

BLOW-UP CRITERIA BELOW SCALING FOR DEFOCUSING ENERGY-SUPERCRITICAL NLS AND QUANTITATIVE GLOBAL SCATTERING BOUNDS

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The author dedicates this work to the memory of Jean Bourgain.

ABSTRACT. We establish quantitative blow-up criteria below the scaling threshold for radially symmetric solutions to the defocusing nonlinear Schrödinger equation with nonlinearity $|u|^6u$. This provides to our knowledge the first generic results distinguishing potential blow-up solutions of the defocusing equation from many of the known examples of blow-up in the focusing case. Our main tool is a quantitative version of a result showing that uniform bounds on L^2 -based critical Sobolev norms imply scattering estimates.

As another application of our techniques, we establish a variant which allows for slow growth in the critical norm. We show that if the critical Sobolev norm on compact time intervals is controlled by a slowly growing quantity depending on the Strichartz norm, then the solution can be extended globally in time, with a corresponding scattering estimate.

1. INTRODUCTION

We consider an energy-supercritical instance of the defocusing Nonlinear Schrödinger (NLS) equation on \mathbb{R}^3 ,

$$\begin{cases} iu_t + \Delta u &= |u|^6u, & (t, x) \in I \times \mathbb{R}^3, \\ u|_{t=0} &= u_0 \in \dot{H}_x^{s_c}, \end{cases} \quad (1)$$

with radially symmetric initial data in the scaling critical Sobolev space $\dot{H}^{s_c}(\mathbb{R}^3)$, $s_c = 7/6$, posed on a time interval $0 \in I \subset \mathbb{R}$. Here, $\dot{H}_x^{s_c}$ denotes the usual homogeneous Sobolev space, with norm given by

$$\|f\|_{\dot{H}^{s_c}}^2 = \int |\xi|^{2s_c} |\hat{f}(\xi)|^2 d\xi,$$

where \hat{f} denotes the Fourier transform of f .

It is well known (see, for instance, [8, 21] and [26]) that initial value problems of type (1) are equipped with a robust local well-posedness theory whenever the initial data belongs to $\dot{H}_x^s(\mathbb{R}^3)$ with $s \geq s_c$, where the space $\dot{H}_x^{s_c}(\mathbb{R}^3)$, $s_c = 7/6$ is distinguished as the only homogeneous L^2 -based Sobolev space preserved by the

scaling symmetry of the equation,

$$u(t, x) \rightsquigarrow \lambda^{-1/3} u(\lambda^{-2}t, \lambda^{-1}x).$$

On the other hand, the status of global-in-time well-posedness properties of solutions to (1) is largely unresolved, even for radially symmetric smooth initial data with compact support. While the mass

$$M[u(t)] := \int_{\mathbb{R}^3} |u(t)|^2 dx,$$

and energy,

$$E[u(t)] := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x u(t)|^2 dx + \frac{1}{8} \int_{\mathbb{R}^3} |u(t)|^8 dx,$$

are preserved by the evolution (that is, $M[u(t)] = M[u_0]$ and $E[u(t)] = E[u_0]$ hold for all $t \in I$), when the nonlinearity is in the energy-supercritical regime as in (1) these quantities have subcritical scaling, and do not provide sufficient control to allow for iteration of the local theory.

Nevertheless, some partial results concerning possible blow-up scenarios for defocusing energy-supercritical nonlinearities are known. First steps in this direction are the uniform *a priori* critical bound implies scattering results of [18] and [25, 26], based on concentration compactness-based methods developed in the context of the nonlinear Schrödinger and nonlinear wave equations (see, for instance [14, 15, 16] and [17, 19, 20, 3, 4, 5]), as well as the references cited in these works). These results show that that solutions cannot blow-up in finite time whenever

$$E_{s_c} := \|u(t)\|_{L_t^\infty(I_{\max}; \dot{H}_x^{s_c})}$$

is finite, with $I_{\max} \subset \mathbb{R}$ denoting the maximal interval of existence.

Other results concerning possible blow-up in the energy-supercritical regime include studies of global well-posedness for logarithmically supercritical nonlinearities [28] (see also related works [32, 27, 29, 30, 6, 11] for the nonlinear wave equation), and constructions of large data global solutions [1]. We remark that there are also well-developed constructions of blow-up solutions for focusing nonlinearities (where $|u|^6 u$ is replaced by $-|u|^p u$ for $p > \frac{4}{d-2}$, with the problem posed on \mathbb{R}^d); see, e.g. [9, 13, 23], and the references cited therein.

Very recently, in a groundbreaking work [24], Merle, Raphaël, Rodnianski, and Szeftel have constructed the first instance of finite-time blow-up for the defocusing energy-supercritical nonlinear Schrödinger equation (earlier blow-up results for systems were obtained by Tao in [33, 34]). The construction in [24] is based on a reduction to the formation of shocks in an instance of the compressible Euler equation, and produces a blow-up solution for which

$$\lim_{t \nearrow T_*} \|u(t)\|_{H_x^s} = \infty, \quad s_* < s < s_c, \quad (2)$$

for some $s_* \in (1, s_c)$, where T_* is the blowup time. This distinguishes the *defocusing* blow-up scenario constructed in [24] from the soliton-focusing and self-similar scenarios constructed in the focusing setting. We remark that the current understanding of the hydrodynamic reduction is tied to the high-dimensional (\mathbb{R}^d with

$d \geq 5$) setting, and [24] therefore does not directly address the issue of blow-up in the three-dimensional setting of (1).

Our first main result is that any potential blow-up solution to (1) must exhibit growth of subcritical Sobolev norms as in the higher-dimensional blow-up scenario constructed in [24]. In particular, we show that when the initial data u_0 has some additional regularity, e.g.

$$u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^3) \cap \dot{H}_x^{s_c+1}(\mathbb{R}^3),$$

global well-posedness remains true under an assumption on “slightly subcritical norms,”

$$\sup_{t \in I} \|u(t)\|_{\dot{H}_x^{s_c-\delta}(\mathbb{R}^3)} \leq E,$$

for $\delta > 0$ sufficiently small, with the required smallness depending on $\|u_0\|_{\dot{H}_x^{s_c+1}}$ and E . The precise statement is given in the theorem below.

Theorem 1.1. *There exists $C > 0$ such that for each $E \geq 1$ and $M > 0$ there exists $\delta_0 = \delta_0(E, M) > 0$ with the following property: For all radially symmetric initial data $u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^3) \cap \dot{H}_x^{s_c+1}(\mathbb{R}^3)$, with*

$$\|u_0\|_{\dot{H}_x^{s_c}(\mathbb{R}^3) \cap \dot{H}_x^{s_c+1}(\mathbb{R}^3)} \leq M,$$

if $0 < \delta < \delta_0$ and

$$u \in C_t(I; \dot{H}_x^{s_c}(\mathbb{R}^3)) \cap L_{t,x}^{15}(I \times \mathbb{R}^3) \quad \text{for all } I \subset\subset I_{max},$$

is a solution with maximal-lifespan I_{max} to (1) which satisfies,

$$\|u\|_{L_t^\infty(I_{max}; \dot{H}_x^{s_c-\delta}(\mathbb{R}^3))} \leq E, \tag{3}$$

then $I_{max} = \mathbb{R}$ and

$$\|u\|_{L_{t,x}^{15}(\mathbb{R} \times \mathbb{R}^3)} \leq C \exp(C(EM^\delta)^C).$$

In this statement and the rest of this paper, a *solution* to (1) on a time interval I will always be understood as a function in $C_t(I; \dot{H}_x^{s_c}(\mathbb{R}^3)) \cap L_{t,x}^{15}(I \times \mathbb{R}^3)$ for all intervals $J \subset\subset I$, which satisfies the initial value problem in the sense of the associated integral equation,

$$u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-t')\Delta} [|u(t')|^6 u(t')] dt'.$$

We remark that the restriction to (1) in our work—that is, considering the case of three spatial dimensions and $p = 6$ in the nonlinearity $|u|^p u$ —is not essential (we have chosen to work with this case to highlight the core aspects of the arguments; it is expected that standard techniques would allow to treat more general energy-supercritical defocusing problems, with the associated technical subtleties, for instance in treating non-smooth nonlinearities, well understood in related energy-critical settings, see, e.g. [35, 36, 22]). In light of this, Theorem 1.1 is a strong indication that properties such as those shown in [24, Appendix D] are indeed *universal* properties of any defocusing energy-supercritical blow-up for NLS.

The first indication that a result like Theorem 1.1 should hold arises from [29], where Roy studied the nonlinear wave equation with log-supercritical defocusing nonlinearities of type $|u|^p u g(|u|)$. The main result of [29] is that a robust global well-posedness and scattering theory for the energy-supercritical nonlinearity $|u|^p u$ with data in the critical space $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ would imply that solutions to the log-supercritical problem which are uniformly bounded in $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ can be extended globally in time. A second motivation, accounting for the transition from uniform $\dot{H}_x^{s_c}$ bounds to uniform control in $\dot{H}_x^{s_c-\delta}$, is found in a recent paper of Colombo and Haffter [11], also studying the nonlinear wave equation, where the authors revisit the log-supercritical theory of [28], replacing log-supercriticality by small power-type supercriticality for bounded sets of initial data, with the level of supercriticality depending on the initial-data bound. The authors of [11] were motivated by recent results for supercritical problems in fluid dynamics, where supercriticality is compensated for in a similar way (see, e.g. [12] for the surface quasi-geostrophic equation and the associated log-supercritical result in [7, Theorem 1.3], as well as [10] for the hyperdissipative Navier-Stokes system).

The main tool used in the proof of Theorem 1.1 is the following proposition, which shows that *a priori* uniform control on the critical norm implies an explicit quantitative bound on the scattering norm (and therefore uniform control over the critical $\dot{H}_x^{s_c}$ norm on a solution's maximal interval of existence leads to global well-posedness).

