in D_+(q) \setminus v} K_{v'}] \times K_{-,q}$, for $i = 0, 1, \Delta$, and let

$$K_{i,Q} = \prod_{q \notin S(Q)} K_q \times \prod_{q \in S(Q)} K_{i,q}.$$

Let $w \notin S$ be a rational prime, fix $t \in G_w$, and let $K_w(t) = K_w \cap tK_w t^{-1}$. Suppose $w \notin S \amalg S(Q)$, and $t \in G_w$. Define the *modification* $K_{i,Q}(t)$ of $K_{i,Q}$, with $i = 0, 1, \Delta$, to be the product

$$K_{i,Q}(t) = \prod_{q \notin S(Q) \cup w} K_q \times \prod_{q \in S(Q)} K_{i,q} \times K_w(t).$$

If Q is empty, one writes $K(t) = K_{0,Q}(t) = K_{1,Q}(t)$.

The following theorem is proved in §A.6 of the first author's thesis [3].

Theorem 16. *Suppose $K = \prod_q K_q \subseteq G(\mathbf{A}_f)$ is a neat subgroup, such that K_q are maximal hyperspecial for almost all q . Suppose further that $K \supset K_p$, a fixed hyperspecial maximal $K_p \subseteq G(\mathbb{Q}_p)$. Fix a set Q of primes as above. Let \mathcal{B} be the set of modifications $K_{i,Q}(t)$ of $K_{i,Q}$, with $i = 0, 1, \Delta$. Then for every $K_{i,Q}(t)$ there is a toroidal datum $\Sigma(K_{i,Q}(t))$ such that*

- (1) *The toroidal compactification $S_{K_{i,Q}(t)}(G, X) \hookrightarrow S_{K_{i,Q}(t)}(G, X)_{\Sigma(K_{i,Q}(t))}$ is smooth and projective;*
- (2) *If $K' \subset K$ is an inclusion in \mathcal{B} then the natural covering map*

$$S_{K'}(G, X) \rightarrow S_K(G, X)$$

extends to a map

$$S_{K'}(G, X)_{\Sigma(K')} \rightarrow S_K(G, X)_{\Sigma(K)}$$

of toroidal compactifications.

- (3) *The map*

$$S_{K_{\Delta,Q}}(G, X)_{\Sigma_{\Delta,Q}} \rightarrow S_{K_{0,Q}}(G, X)_{\Sigma_{0,Q}}$$

is an étale covering with group Δ_Q .

- (4) *Conditions (1)-(3) hold when the $S_{K_{i,Q}}(G, X)$ are replaced by their integral models $\mathbb{S}_{K_{i,Q}}$.*

The first two assertions follow from the classical theory of toroidal compactifications and are used without comment in defining Hecke correspondences in §2.2. The third one requires more delicate analysis of the analytic construction in [2]. Atanasov uses the observation that the intersection of $U_F(\mathbb{Q}_q)$ with any compact open subgroup is contained in a pro- q group for each F and each unramified prime

$q \neq p$. Since Δ_Q is a p -group, it then suffices to show that $K_{\Delta,Q}$ and $K_{0,Q}$ restrict to the same subgroup of $U_F(\mathbb{Q})$, and this is a simple calculation. The last claim follows by purity of the branch locus applied to each irreducible boundary component, which reduces it to the characteristic zero statement in (3).

1.5. Automorphic vector bundles over toroidal compactifications. In the next few sections we will develop the theory of coherent cohomology and Hecke algebras without controlling ramification at the places in $S \setminus S(G)$, in the notation of §1.1, in order to avoid unnecessary complications. Sections 2.3 and 1.1.1 will indicate the adjustments that are needed in order to accommodate ramification.

Fix an automorphic vector bundle $\mathcal{F} = \mathcal{F}_\tau$ on $S_{K_p}(G, X)$, and identify \mathcal{F} with the family of \mathcal{F}_K , as in 1.1. For each open compact $K \supset K_p$, let Σ_K be a cone decomposition defining a toroidal compactification $S_K(G, X) \hookrightarrow S_K(G, X)_{\Sigma_K}$. Provided K is neat, each \mathcal{F}_K admits a *canonical extension* \mathcal{F}_K^{can} over $S_K(G, X)_{\Sigma_K}$, constructed in [25] following Mumford. The canonical extensions are functorial with respect to pullback and direct image: if $K' \subset K$ is an open subgroup, and if $\Sigma_{K'}$ is a cone decomposition for $S_{K'}(G, X)$ compatible with Σ_K , so that the natural finite covering map

$$\pi_{K,K'} : S_{K'}(G, X) \rightarrow S_K(G, X)$$

extends to a morphism of toroidal compactifications

$$\pi_{K,K'}^{tor} : S_{K'}(G, X)_{\Sigma_{K'}} \rightarrow S_K(G, X)_{\Sigma_K}$$

then there are canonical isomorphisms

$$\pi_{K,K'}^{tor,*}(\mathcal{F}_K^{can}) \xrightarrow{\sim} \mathcal{F}_{K'}^{can} \quad \text{and} \quad (\pi_{K,K'}^{tor})_*(\mathcal{F}_{K'}^{can}) \xrightarrow{\sim} \mathcal{F}_K^{can},$$

and moreover

$$R^i(\pi_{K,K'}^{tor})_*(\mathcal{F}_{K'}^{can}) = 0, \quad i > 0.$$

Note that this applies in particular when $K' = K$ but $\Sigma_{K'}$ is a refinement of Σ_K .

Similarly, suppose the complement of $S_K(G, X)$ in $S_K(G, X)_{\Sigma_K}$ is a divisor D_{Σ_K} with normal crossings. Let \mathcal{I}_{Σ_K} be the ideal sheaf of D_{Σ_K} – it is a line bundle on $S_K(G, X)_{\Sigma_K}$ – and let $\mathcal{F}_K^{sub} = \mathcal{F}_K^{can} \otimes \mathcal{I}_{\Sigma_K}$ be the *subcanonical extension* of \mathcal{F}_K . Then, under the hypotheses above, there is a canonical isomorphism

$$(\pi_{K,K'}^{tor})_*(\mathcal{F}_{K'}^{sub}) \xrightarrow{\sim} \mathcal{F}_K^{sub}, \quad R^i(\pi_{K,K'}^{tor})_*(\mathcal{F}_{K'}^{sub}) = 0, \quad i > 0.$$

Moreover,

$$(17) \quad \text{if } \pi_{K,K'}^{tor} \text{ is étale, then one also has } \pi_{K,K'}^{tor,*}(\mathcal{F}_K^{sub}) \xrightarrow{\sim} \mathcal{F}_{K'}^{sub}.$$

Except in a few low-dimensional cases, the action of $G(\mathbf{A}_f^p)$ only extends to the family $S_{K,\Sigma}(G, X)$ if the Σ are also allowed to vary along with K , and in particular the algebra \mathcal{H}_K of Hecke operators of level K do not act geometrically on \mathcal{F}_K^{can} . However, it is explained in [26] that \mathcal{H}_K does act canonically on the finite-dimensional vector space $H^i(S_{K,\Sigma}, \mathcal{F}_K^{can})$, for each i . We reformulate the result of [26], and the integral version proved in [46], in a version better adapted to localization of Hecke algebras at maximal ideals. Let $E(\mathcal{F})$ denote an extension of

the reflex field $E(G, X)$ over which the automorphic vector bundle \mathcal{F} has a rational model. Most of the following Proposition is a consequence of the properties of canonical and subcanonical extensions recalled above.

Proposition 18. (i) *Let (G, X) , K , and Σ_K be as above. Let Σ'_K be a refinement of Σ_K . Then the morphism*

$$\pi_{\Sigma, \Sigma'} : S_K(G, X)_{\Sigma'_K} \rightarrow S_K(G, X)_{\Sigma_K}$$

induces (by pushforward) a canonical quasiisomorphism

$$R\pi_{\Sigma, \Sigma', *}: R\Gamma(S_K(G, X)_{\Sigma'_K}, \mathcal{F}_K^{\text{can}}) \simeq R\Gamma(S_K(G, X)_{\Sigma_K}, \mathcal{F}_K^{\text{can}})$$

in the bounded derived category $D^b(E(\mathcal{F}))$ of complexes of $E(\mathcal{F})$ -vector spaces. (Here the superscript $^{\text{can}}$ is used to designate the canonical extension for any toroidal compactification.) Similarly, $\pi_{\Sigma, \Sigma'}$ induces a canonical quasiisomorphism

$$R\pi_{\Sigma, \Sigma', *}: R\Gamma(S_K(G, X)_{\Sigma'_K}, \mathcal{F}_K^{\text{sub}}) \simeq R\Gamma(S_K(G, X)_{\Sigma_K}, \mathcal{F}_K^{\text{sub}})$$

In particular, the objects

$$R\Gamma(S_K(G, X)_{\Sigma_K}, \mathcal{F}_K^{\text{can}}), \quad R\Gamma(S_K(G, X)_{\Sigma_K}, \mathcal{F}_K^{\text{sub}})$$

of $D^b(E(\mathcal{F}))$ are well-defined and independent of the choice of Σ_K ; we denote them $R\Gamma^{\text{can}}(S_K(G, X), \mathcal{F}_K)$ and $R\Gamma^{\text{sub}}(S_K(G, X), \mathcal{F}_K)$, respectively.

(ii) *Let $t \in G(\mathbf{A}_f)$. For any K as above, and for any \mathcal{F} , the natural isomorphism*

$$*t^{-1} : S_K(G, X) \rightarrow S_{tKt^{-1}}(G, X)$$

defines an isomorphism in $D^b(E(\mathcal{F}))$ by pullback:

$$*t^{-1} : R\Gamma^{\text{can}}(S_{tKt^{-1}}(G, X), \mathcal{F}_{t^{-1}Kt}) \simeq R\Gamma^{\text{can}}(S_K(G, X), \mathcal{F}_K).$$

There are analogous isomorphisms when $^{\text{can}}$ is replaced by $^{\text{sub}}$.

(iii) *Let $K' \subset K$ be an open subgroup. The pullback and direct image functors define canonical morphisms in $D^b(E(\mathcal{F}))$:*

$$\pi_{K, K'}^* R\Gamma^{\text{can}}(S_K(G, X), \mathcal{F}_K) \rightarrow R\Gamma^{\text{can}}(S_{K'}(G, X), \mathcal{F}_{K'});$$

$$\pi_{K, K'}^* R\Gamma^{\text{sub}}(S_K(G, X), \mathcal{F}_K) \rightarrow R\Gamma^{\text{sub}}(S_{K'}(G, X), \mathcal{F}_{K'});$$

$$\pi_{K, K', *}: R\Gamma^{\text{can}}(S_{K'}(G, X), \mathcal{F}_{K'}) \rightarrow R\Gamma^{\text{can}}(S_K(G, X), \mathcal{F}_K)$$

$$\pi_{K, K', *}: R\Gamma^{\text{sub}}(S_{K'}(G, X), \mathcal{F}_{K'}) \rightarrow R\Gamma^{\text{sub}}(S_K(G, X), \mathcal{F}_K)$$

(iv) *Let $\mathcal{O} \subset E(G, X)$ denote the localization at a prime above p of the integers of $E(G, X)$. With K and K' as above, both assumed unramified at p , suppose $S_K(G, X)$, \mathcal{F}_K , $S_{K'}(G, X)$, $\mathcal{F}_{K'}$, and their toroidal compactifications all admit compatible integral models over \mathcal{O} , in the sense of Proposition 13. Denote these models \mathbb{S}_K , \mathbb{S}_{K, Σ_K} , \mathcal{F}_K (no change), etc. Then the object $R\Gamma(\mathbb{S}_{K, \Sigma_K}, \mathcal{F}_K^{\text{can}})$ of $D^b(\mathcal{O})$ is a well-defined perfect complex of \mathcal{O} -modules and independent of the choice of Σ_K ; we denote it $RH(\mathcal{F}_K)$, or when necessary, $RH(\mathcal{F}_K^{\text{can}})$. The conclusions of (iii) hold for*

the corresponding elements of $D^b(\mathcal{O})$. If $t \in G(\mathbf{A}_f^p)$, then the conclusions of (ii) holds for the corresponding elements of $D^b(\mathcal{O})$.

The analogous statements hold when can is replaced by sub .

Proof. Items (i)-(iii) are proved in [26]; item (iv) is due to Lan and is proved in Proposition 1.4.3 of [46]. \square

2. HECKE ALGEBRAS

2.1. Adelic representations and fixed vectors. Let $K = K_p \times K^p$, where K_p is a fixed hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$ as before and K^p is an open compact subgroup of $G(\mathbf{A}_f^p)$. We define

$$RH(\mathcal{F}_{K_p}) = RH(\mathcal{F}_{K_p}^{can}) = \varinjlim_{K^p} RH(\mathcal{F}_{K_p \times K^p}^{can})$$

where the limit is taken as K^p shrinks to the identity. This is a well-defined object of the bounded derived category of \mathcal{O} -modules, and since the colimit is exact on this category we have

$$(19) \quad H^i(RH(\mathcal{F}_{K_p}^{can})) \xrightarrow{\sim} \varinjlim_{K^p} H^i(RH(\mathcal{F}_{K_p \times K^p}^{can}))$$

for all $i \in \mathbb{Z}$.

We define $RH(\mathcal{F}_{K_p}^{sub})$ similarly and note that the analogue of (19) is valid with can replaced by sub .

Proposition 18 (ii) and (iv) have the following Corollary:

Corollary 20. *The objects $RH(\mathcal{F}_{K_p}^{can})$, $RH(\mathcal{F}_{K_p}^{sub})$ have canonical compatible actions of $G(\mathbf{A}_f^p)$. Moreover, the actions are smooth in the sense that any element of $H^i(RH(\mathcal{F}_{K_p}^?))$, where $? = can, sub$ and $i \in \mathbb{Z}$, is fixed by an open compact subgroup of $G(\mathbf{A}_f)$.*

The last sentence follows from (19). However, it is important to note that, for a given open $K \supset K_p$, it is not obviously the case that the canonical map

$$(21) \quad H^i(\mathbb{S}_{K, \Sigma_K}, \mathcal{F}_K^?) \rightarrow H^i(RH(\mathcal{F}_{K_p}^?))^K$$

is an isomorphism, or even necessarily injective. Instead there is a Hochschild-Serre spectral sequence

$$(22) \quad E_2^{r,s} = H^r(K, R^s H(\mathcal{F}_{K_p}^?)) \Rightarrow H^{r+s}(\mathbb{S}_{K, \Sigma_K}, \mathcal{F}_K^?),$$

again with $? = can, sub$. However, we have the following result:

Proposition 23. [7] *With notation as above, $H^r(K, R^s H(\mathcal{F}_{K_p}^?)) = 0$ for all $r > 0$, $? = can, sub$. Thus the canonical map*

$$RH(\mathcal{F}_K^?) \rightarrow RH\text{om}_{K^p}(1, RH(\mathcal{F}_{K_p}^?)),$$

where 1 denotes \mathcal{O} with the trivial action of $K_{\mathbb{Q}}^{\circ}$, is a quasi-isomorphism for $? = can, sub$.

The Proposition is proved in [7] by verifying that the stabilizers in K^p of points in the toroidal boundary have order prime to p .

Let $S = S(K)$ be the set of primes where K does not contain a hyperspecial maximal compact subgroup, plus the prime p , and let $G(\mathbf{A}_f^S)$ be the set of adèles of G with entry 1 at primes in S ; let $K_S = K \cap G(\mathbf{A}_f^S)$. Let \mathcal{O} be as above and define the Hecke algebra $\mathcal{H}_{K,\mathcal{O}}$ to be the \mathcal{O} -algebra of compactly-supported K_S -biinvariant functions on $G(\mathbf{A}_f^S)$, with multiplication given by convolution. Since K_S contains a hyperspecial maximal compact subgroup at each prime, $\mathcal{H}_{K,\mathcal{O}}$ is a commutative algebra. Corollary 20 and Proposition 23 thus imply that $\mathcal{H}_{K,\mathcal{O}}$ acts on $RH(\mathcal{F}_K^?)$ for $? = can, sub$. We let $T_{S,K} = T_{S,K,\mathcal{F}}$ denote the \mathcal{O} -subalgebra of $\text{End}(RH(\mathcal{F}_K))$ generated by this action.

