A HIGHER DIMENSIONAL HILBERT IRREDUCIBILITY THEOREM

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ABSTRACT. Assuming the weak Bombieri-Lang conjecture, we prove that a generalization of Hilbert's irreducibility theorem holds for families of geometrically mordellic varieties (for instance, families of hyperbolic curves). As an application we prove that, assuming Bombieri-Lang, there are no polynomial bijections $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, completing a strategy originally suggested by T. Tao.

Serre reformulated Hilbert's irreducibility theorem as follows [Ser97, Chapter 9].

Theorem (Hilbert's irreducibility, Serre's form)**.** *Let X be a scheme of finite type over a field k finitely generated over $\mathbb{Q}$ and $U \subseteq \mathbb{P}^1$ a non-empty open subset. If $f : X \to U$ is a generically finite morphism with no generic sections $\operatorname{Spec} k(\mathbb{P}^1) \to X$, then $X(k) \to U(k)$ is not surjective.*

Recall that a variety X over a field k is *geometrically mordellic*, or *GeM*, if every subvariety of $X_{\bar{k}}$ is of general type. For instance, a subvariety of a product of curves of genus at least 2 is GeM. The geometric Lang conjecture predicts that every variety of general type contains an open subset which is GeM. Let us generalize GeM varieties by defining a scheme X as geometrically mordellic, or GeM, if it is of finite type over k and every subvariety of $X_{\bar{k}}$ is of general type.

Recall that the weak Bombieri-Lang conjecture states that, if X is a positive dimensional variety of general type over a field k finitely generated over $\mathbb{Q}$, then $X(k)$ is not dense in X. If X is a GeM scheme and the weak Bombieri-Lang conjecture holds, then $X(k)$ is finite, since its Zariski closure cannot have positive dimension.

Assuming Bombieri-Lang, we prove that Hilbert's irreducibility theorem generalizes to morphisms with GeM fibers.

Theorem A. *Let X be a scheme of finite type over a field k finitely generated over $\mathbb{Q}$ and $U \subseteq \mathbb{P}^1$ a non-empty open subset. Let $f : X \to U$ be a morphism with GeM fibers and no generic sections $\operatorname{Spec} k(\mathbb{P}^1) \to X$.*

Assume either that the weak Bombieri-Lang conjecture holds over k in every dimension, or that it holds up to dimension equal to $\dim X$ and that there exists an N such that $|X_v(k)| \leq N$ for every rational point $v \in U(k)$. Then $X(k) \to U(k)$ is not surjective.

Theorem A has an application to Grothendieck's section conjecture, see [Bre23]. Theorem A uses the weak Bombieri-Lang conjecture for a field k finitely generated over $\mathbb{Q}$: if we assume the geometric Lang conjecture, it is actually enough to know the conjecture over $\mathbb{Q}$, see [Bre22]. In the particular case in which X is an open subset of an abelian variety, P. Corvaja and U. Zannier have proved the statement

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of Theorem A unconditionally [CZ18, Theorem 3.47], and without the assumption that the generic fiber is GeM.

It is natural to ask whether it is possible to strengthen Theorem A and prove that $f(X(k)) \subset \mathbb{P}^1(k)$ is thin in the sense of Serre. The following is a crucial ingredient of our proof: if $S \subset \mathbb{P}^1(k)$ is thin, for a generic finite map $c : \mathbb{P}^1 \to \mathbb{P}^1$ the inverse image $c^{-1}(S)$ is thin as well, see Lemma 2.2. If one could show that non-thin sets satify a similar property, then it would be possible to modify our proof and obtain that $f(X(k))$ is thin. We do not know whether non-thin sets satisfy such a property.

The proof of Theorem A relies on two unconditional geometric results we prove, let us briefly discuss them. First, in Corollary 1.12 we prove that if $f : X \to \mathbb{P}^1$ is a morphism of smooth projective varieties whose generic fiber is of general type and X is not birational to a product $F \times \mathbb{P}^1$, then there exists an injective homomorphism $\mathcal{O}_{\mathbb{P}^1}(1) \to f_*\omega_f^m$ for some $m > 0$. Kollár and Viehweg proved this in the case in which two generic fibers are not birational one to another, we complete the analysis by treating the isotrivial case. Second, in Proposition 1.13 we prove that for $f : X \to \mathbb{P}^1$ as above X is of general type if and only if there exists an injective homomorphism $\mathcal{O}_{\mathbb{P}^1}(1) \to f_*\omega_X^m$ for some $m > 0$, or equivalently an injective homomorphism $\mathcal{O}_{\mathbb{P}^1}(2m+1) \to f_*\omega_f^m$. Half of the paper is devoted to proving these geometric results.

As an application of Theorem A we give a conditional answer to a long-standing Mathoverflow question [ZH11] which asks whether there exists a polynomial bijection $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$.

Theorem B. *Assume that the weak Bombieri-Lang conjecture for surfaces holds, and let k be a field finitely generated over $\mathbb{Q}$. There are no polynomial bijections $k \times k \to k$.*

We remark that B. Poonen has proved that, assuming the weak Bombieri-Lang conjecture for surfaces, there are polynomials giving *injective* maps $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, see [Poo10].

In 2019, T. Tao suggested on his blog [Tao19] a strategy to try to solve the problem of polynomial bijections $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ conditional on Bombieri-Lang, let us summarize it. Given a morphism $\mathbb{A}^2 \to \mathbb{A}^1$ and a cover $c : \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$, denote by P_c the pullback of $\mathbb{A}^2$. If P_c is of general type, by Bombieri-Lang $P_c(\mathbb{Q})$ is not dense in P_c and hence by Hilbert irreducibility a generic section $\mathbb{A}^1 \dashrightarrow P_c$ exists. If P_c is of general type for "many" covers c, one might expect this to force the existence a generic section $\mathbb{A}^1 \dashrightarrow \mathbb{A}^2$, which would be in contradiction with the injectivity of $\mathbb{A}^2(\mathbb{Q}) \to \mathbb{A}^1(\mathbb{Q})$.

The strategy had some gaps, though. There were no results showing that the pullback P_c is of general type for "many" covers c, and it was not clear how this would force a generic section of $\mathbb{A}^2 \to \mathbb{A}^1$. Tao started a so-called "polymath project" in order to crowdsource a formalization. The project was active for roughly one week in the comments section of the blog but did not reach a conclusion. Partial progress was made, we cite the two most important contributions. W. Sawin showed that $\mathbb{A}^2(\mathbb{Q}) \to \mathbb{A}^1(\mathbb{Q})$ cannot be bijective if the generic fiber has genus 0 or 1. H. Pasten showed that, for some morphisms $\mathbb{A}^2 \to \mathbb{A}^1$ with generic fiber of genus at least 2, the base change of $\mathbb{A}^2$ along the cover $z^2 - b : \mathbb{A}^1 \to \mathbb{A}^1$ is of general type for a generic b.