Proposition 1.2. *Suppose that u is a radially symmetric solution to (1) which has maximal interval of existence I_{max} and satisfies*

$$\|u\|_{L_t^\infty(I; \dot{H}_x^{s_c}(\mathbb{R}^3))} \leq E = E_{s_c} < +\infty \quad (4)$$

for some $I \subset I_{max}$. Then $\|u\|_{L_{t,x}^{15}(\mathbb{R} \times \mathbb{R}^3)} \leq C \exp(CE^C)$.

In particular, if (4) holds with $I = I_{max}$ then $I_{max} = \mathbb{R}$, and the associated $L_{t,x}^{15}(I \times \mathbb{R}^3)$ bound holds.

A related statement (without quantitative dependence on the *a priori* bound) was obtained for NLS with a class of energy-supercritical nonlinearities in [26], based on concentration compactness methods (the paper [26] also treats the energy-subcritical problem; see also the discussion there for an overview of the related preceding literature). We follow the approach of [31], which is in turn related to the induction on energy ideas of Bourgain [2]. As we remarked above, the restriction to three spatial dimensions (and algebraic nonlinearity) allows us to focus on the simplest form of the argument.

The quantitative bounds derived in Proposition 1.2 are of substantial independent interest. As a second application of our techniques, we prove the following corollary, which shows that slowly-growing control of the critical norm $\dot{H}_x^{s_c}$ by the scale-invariant Strichartz norm $L_{t,x}^{15}(I \times \mathbb{R}^3)$ implies that a solution can be extended globally (and scatters at $\pm\infty$).

Corollary 1.3. *Let C be the constant in Proposition 1.2, and let $g : [0, \infty) \rightarrow [0, \infty)$ be defined by*

$$g(t) = [C^{-1} \log(\log^{1/2}(t))]^{1/C}.$$

Suppose that u is a radially symmetric solution to (1) with maximal interval of existence $I_{max} \subset \mathbb{R}$ such that for every interval $I \subset\subset I_{max}$

$$\|u\|_{L_t^\infty(I; \dot{H}_x^{sc}(\mathbb{R}^3))} \leq g(\|u\|_{L_{t,x}^{15}(I \times \mathbb{R}^3)}).$$

Then there exists $M_0 > 0$ so that $\|u\|_{L_{t,x}^{15}(I)} \leq M_0$ for all $I \subset I_{max}$. In particular, $I_{max} = \mathbb{R}$.

Outline of the paper. We briefly summarize the structure of this paper. In Section 2 we recall some preliminaries, including the standard homogeneous and inhomogeneous Strichartz estimates for the linear Schrödinger equation. In Section 3 we prove Proposition 1.2. Then, in Section 4 and 5 we use a standard continuity argument to give the proofs of Theorem 1.1 and Corollary 1.3.

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2. PRELIMINARIES AND STRICHARTZ ESTIMATES

In this section, we fix our choice of function spaces and recall the corresponding Strichartz estimates which will be used in the rest of this paper. We say that a pair (q, r) with $2 \leq q < \infty$ and $2 \leq r < \infty$ is (Schrödinger) admissible on \mathbb{R}^3 if $\frac{2}{q} + \frac{3}{r} = \frac{3}{2}$. For any such pair, we have

$$\|e^{it\Delta} u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^3)}$$

and

$$\left\| \int_0^t e^{i(t-t')\Delta} F(t') dt' \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{q'_2} L_x^{r'_2}}$$

for every admissible pair (q_2, r_2) , where q'_2 and r'_2 denote conjugate exponents.

Let $I \subset \mathbb{R}$ be an open time interval, and suppose that $u \in C_t(I_0; \dot{H}_x^{sc}(\mathbb{R}^3)) \cap L_{t,x}^{15}(I_0 \times \mathbb{R}^3)$ for all compact $I_0 \subset I$. For $I_0 \subset I$, define

$$\begin{aligned} \|u\|_{S(I_0)} &= \|u\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} \\ \|u\|_{W(I_0)} &= \max\left\{ \|\nabla|^{sc} u\|_{L_{t,x}^{10/3}(I_0 \times \mathbb{R}^3)}, \|\nabla|^{sc} u\|_{L_t^{15}(I_0; L_x^{\frac{90}{11}}(\mathbb{R}^3))} \right\} \\ \|u\|_{N(I_0)} &= \|\nabla|^{sc} u\|_{L_{t,x}^{10/7}(I_0 \times \mathbb{R}^3)}. \end{aligned}$$

The norms $\|\cdot\|_{S(I_0)}$ and $\|\cdot\|_{W(I_0)}$ are chosen so that they can be controlled by appropriate Strichartz estimates, with the norm $\|\cdot\|_{N(I_0)}$ chosen by duality as an admissible choice of norm on the right-hand side of the inhomogeneous Strichartz bound.

In particular, one has (also using the Sobolev embedding)

$$\|e^{it\Delta}u_0\|_{L_{t,x}^{15}} \lesssim \| |\nabla|^{s_c} e^{it\Delta}u_0 \|_{L_t^{15} L_x^{\frac{90}{41}}} \lesssim \|u_0\|_{\dot{H}_x^{s_c}}. \quad (5)$$

3. PROOF OF PROPOSITION 1.2

In this section we prove Proposition 1.2. We begin by recalling some estimates for the localized L^2 norm $M(u; x_0, R)$, where u solves either (1) or the linear Schrödinger equation (we follow the treatment of [31]). These are based on the formal identity $\partial_t |u(t)|^2 = -2 \operatorname{div}(\operatorname{Im}(\overline{u(t)}(\nabla u(t))))$ satisfied by sufficiently smooth solutions of either equation.

Let $\chi \in C_c^\infty(\mathbb{R}^3)$ be such that $\chi \equiv 1$ on $B(0; 1/2)$ and $\operatorname{supp} \chi \subset B(0; 1)$. For each $x \in \mathbb{R}^3$, $R > 0$, set

$$M(u; x_0, R) = \left(\int \left| \chi\left(\frac{x-x_0}{R}\right) u(x) \right|^2 dx \right)^{1/2}.$$

Then, using the identity mentioned above, one obtains

$$\partial_t [M(u(t); x_0, R)^2] = \frac{4}{R} \int \chi\left(\frac{x-x_0}{R}\right) (\nabla \chi)\left(\frac{x-x_0}{R}\right) \cdot \operatorname{Im}(\overline{u} \nabla u) dx$$

and thus

$$|\partial_t [M(u(t); x_0, R)^2]| \lesssim R^{-5/6} M(u(t); x_0, R) \|\nabla u\|_{L_x^{9/4}}$$

so that

$$\partial_t M(u(t); x_0, R) \lesssim R^{-5/6} \|u\|_{L_t^\infty(I; \dot{H}_x^{s_c}(\mathbb{R}^3))}.$$

Moreover, one also has

$$|M(u(t); x_0, R)| \leq \|u\|_{L_x^9(\mathbb{R}^3)} \|\chi(x/R)\|_{L_x^{18/7}} \lesssim R^{7/6} \|u\|_{L_t^\infty(I; \dot{H}_x^{s_c}(\mathbb{R}^3))} \quad (6)$$

We also recall a spatially localized form of the Morawetz estimate in our energy-supercritical setting. Estimates of this type were originally obtained by Bourgain [2] (see also the treatment in [31] and related earlier work of Lin and Strauss). In the energy-supercritical setting, we refer to the treatment of Murphy in [26], which covers the case we need. In the interest of completeness, we sketch the argument in our setting (following the presentation of [2]).

Proposition 3.1 (Spatially localized Morawetz estimate). *Suppose that u solves (1). We then have*

$$\int_I \int_{|x| < C|I|^{1/2}} \frac{|u(t, x)|^8}{|x|} dx dt \lesssim (C|I|)^{2/3} \|u\|_{L_t^\infty(I; \dot{H}_x^{s_c})}$$

for every time interval $I \subset \mathbb{R}$.

Proof. We argue as in [2, Lemma 2.1]. Setting $r = |x|$ and $v = \operatorname{Re} u$, $w = \operatorname{Im} u$, we multiply (NLS) by

$$\overline{\left(u_r + \frac{1}{r}u\right)}$$

and take the real part of both sides to obtain

$$\partial_t X + \operatorname{div} Y + Z = 0 \quad (7)$$

with

$$\begin{aligned} X(t, x) &= -w(t, x) \left(\partial_r v(t, x) + \frac{1}{r} v(t, x) \right), \\ Y(t, x) &= \frac{x}{r} (\partial_t v)(t, x) w(t, x) - (\nabla v)(t, x) \left(\partial_r v(t, x) + \frac{1}{r} v(t, x) \right) \\ &\quad - (\nabla w)(t, x) \left(\partial_r w(t, x) + \frac{1}{r} w(t, x) \right) + \frac{x}{2r} |\nabla u(t, x)|^2 \\ &\quad + \frac{x}{8r} |u(t, x)|^8 - \frac{x}{2r^3} |u(t, x)|^2 \\ Z(t, x) &= \frac{1}{r} (|\nabla u|^2 - |\partial_r u|^2) + \frac{2}{3r} |u|^8 \end{aligned}$$

Now, consider $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}$ radial such that $\phi = 1$ on $\{x : |x| < \delta\}$, $\operatorname{supp} \phi \subset \{x : |x| < 2\delta\}$ and

$$|D^j \phi| \lesssim \delta^{-j}, \quad j \geq 1.$$

Multiplying (7) by ϕ and integrating, it suffices to bound the quantities

$$\int |X| \phi dx \quad \text{and} \quad \left| \int Y \cdot \nabla \phi dx \right|$$

