STABLE SOLUTIONS TO THE FRACTIONAL ALLEN-CAHN EQUATION IN THE NONLOCAL PERIMETER REGIME

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Abstract. We study stable solutions to the fractional Allen-Cahn equation $(-\Delta)^{s/2}u = u - u^3$, $|u| < 1$ in $\mathbb{R}^n$. For every $s \in (0,1)$ and dimension $n \geq 2$, we establish sharp energy estimates, density estimates, and the convergence of blow-downs to stable nonlocal s-minimal cones. As a consequence, we obtain a new classification result: if for some pair (n,s), with $n \geq 3$, hyperplanes are the only stable nonlocal s-minimal cones in $\mathbb{R}^n \setminus \{0\}$, then every stable solution to the fractional Allen-Cahn equation in $\mathbb{R}^n$ is 1D, namely, its level sets are parallel hyperplanes.

Combining this result with the classification of stable s-minimal cones in $\mathbb{R}^3 \setminus \{0\}$ for $s \sim 1$ obtained by the authors in a recent paper, we give positive answers to the "stability conjecture" in $\mathbb{R}^3$ and to the "De Giorgi conjecture" in $\mathbb{R}^4$ for the fractional Allen-Cahn equation when the order $s \in (0,1)$ of the operator is sufficiently close to 1.

Contents

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1. Introduction

1.1. **Two classical models in phase transitions vs. minimal surfaces.** The Peierls-Nabarro model was introduced in the early 1940s in the context of crystal dislocations [52, 51] and also arises in the study of phase transitions with line-tension effects [2] and boundary vortices in thin magnetic films [49]. This model concerns the energy functional

$$I_\varepsilon(u) := \frac{\varepsilon}{4}[u]^2_{H^{1/2}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} W(u)\,dx, \qquad [u]^2_{H^{1/2}(\mathbb{R}^n)} := \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big|u(x)-u(\bar x)\big|^2}{|x-\bar x|^{n+1}}\,dx\,d\bar x,$$

where $u : \mathbb{R}^n \to (-1,1)$ and $W(u) := 1 + \cos(\pi u)$.

The very related Allen-Cahn functional, introduced later, in 1958, within the context of the Van Der Walls-Cahn-Hilliard theory for phase transitions in fluids [24], is also tightly connected to the Ginzburg-Landau theory of superconductivity. It is defined as

$$J_\varepsilon(u) := \frac{\varepsilon^2}{2}[u]^2_{H^1(\mathbb{R}^n)} + \int_{\mathbb{R}^n} W(u)\,dx, \qquad [u]^2_{H^1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\nabla u|^2\,dx,$$

where $u : \mathbb{R}^n \to (-1,1)$ and $W(u) := \frac{1}{4}(1-u^2)^2$.

In both models $\varepsilon > 0$ is a parameter and $W(u)$ is a so-called *double-well potential*, namely, a function with two minima (or wells) at the values $u = -1$ and $u = +1$ which correspond to two "stable phases".

Critical points $u \in C^2(\mathbb{R}^n)$ of I_ε and J_ε (more precisely, of their localized versions presented later in Subsection 1.6) solve,[1] respectively, the Peierls-Nabarro and Allen-Cahn equations:

$$\varepsilon(-\Delta)^{1/2}u + W'(u) = 0 \qquad \text{and} \qquad \varepsilon^2(-\Delta)u + W'(u) = 0.$$

A deep link between any of the two models and minimal surfaces is found when investigating the asymptotic behavior of sequences of minimizers of I_ε and J_ε as $\varepsilon \downarrow 0$. Indeed, as a consequence of Γ-convergence results of Alberti, Bouchitté, and Seppecher [2] and of Modica and Mortola [50], respectively, the following holds:

[1]Up to modifying the multiplicative constants in the definitions of $[\,\cdot\,]_{H^s}$ and $(-\Delta)^s$ in order to make them consistent.

If $u_{\varepsilon_k} : \mathbb{R}^n \to \mathbb{R}$ is a sequence of minimizers (in every bounded set) of either I_{ε_k} or J_{ε_k} and $\varepsilon_k \downarrow 0$, then (up to a subsequence)

$$u_{\varepsilon_k} \xrightarrow{L^1_{\mathrm{loc}}} \chi_E - \chi_{\mathbb{R}^n \setminus E}, \qquad \textit{where } E \subset \mathbb{R}^n \textit{ is a minimizer of the perimeter.} \tag{1.1}$$

In other words, for every $\lambda \in (-1, 1)$ the level sets $\{u_\varepsilon = \lambda\}$ converge, as $\varepsilon \downarrow 0$ and up to a subsequence, to an area minimizing hypersurface (in particular minimal, i.e., with zero mean curvature).

1.2. **The fractional Allen-Cahn energies.** The classical functionals I_ε and J_ε introduced above belong to the more general family of Allen-Cahn energies

$$E_{s,\varepsilon}(u) := \frac{\varepsilon^s}{4}[u]^2_{H^{s/2}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} W(u)\,dx,$$

where $s \in (0, 2]$ and

$$[u]^2_{H^{s/2}(\mathbb{R}^n)} := (2 - s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big|u(x) - u(\bar{x})\big|^2}{|x - \bar{x}|^{n+s}}\,dx\,d\bar{x}.$$

Note that the normalization factor $2 - s$ in the definition of $[u]^2_{H^{s/2}(\mathbb{R}^n)}$ guarantees that

$$[u]^2_{H^{s/2}(\mathbb{R}^n)} \to \int_{\mathbb{R}^n} |\nabla u|^2\,dx \qquad \text{as } s \uparrow 2$$

(up to a dimensional multiplicative constant). The functionals $E_{s,\varepsilon}$ have been extensively studied in the literature; see among others [18, 60, 16, 17, 13, 55] and references therein. Note that the classical functionals I_ε and J_ε correspond to the cases $s = 1$ and $s = 2$ of $E_{s,\varepsilon}$.

Critical points $u : \mathbb{R}^n \to (-1, 1)$ of $E_{s,\varepsilon}$ (in fact, of the natural localized version of $E_{s,\varepsilon}$ given in Subsection 1.6) satisfy the fractional Allen-Cahn equation

$$\varepsilon^s(-\Delta)^{s/2}u + W'(u) = 0, \quad |u| \le 1, \tag{1.2}$$

in $\mathbb{R}^n$ (up to a positive multiplicative constant in front of $(-\Delta)^{s/2}u$). Similarly to the cases $s = 1$ and $s = 2$, for every $s \in [1, 2]$ the energies $E_{s,\varepsilon}$ (suitably localized and renormalized) Γ-converge to the classical perimeter as $\varepsilon \downarrow 0$ —see [57]. As a consequence, if u_{ε_k} is a sequence of minimizers (in every bounded set) of E_{s,ε_k} with $s \in [1, 2]$, then (1.1) holds (up to a subsequence). Interestingly, a new qualitative behaviour is found for $s \in (0, 1)$, where the asymptotic analysis of $E_{s,\varepsilon}$ as $\varepsilon \downarrow 0$ leads to the so-called *s-minimal surfaces* (or *nonlocal s-minimal surfaces*), introduced by Caffarelli, Roquejoffre, and Savin in [19]. These new geometric objects generalize classical minimal surfaces (which are recovered as a limit case when $s \uparrow 1$) and share with them several structural properties; see [19, 42, 29, 14] for the precise definition and several results on s-minimal surfaces.

1.3. **Bernstein's and De Giorgi's conjectures.** Let us recall the classical Bernstein's problem:

If the graph of a function defined in all of $\mathbb{R}^{n-1}$ is a minimal surface in $\mathbb{R}^n$, is the function necessarily linear?

This question has deep connections with the regularity theory for the (co-dimension one) Plateau problem in any dimension and, more precisely, with the optimal dimensional bounds on the singular set of minimal surfaces —as shown by De Giorgi in [32]. Moreover, its study contributed to striking developments in the theory of minimal surfaces. The question was completely answered in the late 1960's in a series of groundbreaking works, which established that:

(i) Hyperplanes are the only graphs of functions defined on $\mathbb{R}^{n-1}$ which are minimal surfaces in $\mathbb{R}^n$ as long as $n \le 8$ (Bernstein [8], Fleming [44], De Giorgi [31, 32], Almgren [3], and Simons [63]).
(ii) There exist entire minimal graphs in $\mathbb{R}^n$ which are not hyperplanes in dimensions $n \ge 9$ (Bombieri, De Giorgi, and Giusti [9]).

Bernstein's problem finds a counterpart for certain "critical points" of J_ε in a famous conjecture by De Giorgi (1978):

Conjecture 1.1 ([33])**.** *Let $u \in C^2(\mathbb{R}^n)$, $|u| < 1$, be a solution of $-\Delta u = u - u^3$ in $\mathbb{R}^n$ satisfying $\partial_{x_n} u > 0$. Then, if $n \le 8$, u must be 1D, that is, all its level sets $\{u = \lambda\}$ must be hyperplanes.*

Conjecture 1.1 was first proved, about twenty years after it was raised, in dimensions $n = 2$ and $n = 3$, by Ghoussoub and Gui [46] and Ambrosio and Cabré [4], respectively. Almost ten years later, in the celebrated paper [54], Savin adressed the conjecture in the dimensions $4 \le n \le 8$, and he succeeded in proving it under the additional assumption

$$\lim_{x_n \to \pm\infty} u = \pm 1. \tag{1.3}$$

Short after, Del Pino, Kowalczyk, and Wei [34] established the existence of a counterexample to the conjecture in dimensions $n \ge 9$.

Since for $s \in [1, 2)$ the functionals J_ε and $E_{s,\varepsilon}$ have the identical asymptotic behaviour (1.1), there are no heuristic reasons to prefer J_ε to $E_{s,\varepsilon}$ when $s \in [1, 2)$, in the statement of the De Giorgi conjecture. This motivates

Conjecture 1.2. *Conjecture 1.1 also holds for $(-\Delta)^{s/2} u = u - u^3$ when $s \in [1, 2)$.*

On the other hand, for nonlocal s-minimal surfaces the analogue of the Bernstein's problem is well understood for $s \in (0, 1)$ sufficiently close to 1. Indeed, it follows, by combining the results in [19], [23], [58], and [42] that:

For $n \le 8$, there is a dimensional constant $s_* \in [0, 1)$ such that, if $s \in (s_*, 1)$, hyperplanes are the only graphs $\{x_n = \phi(x_1, \ldots, x_{n-1})\}$ which are s-minimal surfaces in $\mathbb{R}^n$ —for $n = 2$ and $n = 3$ this is known with $s_* = 0$.

The heuristics thus suggest that the De Giorgi conjecture should be true for monotone critical points of $E_{\varepsilon,s}$ also in the range $s \in (s_*, 1)$. That is:

Conjecture 1.3. *Conjecture 1.1 also holds for* $(-\Delta)^{s/2}u = u - u^3$ *when* $s \in (s_*, 1)$*, where* $s_* \in [0, 1)$ *is a dimensional constant.*

The articles by Cabré and Solà-Morales [18], Cabré and Sire [17], and Sire and Valdinoci [64] proved Conjectures 1.2 and 1.3 in dimension $n = 2$, for s in the whole range $(0, 2)$ —that is $s_* = 0$. Later, Cabré and Cinti [12, 13] established Conjecture 1.2 for $n = 3$, $s \in [1, 2)$. In [36], Dipierro, Farina, and Valdinoci proved Conjecture 1.3 for $n = 3$ and $s_* = 0$. Very recently, Figalli and Serra [41] have proved it for $n = 4$, $s = 1$. For all $s \in (1, 2)$, the existence of monotone solutions to (1.2) which are not $1D$ in dimensions $n \geq 9$ has been announced in [27] —to appear in a work by Chan, Dávila, del Pino, Liu, and Wei [26].

In this paper we will prove Conjecture 1.3 in dimension $n = 4$ for $s \in (s_*, 1)$, for some $s_* \in (0, 1)$. We will also give a sufficient condition (in terms of a rigidity statement on s-minimal cones) guaranteeing that Conjecture 1.3 holds for some pair (n, s), with $s \in (0, 1)$ and $n \geq 3$.

Under the additional assumption (1.3), Conjecture 1.2 has been proved by Savin [55, 56] for $4 \leq n \leq 8$, $s \in [1, 2)$. Finally, Conjectures 1.2 and 1.3 have been proven, also under the additional assumption (1.3), in Dipierro, Serra, and Valdinoci [37] in two situations: for $n = 3$ and $s_* = 0$, as well as for $4 \leq n \leq 8$ and $s \in (s_*, 1)$ with $s_* < 1$ is sufficiently close to 1.

1.4. **Monotone vs. stable solutions.** The assumption $\partial_{x_n} u > 0$ easily yields that u is a *stable solution*, i.e., a critical point of the localized version of $E_{s,\varepsilon}$ (presented below in Subsection 1.6) with nonnegative second variation. The additional assumption (1.3) on limits at infinity is only used in [54, 55, 56, 37] to guarantee that u is, in addition, a minimizer of $E_{s,\varepsilon}$ in every bounded set of $\mathbb{R}^n$. To address the conjecture without the additional assumption (1.3), it is natural to introduce the two limit functions $u^{\pm} := \lim_{x_n \to \pm\infty} u$. These functions depend only on the first $n-1$ variables $x_1, \dots, x_{n-1}$ and are stable solutions of (1.2) in $\mathbb{R}^{n-1}$. It is not difficult to prove (see Proposition 7.9), that if $u^{\pm}$ are 1D then u is a minimizer. As a consequence, the following implication holds for all $s \in (0, 2]$:

$$\left.\begin{array}{c}\text{Stable sol'ns of (1.2) in } \mathbb{R}^{n-1} \text{ are 1D} \\ \text{and} \\ \text{Minimizers of (1.2) in } \mathbb{R}^{n} \text{ are 1D}\end{array}\right\} \Rightarrow \text{Monotone sol'ns of (1.2) in } \mathbb{R}^n \text{ are 1D.}$$

The difficult problem of classifying stable solutions to (1.2) is connected to the following well-known conjecture for minimal surfaces:

Conjecture 1.4. *Stable embedded minimal hypersurfaces in* $\mathbb{R}^n$ *are hyperplanes as long as* $n \leq 7$*.*

A positive answer to this conjecture is known for $n = 3$, by a result of Fischer-Colbrie and Schoen [43] and of Do Carmo and Peng [38] from the late seventies, and for $n = 4$ by a very recent paper of Chodosh and Li [28]. It remains open for $5 \leq n \leq 7$. Instead, the analogue of Conjecture 1.4 for area-minimizing (a stronger notion than stability) hypersurfaces is completely understood in every dimension: it holds, indeed, if and only if $n \leq 7$. This shows that extending a result for minimizers to stable solutions may be a very difficult problem. Still, Conjecture 1.4 suggests the so-called "stability conjecture":

Conjecture 1.5. *Let $u \in C^2(\mathbb{R}^n)$, $|u| < 1$, be a stable critical point of $E_{s,\varepsilon}$. Then, if $n \leq 7$, u must be 1D provided $s \in (s_*, 2]$, where $s_* \in [0, 1)$ is a dimensional constant.*

As explained before, once Conjecture 1.5 is known to hold for some $(n-1, s)$ then Conjecture 1.1 / 1.2 / 1.3 for (n, s) can be reduced to the question of proving 1D symmetry of minimizers, which is well-understood thanks to the results in [54, 55, 56, 37].

With the exception of the case $n = 3$ and $s = 1$ considered in [41], before our work Conjecture 1.5 was open in every dimension $n \geq 3$, with the case of the Laplacian ($s = 2$) being a long standing open problem. The case $n = 2$ is simpler (thanks to a certain "parabolic manifold" type property, it can be proved through some Liouville theorems initiated in [7, 4]) and has been established for the whole range $s \in (0, 2]$: for $s = 2$ in [46, 4, 1], for $s = 1$ in [18], and for $0 < s < 2$ in [17, 64].

In this paper we establish Conjecture 1.5 when $n = 3$ and $s \in (s_*, 1)$, for some $s_* < 1$. From this and a result for minimizers from [37], we deduce Conjecture 1.3 when $n = 4$ and $s \in (s_*, 1)$.

1.5. **Main result: a new classification theorem.** We study stable solutions of the fractional Allen-Cahn equation

$$(-\Delta)^{s/2} u + W'(u) = 0, \quad |u| < 1 \quad \text{in } \mathbb{R}^n, \tag{1.4}$$

with $s \in (0, 1)$, where

$$W(u) := \frac{1}{4}(1 - u^2)^2 \tag{1.5}$$

is the standard quartic double-well potential with wells at ± 1. Note that this is equation (1.2) with $\varepsilon = 1$ (we can always assume this value of the parameter after scaling). For the precise definition of *stable solution* to (1.4), see Subsection 1.6 below.

The main goal of the paper is to establish the following classification result. For brevity, we use in its statement the following terminology. Recall that, as mentioned before, stable solutions to the fractional Allen-Cahn equation in all of $\mathbb{R}^2$ have been already classified.

Definition 1.6. For $n \geq 3$, we say that *hyperplanes are the only stable s-minimal cones in $\mathbb{R}^n \setminus \{0\}$* when the following holds: if $3 \leq m \leq n$ and $\Sigma \subset \mathbb{R}^m$ is a stable s-minimal cone in $\mathbb{R}^m \setminus \{0\}$, whose boundary $\partial\Sigma$ is nonempty and smooth away from 0, then necessarily $\partial\Sigma$ is a hyperplane (up to sets of measure zero).

