BOUNDARY UNIQUE CONTINUATION FOR ELLIPTIC REAL ANALYTIC DIFFERENTIAL OPERATORS

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ABSTRACT. We establish results on unique continuation at the boundary for the solutions of elliptic, partial differential operators of any order with real analytic coefficients. The second order case settles a conjecture of M. S. Baouendi and L. P. Rothschild in [6] and has new applications to boundary unique continuation for holomorphic functions of several variables. The work is motivated by the results of X. Huang et al in [34] and [35], and M.S. Baouendi and L.P. Rothschild in [6].

1. INTRODUCTION

A harmonic function on the upper half-space $\mathbb{R}^n_+$ which is smooth up to the boundary and vanishing to infinite order at the origin may not be constant. In [6] Baouendi and Rothschild (see also [35]) proved that if a harmonic function u in a half ball

$$B_r^+ = \{x = (x', x_n) \in \mathbb{R}^n : |x| < r, x_n > 0\}$$

vanishes to infinite order at the origin and $u(x', 0) \geq 0$, then $u \equiv 0$. In the same paper, they conjectured that similar results will hold for any real analytic second order elliptic differential operator on a domain with real analytic boundary. This paper provides a positive answer to the conjecture and a generalization for operators of higher order. Our result for general second order operators has an application to unique continuation for CR functions. These uniqueness phenomena extend the classical Hopf lemma about the nonvanishing of the normal derivative at a boundary point where a nonconstant solution attains an

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extremum: the assumption is local in nature and imposes conditions only at the boundary.

For holomorphic functions of one variable with nonnegative real part on a piece of the boundary, unique continuation and local forms of Hopf's lemma were proved in [8], [34], [35], and [37]. The results were applied to establish unique continuation for CR mappings for certain classes of CR manifolds.

Results along this line appeared also in the works [3], [4], [5], [11], [25], and [29]. Further extensions of the results of Baouendi and Rothschild were proved in [40] and [41].

In the article [42], N. Suzuki established a local Hopf lemma in the spirit of [6] for the one-dimensional heat equation.

In our recent work ([24]) results on unique continuation at the boundary were proved at the flat piece of the boundary of the half ball B_r^+ for the class of real analytic second order operators whose principal part is the Laplacian. That work used the ideas and methods of Hadamard for the construction of a fundamental solution in ([31]). This article uses the pseudodifferential calculus developed by Boutet de Monvel for studying boundary value problems.

The article is organized as follows: Section 2 contains the statements of the results in this work. Section 3 is devoted to a brief description of an algebra of boundary pseudodifferential operators due to Boutet de Monvel ([27]). In sections 4 and 5 we present the proofs of our results.

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2. Statements of the results

We will say that a continuous function u defined on a domain $D \subset \mathbb{R}^n$ is flat at a boundary point $p \in \partial D$ if for every positive integer N, there is a constant $C_N > 0$ such that

$$|u(x)| \leq C_N |x-p|^N.$$

We also say u vanishes to infinite order (or is flat) at p on a non-singular smooth curve $\gamma = \gamma(t)$ in D passing through $p = \gamma(0) \in \partial D$

and transversal to ∂D if for every N there is $C_N > 0$ such that

$$|u(\gamma(t))| \leq C_N t^N.$$

Clearly, this property is independent of the parametrization.
We recall the main result of [6]:

Theorem 2.1. *Let u be harmonic on the half ball $B_r^+ = \{x = (x', x_n) \in \mathbb{R}^n : |x| < r, x_n > 0\}$, continuous on the closure. Suppose*
(i) $u(x', 0) \geq 0$ for $|x'| \leq r$, $x' \in \mathbb{R}^{n-1}$;
(ii) the function $x_n \mapsto u(0', x_n)$ is flat at $x_n = 0$;
Then $u(x', 0) \equiv 0$ for x' near the origin in $\mathbb{R}^{n-1}$.

A somewhat similar but weaker result under the stronger hypothesis that u is harmonic in the upper half plane and decays exponentially along the $y-$axis was obtained in [38].

The theorem of Baouendi and Rothschild has the following immediate consequence on boundary unique continuation for harmonic functions (see also [35]):

Corollary 2.1. *Let u be harmonic in B_r^+, continuous on the closure of B_r^+. Assume that*
(i) $u(x', 0) \geq 0$ for $|x'| \leq r$;
(ii) u is flat at 0.
Then $u \equiv 0$.

In [6] it was conjectured that similar results are valid for general second order elliptic operators with real analytic coefficients. Our first main result which confirms this conjecture is as follows:

Theorem 2.2. *Let u be a solution of*

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(x) \frac{\partial u}{\partial x_k} + c(x)u = 0$$

on the half ball B_r^+, C^2 on $\overline{B_r^+}$. Suppose L is elliptic and the coefficients are real analytic on $\overline{B_r^+}$. Let $\gamma(t)$ be a real analytic curve transversal to the flat piece of B_r^+ , $\gamma(0) = 0$, $\gamma(t) \subseteq B_r^+$ for $t > 0$. Assume
(i) $u(x', 0) \geq 0$ for $|x'| \leq r$;

(ii) the function $t \mapsto u(\gamma(t))$, $t \geq 0$ is flat at $t = 0$;
(iii) for every positive integer N, the function

$$|x'|^{-N} u(x', 0)$$

is integrable on $|x'| \leq r$.
Then $u(x', 0) \equiv 0$ for x' small.

Observe that by the results in [39], u then extends as a solution, hence as a real analytic function in a neighborhood of the origin in $\mathbb{R}^n$. We note that the results in [6] and [35] do not apply to the Laplace operator on a general domain in $\mathbb{R}^n$ with a real analytic boundary. In general, the examples in [41] for harmonic functions on a general domain with a real analytic boundary show that Theorem 2.2 may not be valid if hypothesis (iii) is dropped.

Theorem 2.2 has the following consequence on boundary unique continuation:

Corollary 2.2. *Let u be a solution of $Lu = 0$ on a domain D, C^2 on $\overline{D}$. Suppose $\Sigma \subseteq \partial D$ is a real analytic hypersurface of $\mathbb{R}^n$, $p \in \Sigma$ and u satisfies:*
(i) $u \geq 0$ on Σ;
(ii) u is flat at p.
Then $u \equiv 0$.

To describe our results for operators of higher order, given a domain $D \subseteq \mathbb{R}^n$ with a real analytic hypersurface $\Sigma \subseteq \partial D$, and given an operator $P(x, \partial_x)$ of order $2m$ on $\overline{D}$, consider linear partial differential operators with real analytic coefficients on Σ:

$$B_j(x, \partial_x) = \sum_{|\alpha| \leq \mu_j} b_{j,\alpha}(x) \partial_x^\alpha, \quad j = 1, \ldots, m,$$

where the μ_j are the orders of B_j, $\mu_j < 2m$ and and we denote the principal symbols of the B_j by $Q_j(x, \xi) = \sum_{|\alpha| = \mu_j} b_{j,\alpha}(x) \xi^\alpha$. For each $x \in \partial D$ and each pair of vectors $\tau \neq 0$ tangent to ∂D at x and $\nu \neq 0$ normal to D at x, the polynomials in z, $Q_j(z) = Q_j(x, \tau + z\nu)$, $j = 1, \ldots, m$ are linearly independent modulo the polynomial $\prod_{k=1}^m (\tau - \lambda_k(\tau, \nu))$, where $\lambda_1(\tau, \nu), \ldots, \lambda_m(\tau, \nu)$ are the roots of the principal symbol $p_{2m}(x, \xi)$ of

$P(x, \partial_x)$ with positive imaginary parts. In what follows, we will assume that $B_1 w(x) = w(x)$ for $x \in \Sigma$.