(see the proof of [2, Lemma 2.1] for more details). To fix ideas, we give the first estimate:

$$\begin{aligned} \int_{\mathbb{R}^d} |X| \phi dx &\leq \int_{\mathbb{R}^d} |u| \left(|\nabla u| + \frac{1}{r} |u| \right) \phi(x) dx \\ &\leq \| |\nabla u(t)| \phi^{1/2} \|_{L_x^2} + \left\| \frac{1}{r} |u(t)| \phi^{1/2} \right\|_{L_x^2} \|u(t) \phi^{1/2}\|_{L_x^2} \\ &\leq \left(\|\nabla u(t)\|_{L_x^{9/4}} \|\phi\|_{L_x^9}^{1/2} \right. \\ &\quad \left. + \left\| \frac{1}{r} u(t) \right\|_{L_x^{9/4}} \|\phi\|_{L_x^9}^{1/2} \right) \|u(t)\|_{L_x^9} \|\phi\|_{L_x^9}^{1/2} \\ &\leq \delta^{2-\frac{2}{3}} \|u\|_{L_t^\infty \dot{H}_x^{s_c}}^2 \end{aligned}$$

The other estimates follow similarly. \square

3.1. Preliminary construction. Let u be a radial solution to (1) on a time interval $I = [t_-, t_+]$. Without loss of generality, we may assume $E := E_{s_c} \geq c_0$ for some $c_0 > 0$ (since E sufficiently small implies the desired bound by the local theory of the previous subsection). Fix parameters $1 \leq C_0 \leq C_1 \leq C_2$ to be determined later in the argument, and set

$$\eta = \frac{1}{C_2} (1 + E)^{-C_2}.$$

Partition I into consecutive disjoint intervals $\{I_1, I_2, \dots, I_J\}$ with each $I_j = [t_j, t_{j+1}]$ so that

$$\int_{t_j}^{t_{j+1}} \int |u(t, x)|^{15} dx dt \in [\eta, 2\eta].$$

Remark 3.2 (Absorbing E into expressions involving η). Let $p > 0$ be given. For each $\epsilon > 0$ there exists $\overline{C_2} = \overline{C_2}(p)$ so that for all $C_2 > \overline{C_2}$ we have $E\eta^p < \epsilon$. Indeed, writing $E\eta^p = (E/(1+E)^{C_2 p})C_2^{-p} < (1+E)^{1-C_2 p}C_2^{-p}$, the condition $1 - C_2 p < 0$ leads to $E\eta^p \leq C_2^{-p}$, and thus the additional condition $C_2 > \epsilon^{-1/p}$ implies the desired bound.

We also have that for each $C > 0$ and $p > 0$ there exists $C' > 0$ so that $\eta^C E^{-p} \geq \eta^{C'}$. Indeed, writing $\eta^C E^{-p} \geq \eta^C (1+E)^{-p} = \eta^C (C_2 \eta)^{p/C_2}$ and recalling $C_2 \geq 1$, one obtains the desired bound with $C' = C + (p/C_2)$.

Similarly, for fixed $p > 0$, $C > 0$, and $\epsilon > 0$, we can choose C_2 sufficiently large to ensure $E^p \eta^{-C} \leq \eta^{-C-\epsilon}$. This follows by writing $E^p \eta^{-C} \leq (1+E)^p \eta^{-C} = C_2^{-p/C_2} \eta^{-C-(p/C_2)}$, and noting that $C_2 \geq 1$ therefore implies $E^p \eta^{-C} \leq \eta^{-C-(p/C_2)}$. The condition $C_2 > p/\epsilon$ now implies (in view of $0 < \eta < 1$) $\eta^{C+\epsilon} < \eta^{C+(p/C_2)}$, and thus one has $E^p \eta^{-C} \leq \eta^{-C-\epsilon}$ as desired.

Lemma 3.3. *There exists $C > 0$ such that for each $I_0 \subset I$, if*

$$\int_{I_0} \int |u(t, x)|^{15} dx dt \leq 2\eta$$

then

$$\|u\|_{W(I_j)} \leq CE.$$

Proof.

$$\begin{aligned} \|u\|_{W(I_j)} &\lesssim \|u(t_j)\|_{\dot{H}_x^{sc}} + \| |u|^6 u \|_{N(I_j)} \\ &\lesssim E + \| |\nabla|^{sc} u \|_{L_{t,x}^{10/3}(I_j)} \|u\|_{S(I_j)}^6 \\ &\lesssim E + \|u\|_{W(I_j)} \eta^{6/15}. \end{aligned}$$

Choosing η sufficiently small now ensures $\|u\|_{W(I_j)} \lesssim E$ as desired. \square

Corollary 3.4. *There exists $C > 0$ and $\overline{C_2} > 0$ so that if $C_2 \geq \overline{C_2}$ then for each $1 \leq j \leq J$, then one has $\|u\|_{W(I_j)} \leq CE$.*

The next lemma is not used in our subsequent argument (it is the analogue of Lemma 3.2 in [31], and is not needed since we are working only in dimension 3), but we record it here as a preliminary version of a similar argument used in the proof of Proposition 3.7 below to estimate a suitable norm of the function v .

Lemma 3.5. *There exists $c > 0$ such that for each $I_0 = [t_1, t_2] \subset I$ satisfying*

$$\int_{t_1}^{t_2} \int |u(t, x)|^{15} dx dt \in \left[\frac{\eta}{2}, 2\eta\right],$$

we have

$$\int_{t_1}^{t_2} \int |e^{i(t-t_i)\Delta} u(t_i)|^{15} dx dt \geq c\eta$$

for $i \in \{1, 2\}$.

Proof. Using the Sobolev embedding

$$\|u - e^{i(t-t_i)\Delta} u(t_i)\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} \lesssim \| |\nabla|^{s_c} (u - e^{i(t-t_i)\Delta} u(t_i)) \|_{L_t^{15}(I_0; L_x^{\frac{90}{41}}(\mathbb{R}^3))}$$

one has

$$\begin{aligned} \|u - e^{i(t-t_i)\Delta} u(t_i)\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} &\lesssim \|u - e^{i(t-t_i)\Delta} u(t_i)\|_{W(I_0)} \\ &\lesssim \| |u|^6 u \|_{N(I_0)} \\ &\lesssim \| |\nabla|^{s_c} u \|_{W(I_0)} \|u\|_{S(I_0)}^6 \end{aligned}$$

so that by Lemma 3.3 one gets the bound

$$\|u - e^{i(t-t_i)\Delta} u(t_i)\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} \leq CE\eta^{6/15}.$$

Using this, we now get

$$\begin{aligned} (\eta/2)^{1/15} &\leq \|u\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} \\ &\leq \|u - e^{i(t-t_i)\Delta} u(t_i)\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} + \|e^{i(t-t_i)\Delta} u(t_i)\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} \\ &\leq CE\eta^{6/15} + \|e^{i(t-t_i)\Delta} u(t_i)\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} \end{aligned}$$

and thus choosing C_2 large enough ensures

$$\|e^{i(t-t_i)\Delta} u(t_i)\|_{L_{t,x}^{15}(I_0 \times \mathbb{R}^3)} \geq c\eta^{1/15}$$

as desired. \square

3.2. Classification of intervals. Set $u_- := e^{i(t-t_-)\Delta} u(t_-)$ and $u_+ = e^{i(t-t_+)\Delta} u(t_+)$. Let B denote the set of $j \in \{1, \dots, J\}$ such that

$$\max\left\{ \int_{t_j}^{t_{j+1}} \int |u_-(t)|^{15} dx dt, \int_{t_j}^{t_{j+1}} \int |u_+(t)|^{15} dx dt \right\} > \eta^{C_1}$$

and set $G = \{1, \dots, J\} \setminus B$. In the language of [31], the intervals I_j , $j \in G$ are *unexceptional* intervals, and the intervals I_j , $j \in B$ are *exceptional* intervals.

Remark 3.6. One immediately has a bound on $\#B$. Indeed, first note that

$$\sum_{j \in B} \int_{I_j} \int |u_+(t)|^{15} + |u_-(t)|^{15} dx dt > (\#B)\eta^{C_1}.$$

On the other hand, by the Strichartz estimate (5), we have

$$\begin{aligned} \sum_{j \in B} \int_{I_j} \int |u_+(t)|^{15} + |u_-(t)|^{15} dx dt &\leq \int_I \int |u_+(t)|^{15} dx dt + \int_I \int |u_-(t)|^{15} dx dt \\ &\lesssim \|u(t_+)\|_{H_x^{s_c}}^{15} + \|u(t_-)\|_{H_x^{s_c}}^{15} \lesssim E^{15}, \end{aligned}$$

so that $\#B \leq CE^{15}/\eta^{C_1}$.

3.3. Concentration bound.

Proposition 3.7. *There exist $c, C > 0$ such that for each $j \in G$ there exists $x_j \in \mathbb{R}^3$ such that*

$$M(u(t); x_j, C\eta^{-C}|I_j|^{1/2}) \geq c\eta^C E^{-3/2} |I_j|^{7/12}.$$

Proof. Let $j \in G$ be given. For simplicity of notation, set $J = I_j$ and recall that

$$\int_J \int |u(t, x)|^{15} dx dt > \eta.$$

Then, setting $J = I_j$, $t_* = (t_j + t_{j+1})/2$, and writing $J = J_1 \cup J_2$ with $J_1 = [t_j, t_*]$, $J_2 = [t_*, t_{j+1}]$, we can choose $k \in \{1, 2\}$ so that

$$\int_{J_k} \int |u(t, x)|^{15} dx dt > \eta/2.$$

Moreover, $j \in G$ implies

$$\int_J \int |u_k(t, x)|^{15} dx dt \leq \eta^{C_1}.$$

where $u_1 = u_+$ and $u_2 = u_-$.