The notion of (smooth away from 0) stable s-minimal cone in $\mathbb{R}^m \setminus \{0\}$ is exactly that of [14, Definition 1.1], which we recall in Definition 2.8 below.[2]

[2] In this paper and in [14] the perturbations to define stability do not modify the cone in a neighborhood of the origin (even if in [14] we used the terminology "stable s-minimal cones in $\mathbb{R}^m$").

Remark 1.7. Let us comment on Definition 1.6, which will be an assumption of our main result, Theorem 1.8 below.

- First, we will not be assuming that stable s-minimal cones in $\mathbb{R}^2 \setminus \{0\}$ must be flat: the integer m in the definition is at least 3. This is important because (similarly as it happens in the classical case $s = 1$) certain s-minimal symmetric cones like the "cross" $\{x_1 x_2 > 0\}$ are stable in $\mathbb{R}^2 \setminus \{0\}$, at least if s is sufficiently close to 1.
- Second, the reason why, given a dimension n we need an hypothesis in lower dimensions $3 \le m \le n$, is the use of a dimension reduction argument of Federer type in the proof of Theorem 1.8. This is also why Definition 1.6 concerns only cones with smooth trace on the sphere (the dimension reduction argument will always allow us to suppose this). Now, in this dimension reduction, one needs to consider the case $m = 2$ (and not only $m \ge 3$). However, for the case $m = 2$ we do not need to assume any flatness hypothesis since we will actually prove in Lemma 7.7 the following property, which holds true for all $s \in (0,1)$: any cone in $\mathbb{R}^n$ of the form $\tilde{\Sigma} \times \mathbb{R}^{n-2}$ and such that $\chi_\Sigma - \chi_{\Sigma^c}$ can be obtained as limit of some sequence of stable critical points of $E_{s,\varepsilon}$ (with vanishing parameter ε) must be a half-space.

The following is our main result.

Theorem 1.8. *Assume that, for some pair (n, s) with $n \ge 3$ and $s \in (0,1)$, hyperplanes are the only stable s-minimal cones in $\mathbb{R}^n \setminus \{0\}$.*

Then, every stable solution of (1.4)-(1.5) is a 1D layer solution, namely, $u(x) = \phi(e \cdot x)$ for some direction $e \in S^{n-1}$ and increasing function $\phi : \mathbb{R} \to (-1,1)$.

Remark 1.9. As it will be clear from the proofs, throughtout the paper we do not need W to be given exactly by $W(u) = \frac{1}{4}(1-u^2)^2$, nor to be even. It can be replaced everywhere by any function $W \in C^3([-1,1])$ satisfying $W(\pm 1) = 0$, $W > 0$ in $(-1,1)$, $W \le W(t_0)$ in $(-1,1)$ for some $t_0 \in (-1,1)$, $\{t \in [-1,1] : W'(t) = 0\} = \{-1, t_0, 1\}$, $W''(\pm 1) > 0$, and $W''(t_0) < 0$. For instance, W can be replaced everywhere by the Peierls-Nabarro potential $1 + \cos(\pi u)$. However, for simplicity, we will write the results for the potential (1.5).

Note also that the results of Subsection 2.1 apply to any C^3 potential W.

Thanks to the theorem, to establish Conjecture 1.5 in $\mathbb{R}^n$, $n \ge 3$, it suffices to prove that hyperplanes are the only stable s-minimal cones in $\mathbb{R}^n \setminus \{0\}$ for $s \in (s_*, 1)$. Recently, in [14] we have proved that planes are the only stable s-minimal cones in $\mathbb{R}^3 \setminus \{0\}$ for $s \in (s_*, 1)$, for some $s_* < 1$. As a consequence of this last result, we obtain the two following corollaries. First, we can prove the stability conjecture in $\mathbb{R}^3$ for $s \in (0,1)$ sufficiently close to 1.

Corollary 1.10. *Conjecture 1.5 holds in $\mathbb{R}^3$ —that is, every stable critical point of $E_{s,\varepsilon}$ in $\mathbb{R}^3$ is 1D for $s \in (s_*, 1)$ with $s_* < 1$ sufficiently close to 1.*

Using Corollary 1.10, we establish the De Giorgi type conjecture in $\mathbb{R}^4$ for $s \in (0,1)$ sufficiently close to 1.

Corollary 1.11. *Conjecture 1.3 holds in $\mathbb{R}^4$ —that is, every monotone solution of* (1.4) *in $\mathbb{R}^4$ is 1D for $s \in (s_*, 1)$ with $s_* < 1$ sufficiently close to 1.*

We recall that in [41] Figalli and Serra proved Conjecture 1.3 in $\mathbb{R}^4$ for $s = 1$.

1.6. **Definitions: localized energy, minimizers, and stable solutions.** Throughout the paper we consider solutions $u : \mathbb{R}^n \to (-1, 1)$ to integro-differential equations

$$L_K u + W'(u) = 0 \quad \text{in } \Omega, \tag{1.6}$$

where $W \in C^3(\mathbb{R})$, $\Omega \subset \mathbb{R}^n$ is an open set, and

$$L_K u(x) := \int_{\mathbb{R}^n} \big(u(x) - u(\bar{x})\big) K(x - \bar{x})\, d\bar{x}. \tag{1.7}$$

Although in some parts of the paper we will take $L_K = (-\Delta)^{s/2}$, i.e., $K(z) = (2-s)|z|^{-n-s}$, some other parts will hold and will be stated for more general kernels K. Throughout the article, K belongs to the ellipticity class $\mathcal{L}_2$ of Caffarelli and Silvestre [22]. That is, we assume $s \in (0, 2)$,

$$K(z) = K(-z), \quad \frac{(2-s)\lambda}{|z|^{n+s}} \le K(z) \le \frac{(2-s)\Lambda}{|z|^{n+s}}, \tag{1.8}$$

and that K is of class C^2 away from the origin with derivatives satisfying

$$\max\big\{|z|\,|\partial_e K(z)|\,,\,|z|^2|\partial_{ee}K(z)|\big\} \le \frac{(2-s)\Lambda}{|z|^{n+s}}, \tag{1.9}$$

for all $z \in \mathbb{R}^n \setminus \{0\}$ and $e \in S^{n-1}$. Here λ and Λ are given positive constants.

By a solution of (1.6), we mean a bounded function in all of $\mathbb{R}^n$, which is, in Ω, a C^2 strong solution. Note that, since K is in the class $\mathcal{L}_2$, $W \in C^3(\mathbb{R})$, and L_K is translation invariant, any measurable function $u : \mathbb{R}^n \to (-1, 1)$ which solves (1.6) in the sense of distributions, i.e., $\int u L_K \xi + W'(u)\xi = 0$ for all $\xi \in C_c^\infty(\Omega)$, belongs to $C^2(\Omega)$ and hence it is a strong solution. Indeed, this follows from the existing regularity theory for nonlocal elliptic equations, as explained in Appendix C.

We now recall the standard definitions of localized energy, minimizers, and stable solution in this more general framework.

We first point out that (1.6) is the Euler-Lagrange equation of the localized energy functional

$$\mathcal{E}_\Omega(v) := \mathcal{E}_\Omega^{\mathrm{Sob}}(v) + \mathcal{E}_\Omega^{\mathrm{Pot}}(v), \tag{1.10}$$

where

$$\mathcal{E}_\Omega^{\mathrm{Sob}}(v) := \frac{1}{4} \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} |v(x)-v(\bar{x})|^2 K(x-\bar{x})\, dx\, d\bar{x}, \quad \mathcal{E}_\Omega^{\mathrm{Pot}}(v) := \int_\Omega W(v)\, dx.$$

Here and throughout the paper, Ω^c denotes the complement of Ω in $\mathbb{R}^n$. Note that $\mathcal{E}_\Omega(v)$ is finite for every function v which is bounded in $\mathbb{R}^n$ and Lipschitz in a neighborhood of $\overline{\Omega}$, since $s \in (0, 2)$.

We say that u is a *minimizer* of $\mathcal{E}_\Omega$ (or a minimizer of (1.6) in Ω) if $\mathcal{E}_\Omega(u) < \infty$ and $\mathcal{E}_\Omega(u) \le \mathcal{E}_\Omega(v)$ for all v satisfying $v = u$ outside of Ω.

If u is a minimizer of $\mathcal{E}_\Omega$, then

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{E}_\Omega(u+t\xi)\geq 0 \quad \text{for all Lipschitz function } \xi \text{ with } \xi=0 \text{ in } \Omega^c. \tag{1.11}$$

This motivates the definition of stable solution. Within this framework, u is a stable solution if and only if u is a critical point of $\mathcal{E}_\Omega$ for which the second variation of $\mathcal{E}_\Omega$ at u —with respect to compactly supported perturbations vanishing outside Ω— is nonnegative. In other words:

Definition 1.12 (Stability). Given an open set $\Omega \subset \mathbb{R}^n$, we say that a solution u of (1.6) is *stable* if (1.11) holds, or equivalently,

$$\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}|\xi(x)-\xi(\bar{x})|^2K(x-\bar{x})\,dx\,d\bar{x}+\int_{\mathbb{R}^n}W''(u)\xi^2\,dx\geq 0 \tag{1.12}$$

for all Lipschitz functions ξ in $\mathbb{R}^n$ with compact support in $\mathbb{R}^n$ and such that $\xi=0$ in Ω^c. As a particular case, we say that u is a stable solution of (1.4) if (1.12) holds with $K(z)=(2-s)|z|^{-n-s}$ for all Lipschitz functions ξ with compact support in $\mathbb{R}^n$.

We conclude this section giving the notion of fractional s-perimeter, as introduced by Caffarelli, Roquejoffre, and Savin in [19]. Let $s\in(0,1)$. Given a set E in $\mathbb{R}^n$ and a bounded open set $\Omega\subset\mathbb{R}^n$, we define the fractional s-perimeter of E in Ω as

$$P_s(E,\Omega):=\iint_{(E\cap\Omega)\times(\mathbb{R}^n\setminus E)}\frac{1}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x}+\iint_{(\Omega\setminus E)\times(E\setminus\Omega)}\frac{1}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x}. \tag{1.13}$$

Observe that, in our notation, $P_s(E,\Omega)=2\mathcal{E}^{\mathrm{Sob}}_\Omega(\chi_E)$, where χ_E denotes the characteristic function of the set E and we choose here $K(z)=|z|^{-n-s}$.

2. Further new results and organization of the paper

Theorem 1.8 will follow as a combination of several new results —which in addition are of independent interest— on stable solutions to integro-differential semilinear equations. They are valid in any dimension $n\geq 2$ and include:

- A sharp BV estimate;
- A bound of $\mathcal{E}^{\mathrm{Pot}}$ by $\mathcal{E}^{\mathrm{Sob}}$, a cube decomposition, and an energy estimate;
- A density estimate;
- The convergence of blow-downs to stable s-minimal cones.

These results are presented in detail in the next subsections.

2.1. BV estimate.

For $s\in(0,1)$, in Section 3 we establish BV and Sobolev estimates for stable solutions to the semilinear equation $L_K(u)+W'(u)=0$ in a ball B_{2R}. Their proof follows the techniques introduced in [29] within the context of stable nonlocal s-minimal surfaces. Our estimates are optimal in their dependence on the radius R and universal in the sense that they are independent of the potential W. Note that a universal BV estimate is somehow surprising since it corresponds to a quantity of higher order than the one of the equation, which is s.

By the same method and independently of our work, the BV and Sobolev estimates (2.1) and (2.3) have also been established by Gui and Li [48, Proposition 1.7].

In the current paper we additionally keep track of how the constants in the estimates blow-up as $s \uparrow 1$. Such information could be useful in some applications.[3]

Theorem 2.1. *Given $s_0 > 0$, let $n \geq 2$, $s \in (s_0, 1)$, W be any $C^3(\mathbb{R})$ function, and K satisfy (1.8) and (1.9). Let $R > 0$ and $u : \mathbb{R}^n \to (-1, 1)$ be a stable solution of $L_K u + W'(u) = 0$ in $B_{2R} \subset \mathbb{R}^n$.*

Then,

$$\int_{B_R} |\nabla u| \, dx \leq \frac{C}{1-s} R^{n-1}, \tag{2.1}$$

where C is a constant which depends only on n, λ, Λ, and s_0.

Two remarks on this result are in order.

Remark 2.2. We emphasize that, in Theorem 2.1, the constant C is completely independent of the potential W, which is an arbitrary C^3 function —in particular it is not necessarily a double-well potential. As a consequence, the estimate (2.1) is scale invariant: the estimate for any $R > 0$ follows from the case $R = 1$ applied to the rescaled function $\tilde{u} = u(R\,\cdot)$, which solves $L_K \tilde{u} + \widetilde{W}'(\tilde{u}) = 0$ in B_2, for $\widetilde{W} := R^s W$.

Remark 2.3. In the case of solutions $u_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ of the fractional Allen-Cahn equation (1.2) (recall that $\varepsilon > 0$ is a small parameter), Theorem 2.1 reads:

$$u_\varepsilon \text{ stable in } B_2 \quad \Rightarrow \quad \int_{B_1} |\nabla u_\varepsilon| \, dx \leq \frac{C}{1-s}.$$

We emphasize that the constant in the right-hand side of the previous estimate is independent of ε. We do not expect the blow-up rate $(1-s)^{-1}$ as $s \uparrow 1$ to be sharp, but for such uniform in ε estimate to hold, the right-hand side needs to blow up at least at the rate $(1-s)^{-1/2}$ —we actually expect the power $-1/2$ to be the sharp rate. Indeed, by the argument sketched next, for all $n \geq 2$ and $s \in (0, 1)$ there exists $\varepsilon = \varepsilon(s) > 0$ and a stable solution $u_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ of (1.2) in B_2 satisfying $\int_{B_1} |\nabla u_\varepsilon| \, dx \geq c(1-s)^{-1/2}$, where c stays bounded away from zero as $s \uparrow 1$.

To show this, consider the set

$$E_* := \bigcup_{k \in \mathbb{Z}} \left\{ 2kC_*(1-s)^{1/2} \leq x_n \leq (2k+1)C_*(1-s)^{1/2} \right\},$$

whose boundary consists of parallel hyperplanes arranged in a $C_*(1-s)^{1/2}$-periodic fashion. Let us prove first that E_* is a stable s-minimal set in $(-1, 1)^n \supset B_1$, provided C_* is chosen large enough. To verify this, we can use the stability criterium found in [30, 40] involving the nonlocal analog c_s^2 of the squared norm of the second fundamental form of ∂E_*. To this end, one first checks that c_s^2 behaves as the

[3] This was the case in our previous paper [14], where we needed the dependence on s (as $s \uparrow 1$) in the BV estimate from [29] for stable s-minimal sets.

quantity $(C_*(1-s)^{1/2})^{-1-s}$. Hence, the stability inequality in $(-1,1)^n$ reads

$$\sum_j C_*^{-1-s}\int_{\Gamma_j}\eta^2 d\mathcal{H}^{n-1}_x \le C(1-s)^{\frac{1+s}{2}}\sum_{i,j}\int_{\Gamma_i}\int_{\Gamma_j}\frac{\big|\eta(x)-\eta(\bar x)\big|^2}{|x-\bar x|^{n+s}}d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_{\bar x}, \quad (2.2)$$

for all $\eta\in C^{0,1}_c([-1,1]^n)$ where $\Gamma_j := \{x_n = jC_*(1-s)^{1/2}\}$.

By the fractional Poincaré inequality for the $H^{(1+s)/2}$ seminorm in $[-1,1]^{n-1}$ (applied to the functions $\eta_{|\Gamma_j}$, $\Gamma_j\cong\mathbb{R}^{n-1}$) we have

$$\int_{\Gamma_j}\eta^2 d\mathcal{H}^{n-1}_x \le C(1-s)\int_{\Gamma_j}\int_{\Gamma_j}\frac{\big|\eta(x)-\eta(\bar x)\big|^2}{|x-\bar x|^{n+s}}d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_{\bar x},$$

for all $\eta\in C^{0,1}_c([-1,1]^n)$, where C is uniformly bounded as $s\uparrow 1$. Therefore, since $(1-s)/(1-s)^{\frac{1+s}{2}}\to 1$ as $s\uparrow 1$, (2.2) will hold provided that we choose C_* large enough. This makes E_* stable.

We also notice that a similar computation (for any fixed smooth cutoff η) shows that the previous configuration is unstable provided $C_*>0$ is chosen small enough.

At the same time, it is known [15] that the fractional Allen-Cahn equation (1.2) admits periodic 1D solutions for all large enough periods (relatively to ε). Hence, given $s\in(0,1)$, for every $0<\varepsilon\le c(1-s)^{1/2}$ there exists a periodic 1D solution $u_{*,\varepsilon}(x_n)$ of (1.2) such that $\{u_{*,\varepsilon}=0\}=\partial E_*$. Now, as $\varepsilon\downarrow 0$ we have, by construction, $u_{*,\varepsilon}\to\chi_{E_*}-\chi_{E_*^c}$ and hence one can show that, for ε small enough depending on s, $u_{*,\varepsilon}$ inherits from E_* the stability in $(-1,1)^n$. Finally, note that for such construction $\int_{B_1}|\nabla u_\varepsilon|\,dx$ blows up at the rate $(1-s)^{-1/2}$ (by the coarea formula), as claimed.

As a simple consequence of Theorem 2.1, we deduce the following estimate for the Sobolev part of the energy of stable solutions to (1.6).