Theorem A is far more general than Theorem B, but it is possible to extract from the proof of the former the minimal arguments needed in order to prove the latter. These minimal arguments are a formalization of the ideas described above, hence as far as Theorem B is concerned we have essentially filled in the gaps in Tao's strategy.

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Conventions. A variety over k is a geometrically integral scheme of finite type over k. A smooth, projective variety is of general type if its Kodaira dimension is equal to its dimension: in particular, a point is a variety of general type.

We say that a variety is of general type if it is birational to a smooth, projective variety of general type. More generally, we define the Kodaira dimension of any variety X as the Kodaira dimension of any smooth projective variety birational to X.

Curves are assumed to be smooth, projective and geometrically connected. Given a variety X (resp. a scheme of finite type X) and C a curve, a morphism $X \to C$ is a family of varieties of general type (resp. of GeM schemes) if a generic fiber is a variety of general type (resp. a GeM scheme). Given a morphism $f : X \to C$, a generic section of f is a morphism $s : \operatorname{Spec} k(C) \to X$ (equivalently, a rational map $s : C \dashrightarrow X$) such that $f \circ s$ is the natural morphism $\operatorname{Spec} k(C) \to C$ (equivalently, the identity $C \dashrightarrow C$).

1. The Kodaira dimension of a family of varieties of general type

This section is of purely geometric nature, thus we may assume that k is algebraically closed of characteristic 0 for simplicity. The results then descend to non-algebraically closed fields with standard arguments.

Given a family $f : X \to \mathbb{P}^1$ of varieties of general type and $c : \mathbb{P}^1 \to \mathbb{P}^1$ a finite covering, let $f_c : X_c \to \mathbb{P}^1$ be the fiber product and, by abuse of notation, $c : X_c \to X$ the base change of c. The goal of this section is to obtain sufficient conditions on c such that X_c is of general type. This goal will be reached in Corollary 1.15, which contains all the geometry we need for arithmetic applications.

Let us say that $X \to \mathbb{P}^1$ is birationally trivial if there exists a birational morphism $X \dashrightarrow F \times \mathbb{P}^1$ which commutes with the projection to $\mathbb{P}^1$. If f is birationally trivial, then clearly our goal is unreachable, since X_c will have Kodaira dimension $-\infty$ no matter which cover $c : \mathbb{P}^1 \to \mathbb{P}^1$ we choose. We will show that this is in fact the only exception.

Assume that X is smooth and projective (we can always reduce to this case), then the relative dualizing sheaf ω_f exists [Kle80, Corollary 24]. First, we show that for *every* non-birationally trivial family there exists an integer m such that $f_*\omega_f^m$ has *some* positivity 1.12. Second, we show that if $f_*\omega_f^m$ has *enough* positivity, then X is of general type 1.13. We then pass from "some" to "enough" positivity by base changing along a cover $c : \mathbb{P}^1 \to \mathbb{P}^1$.

1.1. **Positivity of $f_*\omega_f^m$ for non-trivial families.** There are two cases: either there exists some finite cover $c : C \to \mathbb{P}^1$ such that $X_d \to C$ is birationally trivial, or not. Let us say that $f : X \to \mathbb{P}^1$ is *birationally isotrivial* in the first case, and non-birationally isotrivial in the second case.

The non-birationally isotrivial case has been extensively studied by Viehweg and Kollár, we do not need to do any additional work.

Proposition 1.1 (Kollár, Viehweg [Kol87, Theorem p.363])**.** *Let $f : X \to \mathbb{P}^1$ be a non-birationally isotrivial family of varieties of general type, with X smooth and projective. There exists an $m > 0$ such that, in the decomposition of $f_*\omega_f^m$ in a direct sum of line bundles, each factor has positive degree.* □

We are thus left with studying the positivity of $f_*\omega_f^m$ in the birationally isotrivial, non-birationally trivial case. We'll have to deal with various equivalent birational models of families, not always smooth, so let us first compare their relative pluricanonical sheaves.

1.1.1. *Morphisms of pluricanonical sheaves.* In this subsection, fix a base scheme S. If a morphism to S is given, it is tacitly assumed to be flat, locally projective, finitely presentable, with Cohen-Macauley equidimensional fibers of dimension n. For such a morphism $f : X \to S$, the relative dualizing sheaf ω_f exists and is coherent, see [Kle80, Theorem 21]. Recall that ω_f satisfies the functorial isomorphism

$$f_*\underline{\mathrm{Hom}}_X(F, \omega_f \otimes_X f^*N) \simeq \underline{\mathrm{Hom}}_S(R^n f_*F, N)$$

for every quasi-coherent sheaf F on X and every quasi-coherent sheaf N on S. Write $\omega_f^{\otimes m}$ for the m-th tensor power, we may drop the superscript $\_^{\otimes}$ and just write ω_f^m if ω_f is a line bundle.

Every flat, projective map $f : X \to S$ of smooth varieties over k satisfies the above, see [Kle80, Corollary 24], and in this case we can compute ω_f as $\omega_X \otimes f^*\omega_S^{-1}$, where ω_X and ω_S are the usual canonical bundles. Moreover, the relative dualizing sheaf behaves well under base change along morphisms $S' \to S$, see [Kle80, Proposition 9.iii].

Given a morphism $g : Y \to X$ over S and a quasi-coherent sheaf F over Y, then $R^n f_*(g_*F)$ is the $E_2^{n,0}$ term of the Grothendieck spectral sequence $(R^p f_* \circ R^q g_*)(F) \Rightarrow R^{p+q}(f \circ g)_*(F)$, thus there is a natural morphism $R^n f_*(g_*F) \to R^n(fg)_*F$. This induces a natural map

$$\mathrm{Hom}_Y(F, \omega_{fg}) = \mathrm{Hom}_S(R^n(fg)_*F, \mathcal{O}_S) \to \mathrm{Hom}_S(R^n f_*(g_*F), \mathcal{O}_S) = \mathrm{Hom}_X(g_*F, \omega_f).$$

Definition 1.2. If $g : Y \to X$ is a morphism over S, define $g_{\triangle,f} : g_*(\omega_{fg}) \to \omega_f$ as the sheaf homomorphism induced by the identity of ω_{fg} via the homomorphism

$$\mathrm{Hom}_Y(\omega_{fg}, \omega_{fg}) \to \mathrm{Hom}_X(g_*\omega_{fg}, \omega_f)$$

given above for $F = \omega_{fg}$. With an abuse of notation, call $g_{\triangle,f}$ the induced sheaf homomorphism $g_*(\omega_{fg}^{\otimes m}) \to \omega_f^{\otimes m}$ for every $m \geq 0$. If there is no risk of confusion, we may drop the subscript $\_f$ and just write $g_\triangle$.