Theorem 2.3. *Let u be a solution of*

$$Pu = \sum_{|\alpha| \leq 4} a_\alpha(x) \partial^\alpha u = 0$$

on the half ball B_r^+, C^4 on $\overline{B_r^+}$. Suppose P is elliptic and the coefficients are real analytic on $\overline{B_r^+}$. Let $\gamma(t)$ be a real analytic curve transversal to the flat piece of B_r^+ , $\gamma(0) = 0$, $\gamma(t) \subseteq B_r^+$ for $t > 0$. Assume that
(i) the function $t \mapsto u(\gamma(t))$, $t \geq 0$ is flat at $t = 0$;
(ii) Fix $1 < j \leq m$. Assume that for every positive integer N, the functions $|x'|^{-N} u(x', 0)$ and $|x'|^{-N} B_j u(x', 0)$ are locally integrable.
Then there exists $\epsilon > 0$ such that if $u(x', 0) \geq 0$ then $u(x', 0) \equiv 0$ for $|x'| \leq \epsilon$ and if $B_j u(x', 0) \geq 0$, $B_j u(x', 0) \equiv 0$ for $|x'| \leq \epsilon$. In particular, if $u(x', 0) \geq 0$ and $B_j u(x', 0) \geq 0$, then u extends as a solution to a neighborhood of the origin.

In the following corollary, P is of order 4 as above:

Corollary 2.3. *Let u be a solution of $Pu = 0$ on a domain D, C^4 on $\overline{D}$. Suppose $\Sigma \subseteq \partial D$ is a real analytic hypersurface of $\mathbb{R}^n$, $p \in \Sigma$ and u satisfies:*
(i) $u \geq 0$ and the normal derivative $\partial_\nu u \geq 0$ on Σ; assume also that for every N, $|x - p|^{-N} \partial_\nu u(x)$ is integrable on Σ.
(ii) u is flat at p.
Then $u \equiv 0$.

The unique continuation results of [6] and [35] as well as Theorem 2.2 were motivated by the unique continuation problem for CR functions.

Theorem 2.2 in turn leads to the following application to CR functions and holomorphic functions:

Corollary 2.4. *Let h be a CR function on a connected real analytic hypersurface M in $\mathbb{C}^n$ that is the boundary value of a holomorphic function $f = u + iv$ defined on a connected side M^+ of M. Let $p \in M$. Let $\gamma(t)$ be a real analytic curve transversal to M with $\gamma(0) = p$, $\gamma(t) \subset M^+$ for $t > 0$. If $\Re(h) \geq 0$, $\Re h(z) d(z, p)^N \in L^1_{loc}(M)$, $\forall N$, and*

$u(\gamma(t))$ is flat at $t = 0$, then $\Re(h) \equiv 0$ near p. If in addition M is not Levi flat, then f is constant on M^+.

EXAMPLE 2.1: The preceding corollary applies to the following examples: consider the hypersurfaces

$$M = \{(z', s + i\varphi(z', s) : z' \in \mathbb{C}^{n-1}, s \in \mathbb{R}\},$$

where z', s vary near the origin in $\mathbb{C}^{n-1}$ and $\mathbb{R}$ and φ is a real-valued, real analytic function near the origin, $\varphi(0) = 0$, and $d\varphi(0) = 0$. Suppose M is not Levi flat. Taking $\gamma(t) = (0, \dots, it)$, the corollary leads to: if $\Re(h) \geq 0$, $\Re h(z) d(z, p)^N \in L^1_{\text{loc}}(M)$, $\forall N$ and $u(\gamma(t))$ is flat at $t = 0$, then f is constant on $M^+ = \{z : \Im(z_n) > \varphi(z', s)\}$.

We remark that unlike previous works, the preceding unique continuation result on unique continuation for CR functions doesn't follow from the boundary unique continuation result for holomorphic functions of one variable in [35] or [4].

For operators of arbitrary order, we have:

Theorem 2.4. *Let u be a solution of*

$$Pu = \sum_{|\alpha| \leq 2m} a_\alpha(x) \partial^\alpha u = 0$$

on the half ball B_r^+, C^{2m} on $\overline{B_r^+}$. Suppose P is elliptic and real analytic on $\overline{B_r^+}$ and B_j, $1 \leq j \leq m$ with $B_1 h(x', 0) = h(x', 0)$ satisfy the complementing boundary conditions on the flat piece of ∂B_r^+. Let $\gamma(t)$ be a real analytic curve transversal to the flat piece of B_r^+ , $\gamma(0) = 0$, $\gamma(t) \subseteq B_r^+$ for $t > 0$. Assume that
(i) the function $t \mapsto u((\gamma(t))$ is flat at $t = 0$;
(ii) for every positive integer N, the functions $|x'|^{-N} B_j u(x', 0)$ are locally integrable for $1 \leq j \leq m$;
(iii) $B_1 u(x', 0) = u(x', 0) \geq 0$, $B_2 u(x', 0) \geq 0$ and $B_j u(x', 0) \equiv 0$ for $3 \leq j \leq m$.
Then there exists $\epsilon > 0$ such that $u(x', 0) \equiv B_2 u(x', 0) \equiv 0$ for $|x'| \leq \epsilon$, and u extends as a solution to a neighborhood of the origin.

Corollary 2.5. *Let u be a solution of $Pu = 0$ on a domain D, C^{2m} on $\overline{D}$. Suppose $\Sigma \subseteq \partial D$ is a real analytic hypersurface, $p \in \Sigma$ and u*

satisfies:
(i) $u \geq 0$, $\partial_\nu u \geq 0$, $\partial_\nu^j u \equiv 0$ *for* $2 \leq j \leq m-1$ *on* Σ*;*
(ii) for every positive integer N, $\partial_\nu^j u(x)|x-p|^{-N}$ *is locally integrable on* Σ *for* $1 \leq j \leq m-1$.
(iii) u *is flat at* p.
Then $u \equiv 0$ *on* D.

3. Boutet de Monvel's Boundary Pseudodifferential operators

It is well known that for an elliptic operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(x)\frac{\partial u}{\partial x_k} + c(x)u = 0$$

with smooth coefficients on the closure of a smoothly bounded domain $D \subseteq \mathbb{R}^n$, a Green's function, and hence a Poisson kernel exists if the zeroth order term $c(x) \leq 0$. When the coefficients of L are real analytic, without any sign assumption on $c(x)$, we will exhibit a local Poisson kernel $K(x,y)$ on a neighborhood $p \in \Sigma \subseteq \partial D$ where Σ is a real analytic hypersurface. The local Poisson kernel will have the form:

$$K(x,y) = d(x)\,\frac{A(x,y)}{|x-y|^n} + d(x)B(x,y)\log|x-y|$$

where $x \in D$ near p, $y \in \Sigma$, $d(x)$ denotes distance to ∂D, A and B are real analytic functions.

In this connection, we mention that we were inspired by S. Bell's works (see [12], [13], [14], [15], [16], [17]) for example) on expressing Poisson kernels for domains in the plane in terms of the Szego kernel and the Bergman kernel. We were also inspired by S. Bergman (and S. Bergman and M. Schiffer) who in a series of works (see for example [18], [19], [20], [21]) established a link between the Bergman kernel (or its analogue) and the Poisson kernel of second order operators of the type

$$L = \Delta u + c(x)u, \quad c(x) \text{ real analytic and } c(x) < 0.$$

Our strategy will be to give an explicit representation of the kernels in [36] by employing certain boundary operators developed by Boutet de Monvel in [27]. In this section we present the basic ingredients of Boutet de Monvel's calculus (see [27] and [30]).

The operators to be defined are invariant under coordinate changes in $\overline{\mathbb{R}^n_+}$ preserving the boundary $\{x_n = 0\}$ and so we will work in $\overline{\mathbb{R}^n_+}$.