Corollary 2.4. *Given $s_0>0$, let $n\ge 2$, $s\in(s_0,1)$, W be any $C^3(\mathbb{R})$ function, and K satisfy (1.8) and (1.9). Let $R>0$ and $u:\mathbb{R}^n\to(-1,1)$ be a stable solution of $L_Ku+W'(u)=0$ in $B_{2R}\subset\mathbb{R}^n$.*

Then,

$$\mathcal{E}^{\rm Sob}_{B_R}(u)\le\frac{C}{(1-s)^2}R^{n-s}, \quad (2.3)$$

where C is a constant which depends only on n, λ, Λ, and s_0.

2.2. **Control of $\mathcal{E}^{\rm Pot}$ by $\mathcal{E}^{\rm Sob}$.** In Section 4 we need to assume W to be a double-well potential. For simplicity, we take W to be the quartic potential (1.5), although all proofs apply to the more general class of double-well potentials described in Remark 1.9.

The following result states that $\mathcal{E}^{\rm Sob}$ in B_R controls $\mathcal{E}^{\rm Pot}$ in a slightly smaller ball. Here we can take $s\in(0,2]$ and thus we include the case of the Laplacian. To our knowledge, this result is new even for the Laplacian.[4] To state our result for $s=2$, we recall that if $K_s(z)\asymp(2-s)|z|^{-n-s}$ (where $X\asymp Y$ here means $X\le CY$ and

[4]For $s=2$ and in the particular case of stable solutions in all of $\mathbb{R}^n$, Villegas [66, Proposition 1.5] has recently proved that the quotient of Sobolev and potential energies in B_R tends to 1 as $R\uparrow\infty$.

$Y \le CX$) we have $L_{K_s} v \to a_{ij}\partial_{ij} v$ for all $v \in C^2_c(\mathbb{R}^n)$ as $s \uparrow 2$ (see the computations in [21, Section 6]), where $a_{ij} \asymp \mathrm{Id}$ (that is $a_{ij}\partial_{ij}$ is some translation invariant second order elliptic operator). In the following proposition, L_K is meant to be, when $s = 2$, this limiting second-order elliptic operator.

Proposition 2.5. *Given $s_0 > 0$, let $n \ge 2$, $s \in (s_0, 2]$, $W(u) = \frac{1}{4}(1-u^2)^2$, and K satisfy* (1.8) *and* (1.9)*. Let $R > 0$ and $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $L_K u + W'(u) = 0$ in B_{2R}.*

Then,

$$\mathcal{E}^{\mathrm{Pot}}_{B_{R-R_0}}(u) \le C\mathcal{E}^{\mathrm{Sob}}_{B_R}(u) \tag{2.4}$$

whenever $R > R_0$, where C and R_0 are positive constants which depend only on n, λ, Λ, and s_0.

To prove Proposition 2.5, we cover the ball B_{R-R_0} by cubes of size comparable to R_0 and with controlled overlapping, and we show that the contribution of each of these cubes to the total Sobolev energy controls its contribution to the potential energy. We believe that this cube decomposition, which also holds for the classical Allen-Cahn equation, may be of independent interest.

In Appendix B, we present an alternative proof —only in the case $s = 2$, for brevity— of a weaker version of the estimate of Proposition 2.5 in which an error term appears on the right-hand side of (2.4). The error is of order R^{n-s} when $s \in (0,1)$, R^{n-1} for $s \in (1,2]$, and $R^{n-1}\log R$ for $s = 1$. Even if the result is weaker, we give this alternative proof since it is new even for the Laplacian (to the best of our knowledge), very short, and interesting in itself. It relies on taking a suitable test function on the stability inequality and on elementary estimates.

Combining Proposition 2.5 with Corollary 2.4, we obtain the following bound for the total energy of stable solutions in a ball.

Theorem 2.6. *Given $s_0 > 0$, let $n \ge 2$, $s \in (s_0, 1)$, $W(u) = \frac{1}{4}(1-u^2)^2$, and K satisfy* (1.8) *and* (1.9)*. Let $R \ge 1$ and $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $L_K u + W'(u) = 0$ in $B_{2R} \subset \mathbb{R}^n$.*

Then,

$$\mathcal{E}_{B_R}(u) \le \frac{C}{(1-s)^2} R^{n-s}, \tag{2.5}$$

where C is a constant which depends only on n, λ, Λ, and s_0.

For $s \in [1,2]$, [12, 13] established the related energy estimates $\mathcal{E}_{B_R}(u) \le CR^{n-1}$, for $s \in (1,2]$, and $\mathcal{E}_{B_R}(u) \le CR^{n-1}\log R$, for $s = 1$, in the case of minimizers in $B_{2R} \subset \mathbb{R}^n$. However, such strong estimates are not expected to hold if one considers stable solutions —assuming stability only in the double ball B_{2R} as in Theorem 2.6, not in the whole $\mathbb{R}^n$— instead of minimizers.

2.3. **Density estimates.** Section 5 deals with density estimates for stable solutions of $(-\Delta)^{s/2}u + W'(u) = 0$ in all of $\mathbb{R}^n$. We need to assume the operator L_K to be the fractional Laplacian $(-\Delta)^{s/2}$ since an essential ingredient in their proof is the monotonicity formula from [13], which uses the extension property of $(-\Delta)^{s/2}$ and thus is not available for other kernels. In addition, our proof uses crucially (2.1) and therefore needs $s < 1$.

Proposition 2.7. *Let $n \geq 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. Let $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$.*

Then, for every $\bar{c} \in (0,1)$ there exist positive constants ω_0 and R_0, which depend only on $\bar{c}$, n, and s, such that the following holds. For every $R \geq R_0$, if

$$R^{-n}\int_{B_R} |1+u|\,dx \leq \omega_0 \qquad \left(\textit{respectively,}\quad R^{-n}\int_{B_R} |1-u|\,dx \leq \omega_0\right), \tag{2.6}$$

then

$$\{u \geq -\bar{c}\} \cap B_{R/2} = \varnothing \qquad \big(\textit{respectively,}\quad \{u \leq \bar{c}\} \cap B_{R/2} = \varnothing\big)\,. \tag{2.7}$$

2.4. **Convergence of blow-downs.** The goal of Section 6 is to establish that the blow-downs of entire stable solutions to $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$ converge, up to a subsequence, to the characteristic function of a stable nonlocal s-minimal cone. This convergence is local, both in the L^1-sense and in the sense of the Hausdorff distance between the level sets and the (boundary of the) cone. To be precise, following [14, Definitions 1.1 and 2.2], we recall now two notions of stability for the fractional perimeter P_s defined in (1.13).

The first one is the notion of stability for a cone that is smooth away from the origin. It is the notion used in Definition 1.6 above.

Definition 2.8. Let $\Sigma \subset \mathbb{R}^n$ be a cone with nonempty boundary of class C^2 away from the origin. We say that Σ is a stable cone for the s-perimeter in $\mathbb{R}^n \setminus \{0\}$, or stable s-minimal cone in $\mathbb{R}^n \setminus \{0\}$, if

$$\liminf_{t\to 0} \frac{1}{t^2}\big(P_s(\phi_X^t(\Sigma), B_1) - P_s(\Sigma, B_1)\big) \geq 0 \tag{2.8}$$

for all vector fields $X \in C_c^\infty(B_1 \setminus \{0\}, \mathbb{R}^n)$. Here $\phi_X^t : \mathbb{R}^n \to \mathbb{R}^n$ denotes the integral flow of X at time t (a smooth diffeomorphism for t small).

The following is the notion of *weak stability* for a general set E with finite s-perimeter, as given in [14, Definition 2.2].

Definition 2.9. A set $E \subset \mathbb{R}^n$ with $P_s(E,\Omega) < \infty$ is said to be *weakly stable* in Ω for the s-perimeter if for every given vector field $X = X(x,t) \in C_c^\infty(\Omega \times (-1,1); \mathbb{R}^n)$ we have

$$\liminf_{t\to 0} \frac{1}{t^2}\big(P_s(\phi_X^t(E), \Omega) - P_s(E,\Omega)\big) \geq 0,$$

where ϕ_X^t denotes the integral flow of X at time t (with 0 as initial time).

In contrast with this last notion, Definition 2.8 assumes the vector field $X = X(x)$ to be autonomous. However, since a cone Σ in Definition 1.6 is smooth outside of the origin, it is simple to see that Σ is a stable s-minimal cone in $\mathbb{R}^n \setminus \{0\}$ (in the sense of Definition 2.8) if and only if Σ is a weakly stable set in $\mathbb{R}^n \setminus \{0\}$ for the fractional perimeter P_s (in the sense of Definition 2.9) .

In our following result, we will prove the limiting cone Σ to be weakly stable not only in $\mathbb{R}^n \setminus \{0\}$, but in $\mathbb{R}^n$. In its statement, we use the notation

$$\overline{D}(E;x) = \limsup_{r\to 0} \frac{|E\cap B_r(x)|}{|B_r|} \qquad \text{and} \qquad \underline{D}(E;x) = \liminf_{r\to 0} \frac{|E\cap B_r(x)|}{|B_r|}$$

for the upper and lower densities of a set $E \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$.

Theorem 2.10. *Let $n \geq 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. Let $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$.*

Then, for every given blow-down sequence $u_{R_j}(x) = u(R_j x)$ with $R_j \uparrow \infty$, there is a subsequence R_{j_k} such that

$$u_{R_{j_k}} \to \chi_\Sigma - \chi_{\Sigma^c} \qquad \text{in } L^1(B_1)$$

for some cone Σ which is a weakly stable set in $\mathbb{R}^n$ for the fractional perimeter P_s and which is nontrivial (not equal to $\mathbb{R}^n$ or $\varnothing$ up to sets of measure zero).

In addition, up to changing Σ in a set of measure zero, we have

$$x \in \partial\Sigma \qquad \Leftrightarrow \qquad 0 < \underline{D}(x;\Sigma) \leq \overline{D}(x;\Sigma) < 1 \tag{2.9}$$

and, for all given $c \in (-1,1)$ and $\rho \geq 1$, we have

$$d_{\mathrm{Hausdorff}}\big(\{u_{R_{j_k}} \geq c\} \cap B_\rho\,,\, \Sigma \cap B_\rho\big) \to 0 \qquad \text{as } k \uparrow \infty, \tag{2.10}$$

where $d_{\mathrm{Hausdorff}}(X,Y) = \inf\{d > 0 \,:\, X \subset Y + B_d \text{ and } Y \subset X + B_d\}$ denotes the standard Hausdorff distance between subsets of $\mathbb{R}^n$.

The proof of Theorem 2.10 puts together all our results stated above in this section. More precisely, the first part of Theorem 2.10 (L^1-convergence) is the content of Proposition 6.1, the proof of which uses the BV and energy estimates from Theorems 2.1 and 2.6, as well as the monotonicity formula from [13]. The second part of the statement (uniform convergence) follows then from the density estimates of Proposition 2.7.

2.5. **Proofs of the main result and its corollaries.** In Section 7 we prove Theorem 1.8 by combining Theorem 2.10 and the “improvement of flatness” results established by Dipierro, Serra, and Valdinoci in [37]. Some nontrivial technical details are involved in our proof, like a dimension reduction argument that allows us to assume that the stable cones obtained after blow-down are smooth away from 0 (as required in Definition 1.6). It is important to deal with this smoothness issue to guarantee the applicability of Theorem 1.8 in concrete cases. For instance, from [14], in $\mathbb{R}^3$ we only know how to classify stable cones that are smooth away from 0.

Finally, also in Section 7, we will give the straightforward proofs of Corollaries 1.10 and 1.11.

2.6. **Smooth stable s-minimal surfaces in $\mathbb{R}^3$ are planes when $s \sim 1$.** In Appendix A, the arguments of Section 7 will be easily modified to establish the flatness of C^2 stable s-minimal surfaces in all of $\mathbb{R}^n$ whenever all stable s-minimal cones are known to be hyperplanes up to dimension n. As a consequence we deduce that every C^2 stable s-minimal surface in $\mathbb{R}^3$ must be a plane if $s \in (s_*, 1)$ for some $s_* < 1$, since our main result in [14] states that planes are the only stable s-minimal cones in $\mathbb{R}^3 \setminus \{0\}$ (smooth away from 0) for all $s \in (s_*, 1)$. This was already announced without proof in [14, Corollary 1.3].

More precisely, we have the following result:

Theorem 2.11. *Assume that, for some pair (n, s) with $n \geq 3$ and $s \in (0, 1)$, hyperplanes are the only stable s-minimal cones in $\mathbb{R}^n \setminus \{0\}$.*

Let E be an open subset of $\mathbb{R}^n$, with ∂E nonempty and of class C^2, and such that E is a weakly stable set for the s-perimeter in $\mathbb{R}^n$ (as defined in Definition 2.9).

Then, E is a half-space.

Corollary 2.12. *Let $E \subset \mathbb{R}^3$ be a C^2 open subset, with $\partial E \neq \varnothing$, which is weakly stable for the s-perimeter in $\mathbb{R}^3$ (as defined in Definition 2.9).*

If s is sufficiently close to 1, then E is a half-space.

Theorem 2.11 will follow from a blow-down procedure. Although the general strategy is similar to the classical one for minimizers of the perimeter, to deal with stable solutions, one finds several technical issues that are completely analogous to those in Section 7 for the semilinear equation. This close analogy was our reason for postponing the proof of Corollary 1.3 in [14] (i.e., Corollary 2.12 here) to Appendix A of the current paper.

3. BV and Sobolev energy estimates

Throughout the paper, L_K denotes the operator (1.7), where K satisfies (1.8) and (1.9).

This section extends some results and techniques introduced in [29], within the context of stable nonlocal minimal surfaces, to stable solutions of semilinear nonlocal problems.

Following [59, 29], we consider translations of the solution u in some direction $\boldsymbol{v} \in S^{n-1}$, and we compare the energies of u and of $u(\cdot + t\boldsymbol{v})$. Since we need to consider perturbations vanishing outside the domain, following the notation of [29] we introduce a Lipschitz radial cut-off function φ_4 such that $\varphi_4 \equiv 1$ in B_2, $\varphi_4 \equiv 0$ outside B_4, and $|\nabla \varphi_4| \leq 1/2$. For this, we take φ_4 to be linear, as a function of the radius $|x|$, in $B_4 \setminus B_2$.

For $\boldsymbol{v} \in S^{n-1}$ and $t \in (-1, 1)$ we consider the map

$$\Psi_t(y) := y + t\varphi_4(y)\boldsymbol{v} \qquad \text{for } y \in \mathbb{R}^n. \tag{3.1}$$

Clearly, Ψ_t is a Lipschitz diffeomorphism from $\mathbb{R}^n$ onto itself. In particular, both Ψ_t and Ψ_t^{-1} are Lipschitz and coincide with the identity outside B_4. Given a function

u defined in all of $\mathbb{R}^n$, we introduce

$$u_t(x) := u\big(\Psi_t^{-1}(x)\big).$$

Even if we do not denote its dependence on $\boldsymbol{v}$, the function u_t depends on both $\boldsymbol{v}$ and t. We finally set

$$M_t(x) := \max\{u(x), u_t(x)\} \quad \text{ and } \quad m_t(x) := \min\{u(x), u_t(x)\}.$$

Notice that u_t, M_t, and m_t all coincide with u outside B_4.

We now prove the analog of Lemma 2.1 in [29] for the energy (1.10) associated with (1.6).

Lemma 3.1. *Let $n \ge 2$, $s \in (0,1)$, let W be any continuous function in $\mathbb{R}$, and K satisfy (1.8) and (1.9). Let $u \in C^1(B_6) \cap L^\infty(\mathbb{R}^n)$ be a given function.*

Then, for all $t \in (-1,1)$ we have that

$$\mathcal{E}_{B_4}(u_t) + \mathcal{E}_{B_4}(u_{-t}) - 2\mathcal{E}_{B_4}(u) \le Ct^2 \mathcal{E}^{\mathrm{Sob}}_{B_4}(u), \tag{3.2}$$

where C is a constant which depends only on n, λ, and Λ.

Proof. We set $A_4 := (\mathbb{R}^n \times \mathbb{R}^n) \setminus (B_4^c \times B_4^c)$. We have

$$\mathcal{E}_{B_4}(u_{\pm t}) = \frac{1}{4} \iint_{A_4} |u(\Psi_{\pm t}^{-1}(x)) - u(\Psi_{\pm t}^{-1}(\bar{x}))|^2 K(x - \bar{x})\, dx\, d\bar{x} + \int_{B_4} W(u(\Psi_{\pm t}^{-1}(x)))\, dx.$$

Changing variables $y = \Psi_{\pm t}^{-1}(x)$, $\bar{y} = \Psi_{\pm t}^{-1}(\bar{x})$ in the integrals above (recall that $\Psi_{\pm t}^{-1}$ sends B_4 and B_4^c onto themselves), we obtain the following expressions for the Sobolev and potential energies:

$$\mathcal{E}^{\mathrm{Sob}}_{B_4}(u_{\pm t}) = \frac{1}{4} \iint_{A_4} |u(y) - u(\bar{y})|^2 K\big(\Psi_{\pm t}(y) - \Psi_{\pm t}(\bar{y})\big)\, J_{\pm t}(y)\, J_{\pm t}(\bar{y})\, dy\, d\bar{y}$$

and

$$\mathcal{E}^{\mathrm{Pot}}_{B_4}(u_{\pm t}) = \int_{B_4} W(u(y))\, J_{\pm t}(y)\, dy,$$

where $J_{\pm t}$ are the Jacobians. They are easily seen to be

$$J_{\pm t}(y) = 1 \pm t \partial_{\boldsymbol{v}} \varphi_4(y),$$

which are positive quantities since $|t| < 1$ and $|\partial_{\boldsymbol{v}} \varphi_4| \le 1/2$.

