GLOBAL L-PACKETS OF QUASISPLIT $GSp(2n)$ AND $GO(2n)$

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ABSTRACT. This is a sequel to [Xu18] on the L-packets of quasisplit general symplectic and even orthogonal groups. We show the existence of global L-packets and establish the functoriality of endoscopic transfer for them in many cases.

1. INTRODUCTION

Let F be a number field and G be a quasisplit symplectic or special even orthogonal group over F. Let $\mathbb{A}_F$ be the adèle ring of F. We fix an automorphism θ_0 of G preserving an F-splitting. It induces a dual automorphism $\widehat{\theta_0}$ on the dual group $\widehat{G}$. When G is symplectic, θ_0 is trivial. When G is special even orthogonal, we require θ_0 to be the unique nontrivial outer automorphism induced from the conjugation of the full orthogonal group. We say two irreducible admissible representations of $G(\mathbb{A}_F)$ are θ_0-conjugate if they are θ_0-conjugate at every place. In [Art13] Arthur proved the discrete automorphic spectrum of $G(\mathbb{A}_F)$ can be decomposed

$$L^2_{disc}(G(F)\backslash G(\mathbb{A}_F)) = \bigoplus_{\psi\in\bar{\Psi}_2(G)} L^2_{disc,\psi}(G(F)\backslash G(\mathbb{A}_F))$$

according to the set $\bar{\Psi}_2(G)$ of $\widehat{\theta_0}$-conjugacy classes of discrete Arthur parameters of G. For each $\psi\in\bar{\Psi}_2(G)$, Arthur associated a multi-set $\bar{\Pi}_\psi$ of isomorphism classes of irreducible admissible representations of $G(\mathbb{A}_F)$ modulo θ_0-conjugacy, and proved that

$$L^2_{disc,\psi}(G(F)\backslash G(\mathbb{A}_F)) = \bigoplus_{\substack{[\pi]\in\bar{\Pi}_\psi\\ \langle\cdot,\pi\rangle=\epsilon_\psi}} m_\psi\,\pi$$

modulo θ_0-conjugacy, where ϵ_ψ is a character of certain two-group $\mathcal{S}_\psi$ associated with ψ and there is a map

$$\bar{\Pi}_\psi \to \widehat{\mathcal{S}_\psi},\quad [\pi]\mapsto\langle\cdot,\pi\rangle.$$

Let $\widetilde{G}$ be the group of symplectic or orthogonal similitudes over F, whose derived group is G. We can extend θ_0 to $\widetilde{G}$. Let $\tilde{\zeta}$ be a character of $Z_{\widetilde{G}}(\mathbb{A}_F)/Z_{\widetilde{G}}(F)$ and ζ be the restriction to $Z_G(\mathbb{A}_F)$. Then we have

$$\mathrm{Ind}_{\widetilde{G}(F)Z_{\widetilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\widetilde{G}(\mathbb{A}_F)} L^2_{disc}(G(F)\backslash G(\mathbb{A}_F),\zeta)\cong L^2_{disc}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\tilde{\zeta})$$

(cf. [Xu18, Lemma 5.3]). So we can decompose

$$L^2_{disc}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\tilde{\zeta}) = \bigoplus_{\psi\in\bar{\Psi}_2(G,\zeta)} L^2_{disc,\psi}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\tilde{\zeta})$$

where $\bar{\Psi}_2(G,\zeta)\subseteq\bar{\Psi}_2(G)$ is associated with the central character ζ and

$$L^2_{disc,\psi}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\tilde{\zeta}) := \mathrm{Ind}_{\widetilde{G}(F)Z_{\widetilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\widetilde{G}(\mathbb{A}_F)} L^2_{disc,\psi}(G(F)\backslash G(\mathbb{A}_F),\zeta).$$

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In this paper, we will focus on the subset of generic parameters $\bar{\Phi}_2(G,\zeta)=\bar{\Psi}_2(G,\zeta)\cap\bar{\Phi}(G)$. Let $\tilde{\bar{\Pi}}_{\phi,\widetilde{\zeta}}$ be the set of isomorphism classes of irreducible admissible representations of $\widetilde{G}(\mathbb{A}_F)$ modulo θ_0-conjugacy with central character $\widetilde{\zeta}$, whose restriction to $G(\mathbb{A}_F)$ have irreducible constituents contained in $\bar{\Pi}_\phi$. Let

$$Y=\mathrm{Hom}(\widetilde{G}(\mathbb{A}_F)/\widetilde{G}(F)Z_{\widetilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F),\mathbb{C}^\times),\quad X=\mathrm{Hom}(\widetilde{G}(\mathbb{A}_F)/Z_{\widetilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F),\mathbb{C}^\times).$$

There is a natural group homomorphism $\alpha:\mathcal{S}_\phi\to Y$ (cf. (2.19)), so that $\pi\otimes\omega\cong\pi$ for any $\omega\in\alpha(\mathcal{S}_\phi)$ and $[\pi]\in\tilde{\bar{\Pi}}_{\phi,\widetilde{\zeta}}$. Define $\mathcal{S}_{\tilde{\phi}}:=\mathrm{Ker}\,\alpha$, and

$$\tilde{\bar{\Pi}}_{\phi,\widetilde{\zeta}}\to\widehat{\mathcal{S}_{\tilde{\phi}}},\quad[\tilde{\pi}]\mapsto\langle\cdot,\tilde{\pi}\rangle$$

by

$$\langle\cdot,\tilde{\pi}\rangle=\langle\cdot,\pi\rangle|_{\mathcal{S}_{\tilde{\phi}}}$$

where π is any irreducible constituent in the restriction of $\tilde{\pi}$ to $G(\mathbb{A}_F)$. By [Xu18, Proposition 5.11],

$$L^2_{disc,\phi}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\widetilde{\zeta})=m_\phi\sum_{\omega\in Y/\alpha(\mathcal{S}_\phi)}\sum_{\substack{[\tilde{\pi}]\in\tilde{\bar{\Pi}}_{\phi,\widetilde{\zeta}}/X\\ \langle\cdot,\tilde{\pi}\rangle=1}}\tilde{\pi}\otimes\omega,\tag{1.1}$$

modulo θ_0-conjugacy, where $\tilde{\pi}$ are taken to be the representatives of $\tilde{\bar{\Pi}}_{\phi,\widetilde{\zeta}}/X$ in the discrete automorphic spectrum of $\widetilde{G}(\mathbb{A}_F)$. Our first main result gives a refined decomposition of (1.1).

Theorem 1.1. *For any $\phi\in\bar{\Phi}_2(G)$, there exists a global packet $\bar{\Pi}_{\tilde{\phi}}$ of isomorphism classes of irreducible admissible representations for $\widetilde{G}(\mathbb{A}_F)$ modulo θ_0-conjugacy, unique up to twisting by Y, such that*

$$\bar{\Pi}_{\tilde{\phi}}=\otimes'_v\bar{\Pi}_{\tilde{\phi}_v}$$

where $\bar{\Pi}_{\tilde{\phi}_v}$ is defined in [Xu18, Theorem 4.6]*, and*

$$L^2_{disc,\phi}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\widetilde{\zeta})=m_\phi\bigoplus_{\omega\in Y/\alpha(\mathcal{S}_\phi)}\bigoplus_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega\\ \langle\cdot,\tilde{\pi}\rangle=1}}\tilde{\pi}$$

modulo θ_0-conjugacy.

This result was originally announced in the author's thesis [Xu14, Theorem 5.3.1], but there is an error in the proof. It is due to the application of a comparison formula (cf. [Xu14, Lemma 5.4.5], [Xu18, Lemma 5.25]) in the induction argument, whose validity depends on the assumption of Conjecture 4.7. This conjecture was only proved in certain cases (cf. [Xu14, Theorem 5.3.5], [Xu18, Theorem 5.21]). In [Xu18], we reproduce the proof of Theorem 1.1 in the stable case, i.e., $\mathcal{S}_{\tilde{\phi}}=1$. In the current work, we are able to prove this in all cases by establishing a weak form of this conjecture (cf. Theorem 4.8).

When G is special even orthogonal, Theorem 1.1 only construct the global L-packets of $\widetilde{G}$ modulo some equivalence relation. In this case, let $\widetilde{G}^{\Sigma_0}:=\widetilde{G}\rtimes\langle\theta_0\rangle$, which is isomorphic to the even orthogonal similitude group. Inspired by the works of Atobe-Gan [AG17] for even orthogonal groups, we can define the local L-packet $\Pi^{\Sigma_0}_{\tilde{\phi}_v}$ of $\widetilde{G}^{\Sigma_0}(F_v)$ to be the set of irreducible representations, whose restrictions to $\widetilde{G}(F_v)$ are contained in $\bar{\Pi}_{\tilde{\phi}_v}$. Then the global L-packets of $\widetilde{G}^{\Sigma_0}$ can be defined to be $\Pi^{\Sigma_0}_{\tilde{\phi}}=\otimes'_v\Pi^{\Sigma_0}_{\tilde{\phi}_v}$ and they are expected to appear in some version of Arthur's multiplicity formula for the automorphic representations of $\widetilde{G}^{\Sigma_0}$ (cf. [AG17, Theorem 7.1]). This means that $\bar{\Pi}_{\tilde{\phi}}$ determines a global L-packet for $\widetilde{G}^{\Sigma_0}$ and hence justifies our title.

Our second main result is on the functoriality of endoscopic transfer. Let $\widetilde{H}$ be an elliptic endoscopic group of $\widetilde{G}$. Then the derived group H of $\widetilde{H}$ is an elliptic endoscopic group of G and it is isomorphic to $G_1\times G_2$, where G_i is a quasisplit symplectic or special even orthogonal group. Here we also allow G_i to

be trivial. Let $\widetilde{G}_i$ be the corresponding group of similitudes and λ_i the similitude character. In case G_i is trivial, let $\widetilde{G}_i = \mathbb{G}_m$ and $\lambda_i = id$. Then

$$\widetilde{H} \cong \Big\{(g_1, g_2) \in \widetilde{G}_1 \times \widetilde{G}_2 \,|\, \lambda_1(g_1) = \lambda_2(g_2)\Big\}.$$

For $\phi_i \in \bar{\Phi}_2(G_i)$, let $\phi_H := \phi_1 \times \phi_2$ and we define a global L-packet $\bar{\Pi}_{\tilde{\phi}_H}$ of $\tilde{H}$ to be the restriction of $\bar{\Pi}_{\tilde{\phi}_1} \otimes \bar{\Pi}_{\tilde{\phi}_2}$. Let $\phi := \phi_1 \boxplus \phi_2 \in \bar{\Psi}_2(G)$ (cf. (2.15)). Then we define the global endoscopic transfer of $\bar{\Pi}_{\tilde{\phi}_H}$ to be

$$\mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}_H} := \otimes'_v \mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}_{H,v}}$$

where $\mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}_{H,v}} := \bar{\Pi}_{\tilde{\phi}_v}$ is the local L-packet transfered from $\bar{\Pi}_{\tilde{\phi}_{H,v}}$ through the local character relation (cf. [Xu18, Theorem 4.6]). It follows from Theorem 1.1 that

$$\mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}_H} = \bar{\Pi}_{\tilde{\phi}} \otimes \omega$$

for some $\omega \in X$. The functoriality of endoscopic transfer in this case means that ω can be chosen in Y. We prove this under certain technical assumption.

Theorem 1.2. *Suppose either ϕ_1 or ϕ_2 does not contain GL-type orthogonal simple parameters (cf. Definition 4.4), then*

$$\mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}_H} = \bar{\Pi}_{\tilde{\phi}} \otimes \omega$$

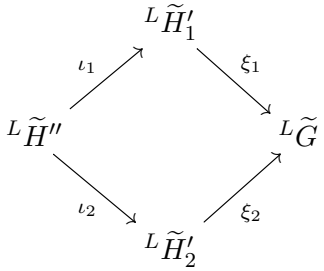
for some $\omega \in Y$.

This result strengthens [Xu14, Theorem 7.4.3] and [Xu18, Lemma 6.27]. To give a few examples, the assumption in this theorem is always satisfied in the following cases:

- $\widetilde{G} = GSp(4), GSp(6), GSO(4,\eta), GSO(6,\eta)$, where η is a quadratic idele class character;
- $\widetilde{G} = GSO(8,\eta)$ for $\eta \neq 1$;
- $\widetilde{G} = GSp(2n)$ and $\widetilde{H} = GSO(2n)$.

At the request of the referee, we make a remark on the non-generic case. One of the main difficulties for extending these theorems to the non-generic case is the construction of local Arthur packets. This has been solved in the nonarchimedean case in the author's recent work [Xu21]. The solution requires fine structural information of the local Arthur packets for the classical groups, which is still not available in the archimedean case. We also expect certain knowledge of the residual spectrum would be helpful.

This paper is organized as follows. In Section 2, we review our earlier work [Xu18] on both the local and global theory about $\widetilde{G}$. In Section 3, we review the relevant stabilized twisted trace formulas. In Section 4, we formulate the main results of this paper in their most general forms. In particular, we also consider the twisted version of Theorem 1.2. In Section 5, we formulate some comparison formulas. In Section 6, we prove the main results for parameters consisting of orthogonal simple parameters of GL-type (cf. Definition 4.4). In Section 7, we prove the comparison formulas following the same strategy as that of [Xu18, Lemma 5.25]. In Section 8, we show that the global L-packets of G has compatible lifting to $\widetilde{G}$ (cf. Corollary 8.8). The notion of compatible lifting has been introduced in [Xu14, Section 7.3]. It has the following implication. Let $\widetilde{H}'_1, \widetilde{H}'_2$ be endoscopic groups of $\widetilde{G}$. If $\widetilde{H}''$ is an endoscopic group for both $\widetilde{H}'_1, \widetilde{H}'_2$ such that the following diagram of endoscopic embeddings commutes.



Then

$$\mathrm{Tran}_{\xi_1}\circ\mathrm{Tran}_{\iota_1}\bar{\Pi}_{\tilde{\phi}_{H''}}=\mathrm{Tran}_{\xi_2}\circ\mathrm{Tran}_{\iota_2}\bar{\Pi}_{\tilde{\phi}_{H''}} \tag{1.2}$$

for any global L-packet $\bar{\Pi}_{\tilde{\phi}_{H''}}$ of $\widetilde{H}''$. This is essentially a local statement. It is not trivial since one can not compose the geometric endoscopic transfer maps. It provides a reduction for the proof of Theorem 1.2. As another application, we construct a bijection between local Langlands parameters and local L-packets for $\widetilde{G}$ in the nonarchimedean case such that it is compatible with the endoscopic transfers (cf. Theorem 8.12 and [Xu14, Theorem 7.3.5]). This construction depends on certain choices, so it is not canonical. Nevertheless, it will suffice for our construction of the local Arthur packets for $\widetilde{G}$ in a forthcoming work. It is also necessary to consider (1.2) for the twisted endoscopic groups. To do so, we have claimed incorrectly in [Xu14, Section 7.3] that there is a canonical lift of endoscopic embedding from an endoscopic group of G to a twisted endoscopic group of $\widetilde{G}$. In fact, that construction has an obstruction by a 2-cycle of the Galois group in the central torus $\widehat{D}$ of the dual group in the case when $\widetilde{G}=GSO(2n,\eta)$ for $\eta\neq 1$. In the current paper, we fix this error by introducing a 1-cochain of the Weil group in $\widehat{D}$, which splits the 2-cocycle. It is the fact that we can not make a canonical choice for the twisted endoscopic embedding for $\widetilde{G}$ makes the discussion in this section complicated. In the last section, we prove the main results in the cases left by Section 6. The proof of Theorem 1.1 is similar to that of [Xu18, Theorem 5.21]. In the proof of Theorem 1.2, we introduce some auxiliary global parameters by doubling certain simple parameter. This is motivated by some arguments in [Art13]. The proof is similar to that of [Xu14, Theorem 7.4.3].

2. L-packets

Let F be a local or global field of characteristic zero and $\bar{F}$ be its algebraic closure. When F is global, let us denote the adèle ring of F by $\mathbb{A}_F$, the idèle group by I_F. Let Γ_F or Γ be the absolute Galois group of F and W_F the Weil group. For any quasisplit connected reductive group G over F, we denote by Z_G its center, by A_G the split connected component of Z_G, by $\widehat{G}$ its complex dual group, by $Z(\widehat{G})$ the centre of $\widehat{G}$, and by ${}^LG$ its L-group, which is a semidirect product of $\widehat{G}$ with the Weil group W_F, i.e., $\widehat{G}\rtimes W_F$. Let $X^*(G)$ be the group of algebraic characters of G over F and $\mathfrak{a}_G=\mathrm{Hom}_{\mathbb{Z}}(X^*(G),\mathbb{R})$.

Let $G\subseteq\widetilde{G}$ be two quasisplit connected reductive groups over F such that they have the same derived group. Then $\widetilde{G}/G$ is a torus, denoted by D. There is an exact sequence

$$1\longrightarrow G\longrightarrow\widetilde{G}\xrightarrow{\lambda}D\longrightarrow 1. \tag{2.1}$$

On the dual side, we have

$$1\longrightarrow\widehat{D}\longrightarrow\widehat{\widetilde{G}}\xrightarrow{\mathbf{p}}\widehat{G}\longrightarrow 1,$$

where all the homomorphisms can be extended to L-homomorphisms of L-groups. Let Σ be a finite abelian group of F-automorphisms of $\widetilde{G}$ preserving a fixed F-splitting of $\widetilde{G}$. We will always assume that λ is Σ-invariant. This implies that Σ also acts on G. Let $\widetilde{G}^{\Sigma}=\widetilde{G}\rtimes\Sigma$ and $G^{\Sigma}=G\rtimes\Sigma$. Since Σ induces dual automorphisms on $\widehat{\widetilde{G}}$ and $\widehat{G}$, we denote this group by $\widehat{\Sigma}$ and define $\widehat{\widetilde{G}}^{\Sigma}=\widehat{\widetilde{G}}\rtimes\widehat{\Sigma}$ and $\widehat{G}^{\Sigma}=\widehat{G}\rtimes\widehat{\Sigma}$.

As in [Xu18] we are mainly concerned with the case that G is a quasisplit symplectic or special even orthogonal group and $\widetilde{G}$ is the corresponding similitude group. We denote the split symplectic and special even orthogonal group of rank n by $Sp(2n)$ and $SO(2n)$ respectively. We also denote the outer twist of $SO(2n)$ with respect to a quadratic extension E/F by $SO(2n,\eta_{E/F})$, where $\eta_{E/F}$ is the quadratic (idèle class) character associated to E/F by the local (global) class field theory. Denote $\eta_G=\eta_{E/F}$. The corresponding similitude groups can be defined as follows.

$$GSp(2n)=(\mathbb{G}_m\times Sp(2n))/(\mathbb{Z}/2\mathbb{Z})\ \text{ and }\ GSO(2n,\eta_{E/F})=(\mathbb{G}_m\times SO(2n,\eta_{E/F}))/(\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z}$ is embedded diagonally into the centre of each factor. The similitude character λ is square on $\mathbb{G}_m$ and trivial on the other factor. We fix an automorphism θ_0 of G preserving an F-splitting. When G is symplectic, we require θ_0 to be trivial. When G is special even orthogonal, we require θ_0 to be the

unique nontrivial outer automorphism induced from the conjugation of the full orthogonal group. Clearly, $\theta_0^2 = 1$, θ_0 extends to $\widetilde{G}$ by acting trivially on $Z_{\widetilde{G}}$, and λ is θ_0-invariant. Let $\Sigma_0 = \langle \theta_0 \rangle$. For our induction arguments in the proofs, we will also need to consider

$$G = G_1 \times \cdots \times G_q \tag{2.2}$$

where G_i is a quasisplit symplectic or special even orthogonal group. We define

$$\widetilde{G} = (\mathbb{G}_m \times G_1 \times G_2 \times \cdots \times G_q)/(\mathbb{Z}/2\mathbb{Z}),$$

where $\mathbb{Z}/2\mathbb{Z}$ is embedded diagonally into the center of each factor. We also define a character λ of $\widetilde{G}$, which is square on $\mathbb{G}_m$ and trivial on the other factors. We define a group of automorphisms of G by taking the product of Σ_0 on each factor, and we denote this group again by Σ_0. We can extend Σ_0 to $\widetilde{G}$ by the trivial action on $Z_{\widetilde{G}}$. We can also view $\widetilde{G}$ as a subgroup of $\widetilde{G}_1 \times \cdots \times \widetilde{G}_q$ by

$$\widetilde{G} \cong \Big\{ (g_i) \in \prod_i \widetilde{G}_i \,|\, \lambda_1(g_1) = \cdots = \lambda_q(g_q) \Big\}.$$

For admissible representations $\tilde{\pi}_i$ of $\widetilde{G}_i(F)$ in the local case (or $\widetilde{G}_i(\mathbb{A}_F)$ in the global case), we define the restriction of $\otimes_i \tilde{\pi}_i$ to $\widetilde{G}$ by $\widetilde{\otimes}_i \tilde{\pi}_i$.

2.1. **Local theory.** Suppose F is local. The local Langlands group is defined as follows

$$L_F = \begin{cases} W_F & \text{if } F \text{ is archimedean,} \\ W_F \times SL(2,\mathbb{C}) & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

Let G be a quasisplit connected reductive group over F. A local Langlands parameter ϕ is a $\widehat{G}$-conjugacy class of admissible homomorphisms from L_F to ${}^L G$ (cf. [Bor79]). Let $\Phi(G)$ be the set of local Langlands parameters and $\Phi_{bdd}(G)$ be the subset of bounded parameters, i.e., the closure of $\phi(W_F)$ is compact. Let $\Pi(G(F))$ be the set of isomorphism classes of irreducible admissible representations of $G(F)$ and $\Pi_{temp}(G(F))$ be the subset of tempered representations. If χ is a quasicharacter of a closed subgroup Z_F of $Z_G(F)$, we define $\mathcal{H}(G,\chi)$ to be the space of χ^{-1}-equivariant smooth functions on $G(F)$ with compact support modulo Z_F.

For $G \subseteq \widetilde{G}$ as in (2.1), the projection $\mathbf{p}$ induces a surjection $\Phi(\widetilde{G}) \to \Phi(G)$ (cf. [Xu18, Section 2.2]). For $\phi \in \Phi(G)$, let L_F act on $\widehat{D}$, $\widehat{G}^\Sigma$, and $\widehat{\widetilde{G}}^\Sigma$ by conjugation through ϕ. We denote the corresponding group cohomology by $H^*_\phi(L_F, \cdot)$. Then $H^0_\phi(L_F, \widehat{D}) = \widehat{D}^\Gamma$ and $H^1_\phi(L_F, \widehat{D}) = H^1(W_F, \widehat{D})$. We define

$$S^\Sigma_\phi := \mathrm{Cent}(\mathrm{Im}\phi, \widehat{G}^\Sigma) = H^0_\phi(L_F, \widehat{G}^\Sigma), \quad S^\Sigma_{\tilde{\phi}} := \mathrm{Cent}(\mathrm{Im}\tilde{\phi}, \widehat{\widetilde{G}}^\Sigma) = H^0_\phi(L_F, \widehat{\widetilde{G}}^\Sigma).$$

The short exact sequence

$$1 \longrightarrow \widehat{D} \longrightarrow \widehat{\widetilde{G}}^\Sigma \longrightarrow \widehat{G}^\Sigma \longrightarrow 1$$

induces a long exact sequence

$$1 \longrightarrow \widehat{D}^\Gamma \longrightarrow S^\Sigma_{\tilde{\phi}} \longrightarrow S^\Sigma_\phi \xrightarrow{\delta} H^1(W_F, \widehat{D}),$$

and hence we get

$$1 \longrightarrow S^\Sigma_{\tilde{\phi}}/\widehat{D}^\Gamma \xrightarrow{\iota} S^\Sigma_\phi \xrightarrow{\delta} H^1(W_F, \widehat{D}). \tag{2.3}$$

Taking the quotient of (2.3) by $Z(\widehat{G})^\Gamma$, we get

$$1 \longrightarrow \bar{S}^\Sigma_{\tilde{\phi}} \xrightarrow{\iota} \bar{S}^\Sigma_\phi \xrightarrow{\bar{\delta}} \bar{H}^1(W_F, \widehat{D}), \tag{2.4}$$

where $\bar{S}_{\tilde{\phi}}^{\Sigma}=S_{\tilde{\phi}}^{\Sigma}/Z(\widehat{\widetilde{G}})^{\Gamma}$, $\bar{S}_{\phi}^{\Sigma}=S_{\phi}^{\Sigma}/Z(\widehat{G})^{\Gamma}$ and $\bar{H}^1(W_F,\widehat{D})=H^1(W_F,\widehat{D})/\delta(Z(\widehat{G})^{\Gamma})$. Since $\mathrm{Im}\delta$ is finite (cf. [Xu18, Lemma 2.1]), then $\bar{S}_{\tilde{\phi}}^{0}=\bar{S}_{\phi}^{0}$, which are the identity components. After taking the quotient of (2.4) by the identity components, we get

$$1\longrightarrow\mathcal{S}_{\tilde{\phi}}^{\Sigma}\xrightarrow{\iota}\mathcal{S}_{\phi}^{\Sigma}\xrightarrow{\bar{\delta}}\bar{H}^1(W_F,\widehat{D}),\tag{2.5}$$

where $\mathcal{S}_{\tilde{\phi}}^{\Sigma}=\bar{S}_{\tilde{\phi}}^{\Sigma}/\bar{S}_{\tilde{\phi}}^{0}$ and $\mathcal{S}_{\phi}^{\Sigma}=\bar{S}_{\phi}^{\Sigma}/\bar{S}_{\phi}^{0}$. There are natural maps from $S_{\phi}^{\Sigma},\bar{S}_{\phi}^{\Sigma}$, and $\mathcal{S}_{\phi}^{\Sigma}$ to $\widehat{\Sigma}$, and for $\theta\in\Sigma$, we denote the preimages of $\widehat{\theta}\in\widehat{\Sigma}$ by $S_{\phi}^{\theta},\bar{S}_{\phi}^{\theta}$ and $\mathcal{S}_{\phi}^{\theta}$ respectively. To understand $\bar{H}^1(W_F,\widehat{D})$, we also take

$$1\longrightarrow\widehat{D}\longrightarrow Z(\widehat{\widetilde{G}})\longrightarrow Z(\widehat{G})\longrightarrow 1$$

and it induces

$$Z(\widehat{G})^{\Gamma}\xrightarrow{\delta}H^1(W_F,\widehat{D})\longrightarrow H^1(W_F,Z(\widehat{\widetilde{G}}))\longrightarrow H^1(W_F,Z(\widehat{G}))\longrightarrow H^2(W_F,\widehat{D})=1.$$

So

$$\bar{H}^1(W_F,\widehat{D})\cong\mathrm{Im}\{H^1(W_F,\widehat{D})\to H^1(W_F,Z(\widehat{\widetilde{G}}))\}$$

and

$$1\longrightarrow\bar{H}^1(W_F,\widehat{D})\longrightarrow H^1(W_F,Z(\widehat{\widetilde{G}}))\longrightarrow H^1(W_F,Z(\widehat{G}))\longrightarrow 1.\tag{2.6}$$

By the isomorphism $H^1(W_F,Z(\widehat{\widetilde{G}}))\longrightarrow\mathrm{Hom}(\widetilde{G}(F),\mathbb{C}^{\times})$ [Xu16, Appendix A], we get an isomorphism $r:\bar{H}^1(W_F,\widehat{D})\to\mathrm{Hom}(\widetilde{G}(F)/G(F),\mathbb{C}^{\times})$ [Xu18, Section 2.2]. Let us denote $r\circ\bar{\delta}$ by α.

Suppose G is (2.2). We denote the set of Σ_0-orbits in $\Pi(G(F))$ by $\bar{\Pi}(G(F))$ and the set of $\widehat{\Sigma}_0$-orbits in $\Phi(G)$ by $\bar{\Phi}(G)$. Similarly, we can define $\bar{\Pi}_{temp}(G(F))$, $\bar{\Phi}_{bdd}(G)$, and analogues of these sets for $\widetilde{G}$. We denote by $\bar{\mathcal{H}}(G,\chi)$ (resp. $\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi})$) the subspace of Σ_0-invariant functions in $\mathcal{H}(G,\chi)$ (resp. $\mathcal{H}(\widetilde{G},\widetilde{\chi})$). By the local Langlands correspondence for classical groups cf. [Art13, Theorem 1.5.1], we can associate any $[\phi]\in\bar{\Phi}_{bdd}(G)$ with a finite subset $\bar{\Pi}_{\phi}$ of $\bar{\Pi}_{temp}(G(F))$ such that

$$f(\phi):=\sum_{[\pi]\in\bar{\Pi}_{\phi}}f_G(\pi),\qquad f\in\bar{\mathcal{H}}(G)$$

is stable and

$$\bar{\Pi}_{temp}(G(F))=\bigsqcup_{[\phi]\in\bar{\Phi}_{bdd}(G)}\bar{\Pi}_{\phi}.\tag{2.7}$$

Moreover, with respect to a fixed Σ_0-stable Whittaker datum of G, there is a canonical pairing between $\bar{\Pi}_{\phi}$ and $\mathcal{S}_{\phi}$, which induces an inclusion of $\bar{\Pi}_{\phi}$ into the set of irreducible characters $\widehat{\mathcal{S}_{\phi}}$ of $\mathcal{S}_{\phi}$

$$\bar{\Pi}_{\phi}\longrightarrow\widehat{\mathcal{S}_{\phi}},\quad\pi\mapsto\langle\cdot,\pi\rangle$$

such that the generic representation is sent to the trivial character. For $[\phi]\in\bar{\Phi}_{bdd}(G)$, the lift $\tilde{\phi}$ is unique up to twists by $\bar{H}^1(W_F,\widehat{D})$ which will be identified with $\mathrm{Hom}(\widetilde{G}(F)/G(F),\mathbb{C}^{\times})$. For any subgroup $\Sigma\subseteq\Sigma_0$, we get from (2.5)

$$1\longrightarrow\mathcal{S}_{\tilde{\phi}}^{\Sigma}\xrightarrow{\iota}\mathcal{S}_{\phi}^{\Sigma}\xrightarrow{\alpha}\mathrm{Hom}(\widetilde{G}(F)/G(F),\mathbb{C}^{\times})\ .\tag{2.8}$$

In [Xu18, Theorem 4.6] we have shown that there exists a lift $\bar{\Pi}_{\tilde{\phi}}\subseteq\bar{\Pi}_{temp}(\widetilde{G}(F))$ unique up to twisting by $\mathrm{Hom}(\widetilde{G}(F)/G(F),\mathbb{C}^{\times})$ such that

$$\tilde{f}(\tilde{\phi}):=\sum_{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}}\tilde{f}_{\widetilde{G}}(\tilde{\pi}),\qquad\tilde{f}\in\bar{\mathcal{H}}(\widetilde{G})$$

is stable. Moreover, we have shown for $\omega \in \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times)$ and any $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$,

$$[\tilde{\phi} \otimes \omega] = [\tilde{\phi}] \Leftrightarrow [\tilde{\pi} \otimes \omega] = [\tilde{\pi}] \Leftrightarrow \omega \in \alpha(\mathcal{S}_\phi^{\Sigma_0}).$$

As a consequence,

$$\bar{\Pi}_{temp}(\widetilde{G}(F)) = \bigsqcup_{[\phi] \in \bar{\Phi}_{bdd}(G)} \bigsqcup_{\omega \in \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times)/\alpha(\mathcal{S}_\phi^{\Sigma_0})} \bar{\Pi}_{\tilde{\phi}} \otimes \omega. \tag{2.9}$$

We can define

$$\bar{\Pi}_{\tilde{\phi}} \longrightarrow \widehat{\mathcal{S}_{\tilde{\phi}}}, \quad \langle \cdot, \tilde{\pi} \rangle := \langle \cdot, \pi \rangle|_{\mathcal{S}_{\tilde{\phi}}}$$

for any $\pi \subseteq \tilde{\pi}|_G$. For $[\phi] \in \bar{\Phi}_{bdd}(G)$, $\theta \in \Sigma_0$ and any semisimple element $s \in \bar{S}_\phi^\theta$, let $\widehat{G}' := \mathrm{Cent}(s, \widehat{G})^0$ and it can be equipped with a Galois action given by ϕ. This determines a quasisplit connected reductive group G', and ϕ will factor through ${}^L G'$ for some θ-twisted endoscopic datum (G', s, ξ) of G, and hence we get a parameter $\phi' \in \bar{\Phi}(G')$. In this way, we call (G', ϕ') corresponds to (ϕ, s), and denote this relation by $(G', \phi') \to (\phi, s)$. By [Xu18], (G', s, ξ) can be lifted to a (θ, ω)-twisted endoscopic datum $(\widetilde{G}', \widetilde{s}, \widetilde{\xi})$ of $\widetilde{G}$ for some $\omega \in \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times)$. In sum, we have the following diagram

$$\begin{array}{ccc} {}^L\widetilde{G}' & \xrightarrow{\widetilde{\xi}} & {}^L\widetilde{G} \\ \downarrow & & \downarrow \\ {}^L G' & \xrightarrow{\xi} & {}^L G. \end{array} \tag{2.10}$$

There is a decomposition $G' \cong M_l \times G'_-$, where M_l is a product of general linear groups and G'_- is as (2.2) (cf. [Xu18, Section 2]). Correspondingly, we can decompose $\phi' = \phi_l \times \phi'_-$ and define $\bar{\Pi}_{\phi'} = \Pi_{\phi_l} \otimes \bar{\Pi}_{\phi'_-}$, where Π_{ϕ_l} is the singleton L-packet given by the local Langlands correspondence for general linear groups [HT01] [Hen00] [Sch13]. Let x be the image of s in $\mathcal{S}_\phi^\theta$. Arthur established the following character relation

$$f^{G'}(\phi') = \sum_{[\pi] \in \bar{\Pi}_\phi} \langle x, \pi^+ \rangle f_{G^\theta}(\pi), \quad f \in \bar{\mathcal{H}}(G)$$

(cf. [Art13, Theorem 2.2.1 and Theorem 2.2.4]), where $f^{G'}$ the Langlands-Shelstad-Kottwitz transfer of f, $f_{G^\theta}(\pi) := \mathrm{tr}\, \pi(f) \circ A_\pi(\theta)$ for an intertwining operator $A_\pi(\theta)$ between π and π^θ of order two. The choice of $A_\pi(\theta)$ determines an extension π^+ of π to $G(F)^+ := G(F) \rtimes \langle \theta \rangle$, which corresponds to the extension $\langle \cdot, \pi^+ \rangle$ of $\langle \cdot, \pi \rangle$ to $\mathcal{S}_\phi^+$ generated by $\mathcal{S}_\phi^\theta$. We define $\bar{\Pi}_{\tilde{\phi}'}$ to be $\Pi_{\phi_l} \otimes \bar{\Pi}_{\tilde{\phi}'_-}$. Then we can choose $\bar{\Pi}_{\tilde{\phi}}$ such that

$$\tilde{f}^{\widetilde{G}'}(\tilde{\phi}') = \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} \tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi}, \omega), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}) \tag{2.11}$$

(cf. [Xu18, Theorem 4.6]), where $\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi}, \omega) = \mathrm{tr}(\tilde{\pi}(\tilde{f}) \circ A_{\tilde{\pi}}(\theta, \omega))$, and $A_{\tilde{\pi}}(\theta, \omega)$ is the intertwining operator between $\tilde{\pi} \otimes \omega$ and $\tilde{\pi}^\theta$ normalized by

$$A_{\tilde{\pi}}(\theta, \omega)|_\pi = \langle x, \pi^+ \rangle A_\pi(\theta)$$

for any $\pi \subseteq \tilde{\pi}|_G$. In this case, we say the transfer of $\bar{\Pi}_{\tilde{\phi}'}$ with respect to $\widetilde{\xi}$ is $\bar{\Pi}_{\tilde{\phi}}$ and write $\mathrm{Tran}_{\widetilde{\xi}} \bar{\Pi}_{\tilde{\phi}'} = \bar{\Pi}_{\tilde{\phi}}$.