The following facts are formal consequences of the definition of $g_\triangle$, we omit proofs.

Lemma 1.3. *Let $g : Y \to X$ be a morphism over S and $s : S' \to S$ any morphism, $f' : X' \to S'$, $g' : Y' \to X'$ the pullbacks to S'. By abuse of notation, call s the morphisms $Y' \to Y$, $X' \to X$, too. Then*

$$g'_\triangle = g_\triangle|_{X'} \in \mathrm{Hom}_{X'}(g'_*\omega_{f'g'}, \omega_{f'}) = \mathrm{Hom}_{X'}(s^*g_*\omega_{fg}, s^*\omega_f).$$ □

Lemma 1.4. *For every quasi-coherent sheaf F on Y, the natural map*

$$\mathrm{Hom}_Y(F, \omega_{fg}) \to \mathrm{Hom}_X(g_*F, \omega_f)$$

constructed above is given by

$$\varphi \mapsto g_\triangle \circ g_*\varphi : g_*F \to g_*\omega_{fg} \to \omega_f. \qquad \square$$

Corollary 1.5. *Let $h : Z \to Y$, $g : Y \to X$ be morphisms over S. Then, for every $m \geq 0$,*

$$g_\triangle \circ g_*h_\triangle = (gh)_\triangle : gh_*\omega_{fgh}^{\otimes m} \to g_*\omega_{fg}^{\otimes m} \to \omega_f^{\otimes m}. \qquad \square$$

Corollary 1.6. *Let $g : Y \to X$ be a morphism over S. Suppose that a group H acts on Y, X, S and g, f are H-equivariant. Then $g_*\omega_{fg}^{\otimes m}, \omega_f^{\otimes m}$ are H-equivariant sheaves and $g_\triangle : g_*\omega_{fg}^{\otimes m} \to \omega_f^{\otimes m}$ is H-equivariant.* $\square$

Lemma 1.7. *Let $g : Y \to X$ be a morphism over S. Assume that Y, X are smooth varieties over a field k, and that g is birational. Then $g_\triangle$ is an isomorphism.*

Proof. We have $\omega_f = \omega_X \otimes f^*\omega_S^{-1}$ and $\omega_{fg} = \omega_Y \otimes (fg)^*\omega_S^{-1}$. Moreover, $\omega_Y = g^*\omega_X \otimes \mathcal{O}_Y(R)$ where R is some effective divisor whose irreducible components are contracted by g, hence $\omega_{fg} = g^*\omega_f \otimes \mathcal{O}_Y(R)$. Since $g_*\mathcal{O}_Y(mR) \simeq \mathcal{O}_X$, we have a natural isomorphism $g_*(\omega_{fg}^m) \simeq \omega_f^m$ by projection formula. This can be checked to correspond to $g_\triangle$, which is then an isomorphism. $\square$

1.1.2. *Birationally isotrivial families.* Let C be a smooth projective curve and $f : X \to C$ a birationally isotrivial family of varieties of general type, and let F/k be a smooth projective variety such that the generic fiber of f is birational to F.

Since the family is birationally isotrivial, there exists a ramified Galois cover $\varphi : C' \to C$ with Galois group Γ and a rational map $\psi : F \times C' \dashrightarrow X$ such that the induced map $F \times C' \dashrightarrow X_{C'}$ is birational. Denote by H the finite group of birational automorphisms of F.

Lemma 1.8. *For every $g \in \Gamma$ and generic $p, q \in C'$, we have*

$$\psi_p^{-1} \circ \psi_{g(p)} = \psi_q^{-1} \circ \psi_{g(q)} \in H.$$

Proof. Denote by ψ_g the composition $\psi \circ (\mathrm{id}_F, g) : F \times C' \to F \times C' \dashrightarrow X$, it satisfies

$$\psi_{g,p} = \psi_{g(p)} : F \dashrightarrow X_{\varphi(p)}.$$

Consider the composition

$$\pi \circ \psi_{g,C'}^{-1} \circ \psi_{C'} : F \times C' \dashrightarrow X_{C'} \dashrightarrow F \times C' \to F,$$

it is sufficient to prove that this rational map is the composition of the projection $F \times C' \to F$ and a birational automorphism of F. Let us show that this is true for any rational map $\tau : F \times C' \dashrightarrow F$ which is birational on the generic fibers.

Up to a birational transformation, we may assume that H acts with isomorphisms on F. For a generic point $x \in F$, all the points $h(x)$ for $h \in H$ are different, and the composition $C' \xrightarrow{(x,\mathrm{id})} F \times C' \dashrightarrow F$ is well defined. By construction, the image of this composition is contained in the set $\{h(x)\}_{h \in H}$, because for a generic $c \in C'$ the restriction τ_c is an element of h. Since the set $\{h(x)\}_{h \in H}$ is finite, then the composition $C' \xrightarrow{(x,\mathrm{id})} F \times C' \dashrightarrow F$ is constantly equal to $h_0(x)$ for some $h_0 \in H$. Since $h(x) \neq h'(x)$ for $h \neq h'$, it follows that the restriction of τ to a generic fiber is equal to h_0, and hence the rational map $F \times C' \dashrightarrow F$ is the composition of the projection to F with h_0. $\square$

Corollary 1.9. *The map $\Gamma \to H$ defined by $g \mapsto \psi_p^{-1} \circ \psi_{g^{-1}(p)}$ for generic p is a homomorphism.*

Proof. First, the map is well-defined thanks to Lemma 1.8. To check that it is a homomorphism, it is sufficient to prove that

$$\psi_c^{-1} \circ \psi_{g'^{-1}(c)} = \psi_{g^{-1}(c)}^{-1} \circ \psi_{g'^{-1}(g^{-1}(c))},$$

and this follows again from Lemma 1.8. $\square$

Let $\Gamma' \subset \Gamma$ be the kernel of the homomorphism $\Gamma \to H$; notice that it coincides with the subgroup of elements $g \in \Gamma$ such that $\psi \circ (\mathrm{id}_F, g) = \psi$. Write $B_f = C'/\Gamma'$ and $G_f = \Gamma/\Gamma'$; by descent, we get a rational map $F \times B_f \dashrightarrow X$ such that $F \times B_f \dashrightarrow X_{B_f}$ is birational, and an injective homomorphism $G_f \hookrightarrow H$. Denote by f_b the base change $X_{B_f} \to B_f$ of f.

Since the homomorphism $G_f \to H$ has the form given in the statement of Corollary 1.9 (we may repeat everything after replacing C' with B_f), it is straightforward to check that the rational map $F \times B_f \dashrightarrow X$ is G_f-invariant with respect to the diagonal action of G_f on $F \times B_f$; it follows that X is birational to $(F \times B_f)/G_f$.

Definition 1.10. We call $B_f \to C$ and G_f the *monodromy cover* and the *monodromy group* of f respectively.