Clearly,

$$\mathcal{E}^{\mathrm{Pot}}_{B_4}(u_t) + \mathcal{E}^{\mathrm{Pot}}_{B_4}(u_{-t}) = \int_{B_4} W(u(y))(J_t(y) + J_{-t}(y))\, dy = 2\mathcal{E}^{\mathrm{Pot}}_{B_4}(u). \tag{3.3}$$

To estimate the sum of the two Sobolev energies $\mathcal{E}^{\mathrm{Sob}}_{B_4}(u_t) + \mathcal{E}^{\mathrm{Sob}}_{B_4}(u_{-t})$, we first observe that, since $K \in \mathcal{L}_2$, then it satisfies the following estimates for its first and second derivatives:

$$\max\left\{ |z||\partial_e K(z)|, |z|^2 \sup_{|y-z| \le |z|/2} |\partial_{ee} K(y)| \right\} \le 2^{n+s+2} \frac{\Lambda}{\lambda} K(z) \le 2^{n+3} \frac{\Lambda}{\lambda} K(z)$$

for every $z \in \mathbb{R}^n \setminus \{0\}$ and $e \in S^{n-1}$. Using this fact and performing the same computations as in the proof of Lemma 2.1 in [29] (we are taking there, with the notations of [29], $R = 4$ and $K^* = 2^{n+3}(\Lambda/\lambda)K$), we deduce that

$$\mathcal{E}^{\mathrm{Sob}}_{B_4}(u_t) + \mathcal{E}^{\mathrm{Sob}}_{B_4}(u_{-t}) \le 2\mathcal{E}^{\mathrm{Sob}}_{B_R}(u) + Ct^2 \iint_{A_4} |u(y) - u(\bar y)|^2 K(y-\bar y)\, dy\, d\bar y,$$

where C depends only on n, λ, and Λ. This, combined with (3.3), concludes the proof of the lemma. $\square$

Lemma 3.2. *Let $n \ge 2$, $s \in (0,1)$, let W be any $C^3([-1,1])$ function, and K satisfy (1.8) and (1.9). Let $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $L_K u + W'(u) = 0$ in $B_6 \subset \mathbb{R}^n$.*

Then, given $\nu > 0$, there exists $t_0 \in (0,1)$ (possibly depending on ν, W, and u), such that

$$\mathcal{E}_{B_4}(M_t) + \mathcal{E}_{B_4}(m_t) - 2\mathcal{E}_{B_4}(u) \ge -\nu t^2 \qquad \text{for all } t \in (-t_0, t_0). \tag{3.4}$$

Proof. As shown in Appendix C, we know that $u \in C^2(B_6)$. Thus, the fractional Sobolev semi-norm $\mathcal{E}_{B_4}$ of u is finite.

We claim that, given any Lipschitz function ξ vanishing outside B_4 and $t \in (-1,1)$ such that $|u + t\xi| \le 1$, we have

$$\mathcal{E}_{B_4}(u + t\xi) - \mathcal{E}_{B_4}(u) \ge -C\|\xi\|^3_{L^\infty(B_4)}|t|^3 \tag{3.5}$$

for some constant C depending only on W. Indeed, we have

$$\begin{aligned}
&\mathcal{E}_{B_4}(u+t\xi) - \mathcal{E}_{B_4}(u) = \\
&\quad = \frac{t}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(\bar x))(\xi(x)-\xi(\bar x))K(x-\bar x)\,dx\,d\bar x \\
&\qquad + \frac{t^2}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n} |\xi(x)-\xi(\bar x)|^2 K(x-\bar x)\,dx\,d\bar x + \int_{B_4}\big(W(u+t\xi)-W(u)\big)\,dx \\
&\quad = \frac{t}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} (u(x)-u(\bar x))(\xi(x)-\xi(\bar x))K(x-\bar x)\,dx\,d\bar x + \int_{B_4} W'(u)t\xi\,dx \\
&\qquad + \frac{t^2}{4}\iint_{\mathbb{R}^n\times\mathbb{R}^n} |\xi(x)-\xi(\bar x)|^2 K(x-\bar x)\,dx\,d\bar x + \int_{B_4} W''(u)\frac{(t\xi)^2}{2}\,dx \\
&\qquad + \int_{B_4} W'''(u^*(t,x))\frac{(t\xi)^3}{6}\,dx
\end{aligned}$$

for some $u^*(t,x)$ in the interval with endpoints $u(x)$ and $u(x) + t\xi(x)$ (and thus satisfying $|u^*| \le 1$). Hence, using that u is a stable critical point and $W \in C^3$, we deduce (3.5).

We now choose $\xi := \frac{(u_t-u)_+}{t}$ and $\tilde\xi := -\frac{(u_t-u)_-}{t}$ in the above computations, where $(\cdot)_+$ and $(\cdot)_-$ denote the positive and negative parts. Observe that $M_t = u + t\xi$ and $m_t = u + t\tilde\xi$. Also, since u and u_t agree outside B_4 and u is C^1 in $\overline{B}_4$, the L^∞-norms of ξ and $\tilde\xi$ are bounded by a constant independent of t (in fact, depending only on u, given the cut-off function φ_4 defining the diffeormorphism Ψ_t). Therefore, from (3.5) applied to these choices ξ and $\tilde{\xi}$, adding both inequalities, we conclude (3.4). $\square$

With this consequence of stability in hand, we can now state the following result, which is the analog of Lemma 2.4 in [29]. We recall that we have set

$$A_4 = (\mathbb{R}^n \times \mathbb{R}^n) \setminus (B_4^c \times B_4^c).$$

Lemma 3.3. *Let $n \geq 2$, $s \in (0,1)$, let W be any $C^3([-1,1])$ function, and K satisfy (1.8) and (1.9). Let $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $L_K u + W'(u) = 0$ in $B_6 \subset \mathbb{R}^n$.*

Then, for every $\nu > 0$ there exists $t_0 > 0$ (possibly depending on ν, W, and u) such that

$$\begin{aligned}\min\Big\{ &\iint_{A_4} (u(x) - u_t(x))_+ (u(\bar{x}) - u_t(\bar{x}))_- K(x - \bar{x})\, dx\, d\bar{x},\\ &\iint_{A_4} (u(x) - u_{-t}(x))_+ (u(\bar{x}) - u_{-t}(\bar{x}))_- K(x - \bar{x})\, dx\, d\bar{x} \Big\} \leq (\eta + 2\nu)\, t^2\end{aligned}$$

holds for $t \in (-t_0, t_0)$, where $\eta = C\mathcal{E}^{\mathrm{Sob}}_{B_4}(u)$ and C depends only on n, λ, and Λ. Here, $(\cdot)_+$ and $(\cdot)_-$ denote a function's positive and negative parts.

Proof. We first observe that, since u is stable and is $C^2(B_6)$ (by Appendix C), it satisfies both estimates (3.2) and (3.4). Recall also that M_t, m_t (as defined at the beginning of the section), u_t, and u all coincide outside B_4. Moreover, it holds

$$\mathcal{E}^{\mathrm{Pot}}_{B_4}(M_t) + \mathcal{E}^{\mathrm{Pot}}_{B_4}(m_t) = \mathcal{E}^{\mathrm{Pot}}_{B_4}(u) + \mathcal{E}^{\mathrm{Pot}}_{B_4}(u_t). \tag{3.6}$$

We now consider the Sobolev energy. We claim that

$$\begin{aligned}&|M_t(x) - M_t(\bar{x})|^2 + |m_t(x) - m_t(\bar{x})|^2 - |u(x) - u(\bar{x})|^2 - |u_t(x) - u_t(\bar{x})|^2\\ &\quad = -2(u(x) - u_t(x))_+ (u(\bar{x}) - u_t(\bar{x}))_-.\end{aligned} \tag{3.7}$$

Indeed, we first observe that, if $(u(x)-u_t(x))(u(\bar{x})-u_t(\bar{x})) \geq 0$, then the right-hand side of (3.7) vanishes and that the equality is clear.

Assume now that $(u(x) - u_t(x))(u(\bar{x}) - u_t(\bar{x})) < 0$. Then, by symmetry between x and $\bar{x}$, we may assume $u(x) > u_t(x)$, $u(\bar{x}) < u_t(\bar{x})$. Now, a simple computation shows that

$$\begin{aligned}&|M_t(x) - M_t(\bar{x})|^2 + |m_t(x) - m_t(\bar{x})|^2 - |u(x) - u(\bar{x})|^2 - |u_t(x) - u_t(\bar{x})|^2\\ &\quad = -2u(x)u_t(\bar{x}) - 2u_t(x)u(\bar{x}) + 2u(x)u(\bar{x}) + 2u_t(x)u_t(\bar{x})\\ &\quad = -2(u(x) - u_t(x))(u_t(\bar{x}) - u(\bar{x})).\end{aligned}$$

This concludes the proof of (3.7).

Using (3.6) and (3.7), we deduce that

$$\begin{aligned}&\mathcal{E}_{B_4}(M_t) + \mathcal{E}_{B_4}(m_t) + \frac{1}{2}\iint_{A_4} (u(x) - u_t(x))_+ (u(\bar{x}) - u_t(\bar{x}))_- K(x - \bar{x})\, dx\, d\bar{x}\\ &\quad = \mathcal{E}_{B_4}(u) + \mathcal{E}_{B_4}(u_t)\end{aligned}$$

for every $t \in (0,1)$. As noticed in [29], the third term on the left-hand side is the important novelty compared to the analog equality in the local case (in which it does not appear). It will be responsible for the BV estimate.

Indeed, analogously we have

$$\begin{aligned}\mathcal{E}_{B_4}(M_{-t}) + \mathcal{E}_{B_4}(m_{-t}) + \frac{1}{2}\iint_{A_4} (u(x) - u_{-t}(x))_+ (u(\bar{x}) - u_{-t}(\bar{x}))_- K(x-\bar{x}) dx d\bar{x} \\ = \ \mathcal{E}_{B_4}(u) + \mathcal{E}_{B_4}(u_{-t}).\end{aligned}$$

Adding the last two equalities, and using the key bounds (3.2) and (3.4), we deduce

$$\begin{aligned}&\mathcal{E}_{B_4}(M_t) + \mathcal{E}_{B_4}(m_t) + \mathcal{E}_{B_4}(M_{-t}) + \mathcal{E}_{B_4}(m_{-t}) + \\ &\quad + \frac{1}{2}\iint_{A_4} (u(x) - u_t(x))_+ (u(\bar{x}) - u_t(\bar{x}))_- K(x-\bar{x})\, dx\, d\bar{x} + \qquad (3.8)\\ &\quad + \frac{1}{2}\iint_{A_4} (u(x) - u_{-t}(x))_+ (u(\bar{x}) - u_{-t}(\bar{x}))_- K(x-\bar{x})\, dx\, d\bar{x} \\ &= 2\mathcal{E}_{B_4}(u) + \mathcal{E}_{B_4}(u_t) + \mathcal{E}_{B_4}(u_{-t}) \\ &\le 4\mathcal{E}_{B_4}(u) + \eta t^2 \\ &\le \mathcal{E}_{B_4}(M_t) + \mathcal{E}_{B_4}(m_t) + \mathcal{E}_{B_4}(M_{-t}) + \mathcal{E}_{B_4}(m_{-t}) + \big(\eta + 2\nu\big)t^2 \qquad (3.9)\end{aligned}$$

for $t \in (-t_0, t_0)$, with $t_0 > 0$ small enough (possibly depending on ν, W, and u). From this, the lemma follows immediately. $\square$

The following result is the analog of Lemma 2.5 in [29].

Lemma 3.4. *Let $n \ge 2$, $\eta > 0$, and $u \in C^1(B_2)$ satisfy $\|u\|_{L^\infty(\mathbb{R}^n)} \le 1$. Assume that for every $\boldsymbol{v} \in S^{n-1}$, there exists a sequence $t_k \to 0$, such that*

$$\limsup_{k\to\infty} \frac{\|\big(u(\cdot) - u(\cdot - t_k\boldsymbol{v})\big)_+\|_{L^1(B_1)}\ \|\big(u(\cdot) - u(\cdot - t_k\boldsymbol{v})\big)_-\|_{L^1(B_1)}}{t_k^2} \le \eta. \qquad (3.10)$$

Then, the following estimates hold:

$$\min\left\{\int_{B_1} (\partial_{\boldsymbol{v}} u)_+\, dx\,,\ \int_{B_1} (\partial_{\boldsymbol{v}} u)_-\, dx\right\} \le \sqrt{\eta}, \qquad (3.11)$$

$$\max\left\{\int_{B_1} (\partial_{\boldsymbol{v}} u)_+\, dx\,,\ \int_{B_1} (\partial_{\boldsymbol{v}} u)_-\, dx\right\} \le 2|B_1^{(n-1)}| + \sqrt{\eta}, \qquad (3.12)$$

and

$$\int_{B_1} |\nabla u|\, dx \le 2n\left(|B_1^{(n-1)}| + \sqrt{\eta}\right), \qquad (3.13)$$

where $B_1^{(n-1)}$ denotes the unit ball of $\mathbb{R}^{n-1}$.

Proof. The proof is the same as that of Lemma 2.5 in [29]. Here the situation is even simpler since, being $u \in C^1$, the directional derivatives of u exist in the classical sense. We sketch the main steps of the proof, for the reader's convenience.

By assumption (3.10), we have that

$$\limsup_{k\to\infty} \frac{\min\left\{\|\big(u(\cdot)-u(\cdot-t_k\boldsymbol{v})\big)_+\|_{L^1(B_1)},\ \|\big(u(\cdot)-u(\cdot-t_k\boldsymbol{v})\big)_-\|_{L^1(B_1)}\right\}}{|t_k|} \le \sqrt{\eta}.$$

Passing to the limit as $t_k \to 0$, we immediately get (3.11).

To prove (3.12), we simply observe that

$$\int_{B_1} \partial_{\boldsymbol{v}} u\,dx = \int_{B_1} \lim_{k\to\infty} \frac{u(x+t_k\boldsymbol{v})-u(x)}{t_k}\,dx = \lim_{k\to\infty} \frac{\int_{B_1+t_k\boldsymbol{v}} u\,dx - \int_{B_1} u\,dx}{t_k}$$

and hence, since $\|u\|_{L^\infty(\mathbb{R}^n)} \le 1$,

$$\left|\int_{B_1} \partial_{\boldsymbol{v}} u\,dx\right| \le \limsup_{k\to\infty} \frac{\big|(B_1+t_k\boldsymbol{v})\setminus B_1\big| + \big|B_1\setminus(B_1+t_k\boldsymbol{v})\big|}{|t_k|} \le 2|B_1^{(n-1)}|.$$

This, together with (3.11), leads to (3.12), since $\partial_{\boldsymbol{v}} u = (\partial_{\boldsymbol{v}} u)_+ - (\partial_{\boldsymbol{v}} u)_-$. Moreover, using that $|\partial_{\boldsymbol{v}} u| = (\partial_{\boldsymbol{v}} u)_+ + (\partial_{\boldsymbol{v}} u)_-$, we also deduce that

$$\int_{B_1} |\partial_{\boldsymbol{v}} u|\,dx \le 2\left(|B_1^{(n-1)}| + \sqrt{\eta}\right).$$

Finally, since $|\nabla u| \le |\partial_{e_1} u| + \ldots + |\partial_{e_n} u|$, we conclude (3.13). □

Before giving the proof of Theorem 2.1, we recall the following standard "covering lemma" due to L. Simon [62] (see also Lemma 3.1 in [29]). In scaling invariant situations, when estimating a function's norm in a ball, this useful lemma allows one to "absorb" a term in the right-hand side where the same norm appears computed in a bigger ball, provided it appears with a small enough constant in front.

Lemma 3.5 ([62])**.** *Let $\beta \in \mathbb{R}$ and $C_0 > 0$. Let $S : \mathcal{B} \to [0,+\infty)$ be a nonnegative function defined on the class $\mathcal{B}$ of open balls contained in the unit ball B_1 of $\mathbb{R}^n$ and satisfying the following subadditivity property:*

$$S(B) \le \sum_{j=1}^N S(B^j) \quad \textit{whenever } N \in \mathbb{Z}^+, \{B^j\}_{j=1}^N \subset \mathcal{B}, \textit{ and } B \subset \bigcup_{j=1}^N B^j.$$

It follows that there exists a constant $\delta > 0$, depending only on n and β, such that if

$$\rho^\beta S\big(B_{\rho/4}(x_0)\big) \le \delta\rho^\beta S\big(B_\rho(x_0)\big) + C_0 \quad \textit{whenever } B_\rho(x_0) \subset B_1, \tag{3.14}$$

then

$$S(B_{1/2}) \le CC_0$$

for some constant C which depends only on n and β.

We can now give the proof of Theorem 2.1.

Proof of Theorem 2.1. We divide the proof into two steps.

-Step 1. We show that if $s \in (s_0, 1)$ and $u : B_6 \to (-1, 1)$ is a stable solution of the semilinear equation $L_K u + W'(u) = 0$ in B_6 then, for any given $\delta > 0$, we have the estimate

$$\int_{B_1} |\nabla u| \, dx \le \frac{C_\delta}{1-s} + \delta \int_{B_4} |\nabla u| \, dx \tag{3.15}$$

where C_δ depends only on δ, n, λ, Λ, and s_0 (in particular, it does not depend on W).