$\mathcal{S}_+$ will denote the restriction to $[0, \infty)$ of functions in the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decaying functions.
Let $d \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^{n-1}$ open. The space $S^d_{1,0}(\Omega, \mathbb{R}^{n-1}, \mathcal{S}_+)$ consists of the functions $\tilde{f}(x', x_n, \xi') \in C^\infty(\Omega \times \overline{\mathbb{R}_+}, \mathbb{R}^{n-1})$ lying in $\mathcal{S}_+$ with respect to x_n, such that for all α, β, k, k' and x in a compact set,

$$\sup_{x_n > 0} \left| x_n^k D_{x_n}^{k'} D_{x'}^{\beta} D_{\xi'}^{\alpha} \tilde{f}(x', x_n, \xi') \right| \le C \langle \xi' \rangle^{d+1-k+k'-|\alpha|}$$

where $\langle \xi' \rangle = (1 + |\xi'|^2)^{\frac{1}{2}}$. The subspace $S^d(\Omega, \mathbb{R}^{n-1}, \mathcal{S}_+)$ of polyhomogenous elements consists of the functions $\tilde{f} \in S^d_{1,0}(\Omega, \mathbb{R}^{n-1}, \mathcal{S}_+)$ that have asymptotic expansion $\tilde{f} \sim \sum_{l=0}^{\infty} \tilde{f}_{d-l}$ where the functions $\tilde{f}_{d-l}$ have the quasi-homogeneity property

$$\tilde{f}_{d-l}\left(x', \frac{x_n}{\lambda}, \lambda \xi'\right) = \lambda^{d+1-l} \tilde{f}_{d-l}(x', x_n, \xi').$$

The functions in $S^d_{1,0}(\Omega, \mathbb{R}^{n-1}, \mathcal{S}_+)$ are called symbol-kernels.
If P is a pseudodifferential operator on $\mathbb{R}^n$, its restriction to R^n_+ is defined by

$$P_+ u = r^+ P e^+ u,$$

where r^+ restricts $D'(\mathbb{R}^n)$ to $D'(\mathbb{R}^n_+)$ and e^+ extends locally integrable functions on $\mathbb{R}^n_+$ by zero on $\mathbb{R}^n_-$.
P is said to have the transmission property with respect to $\mathbb{R}^n_+$ when P_+ preserves smoothness up to the boundary, that is, P_+ maps $C_0^\infty(\overline{\mathbb{R}^n_+})$ into $C^\infty(\overline{\mathbb{R}^n_+})$. Such operators are one of the ingredients in the calculus of Boutet de Monvel.

In addition to P_+, which operates on $\mathbb{R}^n_+$, there are four other operators G, T, K, S forming the matrix

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C_0^\infty(\mathbb{R}^n_+)^N \\ \times \\ C_0^\infty(\mathbb{R}^{n-1}_+)^M \end{matrix} \to \begin{matrix} C_0^\infty(\overline{\mathbb{R}^n_+})^{N'} \\ \times \\ C_0^\infty(\mathbb{R}^{n-1}_+)^{M'} \end{matrix}.$$

T is called a trace operator, going from $\mathbb{R}^n_+$ to $\mathbb{R}^{n-1}$; K is called a Poisson operator (or a potential operator), going from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$; S

is a pseudodifferential operator on $\mathbb{R}^{n-1}$; and G is an operator on $\mathbb{R}^n_+$ called a singular Green operator, a non-pseudodifferential term that has to be added in order to have adequate composition rules. The system $\mathcal{A}$ form an algebra and their adjoints are in the algebra.

The trace operators include the operators $\gamma_j : h \mapsto D^j_{x_n} u|_{x_n=0}$ composed with pseudodifferential operators on $\mathbb{R}^{n-1}$.
A Poisson operator of order d is an operator defined by

$$Kv(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix\cdot\xi'} \tilde{k}(x', x_n, \xi')\hat{v}(\xi')d\xi'$$

where the symbol-kernel $\tilde{k} \in S^{d-1}_{1,0}(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}, \mathcal{S}_+)$. The symbol corresponding to $\tilde{k}(x, \xi')$ is

$$k(x', \xi) = \mathcal{F}_{x_n \to \xi_n} e^+ \tilde{k}(x, \xi'), \ \ \mathcal{F} \text{ the Fourier transform.}$$

4. Proof of Theorem 2.2

For elliptic differential operators of any order with constant coefficients, Poisson kernels for the upper half-space of $\mathbb{R}^n$ were constructed by Agmon, Douglis and Nirenberg in the work [2]. For elliptic operators with real analytic coefficients, the existence of local Poisson kernels was proved in [36]. However, this latter kernel is not explicit since it was defined by using Lax-Milgram's theorem. Our goal here is to express the local kernels K_j of [36] more explicitly in terms of the potential operators of Boutet de Monvel. We stress that we make no assumptions on the lower order terms of our operators. In particular, for L as in Theorem 2.2, we don't require that the zeroth order term $c(x)$ is nonpositive. To prove Theorem 2.2, after a real analytic diffeomorphism, we may assume that the real analytic curve $\gamma(t)$ is given by $\gamma(t) = (0, \ldots, 0, t)$.

Let $K(x, y), x \in \mathbb{R}^n, y \in \mathbb{R}^{n-1}$ be the Poisson kernel of [36] for the second order operator L. Then there is a neighborhood Σ of 0 in $\mathbb{R}^{n-1}$ such that for any $\psi \in C^\infty_0(\mathbb{R}^{n-1})$, the function h defined by

$$h(x) = \int_{\mathbb{R}^{n-1}} K(x, y)\psi(y)dy$$

satisfies

$$Lh = 0 \text{ in } B^+_r \ (r \ \text{ small enough})$$

and

$$h = \psi \ \text{ on } \Sigma.$$

Define

$$\tilde{K}(x,y) = \begin{cases} K(x,y), \ x_n > 0 \\ 0, \ x_n < 0. \end{cases}$$

$\tilde{K} \in L^1_{\text{loc}}$ and $L\tilde{K}$ (L acting in x) is supported in the hyperplane $\{x_n = 0\}$. We wish to determine the distribution $L\tilde{K}$. Let $\psi = \psi(x_n) \in C_0^\infty(\mathbb{R})$, $\psi(x_n) \equiv 1$ for $|x_n| \leq 1$ and ψ supported in $(-2, 2)$.
For $\delta > 0$, let $\psi_\delta(x_n) = \psi(\frac{x_n}{\delta})$. Let $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$, $f(y) \in C_0^\infty(\mathbb{R}^{n-1})$.

Since $L\tilde{K}(x,y)$ is supported in $\{x_n = 0\}$, for any $\delta > 0$,

$$\begin{aligned}
\langle L\tilde{K}, \varphi(x)f(y)\rangle &= \langle L\tilde{K}, \psi_\delta(x_n)\varphi(x)f(y)\rangle \\
&= \langle \tilde{K}, \ {}^tL(\psi_\delta\varphi)f(y)\rangle \\
&= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}}\int_0^\infty K(x,y)\ {}^tL(\psi_\delta\varphi)(x)f(y)dxdy \\
&= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^{n-1}}\int_{D_\epsilon} K(x,y)\ {}^tL(\psi_\delta\varphi)(x)f(y)dxdy
\end{aligned}$$

where $D_\epsilon = \mathbb{R}^{n-1} \times (\epsilon, 2\delta)$.
Recall that

$$L = \sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^n b_k(x)\frac{\partial}{\partial x_k} + c(x).$$

We have:

$$\begin{aligned}
(K, \ {}^tL(\psi_\delta\varphi)f)_{D_\epsilon} &= (LK, \psi_\delta\varphi f)_{D_\epsilon} - \sum_{k=1}^n \int_{D_\epsilon}\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_k}\big(b_k K\psi_\delta\varphi\big) f dxdy \\
&- \sum_j\sum_i \int_{D_\epsilon}\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i}\big(a_{ij}\frac{\partial K}{\partial x_j}\psi_\delta\varphi\big) f dxdy \\
&+ \sum_j\sum_i \int_{D_\epsilon}\int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_i}\big(a_{ij}K\frac{\partial}{\partial x_i}(\psi_\delta\varphi)\big) f dxdy.
\end{aligned}$$

Note that $(LK, \psi_\delta\varphi f)_{D_\epsilon} = 0$. We consider each integral above:
When $1 \leq k < n$,

$$\int_{\mathbb{R}^{n-1}}\int_{D_\epsilon} \frac{\partial}{\partial x_k}\big(b_k(x)K(x,y)\psi_\delta(x_n)\varphi(x)\big)f(y)dxdy = 0 \tag{4.1}$$

since $\varphi(x)$ is of compact support.