Now, set $M_1 = [t_{j+1}, t_+]$ and $M_2 = [t_-, t_j]$, and define

$$v(t) := \int_{M_k} e^{i(t-t')\Delta} [|u(t')|^6 u(t')] dt'.$$

We begin by recording some instances of the Duhamel formula which will be useful in our estimates: for each $t \in J_k$, one has

$$u(t) = e^{i(t-t_+)\Delta} u(t_+) - iv(t) - i \int_t^{t_{j+1}} e^{i(t-t')\Delta} [|u(t')|^6 u(t')] dt' \quad (8)$$

if $k = 1$, and

$$u(t) = e^{i(t-t_-)\Delta} u(t_-) - iv(t) - i \int_{t_j}^t e^{i(t-t')\Delta} [|u(t')|^6 u(t')] dt' \quad (9)$$

if $k = 2$.

The Duhamel formulas (8) and (9) combined with the Strichartz estimates yield an upper bound on the $L_t^\infty \dot{H}_x^{s_c}$ norm of v (on the interval J_k)

$$\begin{aligned} \|v\|_{L_t^\infty(J_k; \dot{H}_x^{s_c}(\mathbb{R}^3))} &\lesssim \|u\|_{L_t^\infty(J_k; \dot{H}_x^{s_c}(\mathbb{R}^3))} + \|u(t_+)\|_{\dot{H}_x^{s_c}} + \| |u|^6 u \|_{N(J_k)} \\ &\lesssim 2E + \| |\nabla|^{s_c} u \|_{W(J)} \|u\|_{S(J)}^6 \\ &\lesssim 2E + E\eta^{6/15} \lesssim E. \end{aligned} \quad (10)$$

We now show the bound

$$\|v\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} \geq c\eta^{1/15}. \quad (11)$$

For this, we note that when $k = 1$, (8) leads to

$$(\eta/2)^{1/15} < \|u\|_{L_{t,x}^{15}(J_1 \times \mathbb{R}^3)} \leq \eta^{C_1/15} + \|u(t) - e^{i(t-t_{j+1})\Delta}u(t_{j+1})\|_{S(J_1)} + \|v\|_{S(J_1)},$$

while when $k = 2$, (9) gives

$$(\eta/2)^{1/15} < \|u\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} \leq \eta^{C_1/15} + \|u(t) - e^{i(t-t_j)\Delta}u(t_j)\|_{S(J_2)} + \|v\|_{S(J_2)}.$$

On the other hand, the Sobolev embedding, Strichartz estimates, and fractional product rules imply

$$\begin{aligned} \|u(t) - e^{i(t-t_j)\Delta}u(t_j)\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} &\lesssim \|u - e^{i(t-t_j)\Delta}u(t_j)\|_{W(J_k)} \\ &\lesssim \| |u|^6 u \|_{N(J_k)} \\ &\lesssim \| |\nabla|^{s_c} u \|_{W(J)} \|u\|_{S(J)}^6 \\ &\lesssim E\eta^{6/15}. \end{aligned}$$

in both of the cases $k = 1$ and $k = 2$. Combining this with the above estimates from below yields the claim.

Now, let $\chi_0 \in C_c^\infty(\mathbb{R}^3)$ be such that $\chi_0(x) \geq 0$ for all $x \in \mathbb{R}^3$, $\text{supp } \chi_0 \subset B(0;1)$, and $\int \chi_0(x)dx = 1$, and define

$$v_{\text{av}}(t, x) := \int \chi(y)v(t, x + ry)dy,$$

where $r > 0$ is a fixed parameter to be determined later in the argument.

We now claim that there exist $c, C > 0$ so that

$$\|v_{\text{av}} - v\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} \leq CE^7 |J_k|^{-3/10} r^c. \quad (12)$$

For this, for each $h \in \mathbb{R}^3$, let $T_h : f \mapsto T_h f$ be the operator defined by $(T_h f)(x) = f(x+h)$ for $f \in \mathcal{S}(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$. Then, since the Fourier multiplier operator $e^{is\Delta}$ commutes with translations,

$$\begin{aligned} &\|v - T_h v\|_{L_t^\infty(J_k; L_x^{15}(\mathbb{R}^3))} \\ &= \left\| \int_{M_k} e^{i(t-t')\Delta} [|u(t')|^6 u(t') - |T_h u(t')|^6 T_h u(t')] dt' \right\|_{L_t^\infty(J_k; L_x^{15}(\mathbb{R}^3))} \\ &\leq \left\| \int_{M_k} \|e^{i(t-t')\Delta} [|u(t')|^6 u(t') - |T_h u(t')|^6 T_h u(t')]\|_{L_x^{15}(\mathbb{R}^3)} dt' \right\|_{L_t^\infty(J_k)} \\ &\lesssim \left\| \int_{M_k} |t-t'|^{-13/10} \| |u(t')|^6 u(t') - |T_h u(t')|^6 T_h u(t') \|_{L_x^{15/14}(\mathbb{R}^3)} dt' \right\|_{L_t^\infty(J_k)} \\ &\lesssim \left\| \int_{M_k} |t-t'|^{-13/10} \|u(t') - T_h u(t')\|_{L_x^{15/4}(\mathbb{R}^3)} \|u\|_{L_t^\infty(I; L_x^9(\mathbb{R}^3))}^6 dt' \right\|_{L_t^\infty(J_k)}. \end{aligned}$$

Now, using the difference quotient bound

$$\|u - T_h u\|_{L_x^{9/4}} \lesssim \|\nabla u\|_{L_x^{9/4}} |h|,$$

interpolation and the Sobolev embedding give

$$\|u(t') - T_h u(t')\|_{L_x^{15/4}} \leq 2 \|u(t')\|_{L_x^9}^{8/15} \|\nabla u(t')\|_{L_x^{9/4}}^{7/15} |h|^{7/15}$$

$$\lesssim \|\nabla|^{s_c} u\|_{L_t^\infty L_x^2} |h|^{7/15}$$

for all $t' \in M_k$. Recalling once more the Sobolev embedding $\|u\|_{L_x^9} \lesssim \|\nabla|^{s_c} u\|_{L_x^2}$, we therefore get

$$\|v - T_h v\|_{L_t^\infty(J_k; L_x^{15}(\mathbb{R}^3))} \lesssim E^7 |h|^{7/15} \sup_{t \in J_k} \int_{M_k} \frac{1}{|t - t'|^{13/10}} dt' \lesssim E^7 |h|^{7/15} |J_k|^{-3/10},$$

where in obtaining the last bound we have used that for $t \in J_k$, $\int_{M_k} |t - t'|^{-13/10} dt' \lesssim \text{dist}(t, M_k)^{-3/10} \leq (|J_k|/2)^{-3/10}$.

To obtain (12), we now write

$$\begin{aligned} \|v_{\text{av}} - v\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} &= \left\| \int \chi(y) (v(t, x + ry) - v(t, x)) dx \right\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} \\ &\leq |J_k|^{1/15} \int \chi(y) \|T_{ry} v - v\|_{L_t^\infty(J_k; L_x^{15}(\mathbb{R}^3))} dy \\ &\lesssim E^7 r^{7/15} |J_k|^{-7/30} \int \chi(y) |y|^{7/15} dy. \end{aligned}$$

Combining (11) with (12), we get

$$\begin{aligned} c\eta^{1/15} \leq \|v\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} &\leq \|v - v_{\text{av}}\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} + \|v_{\text{av}}\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} \\ &\leq CE^7 r^{7/15} |J_k|^{-7/30} + C \|v_{\text{av}}\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} \end{aligned}$$

so that choosing r sufficiently small to ensure $CE^7 |J_k|^{-7/30} r^{7/15} \leq c\eta^{1/15}/2$ gives

$$\|v_{\text{av}}\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} \geq c\eta^{1/15}. \quad (13)$$

In what follows, we fix

$$r := \eta^C |J_k|^{1/2}$$

with C sufficiently large to ensure that (13) holds (in particular, note that the condition (13) is equivalent to $\eta^{(7C-1)/15} E^7 \leq c$ for suitable $c > 0$, so that it suffices to choose $C > 2/7$ and C_2 sufficiently large to ensure $E\eta^{1/105} \leq c^{1/7}$).

To proceed, we now obtain a complementary estimate from above on $\|v_{\text{av}}\|_{L_{t,x}^{15}}$. In particular, by interpolation,

$$\begin{aligned} \|v_{\text{av}}\|_{L_{t,x}^{15}(J_k \times \mathbb{R}^3)} &\leq \|v_{\text{av}}\|_{L_{t,x}^9(J_k \times \mathbb{R}^3)}^{3/5} \|v_{\text{av}}\|_{L_{t,x}^\infty(J_k \times \mathbb{R}^3)}^{2/5} \\ &\lesssim |J_k|^{1/15} \|v_{\text{av}}\|_{L_t^\infty(J_k; L_x^9(\mathbb{R}^3))}^{3/5} \|v_{\text{av}}\|_{L_{t,x}^\infty(J_k \times \mathbb{R}^3)}^{2/5} \\ &\lesssim |J_k|^{1/15} \|v\|_{L_t^\infty(J_k; L_x^9(\mathbb{R}^3))}^{3/5} \|v_{\text{av}}\|_{L_{t,x}^\infty(J_k \times \mathbb{R}^3)}^{2/5}, \end{aligned} \quad (14)$$

where to obtain the last inequality we have defined $\chi_r(x) = r^{-3} \chi(x/r)$ and used (via Young's inequality)

$$\begin{aligned} \|v_{\text{av}}\|_{L_t^\infty(J_k; L_x^9(\mathbb{R}^3))} &= \text{esssup}_{t \in J_k} \|\chi_r \star v\|_{L_x^9(\mathbb{R}^3)} \\ &\lesssim \text{esssup}_{t \in J_k} \|\chi_r\|_{L_x^1(\mathbb{R}^3)} \|v\|_{L_x^9(\mathbb{R}^3)} \end{aligned}$$

and noted that $\|\chi_r\|_{L_x^1} = \|\chi\|_{L_x^1} = 1$.