Indeed, note that in this setting, Lemmas 3.1, 3.3, and 3.4 apply to u.

Hence, by Lemma 3.3, for every $\nu > 0$ there exists $t_0 > 0$ such that

$$\min \Bigg\{ \iint_{A_4} (u(x) - u_t(x))_+ (u(\bar{x}) - u_t(\bar{x}))_- K(x - \bar{x}) \, dx \, d\bar{x}, \\ \iint_{A_4} (u(x) - u_{-t}(x))_+ (u(\bar{x}) - u_{-t}(\bar{x}))_- K(x - \bar{x}) \, dx \, d\bar{x} \Bigg\} \le (\eta + 2\nu) \, t^2$$

holds for every $t \in (0, t_0)$, where

$$\eta = C \mathcal{E}^{\mathrm{Sob}}_{B_4}(u) \tag{3.16}$$

and C depends only on n, λ, and Λ. Now, by (1.8) and since $s \in (0, 1)$, we have $K \ge (2 - s) 2^{-n-s} \lambda \ge 2^{-n-1} \lambda$ in B_2. We deduce that there is some sequence $t_k \in (-1, 1)$ with $t_k \to 0$ such that

$$\limsup_{k \to \infty} \frac{\big\| \big(u(\cdot) - u(\cdot - t_k \boldsymbol{v})\big)_+ \big\|_{L^1(B_1)} \, \big\| \big(u(\cdot) - u(\cdot - t_k \boldsymbol{v})\big)_- \big\|_{L^1(B_1)}}{t_k^2} \le \eta,$$

after changing the value of C in (3.16).

We can now apply Lemma 3.4 and, thanks to (3.13), we arrive at

$$\int_{B_1} |\nabla u| \, dx \le C \left(1 + \sqrt{\mathcal{E}^{\mathrm{Sob}}_{B_4}(u)} \right) < \infty, \tag{3.17}$$

where C depends on n, λ, and Λ.

To keep track of the precise dependence of the constants on s, as $s \uparrow 1$, in what follows C will denote (possibly different) positive constants which depend only on n, λ, Λ, and s_0.

Defining $V(z) := |\nabla u(z)|$ for $z \in B_4$ and $V(z) := 0$ for $z \in \mathbb{R}^n \setminus B_4$, and given x and $\bar{x}$ both in B_4, note that we have

$$|u(x) - u(\bar{x})| = \left| \int_0^1 (\bar{x} - x) \cdot \nabla u(x + t(\bar{x} - x)) \, dt \right| \le |x - \bar{x}| \int_0^1 V(x + t(\bar{x} - x)) \, dt.$$

Using this, that $K \in \mathcal{L}_2$, and $\|u\|_{L^\infty(\mathbb{R}^n)} \le 1$, we deduce (thanks to the presence of the factor $1-s$ on the next left-hand side) that

$$
\begin{aligned}
(1-s)\mathcal{E}^{\mathrm{Sob}}_{B_4}(u) &\le \frac{1}{2}\Lambda(1-s)\Bigg(\iint_{B_4\times B_4} \frac{|u(x)-u(\bar x)|^2}{|x-\bar x|^{n+s}}\,dx\,d\bar x \\
&\qquad\qquad + 2\iint_{B_4\times B_4^c} \frac{2^2}{|x-\bar x|^{n+s}}\,dx\,d\bar x\Bigg) \\
&\le \frac{1}{2}\Lambda(1-s)\iint_{B_4\times B_4} \frac{|u(x)-u(\bar x)|^2}{|x-\bar x|^{n+s}}\,dx\,d\bar x + C \\
&\le \Lambda(1-s)\iint_{B_4\times B_4} \frac{|u(x)-u(\bar x)|}{|x-\bar x|^{n+s}}\,dx\,d\bar x + C \\
&\le \Lambda(1-s)\int_{B_4} dx \int_{B_8} dy\,|y|^{1-n-s}\int_0^1 dt\,V(x+ty) + C \\
&\le \Lambda(1-s)\int_{B_8} dy\,|y|^{1-n-s}\int_0^1 dt\int_{\mathbb{R}^n} dx\,V(x+ty) + C \\
&= \Lambda(1-s)\int_{B_8} dy\,|y|^{1-n-s}\int_{\mathbb{R}^n} dz\,V(z) + C \\
&\le C\int_{\mathbb{R}^n} V(z)\,dz + C \\
&= C\left(1+\int_{B_4}|\nabla u|\,dx\right),
\end{aligned}
\tag{3.18}
$$

where C depends only on n, λ, Λ, and s_0; in particular, it stays bounded as $s \uparrow 1$.

Hence, (3.17), (3.18), and the Cauchy-Schwarz inequality lead to

$$
\begin{aligned}
\int_{B_1}|\nabla u|\,dx &\le C\left(1+\frac{1}{(1-s)^{1/2}}\left(1+\int_{B_4}|\nabla u|\,dx\right)^{1/2}\right) \\
&\le C\left(1+\frac{1}{\delta(1-s)}+\delta\right)+\delta\int_{B_4}|\nabla u|\,dx
\end{aligned}
\tag{3.19}
$$

for all $\delta > 0$.

-Step 2. Now, to conclude the proof of the theorem, we observe that, since its statement is scaling invariant (it is independent of W), we may assume without loss generality that $R = 1$. So, let now $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $L_K u + W'(u) = 0$ in $B_{2R} = B_2$. Given $B_\rho(x_0) \subset B_1$ then the rescaled function $\tilde u(y) = u\left(x_0 + \rho y/4\right)$ is a stable solution in B_6 (since $x_0 + \rho B_1 \subset B_1 \Rightarrow x_0 + \frac{\rho}{4}B_6 \subset B_2$) of the equation $L_{\tilde K}\tilde u + (\rho/4)^s W'(\tilde u) = 0$, where

$$
\tilde K(z) := (\rho/4)^{n+s}K(\rho z/4) \quad \text{belongs again to } \mathcal{L}_2(s,\lambda,\Lambda).
$$

Thus, rescaling the estimate (3.15), which applies to $\tilde{u}$, we obtain

$$\rho^{1-n}\int_{B_{\rho/4}(x_0)}|\nabla u|\,dx \le \delta\,\rho^{1-n}\int_{B_\rho(x_0)}|\nabla u|\,dx + \frac{C_\delta}{1-s} \tag{3.20}$$

for all $\delta > 0$, where C_δ depends only on δ, n, λ, Λ, and s_0.

Therefore, considering the subadditive function

$$S(B) := \int_B |\nabla u|\,dx$$

on the class of balls, and taking $\beta := 1-n$ and δ as given by Lemma 3.5 (and hence depending only on n), we find that

$$\int_{B_{1/2}}|\nabla u|\,dx = S(B_{1/2}) \le \frac{C}{1-s},$$

where C is a constant which depends only on n, λ, Λ, and s_0.

By scaling and using a standard covering argument, we obtain the same estimate with $B_{1/2}$ replaced by $B_1 = B_R$, concluding the proof. □

Proof of Corollary 2.4. Let $s \in (0,1)$. Let $u_R := u(R\,\cdot\,)$. We combine the estimate $\int_{B_1}|\nabla u_R|\,dx \le C(1-s)^{-1}$ of Theorem 2.1 with the interpolation type inequality

$$(1-s)\mathcal{E}^{\mathrm{Sob}}_{B_1}(u_R) \le C\big(1 + \int_{B_1}|\nabla u_R|\,dx\big)$$

—which we proved, in a different ball, in (3.18)— to obtain $\mathcal{E}^{\mathrm{Sob}}_{B_1}(u_R) \le C(1-s)^{-2}$. Now, the corollary follows after rescaling. □

4. Sobolev energy controls potential energy

In this section, we give the proof of Proposition 2.5. In particular, we will have $s \in (0,2]$, and we will include the case of the classical Allen-Cahn equation. All results in this section require the cubes to have large enough diameter and, in particular, the equation to be posed in a sufficiently large ball (as in the statement Proposition 2.5).

The proof is based on a suitable cube decomposition —which may be of independent interest— and covering arguments. We identify two types of cubes: of 'type I" and of "type II". Type I cubes will contain at least one point where $u \sim 0$. Cubes of type II are cubes in which either $u \sim 1$ or $u \sim -1$ in the whole cube. By cube, we always mean a set of the form $Q = x_0 + (-l,l)^n$ for some $x_0 \in \mathbb{R}^n$ and $l > 0$.

The following three lemmas are used in the proof of Proposition 2.5. Essentially, the first one (Lemma 4.1) is used to show that the contribution of a cube of type I to the total Sobolev energy controls its contribution to the potential energy. The third lemma (Lemma 4.3) will prove this same property for cubes of type II.

The second lemma states that if a cube is not of type I (i.e., it contains no points where $u \sim 0$), then the corresponding half-cube is of type II (i.e., we have either $u \sim 1$ or $u \sim -1$ in all of it). Thus, through an appropriate covering argument, this second lemma establishes that there is essentially a dichotomy between cubes of type I and cubes of type II.

Throughout this section W is the double-well potential $\frac{1}{4}(1-u^2)^2$. It satisfies

$$\begin{cases} -W''(t) = -W''(-t) \geq \nu_0 & \text{for } 0 \leq t \leq c_0 \\ W''(t) = W''(-t) \geq \nu_0 & \text{for } 1 - c_0 \leq t \leq 1 \\ -W'(t) = W'(-t) \geq \nu_1 & \text{for } \frac{c_0}{2} \leq t \leq 1 - c_0 \end{cases} \tag{4.1}$$

for some constants $c_0 \in (0, \frac{2}{3})$, $\nu_0 > 0$, and $\nu_1 > 0$ which are totally universal. To check their existence quickly, note that once c_0 and ν_0 have been chosen small enough to guarantee the first two properties, the third condition will be satisfied for ν_1 small enough.

Lemma 4.1. *Given $s_0 > 0$, let $n \geq 2$, $s \in (s_0, 2]$, $W(u) = \frac{1}{4}(1-u^2)^2$, and K satisfy* (1.8) *and* (1.9). *Let $R > 0$ and $u : \mathbb{R}^n \to (-1, 1)$ be a stable solution of $L_K u + W'(u) = 0$ in B_{2R}.*

Then, there exist constants $D_0 \geq 1$ and $d_0 > 0$, which depend only on n, λ, Λ, and s_0, such that the following statement holds true.

For every cube $Q \subset B_R$ satisfying $\mathrm{diam}(Q) \geq D_0$ *and*

$$\{|u| \leq c_0/2\} \cap Q \neq \varnothing$$

for the constant c_0 in (4.1), *there are two cubes $Q^{(1)}$, $Q^{(2)}$ contained in Q and with diameters d_0 for which*

$$\inf_{Q^{(1)}} u - \sup_{Q^{(2)}} u \geq c_0/4. \tag{4.2}$$

Proof. Within the proof, C will denote different positive constants which depend only n, λ, Λ, and s_0. The constants D_0 and d_0 will be chosen to have this same dependence.

Let $Q \subset B_R$ be some cube satisfying $\mathrm{diam}(Q) = D \geq D_0$.

-*Step 1.* Let us prove that $\mathrm{osc}_Q u \geq c_0/2$. Indeed, arguing by contradiction assume that

$$\mathrm{osc}_Q u < c_0/2, \quad \text{and recall that we have } \{|u| \leq c_0/2\} \cap Q \neq \varnothing.$$

Then, $|u| \leq c_0$ in Q and using (4.1) and the stability inequality (1.12), we obtain

$$\begin{aligned} \nu_0 \int_{\mathbb{R}^n} \xi^2\, dx &\leq -\int_{\mathbb{R}^n} W''(u)\xi^2\, dx \\ &\leq \frac{1}{2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \big|\xi(x) - \xi(\bar{x})\big|^2 K(x - \bar{x})\, dx\, d\bar{x} \\ &\leq C(2-s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\big|\xi(x) - \xi(\bar{x})\big|^2}{|x - \bar{x}|^{n+s}}\, dx\, d\bar{x} \end{aligned} \tag{4.3}$$

for every $\xi \in C^1_c(Q)$.

Now, if the diameter D of Q is large enough, we may contradict this inequality just by scaling. Indeed, denote by Q_d the cube centered at 0 and with diameter d, and let x_0 be the center of Q and D its diameter. Choose (universally) a function $\eta \not\equiv 0$ in $C_c^\infty(Q_1)$. Take now $\xi(x) = \eta((x - x_0)/D)$ and notice that

$$(2-s) \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{\left|\xi(x) - \xi(\bar{x})\right|^2}{|x - \bar{x}|^{n+s}}\, dx\, d\bar{x} \le C D^{n-s} \|\nabla \eta\|^2_{L^\infty(Q_1)}. \tag{4.4}$$

while

$$\int_{\mathbb{R}^n} \xi^2\, dx = cD^n \tag{4.5}$$

Choosing $D \ge D_0 \ge 1$ large enough (note that $D_0^{n-s} \le D_0^{n-s_0}$), we contradict (4.3). This proves Step 1.

Note that the same argument applies in the case $s = 2$ when we replace $2 - s$ times the double integral by the classical Dirichlet norm.

-*Step 2.* We now use Step 1 and regularity estimates for the equation to show that there are two cubes $Q^{(1)}$, $Q^{(2)}$ contained in Q and with diameters d_0 such that $\inf_{Q^{(1)}} u - \sup_{Q^{(2)}} u \ge c_0/4$.

Indeed, first note that, as shown in Appendix C, we have the estimate $|\nabla u| \le \bar{C}$ in B_R, for some constant $\bar{C}$ depending only on the quantities stated at the beginning of the proof; here we use that $Q \subset B_R$ and thus $\operatorname{diam}(B_R) \ge \operatorname{diam}(Q) = D \ge D_0 \ge 1$. Let

$$d_0 = \min\left\{D_0, \frac{c_0}{8\bar{C}}\right\}.$$

Now, by Step 1 there are two points $x_1, x_2 \in \overline{Q}$ such that $u(x_1) - u(x_2) \ge c_0/2$. Let $Q^{(i)}$ be any two cubes with diameter d_0 such that $x_i \in \overline{Q^{(i)}} \subset \overline{Q}$ (recall that $d_0 \le D_0 \le \operatorname{diam}(Q)$). Then we readily show that (4.2) is satisfied by these cubes. □

We can now state the second lemma.

Lemma 4.2. *Given $s_0 > 0$, let $n \ge 2$, $s \in (s_0, 2]$, $W(u) = \frac{1}{4}(1-u^2)^2$, and K satisfy* (1.8) *and* (1.9). *Let $R > 0$ and $u : \mathbb{R}^n \to (-1, 1)$ be a stable solution of $L_K u + W'(u) = 0$ in B_{2R}.*

Then, there exists a constant $D_0 \ge 1$, which depends only on n, λ, Λ, and s_0, such that the following statement holds true.

For every cube $Q \subset B_R$ with $\operatorname{diam}(Q) \ge D_0$ *we have*

$$u \ge 1 - c_0 \text{ in } Q' \quad \text{ if } u > c_0/2 \text{ in } Q \tag{4.6}$$

and

$$u \le -1 + c_0 \text{ in } Q' \quad \text{ if } u < -c_0/2 \text{ in } Q, \tag{4.7}$$

where c_0 is the constant in (4.1) *and Q' is the cube with the same center as Q and half its diameter.*

Proof. Within the proof, the constant C will depend only n, λ, Λ, and s_0. The constant D_0 will be chosen to have this same dependence.

Let $Q \subset B_R$ be some cube satisfying $\operatorname{diam}(Q) = D \ge D_0$.

We prove only (4.6) since, by the even symmetry of W, (4.7) follows applying (4.6) to $-u$.

Assume that $u > c_0/2$ in Q. Using (4.1) we have

$$L_K u = -W'(u) \geq \nu_1 \quad \text{in } Q \setminus \{u \geq 1 - c_0\}.$$

Let $\eta \in C^\infty(\mathbb{R}^n)$ with $\eta \equiv 1 - c_0$ in $Q_{1/2}$, $-1 \leq \eta \leq 1 - c_0$ in Q_1, and $\eta \equiv -1$ outside Q_1 (recall that Q_r denotes the cube centered at 0 and with diameter r). Consider

$$\tilde{\eta}(x) = \eta((x - x_0)/D)$$

where x_0 is the center of Q and D its diameter. Note that $D \geq D_0$ and that, by taking D_0 large enough, we will have

$$\left|L_K \tilde{\eta}\right| \leq CD^{-s} \leq CD_0^{-s} \leq \nu_1$$

with C and D_0 as stated at the beginning of the proof.[5] Let us then show that

$$u \geq \tilde{\eta} \quad \text{in } Q. \tag{4.8}$$

Indeed, let $U = Q \setminus \{u \geq 1 - c_0\}$. Since $\tilde{\eta} \leq 1 - c_0$ in Q and $\tilde{\eta} \equiv -1$ outside of Q we have $\tilde{\eta} \leq u$ outside of U. On the other hand

$$L_K(u - \tilde{\eta}) \geq L_K u - \left|L_K \tilde{\eta}\right| \geq 0 \quad \text{in } U,$$

and thus, the maximum principle leads to (4.8). Finally, since by construction $\tilde{\eta} \equiv 1 - c_0$ in Q', the lemma follows. □

Finally, we can state our last auxiliary lemma in this section.