Since [7] is not yet available, we give an alternative construction of $T_{S,K}$ in the next section.

2.2. Hecke operators. Notation is as in Proposition 18. Fix K hyperspecial at p and let $S = S(K)$ be as above. For $t \in G(\mathbf{A}_f^S)$, a Hecke operator $T(t) \in \text{End}(RH(\mathcal{F}_K^?))$, denoted T_t^Σ , is defined in [18, §8.1.5], for $? = can, sub$. An alternative definition is given in [8, §4.2], where it is also proved that the subalgebra $T_{S,K}$ of $\text{End}(RH(\mathcal{F}_K^?))$ generated by the $T(t)$ is commutative. The algebra $T_{S,K}$ naturally maps to $\text{End}(\oplus_i H^i(RH(\mathcal{F}_K)))$ but the map to its image is not necessarily an isomorphism. In particular, there is no reason to assume $T_{S,K}$ to be reduced.

In the setting of (iv) of Proposition 18, the natural inclusion $\mathcal{F}_K^{sub} \rightarrow \mathcal{F}_K^{can}$ (the cone decomposition is implicit and omitted from the notation) determines a morphism in $D^b(\mathcal{O})$

$$(24) \quad R\Gamma^{sub}(\mathbb{S}_K, \mathcal{F}_K) \rightarrow R\Gamma^{can}(\mathbb{S}_K, \mathcal{F}_K)$$

Let $R\Gamma^\partial(\mathbb{S}_K, \mathcal{F}_K)$ denote the cone on the morphism (24).

The following theorem summarizes the state of the art:

Theorem 25. (i) *Let κ be an algebraically closed field and let $\nu : T_{S,K,\mathcal{F}} \rightarrow \kappa$ be a continuous homomorphism. Then there is a semisimple n -dimensional representation $\rho_\nu : \Gamma_F \rightarrow GL(n, \kappa)$ that is characterized, up to equivalence, by the following property: If v is a prime of F not in $S \cup \{p\}$, then ρ_ν is unramified at v . Let $\Gamma_v \subset \Gamma_F$ be a decomposition group at v ; then the semisimplification of the restriction $\rho_{\nu,v}$ of ρ_ν to Γ_v corresponds to the restriction to ν to the image of the Hecke operators at v by the unramified Langlands correspondence, in the following sense: there is an equality of polynomials in $\kappa[X]$*

$$(26) \quad \det(1 - \rho_\nu(\text{Frob}_v)X) = 1 + \sum_{i=1}^n (-1)^i \nu(q^{\frac{(n+1)i}{2}} T_{i,v}) X^i$$

where the $T_{i,v}$ are the standard Hecke operators at v , normalized as in [57].

(ii) *Suppose κ is of characteristic zero and ν is the homomorphism attached to a cuspidal automorphic representation of G . Then there is a homomorphism*

$$r_\nu : \Gamma_{F+} \rightarrow \mathcal{G}_n(\kappa)$$

where \mathcal{G}_n is the disconnected algebraic group defined in ([11], §2.1) that corresponds to the pair (ρ_ν, β) , for some character β of Γ_F , under the correspondence of Lemma 2.1.1 of [11]. In particular, the identity component \mathcal{G}_n^0 is isomorphic to $GL(n) \times GL(1)$, and the restriction of r_ν to $\Gamma_F \subset \Gamma_{F+}$ is given by (ρ_ν, β) .

(iii) Write $R\Gamma^\partial(\mathcal{F}_K) := R\Gamma^\partial(\mathbb{S}_K(G, X), \mathcal{F}_K)$. Suppose ν occurs in the representation on some subquotient of $R\Gamma^\partial(\mathcal{F}_K)$; we say that ν occurs in the support of $R\Gamma^\partial$ (for some \mathcal{F}_K). Then ρ_ν is reducible.

Part (ii) is the familiar association of Galois representations to polarized cohomological cuspidal automorphic representations of $GL(n)$. Part (i) includes the results of [30] and the extension to torsion cohomology by Scholze, Boxer, Pilloni-Stroh, and Goldring-Koskivirta [57, 6, 54, 18].

It remains to prove Part (iii), which follows from part (i) and an analysis of the non-cuspidal coherent cohomology of $\mathbb{S}_K(G, X)$, based primarily on the considerations of [32], [50], and [45]. The details are postponed until section 7; the proof will be particularly long, because it requires a review of the structure of the toroidal boundary.

Let ν be as in Theorem 25, with k a field of characteristic p . Let $\mathfrak{m} = \ker \nu$; we write $\mathfrak{m} = \mathfrak{m}_\nu$ or $\mathfrak{m} = \mathfrak{m}_\rho$ with $\rho = \rho_\nu$. Let \mathbb{T}_ν be the localization of $T_{S,K}$ at \mathfrak{m}_ν , and let

$$RH(\mathcal{F}_K)_\nu = RH(\mathcal{F}_K) \otimes_{T_{S,K}} \mathbb{T}_\nu$$

denote the localization of $RH(\mathcal{F}_K)$ at \mathfrak{m}_ν . Recall that (τ, W_τ) is the representation of the parabolic P_x corresponding to a point $x \in \check{X}$. Let $\mu_\tau \in X_x^+(T)$ denote the highest weight of τ , and assume

$$(27) \quad \mu_\tau = w \cdot \mu, w \in W^x, \mu \in X^+(T).$$

Proposition 28. *We assume \mathcal{F}_K is the automorphic vector bundle attached to a character $w \cdot \mu$, as in (27). Suppose (a) ρ_ν is irreducible; (b) the highest weight $\mu_\mathcal{F}$ of \mathcal{F} satisfies the Lan-Suh vanishing conditions:*

- (i) *The character μ of (27) is p -small (Definition 10);*
- (ii) *μ is sufficiently regular in the sense of [49, 7.18]*
- (iii) *μ satisfies the inequality [49, 7.22].*

Then the cohomology of $RH(\mathcal{F}_K)_\nu$ is concentrated in a single degree $i_0 = i_0(\mathcal{F})$ and is torsion-free over \mathcal{O} .

Proof. Write $\mathbb{S} = \mathbb{S}_K(G, X)$. For $q \geq 0$, define the interior cohomology of \mathcal{F}_K to be

$$H_!^q(R\Gamma(\mathbb{S}, \mathcal{F}_K)) := \text{Im}[H^q(\mathbb{S}, \mathcal{F}_K^{\text{sub}}) \rightarrow H^q(\mathbb{S}, \mathcal{F}_K^{\text{can}})].$$

It follows from hypothesis (a) and (iii) of Theorem 25 that the localization at \mathfrak{m}_ν of $R\Gamma^\partial(\mathbb{S}, \mathcal{F}_K)$ is acyclic. Thus for any q ,

$$H^q(\mathbb{S}, \mathcal{F}_K^{\text{can}})_\nu = H_!^q(R\Gamma(\mathbb{S}, \mathcal{F}_K))_\nu;$$

in other words, the localization at \mathfrak{m}_ν of the cohomology of the canonical extension coincides with the localization at \mathfrak{m}_ν of the interior cohomology. In view of hypothesis (b), it follows from Corollary 7.25 and Theorem 8.2 (3) of [50] that

$H_!^q(R\Gamma(\mathbb{S}, \mathcal{F}_K))_\nu$ is concentrated in a single degree $i_0(\mathcal{F})$ and is torsion-free over \mathcal{O} . The Proposition is an immediate consequence. \square

The Lan-Suh vanishing conditions are written out more explicitly in [28, §6.10]. We write $H_!^q(\mathbb{S}, \mathcal{F}_K)$ instead of $H_!^q(R\Gamma(\mathbb{S}, \mathcal{F}_K))$.

2.3. Ramification and types. The modules $H_!^q(R\Gamma(\mathbb{S}, \mathcal{F}_K))$ of coherent cohomology give rise to Galois representations that may be ramified at places v where K_v is not hyperspecial maximal. In order to restrict attention to minimal deformation problems, we introduce spaces of cohomology with coefficients in *semisimple Bushnell-Kutzko types* [BK99]. This was explained in the unpublished manuscript [31], in the case of supercuspidal types, but seems have never been used subsequently, although the definition we give below of automorphic forms with coefficients in Bushnell-Kutzko types is essentially the one that appeared in [31] as well as [65, 38]; for automorphic forms on $GL(2)$ it has been standard for some time. A version based on finite-dimensional representations of the multiplicative groups of local division algebras is developed in [11, p. 97]; this is essentially equivalent for purposes of proving modularity but it cannot be applied to prove that the coherent cohomology is free over the localized Hecke algebra, which is our main theorem.

Let $S = S(K)$ be as above, and assume the local groups K_v and $K_{-,q}$ are as in §1.1. Let $(K_v^+, K_v, \mathfrak{I}_{\mathcal{O}_v}, \Lambda_{\mathcal{O}_v})$ be type data as in §1.1.1.

Definition 29. The pair $(K_v^+, \mathfrak{I}_{\mathcal{O}_v})$ is a semisimple type if it determines a component of the Bernstein center of $GL(n, F_v^+)$ in the following sense:

- (a) For any irreducible representation π of $GL(n, F_v^+)$,

$$\dim \operatorname{Hom}_{K_v^+}(\mathfrak{I}_{\mathcal{O}_v}, \pi) \leq 1;$$

- (b) If π is an irreducible representation of $GL(n, F_v^+)$ then

$$\operatorname{Hom}_{K_v^+}(\mathfrak{I}_{\mathcal{O}_v}, \pi) \neq 0$$

if and only if there is a parabolic subgroup $P \subset GL(n, F_v^+)$ with Levi quotient L , a (fixed) supercuspidal representation σ of L , and an unramified character α of L , such that π is an irreducible constituent of the induced representation $\operatorname{Ind}_P^{GL(n, F_v^+)}[\sigma \otimes \alpha]$.

Let $\mathfrak{I}_{\mathcal{O}} = (\mathfrak{I}_{\mathcal{O}_v})_{v \in S}$, where for all $v \in S$ (we are now using S to denote a set of primes of F^+ , as in §1.1) we assume

Hypotheses 30. (1) Either $\mathfrak{I}_{\mathcal{O}_v}$ is the trivial representation of the special maximal compact subgroup K_v , in which case we set $K_v^+ = K_v$, or the pair $(K_v^+, \mathfrak{I}_{\mathcal{O}_v})$ is a semisimple type for the irreducible representation π of $GL(n, F_v^+)$. Let $S^+ \subset S$ be the subset for which the second condition holds, and let $K_S^+ = \prod_{v \in S} K_v^+$.

(2) For all $v \in S^+$, p is a banal prime for $GL(n, F_v^+)$: that is, p does not divide the pro-order of any compact open subgroup of $GL(n, F_v^+)$.

We let $\Lambda_{\mathcal{O}}$ denote the representation $\otimes_v \Lambda_{\mathcal{O}_v}$ of $\prod_{v \in S} K_v^+$. Hypothesis 30 (2) implies that the module $\Lambda_{\mathcal{O}}$ is the unique $\prod_{v \in S} K_v^+$ -invariant lattice in $\Lambda_{\mathcal{O}_v} \otimes \mathbb{Q}$,

up to scaling. With \mathcal{F}_K an automorphic vector bundle as above, we let

$$\mathcal{F}_{K,\mathbf{IO}} = \mathcal{F}_K \otimes_{\mathcal{O}} \Lambda_{\mathbf{IO}}$$

and for any cohomology module \mathbf{H} with coefficients in $\mathcal{F}_{K,\mathbf{IO}}$ we define

$$(31) \quad \mathbf{H}_{\mathbf{IO}} := \mathrm{Hom}_{K_S^+}(\Lambda_{\mathbf{IO}}, \mathbf{H}).$$

Thus we can define $H_!^q(R\Gamma(\mathbb{S}, \mathcal{F}_{K,\mathbf{IO}}))$, $H^q(\mathbb{S}, \mathcal{F}_{K,\mathbf{IO}}^{\mathrm{can}})_{\nu,\mathbf{IO}}$, etc.

Theorem 25 (iv) *The assertions of Theorem 25 (iii) hold when \mathcal{F}_K is replaced by $\mathcal{F}_{K,\mathbf{IO}}$.*

2.4. Minimality condition. For applications to the Taylor-Wiles method, we add the following hypotheses on the Galois representations attached to the type data. For each v , we choose a π in the inertial equivalence class corresponding to the quadruple $(K_v^+, K_v, \mathbf{IO}_v, \Lambda_{\mathbf{IO}_v})$ – in other words, $\mathrm{Hom}_{K_v^+}(\mathbf{IO}_v, \pi) \neq 0$. Let $\mathcal{L}_v(\pi)$ denote the n -dimensional representation of the Weil-Deligne group W_v of F_v^+ attached to π by the local Langlands correspondence, with coefficients in the p -adic integer ring \mathcal{O} , and let $\bar{\mathcal{L}}_v(\pi)$ denote its reduction modulo the maximal ideal $\mathfrak{m}_{\mathcal{O}}$. We attach to $\mathcal{L}_v(\pi)$ a deformation problem \mathcal{D}_v over \mathcal{O} , in the sense of [11, Definition 2.2.2], by the condition that, $\mathcal{L}_v(\pi')$ is a deformation of $\mathcal{L}_v(\pi)$ of type \mathcal{D}_v if and only if $\mathrm{Hom}_{K_v^+}(\mathbf{IO}_v, \pi') \neq 0$. To \mathcal{D}_v is attached a subspace $L_v \subset H^1(W_v, \mathrm{ad}(\bar{\mathcal{L}}_v(\pi)))$, as in [11, §2.2], and we assume that the problem \mathcal{D}_v is minimal in the sense that

$$(32) \quad \dim L_v = \dim H^0(W_v, \mathrm{ad}(\bar{\mathcal{L}}_v(\pi))).$$

Since we have already assumed in Hypothesis 30 (2) that p is a banal prime for the local group, some of these hypotheses may be redundant.

3. PERFECT COMPLEXES AND DIAMOND OPERATORS

3.1. Review of the theorem of Nakajima. We begin with a simple generalization of a special case of the theorem of Nakajima [51, Theorem 2] used in [28].