It is possible to characterize B_f (and thus G_f) by the following universal property, so that B_f does not depend on the choice of C': if D is a smooth projective curve with a finite morphism $D \to C$, then $X_D \to D$ is birationally trivial if and only if there exists a factorization $D \to B_f \to C$. However, we do not need this characterization nor the unicity of the monodromy cover, so we do not prove it.

Proposition 1.11. *Let $f : X \to C$ be a birationally isotrivial family of varieties of general type, with X smooth and projective. If $p \in B_f$ is a ramification point of the monodromy cover $b : B_f \to C$, then for some m there exists an injective sheaf homomorphism $\mathcal{O}_{B_f}(p) \to f_{b*}\omega_{f_b}^m$.*

Proof. The statement is equivalent to the existence of a non-trivial section of $\omega_{f_b}^m$ which vanishes on the fiber $X_{b,p}$. Let F be as above, G_f acts faithfully with birational maps on F. By equivariant resolution of singularities, we may assume that G_f acts faithfully by isomorphisms on F. We have that X is birational to $(F \times B_f)/G_f$ where G_f acts diagonally.

By resolution of singularities, let X' be a smooth projective variety with birational morphisms $X' \to X$, $X' \to (F \times B_f)/G_f$: thanks to Lemma 1.7 we may replace X with X' and assume we have a birational morphism $X \to (F \times B_f)/G_f$. By equivariant resolution of singularities again, we may find a smooth projective variety Y with an action of G_f, a birational morphism $g : Y \to X_b$ and a birational, G_f-equivariant morphism $y : Y \to F \times B_f$. Call $\pi : F \times B_f \to B_f$ the projection.

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "y"'] \arrow[ddr, "\pi y"] & X_b \arrow[r, "b"] \arrow[dd, "f_b"] & X \arrow[d] \\
F \times B_f \arrow[rr, dashed] \arrow[dr, "\pi"'] & & (F \times B_f)/G_f \arrow[d] \\
& B_f \arrow[r, "b"] & C
\end{tikzcd}$$

Recall that we are trying to find a global section of $\omega_{f_b}^m$ that vanishes on $X_{b,p}$, where p is a ramification point of b. Thanks to Lemma 1.7, we have that $\pi y_* \omega_{\pi y}^m \simeq \pi_* \omega_\pi^m \simeq \mathcal{O}_{B_f} \otimes \mathrm{H}^0(F, \omega_F^m)$, thus $\mathrm{H}^0(Y, \omega_{\pi y}^m) = \mathrm{H}^0(F, \omega_F^m) = \mathrm{H}^0(Y_p, \omega_{Y_p}^m)$.

The sheaf homomorphism $g_\triangle = g_{\triangle, f_b} : g_* \omega_{\pi y}^m \to \omega_{f_b}^m$ induces a linear map

$$g_\triangle(p) : \mathrm{H}^0(Y_p, \omega_{Y_p}^m) = \mathrm{H}^0(Y, \omega_{\pi y}^m) \xrightarrow{g_\triangle} \mathrm{H}^0(X_b, \omega_{f_b}^m) \xrightarrow{\bullet|_p} \mathrm{H}^0(X_{b,p}, \omega_{X_{b,p}}^m)$$

where the last map is the restriction to the fiber. Let $V \subseteq B_f$ be the étale locus of $b : B_f \to C$. Since $X_b|_V$ is smooth, then $g_\triangle$ restricts to an isomorphism on $X_b|_V$ thanks to Lemma 1.7 and thus the map $\mathrm{H}^0(Y, \omega_{\pi y}^m) \to \mathrm{H}^0(X_b, \omega_{f_b}^m)$ is injective.

We want to show that the restriction map $\mathrm{H}^0(X_b, \omega_{f_b}^m) \to \mathrm{H}^0(X_{b,p}, \omega_{X_{b,p}}^m)$ is not injective for some m, it is enough to show that $g_\triangle(p)$ is not injective. Thanks to Lemma 1.3, we have that $g_\triangle(p) = g_{p,\triangle} : \mathrm{H}^0(Y_p, \omega_{Y_p}^m) \to \mathrm{H}^0(X_{b,p}, \omega_{X_{b,p}}^m)$.

Recall now that G_f acts on Y. Let $G_{f,p}$ be the stabilizer of $p \in B_f$, it is a non-trivial group since p is a ramification point. Thanks to Corollary 1.6, the stabilizer $G_{f,p}$ acts naturally on $\mathrm{H}^0(Y_p, \omega_{Y_p}^m)$, $\mathrm{H}^0(F, \omega_F^m)$, $\mathrm{H}^0(X_{b,p}, \omega_{X_{b,p}}^m)$, and the maps $y_{p,\triangle} : \mathrm{H}^0(Y_p, \omega_{Y_p}^m) \simeq \mathrm{H}^0(F, \omega_F^m)$, $g_{p,\triangle} : \mathrm{H}^0(Y_p, \omega_{Y_p}^m) \to \mathrm{H}^0(X_{b,p}, \omega_{X_{b,p}}^m)$ are $G_{f,p}$-equivariant. Moreover, the action on $\mathrm{H}^0(X_{b,p}, \omega_{X_{b,p}}^m)$ is trivial since the action on $X_{b,p}$ is trivial. It follows that $g_\triangle(p)$ is $G_{f,p}$-invariant, and hence to show that it is not injective for some m it is enough to show that the action of $G_{f,p}$ on $\mathrm{H}^0(F, \omega_F^m)) = \mathrm{H}^0(Y_p, \omega_{Y_p}^m)$ is not trivial for some m.

Since F is of general type, $F \dashrightarrow \mathbb{P}(\mathrm{H}^0(F, \omega_F^m))$ is generically injective for some m, fix such an integer m. Since the action of $G_{f,p}$ on F is faithful, for every non-trivial $g \in G_{f,p}$ there exists a section $s \in \mathrm{H}^0(F, \omega_F^m)$ and a point $v \in F$ such that $s(v) = 0$ and $s(g(v)) \neq 0$, in particular the action of $G_{f,p}$ on $\mathrm{H}^0(F, \omega_F^m)$ is not trivial and we conclude. □

Corollary 1.12. *Let $f : X \to \mathbb{P}^1$ be a non-birationally trivial family of varieties of general type, with X smooth and projective. Then there exists an $m > 0$ with an injective homomorphism $\mathcal{O}_{\mathbb{P}^1}(1) \to f_* \omega_f^m$.*

Proof. If f is not birationally isotrivial, apply Proposition 1.1. Otherwise, f is birationally isotrivial and not birationally trivial, thus the monodromy cover $b : B_f \to \mathbb{P}^1$ is not trivial. Since $\mathbb{P}^1$ has no non-trivial étale covers, we have that $B_f \to \mathbb{P}^1$ has at least one ramification point p. Let m be the integer given by Proposition 1.11, and write $f_* \omega_f^m = \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(d_i)$. Since $\mathcal{O}_{B_f}(p) \subseteq f_{b*} \omega_{f_b}^m$ and $\omega_{f_b} = b^* \omega_f$, see [Kle80, Proposition 9.iii], there exists an i with $d_i > 0$. □

1.2. **Pulling families to maximal Kodaira dimension.** Now that we have established a positivity result for $f_* \omega_f^m$ of any non-birationally trivial family f :

$X \to \mathbb{P}^1$, let us use this to pull families to maximal Kodaira dimension. First, we characterize which families have maximal Kodaira dimension in terms of $f_*\omega_f^m$.