Lemma 4.3. *Given $s_0 > 0$, let $n \geq 2$, $s \in (s_0, 2]$, $W(u) = \frac{1}{4}(1 - u^2)^2$, and K satisfy* (1.8) *and* (1.9). *Let $R > 0$ and $u : \mathbb{R}^n \to (-1, 1)$ be a stable solution of $L_K u + W'(u) = 0$ in B_{2R}.*

Assume that for some cube $Q \subset B_R$, we have that

$$1 - |u| \leq c_0 \quad \text{in } Q,$$

where c_0 is the constant in (4.1).

Then

$$(2 - s) \int_Q \int_{\mathbb{R}^n} \frac{|u(x) - u(\bar{x})|^2}{|x - \bar{x}|^{n+s}} \, dx \, d\bar{x} \geq \kappa_0 \int_Q (1 - |u|)^2 \, dx, \tag{4.9}$$

where $\kappa_0 > 0$ is a constant depending only on n, λ, Λ, s_0, and $\operatorname{diam}(Q)$.

When $s = 2$, the left hand side of (4.9) *must be replaced by $\int_Q |\nabla u|^2 \, dx$.*

Proof. We will do the proof in the case $1 - c_0 \leq u \leq 1$ in Q (the case $-1 \leq u \leq -1 + c_0$ is similar). Since, for simplicity, the lemma is stated for the explicit quartic potential $W(u) = \frac{1}{4}(1 - u^2)^2$, we may take $c_0 := 1 - 1/\sqrt{2}$, and hence

$$W'' = 3u^2 - 1 \in [1/2, 2] \quad \text{for } u \in [1 - c_0, 1]. \tag{4.10}$$

Let $v := 1 - u$. By assumption $v \geq 0$ in all of $\mathbb{R}^n$ and $v \leq c_0$ in Q. Hence, using (4.10) and

$$L_K v = -L_K u = W'(u) = W'(u) - W'(1)$$

[5]Here we use the presence of the factor $2 - s$ on the upper bound for the kernel K as in (4.4). Instead, when $s = 2$, the bound is obvious.

we obtain

$$-2v \le L_K v \le -\frac{1}{2}v \quad \text{in } Q. \tag{4.11}$$

Notice that

$$(2-s)\int_Q\int_{\mathbb{R}^n}\frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x} = (2-s)\int_Q\int_{\mathbb{R}^n}\frac{|v(x)-v(\bar{x})|^2}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x}$$

and

$$\int_Q W(u)dx \le \int_Q \frac{2^2(1-u)^2}{4}dx = \int_Q v^2 dx.$$

Hence, to prove the lemma, we need to show that there exists a constant C depending only on n, λ, Λ, s_0, and $\mathrm{diam}(Q)$, such that

$$C(2-s)\int_Q\int_{\mathbb{R}^n}\frac{|v(x)-v(\bar{x})|^2}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x} \ge \int_Q v^2 dx.$$

To prove this, let $\tilde{v} := v/\|v\|_{L^2(Q)}$ and let

$$\kappa := (2-s)\int_Q\int_{\mathbb{R}^n}\frac{|\tilde{v}(x)-\tilde{v}(\bar{x})|^2}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x}.$$

In the remaining of the proof, we bound κ by below.

First, by the fractional Poincaré inequality, we have

$$\kappa \ge (2-s)\int_Q\int_Q\frac{|\tilde{v}(x)-\tilde{v}(\bar{x})|^2}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x} \ge c_0\int_Q(\tilde{v}-t)^2,$$

where $t = \frac{1}{|Q|}\int_Q \tilde{v}\,dx$ and $c_0 = c_0(n, s_0, \mathrm{diam}(Q)) > 0$.

Note that, by the triangle inequality, we have

$$\begin{aligned} 1-(\kappa/c_0)^{1/2} &\le \|\tilde{v}\|_{L^2(Q)} - \|\tilde{v}-t\|_{L^2(Q)} \\ &\le t|Q|^{1/2} \le \|\tilde{v}\|_{L^2(Q)} + \|\tilde{v}-t\|_{L^2(Q)} \le 1+(\kappa/c_0)^{1/2}. \end{aligned} \tag{4.12}$$

Hence, if κ is sufficiently small (which we may, of course, assume), we have

$$\frac{1}{2}|Q|^{-1/2} \le t \le 2|Q|^{-1/2}. \tag{4.13}$$

On the other hand, $\tilde{v}$ satisfies $L_K\tilde{v} \le -\frac{1}{2}\tilde{v}$. Thus, if we fix $\xi \in C_c^\infty(Q')$ such that $\int_{\mathbb{R}^n}\xi\,dx = 1$ and $\xi \ge 0$, where Q' is the cube with the same center as Q and half its diameter, we have

$$\begin{aligned} \int_{\mathbb{R}^n}\tilde{v}L_K\xi\,dx &\le -\frac{1}{2}\int_{\mathbb{R}^n}\tilde{v}\xi\,dx \le -\frac{1}{2}\int_{\mathbb{R}^n}t\xi\,dx + \int_{\mathbb{R}^n}|\tilde{v}-t|\xi\,dx \\ &\le -\frac{t}{2} + \|\xi\|_{L^2(Q)}\|\tilde{v}-t\|_{L^2(Q)} \le -\frac{t}{2} + C(\kappa/c_0)^{1/2}. \end{aligned} \tag{4.14}$$

At the same time, since $\int_{\mathbb{R}^n}L_K\xi\,dx = 0$ we have

$$\int_{\mathbb{R}^n}\tilde{v}L_K\xi\,dx = \int_Q(\tilde{v}-t)L_K\xi\,dx + \int_{\mathbb{R}^n\setminus Q}(\tilde{v}-t)L_K\xi\,dx.$$

Now similarly to before

$$\left| \int_Q (\tilde v - t) L_K\xi \, dx \right| \le \|L_K\xi\|_{L^2(Q)} \|\tilde v - t\|_{L^2(Q)} \le C(\kappa/c_0)^{1/2}.$$

Also, since $|L_K\xi(x)| \le C(2-s)\mathrm{dist}(x,Q')^{-n-s}$ for $x \in \mathbb{R}^n \setminus Q$, we have

$$\left| \int_{\mathbb{R}^n\setminus Q} (\tilde v - t) L_K\xi \, dx \right| \le C(2-s) \int_{\mathbb{R}^n\setminus Q} \frac{|\tilde v - t|}{\mathrm{dist}(x,Q')^{n+s}} \, dx.$$

Now, since

$$\int_{\mathbb{R}^n\setminus Q} \frac{2-s}{\mathrm{dist}(x,Q')^{n+s}} \, dx \le C$$

and

$$\mathrm{dist}(x,Q')^{-n-s} \le C|x-\bar x|^{-n-s} \quad \text{for all } \bar x \in Q$$

we obtain, also using the definition of t,

$$\begin{aligned}
(2-s)\int_{\mathbb{R}^n\setminus Q} \frac{|\tilde v(x) - t|}{\mathrm{dist}(x,Q')^{n+s}} \, dx &\le (2-s)\frac{1}{|Q|}\int_Q d\bar x \int_{\mathbb{R}^n\setminus Q} dx \, (2-s)\frac{|\tilde v(x) - \tilde v(\bar x)|}{\mathrm{dist}(x,Q')^{n+s}} \\
&\le C\left(\int_Q d\bar x \int_{\mathbb{R}^n\setminus Q} dx\, (2-s)\frac{|\tilde v(x) - \tilde v(\bar x)|^2}{\mathrm{dist}(x,Q')^{n+s}} \right)^{1/2} \\
&\le C\left(\int_Q \int_{\mathbb{R}^n\setminus Q} (2-s)\frac{|\tilde v(x) - \tilde v(\bar x)|^2}{|x-\bar x|^{n+s}} \, dx\, d\bar x \right)^{1/2} \\
&\le C\kappa^{1/2}.
\end{aligned}$$

Hence, we have shown

$$\left| \int_{\mathbb{R}^n} \tilde v L_K\xi \, dx \right| \le C\kappa^{1/2}.$$

Using also (4.14), we deduce

$$\frac{t}{2} \le -\int_{\mathbb{R}^n} \tilde v L_K \xi + C\kappa^{1/2} \le 2C\kappa^{1/2}.$$

Hence, recalling (4.13), this shows the desired lower bound on κ. □

We can now give the

Proof of Proposition 2.5. Throughout the proof, all the constants depend only on n, s, λ, Λ, and s_0. Let D_0 and d_0 be constants for which Lemmas 4.1 and 4.2 hold —for this, take them to be, respectively, the largest of the constants D_0 in the lemmas and the smallest of the constants d_0.

Let $\mathcal{F}$ denote the family of cubes Q with center in the lattice $(D_0/\sqrt{n})\,\mathbb{Z}^n$ and side-length $2D_0/\sqrt{n}$, that is, of the form

$$Q = \frac{D_0}{\sqrt n}\, z_0 + \left(-\frac{D_0}{\sqrt n}, \frac{D_0}{\sqrt n} \right)^n \quad \text{for some } z_0 \in \mathbb{Z}^n.$$

Let $\mathcal{F}_R$ be the family of those cubes $Q \in \mathcal{F}$ such that $Q \subset B_R$. Given a cube Q, denote by Q' the cube with the same center as Q and half its diameter. Finally, define

$$R_0 := \frac{3}{2} D_0,$$

and recall that we assume $R > R_0$.

It is then easy to check that:

(a) For all $Q \in \mathcal{F}_R$, we have $Q \subset B_R$ and $\operatorname{diam}(Q) = 2D_0$.
(b) Each point of $\mathbb{R}^n$ belongs to at most 2^n cubes of the family $\mathcal{F}_R$.
(c) The union $\bigcup_{Q \in \mathcal{F}_R} Q'$ is disjoint and covers B_{R-R_0} except for a set of measure zero.

Throughout the proof, it is easy to check that all arguments hold true when $s = 2$ by replacing, within the double integrals, $2 - s$ times one of the integrals by the pointwise gradient squared.

We first notice that, by properties (a) and (b),

$$\begin{aligned} \mathcal{E}^{\mathrm{Sob}}_{B_R}(u) &\geq \frac{1}{4} \int_{B_R} \int_{\mathbb{R}^n} |u(x) - u(\bar{x})|^2 K(x - \bar{x})\, dx\, d\bar{x} \\ &\geq \frac{\lambda(2-s)}{4} \int_{B_R} \int_{\mathbb{R}^n} \frac{|u(x) - u(\bar{x})|^2}{|x - \bar{x}|^{n+s}}\, dx\, d\bar{x} \\ &\geq \frac{\lambda(2-s)}{4 \cdot 2^n} \sum_{Q \in \mathcal{F}_R} \int_Q \int_{\mathbb{R}^n} \frac{|u(x) - u(\bar{x})|^2}{|x - \bar{x}|^{n+s}}\, dx\, d\bar{x}. \end{aligned} \tag{4.15}$$

Next, for each $Q \in \mathcal{F}_R$, we have the following dichotomy. Either

$$\{|u| \leq c_0/2\} \cap Q \neq \varnothing \tag{4.16}$$

or

$$\{|u| \leq c_0/2\} \cap Q = \varnothing. \tag{4.17}$$

Now, if $Q \in \mathcal{F}_R$ satisfies (4.16) then by Lemma 4.1 (since Q has diameter $2D_0$) there are two cubes $Q^{(1)}$ and $Q^{(2)}$ contained in Q, and with diameters d_0, such that

$$\inf_{Q^{(1)}} u - \sup_{Q^{(2)}} u \geq c_0/4.$$

It thus follows, using the fractional Poincaré inequality and

$$\int_{Q'} W(u)\, dx \leq |Q'| \max_{[-1,1]} W \leq C,$$

that

$$
\begin{aligned}
C(2-s)\int_Q\int_Q \frac{|u(x)-u(\bar x)|^2}{|x-\bar x|^{n+s}}\,dx\,d\bar x &\ge \inf_{t\in\mathbb{R}}\int_Q |u(x)-t|^2\,dx \\
&\ge \left|\frac{(\inf_{Q^{(1)}} u - \sup_{Q^{(2)}} u)^2}{2}\right|^2 \left|Q^{(i)}\right| \qquad (4.18)\\
&\ge \left(\frac{c_0}{8}\right)^2\left(\frac{d_0}{\sqrt{n}}\right)^n \ge c\int_{Q'} W(u)\,dx
\end{aligned}
$$

for some constant $c>0$.

On the other hand, if Q satisfies (4.17) then by Lemma 4.2 we have that

$$|u| \ge 1-c_0 \quad \text{in } Q'.$$

Hence, using Lemma 4.3 (with Q replaced by Q') we have

$$(2-s)\int_Q\int_{\mathbb{R}^n} \frac{|u(x)-u(\bar x)|^2}{|x-\bar x|^{n+s}}\,dx\,d\bar x \ge c\int_{Q'}(1-|u|)^2\,dx \ge c\int_{Q'} W(u)\,dx. \qquad (4.19)$$

Finally, recalling property (c) above, we combine (4.18) and (4.19) with (4.15) to obtain

$$
\begin{aligned}
\mathcal{E}^{\rm Sob}_{Q_R}(u) &\ge \frac{\lambda(2-s)}{2^n}\sum_{Q\in\mathcal{F}_R}\int_Q\int_{\mathbb{R}^n}\frac{|u(x)-u(\bar x)|^2}{|x-\bar x|^{n+s}}\,dx\,d\bar x\\
&\ge c\sum_{Q\in\mathcal{F}_R}\int_{Q''} W(u)\,dx\\
&\ge c\int_{B_{R-R_0}} W(u)\,dx\\
&= c\,\mathcal{E}^{\rm Pot}_{B_{R-R_0}}(u),
\end{aligned}
$$

that finishes the proof. □

Let us now prove the bound for the total energy when $s\in(0,1)$.

Proof of Theorem 2.6. Let $s\in(s_0,1)$. By Corollary 2.4 we have that

$$\mathcal{E}^{\rm Sob}_{B_R}(u) \le \frac{C}{(1-s)^2}R^{n-s}. \qquad (4.20)$$

Now, using Proposition 2.5 we deduce from (4.20) the same estimate for $\mathcal{E}^{\rm Pot}_{B_{R/2}}(u)$, provided $R\ge 2R_0$ (since $R-R_0\ge R/2$). The same bound for $\mathcal{E}^{\rm Pot}_{B_{R/2}}(u)$ is obvious when $1\le R\le 2R_0$.

Finally, a standard covering and scaling argument allows controlling the total energy in B_R instead of $B_{R/2}$. □

We close this section with a useful lemma concerning the decay towards ±1 of solutions u_ε of $(-\Delta)^s u_\varepsilon + \varepsilon^{-s}W'(u_\varepsilon)=0$ for points x away from $\{|u_\varepsilon|\le \frac{9}{10}\}$. It will be used in the proof of Proposition 6.2.

Lemma 4.4. *Given $s_0 > 0$, let $n \ge 2$, $s \in (s_0, 2]$, $W(u) = \frac{1}{4}(1-u^2)^2$, and K satisfy (1.8) and (1.9). Let $u_\varepsilon : \mathbb{R}^n \to (-1,1)$ be a solution of $L_K u + \varepsilon^{-s} W'(u) = 0$ in B_{2r}, with $r \ge \varepsilon > 0$, satisfying $1 - |u_\varepsilon| \le c_0$ in B_{2r}, where c_0 is the constant in (4.1).*

Then,

$$0 \le 1 - |u_\varepsilon| \le C\left(\frac{\varepsilon}{r}\right)^s \quad \text{in } B_r,$$

where C depends only on n, λ, Λ, and s_0.

Proof. By scaling —i.e., replacing u by $u(\varepsilon\,\cdot)$— it is enough to consider the case $\varepsilon = 1$. As in the proof of Lemma 4.3, for the explicit quartic potential W we may take $c_0 := 1 - 1/\sqrt{2}$ and suppose $1 - c_0 \le u \le 1$ in B_{2r}. Hence, (4.10) holds and, similarly as in (4.11), the function $v := 1 - u$ satisfies

$$L_K v \le -\tfrac{1}{2} v \quad \text{in } B_{2r}. \tag{4.21}$$

Fix now some smooth radial "bowl-type" function ξ satisfying $\chi_{\mathbb{R}^n \setminus B_1} \ge \xi \ge \chi_{\mathbb{R}^n \setminus B_2}$, and let $\xi_r(x) := \xi(x/r)$. By scaling we have $|L_K \xi_r| \le C_0 r^{-s}$ in $\mathbb{R}^n$.

Let now $w := 2\xi_r + b$, with $b := 4C_0 r^{-s}$. We have $L_K w \ge -2C_0 r^{-s} = -\frac{1}{2} b \ge -\frac{1}{2} w$ in B_{2r}. Hence, recalling (4.21) and using that $w \ge 2 + b \ge 2 \ge v$ in $\mathbb{R}^n \setminus B_{2r}$ we obtain —by the maximum principle— $w \ge v$ in $\mathbb{R}^n$. Hence using that $\xi_r \equiv 0$ in B_r we have shown $b \ge v$ in B_r, and the lemma follows. $\square$

5. Density estimates

In this section we establish density estimates when $s \in (0,1)$. We need to restrict our attention to the case of the fractional Laplacian since a crucial ingredient in the proof is the monotonicity formula from [13]. This monotonicity formula is known to hold only for equations involving the fractional Laplacian since its proof relies on the so-called Caffarelli-Silvestre extension [20], which we recall here below.