When $k = n$,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \int_{D_\epsilon} \frac{\partial}{\partial x_n} \big(b_n(x) K(x,y) \psi_\delta(x_n) \varphi(x) \big) f(y) dx dy \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \left(\int_\epsilon^{2\delta} \frac{\partial}{\partial x_n} \big(b_n K \psi_\delta \varphi \big) dx_n \right) f(y) dx' dy \\
&= - \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} b_n(x', \epsilon) K(x', \epsilon, y) \varphi(x', \epsilon) \psi_\delta(\epsilon) f(y) dy dx' \\
&= - \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} b_n(x', \epsilon) K(x', \epsilon, y) \varphi(x', \epsilon) f(y) dy dx' \text{ (for } \epsilon < \delta)
\end{aligned}$$

which as $\epsilon \to 0^+$, converges to

$$- \int_{\mathbb{R}^{n-1}} b_n(x', 0) \varphi(x', 0) f(x') dx'. \tag{4.2}$$

For $i < n$,

$$\int_{\mathbb{R}^{n-1}} \int_{D_\epsilon} \frac{\partial}{\partial x_i} \Big(a_{ij}(x) \frac{\partial K}{\partial x_j}(x,y) \varphi(x) \psi_\delta(x_n) \Big) f(y) dx dy = 0 \tag{4.3}$$

since $\varphi(x)$ has compact support.

When $i = n$ and $j < n$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \int_{D_\epsilon} \frac{\partial}{\partial x_n} \Big(a_{nj}(x) \frac{\partial K}{\partial x_j}(x,y) \psi_\delta(x_n) \varphi(x) \Big) f(y) dx dy \\
&= - \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} a_{nj}(x', \epsilon)) \frac{\partial K}{\partial x_j}(x', \epsilon, y) \varphi(x', \epsilon) f(y) dx' dy \ (\epsilon < \delta) \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{\partial}{\partial x_j} \big(a_{nj}(x', \epsilon) \varphi(x', \epsilon) \big) K(x', \epsilon, y) f(y) dy dx'
\end{aligned}$$

which as $\epsilon \to 0^+$, converges to

$$\int_{\mathbb{R}^{n-1}} a_{nj}(x', 0) \varphi(x', 0) \frac{\partial f}{\partial x_j}(x') dx' \tag{4.4}$$

If $i = n$, $j = n$

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \int_{D_\epsilon} \frac{\partial}{\partial x_n} \Big(a_{nn}(x) \frac{\partial K}{\partial x_n}(x,y) \psi_\delta(x_n) \varphi(x) \Big) f(y) dx dy \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} a_{nn}(x', \epsilon) \frac{\partial K}{\partial x_n}(x', \epsilon, y) \varphi(x', \epsilon) f(y) dy dx'
\end{aligned}$$

which as $\epsilon \to 0^+$, converges to

$$- \int_{\mathbb{R}^{n-1}} a_{nn}(x', 0) \varphi(x', 0) A f(x') dx' \tag{4.5}$$

where Af comes from the Dirichlet to Neumann map.

For $1 \leq i < n$, taking $0 < \epsilon < \delta$, we clearly get

$$\int_{\mathbb{R}^{n-1}} \int_{D_\epsilon} \frac{\partial}{\partial x_i}\left(a_{ij}(x)K(x,y)\frac{\partial}{\partial x_i}(\psi^\delta \varphi)(x)\right) f(y)dxdy = 0. \tag{4.6}$$

When $i = n$,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \int_{D_\epsilon} \frac{\partial}{\partial x_n}\left(a_{nj}(x)K(x,y)\frac{\partial}{\partial x_n}(\psi^\delta \varphi)(x)\right) f(y)dxdy \\
&= -\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} a_{nj}(x',\epsilon)K(x',\epsilon,y)\frac{\partial \varphi}{\partial x_n}(x',\epsilon)f(y)dx'dy
\end{aligned}$$

since $\psi'(t) \equiv 0$ when $|t| \leq 1$.

As $\epsilon \to 0^+$, the latter integral converges to

$$= -\int_{\mathbb{R}^{n-1}} a_{nj}(x',0)\frac{\partial \varphi}{\partial x_n}(x',0)f(x')dx'. \tag{4.7}$$

Then from (4.1)-(4.7), we see that

$$\begin{aligned}
\langle L\tilde{K}, \varphi(x)f(y)\rangle &= \int_{\mathbb{R}^{n-1}} b_n(x',0)\varphi(x',0)f(x')dx' \\
&- \sum_{j=1}^{n-1} \int_{\mathbb{R}^{n-1}} a_{nj}(x',0)\varphi(x',0)\frac{\partial f}{\partial x_j}(x')dx' \\
&- \int_{\mathbb{R}^{n-1}} a_{nn}(x',0)\varphi(x',0)Af(x')dx' \\
&+ \sum_{j=1}^{n} \int_{\mathbb{R}^{n-1}} a_{nj}(x',0)\frac{\partial \varphi}{\partial x_n}(x',0)f(x')dx' \\
&= \left(h(x')\frac{\partial \varphi}{\partial x_n}(x',0), f(x')\right) + \left(\varphi(x',0), Vf(x')\right) \\
&= \left(h(x')f(x') \otimes \delta'(x_n)\right)(\varphi) + \left(Vf(x') \otimes \delta(x_n)\right)(\varphi) \\
&+ \left(Af \otimes \delta(x_n)\right)(\varphi),
\end{aligned}$$

where h is real analytic and V is a first order tangential differential operator.

It is well known that the operator A is a first order pseudodifferential operator. This is well known when the underlying Dirichlet problem is uniquely solvable. The map is also known to be a pseudodifferential operator in the absence of uniqueness (see for example [9] and [10]). Let

$$A_1(f) = Af + Vf.$$

Then

$$\begin{aligned}
\langle L\tilde{K}, \varphi(x)f(y)\rangle &= \Big(h(x')f(x') \otimes \delta'(x_n)\Big)(\varphi) + \Big(A_1 f \otimes \delta(x_n)\Big)(\varphi) \\
&= (hf, \gamma_1\varphi) + (A_1 f, \gamma_0\varphi) \\
&= (\gamma_1^*(hf), \varphi) + (\gamma_0^*(A_1 f), \varphi) \qquad (4.8)
\end{aligned}$$

where γ_0 and γ_1 are the trace operators $\gamma_0\varphi(x', 0) = \varphi(x', 0)$ and $\gamma_1\varphi(x', 0) = \partial_{x_n}\varphi(x', 0)$. Let Q be a parametrix of L in a neighborhood of $\overline{B_r^+}$. Then

$$QL = Id + R$$

where R is an analytic regularizing operator.

$$\begin{aligned}
\langle Q(L\tilde{K}), \varphi(x)f(y)\rangle &= \langle L\tilde{K}, Q^t(\varphi(x))f(y)\rangle \\
&(\gamma_1^*(hf), Q^t(\varphi)) + (\gamma_0^*(A_1 f), Q^t(\varphi))
\end{aligned}$$

Thus we have:

$$\int_{\mathbb{R}^{n-1}} Q(L\tilde{K})(x,y)f(y)dy = Q(\gamma_1^*(hf)) + Q(\gamma_0^*(A_1 f)).$$

It was proved by Boutet de Monvel in [27] that when T is a pseudodifferential operator of order d satisfying the transmission condition, then the operator E defined by

$$Ev(x) = r^+ T(v(x') \otimes \delta(x_n))$$

is a Poisson operator of order $d+1$ (see Theorem 10.25 in [30]).