Proposition 1.13. *Let $f : X \to \mathbb{P}^1$ be a family of varieties of general type, with X smooth and projective. Then X is of general type if and only if there exists an injective homomorphism $\mathcal{O}_{\mathbb{P}^1}(1) \to f_*\omega_X^{m_0}$, or equivalently $\mathcal{O}_{\mathbb{P}^1}(2m_0+1) \to f_*\omega_f^{m_0}$, for some $m_0 > 0$.*

Proof. By resolution of singularities, there exists a birational morphism $g : X' \to X$ with X' smooth and projective such that the generic fiber of $X' \to \mathbb{P}^1$ is smooth and projective. We have $\omega_{X'} = g^*\omega_X \otimes \mathcal{O}_{X'}(R)$ where R is some effective divisor whose irreducible components are contracted by g, hence $g_*\omega_{X'}^m = \omega_X^m \otimes g_*O(mR) = \omega_X^m$ for every $m \geq 0$. We may thus replace X with X' and assume that the generic fiber is smooth. This guarantees that $\operatorname{rank} f_*\omega_X^m = \operatorname{rank} f_*\omega_f^m$ has growth $O(m^{\dim X-1})$.

If there are no injective homomorphisms $\mathcal{O}_{\mathbb{P}^1}(1) \to f_*\omega_X^m$ for every $m > 0$, then $\mathrm{h}^0(\omega_X^m) \leq \operatorname{rank} f_*\omega_X^m = \operatorname{rank} f_*\omega_f^m$, and this has growth $O(m^{\dim X-1})$.

On the other hand, let $\mathcal{O}_{\mathbb{P}^1}(1) \to f_*\omega_X^{m_0}$ be an injective homomorphism for some $m_0 > 0$. In particular, X has Kodaira dimension ≥ 0.

For some m, the closure Y of the image of $X \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, \omega_X^{mm_0}))$ has dimension equal to the Kodaira dimension of X and $k(Y)$ is algebraically closed in $k(X)$, see [Iit71, §3]. If X' is a smooth projective variety birational to X, then there is a natural isomorphism $\mathrm{H}^0(X, \omega_X^{mm_0}) = \mathrm{H}^0(X', \omega_{X'}^{mm_0})$, see [Har77, Ch. 2, Theorem 8.19]. Thus, up to replacing X with some other smooth, projective variety birational to X, we may assume that $X \dashrightarrow Y \subseteq \mathbb{P}(\mathrm{H}^0(X, \omega_X^{mm_0}))$ is defined everywhere and has smooth, projective generic fiber Z by resolution of singularities. Iitaka has then shown that Z has Kodaira dimension 0, see [Iit71, Theorem 5]. This is easy to see in the case in which $\omega_X^{mm_0}$ is base point free, since then $\omega_X^{mm_0}$ is the pullback of $\mathcal{O}(1)$ and thus $\omega_Z^{mm_0} = \omega_X^{mm_0}|_Z$ is trivial.

Let us recall briefly Grothendieck's convention that, if V is a vector bundle, then $\mathbb{P}(V)$ is the set (or scheme) of linear quotients $V \to k$ up to a scalar. A non-trivial linear map $W \to V$ thus induces a rational map $\mathbb{P}(V) \dashrightarrow \mathbb{P}(W)$ by restriction. If L is a line bundle with non-trivial global sections, the rational map $X \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, L))$ is defined by sending a point $x \in X$ outside the base locus to the quotient $\mathrm{H}^0(X, L) \to L_x \simeq k$. If L embeds in another line bundle M, then there is a natural factorization $X \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, M)) \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, L))$, and any point of X outside the support of M/L and outside the base locus of L maps to the locus of definition of $\mathbb{P}(\mathrm{H}^0(X, M)) \dashrightarrow \mathbb{P}(\mathrm{H}^0(X, L))$.

Let $F \subseteq X$ be the fiber over any rational point of $\mathbb{P}^1$. The injective homomorphism $\mathcal{O}_{\mathbb{P}^1}(1) \to f_*\omega_X^{m_0}$ induces an injective homomorphism $\mathcal{O}_{\mathbb{P}^1}(m) \to f_*\omega_X^{mm_0}$, choose any embedding $\mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(m)$, these induce an injective homomorphism $\mathcal{O}_X(F) \to \omega_X^{mm_0}$. Since $\mathcal{O}_X(F)$ induces the morphism $f : X \to \mathbb{P}^1$, the composition

$$X \to Y \subseteq \mathbb{P}(\mathrm{H}^0(X, \omega_X^{mm_0})) \dashrightarrow \mathbb{P}^1$$

coincides with f. Observe that the right arrow depends on the choice of the embedding $\mathcal{O}_X(F) \to \omega_X^{mm_0}$, but the composition does not.

Let ξ be the generic point of $\mathbb{P}^1$, $U \subseteq Y$ an open subset such that $U \to \mathbb{P}^1$ is defined, Y_ξ the closure of U_ξ in Y. Then the generic fiber Z of $X \to Y$ is the generic fiber of $X_\xi \to Y_\xi$, too. By hypothesis, X_ξ is of general type, thus by adjunction $\omega_{X_\xi}|_Z = \omega_Z$ is big and hence Z is of general type.

Since Z is a variety of general type of Kodaira dimension 0 over $\operatorname{Spec} k(Y)$, then $Z = \operatorname{Spec} k(Y)$, the morphism $X \to Y$ is generically injective and thus X is of general type. □

Remark 1.14. We do not need the precision of Proposition 1.13: for our purposes it is enough to show that, if $f_*\omega_X^{m_0}$ has a positive *enough* sub-line bundle for some m_0, then X is of general type. This weaker fact has a more direct proof, let us sketch it.

First, let us mention an elementary fact about injective sheaf homomorphisms. Let P, Q be vector bundles on $\mathbb{P}^1$ and M, N vector bundles on X, with P of rank 1. Suppose we are given injective homomorphisms $m \in \operatorname{Hom}(P, f_*M)$, $n \in \operatorname{Hom}(Q, f_*N)$. Then $m^a \otimes n \in \operatorname{Hom}(P^{\otimes a} \otimes Q, f_*(M^{\otimes a} \otimes N))$ is injective for every $a > 0$: this can be checked on the generic point of $\mathbb{P}^1$ and thus on the generic fiber $X_{k(\mathbb{P}^1)}$, where the fact that P has rank 1 allows us to reduce to the fact that the tensor product of non-zero sections of vector bundles is non-zero on an integral scheme.