Remark 2.1. For any proper Levi subgroup $\widetilde{M}'$ of $\widetilde{G}'$, there exists a θ-stable proper Levi subgroup $\widetilde{M}$ of $\widetilde{G}$ and an L-homomorphism $\tilde{\xi}_{\widetilde{M}'}$ such that the following diagram commutes

$$\begin{array}{ccc} {}^L\widetilde{G}' & \xrightarrow{\tilde{\xi}} & {}^L\widetilde{G} \\ \uparrow & & \uparrow \\ {}^L\widetilde{M}' & \xrightarrow{\tilde{\xi}_{\widetilde{M}'}} & {}^L\widetilde{M} \end{array}$$

and $(\widetilde{M}', \tilde{s}, \tilde{\xi}_{\widetilde{M}'})$ is a (θ,ω)-twisted endoscopic datum of $\widetilde{M}$. Suppose ϕ' factors through $\phi'_{M'} \in \bar{\Phi}_{bdd}(M')$ for the Levi subgroup $M' := \widetilde{M}' \cap G'$ of G', then the compatibility of twisted endoscopic transfer with parabolic induction gives

$$\mathrm{Tran}_{\tilde{\xi}} \bar{\Pi}_{\tilde{\phi}'} = \mathrm{Tran}_{\tilde{\xi}} \, \mathrm{Ind}_{\widetilde{M}'}^{\widetilde{G}'} \, \bar{\Pi}_{\tilde{\phi}'_{M'}} = \mathrm{Ind}_{\widetilde{M}}^{\widetilde{G}} \, \mathrm{Tran}_{\tilde{\xi}_{\widetilde{M}'}} \, \bar{\Pi}_{\tilde{\phi}'_{M'}}.$$

If $\widetilde{M} \cong \widetilde{M}'$, then $\mathrm{Tran}_{\tilde{\xi}_{\widetilde{M}'}}$ corresponds to twisting by a character depending on $\tilde{\xi}_{\widetilde{M}'}$, so it also transfers representations. In this case, we can still make sense of $\mathrm{Tran}_{\tilde{\xi}} \bar{\Pi}_{\tilde{\phi}'}$.

2.2. **Global theory.** Suppose F is global and G is a quasisplit symplectic or special even orthogonal group over F. Arthur [Art13, Theorem 1.5.2] showed the following decomposition

$$L^2_{disc}(G(F)\backslash G(\mathbb{A}_F)) = \bigoplus_{\psi \in \bar{\Psi}_2(G)} L^2_{disc,\psi}(G(F)\backslash G(\mathbb{A}_F)),$$

where $\bar{\Psi}_2(G)$ is the set of $\widehat{\Sigma}_0$-conjugacy classes of the discrete global Arthur parameters of G. We will only consider the subset $\bar{\Phi}_2(G)$ of generic parameters in this paper. Let $N = 2n+1$ if $G = Sp(2n)$ and $N = 2n$ if $G = SO(2n, \eta_{E/F})$. Then $\bar{\Phi}_2(G)$ will be defined in terms of automorphic representations of $GL(N, \mathbb{A}_F)$ with respect to the $GL(N,\mathbb{C})$-conjugacy class of twisted endoscopic embedding $\xi_G : {}^L G \longrightarrow GL(N,\mathbb{C})$ satisfying

- $\xi_G|_{\widehat{G}}$ is the standard representation of $\widehat{G}$,
- $\xi_G|_{W_F}$ is trivial if N is odd, and factors through $\Gamma_{E/F}$ with the nontrivial element sent to a reflection if N is even.

We denote by $\Phi_{sim}(m)$ the set of isomorphism classes of irreducible unitary cuspidal automorphic representations of $GL(m, \mathbb{A}_F)$. For $\phi \in \Phi_{sim}(m)$, we denote its dual by $\phi^\vee$. When $\phi = \phi^\vee$, we say ϕ is of orthogonal type (resp. symplectic type) if the symmetric square (resp. skew-symmetric square) L-function $L(s, \phi, S^2)$ (resp. $L(s, \phi, \wedge^2)$) has a pole at $s = 1$. Denote the central character of ϕ by η_ϕ. Suppose $\phi = \phi^\vee \in \Phi_{sim}(m)$, then η_ϕ is quadratic and we denote the associated quadratic extension by E/F. We associate ϕ with a twisted endoscopic datum $(G_\phi, s_\phi, \xi_\phi)$ of $GL(m)$ as follows.

$${}^L G_\phi = \begin{cases} SO(2n+1,\mathbb{C}) \times W_F & \text{if } N = 2n+1 \\ SO(2n,\mathbb{C}) \rtimes W_F & \text{if } N = 2n \text{ and } \phi \text{ is orthogonal type} \\ Sp(2n,\mathbb{C}) \times W_F & \text{if } N = 2n \text{ and } \phi \text{ is symplectic type} \end{cases}$$

where the action of W_F factors through $\Gamma_{E/F}$. The twisted endoscopic embedding $\xi_\phi : {}^L G_\phi \longrightarrow GL(m,\mathbb{C})$ satisfies:

- $\xi_\phi|_{\widehat{G}_\phi}$ is the standard representation,
- $\xi_\phi|_{W_F}$ factors through $\Gamma_{E/F}$ and the image of the nontrivial element of $\Gamma_{E/F}$ is $-I$ if N is odd, and is a reflection if N is even.

It should be noted that ξ_ϕ is not equivalent to ξ_{G_ϕ} if N is odd and $\eta_\phi \neq 1$. For any place v, ϕ_v is an irreducible admissible representation of $GL(m, F_v)$. By the local Langlands correspondence of general linear groups [HT01] [Hen00] [Sch13], we can associate it with a Langlands parameter of $GL(m, F_v)$, still denoted by ϕ_v. We will define ${}^L G_{\phi_v}$ and ξ_{ϕ_v} by restriction. By [Art13, Corollary 6.8.1], ϕ_v factors through ξ_{ϕ_v}. So we have the following diagram.

(2.12)
$$\begin{array}{ccccc} \phi_v : L_{F_v} & \longrightarrow & {}^L G_{\phi_v} & \xrightarrow{\xi_{\phi_v}} & GL(m,\mathbb{C}) \\ & \searrow & \downarrow & & \downarrow \\ & & {}^L G_\phi & \xrightarrow{\xi_\phi} & GL(m,\mathbb{C}) \end{array}$$

We denote by $\Phi(N)$ the set of formal direct sums

(2.13)
$$\phi = l_1\phi_1 \boxplus \cdots \boxplus l_r\phi_r$$

such that $\phi_i \in \Phi_{sim}(N_i)$ and $\sum_{i=1}^r l_i N_i = N$. We can assign a family of semisimple conjugacy classes in $GL(N,\mathbb{C})$ by

$$c(\phi_v) := \underbrace{c(\phi_{1,v}) \oplus \cdots \oplus c(\phi_{1,v})}_{l_1} \oplus \cdots \oplus \underbrace{c(\phi_{r,v}) \oplus \cdots \oplus c(\phi_{r,v})}_{l_r}$$

at unramified places v of ϕ, where $c(\phi_{i,v})$ is the Satake parameter of the local component $\phi_{i,v}$. We define an involution on $\Phi(N)$ by

$$\phi^\vee = l_1\phi_1^\vee \boxplus \cdots \boxplus l_r\phi_r^\vee.$$

Suppose $\phi \in \Phi(N)$ such that $\phi = \phi^\vee$, we also get an involution on the set of indices by requiring $\phi_{i^\vee} = \phi_i^\vee$. Consequently $l_i = l_{i^\vee}$. This gives a disjoint decomposition of these indices

$$I_\phi \bigsqcup J_\phi \bigsqcup J_\phi^\vee \tag{2.14}$$

where I_ϕ indexes the set of self-dual simple parameters. Let $K_\phi = I_\phi \sqcup J_\phi$ and $I_{\phi,O}$ (resp. $I_{\phi,S}$) index the self-dual simple parameters of orthogonal (resp. symplectic) type. For $i \in I_\phi$, let $G_i = G_{\phi_i}$ and $\xi_i = \xi_{\phi_i}$. For $j \in J_\phi$, let $G_j = GL(N_j)$ and define $\xi_j : {}^LG_j \longrightarrow GL(2N_j,\mathbb{C})$ by sending $g \rtimes w$ to $\mathrm{diag}\{g, {}^tg^{-1}\}$. Then we define a substitute global Langlands group for ϕ following [Art13, Section 1.4] by taking the fibre product

$$\mathcal{L}_\phi := \prod_{k\in K_\phi} \{{}^LG_k \longrightarrow W_F\}.$$

It is equipped with $W_F \to \mathcal{L}_\phi$ and $L_{F_v} \longrightarrow \mathcal{L}_\phi$ by (2.12). Let

$$\phi^{\mathcal{E}} := \bigoplus_{k\in K_\phi} l_k\xi_k : \mathcal{L}_\phi \longrightarrow GL(N,\mathbb{C}).$$

We define

$$\bar\Phi(G) := \{\phi \in \Phi(N) \,:\, \phi = \phi^\vee \text{ and } \phi^{\mathcal{E}} \text{ factors through } \xi_G\} \tag{2.15}$$

which substitutes for the set of $\widehat{\Sigma}_0$-conjugacy classes of global Langlands parameters for G. Write $\Phi(G,\phi)$ for the set of $\widehat{G}$-conjugacy classes of L-homomorphisms $\phi_G : \mathcal{L}_\phi \to {}^LG$ such that $\xi_G \circ \phi_G$ is $GL(N,\mathbb{C})$-conjugate to $\phi^{\mathcal{E}}$. So the pair (ϕ,ϕ_G) is a substitute for the global Langlands parameter for G. Let $m_\phi = |\Phi(G,\phi)|$. For $\phi \in \bar\Phi(G)$ and subgroup $\Sigma \subseteq \Sigma_0$, we define

$$S_\phi^\Sigma = \mathrm{Cent}(\mathrm{Im}\phi_G, \widehat{G}^\Sigma), \quad \bar S_\phi^\Sigma = S_\phi^\Sigma/Z(\widehat{G})^{\Gamma_F}, \quad \mathcal{S}_\phi^\Sigma = \bar S_\phi^\Sigma/\bar S_\phi^0.$$

and $S_\phi^\theta, \bar S_\phi^\theta$ and $\mathcal{S}_\phi^\theta$ for $\theta \in \Sigma_0$ as in the local case. Let

$$\bar\Phi_{sim}(G) = \{\phi \in \bar\Phi(G) : \bar S_\phi^{\Sigma_0} = 1\}, \quad \bar\Phi_2(G) = \{\phi \in \bar\Phi(G) : |\bar S_\phi| < \infty\}, \quad \bar\Phi(G^\theta) = \{\phi \in \bar\Phi(G) : S_\phi^\theta \neq \emptyset\},$$
$$\bar\Phi_{ell}(G^\theta) = \{\phi \in \bar\Phi(G^\theta) : |\bar S_{\phi,s}^0| < \infty \text{ for some semisimple } s \in \bar S_\phi^\theta\},$$

where $\theta \in \Sigma_0$. The following lemma is a direct consequence of the computation of S_ϕ (cf. [Xu18, Lemma 3.2]).

Lemma 2.2.

1. $\bar\Phi_{sim}(G) = \Phi_{sim}(N) \cap \bar\Phi(G)$.
2. *Suppose $\phi \in \bar\Phi(G)$, then ϕ is in $\bar\Phi_2(G)$ if and only if $K_\phi = I_{\phi,O}$ and $l_i = 1$ for all $i \in K_\phi$.*
3. *Suppose ϕ is in $\bar\Phi_{ell}(G^\theta)$ for $\theta \in \Sigma_0$, then $K_\phi = I_{\phi,O}$ and $l_i \leqslant 2$ for all $i \in K_\phi$.*
4. *Suppose G is special even orthogonal and $\phi \in \bar\Phi(G)$, then ϕ is in $\bar\Phi(G^{\theta_0})$ if and only if there exists $i \in I_{\phi,O}$ such that N_i is odd.*

At any place v, we define subsets $\bar\Phi_{sim}(G_v), \bar\Phi_2(G_v), \bar\Phi(G_v^\theta), \bar\Phi_{ell}(G_v^\theta)$ of $\bar\Phi_{bdd}(G_v)$ as in the global case. For $[\phi_v] \in \bar\Phi_{bdd}(G_v)$, we can view ϕ_v as a representation of L_{F_v} through the composition with ξ_{G_v}. Then we can decompose it into irreducible subrepresentations

$$\phi_v = l_1\phi_{v,1} \oplus \cdots \oplus l_r\phi_{v,r}.$$

Since ϕ_v is self-dual, we will get a partition of indices as (2.14). If $\phi_{v,i}$ is self-dual, we say it is of orthogonal (resp. symplectic) type if it factors through an orthogonal (symplectic) group. Then Lemma 2.2 is still valid.

For $\phi \in \bar{\Phi}(G)$, we define ϕ_v through the following diagram

$$\begin{array}{ccc} L_{F_v} & \xrightarrow{\phi_v} & {}^L G_v \\ \downarrow & & \downarrow \\ \mathcal{L}_\phi & \xrightarrow{\phi_G} & {}^L G. \end{array} \tag{2.16}$$

The generalized Ramanujan conjecture would imply that $[\phi_v] \in \bar{\Phi}_{bdd}(G_v)$. Since we do not have this conjecture at the moment, we can only assume that $[\phi_v] \in \bar{\Phi}^+_{unit}(G_v)$, which is a subset of $\bar{\Phi}(G_v)$ characterized by the property that

$$\xi_{G_v} \circ \phi_v = \phi_1 \oplus \cdots \oplus \phi_r \oplus (\nu^{a_1}\phi_{r+1} \oplus \nu^{-a_1}\phi_{r+1}) \cdots \oplus (\nu^{a_s}\phi_{r+s} \oplus \nu^{-a_s}\phi_{r+s})$$

where ϕ_i is unitary for $1 \leqslant i \leqslant r+s$ and $0 < a_j < 1/2$ for $1 \leqslant j \leqslant s$ (cf. [Xu18, Proposition 3.10]).

For any $[\phi_v] \in \bar{\Phi}^+_{unit}(G_v^\theta)$ with $\theta \in \Sigma_0$, ϕ_v factors through $\phi_{M_v,\lambda} := \phi_{M_v} \otimes (\lambda \circ |\cdot|_F)$ for some θ-stable parabolic subgroup $P \supseteq M$, where $[\phi_{M_v}] \in \bar{\Phi}_{bdd}(M_v^\theta)$ and $\lambda \in \mathfrak{a}^*_{M_v}$ lies in the open chamber determined by P. The normalized parabolic induction induces bijections $\bar{\Pi}_{\phi_v} \cong \bar{\Pi}_{\phi_{M_v,\lambda}}$ and $\bar{\Pi}_{\tilde{\phi}_v} \cong \bar{\Pi}_{\tilde{\phi}_{M_v,\lambda}}$ (cf. [Xu18, Proposition 3.11]). Moreover, we have the following diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{S}^\Sigma_{\tilde{\phi}_{M_v}} & \xrightarrow{\iota_{M_v}} & \mathcal{S}^\Sigma_{\phi_{M_v}} & \xrightarrow{\alpha^{M_v}} & \mathrm{Hom}(\widetilde{G}(F_v)/G(F_v), \mathbb{C}^\times) \\ & & \downarrow \cong & & \downarrow \cong & & \| \\ 1 & \longrightarrow & \mathcal{S}^\Sigma_{\tilde{\phi}_v} & \xrightarrow{\iota_v} & \mathcal{S}^\Sigma_{\phi_v} & \xrightarrow{\alpha_v} & \mathrm{Hom}(\widetilde{G}(F_v)/G(F_v), \mathbb{C}^\times). \end{array} \tag{2.17}$$

Since the parabolic induction preserves stability and is compatible with the twisted endoscopy, the previous local results for $\bar{\Phi}_{bdd}(G_v)$ extend to $\bar{\Phi}^+_{unit}(G_v)$ except for (2.7) (2.9).

The diagram (2.16) gives rise to an inclusion $S_\phi \hookrightarrow S_{\phi_v}$ at any place v, which induces a homomorphism $\mathcal{S}_\phi \to \mathcal{S}_{\phi_v}$. By the local Langlands correspondence for G_v, we can associate ϕ_v with $\bar{\Pi}_{\phi_v}$ and define the global L-packet by taking the restricted tensor product

$$\bar{\Pi}_\phi := \bigotimes_v{}' \bar{\Pi}_{\phi_v}$$

and define the global pairing by

$$\langle x, \pi \rangle := \prod_v \langle x_v, \pi_v \rangle, \quad x \in \mathcal{S}_\phi,\ [\pi] \in \bar{\Pi}_\phi.$$

For $\phi \in \bar{\Phi}_2(G)$, let $L^2_{disc,\phi}(G(F)\backslash G(\mathbb{A}_F))$ be the subspace of discrete automorphic representations whose Satake parameters are sent to $c(\phi_v)$ under ξ_{G_v} for almost all places. Let $\bar{\mathcal{H}}(G) = \otimes'_v \bar{\mathcal{H}}(G_v)$, then there is a decomposition as $\bar{\mathcal{H}}(G)$-modules

$$L^2_{disc,\phi}(G(F)\backslash G(\mathbb{A}_F)) = m_\phi \sum_{\substack{[\pi] \in \bar{\Pi}_\phi \\ \langle \cdot, \pi \rangle = 1}} \pi \tag{2.18}$$

(cf. [Art13, Theorem 1.5.2]).

Suppose $G = G_1 \times \cdots \times G_q$ as in (2.2) and we define $\bar{\Phi}(G)$ to be the set of $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_q$ such that $\phi_i \in \bar{\Phi}(G_i)$ for $1 \leqslant i \leqslant q$. We also define

$$\mathcal{L}_\phi := \prod_{i=1}^q \{\mathcal{L}_{\phi_i} \longrightarrow W_F\}$$

and

$$\phi^{\mathcal{E}} = \prod_{i=1}^{q} \phi_i^{\mathcal{E}}.$$

Then we can define S_ϕ^Σ and $\Phi(G,\phi)$ as before. Note $S_\phi = \prod_{i=1}^q S_{\phi_i}$ and $m_\phi = \prod_{i=1}^q m_{\phi_i}$. Then (2.18) still holds. Let $\tilde{\zeta}$ be a character of $Z_{\widetilde{G}}(\mathbb{A}_F)/Z_{\widetilde{G}}(F)$ and ζ its restriction to $Z_G(\mathbb{A}_F)$. Then we have

$$\mathrm{Ind}_{\widetilde{G}(F)Z_{\widetilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\widetilde{G}(\mathbb{A}_F)} L^2_{disc}(G(F)\backslash G(\mathbb{A}_F),\zeta) \cong L^2_{disc}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\widetilde{\zeta})$$

(cf. [Xu18, Lemma 5.3]). So we can decompose

$$L^2_{disc}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\widetilde{\zeta}) = \bigoplus_{\psi\in\bar{\Psi}_2(G,\zeta)} L^2_{disc,\psi}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\widetilde{\zeta})$$

where $\bar{\Psi}_2(G,\zeta)\subseteq\bar{\Psi}_2(G)$ is associated with the central character ζ and

$$L^2_{disc,\psi}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\widetilde{\zeta}) := \mathrm{Ind}_{\widetilde{G}(F)Z_{\widetilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F)}^{\widetilde{G}(\mathbb{A}_F)} L^2_{disc,\psi}(G(F)\backslash G(\mathbb{A}_F),\zeta).$$

Our goal is to decompose the right hand side in terms of global L-packets for $\psi=\phi\in\bar{\Phi}_2(G,\zeta)$. In parallel with the local case, we also have (2.5) and an isomorphism $r:\bar{H}^1(W_F,\widehat{D})\to\mathrm{Hom}(\widetilde{G}(\mathbb{A}_F)/\widetilde{G}(F)G(\mathbb{A}_F),\mathbb{C}^\times)$ (cf. [Xu18, Lemma 2.11]). Denote $r\circ\bar{\delta}$ by α. For any $\Sigma\subseteq\Sigma_0$, we have

$$\begin{array}{ccccc} 1 & \longrightarrow & \mathcal{S}^\Sigma_{\tilde{\phi}} & \xrightarrow{\iota} & \mathcal{S}^\Sigma_{\phi} \xrightarrow{\alpha} \mathrm{Hom}(\widetilde{G}(\mathbb{A}_F)/\widetilde{G}(F)G(\mathbb{A}_F),\mathbb{C}^\times) \\ & & \downarrow & & \downarrow \qquad\qquad\qquad \downarrow \\ 1 & \longrightarrow & \mathcal{S}^\Sigma_{\tilde{\phi}_v} & \xrightarrow{\iota_v} & \mathcal{S}^\Sigma_{\phi_v} \xrightarrow{\alpha_v} \mathrm{Hom}(\widetilde{G}(F_v)/G(F_v),\mathbb{C}^\times). \end{array} \tag{2.19}$$

Let $\tilde{\bar{\Pi}}_{\phi,\tilde{\zeta}}$ be the set of isomorphism classes of irreducible admissible representations of $\widetilde{G}(\mathbb{A}_F)$ as $\bar{\mathcal{H}}(\widetilde{G})$-modules with central character $\widetilde{\zeta}$, whose restriction to $G(\mathbb{A}_F)$ have irreducible constituents contained in $\bar{\Pi}_\phi$. It follows from (2.19) that $[\tilde{\pi}\otimes\omega]=[\tilde{\pi}]$ for any $[\tilde{\pi}]\in\tilde{\bar{\Pi}}_{\phi,\tilde{\zeta}}$ and $\omega\in\alpha(\mathcal{S}_\phi^{\Sigma_0})$. Let

$$X=\mathrm{Hom}(\widetilde{G}(\mathbb{A}_F)/Z_{\widetilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F),\mathbb{C}^\times),\quad Y=\mathrm{Hom}(\widetilde{G}(\mathbb{A}_F)/\widetilde{G}(F)Z_{\widetilde{G}}(\mathbb{A}_F)G(\mathbb{A}_F),\mathbb{C}^\times).$$

Then $\alpha(\mathcal{S}_\phi^{\Sigma_0})\subseteq Y$. We define the global pairing as

$$\langle x,\tilde{\pi}\rangle := \prod_v \langle x_v,\tilde{\pi}_v\rangle,\quad x\in\mathcal{S}_{\tilde{\phi}},\ [\tilde{\pi}]\in\tilde{\bar{\Pi}}_{\phi,\tilde{\zeta}}.$$

By [Xu18, Corollary 5.6], we can always choose a representative $\tilde{\pi}$ in the discrete automorphic spectrum of $\widetilde{G}(\mathbb{A}_F)$ for $[\tilde{\pi}]\in\tilde{\bar{\Pi}}_{\phi,\tilde{\zeta}}/X$ with $\langle\cdot,\tilde{\pi}\rangle=1$. Moreover, we have shown the following decomposition as $\bar{\mathcal{H}}(\widetilde{G})$-modules

$$L^2_{disc,\phi}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\widetilde{\zeta}) = m_\phi \sum_{\omega\in Y/\alpha(\mathcal{S}_\phi)} \sum_{\substack{[\tilde{\pi}]\in\tilde{\bar{\Pi}}_{\phi,\tilde{\zeta}}/X \\ \langle\cdot,\tilde{\pi}\rangle=1}} \tilde{\pi}\otimes\omega, \tag{2.20}$$

where $\tilde{\pi}$ are taken to be the representatives of $\tilde{\bar{\Pi}}_{\phi,\tilde{\zeta}}/X$ in the discrete automorphic spectrum of $\widetilde{G}(\mathbb{A}_F)$ (cf. [Xu18, Proposition 5.11]).