It follows that modulo a real analytic regularizing operator, $\tilde{K}$ is given by (for $x_n > 0$) a Poisson operator which we denote by K.

The operator K is of order 0 and hence by the result just mentioned, for some real analytic pseudodifferential operator S of degree -1 with symbol $s(x,\xi)$,

$$\begin{aligned}
Kf(x) &= S\Big(f(x')\otimes\delta(x_n)\Big)(x)\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}s(x,\xi)f(y')\delta(y_n)dyd\xi\\
&= \int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}^n} e^{i(x'-y')\cdot\xi'}e^{ix_n\xi_n}s(x,\xi)d\xi\right)f(y')dy'\\
&= \int_{\mathbb{R}^{n-1}}\tilde{k}(x,x'-y')f(y')dy'
\end{aligned}$$

where for $x\in\mathbb{R}^n$, $t'\in\mathbb{R}^{n-1}$,

$$\tilde{k}(x,t') = \int_{\mathbb{R}^n} e^{i(t'\cdot\xi'+x_n\xi_n)}s(x,\xi)d\xi.$$

The symbol $s(x,\xi)$ is a classical symbol and so

$$s(x,\xi) = \sum_{j=0}^{\infty} s_j(x,\xi)$$

where s_j is homogeneous in ξ of degree $-j-1$, real analytic and for each compact K, $\exists c, A>0$ such that for all j,α,β:

$$\left|\partial_x^\alpha\partial_\xi^\beta s_j(x,\xi)\right| \le c\,A^{j+|\alpha+\beta|}(j+|\alpha|)!\beta!|\xi|^{-1-j-|\beta|}\quad\forall x\in K. \tag{4.9}$$

See [28] for the definition of real analytic pseudodifferential operators. For $j=0,1,2,\dots$ define

$$k_j(x,t') = \int_{\mathbb{R}^n} e^{i(t'\cdot\xi'+x_n\xi_n)}s_j(x,\xi)d\xi.$$

We next argue as in [28] with some modifications. Fix an integer $m < -n+1$ and consider the family

$$\mathcal{F} = \left\{\frac{A^{-j}}{j!}\langle z,\xi\rangle^{m+j}f.p.(s_j(x,\xi)) : z\in\mathbb{C}^n, |z|=1, m+j\ge 0\right\},$$

where $f.p.$ denotes the finite part. For any $h\in\mathcal{F}$, from (4.9), we have the uniform bound

$$|h| \le c|\xi|^{m-1}. \tag{4.10}$$

Let $\varphi_1(\xi)\in C_0^\infty(\mathbb{R}^n)$, $\varphi_1\equiv 1$ for $|\xi|\le 1$, φ_1 supported in $|\xi|\le 2$. Let $\varphi_2 = 1-\varphi_1$ and for $h\in\mathcal{F}$, write

$$h = \varphi_1 h + \varphi_2 h.$$

Inequality (4.10) shows that the Fourier transforms (in the ξ variable)

$$\{\widehat{\varphi_2 h}(x,\eta) : h \in \mathcal{F}\}$$

are uniformly bounded since

$$|\widehat{\varphi_2 h}(x,\eta)| \leq ||\varphi_2 h||_{L^1(\mathbb{R}^n)}, \tag{4.11}$$

Observe that $\varphi_1 h$ is understood in the sense of finite parts (see [33]). We use the results in [33] to estimate the Fourier transform of $\varphi_1 h$. We have:

$$\begin{aligned} \widehat{\varphi_1 h}(x,\eta) &= \langle \varphi_1(y) h(x,y), e^{-iy\cdot\eta}\rangle \\ &= \langle h(x,y), \varphi_1(y) e^{-iy\cdot\eta}\rangle \\ &= \langle h(x,y), \psi(y) R_{m-1}(\varphi_1(y) e^{-iy\cdot\eta})\rangle \qquad (4.12) \end{aligned}$$

where $\psi \in C_0^\infty(\mathbb{R}^n \setminus 0)$ such that

$$\int_0^\infty \frac{\psi(ty_0)}{t} dt = 1 \text{ for some } y_0 \neq 0,$$

and

$$\begin{aligned} R_{m-1}\Big(\varphi_1(y) e^{-iy\cdot\eta}\Big) &= \langle t_+^{m+n-2}, \varphi_1(ty) e^{-ity\cdot\eta}\rangle \\ &= \frac{-1}{(N-1)!} \int_0^\infty (\log t) \Big(\varphi_1(ty) e^{-ity\cdot\eta}\Big)^{(N)}(t) dt \qquad (4.13) \\ &\quad + \frac{(\varphi_1(ty)e^{-ity\cdot\eta})^{(N)}(0)}{(N-1)!} \left(\sum_{j=1}^{N-1} \frac{1}{j}\right) \qquad (4.14) \end{aligned}$$

with $N = -m-n+2$, and the N^{th} derivative

$$\Big(\varphi_1(ty) e^{-ity\cdot\eta}\Big)^{(N)}(t) = \sum_{l=0}^{N} \binom{N}{l} \Big(\sum_{|\alpha|=N-l} \varphi_\alpha(ty) y^\alpha\Big)(-iy\cdot\eta)^l e^{-ity\cdot\eta}.$$

Let $\psi(y)$ be supported in $|y| \geq M$. Then on the support of ψ,

$$|h(y)| \leq \frac{c}{|y|^{1-m}}$$

and when $y \in \text{supp}(\psi)$, and $ty \in \text{supp}(\varphi_1)$,

$$|t| \leq \frac{2}{|y|} \leq \frac{2}{M}$$

and so when $y \in \operatorname{supp}(\psi)$,

$$\begin{aligned}\left|R_{m-1}\big(\varphi_1(y)e^{-iy\cdot\eta}\big)\right| &\leq c_1\bigg(\int_0^{\frac{2}{M}} |\log t|dt\bigg)|y|^N \sum_{j=0}^{N} |\eta|^j \\ &\leq c_2|y|^N \sum_{j=0}^{N} |\eta|^j \qquad (4.15)\end{aligned}$$

for some constants c_1 and $c_2 > 0$.
It follows from (4.12) and (4.15) that for x in a compact set,

$$\left\{\widehat{\varphi_1 h}(x,\eta) : h \in \mathcal{F}\right\}$$

are uniformly bounded on $\{\eta \in \mathcal{C}_\epsilon| : |\eta| \leq \epsilon\}$ if ϵ is small enough. Here

$$\mathcal{C}_\epsilon = \{z \in \mathbb{C}^n : |\Im\, z| < \epsilon|\Re\, z|\}.$$

Let

$$k_j(x,z) = \int_{\mathbb{R}^n} e^{iz\cdot\xi} s_j(x,\xi)d\xi,\ j = 0,1,2,\ldots$$

and set

$$K(x,y') = \sum_{j=0}^{\infty} k_j(x, x'-y', x_n).$$

Observe next that for $j \geq n-1$, since $k_j(x,z)$ is the Fourier transform of the finite part of $s_j(x,\xi)$, by the results in [33],

$$k_j(x,z) = U_{j-n+1}(x,z) + Q_{j-n+1}(x,z)\log|z|$$

where $U_{j-n+1}(x,z)$ is homogeneous in z (away from $z=0$) of degree $j-n+1$ and $Q_{j-n+1}(x,z)$ is homogeneous polynomial in z of degree $j-n+1$. It follows that for $j \geq n+1$, $k_j(x,z)$ and its derivatives of order $< j-n+1$ are continuous and zero at the origin. Therefore, integrating $j-n$ times on the segment from the origin to z in $\mathcal{C}_\epsilon$ for $j \geq n+1$, we get

$$|k_j(x,z)| \leq \frac{cA^j j!}{(j-n)!}|z|^{j-n}$$

for $z \in \mathcal{C}_\epsilon$, $|z| < \epsilon$.
Thus $\sum_{j=n+1}^{\infty} k_j(x,z)$ converges uniformly and has the form

$$\sum_{j=n+1}^{\infty} k_j(x,z) = |z|\ E(x,z),\ z \in \mathcal{C}_\epsilon, |z| < \epsilon \qquad (4.16)$$

where $E(x,z)$ is holomorphic in z.

