A LOCAL NONCOLLAPSING ESTIMATE FOR MEAN CURVATURE FLOW

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Abstract. We prove a local version of the noncollapsing estimate for mean curvature flow. By combining our result with earlier work of X.-J. Wang, it follows that certain ancient convex solutions that sweep out the entire space are noncollapsed.

1. Introduction

A crucial property of mean curvature flow is that embeddedness is preserved under the evolution, and this property is especially consequential in the class of mean-convex flows. Indeed, White [6, 7] proved several important results on singularities of embedded, mean-convex flows. Among other things, White showed certain "collapsed" singularity models, like the grim reaper or the multiplicity-two hyperplane, cannot arise as blow-up limits of embedded mean-convex flows.

In [4], Sheng and Wang introduced a quantitative version of the concept of embeddedness. Let M_t be a family of embedded, mean-convex hypersurfaces evolving by mean curvature flow. We say that the flow M_t is α-noncollapsed if

$$\frac{1}{2}\,\alpha^{-1}\,H(x,t)\,|x-y|^2 - \langle x-y, \nu(x,t)\rangle \geq 0$$

for all points $x, y \in M_t$. This concept has a natural geometric interpretation. A flow is α-noncollapsed if, for every point $x \in M_t$, there exists a ball of radius $\alpha H(x,t)^{-1}$ which lies in the inside of M_t and which touches M_t at x.

It is a consequence of White's work that every compact, embedded, mean-convex solution of mean curvature flow is α-noncollapsed, for some $\alpha > 0$, up to the first singular time. Alternative proofs of this result were given by Sheng and Wang [4] and by Andrews [1]. In [2], the first author proved a sharp version of this noncollapsing estimate. More precisely, if we start from a closed, embedded, mean-convex solution of mean curvature flow, then every blow-up limit is 1-noncollapsed.

In this note, we prove a local version of the noncollapsing estimate for the mean curvature flow.

Theorem 1. *Let us fix radii R and r such that $R \geq \sqrt{1+3n}\,r$. Moreover, let Λ be a positive real number. Let M_t, $t \in [-r^2, 0]$, be an embedded solution*

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of mean curvature flow in the ball $B_{3R}(0)$ satisfying $R^{-1} \leq \Lambda H$ and $|A| \leq \Lambda H$ for all $t \in [-r^2, 0]$ and all $x \in M_t \cap B_{\sqrt{1+3n}\,r}(0)$. Moreover, suppose that

$$\frac{1}{2}\,\Lambda\, H(x, -r^2)\,|x-y|^2 - \langle x-y, \nu(x,-r^2)\rangle \geq 0$$

for all $x \in M_{-r^2} \cap B_{\sqrt{1+3n}\,r}(0)$ and all $y \in M_{-r^2} \cap B_{3R}(0)$. Then

$$\Lambda\,(2+6n)^4\, H(x,t)\,|x-y|^2 - \langle x-y, \nu(x,t)\rangle \geq 0$$

for all $t \in [-r^2, 0]$, all $x \in M_t \cap B_{\frac{r}{2}}(0)$, and all $y \in M_t \cap B_{3R}(0)$.

The proof of Theorem 1 relies on two main ingredients. The first is the evolution equation from [2] for the reciprocal of the inscribed radius. The second is a particular choice of cutoff function inspired in part by the work of Ecker and Huisken [3]. The argument can be readily adapted to fully nonlinear flows given by speeds $G = G(h_{ij}) > 0$ which are homogeneous of degree one, concave, and satisfy $0 < \frac{\partial G}{\partial h_{ij}} \leq K g_{ij}$ for a uniform constant K.

Let us now discuss an application of the main theorem.

Corollary 2. *Let Λ be a positive real number. Let M_t, $t \in (-\infty, 0]$, be an embedded ancient solution of mean curvature flow in $\mathbb{R}^{n+1}$ such that $H > 0$ and $|A| \leq \Lambda H$ at each point in space-time. Suppose that there exists a sequence of times $t_j \to -\infty$ such that*

$$\frac{1}{2}\,\Lambda\, H(x, t_j)\,|x-y|^2 - \langle x-y, \nu(x,t_j)\rangle \geq 0$$

for all $x \in M_{t_j} \cap B_{\sqrt{-(1+3n)t_j}}(0)$ and all $y \in M_{t_j}$. Then

$$\Lambda\,(2+6n)^4\, H(x,t)\,|x-y|^2 - \langle x-y, \nu(x,t)\rangle \geq 0$$

for all $t \in (-\infty, 0]$, all $x \in M_t$, and all $y \in M_t$.