3. Stable trace formula

Suppose F is global and G is (2.2). Let $\widetilde{Z}_{\mathbb{A}_F}=\prod_v'\widetilde{Z}_{F_v}$ be a closed subgroup of $Z_{\widetilde{G}}(\mathbb{A}_F)$ such that $\widetilde{Z}_{\mathbb{A}_F}Z_G(\mathbb{A}_F)=Z_{\widetilde{G}}(\mathbb{A}_F)$. Let $\widetilde{Z}_F=\widetilde{Z}_{\mathbb{A}_F}\cap Z_{\widetilde{G}}(F)$ and $\widetilde{\chi}$ a character of $\widetilde{Z}_{\mathbb{A}_F}/\widetilde{Z}_F$. Let $Z_{\mathbb{A}_F}=\widetilde{Z}_{\mathbb{A}_F}\cap Z_G(\mathbb{A}_F)$ and $Z_F=\widetilde{Z}_F\cap Z_G(F)$. We denote the restriction of $\widetilde{\chi}$ to $Z_{\mathbb{A}_F}$ by χ. Let $\theta\in\Sigma_0$ and $\omega\in Y$, the discrete part of the (θ,ω)-twisted trace formula for $\widetilde{G}$ takes the form

(3.1)
$$I_{disc,t}^{(\widetilde{G}^\theta,\omega)}(\tilde{f})=\sum_{\{\widetilde{M}\}}|W(\widetilde{M})|^{-1}\sum_{w\in W^\theta(\widetilde{M})_{reg}}|\det(w-1)_{\mathfrak{a}_{\widetilde{M}}^{\widetilde{G}^\theta}}|^{-1}tr(M_{\widetilde{P}|\theta\widetilde{P},t}(w,\widetilde{\chi})I_{\widetilde{P},t}^{\theta,\omega}(\widetilde{\chi},\tilde{f})),\quad \tilde{f}\in\mathcal{H}(\widetilde{G},\widetilde{\chi}).$$

We give the explanation of this formula below. The outer sum is taken over $\widetilde{G}(F)$-conjugacy classes of Levi subgroups $\widetilde{M}$ of $\widetilde{G}$, and the inner sum is taken over elements w in the Weyl set

$$W^\theta(\widetilde{M}):=\mathrm{Norm}(A_{\widetilde{M}},\widetilde{G}\rtimes\theta)/\widetilde{M}$$

such that $|\det(w-1)_{\mathfrak{a}_{\widetilde{M}}^{\widetilde{G}^\theta}}|^{-1}\neq 0$. Here $\mathfrak{a}_{\widetilde{M}}^{\widetilde{G}^\theta}$ is the kernel of the canonical projection of $\mathfrak{a}_{\widetilde{M}}\to\mathfrak{a}_{\widetilde{G}}\to\mathfrak{a}_{\widetilde{G}^\theta}$ for $\mathfrak{a}_{\widetilde{G}^\theta}:=\mathfrak{a}_{\widetilde{G}}/\{X-\theta(X):X\in\mathfrak{a}_{\widetilde{G}}\}$. For any Levi subgroup $\widetilde{M}$ of $\widetilde{G}$, we take

$$\bigoplus_{\widetilde{\zeta}_{\widetilde{M}}}L^2_{disc,t}(\widetilde{M}(F)\backslash\widetilde{M}(\mathbb{A}_F),\widetilde{\zeta}_{\widetilde{M}})$$

where the central character $\widetilde{\zeta}_{\widetilde{M}}$ extends $\widetilde{\chi}$ and is invariant under some element of $W^\theta(\widetilde{M})_{reg}$, and the archimedean infinitesimal characters of the irreducible constituents have norm t on their imaginary parts. Then

$$I_{\widetilde{P},t}(\widetilde{\chi},\tilde{f})=\int_{\widetilde{Z}_{\mathbb{A}_F}\backslash\widetilde{G}(\mathbb{A}_F)}\tilde{f}(g)I_{\widetilde{P},t}(\tilde{\chi},g)dg$$

defines an operator on the space $\mathcal{H}_{\widetilde{P},t}(\tilde{\chi})$ of the corresponding normalized induced representation. Let $R(\theta):\mathcal{H}_{\theta\widetilde{P},t}(\tilde{\chi})\to\mathcal{H}_{\widetilde{P},t}(\tilde{\chi})$ be induced by the action of θ on $\widetilde{G}(\mathbb{A}_F)$ and $R(\omega):\mathcal{H}_{\widetilde{P},t}(\tilde{\chi})\to\mathcal{H}_{\widetilde{P},t}(\tilde{\chi})$ induced by multiplying ω. Finally $I_{\widetilde{P},t}^{\theta,\omega}(\widetilde{\chi},\tilde{f})$ is the composition $R(\theta)^{-1}\circ R(\omega)\circ I_{\widetilde{P},t}(\widetilde{\chi},\tilde{f})$ and $M_{\widetilde{P}|\theta\widetilde{P},t}(w,\widetilde{\chi})$ is the standard intertwining operator between $\mathcal{H}_{\theta\widetilde{P},t}(\widetilde{\chi})$ and $\mathcal{H}_{\widetilde{P},t}(\widetilde{\chi})$. For the term corresponding to $\widetilde{M}=\widetilde{G}$, we denote

$$R_{disc,t}^{\widetilde{G}}(\tilde{f}):=I_{\widetilde{G},t}(\widetilde{\chi},\tilde{f}),\quad R_{disc,t}^{(\widetilde{G}^\theta,\omega)}(\tilde{f}):=R(\theta)^{-1}\circ R(\omega)\circ R_{disc,t}^{\widetilde{G}}(\tilde{f}).$$

The stabilization of (3.1) results from the works of Arthur in the ordinary case [Art01] [Art02] [Art03] and the works of Moeglin-Waldspurger in the twisted case [MW16a] [MW16b]. To describe it, let us denote by $\mathcal{E}(\widetilde{G}^\theta,\omega)$ ($\mathcal{E}_{ell}(\widetilde{G}^\theta,\omega)$) the set of equivalence classes of twisted (elliptic) endoscopic data for (G,θ,ω). Then the stabilization takes the following form

$$I_{disc,t}^{(\widetilde{G}^\theta,\omega)}(\tilde{f})=\sum_{\widetilde{G}'\in\mathcal{E}_{ell}(\widetilde{G}^\theta,\omega)}\iota(\widetilde{G},\widetilde{G}')S_{disc,t}^{\widetilde{G}'}(\tilde{f}^{\widetilde{G}'}),\quad \tilde{f}\in\mathcal{H}(\widetilde{G},\widetilde{\chi})\tag{3.2}$$

where $\tilde{f}^{\widetilde{G}'}$ is the Langlands-Shelstad-Kottwitz transfer of $\tilde{f}$. The coefficients $\iota(\widetilde{G},\widetilde{G}')$ are given as follows

$$\iota(\widetilde{G},\widetilde{G}')=|\bar{Z}(\widehat{\widetilde{G}'})^\Gamma|^{-1}|\mathrm{Out}_{\widetilde{G}}(\widetilde{G}')|^{-1}\tag{3.3}$$

(cf. [Xu18, (5.13)]), where

$$\bar{Z}(\widehat{\widetilde{G}'})^\Gamma=Z(\widehat{\widetilde{G}'})^\Gamma Z(\widehat{\widetilde{G}})^\Gamma/Z(\widehat{\widetilde{G}})^\Gamma,\quad \mathrm{Out}_{\widetilde{G}}(\widetilde{G}')=\mathrm{Aut}_{\widetilde{G}}(\widetilde{G}')/\widehat{\widetilde{G}'}Z(\widehat{\widetilde{G}})^\Gamma.$$

For $\phi \in \bar{\Phi}(G)$, we choose $\widetilde{\chi}$ such that χ matches the restriction of the central character of $\bar{\Pi}_\phi$. Then we define the ϕ-component of (3.1) to be

$$I^{(\widetilde{G}^\theta,\omega)}_{disc,\phi}(\tilde{f}) := \sum_{\widetilde{c}\to c(\phi)} I^{(\widetilde{G}^\theta,\omega)}_{disc,t(\phi),\widetilde{c}}(\tilde{f}), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi})$$

which is contributed from automorphic representations $\tilde{\pi}$ of $\widetilde{G}(\mathbb{A}_F)$, whose central characters match $\widetilde{\chi}$ by restriction and Satake parameters $\widetilde{c}_v$ projected to $c(\phi_v)$ and $t(\phi)$ is the norm of of the imaginary part of the archimedean infinitesimal character determined by ϕ. It follows from [Xu18, Lemma 5.1] that the stabilization of the ϕ-component of the twisted stable trace formula (3.2) for $\widetilde{G}$ is

$$I^{(\widetilde{G}^\theta,\omega)}_{disc,\phi}(\tilde{f}) = \sum_{\widetilde{G}'\in\mathcal{E}_{ell}(\widetilde{G}^\theta,\omega)} \iota(\widetilde{G},\widetilde{G}') S^{\widetilde{G}'}_{disc,\phi}(\tilde{f}^{\widetilde{G}'}), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}) \tag{3.4}$$

where

$$S^{\widetilde{G}'}_{disc,\phi}(\tilde{f}^{\widetilde{G}'}) = \sum_{\widetilde{c}'\to c(\phi)} S^{\widetilde{G}'}_{disc,t(\phi),\widetilde{c}'}(\tilde{f}^{\widetilde{G}'})$$

can be defined recursively by using the ordinary stable trace formula.

4. Statements of main results

Let F be a global field of characteristic zero and G be (2.2). The following theorem proves [Xu18, Conjecture 5.16], which concerns the existence of global L-packets of $\widetilde{G}(\mathbb{A}_F)$.

Theorem 4.1.

(1) *For $\phi \in \bar{\Phi}(G)$, there exists a global packet $\bar{\Pi}_{\tilde{\phi}}$ of $\bar{\mathcal{H}}(\widetilde{G})$-modules of irreducible admissible representations of $\widetilde{G}(\mathbb{A}_F)$ satisfies the following properties:*
   (a) *$\bar{\Pi}_{\tilde{\phi}} = \bigotimes'_v \bar{\Pi}_{\tilde{\phi}_v}$, where $\bar{\Pi}_{\tilde{\phi}_v}$ is some lift of $\bar{\Pi}_{\phi_v}$;*
   (b) *there exists $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$ isomorphic to an automorphic representation of $\widetilde{G}$ as $\bar{\mathcal{H}}(\widetilde{G})$-modules.*
   Moreover, $\bar{\Pi}_{\tilde{\phi}}$ is unique up to twisting by characters of $\widetilde{G}(\mathbb{A}_F)/\widetilde{G}(F)G(\mathbb{A}_F)$.

(2) *Suppose $\phi \in \bar{\Phi}_2(G)$, we have the following decomposition as $\bar{\mathcal{H}}(\widetilde{G})$-module*

$$L^2_{disc,\phi}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}_F),\widetilde{\zeta}) = m_\phi \bigoplus_{\omega\in Y/\alpha(\mathcal{S}_\phi)} \bigoplus_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega \\ \langle\cdot,\tilde{\pi}\rangle=1}} \tilde{\pi}. \tag{4.1}$$

   where $\widetilde{\zeta}$ is an extension of the central character of $\bar{\Pi}_\phi$.

The next theorem proves [Xu18, Conjecture 5.19], which describes the discrete spectrum under the twists by an outer automorphism $\theta \in \Sigma_0$ and a character $\omega \in Y$.

Theorem 4.2. *Suppose $\phi \in \bar{\Phi}_2(G)$ and $x \in \mathcal{S}^\theta_\phi$ with $\alpha(x) = \omega$ for $\theta \in \Sigma_0$ and some character ω of $\widetilde{G}(\mathbb{A}_F)/\widetilde{G}(F)G(\mathbb{A}_F)$. For $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$ with $\langle\cdot,\tilde{\pi}\rangle = 1$, the intertwining operator $R(\theta)^{-1}\circ R(\omega)$ restricted to the $\tilde{\pi}$-isotypic component $I(\tilde{\pi})$ in the discrete spectrum is equal to the product of the multiplicity $m(\tilde{\pi})$ and the local intertwining operators $A_{\tilde{\pi}_v}(\theta,\omega_v)$ normalized by x_v, i.e.*

$$I^{(\widetilde{G}^\theta,\omega)}_{disc,\phi}(\tilde{f}) = m_\phi \sum_{\omega'\in Y/\alpha(\mathcal{S}_\phi)} \sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega' \\ \langle\cdot,\tilde{\pi}\rangle=1}} \tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}), \tag{4.2}$$

where $\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega) = \prod_v \tilde{f}_{\widetilde{G}^\theta_v}(\tilde{\pi}_v,\omega_v)$, and it does not depend on the choice of x in the $\mathcal{S}_{\tilde{\phi}}$-coset.

Both theorems have been proved in [Xu18] for $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_q \in \bar{\Phi}_2(G)$ with $\phi_i \in \bar{\Phi}_2(G_i)$ such that $\mathcal{S}_{\tilde{\phi}_i} = 1$ for all i. The next result concerns the functoriality of twisted endoscopic transfer, which can

be formulated as follows. For $\phi \in \bar{\Phi}(G)$, $\theta \in \Sigma_0$ and any semisimple element $s \in \bar{S}_\phi^\theta$, we can define the relation $(G', \phi') \to (\phi, s)$ as in the local case. Define

$$\mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}'} := \otimes_v' \mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}'_v}$$

to be the transfer of a global L-packet $\bar{\Pi}_{\tilde{\phi}'}$ with respect to the endoscopic embedding $\tilde{\xi}$ (cf. (2.10)).

Conjecture 4.3. *There exists a global L-packet $\bar{\Pi}_{\tilde{\phi}}$ such that*

$$\mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}'} = \bar{\Pi}_{\tilde{\phi}}.$$

We are not able to prove this conjecture in all cases. This is due to the occurrence of what we call GL-type orthogonal simple parameters. They are defined as follows.

Definition 4.4. For $\phi \in \Phi_{sim}(N)$ such that $\phi = \phi^\vee$, we call ϕ is of **GL-type** if N is even and at every place v, ϕ_v factors through the L-group of a Levi subgroup of $G_{\phi,v}$, which is a product of general linear groups.

Remark 4.5. For GL-type ϕ, it is necessary that $\eta_\phi = 1$ by definition. As a result, there is no GL-type orthogonal simple parameters when $N = 2$.

Suppose G is symplectic or special even orthogonal and $\phi \in \bar{\Phi}(G)$, we will decompose the parameter as

$$\phi = \phi_o \boxplus \phi_b$$

where ϕ_b consists of all GL-type orthogonal simple parameters in ϕ. The parameter ϕ factors through $\phi_H := \phi_o \times \phi_b$ for the endoscopic group

$$H := G_o \times G_b.$$

The embedding ${}^L H \hookrightarrow {}^L G$ induces an isomorphism

$$S_{\phi_o}^{\Sigma_0} \times S_{\phi_b} \cong S_\phi^{\Sigma_0}.$$

Theorem 4.6. *For $\phi \in \bar{\Phi}(G^\theta)$ and semisimple $s \in (S_{\phi_o}^\theta \times Z(\widehat{G}_b))/Z(\widehat{G})^\Gamma$, let $(G', \phi') \to (\phi, s)$ and $\widetilde{G}' \in \mathcal{E}(\widetilde{G}^\theta, \omega)$ be the lift of G', then there exist global packets $\bar{\Pi}_{\tilde{\phi}'}$ and $\bar{\Pi}_{\tilde{\phi}}$ such that*

$$\mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}'} = \bar{\Pi}_{\tilde{\phi}}.$$

This result is closely related to the stable multiplicity formula conjectured by Arthur.

Conjecture 4.7 (Stable multiplicity formula)**.** *Suppose $\phi \in \bar{\Phi}(G)$, then*

$$S_{disc,\phi}^{\widetilde{G}}(\tilde{f}) = m_\phi \sum_{\omega \in Y/\alpha(\mathcal{S}_\phi)} |\mathcal{S}_{\tilde{\phi}}|^{-1} \sigma(\bar{S}_\phi^0) \tilde{f}^{\widetilde{G}}(\tilde{\phi} \otimes \omega), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}), \tag{4.3}$$

where $\sigma(\bar{S}_\phi^0)$ is some constant defined through (5.1) *and $\tilde{f}^{\widetilde{G}}(\tilde{\phi} \otimes \omega) := \prod_v \tilde{f}_v(\tilde{\phi}_v \otimes \omega_v)$.*

In this paper, we will only consider the case that $\phi_b \in \bar{\Phi}_2(G_b)$. For any unordered m-partition

$$\phi_b = \phi_{b,1} \boxplus \cdots \boxplus \phi_{b,m}, \qquad \phi_{b,i} \in \bar{\Phi}_2(G_{b,i})$$

the parameter ϕ_b factors through $\phi_b' := \phi_{b,1} \times \cdots \times \phi_{b,m}$ for $G' := G_{b,1} \times \cdots \times G_{b,m}$ by a sequence of endoscopic embeddings

$$\widehat{G}_{b,1} \times \cdots \times \widehat{G}_{b,m} \xrightarrow{\xi_m} \cdots \xrightarrow{\xi_1} \widehat{G}_b.$$

We can lift each endoscopic embedding to that of similitude groups. For fixed $\bar{\Pi}_{\tilde{\phi}_{b,i}}$, let us define $\bar{\Pi}_{\tilde{\phi}_b'} := \bar{\Pi}_{\tilde{\phi}_{b,1}} \tilde{\otimes} \cdots \tilde{\otimes} \bar{\Pi}_{\tilde{\phi}_{b,m}}$ and

$$\bar{\Pi}(\tilde{\phi}_{b,1}, \cdots, \tilde{\phi}_{b,m}) := \mathrm{Tran}_{\tilde{\xi}_1} \circ \cdots \circ \mathrm{Tran}_{\tilde{\xi}_m} \bar{\Pi}_{\tilde{\phi}_b'}.$$

We also define

$$\bar{\Pi}(\tilde{\phi}_o; \tilde{\phi}_{b,1}, \cdots, \tilde{\phi}_{b,m}) := \mathrm{Tran}\, \bar{\Pi}_{\tilde{\phi}_o} \tilde{\otimes} \bar{\Pi}(\tilde{\phi}_{b,1}, \cdots, \tilde{\phi}_{b,m})$$

and

$$\tilde{f}(\tilde{\phi}_o;\tilde{\phi}_{b,1},\cdots,\tilde{\phi}_{b,m}) := \sum_{\tilde{\pi}\in\bar{\Pi}(\tilde{\phi}_o;\tilde{\phi}_{b,1},\cdots,\tilde{\phi}_{b,m})} \tilde{f}(\tilde{\pi})$$

$$\tilde{f}(\phi_o;\phi_{b,1},\cdots,\phi_{b,m}) := \sum_{\omega\in Y} \sum_{\tilde{\pi}\in\bar{\Pi}(\tilde{\phi}_o;\tilde{\phi}_{b,1},\cdots,\tilde{\phi}_{b,m})} \tilde{f}(\tilde{\pi}\otimes\omega).$$

At last, we need to define some constants by the recursive formula

$$a_1 = 1, \qquad a_m = \frac{1}{4}\sum_{k=1}^{m-1}\binom{m}{k} a_k a_{m-k} \quad (m>1).$$

Theorem 4.8. *Suppose $\phi\in\bar{\Phi}(G)$ and $\phi_b\in\bar{\Phi}_2(G_b)$.*

(1) *If $\phi=\phi_b$, then*

$$I^{\widetilde{G}_b}_{disc,\phi_b}(\tilde{f}) = m_{\phi_b}\tilde{f}(\phi_b), \quad \tilde{f}\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}), \tag{4.4}$$

$$S^{\widetilde{G}_b}_{disc,\phi_b}(\tilde{f}) = m_{\phi_b}\sum_{\phi_{b,1},\cdots,\phi_{b,m}} (-1)^{m-1}a_m\,\tilde{f}(\phi_{b,1},\cdots,\phi_{b,m}), \quad \tilde{f}\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}), \tag{4.5}$$

where the sum is over unordered partition of ϕ.

(2) *If ϕ contains at most one GL-type orthogonal simple parameter, then*

$$S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = m_{\phi}\sum_{\omega\in Y/\alpha(\mathcal{S}_\phi)} |\mathcal{S}_{\tilde{\phi}}|^{-1}\sigma(\bar{S}^0_\phi)\tilde{f}^{\widetilde{G}}(\tilde{\phi}\otimes\omega), \quad \tilde{f}\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}). \tag{4.6}$$

(3) *If $\phi\neq\phi_o,\phi_b$, then*

$$S^{\widetilde{G}}_{disc,\phi} = \frac{1}{4}\,\mathrm{Tran}\,(S^{\widetilde{G}_o}_{disc,\phi_o}\tilde{\otimes}\,S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}), \tag{4.7}$$

where

$$S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}(\tilde{f}) = m_{\phi_b}\sum_{\phi_{b,1},\cdots,\phi_{b,m}} (-1)^{m-1}a_m\,\tilde{f}(\tilde{\phi}_{b,1},\cdots,\tilde{\phi}_{b,m}), \quad \tilde{f}\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}). \tag{4.8}$$

We can generalize Theorem 4.8 for $G=G_1\times\cdots\times G_q$ and $\phi=\phi_1\times\cdots\times\phi_q\in\bar{\Phi}(G)$ with $\phi_{i,b}\in\bar{\Phi}_2(G_{i,b})$. The parameter ϕ factors through $\phi_H:=\phi_o\times\phi_b$ for the endoscopic group $H:=G_o\times G_b$, where

$$G_o = G_{1,o}\times\cdots\times G_{q,o}, \qquad G_b = G_{1,b}\times\cdots\times G_{q,b},$$

and

$$\phi_o = \phi_{1,o}\times\cdots\times\phi_{q,o}, \qquad \phi_b = \phi_{1,b}\times\cdots\times\phi_{q,b}.$$

Theorem 4.9. *Suppose $G=G_1\times\cdots\times G_q$ as (2.2), $\phi\in\bar{\Phi}(G)$ and $\phi_b\in\bar{\Phi}_2(G_b)$.*

(1) *If $\phi=\phi_b$, then*

$$I^{\widetilde{G}_b}_{disc,\phi_b}(\tilde{f}) = m_{\phi_b}\tilde{f}(\phi_b), \quad \tilde{f}\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}), \tag{4.9}$$

and

$$S^{\widetilde{G}}_{disc,\phi} = (\tilde{\otimes}_{i\neq1}S^{\widetilde{G}_i}_{disc,\tilde{\phi}_i})\,\tilde{\otimes}\,S^{\widetilde{G}_1}_{disc,\phi_1} \tag{4.10}$$

where $S^{\widetilde{G}_i}_{disc,\tilde{\phi}_i}$ is defined as in (4.6) and (4.7) by dropping the sum over ω.

(2) *If each ϕ_i contains at most one GL-type orthogonal simple parameter, then*

$$S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = m_\phi \sum_{\omega \in Y/\alpha(\mathcal{S}_\phi)} |\mathcal{S}_{\tilde{\phi}}|^{-1} \sigma(\bar{S}^0_\phi) \tilde{f}^{\widetilde{G}}(\tilde{\phi} \otimes \omega), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}). \tag{4.11}$$

(3) *If $\phi \neq \phi_o, \phi_b$, then*

$$S^{\widetilde{G}}_{disc,\phi} = (\frac{1}{4})^l \,\mathrm{Tran}\,(S^{\widetilde{G}_o}_{disc,\phi_o} \tilde{\otimes} S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}), \tag{4.12}$$

where $l = \sharp\{1 \leqslant i \leqslant q \,|\, \phi_i \neq \phi_{i,o}, \phi_{i,b}\}$ and

$$S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b} = \tilde{\otimes}_i \, S^{\widetilde{G}_{i,b}}_{disc,\tilde{\phi}_{i,b}}. \tag{4.13}$$

4.1. $GSp(4)$ **and** $GSO(6)$**.** At the suggestion of the referee, we will compare our results in the case of $GSp(4)$ with those in [Art04], [GT19]. Arthur [Art04] formulated a series of conjectures on the classification of automorphic representations of $GSp(4)$. They have been proved recently by Gee-Taïbi [GT19], which is independent of the our previous work [Xu18]. Their approach is to use the twisted endoscopic transfer from $GSp(4)$ to $GL(4)\times GL(1)$. We observe that this is almost the same as the twisted endoscopic transfer from $GSp(4)$ to $GSO(6)$ studied in [Xu18] and this paper. Therefore, it is possible to reprove their results based on our works. Since it is a bit long to give all the details in this direction, we will only show that our results are compatible with those in [GT19] and leave it for another project.

Let F be local or global. There is a short exact sequence

$$1 \longrightarrow GL(1) \longrightarrow GL(4) \times GL(1) \longrightarrow GSO(6) \longrightarrow 1$$

$$(g, x) \longmapsto \wedge^2 g \cdot x$$

$$t \longmapsto (tI_4, t^{-2}).$$

Let us denote the image of $GL(1)$ by Z. It follows that $GSO(6, F) = (GL(4, F) \times GL(1, F))/Z(F)$. We can also lift θ_0 to an automorphism of $GL(4) \times GL(1)$:

$$\tilde{\theta}_0 : (g, x) \mapsto (\tilde{\theta}(g), \det(g)x),$$

where $\tilde{\theta}$ is inner to taking transpose inverse and fixes a splitting compatible with that on $GSO(6)$. On the dual side, we have

$$1 \longrightarrow GSpin(6, \mathbb{C}) \longrightarrow GL(4, \mathbb{C}) \times GL(1, \mathbb{C}) \longrightarrow GL(1, \mathbb{C}) \longrightarrow 1$$

$$(g, x) \longmapsto \det(g) \cdot x^{-2}.$$

In this paper, we primarily view $GSp(4)$ as a θ_0-twisted endoscopic group of $GSO(6)$. As in [Art04], [GT19], it can also be viewed as a $\tilde{\theta}_0$-twisted endoscopic group of $GL(4) \times GL(1)$. The two twisted endoscopic embeddings can be related by the following diagram

$$\begin{array}{ccc} GSpin(5, \mathbb{C}) & \xrightarrow{\tilde{\xi}} & GSpin(6, \mathbb{C}) \\ \cong \downarrow & & \downarrow \\ GSp(4, \mathbb{C}) & \xrightarrow{(\mathrm{std}, \lambda)} & GL(4, \mathbb{C}) \times GL(1, \mathbb{C}). \end{array}$$

When F is local, there is a bijection

$$\Pi(GSO(6,F)) \longrightarrow \{\Pi \boxtimes \chi \in \Pi(GL(4,F)\times GL(1,F)) \,|\, \omega_\Pi = \chi^2\}, \tag{4.14}$$

where ω_Π is the central character of Π. By [Xu18, Proposition 4.4], the L-packets of $GSO(6,F)$ are all singletons and this matches the results for $GL(4,F)$. Inverse to (4.14), we have a surjection

$$\mathcal{H}(GL(4,F)\times GL(1,F)) \to \mathcal{H}(GSO(6,F)), \quad f \mapsto \bar{f}$$

defined by integrating over $Z(F)$. One can check by definition that for any $f \in \mathcal{H}(GL(4,F)\times GL(1,F))$, $f, \bar{f}$ will have the same transfer to $GSp(4,F)$. By (2.11), a tempered L-packet of $GSp(4,F)$ can be transferred to a twisted character of an irreducible tempered representation of $GSO(6,F)$, hence also to the twisted character of the corresponding representation of $GL(4,F)\times GL(1,F)$. This shows that the local L-packets of $GSp(4,F)$ defined in [Xu18] are the same as in [GT19].

When F is global, we fix a central character $\tilde{\zeta}$ for $GSO(6,\mathbb{A}_F)$ and pull it back to a central character for $GL(4,\mathbb{A}_F)\times GL(1,\mathbb{A}_F)$, which is just $\tilde{\zeta}^2 \boxtimes \tilde{\zeta}$. Then

$$\begin{aligned} L^2_{disc}(GSO(6,F)\backslash GSO(6,\mathbb{A}_F),\tilde{\zeta}) &\cong L^2_{disc}(GL(4,F)\times GL(1,F)\backslash GL(4,\mathbb{A}_F)\times GL(1,\mathbb{A}_F),\tilde{\zeta}^2\boxtimes\tilde{\zeta}) \\ &\cong L^2_{disc}(GL(4,F)\backslash GL(4,\mathbb{A}_F),\tilde{\zeta}^2), \end{aligned}$$

where the irreducible subrepresentations of $GSO(6,\mathbb{A}_F)$ and $GL(4,\mathbb{A}_F)$ are in bijection with each other. Note the global L-packet of $GSO(6,\mathbb{A}_F)$ are also singletons, since they are restricted tensor products of local L-packets by construction. This matches the results for $GL(4,\mathbb{A}_F)$.

For $\phi \in \Phi_2(Sp(4))$, the global L-packet $\Pi_{\tilde{\phi}}$ of $GSp(4,\mathbb{A}_F)$ with central character $\tilde{\zeta}$ can be characterized by its transfer to $GSO(6,\mathbb{A}_F)$, which is some global L-packet $\bar{\Pi}_{\tilde{\phi}_+}$ with central character $\tilde{\zeta}$, where $\phi_+ = 1\boxplus\phi \in \bar{\Phi}(SO(6))$. As we have seen, $\bar{\Pi}_{\tilde{\phi}_+}$ consists of a single automorphic representation. Depending on whether ϕ_+ is discrete or not, we can divide into two cases:

(1) $\phi_+ \in \bar{\Phi}_2(SO(6))$: This case has the following two equivalent characterizations.
   (a) ϕ does not contain a summand of trivial representation of $GL(1,\mathbb{A}_F)$;
   (b) $\Pi_{\tilde{\phi}}$ is not endoscopic.

   By [JL16] $\bar{\Pi}_{\tilde{\phi}_+}$ consists of a cuspidal automorphic representation, which also corresponds to a cuspidal automorphic representation $\Pi\boxtimes\tilde{\zeta}$ of $GL(4,\mathbb{A}_F)\times GL(1,\mathbb{A}_F)$. It follows from our local discussion that $\Pi_{\tilde{\phi}}$ also transfers to $\Pi\boxtimes\tilde{\zeta}$. Hence $\Pi_{\tilde{\phi}}$ is the same as the global L-packet in [GT19]. This case corresponds to the general type in [Art04].

(2) $\phi_+ \notin \bar{\Phi}_2(SO(6))$: It is necessary that $\phi_+ = 1\boxplus 1\boxplus\phi_-$, where $\phi_- \in \bar{\Phi}_2(SO(4))$ does not contain a summand of the trivial representation of $GL(1,\mathbb{A}_F)$. By our construction, $\bar{\Pi}_{\tilde{\phi}_+}$ is the parabolic induction of $1\boxtimes\bar{\Pi}_{\tilde{\phi}_-}$ on $GL(1,\mathbb{A}_F)\times GSO(4,\mathbb{A}_F)$. As in the case of $GSO(6)$, one can further show that $\bar{\Pi}_{\tilde{\phi}_-}$ consists of a single cuspidal automorphic representation with central character $\tilde{\zeta}$. Then we can pull back $1\boxtimes\bar{\Pi}_{\tilde{\phi}_-}$ to a cuspidal automorphic representation $\Pi_1\boxtimes\Pi_2\boxtimes\tilde{\zeta}$ of a Levi subgroup $GL(2,\mathbb{A}_F)\times GL(2,\mathbb{A}_F)\times GL(1,\mathbb{A}_F)$ of $GL(4,\mathbb{A}_F)\times GL(1,\mathbb{A}_F)$ by

$$GL(2)\times GL(2)\times GL(1) \to GL(1)\times GSO(4), \quad (g_1,g_2,t)\mapsto (t\det(g_1), t(g_1\otimes g_2))$$

   From our local discussion, $\Pi_{\tilde{\phi}}$ also transfers to $(\Pi_1\boxplus\Pi_2)\boxtimes\tilde{\zeta}$. Hence $\Pi_{\tilde{\phi}}$ is the same as the global L-packet in [GT19]. At last, $\tilde{\theta}_0$ restricts on $GL(2)\times GL(2)\times GL(1)$ as $\tilde{\theta}_0(g_1,g_2,t) = (\tilde{\theta}(g_2),\tilde{\theta}(g_1),\det(g_1g_2)t)$. The condition that ϕ_- does not contain a summand of the trivial representation of $GL(1,\mathbb{A}_F)$ is equivalent to that $\bar{\Pi}_{\tilde{\phi}_-}$ is not θ_0-invariant, i.e., $\Pi_1\boxtimes\Pi_2\boxtimes\tilde{\zeta}$ is not $\tilde{\theta}_0$-invariant. Since Π_i has central character $\tilde{\zeta}$ and $\Pi_i^{\tilde{\theta}} = \tilde{\zeta}^{-1}\otimes\Pi_i$ for $i=1,2$, then this condition is also equivalent to $\Pi_1 \ncong \Pi_2$. This case corresponds to the Yoshida type in [Art04].

5. Comparison formulas

We would like to expand both sides of (3.4) in terms of local objects. First we need to introduce some general constructions from [Art13]. Suppose S is a connected complex reductive group with an automorphism θ, we denote $S^\theta = S \rtimes \theta$, which can be viewed as a connected component of the complex reductive group $S^+ := S \rtimes \langle\theta\rangle$. We fix a maximal torus T of S, and define the Weyl set

$$W^\theta(S) = \mathrm{Norm}(T, S^\theta)/T.$$

Let $W^\theta(S)_{reg}$ be the set of Weyl elements w such that $\det(w-1)|_{\mathfrak{a}_T} \neq 0$. Moreover, let $s^0(w)$ denote the sign $(-1)^n$, where n is the number of positive roots of (S,T) mapped by w to negative roots. Now we can assign to S^θ a real number

$$i^\theta(S) := |W(S)|^{-1} \sum_{w \in W^\theta_{reg}(S)} s^0(w) |\det(w-1)|^{-1}_{\mathfrak{a}_T},$$

where $W(S)$ is the Weyl group of S. Let us denote the set of semisimple elements of S^θ by S^θ_{ss}. And for any $s \in S^\theta_{ss}$, we write $S_s = \mathrm{Cent}(s, S)$. Let $S^\theta_{ell} = \{s \in S^\theta_{ss} : |Z(S_s)| < \infty\}$ and $\mathcal{E}^\theta_{ell}(S)$ the S-conjugacy classes in S^θ_{ell}. Arthur proved ([Art13], Proposition 4.1.1) that there exist unique constants $\sigma(S_1)$ defined for all connected complex reductive groups S_1 satisfying

$$\sigma(S_1) = \sigma(S_1/Z_1)|Z_1|^{-1}$$

for any central subgroup Z_1 of S_1, such that the following equality holds

$$\sum_{s \in \mathcal{E}^\theta_{ell}(S)} |\pi_0(S_s)|^{-1} \sigma((S_s)^0) = i^\theta(S) \tag{5.1}$$

for any pairs (S, θ).

Let us assume Theorem 4.1, 4.2, 4.8, 4.9 hold for the proper Levi subgroups and proper twisted endoscopic groups of $\widetilde{G}$. For $\phi \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$, let $\bar{T}_\phi$ be a maximal torus of $\bar{S}^0_\phi$. Then ϕ factors through $\phi_M \in \bar{\Phi}_2(M)$ for a Levi subgroup M of G such that $\widehat{M} := \mathrm{Cent}(\bar{T}_\phi, \widehat{G})$. For $\theta \in \Sigma_0$, let $\bar{S}^+_\phi$ be generated by $\bar{S}_\phi$ and $\bar{S}^\theta_\phi$. Define

$$\mathfrak{N}^\theta_\phi(G) = \mathrm{Norm}(\bar{T}_\phi, \bar{S}^\theta_\phi)/\mathrm{Cent}(\bar{T}_\phi, \bar{S}^0_\phi)^0, \quad \mathfrak{N}^+_\phi(G) = \mathrm{Norm}(\bar{T}_\phi, \bar{S}^+_\phi)/\mathrm{Cent}(\bar{T}_\phi, \bar{S}^0_\phi)^0,$$

and

$$W^+_\phi = W(\bar{T}_\phi, \bar{S}^+_\phi), \quad W^0_\phi = W(\bar{T}_\phi, \bar{S}^0_\phi), \quad R^+_\phi = W^+_\phi / W^0_\phi.$$

They fit into the following diagram.