When $0 \le j \le n-2$, $k_j(x,z)$ is homogeneous in z of degree $j-n+1$. The terms k_{n-1} and k_n have the form

$$k_{n-1}(x,z) = U_0(x,z) + Q_0(x)\log|z|,$$

$$k_n(x,z) = U_1(x,z) + Q_1(x,z)\log|z|,$$

The functions U_0, U_1 are homogeneous in z (away from $z=0$) of degree 0 and 1 respectively, $Q_0(x)$ is real analytic and $Q_1(x,z)$ is a homogeneous polynomial in z of degree 1.

Since

$$\lim_{x_n \to 0^+} K(x,y') = \delta(x'-y'),$$

we have $k_j(x',0,z) = 0$ for all j and so using the homogeneity in z, we can write

$$k_0(x,z) = \frac{x_n A_0(x,z)}{|z|^n},\ k_1(x,y) = \frac{x_n A_1(x,z)}{|z|^{n-1}}, \ldots, k_{n-2}(x,z) = \frac{x_n A_{n-2}(x,z)}{|z|^2}.$$

It is easy to see that for any $f \in C_0^\infty(\mathbb{R}^{n-1})$ and $1 \le j \le n-2$,

$$\lim_{x_n \to 0^+} \int_{\mathbb{R}^{n-1}} \frac{x_n A_j(x, x'-y', x_n)}{(|x'-y'|^2 + x_n^2)^{\frac{n-j}{2}}}\, g(y')dy' = 0$$

while

$$\lim_{x_n \to 0^+} \int_{\mathbb{R}^{n-1}} \frac{x_n A_0(x, x'-y', x_n)}{(|x'-y'|^2 + x_n^2)^{\frac{n}{2}}} g(y')dy' = g(x').$$

It follows that

$$A_0(x',0,0,0) \equiv 1 \ \text{ for } x' \ \text{ near } 0 \in \mathbb{R}^{n-1}. \tag{4.17}$$

We will next estimate the derivatives $\partial_{x_n}^k k_0(0, x_n, -y', x_n)$ of arbitrary order at $x_n = 0, y' \neq 0$.

Observe that for any N,

$$\partial_{x_n}^{N+1} \left\{ x_n (|y'|^2 + x_n^2)^{\frac{-n}{2}} \right\} = N \partial_{x_n}^N (|y'|^2 + x_n^2)^{\frac{-n}{2}}.$$

To compute the latter derivative, we use Faá di Bruno's formula:

$$\frac{d^N}{dt^N} Q(f(t)) = \sum \frac{N!}{N_1! \ldots N_N!} Q^{(N_1 + \ldots + N_N)}(f(t)) \prod_{j=1}^N \left(\frac{f^{(j)(t)}}{j!} \right)^{N_j}$$

where the sum is taken over all $N-$tuples of nonnegative integers $(N_1, \ldots, N_N)$ that satisfy the constraint

$$N_1 + 2N_2 + \ldots + NN_N = N.$$

Let $f(t) = |y'|^2 + t^2$ and $Q(s) = s^{\frac{-n}{2}}$.
At $t = 0$ all the terms in the preceding formula are zero except when $N = 2N_2$ in which case we get

$$\frac{N!}{N_2!} Q^{(N_2)}\big(f(0)\big).$$

Hence at $x_n = 0, y' \neq 0$, if $N = 2N_2$,
(4.18)

$$\partial_{x_n}^{N+1}\left\{x_n(|y'|^2 + x_n^2)^{\frac{-n}{2}}\right\} = \frac{(-1)^{N_2} N(N!)}{N_2!} \frac{\frac{n}{2}(\frac{n}{2}+1)\ldots(\frac{n}{2}+N_2-1)}{|y'|^{N+n}}$$

while when N is even, at $x_n = 0, y' \neq 0$,

$$\partial_{x_n}^{N}\left\{x_n(|y'|^2 + x_n^2)^{\frac{-n}{2}}\right\} = 0. \tag{4.19}$$

With $A_0 = A_0(0, -y', x_n)$, consider

$$\partial_{x_n}^{N}\left\{\frac{x_n A_0}{(|y'|^2 + x_n^2)^{\frac{n}{2}}}\right\} = \sum_{k=0}^{N} \binom{N}{k} \partial_{x_n}^{k}\left(x_n(|y'|^2 + x_n^2)^{\frac{-n}{2}}\right) \cdot \partial_{x_n}^{N-k} A_0.$$

On a given compact set, by real analyticity, $\exists C > 0$ such that

$$|\partial_{x_n}^{j} A_0| \leq C^{j+1} j!.$$

We thus have

$$\partial_{x_n}^{N}\left\{\frac{x_n A_0}{(|y'|^2 + x_n^2)^{\frac{n}{2}}}\right\}$$
(4.20)
$$= A_0 \partial_{x_n}^{N}\left(x_n\big(|y'|^2 + x_n^2\big)^{\frac{-n}{2}})\right) + \sum_{k=0}^{N-1} \partial_{x_n}^{k}\left(x_n\big(|y'|^2 + x_n^2\big)^{\frac{-n}{2}}\right) \cdot \partial_{x_n}^{N-k} A_0.$$

Note that

$$
\begin{aligned}
&\Big|\sum_{k=0}^{N-1}\binom{N}{k}\partial_{x_n}^k\Big(x_n\big(|y'|^2+x_n^2\big)^{\frac{-n}{2}}\Big)\partial_{x_n}^{N-k}A_0\Big| \\
&\le N!\sum_{k=0,k\text{ odd}}\frac{C^{N-k+1}}{k!}\Big|\partial_{x_n}^k\Big(x_n\big(|y'|^2+x_n^2\big)^{\frac{-n}{2}}\Big)\Big| \\
&= N!\sum_{m=0}^{N_2-1}\frac{C^{N-2m}}{(2m+1)!}\Big|\partial_{x_n}^{2m+1}\Big(x_n\big(|y'|^2+x_n^2\big)^{\frac{-n}{2}}\Big)\Big| \\
&= N!\sum_{m=0}^{N_2-1}\frac{C^{N-2m}}{(2m+1)!}\frac{(2m+1)((2m+1)!)(\frac{n}{2}(\frac{n}{2}+1)\dots(\frac{n}{2}+m-1))}{m!|y'|^{2m+1+n}} \\
&= \frac{N!}{|y'|^{N+n-1}}\sum_{m=0}^{N_2-1}\frac{\big(C|y'|\big)^{N-2m+2}}{m!}\ \frac{n}{2}(\frac{n}{2}+1)\dots(\frac{n}{2}+m-1) \\
&\le \frac{N!\ C^2}{|y'|^{N+n-1}}\sum_{m=0}^{N_2-1}\frac{C_1}{2^m}\big(c|y'|\big)^{N-2m+2} \quad (\text{for some } C_1>0) \\
(4.21)& \\
&\le \frac{N!\ C_2}{|y'|^{N+n-1}}
\end{aligned}
$$

where in the last inequality we chose $|y'| < \frac{1}{2C}$.
From (4.18) and (4.21), when $N = 2k+1$, for some constants $C_1, C_2 > 0$:

$$
C_1\frac{N!}{|y'|^{N+n}} \le |\partial_{x_n}^N k_0| \le C_2\frac{N!}{|y'|^{N+n}}. \tag{4.22}
$$