In view of (10) (combined with the Sobolev embedding $\dot{H}_x^{s_c}(\mathbb{R}^3) \hookrightarrow L_x^9(\mathbb{R}^3)$), the estimates (13) and (14) lead to

$$\|v_{\text{av}}\|_{L_{t,x}^\infty} \geq c\eta^{1/2}|J_k|^{-1/6}E^{-3/2}$$

so that we may find $s_* \in J_k$ and $x_* \in \mathbb{R}^3$ with

$$c\eta^{1/2}|J_k|^{-1/6}E^{-3/2} \leq v_{\text{av}}(s_*, x_*) \lesssim r^{-3/2}M(v(s_*); x_0, r),$$

where to obtain the last inequality we have recalled the equality $v_{\text{av}}(s_*, x_*) = \frac{1}{r^3} \int_{\mathbb{R}^3} \chi\left(\frac{y-x_*}{r}\right)v(s_*, y)dy$ (indeed, this is just the definition of v_{av}).

To conclude the argument, we now transfer this lower bound from v back to u . We consider the cases $k = 1$ and $k = 2$ individually. Suppose first that we are in the case $k = 1$ and fix $\lambda > 0$ to be determined later. Then for all $t \in J$,

$$\begin{aligned} M(u(t); x_*, \lambda) &= M(u(t_{j+1}); x_*, \lambda) - \int_t^{t_{j+1}} \partial_s M(u(s); x_*, \lambda) ds \\ &\geq M(u(t_{j+1}); x_*, \lambda) - CE|J_1|\lambda^{-5/6} \end{aligned}$$

so that since (8) implies

$$u(t_{j+1}) = e^{i(t_{j+1}-t_+)\Delta}u(t_+) - iv(t_{j+1}) = u_+(t_{j+1}) - iv(t_{j+1}).$$

By Minkowski's inequality (with the measure $\chi\left(\frac{x-x_0}{\lambda}\right)^2 dx$) this gives

$$M(u(t); x_*, \lambda) \geq M(v(t_{j+1}); x_*, \lambda) - M(u_+(t_{j+1}); x_*, \lambda) - CE|J_1|\lambda^{-5/6}.$$

Since u_+ and v solve the linear Schrödinger equation, we get

$$\begin{aligned} M(v(t_{j+1}); x_*, \lambda) &= M(v(s_*); x_*, \lambda) + \int_{s_*}^{t_{j+1}} \partial_s M(v(s); x_*, \lambda) ds \\ &\geq c\eta^{1/2}|J_1|^{-1/6}E^{-3/2}r^{3/2} - C|J_1|\|v\|_{L_t^\infty(J_1; \dot{H}_x^{s_c}(\mathbb{R}^3))}\lambda^{-5/6} \\ &\geq c\eta^{1/2}|J_1|^{-1/6}E^{-3/2}r^{3/2} - CE|J_1|\lambda^{-5/6} \end{aligned}$$

provided $\lambda \geq r$, and, noting that $\|u_+\|_{S(J_1)} \leq \eta^{C_1/15}$ implies that for some $\sigma \in J_1$ we have $\|u_+(\sigma)\|_{L_x^{15}(\mathbb{R}^3)} \leq (2\eta^{C_1}|J_1|^{-1})^{1/15}$,

$$\begin{aligned} M(u_+(t_{j+1}); x_*, \lambda) &= M(u_+(\sigma); x_*, \lambda) + \int_\sigma^{t_{j+1}} \partial_s M(u_+(s); x_*, \lambda) ds \\ &\leq C\eta^{C_1/15}|J_1|^{-1/15}\lambda^{13/10} + (C|J_1|\lambda^{-5/6})\|u_+\|_{L_t^\infty(J_1; \dot{H}_x^{s_c}(\mathbb{R}^3))} \\ &\lesssim C\eta^{C_1/15}|J_1|^{-1/15}\lambda^{13/10} + (C|J_1|\lambda^{-5/6})E \end{aligned}$$

Fix $C' > 0$ to be determined momentarily. Assembling these estimates, and choosing $\lambda = \eta^{-C'}|J_1|^{1/2} \geq r = \eta^C|J_1|^{1/2}$, one has

$$\begin{aligned} M(u(t); x_*, \lambda) &\geq c\eta^{1/2}|J_1|^{-1/6}E^{-3/2}r^{3/2} - C\eta^{C_1/15}|J_1|^{-1/15}\lambda^{13/10} - 3CE|J_1|\lambda^{-5/6} \\ &\geq \left(c\eta^{(1+3C)/2}E^{-3/2} - \left(C\eta^{\frac{C_1}{15} - \frac{13C'}{10}} + 3CE\eta^{\frac{5C'}{6}} \right) \right) |J_1|^{7/12}. \end{aligned}$$

An identical argument applies in the case $k = 2$. Appropriate choice of C' and C_1 now gives

$$M(u(t); x_*, \eta^{-C'}|J|^{1/2}) \geq c\eta^C|J|^{7/12}.$$

where we have recalled that $|J_k| = \frac{1}{2}|J|$ and used the second observation in Remark 3.2. \square

Since we are in the radial case, we immediately get a ‘‘centered’’ version of the concentration result.

Corollary 3.8. *There exist $c, C > 0$ such that for each $j \in G$*

$$M(u(t); 0, C\eta^{-C}|I_j|^{1/2}) \geq c\eta^C |I_j|^{7/12}.$$

3.4. Morawetz control. Let $j \in G$ be given, and set $R = C\eta^{-C}|I_j|^{1/2}$. The concentration bounds of the previous subsection, together with the estimate

$$\begin{aligned} M(u(t); 0, R) &\leq \|u\|_{L_x^8(\{x:|x|\leq CR\})} \|\chi(\frac{x}{R})\|_{L_x^{8/3}} \\ &\lesssim \left(\int_{\{x \in \mathbb{R}^3: |x| \leq CR\}} \frac{|u(t)|^8}{R} dx \right)^{1/8} R^{10/8}, \end{aligned}$$

now give

$$(c\eta^C |I_j|^{7/12})^8 R^{-10} \leq \int_{|x| \leq C\eta^{-C}|I_j|^{1/2}} \frac{|u(t, x)|^8}{|x|} dx.$$

This in turn leads to

$$\int_{I_j} \int_{|x| \leq C\eta^{-C}|I_j|^{1/2}} \frac{|u(t, x)|^8}{|x|} dx \geq c\eta^{9C} |I_j|^{2/3}. \quad (15)$$

Combining this with the Morawetz estimate (Proposition 3.1), we obtain

Proposition 3.9. *If $I_0 \subset I$ is a union of consecutive intervals I_j , $j_- \leq j \leq j_+$, then there exists j_* with $j_- \leq j_+$ so that*

$$|I_{j_*}| \geq c\eta^{3C_1/2} |I|.$$

Proof. It suffices to show

$$\sup_{j_- \leq j \leq j_+} |I_j| \geq c\eta^{3C_1/2} |I|.$$

For this, we recall (15) above and Remark 3.6, from which one gets the bound

$$\begin{aligned} |I_0| &= \sum_{j_- \leq j \leq j_+} |I_j| \leq (\sup_j |I_j|)^{1/3} \left(\sum_j |I_j|^{2/3} \right) \\ &\leq (\sup_j |I_j|)^{1/3} \left(\sum_{j \in G} |I_j|^{2/3} + \sum_{j \in B} |I_j|^{2/3} \right) \\ &\leq (\sup_j |I_j|)^{1/3} \left(\sum_{j \in G} C\eta^{-9C} \int_{I_j} \int_{\mathbb{R}^3} \frac{|u|^8}{|x|} dx dt + (\#B) |I_0|^{2/3} \right) \\ &\leq (\sup_j |I_j|)^{1/3} (c\eta^{-9C} E + CE^3 \eta^{-C_1}) |I_0|^{2/3} \\ &\leq (\sup_j |I_j|)^{1/3} C\eta^{-C_1} |I_0|^{2/3}. \end{aligned}$$

This implies the desired inequality. \square

3.5. Recursive control over unexceptional intervals. In this subsection, we reproduce an argument originally due to Bourgain [2] which shows that there exists a time $t_* \in [t_-, t_+]$ at which the intervals I_j concentrate. We follow the presentation in [31, Proposition 3.8] (see also a related treatment in [22]). Each step in the process is a consequence of the Morawetz-based control of the previous two subsections, and is expressed in the following pair of lemmas (stated separately for notational convenience).

In what follows, we set $J' = \#G$. For a collection of sets \mathcal{S} , we also use the notation $\cup \mathcal{S} = \bigcup_{S \in \mathcal{S}} S$.

Lemma 3.10. *There exist $j_-, j_+ \in \{1, 2, \dots, J\}$ so that*

- (i) $j_+ - j_- > c\eta^{C_1/2} J'$,
- (ii) $j \in G$ for $j_- \leq j \leq j_+$, and
- (iii) *there exists $j_1 \in \mathbb{N}$ with $j_- \leq j_1 \leq j_+$ so that*

$$|I_{j_1}| \geq c\eta^C \left| \bigcup_{j_- \leq j \leq j_+} I_j \right|.$$

Proof. Recall that $\#B \leq CE^{15}\eta^{-C_1} \leq C\eta^{-C_1/2}$, and partition $G = \{1, \dots, J\} \setminus B$ into a collection $\{G_1, G_2, \dots, G_m\}$ of nonempty sets of consecutive integers, with $m \leq 2C\eta^{-C_1/2}$. Then there exists $i \in \{1, \dots, m\}$ so that G_i contains at least $(4C)^{-1}\eta^{C_1/2}J'$ elements (if this were not the case, then the total number of elements in G would be at most $J'/2$). Set $j_- = \min G_i$ and $j_+ = \max G_i$ and observe that this choice satisfies (i) and (ii).