Theorem 33. *Let Γ be a finite abelian p -group. Let k be a field of characteristic p and let $f : X \rightarrow Y$ a finite étale Galois covering of projective varieties over k with Galois group Γ . Let \mathcal{F} be a coherent sheaf on Y . Let T be a commutative k -algebra of endomorphisms of the cohomology complex $R\Gamma(X, f^*(\mathcal{F}))$ in the derived category of $k[\Gamma]$ -modules, and let $\mathfrak{m} \subset T$ be a maximal ideal with the property that $H^i(R\Gamma(X, f^*(\mathcal{F})))_{\mathfrak{m}} = 0$ for all indices except $i = i_0$, where the subscript \mathfrak{m} denotes localization at \mathfrak{m} . Then $H^{i_0}(X, f^*(\mathcal{F}))_{\mathfrak{m}}$ is a free $k[\Gamma]$ -module.*

Proof. Theorem 1 of [51] implies that $R\Gamma(X, f^*(\mathcal{F}))$ can be represented by a finite complex of projective $k[\Gamma]$ -modules C^\bullet . The localization $C_{\mathfrak{m}}^\bullet$ is a direct summand of C^\bullet and is therefore also a finite complex of projective $k[\Gamma]$ modules. It then follows, as in the proof of Theorem 2 of [51], that $H^{i_0}(X, f^*(\mathcal{F}))_{\mathfrak{m}}$ is a projective $k[\Gamma]$ -module, hence free because $k[\Gamma]$ is a local ring. \square

The following corollary is then proved just as in [28, Corollary 3.5]:

Corollary 34. *Let \mathcal{O} be a p -adic integer ring with residue field k , and let $f : X \rightarrow Y$ be a finite étale Galois covering of projective \mathcal{O} -schemes with Galois group Γ , a finite abelian p -group. Let T be a commutative \mathcal{O} -algebra of endomorphisms of the cohomology complex $R\Gamma(X, f^*(\mathcal{F}))$ in the derived category of $\mathcal{O}[\Gamma]$ -modules, and let $\mathfrak{m} \subset T$ be a maximal ideal with the property that*

$$H^i(R\Gamma(X, f^*(\mathcal{F}))_{\mathfrak{m}}) = 0$$

for all indices except $i = i_0$, where the subscript \mathfrak{m} denotes localization at \mathfrak{m} . Assume moreover that $H^{i_0}(X, f^(\mathcal{F}))_{\mathfrak{m}}$ is \mathcal{O} -torsion-free. Then $H^{i_0}(X, f^*(\mathcal{F}))_{\mathfrak{m}}$ is a free $\mathcal{O}[\Gamma]$ -module.*

3.2. . We apply this to the situation of Proposition 28. Fix a neat level subgroup K and let Q be a collection of primes of F^+ , split in F/F^+ , at which K is hyperspecial maximal. Choose toroidal data $\Sigma_{i,Q}$ as in Theorem 16, and write

$$\mathbb{S}_{i,Q,\Sigma} = \mathbb{S}_{K_{i,Q}(t)}(G, X)_{\Sigma_{i,Q}(t)}, \quad i = 0, 1, \Delta.$$

For $\mu \in X^+(T)$ and $w \in W^x$ we write $\mathcal{E}_w(\mu)$ for the automorphic vector bundle on $\mathbb{S}_{i,Q,\Sigma}$ (any i) attached to the representation of K_x with highest weight $w \cdot \mu$ – this is the bundle that was denoted $\mathcal{E}_w(W)$ in [28], where W is the representation of G with highest weight μ . Let $\mathcal{E}_w(\mu)^{\text{can}}$ denote its canonical extension on $\mathbb{S}_{i,Q,\Sigma}$. Let $i_0(w, \mu) = i_0(\mathcal{E}_w(\mu))$, in the notation of Proposition 28. We let $T_{S,K}(w, \mu)$ be the algebra denoted $T_{S,K,\mathcal{F}}$ in §2, for the automorphic bundle $\mathcal{F}_K = \mathcal{E}_w(\mu)$.

The following Corollary extends [28, Proposition 3.7] to the noncompact case.

Corollary 35. *Let $\mu \in X^+(T)$ be a character and let $\mathfrak{m} \subset T_{S,K}(w, \mu)$ be a maximal ideal that satisfy the hypotheses of Proposition 28. Then for all $w \in W^x$,*

$$H^{i_0(w,\mu)}(\mathbb{S}_{\Delta,Q,\Sigma}, \mathcal{E}_w(\mu)^{\text{can}})_{\mathfrak{m}}$$

is a free $\mathcal{O}[\Delta_Q]$ -module and

$$H^i(\mathbb{S}_{\Delta,Q,\Sigma}, \mathcal{E}_w(\mu)^{\text{can}})_{\mathfrak{m}} = 0 \quad \text{if } i \neq i_0(w, \mu).$$

Proof. In view of Theorem 16, we can apply the same proof as in [28, Proposition 3.7]. \square

More generally, we introduce a set of type data as in §1.1.1. We let $T_{S,K}(w, \mu, \text{IO})$ be the analogue of $T_{S,K}(w, \mu)$ for the automorphic bundle $\mathcal{F}_{K,\text{IO}} = \mathcal{E}_w(\mu) \otimes_{\mathcal{O}} \Lambda_{\text{IO}}$. Then we have

Corollary 36. *Let $\mu \in X^+(T)$ be a character and let $\mathfrak{m} \subset T_{S,K}(w, \mu)$ be a maximal ideal that satisfy the hypotheses of Proposition 28. Then for all $w \in W^x$,*

$$H^{i_0(w,\mu)}(S_{\Delta}(Q), \mathcal{E}_w(\mu)^{\text{can}} \otimes_{\mathcal{O}} \Lambda_{\text{IO}})_{\mathfrak{m}}$$

is a free $\mathcal{O}[\Delta_Q]$ -module and

$$H^i(S_{\Delta}(Q), \mathcal{E}_w(\mu)^{\text{can}} \otimes_{\mathcal{O}} \Lambda_{\text{IO}})_{\mathfrak{m},\text{IO}} = 0 \quad \text{if } i \neq i_0(w, \mu).$$

The proof is the same as in the case without type data; we only need to add that the functor $\mathrm{Hom}_{K_S^+}(\Lambda_{\mathrm{IO}}, \bullet)$ is exact because we have assumed that p is a banal prime for all places in S^+ .

In the applications we impose the conditions of §2.4 on our choice of type data to ensure that the local deformation conditions are minimal.

4. APPLICATION OF THE TAYLOR-WILES METHOD

We follow the axiomatic treatment of the Taylor-Wiles method described in §4 of [28]. Fix μ and $w \in W^x$ as in Corollary 35, and let $i_0 = i_0(w, \mu)$ for the duration of this section. Fix a neat level subgroup K , define $S = S(K)$ as above, and consider sets Q and a maximal ideal \mathfrak{m} as in §3.2.

Let

$$H_\emptyset = H^{i_0}(\mathbb{S}_{K, \Sigma}, \mathcal{E}_w(\mu)^{\mathrm{can}}), H_{0, Q} = H^{i_0}(\mathbb{S}_{0, Q, \Sigma}, \mathcal{E}_w(\mu)^{\mathrm{can}}),$$

$$H_{\Delta, Q} = H^{i_0}(\mathbb{S}_{\Delta, Q, \Sigma}, \mathcal{E}_w(\mu)^{\mathrm{can}}).$$

Note that by Proposition 28, these are torsion-free \mathcal{O} -modules. The Hecke algebras $\mathbb{T}_{\emptyset, w}, \mathbb{T}_{0, Q, w}, \mathbb{T}_{\Delta, Q, w}$ are defined as in [28, p. 137] with respect to these modules as the images of the relevant Hecke algebras $\mathbb{T}_{S, K}(w, \mu)$, initially defined as endomorphisms of the perfect complex, on the cohomology modules. Given a homomorphism

$$\nu : \mathbb{T}_{\emptyset, w} \rightarrow \bar{\mathbb{Q}}_p$$

we let ρ_ν be the corresponding Galois representation, and let $R_{\bar{\rho}_\nu, \emptyset}$ denote the corresponding deformation ring when (as we assume below) $\bar{\rho}_\nu$ is absolutely irreducible.

We have the following analogue of Theorem 6.8 of [28]:

Theorem 37. *Assume μ satisfies the inequalities of Proposition 28. Let $\nu : \mathbb{T}_{\emptyset, w}(\mu) \rightarrow \bar{\mathbb{Q}}_p$ be a non-trivial homomorphism and assume the corresponding Galois representation ρ_ν satisfies conditions*

- (1) *For all $v \in S(F^+)$, $\bar{\rho}_\nu$ is unramified at v and*

$$H^0(\mathrm{Gal}(\bar{F}_v/F_v), (\mathrm{ad} \bar{\rho}_\nu)(1)) = (0);$$

- (2) *The Fontaine-Laffaille condition at primes above p .*

We also assume the residual representation $\bar{\rho}_\nu$ to be absolutely irreducible and adequate in the sense of [62], which requires the absolute irreducibility. Let $\mathfrak{m} = \mathfrak{m}(\bar{\rho}_\nu) \subset \mathbb{T}_{\emptyset, w}(\mu)$ be the corresponding maximal ideal, and let $H_{\bar{\rho}_\nu, \emptyset}$ and $\mathbb{T}_{\bar{\rho}_\nu, \emptyset}$ denote the localizations at \mathfrak{m} . Suppose that the deformation conditions are minimally ramified at all primes $v \in S$ not dividing p . In particular, for $v \in S$, we assume the deformation condition in S at v is unrestricted and minimal (cf. [28, §6.9]).

Then the classifying map

$$\phi_{\bar{\rho}_\nu, \emptyset} : R_{\bar{\rho}_\nu, \emptyset} \rightarrow \mathbb{T}_{\bar{\rho}_\nu, \emptyset}$$

is an isomorphism, and $H_{\bar{\rho}_\nu, \emptyset}$ is a free module over $\mathbb{T}_{\bar{\rho}_\nu, \emptyset}$, which is a local complete intersection.

More generally, suppose we replace \mathcal{F}_K by $\mathcal{F}_{K,\text{IO}}$, for type data as in §1.1.1. Let $\mathbb{T}_{\emptyset,w,\text{IO}}(\mu)$ be the corresponding Hecke algebra, let $\nu : \mathbb{T}_{\emptyset,w,\text{IO}}(\mu) \rightarrow \bar{\mathbb{Q}}_p$ be a homomorphism satisfying the conditions above, plus the minimality condition of (32) at places in S . Define $\mathbb{T}_{\bar{\rho}_\nu,\emptyset}$ and $R_{\bar{\rho}_\nu,\emptyset}$ with respect to this homomorphism ν . Then the conclusions of the theorem stated above remain true.

Remark 38. The reader can check that, since p is odd, the condition on $v \in S(F^+)$ is not strictly necessary; this is the point of [11, Lemma 2.3.3], whose application is buried at the end of the proof of Lemma 2.3.4 of the same paper. In order to avoid distraction we add this as an extra condition.

Proof. We show that the relevant hypotheses in §4 of [28] are satisfied with the necessary adjustment to account for the assumption that our residual representations are adequate. The treatment of [28] follows the Diamond-Fujiwara extension to the Taylor-Wiles method as applied in [11], while we use a modification as in Theorem 3.6.1 in [4].

The Galois Hypothesis 4.3 follows as in the proofs of Corollaries 5.3 and 5.4 in [28] in light of our Theorem 25 (ii). Hypothesis 4.4.8 is verified in Lemma 6.1 therein. It remains to check the correct analog of Hypothesis 4.4.9 for adequate residual representations.

The conditions (4.4.9.1) and (4.4.9.2) in Part (a) of Hypothesis 4.4.9 follow from the constructions of sets of Taylor-Wiles Q_M satisfying the conditions (6.6.1) and (6.6.2) from the beginning of §6.6 in [28]; such a set Q_M is shown to exist in Proposition 4.4, [62]. The hypotheses of § 4.4.5 and § 4.4.7 as well as Hypothesis 4.4.9 (b) are verified for the set Q_M in the proof of Theorem 6.8 in [62]. In the same proof one can find the verification of the local condition in § 4.4.4. as a consequence of Proposition 5.12 in [62]. The modified version of Condition 4.4.9.3, now concerning nonvanishing of trace of Frobenius along projection, is embedded in the construction of Q_M in Proposition 4.4. The claim now follows since under our assumptions (i), (ii) and (iii) in the proof of Theorem 6.8 [62] hold.

Finally, the modifications necessary for incorporating the type data are the same as in the proof of [11, Theorem 3.5.1], which specifically comes down to the use of the minimality hypothesis in [11, Proposition 2.5.9]. \square

5. CONSEQUENCES FOR COHOMOLOGY MODULES OVER THE HECKE ALGEBRA

5.1. Topological and de Rham cohomology. Theorem 37 asserts that the coherent cohomology module $H^{i_0}(\mathbb{S}_{K,\Sigma}, \mathcal{E}_w(\mu)^{\text{can}})$ localized at the maximal ideal $\mathfrak{m} = \mathfrak{m}(\bar{\rho}_\nu)$ is free over the localized Hecke algebra $\mathbb{T}_{\bar{\rho}_\nu,\emptyset}$, provided p and the weight μ satisfy the stated inequalities. As in [28, §7], this implies that analogous localized modules in p -adic étale and de Rham cohomology are also free over $\mathbb{T}_{\bar{\rho}_\nu,\emptyset}$. The proofs are identical, so we merely state the results.

Let now $W = W_\mu$ be the finite-dimensional irreducible representation of G with highest weight μ . As in [28, (7.1.2)] we always assume μ to be p -small. Let the p -adic place v , the p -adic integer ring \mathcal{O} , the hyperspecial maximal compact subgroup $K_p \subset G(\mathbb{Q}_p)$, and the K_p -invariant lattice $W_\mu(\mathcal{O}) \subset W(\text{Frac}(\mathcal{O}))$ be

as in [28, §7.1]. Let $\tilde{W}_{\mu,B}(\mathcal{O})$ denote the topological local system in \mathcal{O} -modules on $S_K(G, X)(\mathbb{C})$, as in [28, §7.3]. Topological cohomology is computed by the complex

$$R\Gamma(S_K(G, X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O})).$$

We let

$$\mathbb{T}_{\emptyset,B}(\mu) \subset \text{End} \left(R\Gamma(S_K(G, X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O})) \right)$$

denote the \mathcal{O} -subalgebra generated by the topological Hecke correspondences $T(t)$ introduced in §2.

The de Rham version $\mathbb{T}_{\emptyset,dR}(\mu)$ of $\mathbb{T}_{\emptyset,B}(\mu)$ requires a bit more work to define. Let $\mathcal{E}_{G,\mathcal{O}}$ be the exact monoidal functor defined in [49, Lemma 1.20] from the tensor category of representations of the group scheme $G_{\mathbb{Z}_p}$ over \mathcal{O} to the category of locally free coherent sheaves on \mathbb{S}_K with integrable connection and compatible $G(\mathbf{A}_f^p)$ action covering the natural action on the family $\mathbb{S}_{K \times K^p}$. Let $W_{\mu,\mathcal{O}}$ denote the representation of $G_{\mathbb{Z}_p}$ introduced in [28, §7.1], so that the \mathcal{O} -lattice $W_{\mu}(\mathcal{O})$ is just $W_{\mu,\mathcal{O}}(\mathcal{O})$. We let

$$(39) \quad R\Gamma_{dR,\log}(\mathbb{S}_K, W_{\mathcal{O}}) = R\Gamma(\mathbb{S}_{K,\Sigma}, \Omega_{\mathbb{S}_{K,\Sigma},\log}^{\bullet} \otimes \mathcal{E}_{G,\mathcal{O}}(W_{\mu,\mathcal{O}})^{can});$$

see [50, §4.3], or [50, §7.4] for a reformulation in terms of dual BGG complexes. We define $\mathbb{T}_{\emptyset,dR}(\mu) \subset \text{End}(R\Gamma_{dR,\log}(\mathbb{S}_K, W_{\mathcal{O}}))$ again to be the \mathcal{O} -subalgebra generated by the algebraic Hecke correspondences $T(t)$.

We let $\nu_B : \mathbb{T}_{\emptyset,B}(\mu) \rightarrow \mathbb{Q}_p$ (resp. $\nu_{dR} : \mathbb{T}_{\emptyset,dR}(\mu) \rightarrow \bar{\mathbb{Q}}_p$) be a non-trivial homomorphism satisfying the hypotheses of Theorem 37. Define $\mathfrak{m}_B = \mathfrak{m}(\bar{\rho}_\nu) \subset \mathbb{T}_{\emptyset,B}(\mu)$ (resp. $\mathfrak{m}_{dR} = \mathfrak{m}(\bar{\rho}_\nu) \subset \mathbb{T}_{\emptyset,dR}(\mu)$) as in the statement of that theorem, and let $\mathbb{T}_{\bar{\rho}_\nu,\emptyset,B}(\mu)$, $\mathbb{T}_{\bar{\rho}_\nu,\emptyset,dR}(\mu)$ denote the respective localizations. We let

$$R\Gamma_{dR,\log}(\mathbb{S}_K, W_{\mathcal{O}})_{\bar{\rho}_\nu}, \quad R\Gamma(S_K(G, X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O}))_{\bar{\rho}_\nu}$$

denote the localizations of the cohomology complexes at the respective ideals.