Consider next $k_1(x,z)\dots k_n(x,z)$ and $\sum_{j=n+1}^{\infty} k_j(x,z)$.
From (4.18) and (4.19), at $x_n = 0, y' \neq 0$, if $N = 2N_2$,
(4.23)

$$
\partial_{x_n}^{N+1}\Big\{x_n\big(|y'|^2+x_n^2\big)^{\frac{-(n-1)}{2}}\Big\} = \frac{(-1)^{N_2}N(N!)}{N_2!}\ \frac{(\frac{n-1}{2})(\frac{n-1}{2}+1)\dots(\frac{n-1}{2}+N_2-1)}{|y'|^{N+n-1}}
$$

while when N is even, at $x_n = 0, y' \neq 0$,

$$
\partial_{x_n}^N\Big\{x_n\big(|y'|^2+x_n^2\big)^{\frac{-(n-1)}{2}}\Big\} = 0. \tag{4.24}
$$

It follows from (4.22), (4.23) and (4.24) that when $N = 2k+1$, for $x_n = 0$ and y' small enough,

$$|\partial_{x_n}^N k_1| \leq \frac{|\partial_{x_n}^N k_0|}{2}.$$

The same inequality holds for $k_2, \ldots, k_n$ and

$$\sum_{j=n+1}^{\infty} k_j(x,z) = |z| E(x,y).$$

Thus for some $c_1, c_2 > 0$, for $y' \neq 0$ and any $N = 2k+1$,

$$c_1 \frac{N!}{|y'|^{N+n}} \leq \left|\partial_{x_n}^N K(0', x_n, y')|_{x_n=0}\right| \leq c_2 \frac{N!}{|y'|^{N+n}} \tag{4.25}$$

Let $\psi = \psi(x') \in C_0^\infty(\mathbb{R}^{n-1})$ have support in $|x'| < r$, $\psi(x') \equiv 1$ on $|x'| \leq \frac{r}{2}$ and $0 \leq \psi \leq 1$, r sufficiently small.
Define the function

$$v(x) = -\int_{\mathbb{R}^{n-1}} K(x,y')\psi(y')u(y',0)dy', \quad x \in B_r^+$$

where u is the solution in Theorem 2.2.

Observe that $Lv = 0$ in B_r^+ and $v(x',0) = u(x',0)$ for $|x'| \leq \frac{r}{2}$. Let $w(x) = u(x) - v(x)$ for $x \in B_r^+$. The function w is a solution of $Lw = 0$ in B_r^+ and $w(x',0) \equiv 0$ for $|x'| \leq \frac{r}{2}$. By the boundary analyticity result in [39], $w(x)$ extends to a real analytic function on $B_\delta(0)$ for some $0 < \delta < r$. The integrability of $|x'|^{-N}u(x',0)$ for all N and estimate (4.25) imply that the function

$$v(0',x_n) = -\int_{\mathbb{R}^{n-1}} K(0',x_n,y')\psi(y')u(y',0)dy'$$

and hence

$$u(0',x_n) = v(0',x_n) + w(0',x_n)$$

are C^∞ up to $x_n = 0$. Since $u(0,x_n)$ is flat at $x_n = 0$, and $u - v = w$ is real analytic on $B_\delta(0)$, we can find a constant $D > 0$ such that for every k,

$$|\partial_{x_n}^{2k+1} v(0)| = |\partial_{x_n}^{2k+1}(u-v)(0)| \leq D^{2k+2}(2k+1)! \tag{4.26}$$

On the other hand, since $u(y',0)\geq 0$, using (4.25), for ϵ small,

$$
\begin{aligned}
|\partial_{x_n}^{2k+1}v(0)| &= \int_{\mathbb{R}^{n-1}} \partial_{x_n}^{2k+1}K(0',x_n,y')|_{x_n=0}\psi(y')u(y',0)dy' \\
&\geq c_1(2k+1)!\int_{\mathbb{R}^{n-1}} \frac{\psi(y')u(y',0)}{|y'|^{2k+n+1}}dy' \\
&\geq c_1(2k+1)!\int_{|y'|<\epsilon} \frac{u(y',0)}{|y'|^{2k+n+1}}dy' \\
&\geq \frac{c_1(2k+1)!}{\epsilon^{2k+n+1}}\int_{|y'|<\epsilon} u(y',0)dy'. \qquad (4.27)
\end{aligned}
$$

The inequalities (4.26) and (4.27) hold for any k. By choosing ϵ small enough (depending only on D) taking the $(2k+n+1)^{th}$ root and letting $k\to\infty$, we conclude that $u(x',0)\equiv 0$ for $|x'|<\epsilon$.

Proof of Corollary 2.2: After flattening ∂D near p by a real analytic diffeomorphism that maps p to the origin, we are in the context of Theorem 2.2 with the additional assumption that u is flat at the origin. By Theorem 2.2, $u(x',0)\equiv 0$ for $|x'|$ small and hence by the result in [39], u extends as a real analytic function on some ball $B_\delta(0)$. Since it is flat at an interior point, $u\equiv 0$ in $B_\delta(0)$ and hence in B_r^+ by analyticity. $\square$

Proof of Corollary 2.4: Let F be a real analytic diffeomorphism from a neighborhood of p to a neighborhood of 0 that maps p to 0 and flattens M near p. Since $\Re(h)$ is the boundary value of the harmonic function u on M^+, in the new coordinates, it is the boundary value of a solution of an elliptic, real analytic differential operator in B_r^+ for some $r>0$. By Theorem 2.2, $\Re(h)\equiv 0$ near p in M.

Suppose now M is not Levi flat. Let $L_j = X_j+\sqrt{-1}Y_j,\ 1\leq j\leq n$ be a basis of the CR vector fields near p. Pick a point q close to p such that M is minimal at q. By minimality, there are two neighborhoods $S_1, S_2,\ S_1\subseteq S_2$ of q in M with the property that every $q'\in S_1$ can be joined to q by a broken path $\Gamma\subseteq S_2$ consisting of a finite number of curves Γ_j where each Γ_j is an integral curve of X_j or Y_j. Since $\Re(h)\equiv 0$ on S_2, $X_j(\Im(h))\equiv 0\equiv Y_j(\Im h)$ on S_2 for all j. It follows that $\Im(h)$ is constant on the Γ_j and hence it is constant on S_1. Thus the boundary value of f is constant near q and so f is constant M^+. $\square$

5. Proofs of Theorem 2.3, Corollary 2.3

Proof of Theorem 2.3*:* In the work [36], the existence of local Poisson kernels K_0, K_1 with the following properties was established: given $\varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^{n-1})$, the functions

$$u_0(x) = \int_{\mathbb{R}^{n-1}} K_0(x, y')\varphi_1(y')dy', \; u_1(x) = \int_{\mathbb{R}^{n-1}} K_1(x, y')\varphi_2(y')dy'$$

satisfy

$$Pu_0 = 0, Pu_1 = 0 \text{ for } x_n > 0; B_1u_0(x', 0) = u_0(x', 0) = \varphi_1(x'), \; B_ju_0(x', 0) = 0;$$

$$B_1u_1(x', 0) = u_1(x', 0) = 0, \; B_ju_1(x', 0) = \varphi_2(x')$$

for x' in a neighborhood of 0 in $\mathbb{R}^{n-1}$.