To establish (iii), we appeal to Proposition 3.9, which gives the existence of j_1 satisfying the stated conditions. This completes the proof of the lemma. \square

If $j_+ - j_-$ is larger than (a suitable multiple of) η^{-C} , this procedure can be iterated, this time removing j_1 and all intervals of comparable or longer length, rather than exceptional intervals.

Lemma 3.11. *Let $j_0, j_1, j_2 \in \{1, 2, \dots, J\}$ be given with $1 \leq j_0 < j_1 < j_2 \leq J$, and suppose $j_2 - j_0 > \tilde{C}\eta^{-C}$ and $|I_{j_1}| \geq \tilde{c}\eta^C \left| \bigcup \{I_j : j_0 \leq j \leq j_2\} \right|$ with \tilde{C} sufficiently large depending on \tilde{c} . Then there exist j_- and j_+ with $j_0 \leq j_- < j_+ \leq j_2$ so that*

- (i) $j_+ - j_- > c\eta^C(j_2 - j_0)$,
- (ii) $j_1 \notin [j_-, j_+]$ and $|I_j| \leq |I_{j_1}|/2$ for $j_- \leq j \leq j_+$, and
- (iii) *there exists $j_* \in \mathbb{N}$ with $j_- \leq j_* \leq j_+$ so that*

$$|I_{j_*}| \geq c\eta^C \left| \bigcup_{j_- \leq j \leq j_+} I_j \right|.$$

Proof. The argument is similar to the proof of the previous lemma. Define $A = \{j_0, j_0 + 1, \dots, j_2\} \setminus (\{j_1\} \cup \{j : j_0 \leq j \leq j_1 \text{ and } |I_j| \geq |I_{j_1}|/2\})$, and partition A into a collection of m nonempty sets of consecutive integers. Note that since the

removed set has size bounded by $1 + C\eta^{-C} \leq 2C\eta^{-C}$ (if the interval $I' = \cup\{I_j : j_0 \leq j \leq j_2\}$ contained more than $2c^{-1}\eta^{-C}$ disjoint intervals of length at least $|I_{j_1}|/2 \geq c\eta^C|I'|/2$, this would imply $|I'| \geq 2|I'|$, a contradiction), one can form such a partition with

$$m \leq 2C\eta^{-C}.$$

One can now choose $c > 0$ so that at least one of these sets contains more than $c\eta^C(j_2 - j_0)$ elements (otherwise the total number of elements in A is bounded by a multiple of $2Cc(j_2 - j_0)$, while $\#A \geq (j_2 - j_0) - 4\tilde{c}^{-1}\eta^{-C} \geq (1 - 4(\tilde{C}\tilde{c})^{-1})(j_2 - j_0)$, which is a contradiction for c large). Denote the set chosen this way by A' . The claim now follows by choosing $j_- = \min A'$ and $j_+ = \max A'$, and applying Proposition 3.9 to obtain the existence of j_* as in (iii). \square

Applying Lemma 3.10 and subsequently repeatedly applying Lemma 3.11 a finite number of times yields the following proposition (which is the analogue in our setting of Proposition 3.8 in [31]).

Proposition 3.12. *There exist constants $c, C > 0$ and values $K > c\eta^C \log(J')$ and $t_* \in [t_-, t_+]$ so that one can select K distinct unexceptional intervals I_{j_1}, \dots, I_{j_K} , with*

$$|I_{j_1}| \geq 2|I_{j_2}| \geq \dots \geq 2^{K-1}|I_{j_K}|,$$

and, for $1 \leq k \leq K$, $\text{dist}(t_*, I_{j_k}) \leq C\eta^{-C}|I_{j_k}|$.

Proof. First, apply Lemma 3.10 to find j_-, j_+ and j_1 satisfying the conditions given by the statement of the lemma. Then for all $t \in \tilde{I}_1 := \cup\{I_j : j_- \leq j \leq j_+\}$ we have

$$\text{dist}(t, I_{j_1}) \leq |\tilde{I}_1| \leq c^{-1}\eta^{-C}|I_{j_1}|.$$

Let \tilde{C} be a constant admissible for the statement of Lemma 3.11 where \tilde{c} is chosen as c in (iii) of Lemma 3.10. If $j_+ - j_- \leq \tilde{C}\eta^{-C}$, we can set $K = 1$ and choose any $t_* \in I_{j_1}$ to obtain the conclusion.

Alternatively, if $j_+ - j_- > \tilde{C}\eta^{-C}$ then we can apply Lemma 3.11 with $j_0 = j_-$ and $j_2 = j_+$ to find new values of j_- and j_+ satisfying the conditions stated in that lemma. For all $t \in \tilde{I}_2 := \cup\{I_j : j_- \leq j \leq j_+\} \subset \tilde{I}_1$ we have $\text{dist}(t, I_{j_2}) \leq |\tilde{I}_2| \leq c^{-1}\eta^{-C}|I_{j_2}|$. If $j_+ - j_- \leq \tilde{C}\eta^{-C}$, the conclusion follows by setting $K = 2$. Otherwise, we repeats the argument with another application of Lemma 3.11. This procedure can be iterated $K \gtrsim \eta^C \log J'$ times, until one of the constructed pairs (j_-, j_+) satisfies $j_+ - j_- \leq \tilde{C}\eta^{-C}$, or until no more intervals are left to remove. \square

3.6. Conclusion of the argument. We now complete the proof of Proposition 1.2. Apply Proposition 3.12 to choose t_* , K , and I_{j_1}, \dots, I_{j_K} satisfying the stated properties. Then, for each $k = 1, \dots, K$, we have, by Corollary 3.8,

$$M(u(t); 0, C\eta^{-C}|I_{j_k}|^{1/2}) \geq c\eta^C|I_{j_k}|^{7/12}.$$

for all $t \in I_{j_k}$. This gives, with $t \in I_{j_k}$,

$$M(u(t_*); 0, C\eta^{-C}|I_{j_k}|^{1/2})$$

$$\begin{aligned}
&= M(u(t); 0, C\eta^{-C}|I_{j_k}|^{1/2}) - \int_t^{t_*} \partial_s M(u(s); 0, C\eta^{-C}|I_{j_k}|^{1/2}) ds \\
&\geq c\eta^C |I_{j_k}|^{7/12} - (|I_{j_k}| + \text{dist}(t_*, I_{j_k})) C\eta^{5C/6} |I_{j_k}|^{-5/12} E \\
&\geq (c\eta^C - C\eta^{5C/6} E) |I_{j_k}|^{7/12} \\
&\geq c\eta^C |I_{j_k}|^{7/12}
\end{aligned}$$

for suitable choices of the constants, while the mass bound (6) gives

$$M(u(t_*); 0, 2C\eta^{-C}|I_{j_k}|^{1/2}) \lesssim \eta^{-7C/6} |I_{j_k}|^{7/12} E \lesssim \eta^{-C} |I_{j_k}|^{7/12}.$$

Fix $1 \leq k \leq K$. Now, setting $B_k = B(0, C\eta^{-C}|I_{j_k}|^{1/2})$ and fixing $N \geq 1$, it follows that

$$\begin{aligned}
&\int_{B_k \setminus \cup\{B_\ell: k+N \leq \ell \leq K\}} |u(t_*)|^2 dx \\
&\geq \int_{B_k} |u(t_*)|^2 dx - \sum_{\ell=k+N}^K \int_{B_\ell} |u(t_*)|^2 dx \\
&\geq M(u(t_*); 0, C\eta^{-C}|I_{j_k}|^{1/2})^2 - \sum_{\ell=k+N}^K M(u(t_*); 0, 2C\eta^{-C}|I_{j_\ell}|^{1/2})^2 \\
&\geq c\eta^C |I_{j_k}|^{7/6} - \sum_{\ell=k+N}^K C\eta^{-C} |I_{j_\ell}|^{7/6} \\
&\geq c\eta^C |I_{j_k}|^{7/6},
\end{aligned}$$

provided $N = C \log(\eta^{-1})$ with C sufficiently large, where we have used the construction of the sequence (I_{j_k}) to obtain the inequality $\sum_{\ell=k+N}^K |I_{j_\ell}|^{7/6} \leq 2^{-7N/12} |I_{j_k}|^{7/6}$.

This in turn gives

$$\begin{aligned}
c\eta^C |I_{j_k}|^{7/6} &\leq C\eta^{-7C/3} |I_{j_{k+N}}|^{7/6} \int_{B_k \setminus \cup\{B_\ell: k+N \leq \ell \leq K\}} \frac{|u(t_*, x)|^2}{|x|^{7/3}} dx \\
&\lesssim \eta^{-7C/3} |I_{j_k}|^{7/6} \int_{B_k \setminus \cup\{B_\ell: k+N \leq \ell \leq K\}} \frac{|u(t_*, x)|^2}{|x|^{7/3}} dx
\end{aligned}$$

where in the last inequality we have used the bound $|I_{j_{k+N}}| \leq |I_{j_k}|$.

We now sum over values of k in the set $\{k : 1 \leq k \leq K \text{ and } k = 1 + mN \text{ for some } m \geq 0\}$. Since this set has size comparable to K/N , we obtain (via the Hardy inequality $\|u/|x|^\alpha\|_{L^2} \lesssim \|\nabla|^\alpha u\|_{L^2}$ on \mathbb{R}^3 for $0 \leq \alpha < 3/2$)

$$c\eta^C (K/N) \leq \eta^{-7C/3} \int_{B_1} \frac{|u(t_*, x)|^2}{|x|^{7/3}} dx \lesssim \eta^{-7C/3} \int_{\mathbb{R}^3} \|\nabla|^{s_c} u(t_*)\|^2 dx \lesssim \eta^{-7C/3} E^2,$$

i.e.

$$K \lesssim N\eta^{-C} \leq C\eta^{-C}.$$

Recalling the bound $K \gtrsim \eta^C \log(J')$ then gives $\log(J') \lesssim \eta^{-C}$, so that $J' \leq \exp(C\eta^{-C}) = \exp(CE^C)$. Moreover, recalling the estimate $\#B \lesssim \eta^{-C}$ given by

Remark 3.6, we have $J \leq \exp(CE^C)$. We therefore get

$$\int_{t_-}^{t_+} \int_{\mathbb{R}^3} |u(t, x)|^{15} dx dt = \sum_{j=1}^J \int_{I_j} \int_{\mathbb{R}^3} |u(t, x)|^{15} dx dt \leq 2J\eta \lesssim \exp(CE^C)$$

which is the desired uniform bound. This completes the proof of Proposition 1.2.

4. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. Fix $\delta > 0$ to be determined later in the argument. Let u be a radial solution to (1) on a time interval $I = [t_-, t_+]$.

For each interval $J \subset I_{\max}$, set

$$\begin{aligned} S(u, J) &= \|u\|_{L_t^\infty(J; \dot{H}_x^{s_c}(\mathbb{R}^3))} + \|u\|_{L_t^\infty(J; \dot{H}_x^{s_c+1}(\mathbb{R}^3))} \\ &\quad + \|u\|_{L_{t,x}^{15}(J \times \mathbb{R}^3)} + \|\nabla|^{s_c} u\|_{L_{t,x}^{10/3}(J \times \mathbb{R}^3)} + \|\nabla|^{s_c+1} u\|_{L_{t,x}^{10/3}(J \times \mathbb{R}^3)}, \end{aligned}$$

and for ease of notation set $S(u, T) = S(u, [0, T])$ for each $T > 0$. Fix a parameter $R_0 > 0$ to be determined later in the argument.

We use a continuity argument to show that if R_0 is chosen to be sufficiently large then $S(u, T) \leq R_0$ for all T . Suppose that $T > 0$ is such that $S(u, T) \leq R_0$. By the local theory, we can find $T' > T$ so that $S(u, T') \leq 2R_0$. This, combined with our hypothesis on the $L_t^\infty \dot{H}_x^{s_c-\delta}$ norm, gives

$$\|u\|_{L_t^\infty([0, T']; \dot{H}_x^{s_c}(\mathbb{R}^3))} \leq \sup_{t \in [0, T']} \|u\|_{\dot{H}_x^{s_c-\delta}}^{1-\delta} \|u\|_{\dot{H}_x^{s_c+1-\delta}}^\delta \leq E_0^{1-\delta} (2R_0)^\delta,$$

and thus, as a consequence of Proposition 1.2,

$$\|u\|_{L_{t,x}^{15}([0, T'] \times \mathbb{R}^3)} \leq C \exp(CE_0^C R_0^{C\delta}), \quad (16)$$

where we have set $E_0 := E$ with $E \geq 1$ as in (3).

Now, by the Sobolev and Strichartz estimates, one has, for any $J = [t_1, t_2] \subset I$,

$$\begin{aligned} S(u, J) &\lesssim \|u(t_1)\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}} + \| |u|^6 u \|_{N(J)} + \|\nabla| [|u|^6 u] \|_{N(J)} \\ &\lesssim \|u(t_1)\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}} + \|\nabla|^{s_c} u\|_{L_{t,x}^{10/3}(J \times \mathbb{R}^3)} \|u\|_{L_{t,x}^{15}(J \times \mathbb{R}^3)}^6 \\ &\quad + \|\nabla|^{s_c+1} u\|_{L_{t,x}^{10/3}(J \times \mathbb{R}^3)} \|u\|_{L_{t,x}^{15}(J \times \mathbb{R}^3)}^6 \\ &\lesssim \|u(t_1)\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}} + 2S(u, J) \|u\|_{L_{t,x}^{15}(I \times \mathbb{R}^3)}^6 \end{aligned} \quad (17)$$

Let $\tilde{C} \geq 1$ denote the implicit constant in the inequality (17). Fix $\epsilon > 0$ and partition the interval $[0, T]$ into intervals I_1, I_2, \dots, I_m with

$$\|u\|_{L_{t,x}^{15}(I_j \times \mathbb{R}^3)}^6 = \epsilon, \quad 1 \leq j < m,$$

and

$$\|u\|_{L_{t,x}^{15}(I_m \times \mathbb{R}^3)}^6 < \epsilon.$$

Note that (16) implies

$$m \leq (C/\epsilon) \exp(CE_0^C(2R_0)^{C\delta}). \quad (18)$$

Applying (17) to each of the intervals $I_j = [t_j, t_{j+1}]$, $1 \leq j \leq m$ now gives $S(u, I_1) \leq 2\tilde{C}\|u(0)\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}}$ and

$$S(u, I_j) \leq 2\tilde{C}\|u(t_j)\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}} \leq 2\tilde{C}S(u, I_{j-1})$$

for all $1 < j \leq m$, provided $\epsilon \leq (4\tilde{C})^{-1}$. We therefore get $S(u, I_j) \leq (2\tilde{C})^{j-1}S(u, I_1)$ for $1 < j \leq m$, which in turn (combined with the above) gives

$$S(u, I_j) \leq (2\tilde{C})^j\|u(0)\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}}$$

for all $1 \leq j \leq m$.

Putting this together, we get

$$\begin{aligned} S(u, T') &\leq (2\tilde{C})^m\|u_0\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}} + \left(\sum_{j=1}^m \|u\|_{L_{t,x}^{15}(I_j \times \mathbb{R}^3)}^{15} \right)^{1/15} \\ &\quad + \left(\sum_{j=1}^m \|\nabla|^{s_c} u\|_{L_{t,x}^{10/3}(I_j \times \mathbb{R}^3)}^{10/3} \right)^{3/10} + \left(\sum_{j=1}^m \|\nabla|^{s_c+1} u\|_{L_{t,x}^{10/3}(I_j \times \mathbb{R}^3)}^{10/3} \right)^{3/10} \\ &\leq 4C'(2\tilde{C})^m\|u(0)\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}} \end{aligned}$$

for some $C' = C'(C) > 0$.

Now, choose

$$R_0 \geq 4C'(2\tilde{C})^{(C/\epsilon) \exp(2CE_0^C)}\|u_0\|_{\dot{H}_x^{s_c} \cap \dot{H}_x^{s_c+1}}.$$

and $\delta_0 > 0$ small enough to ensure

$$(2R_0)^{C\delta} \leq 2.$$

Recalling (18), we therefore obtain $S(u, T') \leq R_0$. Since $T > 0$ was an arbitrary value for which $S(u, T) \leq R_0$, this establishes the desired uniform bound. \square

Remark. A close examination of the proof shows that the regularity threshold $s_c + 1$ in the statement of Theorem 1.1 can be relaxed to $s_c + \epsilon$, as given in the statement below (with δ now dependent on ϵ).

Corollary 4.1. *There exists $C > 0$ such that for each $E \geq 1$, $M > 0$ and $\epsilon > 0$ there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and $0 \in J \subset \mathbb{R}$, if $u \in C_t(J; \dot{H}_x^{s_c-\delta}(\mathbb{R}^3) \cap \dot{H}_x^{s_c}(\mathbb{R}^3))$ is a radially symmetric solution to (1) which satisfies*

$$\|u_0\|_{\dot{H}_x^{s_c}(\mathbb{R}^3) \cap \dot{H}_x^{s_c+\epsilon}(\mathbb{R}^3)} \leq M,$$

and,

$$\|u\|_{L_t^\infty(I_{max}; \dot{H}_x^{s_c-\delta}(\mathbb{R}^3))} \leq E,$$

then $I_{max} = \mathbb{R}$ and

$$\|u\|_{L_{t,x}^{15}(\mathbb{R} \times \mathbb{R}^3)} \leq C \exp(C(EM^\delta)^C).$$

We remark that elementary scaling considerations show that the restriction $\epsilon > 0$ is essential, at least for a statement in this form.

5. PROOF OF COROLLARY 1.3: ALLOWING FOR GROWTH

In this section we complete the proof of Corollary 1.3.

Proof of Corollary 1.3. For each interval $J \subset I_{\max}$, set

$$T(u, J) = \|u\|_{L_t^\infty(J; \dot{H}_x^{s_c}(\mathbb{R}^3))} + \|u\|_{L_{t,x}^{15}(J \times \mathbb{R}^3)} + \| |\nabla|^{s_c} u \|_{L_{t,x}^{10/3}(J \times \mathbb{R}^3)},$$

and for ease of notation set $T(u, t_+) = T(u, [0, t_+])$ for each $t_+ > 0$. Fix a parameter $M_0 > 0$ to be determined later in the argument. As in the proof of Theorem 1.1, we use a continuity argument to show that if M_0 is sufficiently large then $T(u, t_+) \leq M_0$ for all t_+ .