Hypotheses 40. *In what follows, we assume*

- (a) μ satisfies the inequalities of Proposition 28;
- (b) $\bar{\rho}_\nu$ satisfies the hypotheses of Theorem 37.

Recall that $d_V = \dim S_K(G, X)$.

Corollary 41. *Under Hypotheses 40 we have (i)*

$$R\Gamma_{dR,\log}(\mathbb{S}_K, W_{\mathcal{O}})_{\bar{\rho}_\nu}$$

is concentrated in degree d_V , and

$$H_{dR,\log}^{d_V}(\mathbb{S}_K, W_{\mathcal{O}})_{\bar{\rho}_\nu} := H^{d_V}(R\Gamma_{dR,\log}(\mathbb{S}_K, W_{\mathcal{O}})_{\bar{\rho}_\nu})$$

is a free \mathcal{O} -module of finite rank. Moreover, there is a natural decreasing (Hodge) filtration $F^\bullet H_{dR,\log}^{d_V}(\mathbb{S}_K, W_{\mathcal{O}})_{\bar{\rho}_\nu}$ by \mathcal{O} -direct summands satisfying the analogue of [28, Theorem 7.2.2].

(ii) Assume in addition the inequality $|\mu_{comp}| \leq p - 2$, as in the statement of [28, Theorem 7.3.3]. Then

$$R\Gamma(S_K(G, X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O}))_{\bar{\rho}_\nu}$$

is concentrated in degree d_V . Moreover $H^{d_V}(S_K(G, X)(\mathbb{C}), \tilde{W}_{\mu, B}(\mathcal{O}))_{\bar{\rho}_\nu}$ is a free \mathcal{O} -module of finite rank (the same as in (ii)).

Proof. This follows from the earlier results in the same way as in [28], except that here we need to localize at the non-Eisenstein maximal ideal $\mathfrak{m}(\bar{\rho}_\nu)$. \square

The next two theorems are then proved exactly as in [28, §7.2, §7.3].

Theorem 42. *Assume Hypotheses 40 and the inequality $|\mu_{\text{comp}}| \leq p - 2$, as in the statement of [28, Theorem 7.3.3]. Then there is an isomorphism*

$$\phi_{\bar{\rho}_\nu, \emptyset, B} : R_{\bar{\rho}_\nu, \emptyset} \rightarrow \mathbb{T}_{\bar{\rho}_\nu, \emptyset, B}(\mu)$$

of local complete intersections, and $H^{d_V}(S_K(G, X)(\mathbb{C}), \tilde{W}_{\mu, B}(\mathcal{O}))_{\bar{\rho}_\nu}$ is a free module of finite rank over $\mathbb{T}_{\bar{\rho}_\nu, \emptyset, B}$.

We remind the reader that the inequality $|\mu_{\text{comp}}| \leq p - 2$ is needed in order to apply a comparison theorem for integral p -adic cohomology.

Theorem 43. *Assume μ satisfies the inequalities of Proposition 28. Then there is an isomorphism*

$$\phi_{\bar{\rho}_\nu, \emptyset, dR} : R_{\bar{\rho}_\nu, \emptyset} \rightarrow \mathbb{T}_{\bar{\rho}_\nu, \emptyset, dR}(\mu)$$

of local complete intersections, and $H_{dR, \log}^{d_V}(\mathbb{S}_K, W_{\mathcal{O}})_{\bar{\rho}_\nu}$ is a free module of finite rank over $\mathbb{T}_{\bar{\rho}_\nu, \emptyset, dR}(\mu)$.

5.2. Assumptions. Let \mathbb{T} be a finite free reduced local \mathcal{O} -algebra, M a finite free \mathcal{O} -module, with \mathbb{T} action defined by a faithful homomorphism $\iota : \mathbb{T} \hookrightarrow \text{End}_{\mathcal{O}}(M)$. Then $\mathbb{T}_{\mathbb{Q}} = \mathbb{T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a product of finite extensions of \mathbb{Q}_p that acts faithfully on $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We assume $M_{\mathbb{Q}}$ is free over $\mathbb{T}_{\mathbb{Q}}$ of rank m . For $0 \leq i \leq m$ we consider

$$\wedge^i M_{\mathbb{Q}} = \wedge_{\mathbb{T}_{\mathbb{Q}}}^i M_{\mathbb{Q}};$$

this is naturally a free $\mathbb{T}_{\mathbb{Q}}$ -module of rank $\frac{m!}{i! \cdot (m-i)!}$.

In what follows we assume $p > m$, in order to avoid ambiguity regarding the presence or not of factorials in the definition of exterior powers over \mathcal{O} -algebras.

5.2.1. Multiplicity one. With a bit more work we can show that we can take $m = n$ when $M = H_{dR, \log}^{n-1}(\mathbb{S}_K, W_{\mathcal{O}})_{\bar{\rho}_\nu}$, as in the next section. It would be convenient to assume that cuspidal automorphic representations of G occur that are spherical outside primes that split in F/F^+ satisfy strong multiplicity one: they are determined uniquely, as subspaces of the space of cusp forms, by their local components at all unramified places. Because the ramification is limited to places where G is isomorphic (up to the similitude factor) to $GL(n)$, this follows – almost – from the multiplicity one theorem for L -packets of unitary groups proved in [37] (conditionally on unpublished results of Arthur).

The “almost” refers to the structure of unramified L -packets at primes that ramify in F/F^+ . There the L -packets can have several members, distinguished by the choice of local maximal compact K_q ; so we recover multiplicity one, though not

necessarily strong multiplicity one. This, together with the properties of Bushnell-Kutzko types recalled in Definition 29, is enough to obtain $m = n$ in the situation considered after Proposition 44.

5.3. Exterior powers of modules over Hecke algebras.

Proposition 44. *With notation as in §5.2, assume M is free over \mathbb{T} . Let $M_i \subset \wedge^i M_{\mathbb{Q}}$ be a free \mathbb{T} -submodule of rank $\frac{m!}{i!(m-i)!}$. Then M_i and $\wedge_{\mathbb{T}}^i M$ are isomorphic as \mathbb{T} -modules.*

Proof. Since M is a free \mathbb{T} -module of rank m , then $\wedge_{\mathbb{T}}^i M$ is also a free \mathbb{T} -module of rank $\binom{m}{i}$ (see Theorem 4.2 in [12]). Now, M_i and $\wedge_{\mathbb{T}}^i M$ are finite free modules of the same rank, hence isomorphic. \square

We apply this when $M = H_{dR, \log}^{n-1}(\mathbb{S}_K, W_{\mathcal{O}})_{\bar{\rho}_{\nu}}$ in the notation of Corollary 41, with \mathbb{S}_K an integral model of the Shimura variety $Sh(V_{\sigma})$ attached to an n -dimensional hermitian space V_{σ} over F with signature $(n-1, 1)$ at one place $\sigma \in \tilde{S}_{\infty}$, and definite at the remaining places, and with $\mathbb{T} = \mathbb{T}_{\bar{\rho}_{\nu}, \emptyset, dR}(\mu)$ the corresponding localized Hecke algebra. (If F is not imaginary quadratic the subscript \log is irrelevant.) We also write $M = M_{\sigma}$ and $\mathbb{T} = \mathbb{T}_{\sigma}$ and allow σ to vary over \tilde{S}_{∞} ; we write G_{σ} for the similitude group of V_{σ} and (G_{σ}, X_{σ}) for the corresponding Shimura datum.

The algebra \mathbb{T}_{σ} is the localization of the full Hecke algebra $\mathbb{T}_{\emptyset, dR}(\mu)$ at the maximal ideal containing the kernel of the homomorphism ν_{dR} ; we use the same notation to denote the restriction $\nu_{dR} : \mathbb{T}_{\sigma} \rightarrow \mathbb{Q}_p$ to the localized Hecke algebra. This homomorphism corresponds to the action of \mathbb{T}_{σ} on the K -fixed vectors in an automorphic representation $\pi = \pi_{\nu}$ of $G_{\sigma}(\mathbf{A})$ that is realized in $H_{dR, \log}^{n-1}(\mathbb{S}_K(G_{\sigma}, X_{\sigma}), W_{\mu})$.

Now let V be any n -dimensional (non-degenerate) hermitian space over F , (G_V, X_V) the corresponding Shimura datum, $K_V \subset G_V(\mathbf{A}_f)$ a neat level subgroup that is hyperspecial maximal at p , $Sh_{K_V}(V)$ the corresponding Shimura variety, $\mathbb{S}_{K_V, V}$ a smooth model over \mathcal{O} (we may have to take \mathcal{O} sufficiently large to include the integer rings in all relevant reflex fields). We let $\mathbb{T}_{\emptyset, dR, V}(\mu)$ denote the \mathcal{O} -algebra generated by Hecke operators at places split in F/F^+ acting on $H_{dR, \log}^{d_V}(\mathbb{S}_{K_V, V}, W_{\mathcal{O}})$, with $W = W_{\mu}$; the index K_V is omitted from the notation for the Hecke algebra. We can discard the Hecke operators at a finite number of places without changing $\mathbb{T}_{\emptyset, dR, V}(\mu)$, and thus we can define $\nu : \mathbb{T}_{\emptyset, dR, V}(\mu) \rightarrow \mathbb{Q}_p$ and the localization

$$\mathbb{T}_V := \mathbb{T}_{\bar{\rho}_{\nu}, \emptyset, dR, V}(\mu)$$

for any V such that the representation π_{ν} transfers to $G_V(\mathbf{A})$.

The congruence ideal $C(\nu, \mathbb{T}_V) = C(\pi_{\nu}, \mathbb{T}_V) \subset \mathbb{T}_V$ is defined as in [15, Definition 6.7.2]. More precisely, it is the annihilator of

$$\mathbb{T}_V / (\mathbb{T}_V[\pi_{\nu}] + \mathbb{T}_V[\pi_{\nu}]^{\perp}).$$

Here $\bar{\pi}_{\nu}$ is the dual of π_{ν} , $\mathbb{T}_V[\pi_{\nu}]$ is the localization of \mathbb{T}_V at the prime ideal that is the kernel of the action of the Hecke algebra on the π_{ν} -isotypic subspace in $\mathbb{T}_V \otimes \mathbb{Q}$, and $\mathbb{T}_V[\pi_{\nu}]^{\perp}$ is the intersection of \mathbb{T}_V with the orthogonal complement of $\mathbb{T}_V[\pi_{\nu}]$ in $\mathbb{T}_V \otimes \mathbb{Q}$ with respect to Poincaré duality.

Proposition 45. *Let V and V' be two hermitian spaces of dimension n such that the representation π_ν transfers to $G_V(\mathbf{A})$ and $G_{V'}(\mathbf{A})$. Then we can identify the Hecke algebras $\mathbb{T}_V \xrightarrow{\sim} \mathbb{T}_{V'}$ in such a way that the Hecke operators at places v such that both G_V and $G_{V'}$ are split at v correspond. Moreover, suppose μ and p satisfy the inequalities of Proposition 28 for both d_V and $d_{V'}$. Then with respect to the identification of \mathbb{T}_V with $\mathbb{T}_{V'}$, the congruence ideals $C(\nu, \mathbb{T}_V)$ and $C(\nu, \mathbb{T}_{V'})$ coincide.*

Proof. The first assertion is clear. The claim about the congruence ideals is then a consequence of Theorem 43, which identifies both \mathbb{T}_V and $\mathbb{T}_{V'}$ with the same \mathcal{O} -algebra $R_{\tilde{\rho}_\nu, \emptyset}$. \square

In the same way, we see that the congruence ideals defined with respect to the integral structure on de Rham cohomology coincide with the ideals defined with respect to the graded pieces with respect to the Hodge filtration, which are defined with respect to the integral structure on coherent cohomology. We leave it to the reader to formulate the statement.

Remark 46. *It suffices to say that all the results of this section remain valid when the coefficients $W_{\mu, \mathcal{O}}$ are replaced by $W_{\mu, \mathcal{O}} \otimes \Lambda_{\text{IO}}$ for type data satisfying the minimality conditions, and the (de Rham or p -adic étale) cohomology modules $\bullet = R\Gamma_{dR, \log}(\mathbb{S}_K, W_{\mathcal{O}}), \dots$ are replaced by $\text{Hom}_{K_S^+}(\Lambda_{\text{IO}}, \bullet)$.*

6. ORDINARY MODULAR FORMS

The article [15] constructs p -adic L -functions as elements of Hida's ordinary Hecke algebra when the latter, localized at an appropriate maximal ideal, is known to satisfy a *Gorenstein hypothesis* [15, §6.7.8, §7.3.2]. Here we show how to derive this hypothesis from Theorem 37, when p is sufficiently large and the maximal ideal is as in the statement of that theorem. The first results of this type are due to Hida [35] and Tilouine [64]; our method is essentially the same as Tilouine's.

We make use of the notation of [15] without comment. Thus π is an anti-holomorphic cuspidal automorphic representation of the unitary similitude group G (denoted G_1 in *loc. cit.*) and $\mathbb{T} = \mathbb{T}_\pi$ is the localization of Hida's ordinary Hecke algebra at the corresponding maximal ideal \mathfrak{m}_π . This ideal determines a connected component of weight space, which includes the weight $\kappa(\pi)$ of the holomorphic modular form corresponding to π (more conventionally, to the contragredient of π , a holomorphic cuspidal automorphic representation), which we are allowed to vary. Theorem 37 can be applied provided π can be chosen of a weight that satisfies the inequalities in Proposition 28 as well as those of Hida's Control Theorem, cited as [15, Theorem 7.2.1]. We make these conditions more precise in the following Theorem. We say two weights μ, μ' are congruent modulo p if they define the same characters on the (finite group of) points of the special fiber of the torus T over the residue field k of \mathcal{O} .

Theorem 47. *Let $\kappa(\pi)$ be a weight such that the \mathcal{O} -module $M_{\kappa(\pi)}(K)$ of holomorphic modular forms of weight $\kappa(\pi)$, in the notation of [15], is the module*

$H^{i_0(w, \mu)}(\mathbb{S}_{K, \Sigma}, \mathcal{E}_w(\mu)^{\text{can}})$, in the notation of Corollary 35, with $i_0(w, \mu) = 0$. Suppose μ is congruent modulo p to a character μ' of T that satisfies the regularity condition (ii) of Proposition 28. Suppose the residual Galois representation $\bar{\rho}_\pi$ attached to the maximal ideal \mathfrak{m}_π satisfies the hypotheses of Theorem 37. Then the localized ordinary Hecke algebra $\mathbb{T} = \mathbb{T}_\pi$ is a local complete intersection and satisfies the Gorenstein Hypothesis 7.3.2 of [15].

Proof. The notation $\bar{\rho}_\pi$ corresponds to the notation $\bar{\rho}_\nu$ in the statement of Theorem 37. Applying that theorem, the hypotheses imply that the localized module $S_{\kappa(\pi)}(K)_\pi$ is free over the localized Hecke algebra, which is a local complete intersection. We claim that it suffices to show that Hida's Control Theorem, as stated in [36, Theorem 7.1], applies to $S_{\kappa(\pi)}(K)_\pi$. Indeed, the Hecke algebra $\mathbb{T}_{\bar{\rho}_\nu, \emptyset}$ in weight $\kappa(\pi)$ is the quotient of \mathbb{T}_π by the regular sequence corresponding to the weight $\kappa(\pi)$. Moreover, we are concerned with coherent cohomology in degree 0, so by Koecher's principle the reduced localization hypothesis in Theorem 37 is superfluous.