The arguments in Section 4 show that K_0 and K_1 are Poisson operators and in particular, there exist P_0, P_1 real analytic, classical pseudodifferential operators of order -1 and -2 respectively such that for any $h \in C_0^\infty(\mathbb{R}^{n-1})$,

$$K_0h(x) = r^+P_0\big(h(x') \otimes \delta(x_n)\big), \text{ and } K_1h(x) = r^+P_1\big(h(x') \otimes \delta(x_n)\big),$$

where r^+ is restriction to $x_n > 0$. The argument in Section 4 also show that K_0, K_1 have expansions of the form:

$$K_0(x, y') = \sum_{j=0}^{\infty} k_j^0(x, x' - y', x_n) \text{ and } K_1(x, y') = \sum_{j=0}^{\infty} k_j^1(x, x' - y', x_n).$$

Moreover, since

$$\frac{\partial K^0}{\partial x_n}(x', 0, y') = 0,$$

we get

$$k_0^0 = \frac{x_n^2 A_0(x, z)}{|z|^{n+1}}, \; k_1^0 = \frac{x_n^2 A_1(x, z)}{|z|^n}, \; \ldots, k_{n-2}^0 = \frac{x_n^2 A_{n-2}(x, z)}{|z|^3},$$

$$k_{n-1}^0 = U_0^0(x, z) + Q_0^0(x) \log|z|, \; k_n^0 = U_1^0(x, z) + Q_1^0(x, z) \log|z|,$$

and

$$\sum_{j=n+1}^{\infty} k_j^0(x, z) = |z| E^0(x, z);$$

$$k_0^1(x,z)=\frac{x_nB_0(x,z)}{|z|^{n-1}},\ k_1^1(x,z)=\frac{x_nB_1(x,z)}{|z|^{n-2}},\dots,k_{n-3}^1(x,z)=\frac{x_nB_{n-3}(x,z)}{|z|^2},$$
$$k_{n-2}^1(x,z)=U_0^1(x,z)+Q_0^1(x,z)\log|z|,$$
$$k_{n-1}^1(x,z)=U_1^1(x,z)+Q_1^1(x,z)\log|z|,$$

and

$$\sum_{j=n}^{\infty}k_j^1(x,z)=|z|E^1(x,z).$$

Faá di Bruno's formula this time implies that at $x_n=0, y'\neq 0$,

$$\partial_{x_n}^k\left(\frac{x_n^2}{\left(|y'|^2+x_n^2\right)^{\frac{n-1}{2}}}\right)=0 \text{ when } k \text{ is odd} \tag{5.1}$$

and

$$\partial_{x_n}^{2N}\left(\frac{x_n^2}{\left(|y'|^2+x_n^2\right)^{\frac{n-1}{2}}}\right)=\frac{(2N)!(n-1)(n+1)\dots(n+2N-3)}{(N-1)!2^{N-1}|y'|^{n+2N-3}} \tag{5.2}$$

and so

$$\left|\partial_{x_n}^{2N}\left(\frac{x_n^2}{\left(|y'|^2+x_n^2\right)^{\frac{n-1}{2}}}\right)\right|\geq\frac{(2N)!}{|y'|^{n+2N-3}}. \tag{5.3}$$

This in turn leads to, when $N=2k$,

$$c_1\frac{N!}{|y'|^{n+N-3}}\leq|\partial_{x_n}^N k_0^1|\leq c_2\frac{N!}{|y'|^{n+N-3}}. \tag{5.4}$$

Inequality (5.4) implies that for some $c_1,c_2>0$ for $y'\neq 0$, $x_n=0$, and any $N=2k$,

$$c_1\frac{N!}{|y'|^{n+N-3}}\leq|\partial_{x_n}^N K_1(0',0,y')|\leq c_2\frac{N!}{|y'|^{n+N-3}}. \tag{5.5}$$

Let $\psi=\psi(x')\in C_0^\infty(\mathbb{R}^{n-1})$ be supported in $|x'|<r$, $\psi(x')\equiv 1$ on $|x'|\leq\frac{r}{2}$, $0\leq\psi\leq 1$, r small enough.
Define the function

$$v(x)=\int_{\mathbb{R}^{n-1}}K_0(x,y')\psi(y')u(y',0)dy'+\int_{\mathbb{R}^{n-1}}K_1(x,y')\psi(y')B_ju(y',0)dy'$$

where u is the solution in Theorem 2.3.
We have:

$$Pv=0 \text{ in } B_r^+,\ v(x',0)=u(x',0),\ B_jv(x',0)=B_ju(x',0) \text{ for } |x'|\leq\frac{r}{2}.$$

Then by the results in [39], $u-v$ extends to a real analytic function on $B_\delta(0)$ for some $\delta>0$. We can then argue as in section 4, this time taking derivatives of odd order to conclude $u(x',0) \equiv 0$ (near the origin) and even order to conclude that $B_j u(x',0) \equiv 0$ for x' small. $\square$

Proof of Corollary 2.3: We may assume that $D = B_r^+$, $p = 0$ and Σ is the flat piece of ∂B_r^+. By Theorem 2.3, for x' small,

$$u(x',0) \equiv 0 \equiv \frac{\partial u}{\partial x_n}(x',0).$$

By the results in [39] u extends as a real analytic function to a neighborhood of the origin. By flatness, $u \equiv 0$ in B_r^+. $\square$

Proofs of Theorem 2.4 *and Corollary* 2.5*:* We use the local Poisson kernels of [36] to get Poisson operators K_j, $1 \leq j \leq m$ with the following properties: given $\varphi_j \in C_0^\infty(\mathbb{R}^{n-1})$, $1 \leq j \leq m$, the functions

$$u_j(x) = \int_{\mathbb{R}^{n-1}} K_j(x,y')\varphi_j(y')dy'$$

satisfy

$$Pu_j = 0 \text{ for } x_n > 0; \; B_j u_k(x',0) = \delta_{jk}\varphi_k(x').$$

The rest of the arguments for both the theorem and corollary are the same as those for Theorem 2.3 and Corollary 2.3. $\square$

APPENDIX

In this section, we present a simple proof of the following theorem which can also be proved in the same way for general operators B_j, $0 \leq j \leq m-1$ which satisfy the complementary boundary conditions of [2]:

Theorem 5.1. *Let $D \subseteq \mathbb{R}^n$ be a bounded domain with C^{2m} boundary, $p \in \partial D$. Suppose $\Sigma \subseteq \partial D$ is a neighborhood of p that is a real analytic hypersurface. Let $P = \sum_{\alpha \leq 2m} a_\alpha(x)\partial_x^\alpha$ be a real analytic, elliptic partial differential operator. Let f_j, $0 \leq j \leq m-1$ be C^{2m} functions on ∂D. There is u that is a solution of $Pu = 0$ near p in D and $\partial_\nu^j u = f_j$, $0 \leq j \leq m-1$ near p in Σ.*

Proof. We begin by showing first that given $g \in L^2(D)$, there is a solution $w \in H_0^m(D)$ that solves $Pw = g$ near p in Σ. To see this, define

$$V = \{h \in L^2(D) : \exists u \in H_0^m(D), \, Pu = h\}.$$

Then from the Fredholm Alternative (see [1]), if $N =$ the subset of $H_0^m(D)$ that is in the kernel of the adjoint of P,

$$L^2(D) = V \oplus N.$$

We decompose g as $g = g_1 + g_2 = P(u_1) + g_2$ with $u_1 \in H_0^m(D)$ and $g_2 \in N \cap H_0^m(D)$. Since P is elliptic and real analytic, g_2 is real analytic in D. Moreover, since it belongs to $H_0^m(D)$, by the boundary analyticity theorem of Morrey and Nirenberg, g_2 extends as a real analytic function to a neighborhood of p. We then use the Cauchy-Kovalevska theorem to get u_2 real analytic on a neighborhood of p that solves: $P(u_2) = g_2$, and u_2 extends to a function we still call $u_2 \in H_0^m(D)$. Thus $w = u_1 + u_2$ solves $Pw = g$ near p in D and $w \in H_0^m(D)$. To finish the proof of the theorem, let $F \in C^{2m}(\overline{D})$ such that $\partial_\nu^j F = f_j$, $0 \leq j \leq m-1$ near p in Σ. By what we just established, there is h that satisfies

$$Ph = -PF \quad \text{near } p \in D, \; h \in H_0^m(D).$$

Set $u = h + F$. Then $Pu = 0$ near p in D and $\partial_\nu^j u = f_j$ near p in Σ. $\square$

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