Suppose that $t_+ > 0$ is such that $T(u, t_+) \leq M_0$. By the local theory, we can find $t' > t_+$ so that $T(u, t') \leq 2M_0$. This, combined with our hypothesis on the $L_t^\infty \dot{H}_x^{s_c}$ norm, gives

$$\|u\|_{L_t^\infty([0, t']; \dot{H}_x^{s_c}(\mathbb{R}^3))} \leq g(2M_0),$$

and thus, as a consequence of Proposition 1.2,

$$\|u\|_{L_{t,x}^{15}([0, t'] \times \mathbb{R}^3)} \leq C \exp(C[g(2M_0)]^C). \quad (19)$$

Now, by the Sobolev and Strichartz estimates, one has, for any $J = [t_1, t_2] \subset I$,

$$\begin{aligned} T(u, J) &\lesssim \|u(t_1)\|_{\dot{H}_x^{s_c}} + \| |u|^6 u \|_{N(I)} \\ &\lesssim \|u(t_1)\|_{\dot{H}_x^{s_c}} + \| |\nabla|^{s_c} u \|_{L_{t,x}^{10/3}(I \times \mathbb{R}^3)} \|u\|_{L_{t,x}^{15}(I \times \mathbb{R}^3)}^6 \\ &\lesssim \|u(t_1)\|_{\dot{H}_x^{s_c}} + T(u, J) \|u\|_{L_{t,x}^{15}(I \times \mathbb{R}^3)}^6 \end{aligned} \quad (20)$$

Let \tilde{C} denote the implicit constant in the inequality (20). Fix $\epsilon > 0$ and partition the interval $[0, t']$ into intervals I_1, I_2, \dots, I_m with

$$\|u\|_{L_{t,x}^{15}(I_j \times \mathbb{R}^3)}^6 = \epsilon, \quad 1 \leq j < m,$$

and

$$\|u\|_{L_{t,x}^{15}(I_m \times \mathbb{R}^3)}^6 < \epsilon.$$

Note that (19) implies

$$m \leq C \exp(C[g(2M_0)]^C) / \epsilon \leq (C/\epsilon) \log^{1/2}(2M_0). \quad (21)$$

Applying (20) to each of the intervals $I_j = [t_j, t_{j+1}]$, $1 \leq j \leq m$ now gives $T(u, I_1) \leq 2\tilde{C}\|u(0)\|_{\dot{H}_x^{s_c}}$ and

$$T(u, I_j) \leq 2\tilde{C}\|u(t_j)\|_{\dot{H}_x^{s_c}} \leq 2\tilde{C}\|u\|_{L_t^\infty(I_{j-1}; \dot{H}_x^{s_c})} \leq 2\tilde{C}T(u, I_{j-1})$$

for all $1 < j \leq m$, provided $\epsilon \leq (2\tilde{C})^{-1}$ (in what follows, we make the choice $\epsilon = (2\tilde{C})^{-1}$). We therefore get $T(u, I_j) \leq (2\tilde{C})^{j-1}T(u, I_1)$ for $1 < j \leq m$, which in turn (combined with the above) gives

$$T(u, I_j) \leq (2\tilde{C})^j \|u(0)\|_{\dot{H}_x^{s_c}}$$

for all $1 \leq j \leq m$.

Putting this together, we get

$$\begin{aligned} T(u, t') &\leq (2\tilde{C})^m \|u_0\|_{\dot{H}_x^{s_c}} + \left(\sum_{j=1}^m \|u\|_{L_{t,x}^{15}(I_j \times \mathbb{R}^3)}^{15} \right)^{1/15} \\ &\quad + \left(\sum_{j=1}^m \|\nabla|^{s_c} u\|_{L_{t,x}^{10/3}(I_j \times \mathbb{R}^3)}^{10/3} \right)^{3/10} \\ &\leq 3C'(2\tilde{C})^m \|u(0)\|_{\dot{H}_x^{s_c}} \end{aligned}$$

for some $C' = C'(C) > 0$.

To conclude, recall (21) and note that choosing M_0 large enough to ensure $(C/\epsilon) \log^{1/2}(2M_0) = 2C\tilde{C} \log^{1/2}(2M_0) \leq \log(M_0/(3C'\|u(0)\|_{\dot{H}_x^{s_c}}))/\log(2C)$ therefore gives

$$3C'(2\tilde{C})^m \|u(0)\|_{\dot{H}_x^{s_c}} \leq M_0,$$

so that $T(u, t') \leq M_0$ (to see that this choice of M_0 is possible, note that for every $\lambda > 0$, $\log^{1/2}(t)/\log(t/\lambda) \rightarrow 0$ as $t \rightarrow \infty$).

Since $t_+ > 0$ was an arbitrary value for which $T(u, t_+) \leq M_0$, this establishes the desired uniform bound. \square

REFERENCES

- [1] M. Beceanu, Q. Deng, A. Soffer and Y. Wu. Large global solutions for nonlinear Schrödinger equations III, energy-supercritical cases. Preprint (2019). arXiv:1901.07709.
- [2] J. Bourgain. Global well-posedness of defocusing 3D critical NLS in the radial case. J. Amer. Math. Soc. 12 (1999), no. 1, pp. 145–171.
- [3] A. Bulut. Global well-posedness and scattering for the defocusing energy-supercritical cubic nonlinear wave equation. J. Funct. Anal. 263 (2012), 1609–1660.
- [4] A. Bulut. The radial defocusing energy-supercritical cubic nonlinear wave equation in \mathbb{R}^{1+5} . Nonlinearity 27 (2014), no. 8, 1859–1877.
- [5] A. Bulut. The defocusing energy-supercritical cubic nonlinear wave equation in dimension five. Trans. Amer. Math. Soc. 367 (2015), no. 9, 6017–6061.
- [6] A. Bulut and B. Dodson. Global well-posedness for the logarithmically energy-supercritical nonlinear wave equation with partial symmetry. Int. Math. Res. Not. (2021), no. 8, 5943–5967.
- [7] A. Bulut and H. Dong. Nonlinear instability for the surface quasi-geostrophic equation in the supercritical regime. Comm. Math. Phys. 384 (2021), 1679–1707.
- [8] T. Cazenave. Semilinear Schrödinger Equations. Courant Lecture Notes in Mathematics, 10. American Mathematical Society, Providence, RI, 2003.
- [9] C. Collot. Type II blow up manifolds for the energy supercritical semilinear wave equation, Mem. Amer. Math. Soc. 252 (2018), no. 1205.

- [10] M. Colombo and S. Haffter. Global regularity for the hyperdissipative Navier-Stokes equation below the critical order. *J. Diff. Eq.* 275 (2021), 815–836.
- [11] M. Colombo and S. Haffter. Global regularity for the nonlinear wave equation with slightly supercritical power. Preprint (2019), arXiv:1911.02599.
- [12] M. Coti Zelati and V. Vicol. On the global regularity for the supercritical SQG equation. *Indiana Univ. Math. J.*, 65 (2016), pp. 535–552.
- [13] R. Donninger and B. Schörkhuber. On blowup in supercritical wave equations, *Comm. Math. Phys.* 346 (2016), no. 3, pp. 907–943.
- [14] C. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.* 166 (2006), no. 3, pp. 645–675.
- [15] C. Kenig and F. Merle. Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. *Acta. Math.* 201 (2008), no. 2, pp. 147–212.
- [16] C. Kenig and F. Merle. Nondispersive radial solutions to energy supercritical non-linear wave equations, with applications. *Amer. J. Math.* 133 (2011), no. 4, pp. 1029–1065.
- [17] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. *Amer. J. Math.* 132 (2009), pp. 361–424.
- [18] R. Killip and M. Visan. Energy-supercritical NLS: critical H^s -bounds imply scattering. *Comm. PDE.* 35 (2010), pp. 945–987.
- [19] R. Killip and M. Visan. The defocusing energy-supercritical nonlinear wave equation in three space dimensions. *Trans. Amer. Math. Soc.* 363 (2011), 3893–3934.
- [20] R. Killip and M. Visan. The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions. *Proc. Amer. Math. Soc.* 139 (2011), 1805–1817.
- [21] R. Killip and M. Visan. *Nonlinear Schrödinger Equations at Critical Regularity.* Evolution Equations, Clay Math. Proc. 17, Amer. Math. Soc., Providence, RI, 2013, pp. 325–437.
- [22] R. Killip, M. Visan, and X. Zhang. Energy-critical NLS with quadratic potentials. *Comm. PDE* 34 (2009), no. 10–12, pp. 1531–1565.
- [23] F. Merle, P. Raphaël and I. Rodnianski. Type II blow up for the energy supercritical NLS, *Camb. J. Math.* 3 (2015), no. 4, pp. 439–617.
- [24] F. Merle, P. Raphaël, I. Rodnianski, and J. Szeftel. On blow up for the energy super critical defocusing nonlinear Schrödinger equations. Preprint (2019), arXiv:1912.11005.
- [25] C. Miao, J. Murphy and J. Zheng. The defocusing energy-supercritical NLS in four space dimensions. *J. Funct. Anal.* 267 (2014), no. 6, pp. 1662–1724.
- [26] J. Murphy. The radial defocusing nonlinear Schrödinger equation in three space dimensions. *Comm. PDE* 40 (2015), no. 2, pp. 265–308.
- [27] T. Roy. Global existence of smooth solutions of a 3D log-log energy-supercritical wave equation. *Anal. PDE* 2 (2009), no. 3, pp. 261–280.
- [28] T. Roy. Scattering above energy norm of solutions of a loglog energy-supercritical Schrödinger equation with radial data. *J. Diff. Eq.* 250 (2011), no. 1, pp. 292–319. Corrigendum in *J. Diff. Eq.* 264 (2018), no. 9, pp. 6013–6024.
- [29] T. Roy. A remark on barely \dot{H}^{s_p} -supercritical wave equations. *Anal. and PDE* 5 (2012), no. 1, pp. 199–218.
- [30] H.S. Shih. Some results on scattering for log-subcritical and log-supercritical nonlinear wave equations. *Anal. PDE* 6 (2013), no. 1, pp. 1–24.
- [31] T. Tao. Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data. *New York J. Math.* 11 (2005), pp. 57–80.
- [32] T. Tao. Global regularity for a logarithmically supercritical defocusing nonlinear wave equation for spherically symmetric data. *J. Hyp. Diff. Eq.* 4 (2007), no. 2, pp. 259–265.
- [33] T. Tao. Finite-time blowup for a supercritical defocusing nonlinear wave system, *Anal. PDE* 9 (2016) no. 8, pp. 1999–2030.
- [34] T. Tao. Finite time blowup for a supercritical defocusing nonlinear Schrödinger system, *Anal. PDE* 11 (2018), no. 2, pp. 383–438.
- [35] T. Tao and M. Visan. Stability of energy-critical nonlinear Schrödinger equations in high dimensions. *Electron. J. Diff. Eqns.* 2005 (2005), pp. 1–28.
- [36] M. Visan. The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. *Duke Math. J.* 138 (2007), pp. 281–374.

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