(5.2)

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W^0_\phi & = & W^0_\phi & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{S}_{\phi_M} & \longrightarrow & \mathfrak{N}^+_\phi & \longrightarrow & W^+_\phi & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathcal{S}_{\phi_M} & \longrightarrow & \mathcal{S}^+_\phi & \longrightarrow & R^+_\phi & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

For $[\tilde{\pi}_M] \in \bar{\Pi}_{\tilde{\phi}_M}$,

$$\mathcal{I}_{\widetilde{P}}(\tilde{\pi}_M, \tilde{f}) := \int_{\widetilde{Z}_{\mathbb{A}_F} \backslash \widetilde{G}(\mathbb{A}_F)} \tilde{f}(g) \mathcal{I}_{\widetilde{P}}(\tilde{\pi}_M, g) dg$$

defines an operator on the space $\mathcal{H}_{\widetilde{P}}(\tilde{\pi}_M)$ of normalized induced representation of $\tilde{\pi}_M$. For $\omega \in Y$, let

$$\mathcal{I}_{\widetilde{P}}^{\theta,\omega}(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f}) = R(\theta)^{-1} \circ R(\omega) \circ \mathcal{I}_{\widetilde{P}}(\tilde{\pi}_M \otimes \omega^{-1}, \widetilde{f}).$$

For $u \in \mathfrak{N}_\phi^\theta$, let w_u be the image of u in W_ϕ^+ and θ_u a representative of w_u in $G(F) \rtimes \theta$ preserving the F-splitting of M. Then u gives an element in $\mathcal{S}_{\phi_M}^{\theta_u}$. We define

$$R_{P|\theta P}(\theta_u, \pi_M^+, \phi) := \bigotimes_v R_{P_v|\theta P_v}(\theta_{u_v}, \pi_{M_v}^+, \phi_v)$$

to be the normalized intertwining operator between $\mathcal{H}_{\theta P}(\pi_M^{\theta^{-1}})$ and $\mathcal{H}_P(\pi_M)$. After identifying

$$\mathcal{H}_{\theta\widetilde{P}}(\tilde{\pi}_M^{\theta^{-1}}) \cong \bigoplus_{\pi_M \subseteq \mathrm{Res}\,\tilde{\pi}_M} \mathcal{H}_{\theta P}(\pi_M^{\theta^{-1}})$$

$$\mathcal{H}_{\widetilde{P}}(\tilde{\pi}_M \otimes \omega^{-1}) \cong \bigoplus_{\pi_M \subseteq \mathrm{Res}\,\tilde{\pi}_M} \mathcal{H}_P(\pi_M)$$

as $G(\mathbb{A}_F)$-representations, we define the normalized intertwining operator between $\mathcal{H}_{\theta\widetilde{P}}(\tilde{\pi}_M^{\theta^{-1}})$ and $\mathcal{H}_{\widetilde{P}}(\tilde{\pi}_M \otimes \omega^{-1})$ to be

$$R_{\widetilde{P}|\theta\widetilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) := \bigoplus_{\pi_M \subseteq \mathrm{Res}\,\tilde{\pi}_M} \langle u, \pi_M^+ \rangle R_{P|\theta P}(\theta_u, \pi_M^+, \phi).$$

If $\tilde{\pi}_M$ is a discrete automorphic representation, then it follows from Theorem 4.2 and analogous result for general linear groups (cf. [Art13, Lemma 4.2.3]) that

$$R_{\widetilde{P}|\theta\widetilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) = r_P(w_u, \phi_M)^{-1} M_{\widetilde{P}|\theta\widetilde{P}}(w_u, \tilde{\pi}_M)$$

where

$$r_P(w, \phi) = L(0, \pi_{\phi_M}, \rho_{w_u^{-1}P|P}^\vee)\varepsilon(0, \pi_{\phi_M}, \rho_{w_u^{-1}P|P}^\vee, \psi_F)^{-1} L(1, \pi_{\phi_M}, \rho_{w_u^{-1}P|P}^\vee)^{-1}$$

and $\rho_{w_u^{-1}P|P}^\vee$ is the contragredient of the adjoint representation of ${}^L M$ over $\mathrm{Lie}(w_u^{-1}\widehat{N}_P) \cap \mathrm{Lie}(\widehat{N}_P) \backslash \mathrm{Lie}(w_u^{-1}\widehat{N}_P)$. In [Xu18, 5.4.1] we have incorrectly written the normalizing factor as a product of local normalizing factors. It should be obtained by analytic continuation of the product. This following lemma is [Xu18, Lemma 5.22].

Lemma 5.1. *Suppose $\phi \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$, $\theta \in \Sigma_0$ and $\omega \in Y$, then*

$$I_{disc,\phi}^{\widetilde{G}(\theta,\omega)}(\tilde{f}) = C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} \sum_{x \in \mathcal{S}_\phi^\theta(\omega)} i_\phi^\theta(x) \tilde{f}_{\widetilde{G}^\theta}(\tilde{\phi} \otimes \omega', x), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}) \tag{5.3}$$

where $C_{\tilde{\phi}} = m_\phi |\mathcal{S}_{\tilde{\phi}}|^{-1}$ and

$$\tilde{f}_{\widetilde{G}^\theta}(\tilde{\phi} \otimes \omega', x) = \sum_{[\tilde{\pi}_M] \in \bar{\Pi}_{\tilde{\phi}_M} \otimes \omega'} tr(R_{\widetilde{P}|\theta\widetilde{P}}(u, \tilde{\pi}_M, \tilde{\phi}) I_{\widetilde{P}}^{\theta,\omega}(\tilde{\pi}_M \otimes \omega^{-1}, \tilde{f})),$$

and $i_\phi^\theta(x) := i^s(\bar{S}_\phi^0)$ for any semisimple $s \in \bar{S}_\phi^\theta$.

For the expansion of the right hand side of (3.4), we introduce the following notations. Let $\phi = \phi_1 \times \cdots \times \phi_q \in \bar{\Phi}(G)$ such that $\phi_b \in \bar{\Phi}_2(G_b)$. Let

$$J = \{1 \leqslant i \leqslant q \,|\, \phi_i = \phi_{i,b}\}, \qquad K = \{1 \leqslant i \leqslant q \,|\, \phi_i \neq \phi_{i,b}, \phi_{i,o}\}$$

and

$$\phi_J = \prod_{i \in J} \phi_i, \qquad \phi_K = \prod_{i \in K} \phi_i.$$

The embedding ${}^L H \hookrightarrow {}^L G$ induces an isomorphism

$$S_{\phi_o}^{\Sigma_0} \times S_{\phi_b} \cong S_\phi^{\Sigma_0}$$

where

$$S_{\phi_o}^{\Sigma_0} = S_{\phi_{1,o}}^{\Sigma_{1,0}} \times \cdots \times S_{\phi_{q,o}}^{\Sigma_{q,0}}, \qquad S_{\phi_b} = S_{\phi_{1,b}} \times \cdots \times S_{\phi_{q,b}}.$$

This gives a projection

$$\mathcal{S}_{\phi}^{\Sigma_0} \to \mathcal{S}_{\phi_o}^{\Sigma_0}. \tag{5.4}$$

Let

$$\bar{\Pi}(\tilde{\phi}_o; \tilde{\phi}_b) = \operatorname{Tran} \bar{\Pi}_{\tilde{\phi}_o} \tilde{\otimes} \bar{\Pi}_{\tilde{\phi}_b}.$$

For $x_o \in \mathcal{S}_{\phi_o}^{\theta}(\omega)$, we define

$$\tilde{f}'_{\widetilde{G}^\theta}(\tilde{\phi}_o \otimes \omega', x_o; \tilde{\phi}_b) := \tilde{f}'_{\widetilde{G}^\theta}(\tilde{\phi} \otimes \omega', x) = \tilde{f}^{\widetilde{G}'}(\tilde{\phi}' \otimes \omega')$$

for any

$$x \in (S_{\phi_o}^{\theta}(\omega) \times Z(\widehat{G}_b))/S_{\phi_o}^0 Z(\widehat{G})^\Gamma \subseteq \mathcal{S}_{\phi}^{\theta}(\omega)$$

projecting to x_o. Let

$$e'^{\theta}_{\phi_o}(x_o) = \begin{cases} i^{\theta}_{\phi_o}(x_o) & \text{if } x_o \neq 1, \\ i^{\theta}_{\phi_o}(x_o) - \sigma(\bar{S}^0_{\phi_o}) & \text{if } x_o = 1. \end{cases} \tag{5.5}$$

Lemma 5.2. *Suppose $\phi \in \bar{\Phi}(G)$ such that $\phi_b \in \bar{\Phi}_2(G_b)$ and $\phi \neq \phi_b$. Let $\theta \in \Sigma_0$ and $\omega \in Y$.*

(1) *If $\theta = id, \omega = 1$, then there are two cases.*

(a) *$\phi = \phi_o$:*

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f}) - S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} \sum_{x \in \mathcal{S}_{\tilde{\phi}}} e'_\phi(x) \tilde{f}'_{\widetilde{G}}(\tilde{\phi} \otimes \omega', x), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}). \tag{5.6}$$

(b) *$\phi \neq \phi_o$:*

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f}) - S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = m_{\phi_J} C_{\tilde{\phi}_o} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi_o})} \sum_{x_o \in \mathcal{S}_{\tilde{\phi}_o}} e'_{\phi_o}(x_o) \tilde{f}'_{\widetilde{G}}(\tilde{\phi}_o \otimes \omega', x_o; \tilde{\phi}_b) \tag{5.7}$$

$$+ \operatorname{Tran} S^{\widetilde{G}_o}_{disc,\phi_o} \tilde{\otimes} \Big(\frac{m_{\phi_J}}{m_{\phi_b}} I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b} - (\frac{1}{4})^{|K|} S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b} \Big), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}) \tag{5.8}$$

where

$$I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}(\tilde{f}) = m_{\phi_b} \tilde{f}(\tilde{\phi}_b).$$

(2) *If $\theta \neq id$ or $\omega \neq 1$, then*

$$I^{(\widetilde{G}^\theta, \omega)}_{disc,\phi}(\tilde{f}) = m_{\phi_J} C_{\tilde{\phi}_o} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi_o})} \sum_{x_o \in \mathcal{S}^{\theta}_{\phi_o}(\omega)} e'^{\theta}_{\phi_o}(x_o) \tilde{f}'_{\widetilde{G}^\theta}(\tilde{\phi}_o \otimes \omega', x_o; \tilde{\phi}_b), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}). \tag{5.9}$$

This reduces to [Xu18, Lemma 5.25] when each ϕ_i has at most one GL-type simple orthogonal parameter. The remaining case that $\phi = \phi_b$ will be treated in the next section. To compare (5.7), (5.9) with (5.3), we note that $\tilde{f}_{\widetilde{G}^\theta}(\tilde{\phi} \otimes \omega', x)$ only depends on the image x_o of x under (5.4) and $i^{\theta}_{\phi}(x) = i^{\theta}_{\phi_o}(x_o)$. So we can rewrite (5.3) as

$$I^{(\widetilde{G}^\theta, \omega)}_{disc,\phi}(\tilde{f}) = m_{\phi_J} C_{\tilde{\phi}_o} \sum_{\omega' \in Y/\alpha(\mathcal{S}_{\phi_o})} \sum_{x_o \in \mathcal{S}^{\theta}_{\phi_o}(\omega)} i^{\theta}_{\phi_o}(x_o) \tilde{f}_{\widetilde{G}^\theta}(\tilde{\phi} \otimes \omega', x), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}). \tag{5.10}$$

Moreover,

$$i^{\theta}_{\phi_o}(x) - e'^{\theta}_{\phi_o}(x_o) = \begin{cases} 0 & \text{if } x_o \neq 1, \\ \sigma(\bar{S}^0_{\phi_o}) & \text{if } x_o = 1. \end{cases} \tag{5.11}$$

If we further assume Theorem 4.6, then by the local intertwining relation (cf. [Xu18, Theorem 4.12]) we get

$$\tilde{f}'_{\widetilde{G}^\theta}(\tilde{\phi}_o \otimes \omega', x_o; \tilde{\phi}_b) = \tilde{f}_{\widetilde{G}^\theta}(\tilde{\phi} \otimes \omega', x). \tag{5.12}$$

This is the global intertwining relation for $\widetilde{G}$.

6. GL-type orthogonal parameters

In this section we would like to prove Theorem 4.1, 4.8, 4.9 for $\phi = \phi_b \in \bar{\Phi}_2(G)$. Theorem 4.2 is irrelevant in this case. We can assume $G = G_1 \times \cdots \times G_q$, where $G_i = SO(2n_i)$. Let $\phi = \phi_1 \times \cdots \times \phi_q$, where $\phi_i \in \bar{\Phi}_2(G_i)$ and $\phi_{i,b} = \phi_i$. Then $\mathcal{S}_{\tilde{\phi}} = \mathcal{S}_\phi = \prod_{i=1}^q \mathcal{S}_{\phi_i} = \prod_{i=1}^q \mathcal{S}_{\tilde{\phi}_i}$. For $x = (x_i) \in \mathcal{S}_{\tilde{\phi}}$, ϕ factors through ϕ'_x for the elliptic endoscopic group G'_x. The following lemma treats the remaining case left by Lemma 5.2.

Lemma 6.1.

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f}) - S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = \sum_{1 \neq x \in \mathcal{S}_{\tilde{\phi}}} (m_\phi / m_{\phi'_x})^2 \, S^{\widetilde{G}'_x}_{disc,\phi'_x}(\tilde{f}'), \qquad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}).$$

Proof. We follow the same strategy in [Xu18, Section 5.4.2]. Recall

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f}) - S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = \sum_{\widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}) - \{\widetilde{G}\}} \iota(\widetilde{G}, \widetilde{G}') S^{\widetilde{G}'}_{disc,\phi}(\tilde{f}^{\widetilde{G}'}), \qquad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}),$$

where

$$\iota(\widetilde{G}, \widetilde{G}') = |\bar{Z}(\widehat{\widetilde{G}'})^\Gamma|^{-1} |\mathrm{Out}_{\widetilde{G}}(\widetilde{G}')|^{-1}.$$

First note

$$|\bar{Z}(\widehat{\widetilde{G}'})^\Gamma| = |\bar{Z}(\widehat{G}')^\Gamma| = m_{\phi'}/m_\phi, \quad \mathrm{Out}_{\widetilde{G}}(\widetilde{G}') = \mathrm{Out}_G(G').$$

Secondly, we can turn the right hand side into a sum over $\phi_G \in \Phi(G, \phi)$ and

$$\Big\{ (\widetilde{G}', \phi') : \widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}) - \{\widetilde{G}\} \text{ and } \phi' \in \Phi(G', \phi_G) \Big\}$$

where $\Phi(G', \phi_G)$ is the set $\widehat{G}'$-conjugacy classes of L-homomorphisms $\phi' : \mathcal{L}_\phi \to {}^L G'$ such that $\xi' \circ \phi'$ is $\widehat{G}$-conjugate to ϕ_G. The contribution from each ϕ_G-summand is the same, so we get a constant multiple $|\Phi(G, \phi)| = m_\phi$. Moreover, the contribution of $(\widetilde{G}', \phi')$ only depends on its image under

$$\Big\{ (\widetilde{G}', \phi') : \widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}) - \{\widetilde{G}\} \text{ and } \phi' \in \Phi(G', \phi_G) \Big\} \longrightarrow \mathcal{S}_\phi - \{1\},$$

which is $m_{\phi'_x}^{-1} \, S^{\widetilde{G}'_x}_{disc,\phi'_x}(\tilde{f}')$. The fiber has size $|\mathrm{Out}_G(G'_x)||S_{\phi,x}/S_{\phi,x} \cap \widehat{G}'_x Z(\widehat{G})^\Gamma|^{-1}$. Since $\phi = \phi_b$, we have $S_{\phi,x} = S_\phi \subseteq \widehat{G}'_x$. Hence $|S_{\phi,x}/S_{\phi,x} \cap \widehat{G}'_x Z(\widehat{G})^\Gamma| = 1$. In sum, we get

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f}) - S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = \sum_{1 \neq x \in \mathcal{S}_{\tilde{\phi}}} (m_\phi / m_{\phi'_x})^2 \, S^{\widetilde{G}'_x}_{disc,\phi'_x}(\tilde{f}').$$

□

Before starting the proof of the main theorems, we make the following observations. Theorem 4.1 implies (4.4) (4.9). So for Theorem 4.8, 4.9, it remains to show (4.5) and (4.10). At last, Theorem 4.1 follows from the special case that $G = SO(2n)$. So now we assume Theorem 4.1 holds for $G = SO(2n)$ with $n < N$ and $\phi = \phi_b \in \bar{\Phi}_2(G)$. In particular, we will fix $\bar{\Pi}_{\tilde{\phi}}$. For $G = G_1 \times \cdots \times G_q$ such that $n_i < N$, it follows that Theorem 4.1 and (4.9) holds for G. We define recursively

$$S^{\widetilde{G}}_{disc,\tilde{\phi}}(\tilde{f}) = I^{\widetilde{G}}_{disc,\tilde{\phi}}(\tilde{f}) - \sum_{1 \neq x \in \mathcal{S}_{\tilde{\phi}}} (m_\phi / m_{\phi'_x})^2 \, S^{\widetilde{G}'_x}_{disc,\tilde{\phi}'_x}(\tilde{f}'), \qquad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}).$$

We will show that it admits a tensor product decomposition as (4.13), and matches (4.8) in case $G = SO(2n)$. By induction on $|\mathcal{S}_{\tilde\phi}|$, one can easily show

$$S^{\widetilde{G}}_{disc,\phi}(\tilde f) = \sum_{\omega\in Y} S^{\widetilde{G}}_{disc,\tilde\phi}(\tilde f\otimes\omega). \tag{6.1}$$

Lemma 6.2.

$$S^{\widetilde{G}}_{disc,\tilde\phi} = \tilde\otimes_i\, S^{\widetilde{G}_i}_{disc,\tilde\phi_i} \tag{6.2}$$

$$S^{\widetilde{G}}_{disc,\phi} = (\tilde\otimes_{i\neq 1} S^{\widetilde{G}_i}_{disc,\tilde\phi_i})\,\tilde\otimes\, S^{\widetilde{G}_1}_{disc,\phi_1}. \tag{6.3}$$

Proof. In view of (6.1), (6.3) follows from (6.2). We can write the right hand side of (6.2) as

$$\tilde\otimes_i\, S^{\widetilde{G}_i}_{disc,\tilde\phi_i} = \tilde\otimes_i\, I^{\widetilde{G}_i}_{disc,\tilde\phi_i} - \sum_{1\neq x\in\mathcal{S}_{\tilde\phi}} (m_\phi/m_{\phi'_x})^2\,\mathrm{Tran}\,(\tilde\otimes_i\, S^{\widetilde{G}'_{x_i}}_{disc,\tilde\phi'_{x_i}}).$$

By induction on $|\mathcal{S}_{\tilde\phi}|$, we can assume

$$S^{\widetilde{G}'_x}_{disc,\tilde\phi'_x} = \tilde\otimes_i\, S^{\widetilde{G}'_{x_i}}_{disc,\tilde\phi'_{x_i}}.$$

So it suffices to know

$$I^{\widetilde{G}}_{disc,\tilde\phi} = \tilde\otimes_i\, I^{\widetilde{G}_i}_{disc,\tilde\phi_i}$$

which follows easily from (4.9). □

Lemma 6.3. *For $G = SO(2n)$,*

$$S^{\widetilde{G}}_{disc,\tilde\phi}(\tilde f) = m_\phi \sum_{\phi_1,\cdots,\phi_m} (-1)^{m-1} a_m\, \tilde f(\tilde\phi_1,\cdots,\tilde\phi_m),\qquad \tilde f\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}), \tag{6.4}$$

$$S^{\widetilde{G}}_{disc,\phi}(\tilde f) = m_\phi \sum_{\phi_1,\cdots,\phi_m} (-1)^{m-1} a_m\, \tilde f(\phi_1,\cdots,\phi_m),\qquad \tilde f\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}). \tag{6.5}$$

In both cases, the sum is over unordered partition of ϕ.

Proof. In view of (6.1), (6.5) follows from (6.4). We will prove (6.5) by induction on n. By definition,

$$S^{\widetilde{G}}_{disc,\tilde\phi}(\tilde f) = I^{\widetilde{G}}_{disc,\tilde\phi}(\tilde f) - \sum_{1\neq x\in\mathcal{S}_{\tilde\phi}} (m_\phi/m_{\phi'_x})^2\, S^{\widetilde{G}'_x}_{disc,\tilde\phi'_x}(\tilde f').$$

For $1\neq x\in\mathcal{S}_{\tilde\phi}$, let $G'_x = G_I\times G_{II}$ and $\phi'_x = \phi_I\times\phi_{II}$. By (6.2) and the induction assumption, we can write

$$\begin{aligned} S^{\widetilde{G}'_x}_{disc,\tilde\phi'_x} &= S^{\widetilde{G}_i}_{disc,\tilde\phi_i}\,\tilde\otimes\, S^{\widetilde{G}_{II}}_{disc,\tilde\phi_{II}} \\ &= m_{\phi_I} m_{\phi_{II}} \Big(\sum_{\phi_{I,1},\cdots,\phi_{I,l}} (-1)^{l-1} a_l\, \tilde f_I(\tilde\phi_{I,1},\cdots,\tilde\phi_{I,l})\Big) \\ &\quad \tilde\otimes \Big(\sum_{\phi_{II,1},\cdots,\phi_{II,k}} (-1)^{k-1} a_k\, \tilde f_{II}(\tilde\phi_{II,1},\cdots,\tilde\phi_{II,k})\Big). \end{aligned}$$

Then

$$S^{\widetilde{G}'_x}_{disc,\tilde\phi'_x}(\tilde f') = (-1)^{k+l} m_{\phi_I} m_{\phi_{II}} a_k a_l \tilde f(\tilde\phi_{I,1},\cdots,\tilde\phi_{I,l},\tilde\phi_{II,1},\cdots,\tilde\phi_{II,k}).$$

Note

$$m_\phi/m_{\phi'_x} = \frac{1}{2},\qquad m_{\phi'_x} = m_{\phi_I} m_{\phi_{II}}.$$

So

$$-(m_\phi/m_{\phi'_x})^2\, S^{\widetilde{G}'_x}_{disc,\tilde\phi'_x}(\tilde f') = m_\phi\,(-1)^{k+l-1}\,(\frac{1}{2} a_k a_l)\,\tilde f(\tilde\phi_{I,1},\cdots,\tilde\phi_{I,l},\tilde\phi_{II,1},\cdots,\tilde\phi_{II,k}).$$

At last, the association of x with ϕ_I, ϕ_{II} gives a bijection between $\mathcal{S}_{\tilde{\phi}} - \{1\}$ and unordered 2-partitions of ϕ. So

$$S^{\widetilde{G}}_{disc,\tilde{\phi}}(\tilde{f}) - I^{\widetilde{G}}_{disc,\tilde{\phi}}(\tilde{f})$$

is a sum of distributions

$$\tilde{f}(\tilde{\phi}_1, \cdots, \tilde{\phi}_m)$$

over unordered m-partitions of ϕ for $m > 1$, with coefficients

$$(-1)^{m-1}\frac{1}{2}\sum_{S_1,S_2} a_{|S_1|}a_{|S_2|}$$

where the sum is over unordered 2-partitions (S_1, S_2) of $\{\phi_1, \cdots, \phi_m\}$. It is easy to check that

$$\frac{1}{2}\sum_{S_1,S_2} a_{|S_1|}a_{|S_2|} = a_m.$$

The rest is clear.

□

Lemma 6.2, 6.3 complete the proof of Theorem 4.8, 4.9 in case $n_i < N$. To complete the induction argument, we still need to prove Theorem 4.1 for $G = SO(2N)$. By Lemma 6.1,

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f}) - S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = \sum_{1\neq x\in\mathcal{S}_{\tilde{\phi}}} (m_\phi/m_{\phi'_x})^2\, S^{\widetilde{G}'_x}_{disc,\phi'_x}(\tilde{f}').$$

For $1 \neq x \in \mathcal{S}_{\tilde{\phi}}$, let $G'_x = G_I \times G_{II}$ and $\phi'_x = \phi_I \times \phi_{II}$. By (6.3) and Lemma 6.3, we can write

$$\begin{aligned} S^{\widetilde{G}'_x}_{disc,\phi'_x} &= S^{\widetilde{G}_I}_{disc,\phi_I} \tilde{\otimes}\, S^{\widetilde{G}_{II}}_{disc,\tilde{\phi}_{II}} \\ &= m_{\phi_I} m_{\phi_{II}} \Big(\sum_{\phi_{I,1},\cdots,\phi_{I,l}} (-1)^{l-1} a_l\, \tilde{f}_I(\phi_{I,1}, \cdots, \phi_{I,l}) \Big) \\ &\tilde{\otimes} \Big(\sum_{\phi_{II,1},\cdots,\phi_{II,k}} (-1)^{k-1} a_k\, \tilde{f}_{II}(\tilde{\phi}_{II,1}, \cdots, \tilde{\phi}_{II,k}) \Big). \end{aligned}$$

Then we see $S^{\widetilde{G}'_x}_{disc,\phi'_x}(\tilde{f}')$ is stable. It follows that $I^{\widetilde{G}}_{disc,\phi}(\tilde{f})$ is stable. Then one can argue by stability to show Theorem 4.1. The arguement is as follows. We define $\bar{\Pi}_{\tilde{\phi}} = \otimes'_v \bar{\Pi}_{\tilde{\phi}_v}$ such that it contains a discrete automorphic representation $\tilde{\pi}^0$. Since $I^{\widetilde{G}}_{disc,\phi}(\tilde{f})$ is stable, it is stable at every place. So we can take $\tilde{f} = \otimes_w \tilde{f}_w$ and fix $\otimes_{w\neq v}\tilde{f}_w$ for any place v, then by [Xu18, Corollary 4.8] the coefficient of $\tilde{f}_v(\tilde{\pi}_v)$ in $I^{\widetilde{G}}_{disc,\phi}(\tilde{f})$ must be the same for all $\tilde{\pi}_v \in \bar{\Pi}_{\tilde{\phi}_v}$. By varying $\otimes_{w\neq v}\tilde{f}_w$ and the linear independence of characters of $\otimes_{w\neq v}\bar{\mathcal{H}}(\widetilde{G}_w, \widetilde{\chi}_w)$-modules, we have that

$$[\tilde{\pi}^0] = [\tilde{\pi}^0_v] \otimes (\otimes_{w\neq v}[\tilde{\pi}^0_w])$$

contributes to $I^{\widetilde{G}}_{disc,\phi}(\tilde{f})$ if and only if all elements in

$$\bar{\Pi}_{\tilde{\phi}_v} \otimes (\otimes_{w\neq v}[\tilde{\pi}^0_w])$$

also contribute to $I^{\widetilde{G}}_{disc,\phi}(\tilde{f})$. By repeating this argument, one can show all elements in $\bar{\Pi}_{\tilde{\phi}}$ contribute to $I^{\widetilde{G}}_{disc,\phi}(\tilde{f})$. Then Theorem 4.1 will follow from (2.20) immediately. This finishes the proof of Theorem 4.1, 4.8, 4.9 in the case $\phi = \phi_b \in \bar{\Phi}_2(G)$.

7. Proof of Lemma 5.2

In this section we will prove Lemma 5.2 by assuming Theorem 4.1, 4.2, 4.8, 4.9 for the proper Levi subgroups and proper twisted endoscopic groups of $\widetilde{G}$. We follow the same strategy as in [Xu18, Section 5.4.2]. The goal is to expand the right hand sides of

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f}) - S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = \sum_{\widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}) - \{\widetilde{G}\}} \iota(\widetilde{G}, \widetilde{G}') S^{\widetilde{G}'}_{disc,\phi}(\tilde{f}^{\widetilde{G}'}),$$

and

$$I^{(\widetilde{G}^\theta,\omega)}_{disc,\phi}(\tilde{f}) = \sum_{\widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}^\theta,\omega)} \iota(\widetilde{G}, \widetilde{G}') S^{\widetilde{G}'}_{disc,\phi}(\tilde{f}^{\widetilde{G}'})$$

in case $\theta \neq id$ or $\omega \neq 1$. We can turn the right hand side into a sum over $\phi_G \in \Phi(G,\phi)$ and

$$\Big\{(\widetilde{G}', \phi') : \widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}^\theta) - \{\widetilde{G}\} \text{ and } \phi' \in \Phi(G', \phi_G)\Big\}$$

where $\Phi(G', \phi_G)$ is the set $\widehat{G}'$-conjugacy classes of L-homomorphisms $\phi' : \mathcal{L}_\phi \to {}^L G'$ such that $\xi' \circ \phi'$ is $\widehat{G}$-conjugate to ϕ_G. The contribution from each ϕ_G-summand is the same, so we get a constant multiple $|\Phi(G,\phi)| = m_\phi$. Moreover, the contribution of $(\widetilde{G}', \phi')$ only depends on its image under

$$\Big\{(\widetilde{G}', \phi') : \widetilde{G}' \in \mathcal{E}_{ell}(\widetilde{G}^\theta) - \{\widetilde{G}\} \text{ and } \phi' \in \Phi(G', \phi_G)\Big\} \longrightarrow \bar{S}_\phi \backslash (\bar{S}^\theta_{\phi,ss}(\omega) - \{1\}),$$

which is $\frac{\iota(\widetilde{G}, \widetilde{G}'_x)}{m_{\phi'_x}} S^{\widetilde{G}'_x}_{disc,\phi'_x}(\tilde{f}')$. Here $\bar{S}^\theta_{\phi,ss}(\omega) = \{s \in \bar{S}^\theta_{\phi,ss} : \alpha(s) = \omega\}$ and the fiber has size $|\mathrm{Out}_G(G'_x)||S_{\phi,x}/S_{\phi,x} \cap \widehat{G}'_x Z(\widehat{G})^\Gamma|^{-1}$.