Now it remains to verify that Hida's Control Theorem does apply to weight $\kappa(\pi)$ when μ' satisfies the regularity condition (ii) of Proposition 28. In fact, Boxer and Pilloni have proved a classicality theorem for overconvergent modular forms of small slope [8, Theorem 1.0.15 (3)] that asserts that overconvergent modular forms of weight μ on Shimura varieties of abelian type are classical if they satisfy a small slope condition and if μ' satisfies the regularity condition (ii) of Proposition 28. Since ordinary modular forms automatically satisfy the small slope condition (see [8, Remark 1.0.6]), this completes the proof. \square

Remark 48. *It is important to note that Theorem 47 is really a statement about $\bar{\rho}_\pi$ and the prime p , and not about μ . If p is not too small relative to the group G , then every μ is congruent to a μ' satisfying the regularity condition.*

Remark 49. *Once again, it suffices to say that all the results of this section remain valid when the coefficients $\mathcal{E}_w(\mu)^{\text{can}}$ are replaced by $\mathcal{E}_w(\mu)^{\text{can}} \otimes \Lambda_{\text{IO}}$ for type data satisfying the minimality conditions and the cohomology modules are modified accordingly. We leave the details to the reader.*

7. PROOF OF THEOREM 25 (III) AND (IV)

The proof that the Galois representations attached to the boundary cohomology group $R\Gamma^\partial(\mathcal{F}_K) := R\Gamma^\partial(\mathbb{S}_K(G, X), \mathcal{F}_K)$, or to its generalization

$$R\Gamma^\partial(\mathbb{S}_K(G, X), \mathcal{F}_{K, \text{IO}}),$$

are reducible is based on a lengthy analysis of the toroidal boundary, to show that the Galois representations attached to any piece of the complex that computes the cohomology of the toroidal boundary breaks up as a sum of representations attached to the cohomology of smaller groups. This is a geometric argument and it is identical with or without the introduction of a K -type indicated by the subscript IO . Thus we will write down the proof, which is long enough as it is, in the case where the type data are trivial.

In what follows, R denotes a standard rational proper parabolic subgroup of G . We let $r(R)$ denote the parabolic rank of R ; thus $r(R) = 1$ if and only if R is a maximal proper parabolic. Let L_R denote the Levi quotient of R . It is a connected reductive group that admits a factorization

$$L_R = G_{h,R} \cdot G_{\ell,R}$$

where $G_{h,R} = G(V_R)$ is the similitude group of a hermitian vector space V_R over F , of dimension $n - 2m(R)$, for some $1 \leq m(R) \leq \frac{n}{2}$, and

$$G_{\ell,R} = \prod_{i=1}^{r(R)} GL(m_i(R))_F, \quad \sum_i m_i(R) = m(R).$$

The factorization is defined by choosing a flag

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{r(R)}$$

of totally isotropic subspaces of V , all assumed to be defined over \mathcal{O} . Then $GL(m_i(R))$ is identified with the group scheme $GL(A_i/A_{i-1})$ over $\text{Spec}(\mathcal{O})$. In particular, there is a surjective homomorphism

$$(50) \quad \ell_R : L_R \rightarrow G_{\ell,R}$$

whose kernel is isomorphic to the unitary similitude group $G(V_R)$, where V_R is the quotient of $V/A_{r(R)}$ by the null space of the induced hermitian form.

There is a Shimura datum $(G_{h,R}, X(R))$ attached to R , of the same kind as the pair (G, X) with which we began. As noted in the Appendix, this is *not quite* the boundary Shimura datum that corresponds to R in the theory of [55] or [32, 34]. In those references the group $G_{h,R}$ contains an additional split torus, here relegated to $G_{\ell,R}$, that is needed in order to define the correct mixed Hodge structure on the boundary cohomology. For the purposes of this paper this precision will not be necessary. The absence of this extra factor is compensated by the Tate twists in (67). (Pink's theory of mixed Shimura data (G, X) allows for the possibility that X is a union of a finite set of copies of hermitian symmetric domains in a flag variety attached to G . Taking this into account requires adjustments that are discussed in [34, §1.1.7], and that will be irrelevant for our purposes.) We note for future reference that

$$(51) \quad (G_{h,R}, X(R)) = (G_{h,P(R)}, X(P(R))),$$

where, as usual, $P(R)$ is the maximal parabolic subgroup to which R is subordinate, as in §A.1.

Let $K_{h,R} \subset G_{h,R}(\mathbb{R})$ be the stabilizer of a point $h_R \in X(R)$ – we may as well assume h_R to be a CM point, though this makes no difference – and let $K_{\ell,R}$ be a maximal connected subgroup of $G_{\ell,R}(\mathbb{R})$ that is compact modulo the center. Define $K_{R,\infty} = K_{h,R} \times K_{\ell,R}$. For any compact open subgroup $K_R \subset L_R(\mathbf{A}_f)$, we let ${}_{K_R}Y(R)$ denote the locally symmetric space $L_R(\mathbb{Q}) \backslash L_R(\mathbf{A}) / K_{R,\infty} \times K_R$. We always assume K_R to be *neat*, so that ${}_{K_R}Y(R)$ is smooth and its Borel-Serre compactification is a smooth manifold with corners.

We always assume K_R to have the property that $K_{h,R}^f := K_R \cap G_{h,R}(\mathbf{A}_f)$ contains a hyperspecial maximal compact subgroup of $G_{h,R}(\mathbb{Q}_p)$. Let ${}_{K_R}Y_\ell(R)$ denote the locally symmetric space

$$G_{\ell,R}(\mathbb{Q}) \backslash G_{\ell,R}(\mathbf{A}) / K_{\ell,R} \times \ell_R(K_R^f),$$

with ℓ_R as in (90). We let $\mathbb{S}_{K_R}(G_{h,R}, X(R))$ denote the smooth integral model over \mathcal{O} of the Shimura variety for $(G_{h,R}, X(R))$ of level $K_{h,R}^f$.

The locally symmetric space ${}_{K_R}Y_\ell(R)$ has a family of local coefficient systems attached to finite-dimensional (algebraic) representations of $G_{\ell,R}$, with coefficients in any \mathcal{O} -algebra. Letting $r_{\mathcal{W}} : G_{\ell,R} \rightarrow \text{Aut}(\mathcal{W})$ be such a representation, taken with coefficients in the \mathcal{O} -algebra A , we denote by $\tilde{\mathcal{W}}$ the corresponding local system with coefficients in A :

$$\tilde{\mathcal{W}} = G_{\ell,R}(\mathbb{Q}) \backslash G_{\ell,R}(\mathbf{A}) \times \mathcal{W}(A) / K_{\ell,R} \times K_\ell(R).$$

Fix K_R unramified at p and let $t \in G_{\ell,R}(\mathbf{A}_f)$. Let $S = S(K_R)$ temporarily denote the set of ramified places for K_R . If $t = (t_w)$ with $t_w \in G_w$, set $K_{R,w}(t) = K_{R,w} \cap tK_{R,w}t^{-1}$, and consider

$$K_R(t) := \prod_w K_{R,w}(t).$$

Since $\tilde{\mathcal{W}}$ is algebraic, it is defined over a finite extension $E(\tilde{\mathcal{W}})$, and under our assumption admits a model over any \mathcal{O} -algebra A . For an open subgroup $K'_R \subseteq K_R$, we have a natural finite covering map

$$\pi_{K_R, K'_R}^R : {}_{K'_R}Y_\ell(R) \rightarrow {}_{K_R}Y_\ell(R).$$

Let $R\Gamma({}_{K_R}Y_\ell(R), \tilde{\mathcal{W}})$ be a complex in $D^b(\mathcal{O})$ computing $H^*({}_{K_R}Y_\ell(R), \tilde{\mathcal{W}})$. The map π_{K_R, K'_R}^R induces canonical pullback and direct image functors in $D^b(\mathcal{O})$, while the isomorphism $*t^{-1}$ induces quasiisomorphism in $D^b(\mathcal{O})$ via pullback. Denote by $T_{S, K(R), \mathcal{W}}(t) \in \text{End}(R\Gamma({}_{K_R}Y_\ell(R), \tilde{\mathcal{W}}_{K_R}))$ the element given by the composition

$$(52) \quad R\Gamma({}_{K_R}Y_\ell(R), \tilde{\mathcal{W}}_{K_R}) \xrightarrow{\pi_{K_R, K_R(t)}^*} R\Gamma({}_{K_R(t)}Y_\ell(R), \tilde{\mathcal{W}}_{K_R(t)}) \\ \xrightarrow{\pi_{t^{-1}K_R, K_R(t)}^{*t^{-1}}} R\Gamma({}_{t^{-1}K_R}Y_\ell(R), \tilde{\mathcal{W}}_{t^{-1}K_R}) \xrightarrow{*t^{-1}} R\Gamma({}_{K_R}Y_\ell(R), \tilde{\mathcal{W}}_{K_R}).$$

Denote by $T_{S, K(R), \mathcal{W}}$ the algebra $\text{End}(R\Gamma({}_{K_R}Y_\ell(R), \tilde{\mathcal{W}}_{K_R}))$ spanned by the above operators with $t \in G_q$ with $w \notin S(K) \cup \{p\}$ and with the further stipulation that all $w|q$ in F^+ split in F/F^+ . Recall that $G_{\ell,R} = \prod_{i=1}^{r(R)} GL(m_i(R))_F$ with $\sum_i m_i(R) = m(R)$. The action of the classical Hecke operator $T_{i,j,v}$ for $i = 1, \dots, r(R)$, $j = 1, \dots, m_i(R)$ at unramified place v is recovered by $T_{S, K(R), \mathcal{W}}(t_{i,j,v})$ with

$$t_{i,j,v} = \text{diag}(\underbrace{\varpi_v, \dots, \varpi_v}_{j \text{ times}}, 1, \dots, 1) \times \prod_{k \neq i} \text{Id}_{m_k(R)} \in G_{\ell,R,v},$$

where ϖ_v is the uniformizer and Id_k is the identity $k \times k$ matrix. The following is a consequence of the main theorem of [57].

Theorem 53. *Let κ be an algebraically closed field and let $\nu : T_{S,K(R),\mathcal{W}} \rightarrow \kappa$ be a continuous homomorphism. Then there are semisimple representations*

$$\rho_{i,\nu} : \Gamma_F \rightarrow GL(m_i(R), \kappa), i = 1, \dots, r(R)$$

that are characterized, up to equivalence, by the following property: If v is a prime of F not in $S \cup \{p\}$, then $\rho_{i,\nu}$ is unramified at v . Let $\Gamma_v \subset \Gamma_F$ be a decomposition group at v ; then the semisimplification of the restriction $\rho_{i,\nu,v}$ of $\rho_{i,\nu}$ to Γ_v corresponds to the restriction to ν to the image of the Hecke operators at v by the unramified Langlands correspondence, in the following sense: there is an equality of polynomials in $\kappa[X]$

$$(54) \quad \det(1 - \rho_\nu(\text{Frob}_v)X) = 1 + \sum_{j=1}^{m_i(R)} (-1)^j \nu(q^{\frac{(m_i(R)+1)j}{2}} T_{i,j,v}) X^j$$

where the $T_{i,j,v}$ are the standard Hecke operators at v for $GL(m_i)$, normalized as in [57].

7.1. Eisenstein classes. The boundary cohomology

$$\varinjlim_K H^*(R\Gamma^\partial(S_K(G, X), \mathcal{F}_K))$$

is expressed in [34] by means of a spectral sequence – the *nerve spectral sequence* – whose E_1 terms correspond to contributions of the boundary strata corresponding to various standard rational parabolic subgroups. The result in [34] is as follows:

Proposition 55. [34, Corollary 3.2.9] *There is a spectral sequence abutting to the boundary cohomology in characteristic zero of the automorphic vector bundle \mathcal{F} :*

$$(56) \quad E_1^{r,s} \Rightarrow \varinjlim_K H^{r+s}(R\Gamma^\partial(S_K(G, X), \mathcal{F}_K)).$$

The E_1 term has a natural decomposition $E_1^{r,s} = \bigoplus_{r(R)=r+1} E_1^{r,s}(R)$, where R runs over standard rational parabolic subgroups of G , and

$$E_1^{r,s}(R) = \text{Ind}_{R(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \bigoplus_i \bigoplus_{w \in W^R} E_1^{r,s}(R)_{i,w},$$

where

$$E_1^{r,s}(R)_{i,w} = \tilde{H}^{s-i-\ell(w)}([\mathcal{F}_{\lambda(h,w)}]) \otimes H^i(Y_\ell(R), \tilde{\mathbf{F}}_{\lambda(\ell,w)}).$$

(The operation I^R that appears in [34] is the identity, because the groups $\Delta_{0,R}$ and $\Delta_{1,R}$ are trivial in this case, for the reasons explained in A.2.)

Here $Y_\ell(R) = \varprojlim_{K(R)} K(R) Y_\ell(R)$ is the adelic locally symmetric space attached to the group $G_{\ell,R}$, and $\tilde{H}^\bullet([\mathcal{F}_{\lambda(h,w)}])$ denotes coherent cohomology of the Shimura variety attached to $(G_{h,R}, X(R))$ with coefficients in an automorphic vector bundle $[\mathcal{F}_{\lambda(h,w)}]$.

Unfortunately, this spectral sequence is not adequate to study the cohomology with \mathbb{Z}_p coefficients at fixed level K . First of all, the decomposition over Weyl group elements W^R is based on Kostant's formula for Lie algebra cohomology, which in general fails in mixed characteristic. Moreover, torsion classes at level K cannot in general be identified with the K -invariants in the $G(\mathbf{A}_f)$ -representations $\varinjlim_K H^*(\mathbb{S}, \mathcal{F}_K^{can})$. So we have to replace Proposition 55 with a calculation of the boundary cohomology at level K , and we have to find a less precise expression for the individual terms $E_1^{r,s}(R)$ that does not depend on Kostant's formula. This is not more difficult but it does require additional notation. Our approach is roughly analogous to the study of boundary cohomology in [52, §§3,4], with the additional complication that we are working with coherent rather than topological cohomology, and thus the Levi factor of each parabolic R breaks up into hermitian and linear parts that behave differently.

Denote by

$$i_R : \partial^R \mathbb{S}_{K,\Sigma} \rightarrow \mathbb{S}_{K,\Sigma}$$

the closed immersion of the R -stratum. Then we have the *nerve spectral sequence*

$$(57) \quad E_1^{r,s} = \bigoplus_{r(R)=r+1} H^s(\partial^R \mathbb{S}_{K,\Sigma}, i_R^*(\mathcal{F}_K^{can})) \Rightarrow \varinjlim_K H^{r+s}(R\Gamma^\partial(\mathbb{S}_{K,\Sigma}, \mathcal{F}_K))$$

Each term in (57) is a module over the Hecke algebra $T_{S,K}$, and the differentials in the spectral sequence are all morphisms of $T_{S,K}$ -modules. We can also let the open compact subgroup $K^p \subset G(\mathbf{A}_f^p)$ shrink to the identity, letting $K = K_p \times K^p$ with K_p fixed hyperspecial maximal compact, and take the limit over the sequences (57), while letting Σ vary with K as necessary. Define

$$R\Gamma_\infty^\partial(\mathcal{F}_K) = \varinjlim_{K^p} R\Gamma^\partial(\mathbb{S}_{K,\Sigma}, \mathcal{F}_K)$$

and the R -component as

$$R\Gamma_\infty^R(\mathcal{F}_K) = \varinjlim_{K^p, \Sigma} R\Gamma(\partial^R \mathbb{S}_{K,\Sigma}, i_R^*(\mathcal{F}_K^{can}))$$

We then obtain a spectral sequence of $\mathcal{O}[G(\mathbf{A}_f^p)]$ -modules at the limit:

$$(58) \quad E_1^{r,s} = \bigoplus_{r(R)=r+1} H^s(R\Gamma_\infty^R(\mathcal{F}_K)) \Rightarrow H^{r+s}(R\Gamma_\infty^\partial(\mathcal{F}_K)).$$

For any finite set $T \supset S$ we let $T_{T,K}$ denote the subalgebra of $T_{S,K}$ generated by the Hecke operators at primes outside T . We let $G(\mathbf{A}_f^T) \subset G(\mathbf{A}_f^p)$ denote the subgroup of elements with trivial entries at primes in T , and let $K(T) = K \cap G(\mathbf{A}_f^T)$. The proof of part (iii) of Theorem 25 then comes down to verifying the following three claims:

Claim 59. *Let S' be a finite set of primes containing S and let $\nu : T_{S',K} \rightarrow k$ be a character realized on a subquotient of $H^s(R\Gamma_\infty^\partial(\mathcal{F}_K)) \otimes_{\mathcal{O}} k$ for some $s \geq 0$. Then there is a proper standard rational parabolic subgroup $R \subset G$, a finite $T \supset S'$, an*

$s' \leq s$, and a character $\nu' : T_{T,K} \rightarrow k$ that coincides with the restriction of ν and is realized on a subquotient of the space of $K(T)$ -invariants of

$$\varinjlim_{K^p, \Sigma} H^{s'}(\partial^R \mathbb{S}_{K, \Sigma}, i_R^*(\mathcal{F}_K^{\text{can}})) \otimes_{\mathcal{O}} k.$$

In what follows $T' \supset S$ is a finite set as in Claim 59. Assume $K(R, T') = K(T') \cap L_R(\mathbf{A}_f^{T'})$ is hyperspecial maximal compact at all primes not in T' . We let $T(R)_{T', K}$ denote the product of the unramified Hecke algebras of $L_R(\mathbb{Q}_v)$ relative to $K(R, T')$ for v not in T' and split in F/F^+ , and let

$$(60) \quad s_R : T_{T', K} \rightarrow T(R)_{T', K}$$

be the (unnormalized) partial Satake transform [52, §2.2.6]. Here “unnormalized” means the result of integration along the unipotent radical of R , but without multiplication by the modulus factor $\delta_R^{-1/2}$.