Assume we have an injective homomorphism $\mathcal{O}_{\mathbb{P}^1}(3m_0) \to f_*\omega_X^{m_0}$, or equivalently $\mathcal{O}_{\mathbb{P}^1}(5m_0) \to f_*\omega_f^{m_0}$, we want to prove that X is of general type. Let $r(m)$ be the rank $f_*\omega_f^{mm_0}$ for every m. Since the generic fiber is of general type, up to replacing m_0 by a multiple m_0' we may assume that the growth of $r(m)$ is $O(m^{\dim X - 1})$. The induced morphism $\mathcal{O}_{\mathbb{P}^1}(5m_0') \to f_*\omega_f^{m_0'}$ is injective thanks to the above.

Thanks to [Vie83, Theorem III], every line bundle in the factorization of $f_*\omega_f^{mm_0}$ has non-negative degree, we may thus choose an injective homomorphism $\mathcal{O}_{\mathbb{P}^1}^{r(m)} \to f_*\omega_f^{mm_0}$. Taking the tensor product with the m-th power of the homomorphism given by hypothesis, we get an homomorphism $\mathcal{O}_{\mathbb{P}^1}(5mm_0)^{r(m)} \to f_*\omega_f^{2mm_0}$ which is injective thanks to the above.

Since $f_*\omega_X^{2mm_0} = f_*\omega_f^{2mm_0} \otimes \mathcal{O}_{\mathbb{P}^1}(-4mm_0)$, we thus have an injective homomorphism

$$\mathcal{O}_{\mathbb{P}^1}(mm_0)^{r(m)} \to f_*\omega_X^{2mm_0}.$$

In particular, we have $\mathrm{h}^0(\omega_X^{2mm_0}) \geq (mm_0+1)r(m)$ which has growth $O(m^n)$, hence X is of general type.

Corollary 1.15. *Let $f : X \to \mathbb{P}^1$ be a non-birationally trivial family of varieties of general type. Then there exists an integer d_0 and a non-empty open subset $U \subseteq \mathbb{P}^1$ such that, for every finite cover $c : \mathbb{P}^1 \to \mathbb{P}^1$ with $\deg c \geq d_0$ and such that the branch points of c are contained in U, we have that X_c is of general type. If X is smooth and projective, U can be chosen as the largest open subset such that $f|_{X_U}$ is smooth.*

Proof. By resolution of singularities, we may assume that X is smooth and projective. By generic smoothness, there exists an open subset $U \subseteq \mathbb{P}^1$ be such that $f|_{X_U}$ is smooth. We have that X_c is smooth for every $c : \mathbb{P}^1 \to \mathbb{P}^1$ whose branch points are contained in U since each point of X_c is smooth either over X or over $\mathbb{P}^1$.

Let m_0 be the integer given by Corollary 1.12, we have an injective homomorphism $\mathcal{O}(1) \to f_*\omega_f^{m_0}$. Set $d_0 = 2m_0 + 1$, for every finite cover c of degree $\deg c \geq d_0 = 2m_0 + 1$ we have an induced homomorphism $\mathcal{O}(2m_0 + 1) \to f_{c*}\omega_{f_c}^{m_0}$ and thus $\mathcal{O}(1) \to f_{c*}\omega_{X_c}^{m_0}$. It follows that X_c is of general type thanks to Proposition 1.13. □

2. Higher dimensional HIT

Recall that Serre [Ser97, Chapter 9] defined a subset S of $\mathbb{P}^1(k)$ as *thin* if there exists a morphism $f : X \to \mathbb{P}^1$ with X of finite type over k, finite generic fiber and no generic sections $\operatorname{Spec} k(\mathbb{P}^1) \to X$ such that $S \subseteq f(X(k))$. It is immediate to check that a subset of a thin set is thin, and a finite union of thin sets is thin. Serre's form of Hilbert's irreducibility theorem says that, if k is finitely generated over $\mathbb{Q}$, then $\mathbb{P}^1(k)$ is not thin.

Definition 2.1. A subset $S \subseteq \mathbb{P}^1(k)$ is *fat* if the complement $\mathbb{P}^1(k) \setminus S$ is thin.

Given a subset $S \subseteq \mathbb{P}^1(k)$, a finite set of finite morphisms $D = \{d_i : D_i \to \mathbb{P}^1\}_i$ each of degree > 1 with D_i smooth, projective and geometrically connected is a *scale* for S if $S \cup \bigcup_i d_i(D_i(k)) = \mathbb{P}^1(k)$. The set of branch points of the scale D is the union of the sets of branch points of d_i.

Using the fact that a connected scheme with a rational point is geometrically connected [Sta24, Lemma 04KV], it is immediate to check that a subset of $\mathbb{P}^1$ is fat if and only if it has a scale. The set of branch points of a scale gives valuable information about a fat set.

Lemma 2.2. *Let $S \subseteq \mathbb{P}^1$ be a fat set, and let $D = \{d_i : D_i \to \mathbb{P}^1\}_i$ be a scale for S. Let $c : \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism such that the sets of branch points of c and D are disjoint. Then $c^{-1}(S)$ is fat.*

Proof. Let $d'_i : D'_i \to \mathbb{P}^1$ be the base change of d_i along $c : \mathbb{P}^1 \to \mathbb{P}^1$. By construction, $c^{-1}(S) \cup \bigcup_i d'_i(D'_i(k)) = \mathbb{P}^1(k)$. Since the sets of branch points of c and d_i are disjoint, we have that D'_i is geometrically connected, see for instance [Str21, Lemma 2.8]. Moreover, D'_i is smooth since each point of D'_i is étale either over $\mathbb{P}^1$ or D_i. It follows that d'_i has degree > 1 and $\{d'_i : D'_i \to \mathbb{P}^1\}_i$ is a scale for $c^{-1}(S)$, which is thus fat. □

We will spend the rest of the section proving Theorem A.

2.1. **Decreasing the fiber dimension.** Assuming Bombieri-Lang and using Hilbert's irreducibility, it is easy to check that Theorem A is equivalent to the following statement.

If the fibers of $f : X \to \mathbb{P}^1$ are GeM and $f(X(k))$ is fat, there exists a section $\operatorname{Spec} k(\mathbb{P}^1) \to X$.

We prove this statement by induction on the dimension of the generic fiber. If the generic fiber has dimension 0, this follows from the definition of fat set. Let us prove the inductive step.

We define recursively a sequence of closed subschemes $X_{i+1} \subseteq X_i$ with $X_0 = X$ and such that $f(X_i(k)) \subseteq \mathbb{P}^1_k$ is fat.