Suppose G is $Sp(2n)$ or $SO(2n,\eta)$ and $\phi \neq \phi_o, \phi_b$. We have $m_\phi = m_{\phi_o}$ and $S^{\Sigma_0}_\phi \cong S^{\Sigma_0}_{\phi_o} \times S_{\phi_b}$, which induces

$$1 \to S_{\phi_b} \to \bar{S}^{\Sigma_0}_\phi \to \bar{S}^{\Sigma_0}_{\phi_o} \to 1.$$

Moreover, we have

$$\bar{S}_\phi \backslash (\bar{S}^\theta_{\phi,ss}(\omega) - \{1\}) \longrightarrow \bar{S}_{\phi_o} \backslash \bar{S}^\theta_{\phi_o,ss}(\omega) \quad x \mapsto x_o. \tag{7.1}$$

Note

$$|\mathrm{Out}_G(G')| = \begin{cases} |\mathrm{Out}_{G_o}(G'_o)| & \text{if } \phi'_o \neq \phi_o \\ 2|\mathrm{Out}_{G_o}(G'_o)| & \text{if } \phi'_o = \phi_o \end{cases}$$

and $|S_{\phi,x}/S_{\phi,x} \cap \widehat{G}'_x Z(\widehat{G})^\Gamma| = |S_{\phi_o,x_o}/S_{\phi_o,x_o} \cap \widehat{G}'_{o,x_o} Z(\widehat{G}_o)^\Gamma|$ for $S_{\phi_b} \subseteq S_{\phi,x} \cap \widehat{G}' Z(\widehat{G})^\Gamma$. At last

$$\iota(\widetilde{G}, \widetilde{G}') = \begin{cases} \iota(\widetilde{G}_o, \widetilde{G}'_o) & \text{if } \phi'_o \neq \phi_o \\ \frac{1}{4}\iota(\widetilde{G}_o, \widetilde{G}'_o) & \text{if } \phi'_o = \phi_o \end{cases}$$

$$m_{\phi'} = \begin{cases} m_{\phi'_o} & \text{if } \phi'_o \neq \phi_o \\ 2\, m_{\phi'_o} & \text{if } \phi'_o = \phi_o \end{cases}$$

and

$$S^{\widetilde{G}'}_{disc,\phi'} = (\frac{1}{4})^l \,\mathrm{Tran}\,(S^{\widetilde{G}'_o}_{disc,\phi'_o} \tilde{\otimes} S^{\widetilde{G}'_b}_{disc,\tilde{\phi}'_b}),$$

where

$$l = \begin{cases} 2 & \text{if } \phi'_o \neq \phi_o, \phi'_b \neq \phi_b \\ 0 & \text{if } \phi'_o = \phi_o, \phi'_b = \phi_b \\ 1 & \text{otherwise.} \end{cases}$$

By the compatibility of parabolic induction with endoscopic transfer, we have

$$\mathrm{Tran}\, S^{\widetilde{G}'}_{disc,\phi'} = (\frac{1}{4})^l \,\mathrm{Tran}\,\Big(\mathrm{Tran}\, S^{\widetilde{G}'_o}_{disc,\phi'_o} \tilde{\otimes} \mathrm{Tran}\, S^{\widetilde{G}'_b}_{disc,\tilde{\phi}'_b}\Big).$$

Here the distribution in the bracket may not be stable, so its transfer is in the sense of Remark 2.1. We separate the coefficients for $(\widetilde{G}_o, \widetilde{G}'_o)$ and sum over the fibers of (7.1). Since S_{ϕ_b} is in the center of $\bar{S}_\phi^{\Sigma_0}$, the fiber of

$$\bar{S}_\phi \backslash \bar{S}^\theta_{\phi,ss}(\omega) \longrightarrow \bar{S}_{\phi_o} \backslash \bar{S}^\theta_{\phi_o,ss}(\omega)$$

is an S_{ϕ_b}-torsor. Note $\operatorname{Tran} S^{\widetilde{G}'_o}_{disc,\phi'_o}$ remains the same over the fiber and $\operatorname{Tran} S^{\widetilde{G}'_b}_{disc,\tilde{\phi}'_b}$ only depends on the image under $S_{\phi_b} \to \mathcal{S}_{\phi_b} = \mathcal{S}_{\tilde{\phi}_b}$, whose kernel is $Z(\widehat{G}_b) \cong \mathbb{Z}/2\mathbb{Z}$. First we sum over the fiber of $x_o \neq 1$, in which case $\phi'_o \neq \phi_o$ and the result is

$$\operatorname{Tran}\Big(\operatorname{Tran}(S^{\widetilde{G}'_o}_{disc,\phi'_o}) \tilde{\otimes} |Z(\widehat{G}_b)| \cdot \Big(\sum_{1\neq x \in \mathcal{S}_{\tilde{\phi}_b}} (\frac{1}{4})^2 \operatorname{Tran}(S^{\widetilde{G}'_{b,x}}_{disc,\tilde{\phi}'_{b,x}}) + \frac{1}{4} S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}\Big)\Big)$$

$$= \operatorname{Tran}\Big(\operatorname{Tran}(S^{\widetilde{G}'_o}_{disc,\phi'_o}) \tilde{\otimes} \frac{1}{2} I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}\Big).$$

Next we sum over the fiber of $x_o = 1$, in which case $\phi'_o = \phi_o$ and the result is

$$\operatorname{Tran}\Big(\operatorname{Tran}(S^{\widetilde{G}_o}_{disc,\phi_o}) \tilde{\otimes} \frac{1}{4}\Big(|Z(\widehat{G}_b)| \sum_{1\neq x \in \mathcal{S}_{\tilde{\phi}_b}} \frac{1}{4} \operatorname{Tran}(S^{\widetilde{G}'_{b,x}}_{disc,\tilde{\phi}'_{b,x}}) + S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}\Big)\Big)$$

$$= \operatorname{Tran} S^{\widetilde{G}_o}_{disc,\phi_o} \tilde{\otimes} \Big(\frac{1}{2} I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b} - \frac{1}{4} S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}\Big).$$

In both cases we have used Lemma 6.1.

In general, recall

$$\phi_J = \prod_{i\in J} \phi_i, \qquad \phi_K = \prod_{i\in K} \phi_i$$

and

$$1 \to S_{\phi_{K,b}} \times \bar{S}_{\phi_{J,b}} \to \bar{S}_\phi^{\Sigma_0} \to \bar{S}_{\phi_o}^{\Sigma_0} \to 1$$

which induces

$$\bar{S}_\phi \backslash \bar{S}^\theta_{\phi,ss}(\omega) \longrightarrow \bar{S}_{\phi_o} \backslash \bar{S}^\theta_{\phi_o,ss}(\omega) \quad x \mapsto x_o$$

where the fiber is a $S_{\phi_{K,b}} \times \bar{S}_{\phi_{J,b}}$-torsor. By the decomposition

$$S_{\phi_{K,b}} \times \bar{S}_{\phi_{J,b}} = \prod_{k\in K} S_{\phi_{k,b}} \times \prod_{j\in J} \bar{S}_{\phi_{j,b}}$$

we could sum over $S_{\phi_{k,b}}$ and $\bar{S}_{\phi_{j,b}}$ respectively. Note

$$m_\phi = \prod_i m_{\phi_i} \qquad m_{\phi'} = \prod_i m_{\phi'_i}$$

$$|\operatorname{Out}_G(G')| = \prod_i |\operatorname{Out}_{G_i}(G'_i)| \qquad \iota(\widetilde{G}, \widetilde{G}') = \prod_i \iota(\widetilde{G}_i, \widetilde{G}'_i)$$

and

$$S^{\widetilde{G}'}_{disc,\phi'} = (\frac{1}{4})^l \operatorname{Tran}(S^{\widetilde{G}'_o}_{disc,\phi'_o} \tilde{\otimes} S^{\widetilde{G}'_b}_{disc,\tilde{\phi}'_b})$$

where

$$(\frac{1}{4})^l S^{\widetilde{G}'_b}_{disc,\tilde{\phi}'_b} = \tilde{\otimes}_{k\in K} (\frac{1}{4})^{l_k} S^{\widetilde{G}'_{k,b}}_{disc,\tilde{\phi}'_{k,b}} \tilde{\otimes} \tilde{\otimes}_{j\in J} S^{\widetilde{G}'_{j,b}}_{disc,\tilde{\phi}'_{j,b}}$$

and

$$l_k = \begin{cases} 2 & \text{if } \phi'_{k,o} \neq \phi_{k,o}, \phi'_{k,b} \neq \phi_{k,b} \\ 0 & \text{if } \phi'_{k,o} = \phi_{k,o}, \phi'_{k,b} = \phi_{k,b} \\ 1 & \text{otherwise.} \end{cases}$$

By the compatibility of parabolic induction with endoscopic transfer,

$$\operatorname{Tran} S^{\widetilde{G}'}_{disc,\phi'} = (\frac{1}{4})^l \operatorname{Tran}\Big(\operatorname{Tran} S^{\widetilde{G}'_o}_{disc,\phi'_o} \tilde{\otimes} \operatorname{Tran} S^{\widetilde{G}'_b}_{disc,\tilde{\phi}'_b}\Big)$$

$$= \mathrm{Tran}\Big(\mathrm{Tran}\, S^{\widetilde{G}'_o}_{disc,\phi'_o} \tilde{\otimes} \Big(\tilde{\otimes}_{k\in K}\mathrm{Tran}\,(\frac{1}{4})^{l_k} S^{\widetilde{G}'_{k,b}}_{disc,\tilde{\phi}'_{k,b}}\Big)\tilde{\otimes}\Big(\tilde{\otimes}_{j\in J}\mathrm{Tran}\, S^{\widetilde{G}'_{j,b}}_{disc,\tilde{\phi}'_{j,b}}\Big)\Big).$$

We again separate the coefficients for $(\widetilde{G}_o, \widetilde{G}'_o)$ and sum over the fibers of

$$\bar{S}_\phi\backslash(\bar{S}^\theta_{\phi,ss}(\omega) - \{1\}) \longrightarrow \bar{S}_{\phi_o}\backslash\bar{S}^\theta_{\phi_o,ss}(\omega) \quad x \mapsto x_o.$$

For $x_o \neq 1$, the sum over $\bar{S}_{\phi_{j,b}}$ gives $I^{\widetilde{G}_{j,b}} disc, \tilde{\phi}_{j,b}$ and the sum over $S_{\phi_{k,b}}$ gives $\frac{1}{2} I^{\widetilde{G}_{k,b}}_{disc,\tilde{\phi}_{k,b}}$. So the result is

$$\begin{aligned}&\mathrm{Tran}\Big(\mathrm{Tran}\,(S^{\widetilde{G}'_o}_{disc,\phi'_o})\tilde{\otimes}\Big(\tilde{\otimes}_{k\in K}\frac{1}{2}\, I^{\widetilde{G}_{k,b}}_{disc,\tilde{\phi}_{k,b}}\Big)\tilde{\otimes}\Big(\tilde{\otimes}_{j\in J} I^{\widetilde{G}_{j,b}} disc, \tilde{\phi}_{j,b}\Big)\Big)\\ =&\,\mathrm{Tran}\Big(\mathrm{Tran}\,(S^{\widetilde{G}'_o}_{disc,\phi'_o})\tilde{\otimes}\frac{m_{\phi_J}}{m_{\phi_b}}\, I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}\Big).\end{aligned}$$

For $x_o = 1$, we obtain the same result after adding $\mathrm{Tran}\, S^{\widetilde{G}_o}_{disc,\phi_o}\tilde{\otimes}\,(\frac{1}{4})^{|K|}\, S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}$. At last, the sum over $\bar{S}_{\phi_o}\backslash(\bar{S}^\theta_{\phi_o,ss}(\omega) - \{1\})$ reduces to the case for ϕ_o, which has been treated in [Xu18, Lemma 5.25].

8. Compatible lifting of L-packets

8.1. **Factorization of local and global parameters.** Let F be a local (resp. global) field of characteristic zero. Let G be a quasisplit symplectic or special even orthogonal group over F, and $[\phi] \in \bar{\Phi}^+_{unit}(G)$ (resp. $\bar{\Phi}(G)$). We choose $\xi_G : {}^LG \to GL(V)$, whose image is contained in an orthogonal group $O(V)$ for some nondegenerate symmetric bilinear form B. We choose a decomposition of V into subrepresentations (not necessarily irreducible) of L_F (resp. $\mathcal{L}_\phi$) as follows

$$\phi = \oplus_{i\in I}\phi_i \oplus \oplus_{j\in J}(\phi_{GL,j} \oplus \phi^\vee_{GL,j}), \quad V = \oplus_{i\in I} V_i \oplus \oplus_{j\in J}(W_j \oplus W_j^\vee) \tag{8.1}$$

such that the restriction of B to V_i is nondegenerate and to W_j is zero, and $W_j^\vee$ is dual to W_j under B. Let $N_i = \dim V_i$ $(i \in I)$ and $N_j = \dim W_j$ $(j \in J)$. Let $\eta_i := \eta_{\phi_i}$ $(i \in I)$ associated with an extension E_i/F of degree at most two. We would like to factorize ϕ through a sequence of proper twisted endoscopic groups and proper Levi subgroups

$$\phi : L_F \text{ (resp. } \mathcal{L}_\phi) \xrightarrow{\phi^{(n)}} {}^LG^{(n)} \xrightarrow{\xi^{(n-1)}} {}^LG^{(n-1)} \xrightarrow{\xi^{(n-2)}} \cdots \xrightarrow{\xi^{(0)}} {}^LG \tag{8.2}$$

according to the decomposition (8.1). By this we mean

$$\widehat{G}^{(k)} = \prod_l SO(V_l^{(k)}) \times \prod_m GL(W_m^{(k)}),$$

where

$$V_l^{(k)} = \oplus_{i\in I_l^{(k)}} V_i \oplus \oplus_{j\in J_l^{(k)}}(W_j \oplus W_j^\vee), \qquad W_m^{(k)} = \oplus_{i\in S_m^{(k)}/\langle\sigma\rangle} V^{1/2}_{i,\sigma(i)} \oplus \oplus_{j\in T_m^{(k)}} W_j$$

for $I_l^{(k)}, S_m^{(k)} \subseteq I$ and $J_l^{(k)}, T_m^{(k)} \subseteq J$, and σ is an involution on $S_m^{(k)}$ without fixed points such that $\phi_i \cong \phi_{\sigma(i)}$, and $V^{1/2}_{i,\sigma(i)} \subseteq V_i \oplus V_{\sigma(i)}$ is an L_F (resp. $\mathcal{L}_\phi$)-subrepresentation $\phi^{1/2}_{i,\sigma(i)}$, on which B vanishes. The composition of ${}^LG^{(k)} \to {}^LG$ with ξ_G factors through

$${}^LG^{(k)} \longrightarrow \prod_l O(V_l^{(k)}) \times \prod_m GL(W_m^{(k)}).$$

Let $\phi^{(k)} : L_F$ (resp. $\mathcal{L}_\phi$) $\to {}^LG^{(k)}$ and $\phi^{(0)} = \phi$. We write

$$\phi^{(k)} = \prod_l \phi_l^{(k)} \times \prod_m \phi^{(k)}_{GL,m}. \tag{8.3}$$

Then

$$\phi_l^{(k)} = (\oplus_{i\in I_l^{(k)}}\phi_i \oplus \oplus_{j\in J_l^{(k)}}(\phi_{GL,j} \oplus \phi^\vee_{GL,j})) \otimes \eta_l^{(k)} \in \bar{\Phi}^+_{unit}(G_l^{(k)}) \,(\text{resp. } \bar{\Phi}(G_l^{(k)}))$$

for some quadratic character $\eta_l^{(k)}$. We require that $\phi_l^{(n)} = \phi_i \otimes \eta_l^{(n)}$ for some $i \in I$ such that $N_i \neq 1$, or $\phi_l^{(n)} = (1 \oplus \eta')\eta$ for some quadratic characters η, η'. We also require $\xi^{(k)}$ to be given by the fiber product

$$\begin{array}{ccc} {}^LG^{(k+1)} & \longrightarrow & {}^LG^{(k)} \\ \downarrow & & \downarrow \\ O(V_{l_1}^{(k+1)}) \times O(V_{l_2}^{(k+1)}) & \longrightarrow & O(V_l^{(k)}) \end{array}$$

for some l or

$$\begin{array}{ccc} {}^LG^{(k+1)} & \longrightarrow & {}^LG^{(k)} \\ \downarrow & & \downarrow \\ GL(W_{m_1}^{(k+1)}) \times O(V_{l_1}^{(k+1)}) & \longrightarrow & O(V_l^{(k)}) \end{array}$$

for some l or

$$\begin{array}{ccc} {}^LG^{(k+1)} & \longrightarrow & {}^LG^{(k)} \\ \downarrow & & \downarrow \\ GL(W_{m_1}^{(k+1)}) \times GL(W_{m_2}^{(k+1)}) & \longrightarrow & GL(W_m^{(k)}) \end{array}$$

for some m.

8.2. **Lifting local and global parameters.** When F is global, we will only consider the restriction of the global parameter from $\mathcal{L}_\phi$ to $L_F := W_F$ and denote it still by ϕ. We will give two lifts of ϕ. First of all, we would like to choose lifts of ϕ_i for even N_i and $\phi_i \otimes \eta_i$ for odd $N_i \neq 1$. Then we would like to lift ϕ_i to $GPin(V_i)$. When N is even, we just take the chosen lift $\tilde{\phi}_i$. When N is odd, we take

$$\widetilde{\phi_i \otimes \eta_i} \otimes \tilde{\eta}_i^{\square}$$

where $\tilde{\eta}_i^{\square} : \Gamma_{E_i/F} \to Pin(V_i)$ is a lift of $\eta_i : \Gamma_{E_i/F} \to O(V_i), \tau_i \mapsto -I$. Then we get a lift $\tilde{\phi}_O^\sharp$ of $\phi_O := \prod_{i\in I} \phi_i$ through the multiplication in the Clifford algebra $Cl(\oplus_{i\in I} V_i)$:

$$m : \prod_{i\in I} GPin(V_i) \to GPin(\oplus_{i\in I} V_i).$$

Let $\tilde{\phi}^\sharp := \prod_j \phi_{GL,j} \rtimes \tilde{\phi}_O^\sharp$ be a lift of ϕ through the natural embedding

$$\prod_{j\in J} GL(W_j) \times GPin(\oplus_{i\in I} V_i) \hookrightarrow GPin(V).$$

Note $\tilde{\phi}^\sharp$ is not a homomorphism.

Next, we want to lift ϕ through that of $\phi^{(n)}$ and $\xi^{(k)}$. To lift $\phi^{(n)}$, we first lift $\phi_O^{(n)} := \prod_l \phi_l^{(n)}$ through that of $\phi_l^{(n)}$ (cf. (8.3)). There are three cases.

(1) $\phi_l^{(n)} = \phi_i \otimes \eta_i$ for odd $N_i \neq 1$. Then we can take the chosen lift $\widetilde{\phi_i \otimes \eta_i}$.

(2) $\phi_l^{(n)} = \phi_i \otimes \eta$ for even N_i and some quadratic character η. We can take the chosen lift of ϕ_i

$$\tilde{\phi}_i : L_F \to GSpin(V_i) \rtimes \Gamma_{E_i/F}$$

and twist it by a lift of η

$$\tilde{\eta}^{\square} : \Gamma_{E/F} \to Spin(V_i),$$

where E/F is the quadratic extension associated with η. Note $\tilde{\phi}_i \otimes \tilde{\eta}^{\square}$ is not a homomorphism. The obstruction is given by a 2-cocycle $b_l^{(n)} \in Z^2(\Gamma_{EE_i/F}, \mathbb{Z}/2\mathbb{Z})$, which splits in $Z^2(W_F, \widehat{D})$ (cf. [Lan79, Lemma 4]). In particular, we can choose a 1-cochain $c_l^{(n)} : W_F \to \widehat{D}$ such that $\tilde{\phi}_i \otimes \tilde{\eta}^{\square} c_l^{(n)}$ becomes a homomorphism.

(3) $\phi_l^{(n)} = (1 \oplus \eta')\eta$. Note $G_l^{(n)} = SO(2, \eta')$ and $1 \oplus \eta' \in \bar{\Phi}_{bdd}(G^{(n))_l}$ corresponds to the trivial representation of $G_l^{(n)}(F)$ (resp. $G_l^{(n)}(\mathbb{A}_F)$). We take the trivial lift $\widetilde{1 \oplus \eta'}^{\text{triv}} \in \bar{\Phi}_{bdd}(\widetilde{G}_l^n)$, i.e., it corresponds to the trivial representation of $\widetilde{G}_l^{(n)}(F)$ (resp. $\widetilde{G}_l^{(n)}(\mathbb{A}_F)$), and twist it by $\tilde{\eta}^{\square}$ as in the second case. To make $(\widetilde{1 \oplus \eta'}^{\text{triv}}) \otimes \tilde{\eta}^{\square}$ a homomorphism, we need to further choose a 1-cochain $c_l^{(n)} : W_F \to \widehat{D}$.

Let us denote the resulting lift of $\phi_O^{(n)}$ by $\tilde{\phi}_O^{(n)\square}$. Then $\prod_m \phi_{GL,m}^{(n)} \rtimes \tilde{\phi}_O^{(n)\square}$ is a lift of $\phi^{(n)}$. In order to compare it with $\tilde{\phi}^{\sharp}$, we need to twist it further according to $\phi_{GL,m}^{(n)}$. There are two cases. For $\phi_{GL,j} \otimes \eta$ in $\phi_{GL,m}^{(n)}$, we need to introduce a twist by

$$(\phi_{GL,j} \rtimes 1) \otimes \tilde{\eta}^{\square}/(\phi_{GL,j} \otimes \eta) \rtimes 1. \tag{8.4}$$

For $\phi_{i,\sigma(i)}^{1/2} \otimes \eta$ in $\phi_{GL,m}^{(n)}$, let us take $\phi_{i,\sigma(i)} := \phi_i \oplus \phi_{\sigma(i)}$ as a parameter of $SO(V_i \oplus V_{\sigma(i)})$. Then we need to introduce a twist by

$$\widetilde{(\phi_{i,\sigma(i)} \otimes \eta)}^{\sharp}/(\phi_{i,\sigma(i)}^{1/2} \otimes \eta) \rtimes 1. \tag{8.5}$$

Denote the product of twists from $\phi_{GL,m}^{(n)}$ by $\beta_m^{(n)}$. Then we define $\tilde{\phi}^{(n)\square} := \prod_m \phi_{GL,m}^{(n)} \rtimes (\phi_O^{(n)\square} \otimes \prod_m \beta_m^{(n)})$.

We can calculate the twists (8.4), (8.5) explicitly. First, we compute $(\phi_{i,\sigma(i)}^{1/2} \otimes \eta) \rtimes 1$. Let $V_{i,\sigma(i)} = V_{i,\sigma(i)}^{1/2} \oplus (V_{i,\sigma(i)}^{1/2})^{\vee}$ and $\iota : GL(V_{i,\sigma(i)}^{1/2}) \longrightarrow O(V_{i,\sigma(i)}), x \mapsto (x, {}^t x^{-1})$, where ${}^t x^{-1} \in GL((V_{i,\sigma(i)}^{1/2})^{\vee})$ is defined by

$$B(v, {}^t x^{-1} v^{\vee}) = B(x^{-1} v, v^{\vee}), \qquad v \in V_{i,\sigma(i)}^{1/2}, \quad v^{\vee} \in (V_{i,\sigma(i)}^{1/2})^{\vee}.$$

Since $\phi_{i,\sigma(i)}^{1/2} \oplus (\phi_{i,\sigma(i)}^{1/2})^{\vee}$ preserves V_i, the image of $\phi_{i,\sigma(i)}^{1/2}$ lies in an orthogonal group C such that

$$O(V_i) \xleftarrow{\cong} \iota(C) = \iota(GL(V_{i,\sigma(i)}^{1/2})) \cap (O(V_i) \times O(V_{\sigma(i)})) \xrightarrow{\cong} O(V_{\sigma(i)}).$$

The isomorphism $O(V_i) \cong O(V_{\sigma(i)})$ determines an isometry $V_i \cong V_{\sigma(i)}$ up to multiplication by -1. We fix an isometry and it induces an isomorphism $Pin(V_i) \cong Pin(V_{\sigma(i)})$. We choose sections $x \mapsto \tilde{x}$ so that the diagram commutes

$$\begin{array}{ccc} Pin(V_i) & \xrightarrow{\sim} & Pin(V_{\sigma(i)}) \\ \uparrow{\scriptstyle x \mapsto \tilde{x}} & & \uparrow{\scriptstyle x \mapsto \tilde{x}} \\ O(V_i) & \xrightarrow{\sim} & O(V_{\sigma(i)}). \end{array}$$

Let C^0 be the identity component of C. We would like to describe the image of the composition

$$l : C \hookrightarrow GL(V_{i,\sigma(i)}^{1/2}) \hookrightarrow GL(V_{i,\sigma(i)}^{1/2}) \times \mathbb{C}^{\times} \hookrightarrow GSpin(V_{i,\sigma(i)}).$$

Note

$$\begin{array}{ccc} GSpin(V_{i,\sigma(i)}) & \xrightarrow{\lambda} & \mathbb{C}^{\times} \\ \uparrow & \nearrow_{(g,t) \mapsto \det(g) t^2} & \\ GL(V_{i,\sigma(i)}^{1/2}) \times \mathbb{C}^{\times} & & \end{array}$$

So the image of C^0 is in $Spin(V_{i,\sigma(i)})$.

Lemma 8.1.

(1) *For* $x \in C^0$, $l(x) = \widetilde{\iota(x)|_{V_i}} \cdot \widetilde{\iota(x)|_{V_{\sigma(i)}}}$.

(2) *For* $x \in C \backslash C^0$, $l(x) = \sqrt{-1}\, \widetilde{\iota(x)|_{V_i}} \cdot \widetilde{\iota(x)|_{V_{\sigma(i)}}} \bmod \{\pm 1\}$.

Proof. We first show $C^0 \to Spin(V_{i,\sigma(i)}), x \mapsto \widetilde{\iota(x)|_{V_i}} \cdot \widetilde{\iota(x)|_{V_{\sigma(i)}}}$ defines a group homomorphism. Let $b_i \in Z^2(C^0, \mathbb{Z}/2\mathbb{Z})$ (resp. $b_{\sigma(i)} \in Z^2(C^0, \mathbb{Z}/2\mathbb{Z})$) be the obstruction of $\widetilde{\iota(x)|_{V_i}}$ (resp. $\widetilde{\iota(x)|_{V_{\sigma(i)}}}$) from being a group homomorphism. Since $b_i = b_{\sigma(i)}$ and $\widetilde{\iota(x)|_{V_i}}, \widetilde{\iota(x)|_{V_{\sigma(i)}}}$ commute, then it defines a group homomorphism. It differs from l by a group homomorphism $C^0 \to \widehat{D}$. Since C^0 is semisimple, then this must be trivial. The second part follows from the fact that $\lambda(\iota(x)) = -1$.

□

Lemma 8.2. (1) *The twist* (8.4) *is equal to*

$$c_\eta : \Gamma_{E/F} \to \widehat{D}, \quad \tau \mapsto \begin{cases} \sqrt{-1} & N_j \text{ odd} \\ 1 & N_j \text{ even} \end{cases} \tag{8.6}$$

up to $\mathrm{Hom}(\Gamma_{E/F}, \mathbb{Z}/2\mathbb{Z})$*-twists.*

(2) *The twist* (8.5) *is equal to*

$$\chi_i \cdot \omega_i \cdot c_i \cdot c_\eta \bmod \mathrm{Hom}(\Gamma_{EE_i/F}, \mathbb{Z}/2\mathbb{Z}) \tag{8.7}$$

where $\chi_i = \lambda_i \circ (\widetilde{\phi_i \otimes \eta_i})$ *(resp.* $\chi_i = \lambda_i \circ \tilde{\phi}_i$*),*

$$(\widetilde{\phi_i \otimes \eta_i}) \otimes w_i = (\widetilde{\phi_{\sigma(i)} \otimes \eta_i}) \qquad \text{(resp. } \tilde{\phi}_i \otimes \omega_i = \tilde{\phi}_{\sigma(i)}\text{)}$$

for N_i *odd (resp.* N_i *even), and* $c_i : \Gamma_{E_i/F} \to \widehat{D}, \tau_i \mapsto \sqrt{-1}$.

Proof. For (1) it suffices to to compute the composition of (8.4) with the similitude character λ. For (2) we also get c_η for the same reason. So let us assume $\eta = 1$. Let $\phi_{i,\sigma(i)}(w) = \iota(x)$ for some $x \in C$. Then $(\phi_{i,\sigma(i)}^{1/2} \rtimes 1)(w) = l(x)$. Let $\tilde{\phi}_i^\sharp(w) = u(w)\,\widetilde{\iota(x)|_{V_i}}$ and $\tilde{\phi}_{\sigma(i)}^\sharp(w) = \omega_i(w)\,u(w)\,\widetilde{\iota(x)|_{V_{\sigma(i)}}}$ for $u(w) \in \widehat{D}$. Then

$$(\widetilde{\phi_{i,\sigma(i)}})^\sharp(w) = u(w)^2\,\omega_i(x)\,\widetilde{\iota(x)|_{V_i}} \cdot \widetilde{\iota(x)|_{V_{\sigma(i)}}}.$$

Note $\chi_i(w) = \lambda_i \circ \tilde{\phi}_i(w) = u(w)^2$. So it follows from Lemma 8.1 that

$$(\widetilde{\phi_{i,\sigma(i)}})^\sharp(w) = c_i(w)\chi_i(w)\omega_i(w)(\phi_{i,\sigma(i)}^{1/2} \rtimes 1)(w) \bmod \mathrm{Hom}(\Gamma_{E_i/F}, \mathbb{Z}/2\mathbb{Z}).$$

□

Now we want to lift $\xi^{(k)}$, which is given by a twisted elliptic endoscopic embedding

$${}^L H \to SO(V_l^{(k)}) \rtimes W_F \tag{8.8}$$

for some l, or embedding of maximal Levi subgroup of $SO(V_l^{(k)})$ or $GL(W_m^{(k)})$. In the case of Levi subgroups, there is a natural lifting. So we focus on the endoscopic case. Suppose $\widehat{H} = SO(V_{l_1}^{(k+1)}) \times SO(V_{l_2}^{(k+1)})$.

(1) $\dim V_{l_1}^{(k+1)}$ is odd, $\dim V_{l_2}^{(k+1)}$ is even: (8.8) factors through

$$SO(V_{l_1}^{(k+1)}) \times SO(V_{l_2}^{(k+1)}) \rtimes \Gamma_{E/F} \xrightarrow{\xi_k} SO(V_l^{(k)}), \quad \tau \mapsto z$$

where τ is the nontrivial element of $\Gamma_{E/F}$. We choose a lift by requiring $\widetilde{\xi}_k^\square : \tau \mapsto \tilde{z} \in Spin(V_l^{(k)})$. The obstruction is a 2-cocycle $b_k \in Z^2(\Gamma_{E/F}, \mathbb{Z}/2\mathbb{Z})$, which splits in $Z^2(\Gamma_{E/F}, \widehat{D})$.

(2) $\dim V_{l_1}^{(k+1)}$ and $\dim V_{l_2}^{(k+1)}$ are both even or odd: (8.8) factors through

$$SO(V_{l_1}^{(k+1)}) \times SO(V_{l_2}^{(k+1)}) \rtimes \Gamma_{E_I E_{II}/F} \xrightarrow{\xi_k} SO(V_l^{(k)}) \rtimes \Gamma_{E/F}, \quad \tau_i \mapsto z_i$$

where τ_i is the nontrivial element of $\Gamma_{E_i/F}$ $(i = I, II)$. We choose a lift by requiring $\widetilde{\xi}_k^\square : \tau_i \mapsto \tilde{z}_i \in Spin(V_l^{(k)}) \rtimes \Gamma_{E/F}$. The obstruction is a 2-cocycle $b_k \in Z^2(\Gamma_{E_I E_{II}/F}, \mathbb{Z}/2\mathbb{Z})$, which splits in $Z^2(W_F, \widehat{D})$.

Let us denote the resulting lift by $\tilde{\xi}^{(k)\square} : {}^L\widetilde{G}^{(k+1)} \to {}^L\widetilde{G}^{(k)}$. Now we can define a lift of ϕ

$$\tilde{\phi}^{\square} := \tilde{\xi}^{(0)\square} \circ \cdots \circ \tilde{\phi}^{(n)\square}.$$

To get to a homomorphism, we still need to choose 1-cochains $c^{(n)}, c^{(k)} : W_F \to \widehat{D}$ for $\tilde{\phi}^{(n)\square}$ and $\tilde{\xi}^{(k)\square}$. Take $\tilde{\phi}^{(n)} := c^{(n)}\tilde{\phi}^{(n)\square}$ and $\tilde{\xi}^{(k)} := c^{(k)}\tilde{\xi}^{(k)\square}$. Then we define

$$\tilde{\phi} = \tilde{\xi}^{(0)} \circ \cdots \circ \tilde{\phi}^{(n)}.$$

Let $c = \prod_{k=0}^{n} c^{(k)}$ and $L = \prod_{i\in I} E_i$.