Claim 61. *Let ν' be a character of $T_{T,K}$ as in Claim 59. Then there is an automorphic vector bundle \mathcal{F}^R on $\mathbb{S}_{K_R}(G_{h,R}, X(R))$, a finite-dimensional representation \mathcal{W} of $G_{\ell,R}$ with coefficients in a finite extension k'/k , a finite set $T' \supset T$ as above such that $K(T') \cap L_R(\mathbf{A}_f^{T'})$ is hyperspecial maximal at all primes not in T' , and a character $\nu'_R : T(R)_{T', K} \rightarrow k'$ such that*

(a) ν'_R is realized on a subquotient of

$$H^i(\mathbb{S}_{K_R}(G_{h,R}, X(R))_{\Sigma_R}, \mathcal{F}^{R, \text{can}})_k \otimes H^j(K(R)Y_{\ell}(R), \tilde{\mathcal{W}})$$

for some $i, j \geq 0$;

(b) $\nu'_R \circ s_R$ coincides with the restriction of ν' to $T_{T', K}$.

Claim 62. *Notation is as in Claim 61. Let ν' be a character of $T_{T,K}$ with values in a finite extension k' of k . Suppose R, T' , and ν'_R are as in Claim 61. Then the residual Galois representation $\bar{r}_{\nu'}$ attached to ν' by Theorem 25 (i) is reducible.*

7.1.1. Proof of Claim 62. The proofs of Claims 59 and 61 are postponed until the next section. Assuming both of these Claims, 62 can now be proved right away. The bulk of the work is already contained in §4 of [52]. We introduce some additional notation. By Claim 59 we may assume $K(R, T')$ factors as a product $K_{h,R,T'} \times K_{\ell,R,T'}$ with $K_{h,R,T'} = K(R, T') \cap G_{h,R}(\mathbf{A}_f^{T'})$, $K_{\ell,R,T'} = K(R, T') \cap G_{\ell,R}(\mathbf{A}_f^{T'})$. Let

$$T(R)_{h,T',K} \text{ (resp. } T(R)_{\ell,T',K})$$

denote the product of the unramified Hecke algebras of $G_{h,R}(\mathbb{Q}_v)$ (resp. $G_{\ell,R}(\mathbb{Q}_v)$) relative to $K_{h,R,T'}$ (resp. $K_{\ell,R,T'}$) for v not in T' and split in F/F^+ . The product map $G_{h,R} \times G_{\ell,R} \xrightarrow{\sim} L_R$ defines a canonical isomorphism

$$(63) \quad \varphi_R : T(R)_{T', K} \xrightarrow{\sim} T(R)_{h,T', K} \otimes T(R)_{\ell,T', K}.$$

Let $T_{h,K(R),T',\mathcal{F}^R,k}^i$ (resp. $T_{\ell,K(R),T',\mathcal{W}}^j$) denote the image of the natural map

$$T(R)_{h,T',K} \rightarrow \text{End}(H^i(\mathbb{S}_{K_R}(G_{h,R}, X(R))_{\Sigma_R}, \mathcal{F}^{R, \text{can}})_k)$$

(resp.

$$T(R)_{\ell, T', K} \rightarrow \text{End}(H^j(K(R)Y_\ell(R), \tilde{\mathcal{W}})).$$

Then (possibly replacing k' by a finite extension) there are characters

$$\nu'_h : T_{h, K(R), T', \mathcal{F}^R}^i \rightarrow k'; \quad \nu'_\ell \rightarrow k' : T_{\ell, K(R), T', \mathcal{W}}^j$$

such that

$$(64) \quad \nu' = \nu'_h \otimes \nu'_\ell \circ (\varphi_R \otimes k').$$

Now it follows from Theorem 25 (i), applied to ν'_h , that there exists an $n_h := n - 2m(R)$ -dimensional representation $\bar{r}(\nu'_h)$ of Γ_F such that, for all v split in F/F^+ and outside T' ,

$$(65) \quad \det(1 - \bar{r}(\nu'_h)(\text{Frob}_v)X) = 1 + \sum_{j=1}^{n_h} (-1)^j \nu(q^{\frac{(n_h+1)j}{2}} T_{j,v}) X^j$$

Similarly, with notation as in Theorem 53, there are $m_i(R)$ -dimensional representations $\bar{r}_i(\nu'_\ell)$ of Γ_F with coefficients in k' , $i = 1, \dots, r(R)$, such that, for all v split in F/F^+ and outside T' ,

$$(66) \quad \det(1 - \bar{r}_i(\nu'_\ell)(\text{Frob}_v)X) = 1 + \sum_{j=1}^{m_i} (-1)^j \nu(q^{\frac{(m_i+1)j}{2}} T_{i,j,v}) X^j.$$

Let $\bar{r}_i^d(\nu'_\ell) = \bar{r}_i(\nu'_\ell)^\vee(1 - m_i(R))$ (Tate twist). It then follows from identity (b) of Claim 61, as in the discussion following [52, Lemma 4.6], that there are integers μ_0 and $\mu_1^\pm, \mu_2^\pm, \dots, \mu_{r(R)}^\pm$ such that ν'_R and

$$\bar{r}(\nu'_h)(\mu_0) \oplus \sum_{i=1}^{r(R)} \bar{r}_i(\nu'_R)(\mu_i^+) \oplus \bar{r}_i^d(\nu'_R)(\mu_i^-),$$

(where (μ_i^\pm) denote Tate twist) have the same Frobenius eigenvalues outside of T' . More precisely, these are given by $\mu_0 = n_h - n = -2m(R)$ and

$$(67) \quad \mu_i^+ = -m_i(R) - 2 \sum_{j>i} m_j(R), \quad \mu_i^- = 2 \sum_{j>i} m_j(R)$$

for $i = 1, \dots, r(R)$. By Chebotarev density, this completes the proof.

7.1.2. Eisenstein characters. Let k be an algebraically closed field (of any characteristic). Let $R \subset G$ be a rational parabolic subgroup. Let \mathcal{S} be a finite set of primes of F^+ , and let $\mathcal{T}_R^{\mathcal{S}}$ be the restricted tensor product of the local Hecke algebras of L_R , with coefficients in k , at all primes of F^+ outside \mathcal{S} that split in F ; let $\mathcal{T}^{\mathcal{S}} = \mathcal{T}_G^{\mathcal{S}}$. Here by *restricted* we mean that the local component is the identity at all but finitely many places. Choose a maximal split torus $T_v \subset L_R$ for each prime v of F^+ that splits in F , and let \mathcal{T}_{T_v} denote its Hecke algebra over k . The Satake homomorphism is an injective map

$$(68) \quad s_G : \mathcal{T}^{\mathcal{S}} \rightarrow \bigotimes'_{v \notin \mathcal{S}} \mathcal{T}_{T_v}$$

(again, v runs only over primes that split in F). Similarly, we have a Satake homomorphism for L_R

$$(69) \quad s_R : \mathcal{T}_R^S \rightarrow \bigotimes'_{v \notin \mathcal{S}} \mathcal{T}_{T_v}$$

Corresponding to the factorization $L_R \xrightarrow{\sim} G_{h,R} \times G_{\ell,R}$, we write $\mathcal{T}_R^S = \mathcal{T}_{h,R}^S \otimes \mathcal{T}_{\ell,R}^S$. The character $\beta : \mathcal{T}_R^S \rightarrow k$ is called *cohomological* if it is the tensor product of characters β_h of $\mathcal{T}_{h,R}^S$ and β_ℓ of $\mathcal{T}_{\ell,R}^S$, where β_h occurs as a subquotient of the coherent cohomology of a (smooth projective) toroidal compactification of the integral model $\mathbb{S}_{K_R}(G_{h,R}, X(R))$ with coefficients in some canonically extended automorphic vector bundle, for some level subgroup K_R that is hyperspecial maximal compact at split primes not in \mathcal{S} , and where β_ℓ occurs as a subquotient of the cohomology (with some coefficients) of the locally symmetric space attached to $G_{\ell,R}$, with full level at split primes not in \mathcal{S} .

Definition 70. Let $\nu : \mathcal{T}^S \rightarrow k$ be a character of \mathcal{T}^S . We say ν is *Eisenstein* if, viewing it via (68) as a character of $s_G(\mathcal{T}^S) \subset \bigotimes'_{v \notin \mathcal{S}} \mathcal{T}_{T_v}$, it extends to a character $\beta' : s_R(\mathcal{T}_R^S) \rightarrow k$ so that $\beta = \beta' \circ s_R$ is cohomological in the sense just defined.

Let $K_f \subset G(\mathbf{A}_f)$ be a (neat) level subgroup, and let $K_R \subset K_f \cap L_R(\mathbf{A}_f)$ be a compact open subgroup.

7.2. Proof of Claim 59. Let $K' \subset K \subset G(\mathbf{A}_f)$, with $K'_p = K_p$ our fixed hyperspecial maximal compact and K' normal in K . Then for any fixed toroidal datum Σ , we have a finite morphism

$$\mathbb{S}_{K',\Sigma} \rightarrow \mathbb{S}_{K,\Sigma}$$

of normal schemes. We assume $\mathbb{S}_{K,\Sigma}$ to be smooth and projective for convenience. The proof of [25, Lemma 2.6] applies in the mixed characteristic situation and implies that

Lemma 71. *The above map defines a canonical isomorphism*

$$\mathbb{S}_{K',\Sigma}/(K/K') \xrightarrow{\sim} \mathbb{S}_{K,\Sigma}.$$

In particular, for any automorphic vector bundle \mathcal{F}_\bullet there are canonical isomorphisms (Hochschild-Serre spectral sequence) in the derived category of \mathcal{O} -modules

$$RHom_{K/K'}(\mathcal{O}, R\Gamma(\mathbb{S}_{K',\Sigma}, \mathcal{F}_{K'}^{\text{can}})) \xrightarrow{\sim} R\Gamma(\mathbb{S}_{K,\Sigma}, \mathcal{F}_K^{\text{can}});$$

$$RHom_{K/K'}(\mathcal{O}, R\Gamma(\mathbb{S}_{K',\Sigma}, \mathcal{F}_{K'}^{\text{sub}})) \xrightarrow{\sim} R\Gamma(\mathbb{S}_{K,\Sigma}, \mathcal{F}_K^{\text{sub}});$$

$$RHom_{K/K'}(\mathcal{O}, R\Gamma^\partial(\mathbb{S}_{K',\Sigma}, \mathcal{F}_{K'}^{\text{can}})) \xrightarrow{\sim} R\Gamma^\partial(\mathbb{S}_{K,\Sigma}, \mathcal{F}_K^{\text{can}}).$$

Proof. The first claim, as noted, follows as in the proof of [25, Lemma 2.6]. The first two spectral sequences then follow from Proposition 18 (iii), and the third from the definition of $R\Gamma^\partial$. \square

7.2.1. Proof of Claim 59. Passing to the limit over K' (containing K_p), the last isomorphism yields a Hochschild-Serre spectral sequence of \mathcal{O} -modules:

$$(72) \quad E_2^{a,b} = H^a(K^p, H^b(R\Gamma_\infty^\partial(\mathcal{F}_K))) \Rightarrow H^{a+b}(R\Gamma^\partial(\mathbb{S}_{K,\Sigma}, \mathcal{F}_K^{\text{can}})).$$

For T as in the statement of Claim 59, let K_T^p denote the subgroup of elements of K^p whose entries at each prime v in T belongs to the principal congruence subgroup $K(v) \subset K_v$ of elements congruent to the identity modulo v , $K_T = K_T^p \times K_p$. Since $K(v)$ has pro-order prime to p , we can rewrite

$$(73) \quad H^a(K^p, H^b(R\Gamma_\infty^\partial(\mathcal{F}_K))) = \varinjlim_{T \supset S'} H^a(K^p/K_T^p, H^b(R\Gamma_\infty^\partial(\mathcal{F}_K))^{K_T})$$

(compare [52, Lemma 4.2]). For each T , the Hecke algebra T_{T,K_T} acts on the K_T -invariants on the left-hand side compatibly with the action on $H^\bullet(R\Gamma_\infty^\partial(\mathcal{F}_K))^{K_T}$ on the right hand side. Claim 59 now follows by combining (73) with (58).

8. AUTOMORPHIC VECTOR BUNDLES ON THE TOROIDAL BOUNDARY

The calculation of the boundary coherent cohomology in [32, 34] uses a combination of algebraic, analytic, and representation-theoretic arguments that do not apply to the integral models studied in this paper. In fact, some of the arguments probably fail in small characteristic. For example, if $f : A \rightarrow S$ is the canonical morphism of a universal abelian scheme A (with some level structure) to a Shimura variety S of PEL type, the computation in [32] of the higher direct images $R^i f_*$ of an automorphic vector bundle on A as automorphic vector bundles on S is based on Kostant's theorem on Lie algebra cohomology of unipotent radicals of parabolic subalgebras of a reductive Lie algebra, and this just breaks down in general.

Fortunately, most of what we need has been worked out by Kai-Wen Lan for the toroidal compactifications of [44]. As in [32, 34], the calculation is carried out for individual closed strata of the boundary, which gives coherent cohomology of the boundary strata attached to a Shimura datum $(G_{h,R}, X(R))$ in the minimal compactification; these are then put together according to the configuration of the strata, which introduces the topological cohomology of the factor $G_{\ell,R}$. The first important observation is that canonical extensions of automorphic vector bundles behave well under restriction to boundary strata: this corresponds to the discussion around Corollary 3.4.3 of [32] and the first part of Proposition 5.6 of [47].

8.1. The algebraic part. To begin, we write $\mathbb{S}_{K,\Sigma}$ for the integral model of the toroidal compactification, previously denoted $\mathbb{S}_K(G, X)_\Sigma$. We always choose Σ as in Theorem 13, so that the compactification is proper and smooth and the boundary divisor is a divisor with smooth normal crossings. Fix a rational boundary component F as in §1.3 – we may assume it corresponds to (a component of) a Shimura datum $(G_{h,R}, X(R))$ with $R = P_F$, its stabilizer; in other words $F \subset X(R)$ – let $\sigma \in \Sigma_F$ be a cone and let

$$i_\sigma : Z_\sigma \hookrightarrow \mathbb{S}_{K,\Sigma}$$

denote the inclusion of the corresponding *locally closed* stratum of the toroidal boundary, \bar{Z}_σ its closure in $\mathbb{S}_K(G, X)_\Sigma$. Then we have a diagram as in [32, (1.2.5)],

[47, (4.4),(4.5)] of smooth schemes over $\text{Spec}(\mathcal{O})$:

$$(74) \quad Z_\sigma \xrightarrow{\pi_2} A_F \xrightarrow{\pi_1} M_F \subset \mathbb{S}_{K'}(G_{h,R}, X(R))$$

for some appropriate level subgroup $K' \subset G_{h,R}(\mathbf{A}_f)$, where π_2 is a torus fibration and π_1 is an abelian scheme over a connected component M_F of $\mathbb{S}_{K'}(G_{h,R}, X(R))$.