To deduce Corollary 2 from Theorem 1, we put $r_j = \sqrt{-t_j}$. Moreover, for each j, we choose R_j large enough so that $R_j \geq \sqrt{-(1+3n)t_j}$ and $R_j^{-1} \leq \Lambda H$ for all $t \in [t_j, 0]$ and all $x \in M_t \cap B_{\sqrt{-(1+3n)t_j}}(0)$. If we apply Theorem 1 and take the limit as $j \to \infty$, the assertion follows.

Corollary 3. *Let M_t, $t \in (-\infty, 0]$, be a convex ancient solution of mean curvature flow in $\mathbb{R}^{n+1}$ with $H > 0$. Suppose that there exists a sequence $t_j \to -\infty$ such that the rescaled hypersurfaces $(-t_j)^{-\frac{1}{2}}\, M_{t_j}$ converge in C^∞_{loc} to a cylinder $S^{n-k} \times \mathbb{R}^k$ with multiplicity 1, where $k \in \{0, 1, \ldots, n-1\}$. Then there exists a constant $\Lambda(n)$ such that*

$$\Lambda(n)\, H(x,t)\,|x-y|^2 - \langle x-y, \nu(x,t)\rangle \geq 0$$

for all $t \in (-\infty, 0]$, all $x \in M_t$, and all $y \in M_t$.

In [5], X.-J. Wang considered ancient solutions to mean curvature flow that can be expressed as level sets $M_t = \{u = -t\}$, where u is a convex function which is defined on the entire space $\mathbb{R}^{n+1}$ and satisfies

$$\sum_{i,j=1}^{n+1} \Big(\delta_{ij} - \frac{D_i u\, D_j u}{|\nabla u|^2}\Big)\, D_i D_j u = 1.$$

Wang showed that such ancient solutions admit a blow-down limit which is a cylinder $S^{n-k} \times \mathbb{R}^k$ with multiplicity 1, where $k \in \{0, 1, \ldots, n-1\}$ (see [5], Theorem 1.3). By Corollary 3, such an ancient solution must be noncollapsed.

2. Proof of Theorem 1

By scaling, it suffices to prove the assertion for $r = 1$. Let us fix a radius $R \geq \sqrt{1+3n}$. Moreover, we fix a positive real number Λ. Let M_t, $t \in [-1, 0]$, be an embedded solution of mean curvature flow in the ball $B_{3R}(0)$ satisfying $R^{-1} \leq \Lambda H$ and $|A| \leq \Lambda H$ for all $t \in [-1, 0]$ and all $x \in M_t \cap B_{\sqrt{1+3n}}(0)$. Moreover, we assume that

$$\frac{1}{2}\, \Lambda\, H(x, -1)\, |x - y|^2 - \langle x - y, \nu(x, -1)\rangle \geq 0$$

for all $x \in M_{-1} \cap B_{\sqrt{1+3n}}(0)$ and all $y \in M_{-1} \cap B_{3R}(0)$.

We define a function φ by

$$\varphi(x, t) := (2\Lambda)^{-\frac{1}{4}}\, (1 + 3n)^{-1}\, (1 - |x|^2 - 3nt)$$

for all $t \in [-1, 0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$. Moreover, we define

$$\Phi(x, t) := \varphi(x, t)^{-4}\, H(x, t)$$

for all $t \in [-1, 0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$. In the following, let $\lambda_1 \leq \ldots \leq \lambda_n$ denote the eigenvalues of the second fundamental form.

Lemma 4. *We have $\Phi \geq 2\Lambda H$ for all $t \in [-1, 0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$. Moreover, the eigenvalues of the second fundamental form satisfy $|\lambda_i| \leq \frac{\Phi}{2}$ for all $t \in [-1, 0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$.*

Proof. By definition, $\varphi \leq (2\Lambda)^{-\frac{1}{4}}$ and $\Phi \geq 2\Lambda H$ for all $t \in [-1, 0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$. This proves the first statement.

To prove the second statement, we observe that $|A| \leq \Lambda H \leq \frac{\Phi}{2}$ for all $t \in [-1, 0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$. This completes the proof of Lemma 4.

Lemma 5. *The function φ satisfies*

$$\frac{\partial \varphi}{\partial t} - \Delta \varphi < 0$$

for all $t \in [-1, 0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$.