Lemma 8.3. *$\tilde{\phi}^{\sharp} = u \cdot \tilde{\phi}^{\square}$ for some 1-cochain $u \in C^1(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z})$.*

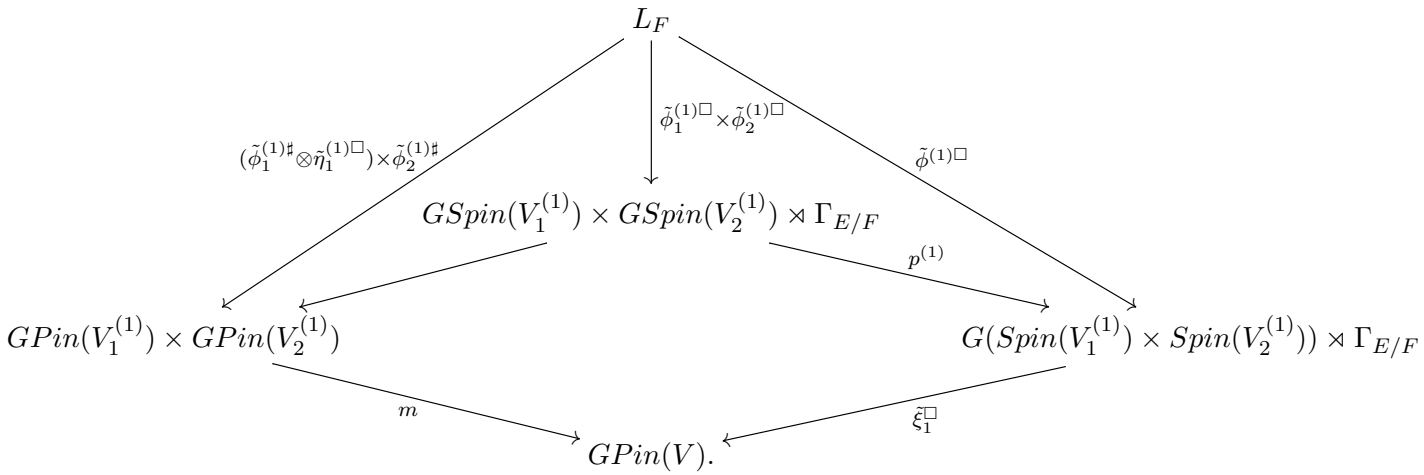
Proof. To use induction in the proof, we define

$$\tilde{\phi}_l^{(k)\sharp} := (\widetilde{\phi_l^{(k)} \otimes \eta_l^{(k)}})^{\sharp} \otimes \tilde{\eta}_l^{(k)\square}$$

and define $\tilde{\phi}_l^{(k)\square}$ as $\tilde{\phi}^{\square}$. By induction, we can assume for $k > 0$

$$\tilde{\phi}_l^{(k)\sharp} = \tilde{\phi}_l^{(k)\square} \bmod C^1(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z}).$$

Suppose $G^{(1)}$ is a twisted elliptic endoscopic group of G as in case (1). We have



Here $\eta_1^{(1)}$ is associated with E/F. Since

$$\tilde{\phi}_i^{(1)\sharp} = \tilde{\phi}_i^{(1)\square} \bmod C^1(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z}),$$

then the left diagram commutes up to twists by $C^1(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z})$. Note that

$$\tilde{\phi}^{\sharp} = m \circ ((\widetilde{\phi_1^{(1)} \otimes \eta_1^{(1)}})^{\sharp} \times \tilde{\phi}_2^{(1)\sharp}) \bmod C^1(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z}).$$

Hence

$$\tilde{\phi}^{\sharp} = \tilde{\phi}^{\square} \bmod C^1(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z}).$$

The argument for case (2) is similar.

Now let us suppose $G^{(1)}$ is a maximal Levi subgroup of G, then by definition

$$\tilde{\phi}^{(1)\square} = \phi_{GL,1}^{(1)} \times (\tilde{\phi}_1^{(1)\square} \otimes \beta_1^{(1)}).$$

We can let $\tilde{\phi}^{(1)\square}$ factor through the Levi subgroup

$$\prod_{j\in T_1^{(1)}} GL(W_j) \times \prod_{i\in S_1^{(1)}/\langle\sigma\rangle} GL(V_{i,\sigma(i)}^{\frac{1}{2}}) \times GSpin(V_1^{(1)}) \rtimes \Gamma_{E/F}.$$

Note this is also the Levi subgroup of

$$^L\widetilde{H} := G(\prod_{j\in T_1^{(1)}} Spin(W_j\oplus W_j^\vee)\times \prod_{i\in S_1^{(1)}/\langle\sigma\rangle} Spin(V_i\oplus V_{\sigma(i)})\times Spin(V_1^{(1)}))\rtimes \Gamma_{E/F}$$

onto which we have a projection from

$$^L\widetilde{\widetilde{H}} := \prod_{j\in T_1^{(1)}} GSpin(W_j\oplus W_j^\vee)\times \prod_{i\in S_1^{(1)}/\langle\sigma\rangle} GSpin(V_i\oplus V_{\sigma(i)})\times GSpin(V_1^{(1)})\rtimes \Gamma_{E/F}.$$

Note

$$\tilde{\phi}^\sharp = m\circ(\prod_{j\in T_1^{(1)}}(\phi_{GL,j}\rtimes 1)\times \prod_{i\in S_1^{(1)}/\langle\sigma\rangle}\tilde{\phi}^\sharp_{i,\sigma(i)}\times \tilde{\phi}_1^{(1)\sharp}) \bmod C^1(\Gamma_{L/F},\mathbb{Z}/2\mathbb{Z}).$$

By induction, we may assume

$$\tilde{\phi}_1^{(1)\square} = \tilde{\phi}_1^{(1)\sharp} \bmod C^1(\Gamma_{L/F},\mathbb{Z}/2\mathbb{Z}).$$

Since

$$\beta_1^{(1)} = \prod_{i\in S_1^{(1)}/\langle\sigma\rangle}\tilde{\phi}^\sharp_{i,\sigma(i)}/\phi^{1/2}_{i,\sigma(i)}\rtimes 1,$$

then the result follows from the commutative diagram

$$\begin{array}{ccccc} L_F & \longrightarrow & ^L\widetilde{H} & \longrightarrow & GPin(V) \\ & \searrow & \uparrow & \nearrow_{m} & \\ & & ^L\widetilde{\widetilde{H}}. & & \end{array}$$

□

As a direct consequence, we have

Proposition 8.4. *Suppose we get two lifts $\tilde{\phi}^\square$ and $'\tilde{\phi}^\square$ from different factorizations of ϕ as* (8.2)*. Then there exists a* 1*-cochain $u\in C^1(\Gamma_{L/F},\mathbb{Z}/2\mathbb{Z})$ such that $\tilde{\phi}^\square = u\cdot{}'\tilde{\phi}^\square$.*

Suppose we construct two lifts $\tilde{\phi},{}'\tilde{\phi}$ by choosing 1-cochains $\{c^{(k)}\}$ and $\{'c^{(k)}\}$ respectively. We would like to define a homomorphism $\chi^{c,'c}: W_F\to\widehat{D}$ such that $\chi^{c,'c} = u\cdot{}'c/c$ for some 1-cochain $u\in C^1(\Gamma_{L/F},\mathbb{Z}/2\mathbb{Z})$. The existence of $\chi^{c,'c}$ follows from Proposition 8.4. From this definition, we also see that $\chi^{c,'c}$ is uniquely determined up to twists by $\mathrm{Hom}(\Gamma_{L/F},\mathbb{Z}/2\mathbb{Z})$. Note $c,{}'c$ split two cocycles $b,{}'b$ of $\Gamma_{L/F}$ in $\mathbb{Z}/2\mathbb{Z}$ respectively by our construction. So the existence of $\chi^{c,'c}$ is also equivalent to $b={}'b$ in $H^2(\Gamma_{L/F},\mathbb{Z}/2\mathbb{Z})$. Since $\mathrm{Hom}(\Gamma_{L/F},\mathbb{Z}/2\mathbb{Z})\subseteq\alpha(\mathcal{S}_\phi^{\Sigma_0})$ (cf. [Xu18, Lemma 6.9]), we can draw the following consequence.

Corollary 8.5. *Suppose F is local and let $\tilde{\phi},{}'\tilde{\phi}$ be two lifts of $\phi\in\bar{\Phi}^+_{unit}(G)$ obtained by choosing* 1*-cochains $\{c^{(k)}\}$ and $\{'c^{(k)}\}$ respectively, then $\tilde{\phi}\otimes\chi^{c,'c} = {}'\tilde{\phi}$ in $\bar{\Phi}^+_{unit}(\widetilde{G})$.*

8.3. **Lifting local and global L-packets.** Let F be local (resp. global), we would like to state the representation theoretic analogue of Corollary 8.5. The idea is to lift $\bar{\Pi}_{\phi^{(n)}}$ and $\xi^{(k)}$. The strategy is parallel with the case of parameters. First of all, we would like to choose lifts of $\bar{\Pi}_{\phi_i}$ for even N_i and $\bar{\Pi}_{\phi_i\otimes\eta_i}$ for odd $N_i\neq 1$. We will fix them no matter of the factorization of ϕ. To lift $\bar{\Pi}_{\phi^{(n)}}$, let $\phi_O^{(n)} = \prod_l\phi_l^{(n)}$. We first lift $\bar{\Pi}_{\phi_O^{(n)}}$ through that of $\bar{\Pi}_{\phi_l^{(n)}}$. There are three cases.

(1) $\phi_l^{(n)} = \phi_i\otimes\eta_i$ for odd $N_i\neq 1$. Then we can take the chosen lift $\bar{\Pi}_{\widetilde{\phi_i\otimes\eta_i}}$.
(2) $\phi_l^{(n)} = \phi_i\otimes\eta$ for even N_i. We can take the chosen lift of $\bar{\Pi}_{\tilde{\phi}_i}$ and twist it by $\tilde{\eta} := \tilde{\eta}^\square c_l^{(n)}$ for some 1-cochain $c_l^{(n)}$ of W_F in $\widehat{D}$.

(3) $\phi_l^{(n)} = (1 \oplus \eta') \otimes \eta$. We take the trivial lift $\bar{\Pi}_{\widetilde{1\oplus\eta'}^{\mathrm{triv}}}$ and twist it by $\widetilde{\eta} := \widetilde{\eta}^{\square} c_l^{(n)}$ for some 1-cochain $c_l^{(n)}$ of W_F in $\widehat{D}$.

Let us denote the resulting lift by $\bar{\Pi}_{\tilde{\phi}_O^{(n)}}$ with 1-cochain $c_O^{(n)} := \prod_l c_l^{(n)}$. Then $\boxtimes_m \pi_{\phi_{GL,m}^{(n)}} \boxtimes \bar{\Pi}_{\tilde{\phi}_O^{(n)}}$ is a lift of $\bar{\Pi}_{\phi^{(n)}}$. For the purpose of comparison, we also need to introduce some twists and 1-cochains according to $\phi_{GL,m}^{(n)}$. There are two cases.

- For $\phi_{GL,j} \otimes \eta$ in $\phi_{GL,m}^{(n)}$, we introduce a 1-cochain c_η of $\Gamma_{E/F}$ in $\widehat{D}$ (cf. (8.6)).
- For $\phi_{i,\sigma(i)}^{1/2} \otimes \eta$ in $\phi_{GL,m}^{(n)}$, we introduce a twist by $\chi_i \omega_i$, where χ_i is the central character of $\bar{\Pi}_{\widetilde{\phi_i \otimes \eta_i}}$ (resp. $\bar{\Pi}_{\tilde{\phi}_i}$), and

$$\bar{\Pi}_{\widetilde{\phi_i \otimes \eta_i}} \otimes \omega_i = \bar{\Pi}_{\widetilde{\phi_{\sigma(i)} \otimes \eta_i}} \qquad (\text{resp. } \bar{\Pi}_{\tilde{\phi}_i} \otimes \omega_i = \bar{\Pi}_{\tilde{\phi}_{\sigma(i)}})$$

for N_i odd (resp. even). We also introduce a 1-cochain $c_\eta c_i$ of $\Gamma_{EE_i/F}$ in $\widehat{D}$ (cf. (8.7)).

We define $c^{(n)}$ to be the product of $c_O^{(n)}$ with these 1-cochains. Denote the product of twists from $\phi_{GL,m}^{(n)}$ by $\chi_m^{(n)}$. Then we define

$$\bar{\Pi}_{\tilde{\phi}^{(n)}} := \boxtimes_m \pi_{\phi_{GL,m}^{(n)}} \boxtimes (\bar{\Pi}_{\tilde{\phi}_O^{(n)}} \otimes \prod_m \chi_m^{(n)}).$$

We lift $\xi^{(k)}$ as before. Then we can define

$$\bar{\Pi}_{\tilde{\phi}} := \mathrm{Tran}_{\tilde{\xi}^{(0)}} \circ \cdots \circ \mathrm{Tran}_{\tilde{\xi}^{(n-1)}} \bar{\Pi}_{\tilde{\phi}^{(n)}}$$

with respect to some 1-cochains $\{c^{(k)}\}$. Let $\bar{\Pi}_{\tilde{\phi}}$ and $\bar{\Pi}_{'\tilde{\phi}}$ be two lifts obtained by choosing 1-cochains $\{c^{(k)}\}$ and $\{'c^{(k)}\}$ respectively. Then $c, 'c$ split two cocycles $b, 'b$ of $\Gamma_{L/F}$ in $\mathbb{Z}/2\mathbb{Z}$ respectively by our construction.

Lemma 8.6. *$b = 'b$ in $H^2(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z})$.*

Proof. The 1-cochains $\{c^{(k)}\}$ and $\{'c^{(k)}\}$ can be taken to be the same as for the lifting of ϕ. Then the result follows from Proposition 8.4 and the discussion after that. □

As a consequence, we can define a homomorphism $\chi^{c,'c} : W_F \to \widehat{D}$ such that $\chi^{c,'c} = u \cdot 'c/c$ for some 1-cochain $u \in C^1(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z})$. It is uniquely determined up to twists by $\mathrm{Hom}(\Gamma_{L/F}, \mathbb{Z}/2\mathbb{Z})$.

Proposition 8.7. *Let $\bar{\Pi}_{\tilde{\phi}}$ and $\bar{\Pi}_{'\tilde{\phi}}$ be two lifts obtained by 1-cochains $\{c^{(k)}\}$ and $\{'c^{(k)}\}$ respectively, then*

$$\bar{\Pi}_{\tilde{\phi}} \otimes \chi^{c,'c} = \bar{\Pi}_{'\tilde{\phi}}. \tag{8.9}$$

We say $\bar{\Pi}_\phi$ has **compatible lifting** with respect to the decomposition (8.1) if (8.9) holds. We can generalize this notion to $G = G(n_1) \times G(n_2) \times \cdots \times G(n_q)$, which is an immediate consequence of Proposition 8.7.

Corollary 8.8. *Compatible lifting of L-packets holds for $\widetilde{G}$.*

Proof. For $[\phi] \in \bar{\Phi}_{unit}^+(G)$ (resp. $\bar{\Phi}(G)$), we write $\phi = \phi_1 \times \phi_2 \times \cdots \times \phi_q$ such that $[\phi_i] \in \bar{\Phi}_{unit}^+(G(n_i))$ (resp. $\bar{\Phi}(G(n_i))$) for $1 \leqslant i \leqslant q$. Note

$$\widetilde{G} \subseteq \widetilde{G}(n_1) \times \widetilde{G}(n_2) \times \cdots \times \widetilde{G}(n_q).$$

Any factorization of ϕ gives rise to a factorization of ϕ_i for $1 \leqslant i \leqslant q$, which determines the packets $\bar{\Pi}_{\tilde{\phi}_i}$, and the corresponding packet $\bar{\Pi}_{\tilde{\phi}}$ will be the restriction of $\bigotimes_{i=1}^q \bar{\Pi}_{\tilde{\phi}_i}$ to $\widetilde{G}$. By Proposition 8.7, $\bar{\Pi}_{\tilde{\phi}_i} \otimes \chi^{c_i,'c_i} = \bar{\Pi}_{'\tilde{\phi}_i}$. Since $\chi^{c,'c} = \prod_i \chi^{c_i,'c_i}$, then $\bar{\Pi}_{\tilde{\phi}} \otimes \chi^{c,'c} = \bar{\Pi}_{'\tilde{\phi}}$. □

8.4. **Proof of Proposition 8.7.** In this section we shall give the proof of Proposition 8.7. First we observe that the local case implies the global case. In the archimedean case, the Local Langlands correspondence is known and is compatible with twisted endoscopic transfer and parabolic induction [Lan89] [She08] [Mez13]. So the result follows from that for lifting parameters (cf. Corollary 8.5).

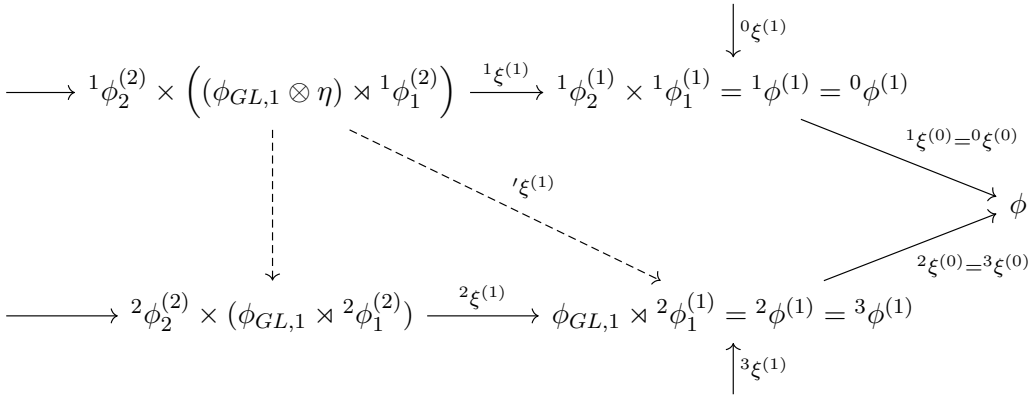
Now let us assume F is nonarchimedean. Note if any ϕ_i or $\phi_{GL,j}$ in (8.2) is not irreducible, we can always factorize them further. This means that we can reduce to the case that they are all irreducible and $\phi_{GL,j}$ are not of orthogonal types. Moreover, when ϕ is unramified, the fundamental lemma for spherical Hecke algebras [Hal95] [LMW18] implies the compatibility of the local Langlands correspondence for unramified representations with the twisted endoscopic transfer. We should point out that the twisted case has been shown only for sufficiently large residue characteristic. The compatibility with parabolic induction is clear. So the result would follow from the same argument as in the archimedean case.

After these reductions, the proof of compatible lifting of L-packets proceeds in the same way as our proof of the main local theorem in [Xu18]. We assume that it holds for $\widetilde{G} = GSp(2n)$ and $GSO(2n+2,\eta)$, when $n < N$. When $n = N$, we first show that the case of non-elliptic parameters follows immediately from this assumption.

Lemma 8.9. *Suppose $[\phi] \in \bar{\Phi}^{+}_{unit}(G) - \bar{\Phi}_{ell}(G)$, then the packet $\bar{\Pi}_\phi$ has compatible lifting.*

Proof. For the decomposition of ϕ in (8.1), we can assume $J \neq \emptyset$ or $\phi_1 \cong \phi_2 \cong \phi_3$. The idea is to compare any factorization of ϕ to certain fixed ones. For a given factorization, we will assume that $G^{(1)}$ is a twisted elliptic endoscopic group of G such that $\dim V_1^{(1)}$ is odd and $\dim V_2^{(1)}$ is even, and $\phi_{GL,1}$ (resp. $\phi_i \oplus \phi_j$ for $\{i,j\} \subseteq \{1,2,3\}$) is in $\phi_1^{(1)}$. We will add superscripts to the left of ϕ to indicate different factorizations. The arguments for the other cases are similar, so we do not include them here.

If $J \neq \emptyset$, we factorize ϕ in the following four ways. From top to bottom, the first is the given factorization and the last is the fixed one.



By our assumption, we have ${}^1\phi_2^{(1)} = {}^1\phi_2^{(2)} = {}^2\phi_2^{(2)}$, ${}^1\phi_1^{(2)} = {}^2\phi_1^{(2)}$ and $\eta = {}^1\eta_1^{(1)} = {}^1\eta_1^{(2)}$. By induction and Corollary 8.8, we can assume $\bar{\Pi}_{{}^1\phi^{(1)}}$ has compatible lifting, then

$$\bar{\Pi}_{{}^0\tilde{\phi}} \otimes \chi^{{}^0c,{}^1c} = \bar{\Pi}_{{}^1\tilde{\phi}}. \tag{8.10}$$

Next we would like to show

$$\bar{\Pi}_{{}^1\tilde{\phi}} \otimes \chi^{{}^1c,{}^2c} = \bar{\Pi}_{{}^2\tilde{\phi}} \tag{8.11}$$

where ${}^2c$ is chosen according to ${}^1c$ as follows. The factorization of ${}^1\phi^{(2)}$ gives rise to a factorization of ${}^2\phi^{(2)}$. We will choose ${}^1c^{(k)} = {}^2c^{(k)}$ if $2 \leqslant k < n$, and ${}^1c^{(n)} = {}^2c^{(n)}c_\eta$. It follows $\bar{\Pi}_{{}^1\tilde{\phi}^{(2)}} = \bar{\Pi}_{{}^2\tilde{\phi}^{(2)}} \otimes (\eta \circ \det_{GL(W_1^{(2)})})$. By the compatibility of twisted endoscopic transfer with parabolic induction,

$$\bar{\Pi}_{{}^1\tilde{\phi}} = \mathrm{Tran}_{{}^2\tilde{\xi}^{(0)}} \mathrm{Tran}_{'\tilde{\xi}^{(1)}} \bar{\Pi}_{{}^1\tilde{\phi}^{(2)}}.$$

Moreover,

$$\mathrm{Tran}_{{}^2\tilde{\xi}^{(1)}} \bar{\Pi}_{{}^2\tilde{\phi}^{(2)}} = \mathrm{Tran}_{'\tilde{\xi}^{(1)}} \bar{\Pi}_{{}^1\tilde{\phi}^{(2)}} \otimes \chi^{{}^1c^{(0)}c_\eta,{}^2c^{(1)}}.$$

Since

$$ {}^2c^{(1)}/{}^1c^{(0)}c_\eta = {}^2c^{(1)}\,{}^2c^{(n)}/{}^1c^{(0)}{}^1c^{(n)} = {}^2c/{}^1c, $$

then

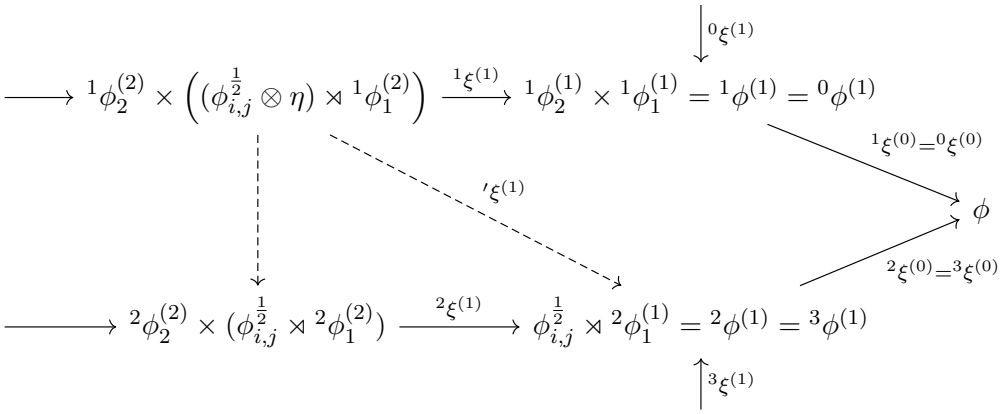
$$ \chi^{{}^1c^{(0)}c_\eta,{}^2c^{(1)}} = \chi^{{}^1c,{}^2c} \bmod \mathrm{Hom}(\Gamma_{L/F},\mathbb{Z}/2\mathbb{Z}). $$

This proves (8.11). By induction, we can also assume $\bar{\Pi}_{{}^2\phi^{(1)}}$ has compatible lifting, then

$$ \bar{\Pi}_{{}^2\tilde{\phi}} \otimes \chi^{{}^2c,{}^3c} = \bar{\Pi}_{{}^3\tilde{\phi}}. \tag{8.12} $$

So the result follows from (8.10), (8.11) and (8.12).

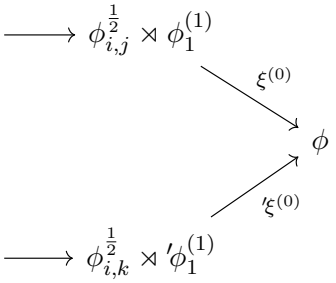
If $\phi_1 \cong \phi_2 \cong \phi_3$, we factorize ϕ in the following four ways. From top to bottom, the first is the given factorization and the last is one of the three fixed ones indexed by $\{i,j\} \subseteq \{1,2,3\}$.



By our assumption, ${}^1\phi_2^{(1)} = {}^1\phi_2^{(2)} = {}^2\phi_2^{(2)}$, ${}^1\phi_1^{(2)} = {}^2\phi_1^{(2)}$ and $\eta = {}^1\eta_1^{(1)} = {}^1\eta_1^{(2)}$. Here $\{i,j\}$ depends on ${}^0\phi^{(1)}$. As in the previous case, we can show

$$ \bar{\Pi}_{{}^0\tilde{\phi}} \otimes \chi^{{}^0c,{}^3c} = \bar{\Pi}_{{}^3\tilde{\phi}}. $$

It remains to compare the three fixed factorizations.



Note $\widehat{G}^{(1)}, {}'\widehat{G}^{(1)}$ are Levi subgroups of $\widehat{G}$ conjugate under S_ϕ, and we have $\phi_{i,j}^{\frac{1}{2}} \cong \phi_{i,k}^{\frac{1}{2}}$ and $\phi_1^{(1)} \cong {}'\phi_1^{(1)}$ under the conjugation. It follows that there exists $g \in G(F)$ such that $g(G^{(1)})g^{-1} = {}'G^{(1)}$ and $\bar{\Pi}^g_{\phi^{(1)}} = \bar{\Pi}_{{}'\phi^{(1)}}$. We can identify $\phi_1^{(1)}$ and ${}'\phi_1^{(1)}$, and take the same factorization and 1-cochains for them. Recall that for $\phi_{i,j}^{\frac{1}{2}}$ (resp. $\phi_{i,k}^{\frac{1}{2}}$), we need to introduce a twist by $\omega_{i,j}\chi_i$ (resp. $\omega_{i,k}\chi_i$), where χ_i is the central character of $\bar{\Pi}_{\tilde{\phi}_i}$ and $\bar{\Pi}_{\tilde{\phi}_i} \otimes \omega_{i,j} = \bar{\Pi}_{\tilde{\phi}_j}, \bar{\Pi}_{\tilde{\phi}_i} \otimes \omega_{i,k} = \bar{\Pi}_{\tilde{\phi}_k}$(resp. $\bar{\Pi}_{\widetilde{\phi_i \otimes \eta_i}} \otimes \omega_{i,j} = \bar{\Pi}_{\widetilde{\phi_j \otimes \eta_j}}, \bar{\Pi}_{\widetilde{\phi_i \otimes \eta_i}} \otimes \omega_{i,k} = \bar{\Pi}_{\widetilde{\phi_k \otimes \eta_k}}$) for N_i even (resp. odd). Then

$$ \bar{\Pi}^g_{\tilde{\phi}_1^{(1)}} = \bar{\Pi}_{{}'\tilde{\phi}_1^{(1)}} \otimes \omega_{k,j} $$

and it follows that

$$ \bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{{}'\tilde{\phi}} \otimes \omega_{j,k} \otimes (\omega_{i,j}\chi_i/\omega_{i,k}\chi_i) = \bar{\Pi}_{{}'\tilde{\phi}}. $$

So we have (8.9) for the three fixed factorizations. This finishes the proof. □

At last, we can treat the elliptic parameters. Suppose

$$\phi = \phi_1 \oplus \cdots \oplus \phi_q \oplus 2\phi_{q+1} \oplus \cdots \oplus 2\phi_r \in \bar{\Phi}_{ell}(G).$$

Lemma 8.10. *If $\phi_i = \eta_i$, and $r \leqslant 3$ (or $r \leqslant 4$ when G is symplectic), then $\bar{\Pi}_\phi$ has compatible lifting.*

The proof is similar to the general case with one difference. So we will consider the general case first.

Lemma 8.11. *Suppose $[\phi] \in \bar{\Phi}_{ell}(G)$, then the packet $\bar{\Pi}_\phi$ has compatible lifting.*

Proof. By applying [Xu18, Lemma 6.18] to ϕ, we get a global lift

$$\dot{\phi} = \dot{\phi}_1 \oplus \cdots \oplus \dot{\phi}_q \oplus 2\dot{\phi}_{q+1} \oplus \cdots \oplus 2\dot{\phi}_r \in \bar{\Phi}_{ell}(\dot{G})$$

satisfying the following properties.

(1) $\dot{\phi}_u = \phi$, $\mathcal{S}_{\dot{\phi}} = \mathcal{S}_\phi$, $\mathcal{S}_{\widetilde{\dot{\phi}}} = 1$.
(2) At nonarchimedean place $v \neq u$, $\dot{\phi}_v$ is a direct sum of quasicharacters of $\dot{F}_v^\times$ with at most one ramified quasicharacter counted without multiplicities modulo the unramified quasicharacters.
(3) Σ_0-strong multiplicity one holds for $\widetilde{\dot{\phi}}$ at the place u, i.e., for any automorphic representation $\tilde{\dot{\pi}}$ of $\widetilde{\dot{G}}$ such that $[\tilde{\dot{\pi}}_v] \in \bar{\Pi}_{\widetilde{\dot{\phi}}_v}$ for all $v \neq u$, $[\tilde{\dot{\pi}}]$ must be also in $\bar{\Pi}_{\widetilde{\dot{\phi}}}$.

Suppose $\bar{\Pi}_{\widetilde{\dot{\phi}}}$ and $\bar{\Pi}_{{}'\widetilde{\dot{\phi}}}$ are obtained from two different factorizations of $\dot{\phi}$ as in (8.2) with 1-cochains $\dot{c}$ and ${}'\dot{c}$ respectively, then they are global L-packets of $\widetilde{\dot{G}}$ by the functoriality of twisted endoscopic transfer (cf. [Xu18, Lemma 6.28]). By Lemma 8.9 and Lemma 8.10, $\bar{\Pi}_{\widetilde{\dot{\phi}}_v} \otimes \chi_v^{\dot{c},{}'\dot{c}} = \bar{\Pi}_{{}'\widetilde{\dot{\phi}}_v}$ for all $v \neq u$. By Σ_0-strong multiplicity one at the place u, we have $\bar{\Pi}_{\widetilde{\dot{\phi}}} \otimes \chi^{\dot{c},{}'\dot{c}} = \bar{\Pi}_{{}'\widetilde{\dot{\phi}}}$ and hence $\bar{\Pi}_{\widetilde{\dot{\phi}}_u} \otimes \chi_u^{\dot{c},{}'\dot{c}} = \bar{\Pi}_{{}'\widetilde{\dot{\phi}}_u}$. At last, one just needs to notice that the factorizations of $\dot{\phi}$ one-to-one correspond to the factorizations of ϕ. □

To apply this proof to Lemma 8.10, one notes $\widetilde{\dot{\phi}}_v$ may be elliptic at nonarchimedean place $v \neq u$. Nevertheless, we have $\bar{\Pi}_{\widetilde{\dot{\phi}}_v} \otimes \chi_v^{\dot{c},{}'\dot{c}} = \bar{\Pi}_{{}'\widetilde{\dot{\phi}}_v}$ for almost all places by the fundamental lemma for spherical Hecke algebras. By [Xu18, Lemma 6.13]), we know that Σ_0-strong multiplicity one holds for $\widetilde{\dot{\phi}}$, i.e., for any automorphic representation $\tilde{\dot{\pi}}$ of $\widetilde{\dot{G}}$ such that $[\tilde{\dot{\pi}}_v] \in \bar{\Pi}_{\widetilde{\dot{\phi}}_v}$ for almost all places v, $[\widetilde{\dot{\pi}}]$ must be also in $\bar{\Pi}_{\widetilde{\dot{\phi}}}$. Hence $\bar{\Pi}_{\widetilde{\dot{\phi}}} \otimes \chi^{\dot{c},{}'\dot{c}} = \bar{\Pi}_{{}'\widetilde{\dot{\phi}}}$. This completes the proof of Lemma 8.10.