Now let $\mathcal{F}_K^{\text{can}}$ be a canonically extended automorphic vector bundle over $\mathbb{S}_{K,\Sigma}$.

Proposition 75. [47, Proposition 5.6] (i) *There is a canonically determined locally free coherent sheaf \mathcal{F}_K^A over A_F and a canonical isomorphism*

$$i_\sigma^*(\mathcal{F}_K^{\text{can}}) \xrightarrow{\sim} \pi_2^*(\mathcal{F}_K^A)$$

compatible with the action of $G(\mathbf{A}_f^p)$ (see below for explanation).

(ii) *The vector bundle \mathcal{F}_K^A is endowed with an increasing $P_F(\mathbf{A}_f^p)$ -invariant filtration by vector bundles*

$$\dots \subset \mathcal{F}_K^{A,j} \subset \mathcal{F}_K^{A,j+1} \subset \dots$$

such that each associated graded piece $\text{gr}^j(\mathcal{F}_K^{A,j})$ is isomorphic to the pullback via π_1 of an automorphic vector bundle on $\mathbb{S}_{K'}(G_{h,R}, X(R))$:

$$\text{gr}^j(\mathcal{F}_K^{A,j}) \xrightarrow{\sim} \pi_1^*(\mathcal{F}_{K',R}^j).$$

Remark 76. (i) *The group $G(\mathbf{A}_f^p)$ acts on the set of toroidal compactifications $\mathbb{S}_{K,\Sigma}$ by acting on the group K as well as the toroidal datum Σ . The subgroup $P_F(\mathbf{A}_f^p) \subset G(\mathbf{A}_f^p)$ fixes the set of toroidal boundary components, as K and Σ vary, over the rational boundary component F , and the isomorphisms are compatible with diagrams (74) and thus with the morphisms π_1, π_2 (which can be labelled with σ). Detail are left to the reader, but see the discussion on pp. 310-11 of [32] and p. 89 of [34].*

(ii) *Part (ii) of Proposition 75 corresponds to the discussion on pp. 325-6 of [32]. In [47] it is not stated explicitly that the graded pieces correspond to automorphic vector bundles on the base M_F , but the proof makes it clear that the “coherent sheaves over \mathbf{Z} ” of the statement of Lan’s Proposition 5.6 are indeed the automorphic vector bundles as constructed by Lan in [45].*

The $P_F(\mathbf{A}_f^p)$ -invariance of the filtration follows from the fact that the construction of automorphic vector bundles in [45, Definition 6.7] is a functor from locally free \mathcal{O} -representations of P_F to $P_F(\mathbf{A}_f^p)$ -equivariant vector bundles over A_F . The $P_F(\mathbf{A}_f^p)$ -action is not mentioned explicitly in [45] but the construction is exactly analogous to that in [46, Proposition 8.1.4.1].

Proposition 77. *After replacing Σ by a refinement if necessary, the morphism π_2 of (74) extends to a morphism*

$$\bar{\pi}_2 = \bar{\pi}_{2,\sigma} : \bar{Z}_\sigma \rightarrow A_{F,\Sigma'}$$

where $A_{F,\Sigma'}$ is an appropriate (smooth projective) toroidal compactification of A_F . Moreover, letting $\bar{i}_\sigma : \bar{Z}_\sigma \hookrightarrow \mathbb{S}_{K,\Sigma}$ denote the inclusion of the closed boundary stratum, the isomorphism of Proposition 75 extends to a canonical isomorphism

$$\bar{i}_\sigma^*(\mathcal{F}_K^{\text{can}}) \xrightarrow{\sim} \bar{\pi}_2^*(\mathcal{F}_K^{A,\text{can}}).$$

Proof. The first part is implicit in [46]; a complete proof will appear in future work of Lan. To prove the second part, note that $\bar{i}_\sigma^*(\mathcal{F}_K^{\text{can}})$ and $\bar{\pi}_2^*(\mathcal{F}_K^{A,\text{can}})$ are both vector bundles on the regular scheme \bar{Z}_σ that are known to be canonically isomorphic away from a subscheme of codimension 2 by Proposition 75 and by Proposition 3.12.2 of [32], together with the results of [43]. Thus there is a canonical isomorphism over \bar{Z}_σ . \square

Proposition 78. [45] *Let $A_F \hookrightarrow A_{F,\Sigma'}$ be the smooth projective toroidal compactification of A_F of Proposition 77. After possibly replacing Σ' by a refinement, say Σ'' , the map π_1 of (74) extends to a morphism*

$$\bar{\pi}_1 : A_{F,\Sigma''} \rightarrow \bar{M}_F \subset \mathbb{S}_{K'}(G_{h,R}, X(R))_{\Sigma_F}$$

where $\mathbb{S}_{K'}(G_{h,R}, X(R))_{\Sigma_F}$ is a smooth projective toroidal compactification of $\mathbb{S}_{K'}(G_{h,R}, X(R))$ and \bar{M}_F is the Zariski closure of M_F .

Proof. The main reference for this fact is Theorem 2.15 of [45]. Since the notation $A_{F,\Sigma'}$ is not quite justified there, Lan advises us to add a reference to Section 1.3.4 of [46]. Although the title of [46, §1.3] is “Algebraic compactifications in characteristic zero,” the constructions in the relevant section are valid in mixed characteristic. \square

We use the same notation $\mathcal{F}_K^{A,\text{can}}$ to denote the canonical extension of \mathcal{F}_K^A on $A_{F,\Sigma''}$.

Proposition 79. [45] (i) *The filtration $\{\mathcal{F}_K^{A,j}\}$ of \mathcal{F}_K^A of Proposition 75 (ii) extends to a filtration $\{\mathcal{F}_{K,\Sigma'}^{A,j}\}$ of $\mathcal{F}_K^{A,\text{can}}$ by vector bundles such that each $\text{gr}^j(\mathcal{F}_K^{A,\text{can}})$ is the pullback via $\bar{\pi}_1$ of the canonical extension of $\mathcal{F}_{K',R}^j$.*

(ii) *The higher direct images $R^k \bar{\pi}_1 \text{gr}^j(\mathcal{F}_K^{A,\text{can}})$, $k \geq 0$, are canonical extensions of the automorphic vector bundles $R^k \pi_1 \text{gr}^j(\mathcal{F}_K^A)$ on M_F .*

Proof. (i) We thank Lan for the following argument. The point is that the construction in §3.B of [45] defines the canonical extensions algebraically in terms of the relative differentials on semiabelian schemes over M_F . More precisely, the semi-abelian scheme over the toroidal compactification $A_{F,\Sigma'}$, denoted \tilde{G} in *loc. cit.*, admits a split subtorus \mathcal{T} and hence a quotient semi-abelian scheme \bar{G} . It is explained in *loc. cit.* that this \bar{G} is the pullback via $\bar{\pi}_1$ from a semi-abelian scheme (denoted G in *loc. cit.* over \bar{M}_F). In particular, since Lan shows that the canonical extension of a given automorphic vector bundle on $A_{F,\Sigma'}$ arises as a subquotient of some tensor power of the relative Lie algebra of \tilde{G} , it admits an extension structure coming from the short exact sequence

$$1 \rightarrow \mathcal{T} \rightarrow \tilde{G} \rightarrow \bar{G} \rightarrow 1$$

of group schemes over $A_{F,\Sigma'}$. This structure induces filtrations on all the canonical extensions in such a way as to guarantee that the graded pieces are defined by the relative Lie algebras of \mathcal{T} and \bar{G} ; but each of these is a pullback from the toroidal compactification \bar{M}_F .

(ii) We first observe that each $R^k \bar{\pi}_1 \mathrm{gr}^j(\mathcal{F}_K^{A,\mathrm{can}})$ is a locally free sheaf on \bar{M}_F . Indeed, by (i), each $\mathrm{gr}^j(\mathcal{F}_K^{A,\mathrm{can}})$ is a pullback via $\bar{\pi}_1$, so by the projection formula the claim reduces to the corresponding assertion for the structure sheaf of $A_{F,\Sigma''}$, where it follows from [45, Theorem 2.15]. The same result of Lan implies that our statement is true outside a divisor on the special fiber, and we know by the results of [32] that the statement is true on the generic fiber, and is true outside a divisor on the special fiber by results of Lan. Thus, as in the proof of Proposition 77, it is true globally, since \bar{M}_F is a regular scheme. \square

8.2. The topological part. In order to derive Claim 61 from Claim 59 we need to understand $H^{s'}(\partial^R \mathbb{S}_{K,\Sigma}, i_R^*(\mathcal{F}_K^{\mathrm{can}})) \otimes_{\mathcal{O}} k$ as K^p and Σ vary. We need to fix Σ (and therefore K^p) in order to state the next Proposition.

Proposition 80. (i) *For each $b \geq 0$, the assignment*

$$\sigma \longmapsto H^b(\bar{Z}_\sigma, i_\sigma^*(\mathcal{F}_K)^{\mathrm{can}})$$

defines a locally constant sheaf $\mathbf{L}^b(\cdot, \mathcal{F}_K)$ of \mathcal{O} -modules on the simplicial complex $\mathfrak{N}_\Sigma(R)$, and an associated spectral sequence

$$E_1^{a,b} = H^a(\mathfrak{N}_\Sigma(R), \mathbf{L}^b(\cdot, \mathcal{F}_K)) \implies H^{a+b}(\partial^R \mathbb{S}_{K,\Sigma}, i_R^*(\mathcal{F}_K)^{\mathrm{can}}).$$

(ii) *Similarly, for each $b \geq 0$, the assignment*

$$\sigma \longmapsto H^b(\bar{Z}_\sigma, \bar{\pi}_{2,\sigma}^*(\mathcal{F}_{K,\Sigma'}^{A,j}))$$

defines a Hecke-equivariant locally constant sheaf $\mathbf{L}^b(\cdot, \mathcal{F}_{K'}^j)$ of \mathcal{O} -modules, written $\sigma \mapsto \mathbf{L}^b(\cdot, \mathcal{F}_{K'}^j)(\sigma)$ with K' as in Propositions 75, 78, on the simplicial complex $\mathfrak{N}_\Sigma(R)$. There is a Hecke-equivariant spectral sequence derived from the filtration in Proposition 79

$$E_1^{a',j} = H^{a'}(\mathfrak{N}_\Sigma(R), \mathbf{L}^b(\cdot, \mathcal{F}_{K'}^j)) \Rightarrow H^{a'+b+j}(\mathfrak{N}_\Sigma(R), \mathbf{L}^b(\cdot, \mathcal{F}_K)).$$

(iii) *For each σ there is a canonical Leray spectral sequence*

$$E_2^{s,t} = H^s(\bar{M}_F, R^t \bar{\pi}_{1,*} \mathcal{O}_{A_{F,\Sigma''}} \otimes \mathcal{F}_{K'}^j) \Rightarrow \mathbf{L}^{s+t}(\cdot, \mathcal{F}_{K'}^j)(\sigma)$$

(iv) *Suppose σ is a face of σ' , and let*

$$i_{\sigma,\sigma'} : \bar{Z}_{\sigma'} \hookrightarrow \bar{Z}_\sigma$$

denote the corresponding closed immersion. Then for each j the pullback map

$$i_{\sigma,\sigma'}^* : \bar{\pi}_{2,\sigma'}^*(\mathcal{F}_{K,\Sigma'}^{A,j}) \rightarrow \bar{\pi}_{2,\sigma}^*(\mathcal{F}_{K,\Sigma'}^{A,j})$$

determines isomorphisms on cohomology

$$\mathbf{L}^b(\cdot, \mathcal{F}_{K'}^j)(\sigma) \rightarrow \mathbf{L}^b(\cdot, \mathcal{F}_{K'}^j)(\sigma')$$

that are the face maps for the local system on $\mathfrak{N}_\Sigma(R)$.

Proof. Part (i) is proved as in [32, 34]. We compute the cohomology by the spectral sequence for the closed cover of $Z_\Sigma(R)$ by the maximal strata \bar{Z}_σ as σ runs over the simplices in $\mathfrak{N}_\Sigma(R)$. Proposition 77 allows us to apply the proof of [34, Proposition 3.1.1 (i)]. Part (ii) is a consequence Proposition 79. Part (iii) follows from the projection formula, in view of the isomorphism

$$(81) \quad H^b(\bar{Z}_\sigma, i_\sigma^*(\mathcal{F}_K)^{can}) \xrightarrow{\sim} H^b(A_{F,\Sigma''}, \mathcal{F}_K^{A,can}),$$

which is proved as [34, Proposition 3.4.1] in light of the results of this section and

$$H^b(A_{F,\Sigma''}, \mathrm{gr}^j \mathcal{F}_K^{A,can}) \xrightarrow{\sim} H^b(A_{F,\Sigma''}, \bar{\pi}_1^*(\mathcal{F}_{K',R}^{j,can})).$$

Part (iv) is just a restatement of the fact that the functor $\sigma \mapsto \mathbf{L}^b(\cdot, \mathcal{F}_{K'}^j)(\sigma)$ defines a locally constant sheaf on $\mathfrak{N}_\Sigma(R)$. \square

Combining the three parts of Proposition 80 we find:

Corollary 82. *Let ν' be a character of $T_{T,K}$ as in Claim 59. In order to prove Claim 61, it suffices to prove it if ν' is realized in the action of $T_{T,K}$ (after possibly increasing T) on $H^a(\mathfrak{N}_\Sigma(R), \mathrm{Tor}_c^{\mathcal{O}}(\mathbf{L}^b(\cdot, \bar{\pi}_1^* \mathcal{J}_{K'}), k))$ for an automorphic vector bundle $\mathcal{J}_{K'}$ on M_F and for $c = 0, 1$, where the cohomology of $\bar{\pi}_1^* \mathcal{J}_{K'}$ is computed via (81).*

8.2.1. *The homotopy type of $\mathfrak{N}_\Sigma(R)$.*

Proposition 83. *The nerve $\mathfrak{N}_\Sigma(R)$ of the R -stratum $\partial^R \mathbb{S}_{K,\Sigma}$ is homotopy equivalent to the $G_{\ell,R}$ -stratum of the Borel-Serre compactification of the locally symmetric space $_{K(R)} Y_\ell(P(R))$.*

Proof. This is [33, Proposition 2.6.4]. \square

8.2.2. *Local systems on adelic and connected locally symmetric spaces.* We let H be a connected reductive algebraic group over \mathbb{Q} with center Z , let $K_H \subset H(\mathbb{R})$ be a maximal connected subgroup that is compact modulo $Z(\mathbb{R})$, and let $Y = H(\mathbb{R})/K_H$ be the corresponding (possibly disconnected) symmetric space. Let $K_f \subset H(\mathbf{A}_f)$ be a compact open subgroup and let

$$S(Y, K_f) = H(\mathbb{Q}) \backslash Y \times H(\mathbf{A}_f) / K_f$$

denote the adelic symmetric space. Write $H(\mathbf{A}_f) = \coprod_j H(\mathbb{Q}) \alpha_j K_f$ and let $\Gamma_j = H(\mathbb{Q}) \cap \alpha_j K_f \alpha_j^{-1}$, $S(\Gamma_j) = \Gamma_j \backslash Y$. Then we can rewrite

$$(84) \quad S(Y, K_f) = \coprod_j S(\Gamma_j).$$

Now let M be a finite abelian group, and let $\lambda : K_f \rightarrow \mathrm{Aut}(M)$ be a homomorphism. Consider the local system over $S(Y, K_f)$

$$M(K_f) = H(\mathbb{Q}) \backslash Y \times H(\mathbf{A}_f) \times M / K_f$$

where K_f (resp. $H(\mathbb{Q})$) acts diagonally on the last two (resp. first two) factors in the product $Y \times H(\mathbf{A}_f) \times M$.