By the result in [20], $u : \mathbb{R}^n \to (-1,1)$ is a solution of $(-\Delta)^{s/2} u + W'(u) = 0$ in $\mathbb{R}^n$, for $s \in (0,2)$, if and only if $u(x) = U(x,0)$ where $U : \mathbb{R}^{n+1}_+ \to (-1,1)$, defined in the half-space $\mathbb{R}^{n+1}_+ := \{(x,y) \,:\, x \in \mathbb{R}^n, y > 0\}$, is a solution of

$$\begin{cases} \operatorname{div}(y^{1-s} \nabla U(x,y)) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -d_s \lim_{y \to 0} y^{1-s} \partial_y U(x,y) = -W'(u(x)) & \text{on } \mathbb{R}^n \end{cases} \tag{5.1}$$

and $d_s = 2^{s-1}\Gamma(\frac{s}{2})/\Gamma(1-\frac{s}{2}) > 0$. Here W is any C^3 potential and, for simplicity of notation, ∇ denotes the full gradient —i.e., with respect to the (x,y) variables— in $\mathbb{R}^{n+1}_+$.

Definition 5.1. Given a bounded function u defined on $\mathbb{R}^n$, we will call the unique bounded extension U of u in $\mathbb{R}^{n+1}_+$ satisfying $\operatorname{div}(y^{1-s} \nabla U(x,y)) = 0$, the *$s$-extension* of u (the existence and properties of this extension were studied in [20]; see also [16]).

Let $\widetilde{B}^+_r((x_0,0)) = \{(x,y) \in \mathbb{R}^{n+1}_+ \,:\, |(x,y) - (x_0,0)| < r\}$ and $\widetilde{B}^+_r = \widetilde{B}^+_r((0,0))$ — the tilde is used to distinguish balls in $\mathbb{R}^{n+1}$ from balls in $\mathbb{R}^n$. The energy associated to problem (5.1) in a half-ball $\widetilde{B}_R^+$ is given by

$$\widetilde{\mathcal{E}}_R(U) = \frac{d_s}{2}\int_{\widetilde{B}_R^+} y^{1-s}|\nabla U(x,y)|^2\,dx\,dy + \int_{B_R} W(U(x,0))\,dx. \tag{5.2}$$

As before, we distinguish between the Sobolev and the potential energies using the following notations:

$$\widetilde{\mathcal{E}}_R^{\mathrm{Sob}}(U) = \frac{d_s}{2}\int_{\widetilde{B}_R^+} y^{1-s}|\nabla U(x,y)|^2\,dx\,dy \quad \text{and} \quad \widetilde{\mathcal{E}}_R^{\mathrm{Pot}}(U) = \int_{B_R} W(U(x,0))\,dx.$$

We recall now a result from [19], which allows one to control the Sobolev energy of the s-extension U of u in $\mathbb{R}^{n+1}_+$ by the local contribution of the $H^{s/2}$-seminorm of u itself. We write the result for the specific case of half-balls in $\mathbb{R}^{n+1}$, since this is what we will need later on. The result is true for a general Lipschitz domain Ω, as seen in Proposition 7.1 of [19].

Lemma 5.2 (Proposition 7.1 in [19])**.** *Let $s \in (0,2)$, $\widetilde{B}_\rho^+((x_0,0))$ be, as before, a half-ball in $\mathbb{R}^{n+1}_+$ centered at $(x_0,0)$ and with radius ρ, and let U be the s-extension of u.*

Then,

$$\int_{\widetilde{B}_\rho^+((x_0,0))} y^{1-s}|\nabla U(x,y)|^2\,dx\,dy \le C \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(B^c_{2\rho}(x_0)\times B^c_{2\rho}(x_0))} \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x}$$

for some constant C which depends only on n and s.

We also recall the monotonicity formula established in [13].

Proposition 5.3 (Proposition 3.2 in [13])**.** *Let $s \in (0,2)$, $W \in C^3([-1,1])$ be nonnegative, and $U : \mathbb{R}^{n+1}_+ \to (-1,1)$ be a solution of (5.1).*

Then,

$$\Phi(R) := \frac{1}{R^{n-s}}\widetilde{\mathcal{E}}_R(U)$$

is a nondecreasing function of $R > 0$.

The main ingredients in the proof of our density estimate are the previous monotonicity formula and the BV estimate for stable solutions established in Theorem 2.1.

Before giving the proof of Proposition 2.7, we state the following easy lemma that allows to interpolate between L^1 and $W^{1,1}$.

Lemma 5.4 (Theorem 1 in [11])**.** *Let $s \in (0,1)$ and $R > 0$. Let u be a C^1 function in $B_R \subset \mathbb{R}^n$ satisfying*

$$R^{-n}\int_{B_R} |u+k|\,dx \le V \quad \textit{and} \quad R^{1-n}\int_{B_R} |\nabla u|\,dx \le P$$

for some $k \in \mathbb{R}$ and constants V and P.

Then,

$$R^{s-n}\int_{B_R}\int_{B_R} \frac{|u(x)-u(\bar{x})|}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x} \le CV^{1-s}P^s$$

for some constant C which depends only on n and s.

Proof. It follows, after scaling, from [11, Theorem 1] used with $p_1 = p_2 = p = 1$, $s_1 = 0 < 1 = s_2$, $\theta = 1 - s$, and $\Omega = B_1$. $\square$

In the proof of Proposition 2.7, we will apply Lemma 5.4 above with $k = \pm 1$, $V = \omega_0$, where ω_0 is given in (2.6), and $P = C(1-s)^{-1}$, where $C(1-s)^{-1}$ comes from the BV bound (2.1).

We can now give the proof of our density estimate.

Proof of Proposition 2.7. We argue by contradiction. Assume that there exists $\bar{c} \in (0,1)$ for which $R^{-n}\int_{B_R}|1+u|\,dx \le \omega_0$ and $\{u \ge -\bar{c}\} \cap B_{R/2} \neq \varnothing$.

Throughout the proof, all constants will depend only on $\bar{c}$, n, and s.

First, by continuity of u and by taking $\omega_0 < R^{-n}|B_{R/2}|$, there will be a point $x_0 \in B_{R/2}$ for which $|u(x_0)| \le \bar{c}$. Moreover, by the uniform continuity of u (recall that $|\nabla u| \le C$ in $\mathbb{R}^n$; see Appendix C), we will have that $|u| \le \frac{1+\bar{c}}{2}$ in some ball of radius $r > 0$ centered at x_0 (we emphasize that we can take r, as well as all other constants in the rest of the proof, to depend only on $\bar{c}$, n, and s). Using that $W(u) = \frac{1}{4}(1-u^2)^2$, we deduce

$$r^{s-n}\int_{B_r(x_0)} W(u)\,dx \ge \theta > 0$$

for some positive constant θ.

Let now U be the s-extension of u in $\mathbb{R}^{n+1}_+$. The previous lower bound on the potential energy in $B_r(x_0)$ leads to

$$r^{s-n}\widetilde{\mathcal{E}}_r(U) = r^{s-n}\left(\frac{d_s}{2}\int_{\widetilde{B}^+_r(x_0)} y^{1-s}|\nabla U|^2\,dx\,dy + \int_{B_r(x_0)} W(u)\,dx\right) \ge \theta,$$

where, for simplicity of notation, we keep denoting by $\widetilde{\mathcal{E}}_r$ the energy in the half-ball $\widetilde{B}^+_r((x_0,0))$ centered at $(x_0,0)$ (instead of at the origin).

Applying the monotonicity formula of Proposition 5.3, we deduce that

$$\rho^{s-n}\widetilde{\mathcal{E}}_\rho(U) \ge \theta \quad \text{for every } \rho \ge r. \tag{5.3}$$

We now use Lemma 5.2 to translate the bound in (5.3) for the energy of the extension U into a lower bound for the energy of u. We get

$$\begin{aligned}
\theta &\le \rho^{s-n}\widetilde{\mathcal{E}}_\rho(U)\\
&\le C\rho^{s-n}\left(\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(B^c_{2\rho}(x_0)\times B^c_{2\rho}(x_0))} \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x} + \int_{B_\rho(x_0)} W(u)\,dx\right)\\
&\le C\rho^{s-n}\iint_{B_{2\rho}(x_0)\times\mathbb{R}^n} \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x},
\end{aligned} \tag{5.4}$$

where in the last inequality, we have used Proposition 2.5 to bound the potential energy by the Sobolev energy in the larger ball, which requires to take $\rho \geq R_0$ (with R_0 being the constant in that proposition) since, then, $\rho + R_0 \leq 2\rho$.

We aim now to use assumption (2.6) and Lemma 5.4 in order to find a contradiction with (5.4). To this end, we need to introduce a larger radius R, with $R \geq 4\rho$, and observe that the set of integration in (5.4) satisfies

$$B_{2\rho}(x_0) \times \mathbb{R}^n \subset \big(B_R(x_0) \times B_R(x_0)\big) \cup \big(B_{2\rho}(x_0) \times B_R^c(x_0)\big).$$

Now, applying Lemma 5.4, we are able to control the $W^{1,s}$-seminorm of u in B_R by a positive power of ω_0. More precisely, by (2.6), the quantity $R^{-n}\|u+1\|_{L^1(B_R)}$ is controlled by ω_0. On the other hand, our BV estimate (2.1) gives a bound for $R^{1-n}\|\nabla u\|_{L^1(B_R)}$ by a constant $C(1-s)^{-1}$. Thus, using Lemma 5.4, $|u| \leq 1$, and $R - 2\rho \geq R/2$ (since we take $R \geq 4\rho$), we deduce

$$\begin{aligned}
\theta &\leq \rho^{s-n}\widetilde{\mathcal{E}}_\rho(U) \leq C\left(\frac{\rho}{R}\right)^{s-n} R^{s-n} \iint_{B_R(x_0)\times B_R(x_0)} \frac{|u(x)-u(\bar{x})|}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x} \\
&\qquad + C\rho^{s-n}\iint_{B_{2\rho}(x_0)\times B_R^c(x_0)} \frac{|u(x)-u(\bar{x})|}{|x-\bar{x}|^{n+s}}\,dx\,d\bar{x} \\
&\leq C\left(\frac{\rho}{R}\right)^{s-n}\omega_0^{1-s} + C\rho^{s-n}\iint_{B_{2\rho}(x_0)\times B_R^c(x_0)} \frac{dx\,d\bar{x}}{|x-\bar{x}|^{n+s}} \\
&\leq C\left(\frac{\rho}{R}\right)^{s-n}\omega_0^{1-s} + C\rho^{s}\int_{\frac{R}{2}}^{+\infty} \frac{r^{n-1}}{r^{n+s}}\,dr \\
&\leq C_1\left[\left(\frac{\rho}{R}\right)^{s-n}\omega_0^{1-s} + \left(\frac{\rho}{R}\right)^{s}\right].
\end{aligned} \tag{5.5}$$

We now take R and ρ such that $\frac{\rho}{R} = (\frac{\theta}{4C_1})^{1/s}$. We may ensure $R \geq 4\rho$ (as required before) by increasing the constant C_1 in (5.5), if necessary. Since we needed $\rho \geq \max\{r, R_0\}$ within the proof, with R_0 being the radius from Proposition 2.5, this gives a lower bound for R, which becomes our final choice of radius R_0 in the statement of Proposition 2.7.

Finally, with this choice of ρ/R, (5.5) becomes

$$\theta \leq C_1\left(\frac{\theta}{4C_1}\right)^{\frac{s-n}{s}}\omega_0^{1-s} + \frac{\theta}{4}. \tag{5.6}$$

Therefore, choosing ω_0 small enough, we obtain a contradiction and conclude the proof. □

6. Convergence results

The goal of this section is to prove the following convergence result, which will allow us to give the Proof of Theorem 2.10.

Proposition 6.1. *Let $n \geq 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. Let $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$.*

Then, for every sequence $R_j \uparrow \infty$ there exists a subsequence R_{j_k} such that, defining $u_R(x) := u(Rx)$, we have

$$u_{R_{j_k}} \to u_\infty := \chi_\Sigma - \chi_{\Sigma^c} \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n)$$

for some cone $\Sigma \subset \mathbb{R}^n$ which is nontrivial (i.e., it is not equivalent to $\mathbb{R}^n$ nor to $\varnothing$ up to sets of zero measure) and which is a weakly stable set in $\mathbb{R}^n$ for the fractional perimeter P_s.

Moreover, we have the following convergence of the localized energies:

$$\mathcal{E}^{\mathrm{Sob}}_{B_{R'}}(u_{R_{j_k}}) \to \mathcal{E}^{\mathrm{Sob}}_{B_{R'}}(u_\infty) \quad \text{and} \quad R^s_{j_k} \int_{B_{R'}} W(u_{R_{j_k}})\, dx \to 0 = \int_{B_{R'}} W(u_\infty)\, dx, \tag{6.1}$$

as $k \uparrow \infty$, in every ball $B_{R'} \subset \mathbb{R}^n$.

Proof. We divide the proof into two steps.

-*Step 1.* We start by proving that some subsequence of u_{R_j} converges in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to $u_\infty := \chi_\Sigma - \chi_{\Sigma^c}$, for some nontrivial cone Σ, and that the convergence of energies (6.1) holds.

Throughout the proof, C will denote (possibly different) positive constants which depend only on n and s.

Let us take a radius $R' \geq 1$. Theorem 2.1 and Corollary 2.4 yield that

$$\int_{B_R} |\nabla u|\, dx \leq C R^{n-1} \quad \text{and} \quad \mathcal{E}_{B_R}(u) \leq C R^{n-s} \quad \text{for all } R \geq 1.$$

Thus, for $R \geq 1$, by rescaling we deduce

$$\int_{B_{R'}} |\nabla u_R|\, dx \leq C_{R'} \quad \text{and} \quad \mathcal{E}^{\mathrm{Sob}}_{B_{R'}}(u_R) \leq C_{R'}, \tag{6.2}$$

where $C_{R'}$ denote constants which depend only on R', n, and s.

Let U be, as in the previous section, the s-extension of u in $\mathbb{R}^{n+1}_+$. By Lemma 5.2, and since the potential energy is nonnegative, we have

$$\widetilde{\mathcal{E}}^{\mathrm{Sob}}_R(U) \leq C \iint_{(\mathbb{R}^n \times \mathbb{R}^n) \setminus (B^c_{2R} \times B^c_{2R})} \frac{|u(x) - u(\bar{x})|^2}{|x - \bar{x}|^{n+s}}\, dx\, d\bar{x} = 4C\, \mathcal{E}^{\mathrm{Sob}}_{B_{2R}}(u) \leq C R^{n-s}$$

for $R \geq 1$. In addition, by the monotonicity formula of Proposition 5.3, the quantity

$$\Phi(R) = R^{s-n} \widetilde{\mathcal{E}}_R(U)$$

is monotone nondecreasing. At the same time, by the previous bound on the Sobolev energy and by Proposition 2.5 (which gives control on the potential energy in B_R by the Sobolev energy in B_{2R} if we take $R + R_0 \leq 2R$), we deduce that Φ is bounded above by a finite constant. We deduce that

$$\Phi(R'R) - \Phi(\bar{R}) \to 0 \quad \text{as } R'R \geq \bar{R} \uparrow \infty. \tag{6.3}$$

Now, from the proof of the monotonicity formula, which is based on a Pohozaev identity —see the proof of Proposition 3.2 in [13]— we have that

$$\Phi'(\rho) = \frac{d_s}{\rho^{n-s}} \int_{\partial^+ \widetilde{B}_\rho^+} y^{1-s} (\partial_r U)^2 \, d\mathcal{H}^n(x,y) + \frac{s}{\rho^{n-s+1}} \int_{B_\rho} W(u)\, dx,$$

where ∂^+ denotes the part of the boundary contained in the open half-space $\{y > 0\}$ and ∂_r denotes the radial derivative in $\mathbb{R}^{n+1}_+$. After rescaling, this becomes

$$\Phi'(R\tilde\rho) = \frac{d_s}{R\tilde\rho^{n-s}} \int_{\partial^+ \widetilde{B}_{\tilde\rho}^+} \tilde y^{1-s} (\partial_r U_R)^2 \, d\mathcal{H}^n(\tilde x,\tilde y) + \frac{s}{R^{1-s}\, \tilde\rho^{n-s+1}} \int_{B_{\tilde\rho}} W(u_R)\, d\tilde x, \tag{6.4}$$

where U_R is the s-extension of u_R. We integrate now with respect to ρ to obtain, for $R'/2 \geq \bar R/R$,

$$\begin{aligned}
\Phi(R'R) - \Phi(\bar R) &= \int_{\bar R}^{R'R} \Phi'(\rho)\, d\rho = R \int_{\bar R/R}^{R'} \Phi'(R\tilde\rho)\, d\tilde\rho \\
&= d_s \int_{\bar R/R}^{R'} d\tilde\rho \, \tilde\rho^{s-n} \int_{\partial^+ \widetilde{B}_{\tilde\rho}^+} \tilde y^{1-s} (\partial_r U_R)^2 \, d\mathcal{H}^n(\tilde x,\tilde y) \\
&\qquad + sR^s \int_{\bar R/R}^{R'} d\tilde\rho \, \tilde\rho^{s-n-1} \int_{B_{\tilde\rho}} W(u_R)\, d\tilde x \\
&\geq d_s (R')^{s-n} \int_{\widetilde{B}_{R'}^+ \setminus \widetilde{B}_{\bar R/R}^+} \tilde y^{1-s} (\partial_r U_R)^2 \, d\tilde x\, d\tilde y, \\
&\qquad + sR^s (R')^{s-n-1} \frac{R'}{2} \int_{B_{R'/2}} W(u_R)\, d\tilde x \qquad \text{if } R \geq 2\bar R/R'.
\end{aligned} \tag{6.5}$$

Note that, given R' and $\bar R$, we have

$$\int_{\widetilde{B}_{\bar R/R}^+} \tilde y^{1-s} (\partial_r U_R)^2 \, d\tilde x\, d\tilde y \leq C R^2 \int_{\widetilde{B}_{\bar R/R}^+} \tilde y^{1-s} \, d\tilde x\, d\tilde y \leq C \bar R^{2-s+n} R^{s-n} \to 0 \qquad \text{as } R \uparrow \infty.$$

This together with (6.3) and (6.5), leads to

$$\int_{\widetilde{B}_{R'}^+} y^{1-s} (\partial_r U_R)^2 \, d\tilde x\, d\tilde y \to 0 \quad \text{as } R \uparrow \infty \tag{6.6}$$

and

$$R^s \int_{B_{R'/2}} W(u_R)\, d\tilde x \to 0 \quad \text{as } R \uparrow \infty \tag{6.7}$$

for every $R' \geq 1$.