8.5. **Towards local Langlands correspondence for similitude groups.** Suppose F is nonarchimedean, and G is a quasisplit symplectic or special even orthogonal group over F. We would like to show

Theorem 8.12. *One can associate any $\tilde{\phi} \in \bar{\Phi}_{bdd}(\widetilde{G})$ with an L-packet $\bar{\Pi}_{\tilde{\phi}}$ such that*

(1) $\bar{\Pi}_{\tilde{\phi}\otimes\omega} = \bar{\Pi}_{\tilde{\phi}} \otimes \omega$ *for any* $\omega \in H^1(W_F, Z(\widehat{\widetilde{G}})) \cong \mathrm{Hom}(\widetilde{G}(F), \mathbb{C}^\times)$;
(2) *the central character of $\bar{\Pi}_{\tilde{\phi}}$ is $\chi_{\tilde{\phi}}$, which is associated with the composition of $\tilde{\phi}$ and $\widehat{\widetilde{G}} \to \widehat{Z}_{\widetilde{G}}$;*
(3) *it is compatible with twisted endoscopic transfer and parabolic induction.*

For the compatibility with twisted endoscopic transfer, we first need to construct the correspondence for the twisted endoscopic groups. It suffices to consider $G = G_1 \times \cdots \times G_q$ (2.2) and $\phi = \phi_1 \times \cdots \times \phi_q$ for $\phi_i \in \bar{\Phi}_2(G_i)$. We define

$$\tilde{\phi} = \mathbf{p} \circ (\tilde{\phi}_1 \times \cdots \times \tilde{\phi}_q) \mapsto \bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{\tilde{\phi}_1} \tilde{\otimes} \cdots \tilde{\otimes} \bar{\Pi}_{\tilde{\phi}_q}.$$

where $\mathbf{p} : {}^L(\prod_{i=1}^q \widetilde{G}_i) \to {}^L\widetilde{G}$ is dual to the inclusion $\widetilde{G} \hookrightarrow \prod_{i=1}^q \widetilde{G}_i$.

Proof. The proof is by construction, which depends on our choices for simple parameters.

- For $N \neq 1$ and $\phi \in \bar{\Phi}_{sim}(G)$, we fix a correspondence $\tilde{\phi} \mapsto \bar{\Pi}_{\tilde{\phi}}$ such that (1) and (2) hold. Here (1) can be achieved by [Xu18, Corollary 4.2] and Theorem A.1, and (2) is due to [Xu16, Proposition 6.27].

- Associate $(\widetilde{1\oplus\eta})^{\mathrm{triv}}\otimes\omega$ with $\bar{\Pi}_{\widetilde{1\oplus\eta}^{\mathrm{triv}}}\otimes\omega$.

For $\phi\in\bar{\Phi}_{bdd}(G)$, we take a decomposition of ϕ as (8.1) such that all $\phi_i,\phi_{GL,j}$ are irreducible and $\phi_{GL,j}$ are not of orthogonal type. Then we also take a factorization of ϕ as (8.2). We can construct a lift $\tilde{\phi}$ with respect to any choice of lifts $\tilde{\phi}_i$ for even N_i (resp. $\widetilde{\phi_i\otimes\eta_i}$ for odd $N_i\neq 1$) and 1-cochains $\{c^{(k)}\}$. Correspondingly, we will construct $\bar{\Pi}_{\tilde{\phi}}$ with respect to the same factorization of ϕ, $\bar{\Pi}_{\tilde{\phi}_i}$ (resp. $\bar{\Pi}_{\widetilde{\phi_i\otimes\eta_i}}$) associated with $\tilde{\phi}_i$ (resp. $\widetilde{\phi_i\otimes\eta_i}$), and the same 1-cochains. Then we define

$$\tilde{\phi}\otimes\omega\mapsto\bar{\Pi}_{\tilde{\phi}}\otimes\omega$$

for $\omega\in\bar{H}^1(W_F,\widehat{D})\cong\mathrm{Hom}(\widetilde{G}(F)/G(F),\mathbb{C}^\times)$. It is not hard to see that this is independent of choices of lifts $\tilde{\phi}_i$ (resp. $\widetilde{\phi_i\otimes\eta_i}$ if N_i is odd) and $\{c^{(k)}\}$. It is also independent of the factorization by Corollary 8.5 and Proposition 8.7. At last, it is independent of the decomposition of ϕ, since any two decompositions are conjugate under S_ϕ.

To show (3), note $\tilde{\phi}=\widetilde{\xi}^{(0)}\circ\tilde{\phi}^{(1)}$ and $\bar{\Pi}_{\tilde{\phi}}=\mathrm{Tran}_{\widetilde{\xi}^{(0)}}\bar{\Pi}_{\tilde{\phi}^{(1)}}$. So it remains to show that $\tilde{\phi}^{(1)}$ is associated with $\bar{\Pi}_{\tilde{\phi}^{(1)}}$, which is clear by our construction and choices for simple parameters. Finally, (1) and (2) follow from (3) by extending the arguments of [Xu16, Lemma 5.2 and Lemma 4.1] to the twisted case, which is straightforward. □

As a consequence of Theorem 8.12 (1), we have

Corollary 8.13. *For $\phi\in\bar{\Phi}_{bdd}(G)$ and $\omega\in H^1(W_F,Z(\widehat{\widetilde{G}}))\cong\mathrm{Hom}(\widetilde{G}(F),\mathbb{C}^\times)$,*

$$[\tilde{\phi}\otimes\omega]=[\tilde{\phi}]\ \textit{if and only if}\ \bar{\Pi}_{\tilde{\phi}}\otimes\omega=\bar{\Pi}_{\tilde{\phi}}.$$

This corollary is an enhancement of [Xu18, Corollary 4.2] when G is orthogonal. It generalizes Theorem A.1.

9. Proof of main theorems

Let F be global. We denote $G(n):=Sp(2n),SO(2n+2,\eta)$ and $\widetilde{G}(n):=GSp(2n),GSO(2n+2,\eta)$ over F. Let $G=G(n_1)\times G(n_2)\times\cdots\times G(n_q)$. We assume Theorem 4.1, 4.2, 4.6, 4.8, 4.9 hold when $n_i<N$ for all $1\leqslant i\leqslant q$. In the proof, we will always treat the symplectic case first, and then take the results to enhance the induction assumptions.

9.1. **Proof of theorem 4.1.**

9.1.1. *Discrete parameters.* Let $G=G(N)$, $\phi=\phi_1\boxplus\cdots\boxplus\phi_r\in\bar{\Phi}_2(G)$ and $\phi\neq\phi_b$. Suppose $\phi\neq\phi_o$, then by (5.7)

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f})-S^{\widetilde{G}}_{disc,\phi}(\tilde{f})=C_{\tilde{\phi}_o}\sum_{\omega'\in Y/\alpha(\mathcal{S}_{\phi_o})}\sum_{x_o\in\mathcal{S}_{\tilde{\phi}_o}-\{1\}}\tilde{f}'_{\widetilde{G}}(\tilde{\phi}_o\otimes\omega',x_o;\tilde{\phi}_b)$$
$$+\mathrm{Tran}\,S^{\widetilde{G}_o}_{disc,\phi_o}\tilde{\otimes}\,(\frac{1}{2}I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}-\frac{1}{4}S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}),\quad\tilde{f}\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}).$$

We add

$$2\cdot C_{\tilde{\phi}_o}\sum_{\omega\in Y/\alpha(\mathcal{S}_{\phi_o})}\sum_{x_o\in\mathcal{S}_{\tilde{\phi}_o}-\{1\}}\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}(\tilde{\phi}_o;\tilde{\phi}_b)\\ \langle x_o,\tilde{\pi}\rangle=-1}}\tilde{f}_{\widetilde{G}}(\tilde{\pi}\otimes\omega)$$

to both sides, then the right hand side becomes stable. It follows that

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f})+2\cdot C_{\tilde{\phi}_o}\sum_{\omega\in Y/\alpha(\mathcal{S}_{\phi_o})}\sum_{x_o\in\mathcal{S}_{\tilde{\phi}_o}-\{1\}}\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}(\tilde{\phi}_o;\tilde{\phi}_b)\\ \langle x_o,\tilde{\pi}\rangle=-1}}\tilde{f}_{\widetilde{G}}(\tilde{\pi}\otimes\omega)\tag{9.1}$$

is also stable. Then we can argue by stability as follows. We define $\bar{\Pi}_{\tilde{\phi}} = \otimes'_v \bar{\Pi}_{\tilde{\phi}_v}$ such that it contains a discrete automorphic representation $\tilde{\pi}^0$. Since (9.1) is stable, it is stable at every place. So we can take $\tilde{f} = \otimes_w \tilde{f}_w$ and fix $\otimes_{w\neq v}\tilde{f}_w$ for any place v, then by [Xu18, Corollary 4.8] the coefficient of $\tilde{f}_v(\tilde{\pi}_v)$ in (9.1) must be the same for all $\tilde{\pi}_v \in \bar{\Pi}_{\tilde{\phi}_v}$. By varying $\otimes_{w\neq v}\tilde{f}_w$ and the linear independence of characters of $\otimes_{w\neq v}\bar{\mathcal{H}}(\widetilde{G}_w, \widetilde{\chi}_w)$-modules, we have that

$$[\tilde{\pi}^0] = [\tilde{\pi}^0_v] \otimes (\otimes_{w\neq v}[\tilde{\pi}^0_w])$$

contributes to (9.1) if and only if all elements in

$$\bar{\Pi}_{\tilde{\phi}_v} \otimes (\otimes_{w\neq v}[\tilde{\pi}^0_w])$$

also contribute to (9.1). By repeating this kind of argument, one can show all elements in $\bar{\Pi}_{\tilde{\phi}}$ contribute to (9.1). Note for any $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}$ such that $\langle \cdot, \tilde{\pi}\rangle = 1$, it can only contribute to $I^{\widetilde{G}}_{disc,\phi}(\tilde{f})$, which means it belongs to the discrete automorphic spectrum. Then (4.1) follows from this observation and (2.20). Moreover, we get $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}(\tilde{\phi_o}; \tilde{\phi_b})$, if there exists $x_o \in \mathcal{S}_{\tilde{\phi}_o} - \{1\}$ and $[\tilde{\pi}] \in \bar{\Pi}(\tilde{\phi_o}; \tilde{\phi_b})$ such that $\langle x_o, \tilde{\pi}\rangle = -1$, i.e.,

$$\mathcal{S}_{\tilde{\phi}_o} \longrightarrow \prod_v \mathcal{S}_{\tilde{\phi}_{o,v}} \tag{9.2}$$

is not trivial. Suppose $\phi = \phi_o$, then by (5.6)

$$I^{\widetilde{G}}_{disc,\phi}(\tilde{f}) - S^{\widetilde{G}}_{disc,\phi}(\tilde{f}) = C_{\tilde{\phi}} \sum_{\omega' \in Y/\alpha(\mathcal{S}_\phi)} \sum_{x\in \mathcal{S}_{\tilde{\phi}} - \{1\}} \tilde{f}'_{\widetilde{G}}(\tilde{\phi}\otimes\omega', x), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}).$$

We add

$$2\cdot C_{\tilde{\phi}} \sum_{\omega\in Y/\alpha(\mathcal{S}_\phi)} \sum_{x\in\mathcal{S}_{\tilde{\phi}}-\{1\}} \sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}_x} \\ \langle x,\tilde{\pi}\rangle = -1}} \tilde{f}_{\widetilde{G}}(\tilde{\pi}\otimes\omega)$$

to both sides, where $\bar{\Pi}_{\tilde{\phi}_x}$ is a global packet transferred from $\bar{\Pi}_{\tilde{\phi}'}$ for $x \in \mathcal{S}_{\tilde{\phi}} - \{1\}$. By the argument on stability again, we get the global L-packet $\bar{\Pi}_{\tilde{\phi}}$, with which (4.1) holds. Moreover,

$$\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{\tilde{\phi}_x} \tag{9.3}$$

if there exists $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}_x}$ such that $\langle x, \tilde{\pi}\rangle = -1$, i.e., x has nontrivial image in $\prod_v \mathcal{S}_{\tilde{\phi}_v}$. By Proposition 8.7, (9.3) holds for all $x \neq 1$ if it holds for one. So it suffices to know that (9.2) is not trivial.

9.1.2. *General case.* Suppose $\phi \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$, then ϕ factors through $\phi_M \in \bar{\Phi}_2(M)$ for some proper Levi subgroup M of G. Then by our induction assumption, we have a global L-packet $\bar{\Pi}_{\tilde{\phi}_M}$ for $\widetilde{M}$, and we can define $\bar{\Pi}_{\tilde{\phi}}$ to be the set of irreducible constituents induced from $\bar{\Pi}_{\tilde{\phi}_M}$. At last, let

$$G = G(n_1)\times G(n_2)\times\cdots\times G(n_q),$$

with $n_i \leqslant N$ for $1 \leqslant i \leqslant q$ and $\phi \in \bar{\Phi}(G)$. The global L-packet is the restriction of the global L-packet $\bigotimes_{i=1}^q \bar{\Pi}_{\tilde{\phi}_i}$ of $\widetilde{G}(n_1)\times\widetilde{G}(n_2)\times\cdots\times\widetilde{G}(n_q)$, with which (4.1) holds.

9.2. **Proof of theorem 4.2.** Next, we consider Theorem 4.2 for G. For $\phi \in \bar{\Phi}_2(G)$,

$$I^{(\widetilde{G}^\theta,\omega)}_{disc,\phi}(\tilde{f}) = trR^{(\widetilde{G}^\theta.\omega)}_{disc,\phi}(\tilde{f}) = \sum_{\omega'} \sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega' \\ \langle\cdot,\tilde{\pi}\rangle = 1}} m(\tilde{\pi},\theta,\omega)\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega), \quad \tilde{f}\in\bar{\mathcal{H}}(\widetilde{G},\widetilde{\chi}).$$

Here the sum of ω' is taken over $Y/\prod_v^{aut}\alpha(\mathcal{S}^{\Sigma_0}_{\phi_v})$ with $\prod_v^{aut}\alpha(\mathcal{S}^{\Sigma_0}_{\phi_v}) := \{\omega\in Y : \omega_v \in \alpha(\mathcal{S}^{\Sigma_0}_{\phi_v}) \text{ for all } v\}$, and $m(\tilde{\pi},\theta,\omega)$ is some integer, whose absolute value is less than or equal to $m(\tilde{\phi}) := m_\phi\,|\prod_v^{aut}\alpha(\mathcal{S}^{\Sigma_0}_{\phi_v})|\,|\alpha(\mathcal{S}_\phi)|^{-1}$.

By (5.9), we have

$$I^{(\widetilde{G}^\theta,\omega)}_{disc,\phi}(\tilde{f}) = |\mathcal{S}_{\tilde{\phi}_o}|^{-1}\sum_{\omega'}\sum_{x'_o\in\mathcal{S}^\theta_{\phi_o}(\omega)} m(\tilde{\phi})\tilde{f}'_{\widetilde{G}^\theta}(\tilde{\phi}_o\otimes\omega', x'_o;\tilde{\phi}_b),$$

where the sum of ω' is again over $Y/\prod_v^{aut}\alpha(\mathcal{S}^{\Sigma_0}_{\phi_v})$. Therefore

$$\sum_{\omega'}\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega'\\ \langle\cdot,\tilde{\pi}\rangle=1}} m(\tilde{\pi},\theta,\omega)\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega) = |\mathcal{S}_{\tilde{\phi}_o}|^{-1}\sum_{\omega'}\sum_{x'_o\in\mathcal{S}^\theta_{\phi_o}(\omega)} m(\tilde{\phi})\tilde{f}'_{\widetilde{G}^\theta}(\tilde{\phi}_o\otimes\omega', x'_o;\tilde{\phi}_b).$$

There are two cases. If $\phi\neq\phi_o$, then

$$\begin{aligned}\sum_{\omega'}\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega'\\ \langle\cdot,\tilde{\pi}\rangle=1}} m(\tilde{\pi},\theta,\omega)\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega) &= |\mathcal{S}_{\tilde{\phi}_o}|^{-1}\sum_{\omega'}\sum_{y_o\in\mathcal{S}_{\tilde{\phi}_o}} m(\tilde{\phi})\sum_{[\tilde{\pi}]\in\bar{\Pi}(\tilde{\phi}_o;\tilde{\phi}_b)\otimes\omega'}\langle y,\tilde{\pi}\rangle\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega)\\ &= m(\tilde{\phi})\sum_{\omega'}\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}(\tilde{\phi}_o;\tilde{\phi}_b)\otimes\omega'\\ \langle\cdot,\tilde{\pi}\rangle=1}}\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega),\end{aligned}$$

where $\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega)$ is normalized by x. By the linear independence of twisted characters of $\bar{\bar{\mathcal{H}}}(\widetilde{G},\widetilde{\chi})$-modules, we get $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}(\tilde{\phi}_o;\tilde{\phi}_b)$. If $\phi=\phi_o$, we define a global packet $\bar{\Pi}_{\tilde{\phi}_{x'}}$ transferred from $\bar{\Pi}_{\tilde{\phi}'}$ for any $x'\in\mathcal{S}^\theta_\phi(\omega)$. Note $\mathcal{S}^\theta_\phi(\omega) = x\cdot\mathcal{S}_{\tilde{\phi}}$, then

$$\sum_{\omega'}\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega'\\ \langle\cdot,\tilde{\pi}\rangle=1}} m(\tilde{\pi},\theta,\omega)\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega) = |\mathcal{S}_{\tilde{\phi}}|^{-1}\sum_{\omega'}\sum_{y\in\mathcal{S}_{\tilde{\phi}}} m(\tilde{\phi})\sum_{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}_{xy}}\otimes\omega'}\langle y,\tilde{\pi}\rangle\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega), \tag{9.4}$$

where $\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega)$ is normalized by x. This implies

$$\sum_{\omega'}\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega'\\ \langle\cdot,\tilde{\pi}\rangle=1}} m(\tilde{\pi},\theta,\omega)\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega) = |\mathcal{S}_{\tilde{\phi}}|^{-1}\sum_{\omega'}\sum_{y\in\mathcal{S}_{\tilde{\phi}}} m(\tilde{\phi})\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}_{xy}}\otimes\omega'\\ \langle\cdot,\tilde{\pi}\rangle=1}}\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega).$$

It follows from the linear independence of twisted characters of $\bar{\bar{\mathcal{H}}}(\widetilde{G},\widetilde{\chi})$-modules that we can choose $\bar{\Pi}_{\tilde{\phi}_{x'}} = \bar{\Pi}_{\tilde{\phi}}$ for all $x'\in\mathcal{S}^\theta_\phi(\omega)$. In both cases, we have

$$\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\\ \langle\cdot,\tilde{\pi}\rangle=1}} m(\tilde{\pi},\theta,\omega)\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega) = m(\tilde{\phi})\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\\ \langle\cdot,\tilde{\pi}\rangle=1}}\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega).$$

So $m(\tilde{\pi},\theta,\omega) = m(\tilde{\phi})$. Hence

$$I^{(\widetilde{G}^\theta,\omega)}_{disc,\phi}(\tilde{f}) = m_\phi\sum_{\omega'\in Y/\alpha(\mathcal{S}_\phi)}\sum_{\substack{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\phi}}\otimes\omega'\\ \langle\cdot,\tilde{\pi}\rangle=1}}\tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi},\omega).$$

As a consequence, we have shown the functoriality for

$$s\in(S^{\Sigma_0}_{\phi_o}\times Z(\widehat{G}_b))/Z(G)^\Gamma - (S_{\tilde{\phi}_o}\times Z(\widehat{G}_b)/Z(G)^\Gamma.$$

By the compatible lifting of global L-packets again (cf. Corollary 8.8), this implies the functoriality for all $s\in(S^{\Sigma_0}_{\phi_o}\times Z(\widehat{G}_b))/Z(G)^\Gamma$, whenever $\bar{S}^{\Sigma_0}_{\phi_o}\neq\bar{S}_{\tilde{\phi}_o}$, i.e. $\mathcal{S}^{\Sigma_0}_{\phi_o}\neq\mathcal{S}_{\tilde{\phi}_o}$.

9.3. **Proof of Theorem 4.6, 4.8, 4.9.** For Theorem 4.8, 4.9, it remains to show part (2) and (3) (cf. Section 6), which are direct consequences of Theorem 4.6 and the comparison formulas (cf. Section 5). So we will mainly focus on Theorem 4.6.

9.3.1. *Non-discrete parameters.*

Lemma 9.1. *Suppose $G = G(N)$ and $\phi \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$, then Theorem 4.6 holds for $\tilde{\phi}$.*

Proof. Suppose $\phi_o \in \bar{\Phi}(G_o) - \bar{\Phi}_2(G_o)$, then ϕ factors through ϕ_M for a Levi subgroup M of G such that $\phi_{M,b} = \phi_b$. Note $\bar{\Pi}_{\tilde{\phi}}$ is obtained by parabolic induction of $\bar{\Pi}_{\tilde{\phi}_M}$. So the theorem follows from compatible lifting of L-packets (cf. Proposition 8.7). Suppose $\phi_b \in \bar{\Phi}(G_b) - \bar{\Phi}_2(G_b)$, then for any $s \in (S^{\theta}_{\phi_o} \times Z(\widehat{G}_b))/Z(\widehat{G})^{\Gamma}$ we have $\phi'_{s,b} \in \bar{\Phi}(G'_{s,b}) - \bar{\Phi}_2(G'_{s,b})$. So ϕ'_s factors through a Levi subgroup of G'_s. Then the theorem follows from the compatibility of parabolic induction with twisted endoscopic transfer. □

To treat the discrete case, we need to strengthen our induction assumptions as follows.

Lemma 9.2. *Suppose $G = G(n)$ and*

$$\phi = \phi_1 \boxplus \cdots \boxplus \phi_q \boxplus (2\phi_{q+1} \boxplus \cdots \boxplus 2\phi_r) \in \bar{\Phi}_{ell}(G^{\theta})$$

for some $\theta \in \Sigma_0$ such that $\sum_{i=1}^{r} N_i < N$, then Theorem 4.1, 4.6 hold for $\tilde{\phi}$.

Proof. The case of discrete parameters follows from our induction assumption. For non-discrete parameters, we can construct the global L-packets by parabolic induction, which shows Theorem 4.1. By repeating the argument in Lemma 9.1, we can reduce Theorem 4.6 to the discrete case. □

Corollary 9.3. *Suppose $G = G(n_1) \times \cdots \times G(n_q)$ and $\phi = \phi_1 \times \cdots \times \phi_q \in \bar{\Phi}(G) - \bar{\Phi}_2(G)$ such that $\phi_i \in \bar{\Phi}(G(n_i))$ satisfies the condition in Lemma 9.2, then Theorem 4.1, 4.6, 4.8, 4.9 hold for $\tilde{\phi}$.*

Proof. For Theorem 4.1, we can construct the L-packets by parabolic induction. Next we show Theorem 4.6. Since the restriction of global L-packet $\bigotimes_{i=1}^{q} \bar{\Pi}_{\tilde{\phi}_i}$ of $\widetilde{G}(n_1) \times \widetilde{G}(n_2) \times \cdots \times \widetilde{G}(n_q)$ to $\widetilde{G}$ is also a global L-packet, then the functoriality of twisted endoscopic transfer for $\tilde{\phi}$ follows from that for $\tilde{\phi}_i$. At last we prove Theorem 4.8, 4.9 by induction on $\sum_{i=1}^{q} n_i$. It remains to show part (2) and (3), which follows from the comparison of (5.6), (5.7) with (5.10) using (5.11) and (5.12). □

9.3.2. *Discrete parameters.*

Lemma 9.4. *Suppose $G = G(N)$ and $\phi \in \bar{\Phi}_2(G)$, then Theorem 4.6 holds for $\tilde{\phi}$.*

Suppose $\phi = \phi_1 \boxplus \cdots \boxplus \phi_r \in \bar{\Phi}_2(G)$ and $\phi \neq \phi_b$. From the discussions in the end of Section 9.1.1 and Section 9.2, it remains to consider the case that (9.2) is trivial and $\mathcal{S}^{\Sigma_0}_{\phi_o} = \mathcal{S}_{\tilde{\phi}_o}$. Without loss of generality, we can make the following assumptions.

(1) If G is symplectic, then N_1 is odd and $\eta_i = 1$ for all i.
(2) If G is special even orthogonal and $\eta_G \neq 1$, then N_i is even for all i and $\eta_1 = \eta_G$.
(3) If G is special even orthogonal and $\eta_G = 1$, then N_i is even and $\eta_i = 1$ for all i, and ϕ_1 is not of GL-type (cf. Definition 4.4).

In all cases we take an auxiliary parameter as follows

$$\phi^+ := 2\phi_1 \boxplus \phi_2 \boxplus \cdots \boxplus \phi_r \in \bar{\Phi}(G^+).$$

Let $\epsilon \in \mathcal{S}^{\Sigma_0}_{\phi^+}$ such that ϕ^+ factors through

$$\phi^+_{\epsilon} := (\phi_1 \boxplus \phi_2 \boxplus \cdots \boxplus \phi_r) \times \phi_1 \in \bar{\Phi}_2(G^+_{\epsilon}).$$

Let $\theta = \theta_0$ in case (1) and $\theta = 1$ in case (2) and (3). By (5.10),

$$I^{(\widetilde{G}^{+,\theta},\eta_1)}_{disc,\phi^+}(\tilde{f}) = C_{\tilde{\phi}^+_o} \sum_{\omega \in Y/\langle \eta_1 \rangle} \sum_{x_o \in \mathcal{S}^{\theta}_{\phi^+_o}(\eta_1)} i^{\theta}_{\phi^+_o}(x_o) \tilde{f}_{\widetilde{G}^{+,\theta}}(\tilde{\phi}^+_o \otimes \omega, x_o; \tilde{\phi}_b), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}^+, \widetilde{\chi}^+). \tag{9.5}$$

The idea is to compare this with (5.9) in case (1), (2) and with (5.6), (5.7) in case (3). The main problem is that we can not apply (5.9), (5.6), (5.7) to ϕ^+ directly, since Theorem 4.9 is only applicable for ϕ^+_x with $x \in \mathcal{S}^{\Sigma_0}_{\phi^+} - \{\epsilon, 1\}$ by our induction assumption and Corollary 9.3. So we need to compute them separately

here. In view of the proof of Lemma 5.2, it suffices to determine the contribution of Tran $S^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon}$. First, the coefficient of this term is

$$d := m_{\phi^+}\frac{\iota(\widetilde{G}^+,\widetilde{G}^+_\epsilon)}{m_{\phi^+_\epsilon}}|\mathrm{Out}_{G^+}(G^+_\epsilon)||S_{\phi^+,\epsilon}/S_{\phi^+,\epsilon}\cap \widehat{G}^+_\epsilon Z(\widehat{G}^+)^\Gamma|^{-1}.$$

Next we will give a formula for $S^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon}$. If $\phi\neq\phi_o$, we can apply (5.7) to ϕ^+_ϵ and (5.6) to $\phi^+_{\epsilon,o}$, and get

$$\begin{aligned}S^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon} &= \frac{1}{4}\,S^{\widetilde{G}^+_{\epsilon,o}}_{disc,\phi^+_{\epsilon,o}}\tilde{\otimes}\, S^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}\\ &\quad+\Big(I^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon}-\frac{1}{2}\mathrm{Tran}(I^{\widetilde{G}^+_{\epsilon,o}}_{disc,\phi^+_{\epsilon,o}}\tilde{\otimes}\, I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b})\Big).\end{aligned}$$

If $\phi=\phi_o$, let us define $\bar\Pi_{\tilde\phi^+_{\epsilon,-}}$ to be the packet $\bar\Pi_{\tilde\phi^+_{\epsilon,x}}$ of $\widetilde{G}^+_\epsilon$ transferred from the L-packet of endoscopic group $\widetilde{G}^+_{\epsilon,x}$ for any $x\in\mathcal{S}_{\tilde\phi^+_\epsilon}-\{1\}$, and we have $\tilde f_{\widetilde{G}^+_\epsilon}(\tilde\phi^+_{\epsilon,-}\otimes\omega')=\tilde f'_{\widetilde{G}^+_\epsilon}(\tilde\phi^+_\epsilon\otimes\omega',x)$. Then we can apply (5.6) to ϕ^+_ϵ, and get

$$\begin{aligned}S^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon}(\tilde f) &= C_{\tilde\phi^+_\epsilon}\sum_{\omega'\in Y'}\tilde f_{\widetilde{G}^+_\epsilon}(\tilde\phi^+_{\epsilon,-}\otimes\omega')\\ &\quad+m_{\phi^+_\epsilon}\Big(\sum_{\omega'\in Y'}\tilde f_{\widetilde{G}^+_\epsilon}(\tilde\phi^+_\epsilon\otimes\omega')-\sum_{\omega'\in Y'}\tilde f_{\widetilde{G}^+_\epsilon}(\tilde\phi^+_{\epsilon,-}\otimes\omega')\Big),\quad \tilde f\in\bar{\mathcal{H}}(\widetilde{G}^+_\epsilon,\widetilde{\chi}^+_\epsilon).\end{aligned}$$

In case (1) and (2), the substitutes for (5.9) can be described as follows. If $\phi\neq\phi_o$, then

$$\begin{aligned}I^{(\widetilde{G}^{+,\theta},\eta_1)}_{disc,\phi^+}(\tilde f) &= C_{\tilde\phi^+_o}\sum_{\omega\in Y/\langle\eta_1\rangle}\sum_{x\in\mathcal{S}^\theta_{\phi^+_o}(\eta_1)}e'^\theta_{\phi^+_o}(x_o)\tilde f'_{\widetilde{G}^{+,\theta}}(\tilde\phi^+_o\otimes\omega,x_o;\tilde\phi_b)\\ &\quad+d\cdot\mathrm{Tran}\Big(I^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon}-\frac{1}{2}\mathrm{Tran}(I^{\widetilde{G}^+_{\epsilon,o}}_{disc,\phi^+_{\epsilon,o}}\tilde{\otimes}\, I^{\widetilde{G}_b}_{disc,\tilde\phi_b})\Big),\quad \tilde f\in\bar{\mathcal{H}}(\widetilde{G}^+,\widetilde{\chi}^+).\end{aligned}$$

If $\phi=\phi_o$, then

$$\begin{aligned}I^{(\widetilde{G}^{+,\theta},\eta_1)}_{disc,\phi^+}(\tilde f) &= C_{\tilde\phi^+}\sum_{\omega\in Y/\langle\eta_1\rangle}\sum_{x\in\mathcal{S}^\theta_{\phi^+}(\eta_1)}e'^\theta_{\phi^+}(x)\tilde f'_{\widetilde{G}^{+,\theta}}(\tilde\phi^+\otimes\omega,x)\\ &\quad+d\cdot m_{\phi^+_\epsilon}\Big(\sum_{\omega'\in Y'}\tilde f^{\widetilde{G}^+_\epsilon}(\tilde\phi^+_\epsilon\otimes\omega')-\sum_{\omega\in Y'}\tilde f^{\widetilde{G}^+_\epsilon}(\tilde\phi^+_{\epsilon,-}\otimes\omega')\Big),\quad \tilde f\in\bar{\mathcal{H}}(\widetilde{G}^+,\widetilde{\chi}^+),\end{aligned}$$

where $\tilde f'_{\widetilde{G}^{+,\theta}}(\tilde\phi^+\otimes\omega,\epsilon)$ is redefined to be the transfer of $\bar\Pi_{\tilde\phi^+_{\epsilon,-}}$ rather than $\bar\Pi_{\tilde\phi^+_\epsilon}$. In case (3), the substitutes for (5.6), (5.7) can be described as follows. If $\phi\neq\phi_o$, then

$$\begin{aligned}I^{\widetilde{G}^+}_{disc,\phi^+}(\tilde f)-S^{\widetilde{G}^+}_{disc,\phi^+}(\tilde f) &= C_{\tilde\phi^+_o}\sum_{\omega\in Y}\sum_{x\in\mathcal{S}_{\phi^+_o}}e'_{\phi^+_o}(x_o)\tilde f'_{\widetilde{G}^+}(\tilde\phi^+_o\otimes\omega,x_o;\tilde\phi_b)\\ &\quad+d\cdot\mathrm{Tran}\Big(I^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon}-\frac{1}{2}\mathrm{Tran}(I^{\widetilde{G}^+_{\epsilon,o}}_{disc,\phi^+_{\epsilon,o}}\tilde{\otimes}\, I^{\widetilde{G}_b}_{disc,\tilde\phi_b})\Big),\quad \tilde f\in\bar{\mathcal{H}}(\widetilde{G}^+,\widetilde{\chi}^+).\end{aligned}$$

If $\phi=\phi_o$, then

$$\begin{aligned}I^{\widetilde{G}^+}_{disc,\phi^+}(\tilde f)-S^{\widetilde{G}^+}_{disc,\phi^+}(\tilde f) &= C_{\tilde\phi^+}\sum_{\omega\in Y}\sum_{x\in\mathcal{S}_{\phi^+}}e'_{\phi^+}(x)\tilde f'_{\widetilde{G}^+}(\tilde\phi^+\otimes\omega,x)\\ &\quad+d\cdot m_{\phi^+_\epsilon}\Big(\sum_{\omega'\in Y'}\tilde f^{\widetilde{G}^+_\epsilon}(\tilde\phi^+_\epsilon\otimes\omega')-\sum_{\omega'\in Y'}\tilde f^{\widetilde{G}^+_\epsilon}(\tilde\phi^+_{\epsilon,-}\otimes\omega')\Big),\quad \tilde f\in\bar{\mathcal{H}}(\widetilde{G}^+,\widetilde{\chi}^+).\end{aligned}$$

To make the comparison with (9.5), we note in all cases

$$i^\theta_{\phi^+_o}(x_o)\tilde f_{\widetilde{G}^{+,\theta}}(\tilde\phi^+_o\otimes\omega,x_o;\tilde\phi_b)=e'^\theta_{\phi^+_o}(x_o)\tilde f'_{\widetilde{G}^{+,\theta}}(\tilde\phi^+_o\otimes\omega,x_o;\tilde\phi_b),\quad \tilde f\in\bar{\mathcal{H}}(\widetilde{G}^+,\widetilde{\chi}^+)$$

for all $x_o \in \mathcal{S}^{\theta}_{\phi_o^+}(\eta_1)$. In case (1) and (2), we get

$$\mathrm{Tran}\Big(I^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon} - \frac{1}{2}\mathrm{Tran}(I^{\widetilde{G}^+_{\epsilon,o}}_{disc,\phi^+_{\epsilon,o}} \tilde{\otimes} I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b})\Big) = 0 \tag{9.6}$$

when $\phi \neq \phi_o$, and

$$\sum_{\omega' \in Y'} \tilde{f}^{\widetilde{G}^+_\epsilon}(\tilde{\phi}^+_\epsilon \otimes \omega') - \sum_{\omega' \in Y'} \tilde{f}^{\widetilde{G}^+_\epsilon}(\tilde{\phi}^+_{\epsilon,-} \otimes \omega') = 0, \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}^+, \widetilde{\chi}^+) \tag{9.7}$$

when $\phi = \phi_o$. In case (3), we get

$$\mathrm{Tran}\Big(I^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon} - \frac{1}{2}\mathrm{Tran}(I^{\widetilde{G}^+_{\epsilon,o}}_{disc,\phi^+_{\epsilon,o}} \tilde{\otimes} I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b})\Big) = \frac{-1}{d} S^{\widetilde{G}^+}_{disc,\phi^+} \tag{9.8}$$

when $\phi \neq \phi_o$, and

$$\sum_{\omega' \in Y'} \tilde{f}^{\widetilde{G}^+_\epsilon}(\tilde{\phi}^+_\epsilon \otimes \omega') - \sum_{\omega' \in Y'} \tilde{f}^{\widetilde{G}^+_\epsilon}(\tilde{\phi}^+_{\epsilon,-} \otimes \omega') = \frac{-1}{d\, m_{\phi^+_\epsilon}} S^{\widetilde{G}}_{disc,\phi^+}(\tilde{f}), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}^+, \widetilde{\chi}^+) \tag{9.9}$$

when $\phi = \phi_o$.