Proposition 85. *Under these hypothesis, the restriction $M(K_f)_j$ of the local system $M(K_f)$ to the subspace $S(\Gamma_j) \subset S(Y, K_f)$ can be written as*

$$\Gamma_j \backslash Y \times M,$$

where Γ_j acts via $q \cdot (y, m) = (qy, \lambda(\alpha_j^{-1}q^{-1}\alpha_j) \cdot m)$.

Proof. For α_j as above, consider

$$[y, m] \mapsto [y, \alpha_j, m] : \Gamma_j \backslash Y \times M \rightarrow H(\mathbb{Q}) \backslash Y \times H(\mathbf{A}_f) \times M/K_f$$

To assert injectivity note that $[y, \alpha_j, m] = [y', \alpha_j, m']$, then $y = qy'$, $\alpha_j = q\alpha_j k$ and $\lambda(k)m' = m$ for some $q \in H(\mathbb{Q})$ and $k \in K_f$. From the second equality we obtain that $q \in \Gamma_j$ and $k = \alpha_j^{-1}q^{-1}\alpha_j$, so that $[y, m] = [y', m']$.

To show that $M(K_f)$ is the disjoint union of the local systems $M(K_f)_j$, pick $(y, h, m) \in Y \times H(\mathbf{A}_f) \times M$. Then $h = q\alpha_j k$ for some $q \in H(\mathbb{Q})$ and $k \in K_f$. Then $[y, h, m] = [q^{-1}y, \alpha_j, \lambda(k)^{-1}m]$, which is in the image of $M(K_f)_j$. Lastly, if $[y, \alpha_j, m] = [y', \alpha_i, m']$, then $y = qy'$, $\alpha_j = q\alpha_i k$, and $m = \lambda(k)m'$. The second equation implies $i = j$. \square

Let $H = G_{\ell, R}^{red}$, $Y_{\ell, R}$ be the corresponding symmetric space, and $\{\alpha_j\}$ as above. Let $\lambda : H \rightarrow \mathbb{M}$ be an irreducible algebraic representation defined over $\text{Spec}(\mathbb{Z}_p)$. Fix an $H(\mathbb{Z}_p)$ -invariant lattice $\Lambda \subset \mathbb{M}(\mathbb{Q}_p)$ and let $M = \Lambda/p\Lambda$. The $H(\mathbb{Z}_p)$ -module M may depend on the lattice but by the Brauer-Nesbitt theorem its semisimplification does not. Applying Proposition 85 to this situation, we find

Corollary 86. *Let $\tilde{\mathbb{M}}$ be the local system*

$$\tilde{\mathbb{M}} = H(\mathbb{Q}) \backslash Y_{\ell, R} \times H(\mathbf{A}_f) \times \mathbb{M}(\mathbb{Q}_p)/K_f.$$

Then there is a local system of free \mathbb{Z}_p -modules $\tilde{\Lambda} \subset \tilde{\mathbb{M}}$ with the property that, for each component $S(\Gamma_j)$ as above, the restriction of $\tilde{\Lambda}/p\tilde{\Lambda}$ to $S(\Gamma_j)$ is isomorphic to

$$\Gamma_j \backslash (Y \times \tilde{\Lambda}/p\tilde{\Lambda}),$$

with the twisted Γ_j -action via $q \cdot (y, \tilde{m}) = (qy, \lambda(\alpha_j^{-1}q^{-1}\alpha_j) \cdot \tilde{m})$.

Consider a compact open $K \subseteq G(\mathbf{A}_f)$ satisfying the properties listed in Appendix A.2. For a standard rational parabolic R with Levi decomposition $R = L_R \cdot U_R$, let $\Gamma_R := R(\mathbf{A}_f) \cap K$, and $\Gamma_{\ell, R} := l_R(\Gamma_R)$ with l_R as in (90). Set $\Gamma_{\ell, R}^{red}$ to be the restriction of $\Gamma_{\ell, R}$ to $G_{\ell, R}^{red}$. Denote by $\mathcal{K}_{\ell, R}$ (resp. $\mathcal{K}_{\ell, R}^{red}$) the set of all $\Gamma_{\ell, R}$ (resp. $\Gamma_{\ell, R}^{red}$) as we let K vary over the neat compacts with the listed properties.

8.2.3. The local system on $X(\Gamma_{\ell, R}^{red})$. For any congruence subgroup $\Gamma_{\ell, R}^{red} \in \mathcal{K}_{\ell, R}^{red}$ the image of $\Gamma_{\ell, R}^{red}$ under the natural map to $G_{\ell, R}^{red}(\mathbb{Q}_p)$ is contained in $G_{\ell, R}^{red}(\mathbb{Z}_p)$.

Lemma 87. *The action of $\Gamma_{\ell, R}^{red}$ on $\text{Tor}_c^{\mathcal{O}}(\mathbf{L}^b(\cdot, \pi_1^* \mathcal{I}_{K'}), k)$ factors through the projection on $G_{\ell, R}^{red}(\mathbb{F}_p)$.*

Proof. Since the tensor category of automorphic vector bundles on M_F is generated by the relative cotangent bundle of A_F over M_F , this follows directly from the description of the latter in [46, (1.3.2.8)], given that the action of $\Gamma_{\ell,R}^{red}$ is linear on the factor denoted X in that formula. Although the reference in [46] is asserted for the compactification in characteristic zero, the statements remain true in mixed characteristic as long as we are in a situation of good reduction, which is the case throughout this paper. \square

Corollary 88. *Every irreducible constituent of the local system*

$$Tor_c^{\mathcal{O}}(\mathbf{L}^b(\cdot, \bar{\pi}_1^* \mathcal{J}_{K'}), k)$$

is a subquotient of the local system attached to an algebraic representation of $G_{\ell,R}^{red}$.

Proof. This follows from Lemma 87 and Steinberg's theorem [60] that every irreducible representation of $G_{\ell,R}^{red}(\mathbb{F}_p)$ is a subquotient of the restriction to the \mathbb{F}_p points of an algebraic representation. \square

8.3. Proof of Claim 61. Let $\nu' : T_{T,K} \rightarrow k$ be a character as in Claim 59. It follows from Corollary 88 that there is an irreducible algebraic representation $r_{\mathcal{W}} : G_{\ell,R}(\mathbb{F}_p) \rightarrow \text{Aut}(\mathcal{W})$ over \mathbb{F}_p , as in the discussion preceding Theorem 53,1 such that ν' is realized in an $r_{\mathcal{W}}$ -isotypic subquotient of

$$H^a(\mathfrak{N}_{\Sigma}(R), Tor_c^{\mathcal{O}}(\mathbf{L}^b(\cdot, \bar{\pi}_1^* \mathcal{J}_{K'}), k))$$

for an automorphic vector bundle $\mathcal{J}_{K'}$ on $\mathbb{S}_{K_R}(G_{h,R}, X(R))$ and for some a, b, c . Part (a) of Claim 61 then follows by filtering $\mathcal{J}_{K'}$ by irreducible subquotients, one of which will be the \mathcal{F}^R of the statement of Claim 61. Part (b) is then standard. More precisely, it comes down to the following observation. Let $\ell \neq p$ be an unramified prime for K and α be an irreducible spherical representation of $L_R(\mathbb{Q}_{\ell})$ with coefficients in k ; let $v_R \in \alpha$ be a non-zero spherical vector. Let $I(\alpha)$ denote the unnormalized induction of the pullback of α to $R(\mathbb{Q}_{\ell})$ to $G(\mathbb{Q}_{\ell})$, and let $v \in I(\alpha)$ be a (non-zero) spherical vector. Let $T(G, K_{\ell})$ and $T_{\ell}(L_R, K_{R,\ell})$ denote the local spherical Hecke algebras of $G(\mathbb{Q}_{\ell})$ and $L_R(\mathbb{Q}_{\ell})$, respectively, relative to the indicated hyperspecial maximal compact subgroups. Then $T(G, K_{\ell})$ (resp. $T_{\ell}(L_R, K_{R,\ell})$) acts on v (resp. on v_R) through a character

$$\nu_{I(\alpha)} : T(G, K_{\ell}) \rightarrow k \text{ (resp. } \nu_{\alpha} : T_{\ell}(L_R, K_{R,\ell}) \rightarrow k)$$

and

$$\nu_{I(\alpha)} = \nu_{\alpha} \circ s_R,$$

where s_R is the unnormalized partial Satake transform, as above. See [52, Proposition 3.8] for the analogous case of topological cohomology, and see [23, Lemma 2.3.2] for the analogous global argument for holomorphic modular forms.

APPENDIX A. ADELIC BOUNDARY COMPONENTS

A.1. Parabolic subgroups. In what follows, R denotes a standard rational proper parabolic subgroup of G . We let $r(R)$ denote the parabolic rank of R ; thus $r(R) =$

1 if and only if R is a maximal proper parabolic. Let L_R denote the Levi quotient of R . It is a connected reductive group that admits a factorization

$$(89) \quad L_R = G_{h,R} \cdot G_{\ell,R}$$

where $G_{h,R} = G(V_R)$ is the similitude group of a hermitian vector space V_R over F , of dimension $n - 2m(R)$, for some $1 \leq m(R) \leq \frac{n}{2}$, and

$$G_{\ell,R} = \prod_{i=1}^{r(R)} GL(m_i(R))_F, \quad \sum_i m_i(R) = m(R).$$

The factorization is defined by choosing a flag

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{r(R)}$$

of totally isotropic subspaces of V , all assumed to be defined over \mathcal{O} . Then $GL(m_i(R))$ is identified with the group scheme $GL(A_i/A_{i-1})$ over $\text{Spec}(\mathcal{O})$. In particular, there is a surjective homomorphism

$$(90) \quad \ell_R : L_R \rightarrow G_{\ell,R}$$

whose kernel is isomorphic to the unitary similitude group $G(V_R)$, where V_R is the quotient of $V/A_{r(R)}$ by the null space of the induced hermitian form.

If P is a standard rational maximal parabolic subgroup then $r(P) = 1$, $G_{\ell,P} = R_{F/\mathbb{Q}}GL(m(P))_F$, and $G_{h,P}$ is the similitude group of a hermitian vector space of dimension $n - 2m(P)$. Intersection with $G_{\ell,P}$ defines a bijection between the set of standard rational parabolics R with $G_{h,R} = G_{h,P}$ and the set of standard rational parabolics $R_{\ell,P} \subset G_{\ell,P}$. In this way we obtain a canonical map from the set of all standard rational proper parabolic subgroups to the set of (standard rational) maximal parabolic subgroups: any R is contained in a unique maximal P with $G_{h,R} = G_{h,P}$, and we let $P(R)$ denote that P ; we say that R is *subordinate* to P .

The (standard rational) maximal parabolic subgroups are totally ordered: $P < P'$ if and only if $m(P) > m(P')$, which is true if and only if $\dim G_{h,P} < \dim G_{h,P'}$. Moreover, $P < P'$ if and only if the standard boundary component F of X stabilized by P , as we discuss in §1.3 below, is smaller than (in fact, is a boundary component of) the boundary component F' stabilized by P' .

A.2. Parametrization of adelic boundary components. The subgroup $G_{h,P}$ is defined differently than in [32, 33, 34]. In those references the split component of $G_{\ell,P}$ was included as a subgroup of $G_{h,P}$ in order to account for the mixed Hodge structure on the boundary cohomology; an extra factor of \mathbb{G}_m is needed to define the appropriate Shimura datum. The twist does not change the algebraic structure of the Shimura variety attached to $G_{h,P}$ but it does affect the weights of the variations of Hodge structure on the local systems, and this is reflected in the Tate twists that appear in (67).

The factorization (89) is a **direct product**. This implies that the groups denoted $\Delta_{1,R}$ and $\Delta_{0,R}$ that appear in the formula [34, (3.2.8)] are trivial. It also implies that the combinatorial structure of the toroidal boundary bears a simple relation to

the topology of the Borel-Serre compactifications of the locally symmetric spaces attached to $G_{\ell,P}$, as P varies.

We begin by recalling the Borel-Serre compactifications for

$$G_{\ell,P} = R_{F/\mathbb{Q}} GL(m(P))_F.$$

Good references for this are [22, §4] (though this only treats totally real fields the method is general), [56, §1.3.9], and [52], who consider torsion coefficients, as well as the unpublished book [21]. Fix once and for all a rational minimal parabolic $P_0 \subset G$ with Levi decomposition $P_0 = L_0 \cdot U_0$. We consider open compact subgroups $K \subset G(\mathbf{A}_f)$ with the following properties:

- $K = K^p \times K_p$ where K_p is a fixed hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$.
- For any standard rational parabolic $R \supset P_0$, with $R = L_R \cdot U_R$, $L_R \supset L_0$, we have

$$K^p \cap R(\mathbf{A}_f^p) = (K^p \cap L_R(\mathbf{A}_f^p)) \cdot (K^p \cap U_R(\mathbf{A}_f^p));$$

Let

$$K_R = K \cap R(\mathbf{A}_f) = K_p \cap R(\mathbb{Q}_p) \times K^p \cap R(\mathbf{A}_f^p).$$

- $K(L_R) := K \cap L_R(\mathbf{A}_f) = K_h(R) \times K_\ell(R)$ where

$$K_h(R) = K(L_R) \cap G_{h,R}(\mathbf{A}_f); K_\ell(R) = K(R) \cap G_{\ell,R}(\mathbf{A}_f).$$

- K is neat.

Such K^p are cofinal in the set of all open compact subgroups of $G(\mathbf{A}_f^p)$.

We define the locally symmetric space

$$X_K(G_{\ell,P}) = G_{\ell,P}(\mathbb{Q}) \backslash G_{\ell,P}(\mathbf{A}_f) / K_{\infty,\ell,P} \cdot K_\ell(P)$$

for a subgroup $K_{\infty,\ell,P}$ containing the center $Z_{G_{\ell,P}}(\mathbb{R})$ of $G_{\ell,P}(\mathbb{R})$ and maximal compact modulo $Z_{G_{\ell,P}}(\mathbb{R})$. We choose maximal compact (mod center) subgroups $K_{\infty,\ell,R}$ compatibly with a fixed maximal compact (mod center) subgroup $K_\infty \subset G(\mathbb{R})$.

The inclusion of $X_K(G_{\ell,P})$ in its Borel-Serre compactification

$$X_K(G_{\ell,P}) \subset X_K(G_{\ell,P})^{\text{BS}}$$

is a homotopy equivalence. The complement

$$\partial_{\ell,P,K} = X_K(G_{\ell,P})^{\text{BS}} \setminus X_K(G_{\ell,P})$$

is the disjoint union of locally closed strata indexed by the rational standard parabolics R of G subordinate to P :

$$\partial_{\ell,P,K} = \coprod_{P(R)=P} \partial_{R,K}.$$

For each such R , or equivalently for each standard rational parabolic $R_{\ell,P} \subset G_{\ell,P}$, we have

$$\partial_{R,K} = R_{\ell,P}(\mathbb{Q}) \cdot U_{R_{\ell,P}}(\mathbf{A}_f) \backslash R_{\ell,P}(\mathbb{R}) \times G_{\ell,P}(\mathbf{A}_f) / K_{\infty,\ell,R} \cdot K_\ell(P),$$

where $R_{\ell,P}(\mathbb{Q})$ acts diagonally on the product, $U_{R_{\ell,P}}(\mathbf{A}_f)$ and $K_{\ell}(P)$ act on the factor $G_{\ell,P}(\mathbf{A}_f)$, and $K_{\infty,\ell,R}$ acts on the right on $R_{\ell,P}(\mathbb{R})$.

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