Proof. We compute

$$\frac{\partial \varphi}{\partial t} - \Delta \varphi = -(2\Lambda)^{-\frac{1}{4}} \, (1+3n)^{-1} \, n < 0$$

for all $t \in [-1,0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$.

Lemma 6. *The function Φ satisfies*

$$\frac{\partial \Phi}{\partial t} - \Delta \Phi - |A|^2 \, \Phi + 2 \sum_{i=1}^n \frac{(D_i \Phi)^2}{\Phi - \lambda_i} > 0$$

for all $t \in [-1,0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$.

Proof. Using Lemma 5, we obtain

$$\begin{aligned} \frac{\partial \Phi}{\partial t} - \Delta \Phi &= \varphi^{-4} \Big(\frac{\partial H}{\partial t} - \Delta H \Big) - 4 \, \varphi^{-5} H \, \Big(\frac{\partial \varphi}{\partial t} - \Delta \varphi \Big) \\ &\quad - \frac{4}{3} \, \varphi^{-4} H \, \Big| \frac{\nabla H}{H} - 4 \, \frac{\nabla \varphi}{\varphi} \Big|^2 + \frac{4}{3} \, \varphi^{-4} H \, \Big| \frac{\nabla H}{H} - \frac{\nabla \varphi}{\varphi} \Big|^2 \\ &> |A|^2 \, \varphi^{-4} H - \frac{4}{3} \, \varphi^{-4} H \, \Big| \frac{\nabla H}{H} - 4 \, \frac{\nabla \varphi}{\varphi} \Big|^2 \\ &= |A|^2 \, \Phi - \frac{4}{3} \, \frac{|\nabla \Phi|^2}{\Phi} \end{aligned}$$

for all $t \in [-1,0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$. Moreover, it follows from Lemma 4 that $\frac{\Phi}{2} \leq \Phi - \lambda_i \leq \frac{3\Phi}{2}$ for all $t \in [-1,0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$. Consequently,

$$\frac{\partial \Phi}{\partial t} - \Delta \Phi > |A|^2 \, \Phi - \frac{4}{3} \, \frac{|\nabla \Phi|^2}{\Phi} \geq |A|^2 \, \Phi - 2 \sum_{i=1}^n \frac{(D_i \Phi)^2}{\Phi - \lambda_i}$$

for all $t \in [-1,0]$ and all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$. This completes the proof of Lemma 6.

We next define

$$Z(x,y,t) := \frac{1}{2} \, \Phi(x,t) \, |x-y|^2 - \langle x-y, \nu(x,t) \rangle$$

for $t \in [-1,0]$, $x \in M_t$, and $y \in M_t$.

Lemma 7. *We have $Z(x,y,t) \geq 0$ for all $t \in [-1,0]$, all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$, and all $y \in M_t \cap B_{3R}(0)$.*

Proof. Suppose that the assertion is false. Let J denote the set of all times $t \in [-1,0]$ with the property that we can find a point $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$ and a point $y \in M_t \cap B_{3R}(0)$ such that $Z(x,y,t) < 0$. Moreover, we define $\bar{t} := \inf J$.

By definition, $Z(x,y,t) \geq 0$ for all $t \in [-1,\bar{t})$, all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$, and all $y \in M_t \cap B_{3R}(0)$. Moreover, we can find a sequence of times $t_j \in J$

such that $t_j \to \bar{t}$. For each j, we can find a point $x_j \in M_{t_j} \cap B_{\sqrt{1-3nt_j}}(0)$ and a point $y_j \in M_{t_j} \cap B_{3R}(0)$ such that $Z(x_j, y_j, t_j) < 0$. Clearly, $x_j \neq y_j$, and

$$\frac{2\,\langle x_j - y_j, \nu(x_j, t_j)\rangle}{|x_j - y_j|^2} > \Phi(x_j, t_j).$$

Using the Cauchy-Schwarz inequality, we obtain

$$\frac{2}{|x_j - y_j|} > \Phi(x_j, t_j) \geq 2\Lambda H(x_j, t_j) \geq 2R^{-1}.$$

In other words, $|x_j - y_j| \leq R$. After passing to a subsequence, the points x_j converge to a point $\bar{x} \in M_{\bar{t}}$ satisfying $|\bar{x}| \leq \sqrt{1 - 3n\bar{t}}$. Moreover, the points y_j converge to a point $\bar{y} \in M_{\bar{t}}$ satisfying $|\bar{x} - \bar{y}| \leq R$. In particular, $|\bar{y}| \leq \sqrt{1 + 3n} + R \leq 2R$.