- Define X'_i as the closure of $X_i(k)$ with the reduced scheme structure, $f(X'_i(k)) = f(X_i(k)) \subseteq \mathbb{P}^1_k$ is fat.
- Define X''_i as the union of the irreducible components of X'_i which dominate $\mathbb{P}^1$, $f(X''_i(k)) \subseteq \mathbb{P}^1_k$ is fat since $f(X'_i(k)) \setminus f(X''_i(k))$ is finite.
- Write $X''_i = \bigcup_j Y_{i,j}$ as union of irreducible components, $Y_{i,j} \to \mathbb{P}^1$ is dominant for every j. Let $C_{i,j} \to \mathbb{P}^1$ be the smooth projective curve corresponding to the algebraic closure of $k(\mathbb{P}^1)$ in $k(Y_{i,j})$, we have a factorization

$Y_{i,j} \dashrightarrow C_{i,j} \to \mathbb{P}^1$ such that $Y_{i,j} \dashrightarrow C_{i,j}$ has geometrically irreducible generic fiber. If $C_{i,j} \to \mathbb{P}^1$ is an isomorphism, define $Z_{i,j} = Y_{i,j}$. Otherwise, there exists a non-empty open subset $V_{i,j} \subseteq Y_{i,j}$ such that $Y_{i,j} \dashrightarrow C_{i,j}$ is defined on $V_{i,j}$. In particular, $f(V_{i,j}(k)) \subseteq \mathbb{P}^1(k)$ is thin. Define $Z_{i,j} = Y_{i,j} \backslash V_{i,j}$ and $X_{i+1} = \bigcup_j Z_{i,j} \subseteq X_i$. By construction, $f(X_{i+1}(k)) \subseteq \mathbb{P}^1(k)$ is fat since $f(X_i''(k)) \setminus f(X_{i+1}(k))$ is thin.

Since X is Noetherian, the sequence is eventually stable, let r be such that $X_{r+1} = X_r$. Since $X_{r+1} = X_r$, then $X_r(k)$ is dense in X_r, thus every irreducible component is geometrically irreducible, see [Sta24, Lemma 0G69]. Moreover, every irreducible component of X_r dominates $\mathbb{P}^1$ with geometrically irreducible generic fiber. Replace X with X_r and write $X = \bigcup_j Y_j$ as union of irreducible components, we may assume that $Y_j \to \mathbb{P}^1$ is a family of GeM varieties for every j and $Y_j(k)$ is dense in Y_j.

If $Y_j \to \mathbb{P}^1$ is birationally trivial for some j, since $Y_j(k)$ is dense in Y_j and a generic fiber of $Y_j \to \mathbb{P}^1$ has a finite number of rational points, then $\dim Y_j = 1$, $Y_j \to \mathbb{P}^1$ is birational and we conclude. Otherwise, thanks to Corollary 1.15, there exists an integer d_0 and a non-empty open subset $U \subseteq \mathbb{P}^1$ such that, for every finite cover $c : \mathbb{P}^1 \to \mathbb{P}^1$ with $\deg c \geq d_0$ such that the branch points of c are contained in U, we have that $Y_{j,c}$ is of general type for every j.

Let $D = \{d_l : D_l \to \mathbb{P}^1\}$ be a scale for $f(X(k))$. Up to shrinking U furthermore, we may assume that the set of branch points of D is disjoint from U. Since we are assuming that the weak Bombieri-Lang conjecture holds up to dimension $\dim X$, the dimension of $\overline{Y_{j,c}(k)} \subseteq Y_{j,c}$ is strictly smaller than $\dim Y_j$ for every j. Moreover, we have that $f_c(X_c(k)) = m_c^{-1}(f(X(k)))$ is fat thanks to Lemma 2.2. It follows that, by induction hypothesis, there exists a generic section $\operatorname{Spec} k(\mathbb{P}^1) \to X_c$ for *every* finite cover c as above. There are a lot of such covers: let us show that we can choose them so that the resulting sections "glue" to a generic section $\operatorname{Spec} k(\mathbb{P}^1) \to X$.

2.2. **Gluing sections.** Choose coordinates on $\mathbb{P}^1$ so that $0, \infty \in U$, let p be any prime number greater than d_0. For any positive integer n, let $m_n : \mathbb{P}^1 \to \mathbb{P}^1$ be the n-th power map. We have shown above that there exists a rational section $\mathbb{P}^1 \dashrightarrow X_{m_p}$ for every prime $p \geq d_0$, call $s_p : \mathbb{P}^1 \dashrightarrow X_{m_p} \to X$ the composition.

We either assume that there exists an integer N such that, for every rational point $v \in \mathbb{P}^1(k)$, we have $|X_v(k)| \leq N$ or that the Bombieri-Lang conjecture holds in every dimension. In the second case, the uniform bound N exists thanks to a theorem of Caporaso-Harris-Mazur and Abramovich-Voloch [CHM97, Theorem 1.1] [AV96, Theorem 1.5] [Abr97]. Choose $N+1$ prime numbers $p_0, \dots, p_N$ greater than d_0, for each one we have a rational map

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \mathbb{P}^1 \arrow[ur, dashed, "s_p"] \arrow[r, "m_p"'] & \mathbb{P}^1 \end{tikzcd}$$

Let $Q = \prod_{i=0}^{N} p_i$, for every $i = 0, \dots, N$, we get a rational section S_{p_i} by composition with s_{p_i}:

$$\begin{tikzcd}
& & X \arrow[d, "f"] \\
\mathbb{P}^1 \arrow[rru, dashed, "S_{p_i}"] \arrow[r, "m_{Q/p_i}"'] \arrow[rr, bend right, "m_Q"'] & \mathbb{P}^1 \arrow[ru, dashed, "s_{p_i}"'] \arrow[r, "m_{p_i}"'] & \mathbb{P}^1
\end{tikzcd}$$

Let $V \subseteq \mathbb{P}^1$ be an open subset such that S_{p_i} is defined on V for every i. For every rational point $v \in V(k)$, we have $|X_v(k)| \leq N$ and thus there exists a couple of different indexes $i \neq j$ such that $S_{p_i}(v) = S_{p_j}(v)$ for infinitely many $v \in V(k)$, hence $S_{p_i} = S_{p_j}$. Let $Z \subseteq X$ be the image $S_{p_i} = S_{p_j}$, by construction we have

$$k(\mathbb{P}^1) = k(t) \subseteq k(Z) \subseteq k(t^{-p_i}) \cap k(t^{-p_j}) \subseteq k(t^{-Q}).$$

Using Galois theory on the cyclic extension $k(t^{-Q})/k(t)$, it is immediate to check that $k(t^{-p_i}) \cap k(t^{-p_j}) = k(t) \subseteq k(t^{-Q})$ since p_i, p_j are coprime, thus $k(Z) = k(t)$ and $Z \to \mathbb{P}^1$ is birational. This concludes the proof of Theorem A.