We will further show that (9.8) and (9.9) both vanish, hence reduce to (9.6) and (9.7) respectively. By compatible lifting of L-packets (cf. Proposition 8.7), the left hand sides of (9.8) and (9.9) become

$$\sum_{\omega \in Y/\langle \eta_1 \rangle} \Big(\sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}^+_-} \otimes \omega} \langle \epsilon, \tilde{\pi} \rangle \tilde{f}_{\widetilde{G}^+}(\tilde{\pi}) - \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}^+} \otimes \omega} \langle \epsilon, \tilde{\pi} \rangle \tilde{f}_{\widetilde{G}^+}(\tilde{\pi}) \Big) \tag{9.10}$$

where $\bar{\Pi}_{\tilde{\phi}^+_-}$ denotes the packet of $\widetilde{G}^+$ transferred from the L-packet $\bar{\Pi}_{\tilde{\phi}^+_\epsilon}$ of $\widetilde{G}^+_\epsilon$. By the argument on stability as in the proof of Theorem 4.1, we can conclude the coefficients of characters of all $[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}^+}$ must be the same. Since ϕ_1 is not of GL-type, then there exist $[\tilde{\pi}_1], [\tilde{\pi}_2] \in \bar{\Pi}_{\tilde{\phi}^+}$ such that $\langle \epsilon, \tilde{\pi}_1 \rangle = 1, \langle \epsilon, \tilde{\pi}_2 \rangle = -1$. So (9.10) would not be stable unless it is zero.

At last, we would like to show $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}(\tilde{\phi}_o; \tilde{\phi}_b)$ if $\phi \neq \phi_o$, and $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}_{\tilde{\phi}_x}$ for $1 \neq x \in \mathcal{S}_{\tilde{\phi}}$ if $\phi = \phi_o$. Since $\alpha(\mathcal{S}^{\Sigma_0}_{\phi_v^+}) = \alpha(\mathcal{S}^{\Sigma_0}_{\phi^+_{\epsilon,v}})$ for all v, then (9.6) implies that

$$I^{\widetilde{G}^+_\epsilon}_{disc,\phi^+_\epsilon} - \frac{1}{2}\mathrm{Tran}(I^{\widetilde{G}^+_{\epsilon,o}}_{disc,\phi^+_{\epsilon,o}} \tilde{\otimes} I^{\widetilde{G}_b}_{disc,\tilde{\phi}_b}) = 0.$$

Since $\bar{\Pi}_{\tilde{\phi}^+_\epsilon}$ (resp. $\bar{\Pi}_{\tilde{\phi}^+_{\epsilon,o}}$) is the restriction of $\bar{\Pi}_{\tilde{\phi}} \otimes \bar{\Pi}_{\tilde{\phi}_1}$ (resp. $\bar{\Pi}_{\tilde{\phi}_o} \otimes \bar{\Pi}_{\tilde{\phi}_1}$) and $\alpha(\mathcal{S}^{\Sigma_0}_{\phi_{1,v}}) \subseteq \alpha(\mathcal{S}^{\Sigma_0}_{\phi_v})$ for all v, then $\bar{\Pi}_{\tilde{\phi}} = \bar{\Pi}(\tilde{\phi}_o; \tilde{\phi}_b)$. The other case follows from (9.7) by the same argument.

Corollary 9.5. *Suppose $G = G(n_1) \times \cdots \times G(n_q)$ and $\phi = \phi_1 \times \cdots \times \phi_q \in \bar{\Phi}(G)$ such that $n_i \leqslant N$, then Theorem 4.6 4.8, 4.9 hold for $\tilde{\phi}$.*

Proof. The proof is the same as Corollary 9.3 under Lemma 9.1, 9.4, except for (2), (3) of Theorem 4.8, 4.9 in the discrete case, where we need to compare (5.6), (5.7) with (4.1) instead. □

Appendix A.

Let F be a p-adic field and $G = SO(2n, \eta')$ and $\phi \in \bar{\Phi}_{sim}(G)$. We would like to show

Theorem A.1. *For $\omega \in H^1(W_F, Z(\widehat{\widetilde{G}})) \cong \mathrm{Hom}(\widetilde{G}(F), \mathbb{C}^\times)$,*

$$[\tilde{\phi} \otimes \omega] = [\tilde{\phi}] \text{ if and only if } \bar{\Pi}_{\tilde{\phi}} \otimes \omega = \bar{\Pi}_{\tilde{\phi}}. \tag{A.1}$$

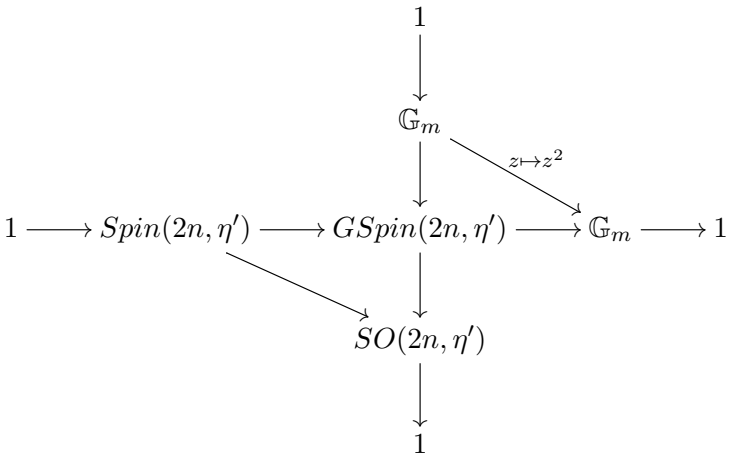
First let us consider $n = 1$. In this case, we have $\eta' = \eta_\phi \neq 1$. Let E'/F be the quadratic extension associated with η'. Then $G \cong U_{E'/F}(1)$ and $\widetilde{G} \cong \mathrm{Res}_{E'/F} GL(1)$, both of which are tori. The local Langlands correspondence for tori is known (cf. [Bor79]) and Theorem A.1 is a consequence of that, provided it is compatible with the one given by Arthur. In other words, we need to show the twisted endoscopic lifting from $U_{E'/F}(1, F)$ to $GL(2, F)$ is compatible with the local Langlands correspondence. In fact, we can break it up into two steps: standard base change from $U_{E'/F}(1, F)$ to $GL(1, E)$ followed

by the automorphic induction from $GL(1,E)$ to $GL(2,F)$. This is because both the standard base change and the automorphic induction are special cases of the twisted endoscopy, and one can check that the geometric endoscopic transfer from $GL(2,F)$ to $U_{E'/F}(1,F)$ is the composition of the other two. The standard base change is just the pull back along the homomorphism $GL(1,E) \to U_{E'/F}(1,F), x \mapsto \bar{x}/x$ for $x \in E^\times$ and it is compatible with the correspondence for tori. For the automorphic induction, the compatibility follows from [HH95].

Next we assume $n > 1$. Dual to (2.6), we have

$$1 \longrightarrow \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times) \longrightarrow \mathrm{Hom}(\widetilde{G}(F), \mathbb{C}^\times) \longrightarrow \mathrm{Hom}(G(F), \mathbb{C}^\times) \longrightarrow 1.$$

By [Xu18, Corollary 4.2], (A.1) holds for $\phi \in \bar{\Phi}_{bdd}(G)$ and $\omega \in \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times)$. Since we have assumed $\phi \in \bar{\Phi}_{sim}(G)$, then any nontrivial $\omega \in \mathrm{Hom}(\widetilde{G}(F), \mathbb{C}^\times)$ such that $[\tilde{\phi} \otimes \omega] = [\tilde{\phi}]$ must be nontrivial on $G(F)$. In order to describe $\mathrm{Hom}(G(F), \mathbb{C}^\times)$, we lift them to characters of $GSpin(2n, \eta')(F)$.



Let us denote $G_{sc} := Spin(2n, \eta')$ and $\widetilde{G}_{sc} := GSpin(2n, \eta')$. Then

$$\mathrm{Hom}(G(F), \mathbb{C}^\times) \cong \mathrm{Hom}(\widetilde{G}_{sc}(F)/F^\times G_{sc}(F), \mathbb{C}^\times) \cong \mathrm{Hom}(F^\times/F^{\times 2}, \mathbb{C}^\times).$$

Lemma A.2. *For $\eta \in \mathrm{Hom}(F^\times/F^{\times 2}, \mathbb{C}^\times)$, $\bar{\Pi}_{\phi\otimes\eta} = \bar{\Pi}_\phi \otimes \eta$.*

Proof. The packet $\bar{\Pi}_\phi$ is determined by its θ_{2n}-twisted endoscopic transfer π_ϕ^{GL} to $GL(2n,F)$ through the character relation

$$f^G(\phi) = f_{GL(2n)^{\theta_{2n}}}(\pi_\phi^{GL}), \quad f \in C_c^\infty(GL(2n,F))$$

(cf. [Art13, Theorem 2.2.1]). Since $\pi_{\phi\otimes\eta}^{GL} \cong \pi_\phi^{GL} \otimes \eta$ and $(f\eta)^G = f^G\eta$, then

$$f^G(\phi \otimes \eta) = f_{GL(2n)^{\theta_{2n}}}(\pi_\phi^{GL} \otimes \eta) = (f\eta)_{GL(2n)^{\theta_{2n}}}(\pi_\phi^{GL}) = (f^G\eta)(\phi).$$

This finishes the proof. □

Suppose $1 \neq \eta \in \mathrm{Hom}(F^\times/F^{\times 2}, \mathbb{C}^\times)$ is associated with a quadratic extension E/F and $[\phi \otimes \eta] = [\phi]$, then $\bar{\Pi}_\phi \otimes \eta = \bar{\Pi}_\phi$ by Lemma A.2. There exists a unique extension $\tilde{\eta} \in \mathrm{Hom}(\widetilde{G}(F), \mathbb{C}^\times)$ of η such that $[\tilde{\phi} \otimes \tilde{\eta}] = [\tilde{\phi}]$. In order to prove Theorem A.1, it suffices to show that $\bar{\Pi}_{\tilde{\phi}} \otimes \tilde{\eta} = \bar{\Pi}_{\tilde{\phi}}$.

Note π_ϕ^{GL} is an essential discrete series representation of $GL(2n,F)$. We denote its Langlands parameter again by ϕ. Since $\pi_\phi^{GL} \otimes \eta \cong \pi_\phi^{GL}$, there exists an essential discrete series representation π_E of $GL(n,E)$, which is not $\Gamma_{E/F}$-conjugation invariant, such that π_ϕ^{GL} is the automorphic induction of π_E (cf. [HH95] [AC89]). Denote the Langlands parameter of π_E by ϕ_E. We can also view π_E as a representation of $\mathrm{Res}_{E/F} GL(n)(F)$ and denote the corresponding Langlands parameter by $\phi_{E/F}$. Since $\phi = \mathrm{Ind}_{L_E}^{L_F} \phi_E$ (cf. [Hen01]), then ϕ factors through $\phi_{E/F}$. Moreover,

$$\phi|_{L_E} = \phi_E^c \oplus \phi_E$$

where ϕ_E^c is the $\Gamma_{E/F}$-conjugate of ϕ_E. Since ϕ is self-dual of orthogonal type, then ϕ_E is either self-dual of orthogonal type or conjugate orthogonal (cf. [Mok14, Section 2]). It follows that $\phi_{E/F}$ factors through $\phi_H \in \bar{\Phi}_{bdd}(H)$ for a twisted elliptic endoscopic group H of $\mathrm{Res}_{E/F} GL(n)$. It also induces an embedding of ${}^L H$ into ${}^L G$, through which we can view H as a twisted elliptic endoscopic group of G.

$$\begin{array}{ccccc} \phi : L_F & \longrightarrow & {}^L G & \longrightarrow & {}^L GL(2n) \\ & \searrow & \uparrow & & \uparrow \\ & & {}^L H & \longrightarrow & {}^L \mathrm{Res}_{E/F} GL(n) \end{array}$$

More precisely, we have the following three cases.

- If $\phi_E^c = \phi_E^\vee$, then $H = U_{E/F}(n)$ is an η-twisted endoscopic group of G. In this case,
$$\eta' = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \eta & \text{if } n \text{ is odd.} \end{cases}$$
- If $\phi_E = \phi_E^\vee$ and n is even, then $H = \mathrm{Res}_{E/F} SO(n, \eta_{\phi_E})$ is an η-twisted endoscopic group of G. In this case, $\eta' = \eta_{\phi_E}|_{F^\times}$.
- If $\phi_E = \phi_E^\vee$ and n is odd, then $H = \mathrm{Res}_{E/F} Sp(n-1)$ is a (θ_0, η)-twisted endoscopic group of G. In this case, $\eta' = \eta \cdot \eta_{\phi_E}|_{F^\times}$.

We can lift H to an $\tilde{\eta}$-twisted (resp. $(\theta_0, \tilde{\eta})$-twisted) elliptic endoscopic group $\widetilde{H}$ of $\widetilde{G}$ (cf. [Xu16, Proposition 3.1]). Then $[\tilde{\phi} \otimes \tilde{\eta}] = [\tilde{\phi}]$.

Next we can construct a globalization of ϕ_H, namely $\dot{\phi}_{\dot{H}} \in \bar{\Phi}_{sim}(\dot{H})$ such that $\dot{\phi}_{\dot{H},u} = \phi_H$ following [Art13, Proposition 6.3.1] [Mok14, Proposition 7.3.1]. It gives rise to a self-dual or conjugate self-dual $\dot{\phi}_{\dot{E}} \in \Phi_{sim}(GL(n)_{\dot{E}})$ such that $\dot{\phi}_{\dot{E},w} = \phi_E$ for $w|u$. By automorphic induction [AC89] [Hen12], we obtain a self-dual $\dot{\phi} \in \Phi_{sim}(GL(2n))$ such that $\dot{\phi} \otimes \dot{\eta} = \dot{\phi}$. Since $\dot{\phi}_u = \phi \in \bar{\Phi}_{sim}(G)$, then $\dot{\phi} \in \bar{\Phi}_{sim}(\dot{G})$, where $\dot{G} = SO(2n, \dot{\eta}')$. Let us lift $\dot{H}$ to an $\tilde{\dot{\eta}}$-twisted (resp. $(\theta_0, \tilde{\dot{\eta}})$-twisted) elliptic endoscopic group $\widetilde{\dot{H}}$ of $\widetilde{\dot{G}}$. The following lemma is the key step of the proof.

Lemma A.3. *If n is even, then*

$$I_{disc,\dot{\phi}}^{(\widetilde{\dot{G}},\tilde{\dot{\eta}})}(\tilde{\dot{f}}) = 4\,\iota(\widetilde{\dot{G}}, \widetilde{\dot{H}}) S_{disc,\dot{\phi}_{\dot{H}}}^{\widetilde{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}), \quad \tilde{\dot{f}} \in \bar{\mathcal{H}}(\widetilde{\dot{G}}, \widetilde{\chi}).$$

If n is odd and $\phi_E^c = \phi_E^\vee$, then

$$I_{disc,\dot{\phi}}^{(\widetilde{\dot{G}},\tilde{\dot{\eta}})}(\tilde{\dot{f}}) = 2\,\iota(\widetilde{\dot{G}}, \widetilde{\dot{H}}) S_{disc,\dot{\phi}_{\dot{H}}}^{\widetilde{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}), \quad \tilde{\dot{f}} \in \bar{\mathcal{H}}(\widetilde{\dot{G}}, \widetilde{\chi}).$$

If n is odd and $\phi_E = \phi_E^\vee$, then

$$I_{disc,\dot{\phi}}^{(\widetilde{\dot{G}}^{\theta_0},\tilde{\dot{\eta}})}(\tilde{\dot{f}}) = 4\,\iota(\widetilde{\dot{G}}, \widetilde{\dot{H}}) S_{disc,\dot{\phi}_{\dot{H}}}^{\widetilde{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}), \quad \tilde{\dot{f}} \in \bar{\mathcal{H}}(\widetilde{\dot{G}}, \widetilde{\chi}).$$

Proof. By the twisted stable trace formula, we have

$$I_{disc,\dot{\phi}}^{(\widetilde{\dot{G}},\tilde{\dot{\eta}})}(\tilde{\dot{f}}) = \sum_{\widetilde{\dot{G}}' \in \mathcal{E}_{ell}(\widetilde{\dot{G}},\tilde{\dot{\eta}})} \iota(\widetilde{\dot{G}}, \widetilde{\dot{G}}') S_{disc,\dot{\phi}}^{\widetilde{\dot{G}}'}(\tilde{\dot{f}}'), \quad \tilde{\dot{f}} \in \bar{\mathcal{H}}(\widetilde{\dot{G}}, \widetilde{\chi})$$

$$\Big(\text{resp. } I_{disc,\dot{\phi}}^{(\widetilde{\dot{G}}^{\theta_0},\tilde{\dot{\eta}})}(\tilde{\dot{f}}) = \sum_{\widetilde{\dot{G}}' \in \mathcal{E}_{ell}(\widetilde{\dot{G}}^{\theta_0},\tilde{\dot{\eta}})} \iota(\widetilde{\dot{G}}, \widetilde{\dot{G}}') S_{disc,\dot{\phi}}^{\widetilde{\dot{G}}'}(\tilde{\dot{f}}'), \quad \tilde{\dot{f}} \in \bar{\mathcal{H}}(\widetilde{\dot{G}}, \widetilde{\chi}). \Big)$$

The $\widehat{\Sigma}_0$-conjugacy class of isomorphism classes of $\dot{\eta}$-twisted (resp. $(\theta_0, \dot{\eta})$-twisted) elliptic endoscopic data of $\dot{G}$ can be classified by $(\dot{G}', \dot{K}/\dot{E})$, where $[\dot{K} : \dot{E}] \leqslant 2$ and $\dot{G}' = U_{\dot{E}/\dot{F}}(n_1) \times \mathrm{Res}_{\dot{E}/\dot{F}} SO(n_2, \eta_{\dot{K}/\dot{E}})$ (resp. $U_{\dot{E}/\dot{F}}(n_1) \times \mathrm{Res}_{\dot{E}/\dot{F}} Sp(n_2 - 1)$) for $n = n_1 + n_2$ and n_2 is even (resp. n_2 is odd), subject to the conditions

that $\dot{\eta}' = \dot{\eta}^n \cdot \eta_{\dot{K}/\dot{E}}|_{\mathbb{A}_{\dot{F}}^\times}$ and if $\dot{G}' = U_{\dot{E}/\dot{F}}(n)$, then $\dot{K} = \dot{E}$. The twisted endoscopic embedding ξ can be chosen to satisfy the following diagram

$$\begin{array}{ccccc} {}^L\dot{G} & & \longrightarrow & & {}^LGL(2n) \\ \uparrow{\scriptstyle \xi} & & & & \uparrow \\ {}^L\dot{G}' & \xrightarrow{\xi_{\dot{E}/\dot{F}}} & {}^L\mathrm{Res}_{\dot{E}/\dot{F}}(GL(n_1)\times GL(n_2)) & \longrightarrow & {}^L\mathrm{Res}_{\dot{E}/\dot{F}}GL(n), \end{array}$$

where $\xi_{\dot{E}/\dot{F}}$ is given by the fiber product over $W_{\dot{F}}$ of

$$\xi_{\chi_{(-1)^{n_1-1}}} : {}^LU_{\dot{E}/\dot{F}}(n_1) \to {}^L\mathrm{Res}_{\dot{E}/\dot{F}}GL(n_1)$$

(cf. [Mok14, Section 2]) and

$$\xi_{\dot{K}/\dot{E}} : {}^L\mathrm{Res}_{\dot{E}/\dot{F}\,SO(n_2,\eta_{\dot{K}/\dot{E}})} \to {}^L\mathrm{Res}_{\dot{E}/\dot{F}GL(n_2)}$$
$$\Big(\text{resp. } \xi_{\dot{K}/\dot{E}} : {}^L\mathrm{Res}_{\dot{E}/\dot{F}\,Sp(n_2-1)} \to {}^L\mathrm{Res}_{\dot{E}/\dot{F}}GL(n_2)\Big).$$

By [Xu18, Proposition 2.7]), there is a one to one correspondence

$$\mathcal{E}_{ell}(\dot{G},\dot{\eta}) \cong \bigsqcup_{\tilde{\dot{\eta}}|_{\dot{G}}=\dot{\eta}} \mathcal{E}_{ell}(\widetilde{\dot{G}},\tilde{\dot{\eta}})$$
$$\Big(\text{resp. } \mathcal{E}_{ell}(\dot{G}^{\theta_0},\dot{\eta}) \cong \bigsqcup_{\tilde{\dot{\eta}}|_{\dot{G}}=\dot{\eta}} \mathcal{E}_{ell}(\widetilde{\dot{G}}^{\theta_0},\tilde{\dot{\eta}})\Big).$$

It is easy to show that if $S^{\widetilde{\dot{G}}'}_{disc,\dot{\phi}} \neq 0$, then $S^{\dot{G}'}_{disc,\dot{\phi}} \neq 0$ (cf. [Xu18, Lemma 5.7]). Suppose $c(\dot{\phi})$ is the image of $c(\dot{\phi}')$ for some $\dot{\phi}' \in \bar{\Phi}_2(\dot{G}')$. Compose $\dot{\phi}'$ with the embeddings

$${}^L\dot{G}' \to {}^L\mathrm{Res}_{\dot{E}/\dot{F}(GL(n_1)\times GL(n_2))} \to {}^L\mathrm{Res}_{\dot{E}/\dot{F}}GL(n),$$

we get $\dot{\phi}_{\dot{E},1} \boxplus \dot{\phi}_{\dot{E},2} \in \Phi(GL(n)_{\dot{E}})$, where $\dot{\phi}_{\dot{E},1}$ is conjugate self-dual and $\dot{\phi}_{\dot{E},2}$ is self-dual. If the image of $\dot{\phi}'$ in $\bar{\Phi}_2(\dot{G})$ is equal to $\dot{\phi}$, then

$$\dot{\phi}_{\dot{E},1} \boxplus \dot{\phi}_{\dot{E},2} = \dot{\phi}_{\dot{E}} \text{ or } \dot{\phi}^c_{\dot{E}}.$$

Hence, it is necessary that $\dot{G}' = \dot{H}$ and $\dot{\phi}' = \dot{\phi}_{\dot{H}}$ or $\dot{\phi}^c_{\dot{H}}$, where $\dot{\phi}^c_{\dot{H}}$ is the $\widehat{\theta}_c$-conjugate of $\dot{\phi}_{\dot{H}}$ for an automorphism θ_c of $\dot{H}$ and it gives rise to $\dot{\phi}^c_{\dot{E}}$.

If n is odd and $\dot{H} = U_{\dot{E}/\dot{F}}(n)$, the corresponding $\widehat{\Sigma}_0$-conjugacy class of isomorphism classes of endoscopic data consists of only one isomorphism class. Moreover, $\mathrm{Out}_{\widetilde{\dot{G}}_v}(\widetilde{\dot{H}}_v) = 1$ and

$$S^{\widetilde{\dot{H}}}_{disc,\dot{\phi}^c_{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}) = S^{\widetilde{\dot{H}}}_{disc,\dot{\phi}_{\dot{H}}}((\tilde{\dot{f}}^{\theta_0})^{\widetilde{\dot{H}}}), \quad \tilde{\dot{f}} \in \mathcal{H}(\widetilde{\dot{G}},\widetilde{\chi}).$$

In other cases, it follows from $\theta_c \in \mathrm{Out}_{\widetilde{\dot{G}}_v}(\widetilde{\dot{H}}_v)$ that

$$S^{\widetilde{\dot{H}}}_{disc,\dot{\phi}^c_{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}) = S^{\widetilde{\dot{H}}}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}), \quad \tilde{\dot{f}} \in \mathcal{H}(\widetilde{\dot{G}},\widetilde{\chi}).$$

Moreover, the corresponding $\widehat{\Sigma}_0$-conjugacy class of isomorphism classes of endoscopic data consists of two isomorphism classes. We can lift them to twisted endoscopic data of $\widetilde{\dot{G}}$ and denote the transfers by $\tilde{\dot{f}}^{\widetilde{\dot{H}}}$ and $\tilde{\dot{f}}^{\widetilde{\dot{H}}_1}$ respectively. Then

$$\tilde{\dot{f}}^{\widetilde{\dot{H}}_1} = (\tilde{\dot{f}}^{\theta_0})^{\widetilde{\dot{H}}}, \quad \tilde{\dot{f}} \in \mathcal{H}(\widetilde{\dot{G}},\widetilde{\chi})$$

and $\iota(\widetilde{\dot{G}},\widetilde{\dot{H}})$ remains the same for the two endoscopic data. So in all cases we can conclude the lemma by restricting to $\bar{\mathcal{H}}(\widetilde{\dot{G}},\widetilde{\chi})$.

□

Remark A.4. We can determine the coefficient $\iota(\widetilde{\dot{G}}, \widetilde{\dot{H}})$ in the lemma using (3.3) as follows.

$$\iota(\widetilde{\dot{G}}, \widetilde{\dot{H}}) = \begin{cases} 1 & \text{if } \dot{\eta}' = \dot{\eta} \text{ and } \dot{H} = U_{\dot{E}/\dot{F}}(n), \\ 1/2 & \text{if } \dot{\eta}' = 1 \text{ and } \dot{H} = \mathrm{Res}_{\dot{E}/\dot{F}} Sp(n-1) \text{ or } U_{\dot{E}/\dot{F}}(n), \\ 1/4 & \text{otherwise.} \end{cases}$$

To finish the proof of Theorem A.1, we still need to show the distributions in Lemma A.3 are nonzero. By the stable trace formula of $\widetilde{\dot{H}}$, we have

$$S^{\widetilde{\dot{H}}}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}) = I^{\widetilde{\dot{H}}}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}) - \sum_{\widetilde{\dot{H}}' \in \mathcal{E}_{ell}(\widetilde{\dot{H}}) - \{\widetilde{\dot{H}}\}} \iota(\widetilde{\dot{H}}, \widetilde{\dot{H}}') S^{\widetilde{\dot{H}}'}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}'), \quad \tilde{\dot{f}} \in \mathcal{H}(\widetilde{\dot{H}}, \widetilde{\chi}_{\widetilde{\dot{H}}}).$$

Since $\dot{\phi}_{\dot{H}}$ does not factor through any proper elliptic endoscopic group of $\dot{H}$, then $S^{\dot{H}'}_{disc,\dot{\phi}_{\dot{H}}}(\dot{f}') = 0$ for $\dot{H}' \neq H$, which implies that $S^{\widetilde{\dot{H}}'}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}') = 0$ (cf. [Xu18, Lemma 5.7]). Therefore,

$$S^{\widetilde{\dot{H}}}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}) = I^{\widetilde{\dot{H}}}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}) = tr R^{\widetilde{\dot{H}}}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}), \quad \tilde{\dot{f}} \in \mathcal{H}(\widetilde{\dot{H}}, \widetilde{\chi}_{\widetilde{\dot{H}}}).$$

So there exists $\tilde{\dot{f}} \in \bar{\mathcal{H}}(\widetilde{\dot{G}}, \widetilde{\chi})$ such that $S^{\widetilde{\dot{H}}}_{disc,\dot{\phi}_{\dot{H}}}(\tilde{\dot{f}}^{\widetilde{\dot{H}}}) \neq 0$, hence $I^{(\widetilde{\dot{G}}, \tilde{\dot{\eta}})}_{disc,\dot{\phi}}(\tilde{\dot{f}}) \neq 0$ (resp. $I^{(\widetilde{\dot{G}}^{\theta_0}, \tilde{\dot{\eta}})}_{disc,\dot{\phi}}(\tilde{\dot{f}}) \neq 0$) by Lemma A.3. It follows that there exists $[\tilde{\dot{\pi}}] \in \bar{\Pi}_{\tilde{\dot{\phi}}}$ such that $\tilde{\dot{\pi}} \cong \tilde{\dot{\pi}} \otimes \tilde{\dot{\eta}}$ (resp. $\tilde{\dot{\pi}}^{\theta_0} \cong \tilde{\dot{\pi}} \otimes \tilde{\dot{\eta}}$). In particular, $\tilde{\dot{\pi}}_u \cong \tilde{\dot{\pi}}_u \otimes \tilde{\eta}$ (resp. $\tilde{\dot{\pi}}_u^{\theta_0} \cong \tilde{\dot{\pi}}_u \otimes \tilde{\eta}$). Hence $\bar{\Pi}_{\tilde{\dot{\phi}}} = \bar{\Pi}_{\tilde{\dot{\phi}}} \otimes \tilde{\eta}$.

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