If $|\bar{x}| = \sqrt{1 - 3n\bar{t}}$ and $\bar{x} \neq \bar{y}$, then

$$\frac{2\,\langle \bar{x} - \bar{y}, \nu(\bar{x}, \bar{t})\rangle}{|\bar{x} - \bar{y}|^2} = \limsup_{j\to\infty} \frac{2\,\langle x_j - y_j, \nu(x_j, t_j)\rangle}{|x_j - y_j|^2} \geq \limsup_{j\to\infty} \Phi(x_j, t_j) = \infty,$$

which is impossible.

If $|\bar{x}| = \sqrt{1 - 3n\bar{t}}$ and $\bar{x} = \bar{y}$, then

$$\lambda_n(\bar{x}, \bar{t}) \geq \limsup_{j\to\infty} \frac{2\,\langle x_j - y_j, \nu(x_j, t_j)\rangle}{|x_j - y_j|^2} \geq \limsup_{j\to\infty} \Phi(x_j, t_j) = \infty,$$

which is impossible.

If $|\bar{x}| < \sqrt{1 - 3n\bar{t}}$ and $\bar{x} = \bar{y}$, then

$$\lambda_n(\bar{x}, \bar{t}) \geq \limsup_{j\to\infty} \frac{2\,\langle x_j - y_j, \nu(x_j, t_j)\rangle}{|x_j - y_j|^2} \geq \limsup_{j\to\infty} \Phi(x_j, t_j) = \Phi(\bar{x}, \bar{t}),$$

which contradicts Lemma 4.

Therefore, we must have $|\bar{x}| < \sqrt{1 - 3n\bar{t}}$ and $\bar{x} \neq \bar{y}$. Moreover,

$$\frac{2\,\langle \bar{x} - \bar{y}, \nu(\bar{x}, \bar{t})\rangle}{|\bar{x} - \bar{y}|^2} = \limsup_{j\to\infty} \frac{2\,\langle x_j - y_j, \nu(x_j, t_j)\rangle}{|x_j - y_j|^2} \geq \limsup_{j\to\infty} \Phi(x_j, t_j) \geq \Phi(\bar{x}, \bar{t}).$$

We claim that $\bar{t} \in (-1, 0]$. Indeed, if $\bar{t} = -1$, then our assumption implies

$$\frac{2\,\langle \bar{x} - \bar{y}, \nu(\bar{x}, \bar{t})\rangle}{|\bar{x} - \bar{y}|^2} \leq \Lambda\, H(\bar{x}, \bar{t}) < \Phi(\bar{x}, \bar{t}),$$

which is impossible. Consequently, $\bar{t} \in (-1, 0]$.

To summarize, we have shown that $\bar{t} \in (-1, 0]$, $Z(\bar{x}, \bar{y}, \bar{t}) \leq 0$, and $Z(x, y, t) \geq 0$ for all $t \in [-1, \bar{t})$, all $x \in M_t \cap B_{\sqrt{1-3nt}}(0)$, and all $y \in M_t \cap B_{3R}(0)$. Arguing as in the proof of Proposition 2.3 in [2], we conclude that

$$\frac{\partial \Phi}{\partial t} - \Delta \Phi - |A|^2\, \Phi + 2 \sum_{i=1}^n \frac{(D_i \Phi)^2}{\Phi - \lambda_i} \leq 0$$

at the point $(\bar{x}, \bar{t})$. This contradicts Lemma 6. This completes the proof of Lemma 7.

We now complete the proof of Theorem 1. By definition, $\varphi(x,t) \geq (2\Lambda)^{-\frac{1}{4}} (2+6n)^{-1}$ and $\Phi(x,t) \leq 2\Lambda\,(2+6n)^4\, H(x,t)$ for all $t \in [-1,0]$ and all $x \in M_t \cap B_{\frac{1}{2}}(0)$. Using Lemma 7, we conclude that

$$\Lambda\,(2+6n)^4\, H(x,t)\,|x-y|^2 - \langle x-y, \nu(x,t)\rangle \geq Z(x,y,t) \geq 0$$

for all $t \in [-1,0]$, all $x \in M_t \cap B_{\frac{1}{2}}(0)$, and all $y \in M_t \cap B_{3R}(0)$. This completes the proof of Theorem 1.

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