2.3. **Non-rational base.** For future reference, we give a version of Theorem A over non-rational curves. This follows directly from Theorem A using Weil's restriction of scalars.

Theorem 2.3. *Assume that the weak Bombieri-Lang conjecture holds in every dimension. Let k be finitely generated over $\mathbb{Q}$, and let $f : X \to C$ be a morphism with X any scheme of finite type over k and C a geometrically connected curve. Assume that the fibers are GeM and that there are no generic sections* $\operatorname{Spec} k(C) \to X$. *There exists a finite extension h/k such that $X(h) \to C(h)$ is not surjective.*

Proof. Assume that $X(h) \to C(h)$ is surjective for every finite extension h/k, we want to prove that there exists a generic section $C \dashrightarrow X$. It's easy to reduce to the case in which C is smooth and projective, so let us make this assumption.

Observe that, up to replacing X with an affine covering, we may assume that X is affine. Choose $C \to \mathbb{P}^1$ any finite map: since X is affine, the Weil restriction $R_{C/\mathbb{P}^1}(X) \to \mathbb{P}^1$ exists [BLR90, §7.6, Theorem 4]. Recall that $R_{C/\mathbb{P}^1}(X) \to \mathbb{P}^1$ represents the functor on $\mathbb{P}^1$-schemes $S \mapsto \operatorname{Hom}_C(S \times_{\mathbb{P}^1} C, X)$.

If $L/k(C)/k(\mathbb{P}^1)$ is a Galois closure and Σ is the set of embeddings $\sigma : k(C) \to L$ as $k(\mathbb{P}^1)$ extensions, the scheme $R_{C/\mathbb{P}^1}(X)_L$ is isomorphic to the product $\prod_\Sigma X \times_{\operatorname{Spec} k(C), \sigma} \operatorname{Spec} L$ and hence is a GeM scheme, see [Wei82, Theorem 1.3.2]. It follows that the generic fiber $R_{C/\mathbb{P}^1}(X)_{k(\mathbb{P}^1)}$ is a GeM scheme, too.

Let $U \subseteq \mathbb{P}^1$ be the image of $V \subseteq C$. The fact that $X|_V(h) \to V(h)$ is surjective for every finite extension h/k implies that $R_{C/\mathbb{P}^1}(X)|_U(k) \to U(k)$ is surjective. By Theorem A, we get a generic section $\mathbb{P}^1 \dashrightarrow R_{C/\mathbb{P}^1}(X)$, which in turn induces generic section $C \dashrightarrow X$ by the universal property of $R_{C/\mathbb{P}^1}(X)$. $\square$

3. Polynomial bijections $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$

Finally, let us prove Theorem B using Theorem A.

Let $f : \mathbb{A}^2 \to \mathbb{A}^1$ be any morphism. Assume by contradiction that f is bijective on rational points.

First, let us show that the generic fiber of f is geometrically irreducible. This is equivalent to saying that $\operatorname{Spec} k(\mathbb{A}^2)$ is geometrically connected over $\operatorname{Spec} k(\mathbb{A}^1)$, or that $k(\mathbb{A}^1)$ is algebraically closed in $k(\mathbb{A}^2)$. Let $k(\mathbb{A}^1) \subseteq L \subseteq k(\mathbb{A}^2)$ a subextension algebraic over $k(\mathbb{A}^1)$. Let $C \to \mathbb{A}^1$ be a finite cover with C regular and $k(C) = L$. The rational map $\mathbb{A}^2 \dashrightarrow C$ is defined in codimension 1, thus there exists a finite subset $S \subseteq \mathbb{A}^2$ and an extension $\mathbb{A}^2 \setminus S \to C$. Since the composition $\mathbb{A}^2 \setminus S(k) \to C(k) \to \mathbb{A}^1(k)$ is surjective up to a finite number of points, by Hilbert's irreducibility theorem we have that $C = \mathbb{A}^1$, i.e. $L = k(\mathbb{A}^1)$.

This leaves us with three cases: the generic fiber is a geometrically irreducible curve of geometric genus 0, 1, or ≥ 2. The first two have been settled by W. Sawin in the polymath project [Tao19], while the third follows from Theorem A. Let us give details for all of them.

Genus 0. Assume that the generic fiber of f has genus 0. By generic smoothness, there exists an open subset $U \subseteq \mathbb{A}^2$ such that $f|_U$ is smooth. For a generic rational point $u \in U(k)$, the fiber $f^{-1}(f(u))$ is birational to a Brauer-Severi variety of dimension 1 and has a smooth rational point, thus it is birational to $\mathbb{P}^1$ and $f^{-1}(f(u))(k)$ is infinite. This is a contradiction.

Genus 1. Assume now that the generic fiber has genus 1. By resolution of singularities, there exists an open subset $V \subseteq \mathbb{A}^1$, a variety X with a smooth projective morphism $g : X \to V$ whose fibers are smooth genus 1 curves and a compatible birational map $X \dashrightarrow \mathbb{A}^2$. Let U be a variety with open embeddings $U \subseteq X$, $U \subseteq \mathbb{A}^2$, replace V with $g(U) \subseteq V$ so that $g|_U$ is surjective.

The morphism $X \setminus U \to V$ is finite, let N be an upper bound for the degree of its fibers. Since the fibers of $U \to V$ have at most one rational point, it follows that $|X_v(k)| \leq N + 1$ for every $v \in V(k)$.

Every smooth genus 1 fibration is a torsor for a relative elliptic curve (namely, its relative $\underline{\mathrm{Pic}}^0$), thus there exists an elliptic curve $E \to V$ such that X is an E-torsor. Moreover, every torsor for an abelian variety is torsion, thus there exists a finite morphism $\pi : X \to E$ over V induced by the n-multiplication map $E \to E$ for some n.

If $v \in V(k)$ is such that $X_v(k)$ is non-empty, then $|X_v(k)| = |E_v(k)| \leq N + 1$. This means that, up to composing π with the $(N+1)!$ multiplication $E \to E$, we may assume that $\pi(X(k)) \subseteq V(k) \subseteq E(k)$, where $V \to E$ is the identity section. In particular, $X(k) \subseteq \pi^{-1}(V(k))$ is not dense. This gives a contradiction, since X is birational to $\mathbb{A}^2$.

Genus ≥ 2. Thanks to Theorem A, there exists an open subset $V \subseteq \mathbb{A}^1$ and a section $s : V \to \mathbb{A}^2$. It follows that $\mathbb{A}^2|_V(k) = s(V(k))$, which gives a contradiction since $s(V)$ is a proper closed subset and $\mathbb{A}^2|_V(k)$ is dense. This concludes the proof of Theorem B.

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