Next, choose any $\sigma \in (s,1)$ and let $N \geq 1$ be an integer. By the $W^{1,1}$ estimate (6.2) applied with $R' = N$, and since $|u| \leq 1$, Lemma 5.4 (applied with s replaced by σ) leads to

$$\int_{B_N \times B_N} \frac{\left|u_R(x) - u_R(\bar x)\right|^2}{|x - \bar x|^{n+\sigma}} \, dx\, d\bar x \leq C_{N,\sigma}$$

for all $R \geq 1$, where $C_{N,\sigma}$ is a constant depending only on N, σ, n, and s. Hence, using that $|u_R| \leq 1$ and the compactness of $W^{\sigma/2,2}$ inside $W^{s/2,2}$, there exists a subsequence $u_{R_{j_k}}$, that we still denote by u_{R_k}, and a function u_∞ such that

$$\left\|u_{R_k} - u_\infty\right\|_{L^2(B_N)} + \int_{B_N\times B_N} \frac{\left|(u_{R_k} - u_\infty)(x) - (u_{R_k} - u_\infty)(\bar{x})\right|^2}{|x-\bar{x}|^{n+s}}\, dx\, d\bar{x} \to 0 \quad (6.8)$$

as $k \uparrow \infty$. In addition, letting $N \uparrow \infty$, taking further subsequences, and using a Cantor diagonal argument, we obtain a new subsequence converging in L^2 in every ball of $\mathbb{R}^n$.

Now, given $R' \geq 1$, we use once more Lemma 5.2 (now applied to $U_{R_k} - U_\infty$ in $\widetilde{B}^+_{R'}$, where U_∞ denotes the s-extension of u_∞) to control

$$\int_{\widetilde{B}^+_{R'}} y^{1-s}\left|\nabla U_{R_k} - \nabla U_\infty\right|^2 dx\, dy$$

by the double integral in (6.8) computed now in $B_{2R'} \times \mathbb{R}^n$. Taking $N > 2R'$, since (6.8) gives control on the integrals computed over $B_{2R'} \times B_N$, it only remains to make the double integral on $B_{2R'} \times (\mathbb{R}^n \setminus B_N)$ arbitrary small. Such bound is obvious, since $|u_{R_k} - u_\infty| \leq 2$, by taking N large enough. Therefore, we conclude

$$\int_{\widetilde{B}^+_{R'}} y^{1-s}\left|\nabla U_{R_k} - \nabla U_\infty\right|^2 dx\, dy \to 0 \qquad (6.9)$$

as $k \uparrow \infty$.

From this strong convergence and the local uniform convergence of u_{R_k}, passing to the limit in (6.6) and (6.7) we obtain

$$\int_{\widetilde{B}^+_{R'/2}} y^{1-s}(\partial_r U_\infty)^2\, dx\, dy = 0 \quad \text{and} \quad \int_{B_{R'/2}} W(u_\infty)\, dx = 0$$

for every $R' \geq 1$. Therefore, $\partial_r U_\infty = 0$ in $\mathbb{R}^{n+1}_+$ and $W(u_\infty) = 0$ in $\mathbb{R}^n$. In other words, U_∞ and its trace u_∞ are homogeneous of degree 0, and u_∞ takes values ± 1 —the two wells of W. Equivalently, we have that

$$u_\infty = \chi_\Sigma - \chi_{\Sigma^c} \quad \text{in } \mathbb{R}^n$$

for some cone Σ. In addition, by the same convergences for u_{R_k} that we have just used, we see that (6.1) holds.

Finally, using the monotonicity of Φ, (6.9), and (6.7), we obtain

$$\begin{aligned} 0 < \Phi(1) \leq \Phi(R) &= R^{s-n}\widetilde{\mathcal{E}}_R(U) = \widetilde{\mathcal{E}}^{\mathrm{Sob}}_1(U_R) + R^s \int_{B_1} W(u_R)\, dx) \\ &\leq \lim_{R\uparrow\infty} \left(\widetilde{\mathcal{E}}^{\mathrm{Sob}}_1(U_R) + R^s \int_{B_1} W(u_R)\, dx\right) \\ &= \widetilde{\mathcal{E}}^{\mathrm{Sob}}_1(U_\infty). \end{aligned}$$

Thus U_∞ and u_∞ have positive energy, and hence Σ is nontrivial (i.e. it is not equal to $\mathbb{R}^n$ nor $\varnothing$ up to sets of measure zero).

-*Step 2*. It remains to prove that Σ is a weakly stable set in $\mathbb{R}^n$ for the fractional perimeter P_s.

To this end, let us take an arbitrary smooth vector field $X = X(x,t)$ that is compactly supported in $B_1 \times (-1,1)$ (by scaling, since Σ is a cone, we may take the support to be the unit ball) and let $\Psi = \Psi_t(x)$ denote the map $(x,t) \mapsto \phi_X^t(x)$, where ϕ_X^t is the integral flow defined by the ODE

$$\frac{d}{dt}\phi_X^t(x) = X(\phi_X^t(x),t) \quad \text{with initial condition} \quad \phi_X^0(x) = x.$$

Note that $\Psi_0 = \mathrm{Id}$ in $\mathbb{R}^n$, $\Psi_t = \mathrm{Id}$ outside of B_1, and that $\Psi_t : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism for $|t|$ small.

Let us introduce the rescaled energy functional (of which u_R is a stable critical point)

$$\mathcal{E}_{B_1}^R(v) := \frac{1}{4}\iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(B_1^c\times B_1^c)} \frac{|v(y)-v(\bar y)|^2}{|y-\bar y|^{n+s}}\,dy\,d\bar y + R^s\int_{B_1} W(v(y))\,dy\,,$$

We first show that the function $u_{R,t} := u_R \circ \Psi_t^{-1}$ satisfies

$$\mathcal{E}_{B_1}^R(u_{R,t}) \ge \mathcal{E}_{B_1}^R(u_R) - C_\Psi t^3, \quad \text{for } t \in (-T_\Psi, T_\Psi), \tag{6.10}$$

where $T_\Psi > 0$ and C_Ψ will be, from now on, different positive constants which depend only on X, n, and s —in particular, they are independent of R.

To prove this, and as in the proof of Lemma 3.1, we make the change of variables $y = \Psi_t^{-1}(x)$, $\bar y = \Psi_t^{-1}(\bar x)$ for $|t|$ small. Since Ψ_t^{-1} sends B_1 and B_1^c onto themselves, setting $A_1 := (\mathbb{R}^n\times\mathbb{R}^n)\setminus(B_1^c\times B_1^c)$ we have

$$\mathcal{E}_{B_1}^R(u_{R,t}) = \frac{1}{4}\iint_{A_1} \frac{|u_R(y)-u_R(\bar y)|^2}{\big|\Psi_t(y)-\Psi_t(\bar y)\big|^{n+s}}\, J_t(y)\,J_t(\bar y)\,dy\,d\bar y + R^{-s}\int_{B_1} W(u_R(y))\,J_t(y)\,dy, \tag{6.11}$$

where J_t is the Jacobian $\det D\Psi_t$.

A Taylor expansion for J_t yields

$$\big|J_t(y) - 1 - h_1(y)t - h_2(y)t^2\big| \le C_\Psi t^3 \quad \text{for } t \in (-T_\Psi, T_\Psi), \tag{6.12}$$

with $\|h_1\|_{L^\infty(\mathbb{R}^n)} + \|h_2\|_{L^\infty(\mathbb{R}^n)} \le C_\Psi$. Similarly, we have

$$\Big|\,\Psi_t(y)-\Psi_t(\bar y) - (y-\bar y) - |y-\bar y|\big(g_1(y,\bar y)t + g_2(y,\bar y)t^2\big)\,\Big| \le C_\Psi|y-\bar y|t^3$$

for $t \in (-T_\Psi, T_\Psi)$, where $\|g_1\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} + \|g_2\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \le C_\Psi$.

Therefore

$$\begin{aligned}\big|\Psi_t(y)-\Psi_t(\bar y)\big|^{-n-s} &= \big|y-\bar y + |y-\bar y|\,\big(g_1(y,\bar y)t + g_2(y,\bar y)t^2 + O(t^3)\big)\big|^{-n-s}\\ &= |y-\bar y|^{-n-s}\big(1 + tk_1(y,\bar y) + t^2k_2(y,\bar y) + O(t^3)\big),\end{aligned} \tag{6.13}$$

where $\|k_1\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} + \|k_2\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}^n)} \le C_\Psi$.

Using (6.12) and (6.13) in (6.11) we obtain

$$\big|\mathcal{E}_{B_1}^R(u_{R,t}) - \big(\mathcal{E}_{B_1}^R(u_R) + a_1t + a_2t^2\big)\big| \le C_\Psi\, t^3\mathcal{E}_{B_1}^R(u_R)$$

for some constants a_1 and a_2, since the quantity

$$t^3 \mathcal{E}^R_{B_1}(u_R) = t^3 \left(\frac{1}{4} \iint_{A_1} \frac{|u_R(y) - u_R(\bar{y})|^2}{|y - \bar{y}|^{n+s}}\, dy\, d\bar{y} + R^s \int_{B_1} W(u_R(y))\, dy \right)$$

controls the error terms, which are cubic in the variable t. Now, since by assumption u_R is a stable solution, it must be $a_1 = 0$ and $a_2 \geq 0$, and thus

$$\mathcal{E}^R_{B_1}(u_{R,t}) \geq \mathcal{E}^R_{B_1}(u_R) - C_\Psi \mathcal{E}^R_{B_1}(u_R) t^3, \quad \text{for } t \in (-T_\Psi, T_\Psi).$$

Finally, thanks to (6.2) and (6.7) we have $\mathcal{E}^R_{B_1}(u_R) \leq C$, with C depending only on n and s. This concludes the proof of (6.10).

Now, recalling the convergence of the energies, (6.1), we have that (6.10) passes to the limit, and we deduce

$$\mathcal{E}^{\mathrm{Sob}}_{B_1}(u_{\infty,t}) \geq \mathcal{E}^{\mathrm{Sob}}_{B_1}(u_\infty) - C_\Psi t^3 \quad \text{for } t \in (-T_\Psi, T_\Psi),$$

where $u_{\infty,t} = u_\infty \circ \Psi_t^{-1}$.

Since $u_\infty = \chi_\Sigma - \chi_{\Sigma^c}$, we have

$$\mathcal{E}^{\mathrm{Sob}}_{B_1}(u_\infty) = 2P_s(\Sigma, B_1) \quad \text{and} \quad \mathcal{E}^{\mathrm{Sob}}_{B_1}(u_{\infty,t}) = 2P_s(\Psi_t(\Sigma), B_1).$$

Thus,

$$P_s(\Psi_t(\Sigma), B_1) \geq P_s(\Sigma, B_1) - C_\Psi t^3, \quad \text{for } t \in (-T_\Psi, T_\Psi).$$

Recalling Definition 2.9 and since the smooth compactly supported vector field X defining Ψ was arbitrary, we have shown that Σ is a weakly stable set in B_1 for the fractional perimeter P_s. Finally, using that Σ is a cone, we easily deduce, by scaling, that it is in fact weakly stable in all of $\mathbb{R}^n$. □

We can now give the

Proof of Theorem 2.10. The first part of the statement on the L^1-convergence has just been proven in Proposition 6.1 above. We have also proved that Σ is nontrivial.

The last part of the statement, i.e., that (2.9) and (2.10) hold after choosing the representative of Σ (for which every point of Σ with density 1 belongs to its interior and every point of density 0 belongs to its complement) follows, as usual, from the local L^1-convergence and the density estimate of Proposition 2.7. □

In Proposition 6.1 we showed that the potential energies of sequences of blow-downs converge to zero. This was a consequence of the monotonicity formula. To end this section we now give a stronger property (which will be useful in Section 7): a quantitative convergence of the potential energy to zero, as $\varepsilon \downarrow 0$, for stable solutions of $(-\Delta)^{s/2} u_\varepsilon + \varepsilon^{-s} W'(u_\varepsilon) = 0$ with $s \in (0,1)$.

Proposition 6.2. *Let $n \geq 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. For $\varepsilon > 0$, let $u_\varepsilon : \mathbb{R}^n \to (-1,1)$ be a stable solution of $(-\Delta)^{s/2} u_\varepsilon + \varepsilon^{-s} W'(u_\varepsilon) = 0$ in $\mathbb{R}^n$.*

If $R \geq \varepsilon$, then

$$\int_{B_R} (\varepsilon/R)^{-s} W(u_\varepsilon)\, dx \leq C R^n (\varepsilon/R)^\beta,$$

where $\beta := \min\left(\frac{1-s}{2}, s\right) > 0$ and C depends only on n and s.

Proof. By scaling we may (and do) assume without loss of generality that $R = 1$.

Given $x \in B_1$, let

$$r_x := \max(\min\left(\tfrac{1}{8}, \tfrac{1}{2}\mathrm{dist}(x, \{|u| \le \tfrac{9}{10}\}\right), C_0\varepsilon),$$

where $C_0 > 0$ is a large enough constant, depending only on n and s, to be chosen later. Note that if ε satisfies $1 \le 8C_0\varepsilon$, then

$$\int_{B_1} \varepsilon^{-s} W(u_\varepsilon)\, dx \le (8C_0)^s (\max_{[-1,1]} W)|B_1| \le C \le CC_0^\beta \varepsilon^\beta.$$

Thus, we may (and do) assume that $\frac{1}{8} > C_0\varepsilon$. In particular, $r_x \in [C_0\varepsilon, \frac{1}{8}]$ for all $x \in B_1$.

We will take C_0 satisfying $C_0 \ge \max(R_0, 1)$, where R_0 is the constant of Proposition 2.7 for $\bar{c} = \frac{9}{10}$. Then, for the constant ω_0 of Proposition 2.7, we claim that

$$\min\left(\int_{B_{4r_x}(x)} |u_\varepsilon - 1|\, dx, \int_{B_{4r_x}(x)} |u_\varepsilon + 1|\, dx\right) \ge \omega_0 (2r_x)^n \tag{6.14}$$

for all $x \in B_1$ such that $r_x < \frac{1}{8}$. Indeed, since $r_x < \frac{1}{8}$, there exists $z \in \{|u| \le \frac{9}{10}\} \cap \overline{B}_{1+\frac{1}{4}}$ such that $|x - z| \le 2r_x$. Hence, by Proposition 2.7 —applied to $u = u_\varepsilon(\varepsilon\,\cdot\,)$—,

$$\min\left(\int_{B_{2r_x}(z)} |u_\varepsilon - 1|\, dx, \int_{B_{2r_x}(z)} |u_\varepsilon + 1|\, dx\right) \ge \omega_0 (2r_x)^n,$$

and (6.14) follows since $B_{2r_x}(z) \subset B_{4r_x}(x)$.

Note also that $B_{4r_x}(x) \subset B_{3/2}$ for all $x \in B_1$.

On the other hand, thanks to the potential energy estimate in Theorem 2.6 (rescaled) we have (using $4r_x/\varepsilon \ge C_0 \ge 1$)

$$C_0^s ⨍_{B_{4r_x}} \frac{1}{4}(1 - |u_\varepsilon|)^2\, dx \le (\varepsilon/r_x)^{-s} ⨍_{B_{4r_x}} W(u_\varepsilon)\, dx \le C, \tag{6.15}$$

where $⨍$ denotes the average.

Let us next show that Poincaré's inequality, (6.14), and (6.15) for C_0 sufficiently large, yield

$$\int_{B_{4r_x}(x)} |\nabla u_\varepsilon| \ge c r_x^{n-1} \quad \text{whenever } r_x < \tfrac{1}{8}, \tag{6.16}$$

for some constant $c > 0$ depending only on n and s. Indeed, if $\int_{B_{4r_x}(x)} |\nabla u_\varepsilon| =: \kappa\, r_x^{n-1}$, then by Poincaré's inequality we have

$$⨍_{B_{4r_x}(x)} |u_\varepsilon - t| \le C\kappa \quad \text{for some } t \in [-1, 1].$$

But then using (6.15), we have

$$\begin{aligned} |1 - |t||^2 = ⨍_{B_{4r_x}(x)} |1 - |t||^2 &\le 2 ⨍_{B_{4r_x}} (1 - |u_\varepsilon|)^2 + 2 ⨍_{B_{4r_x}} |u_\varepsilon - t|^2 \\ &\le CC_0^{-s} + 4 ⨍_{B_{4r_x}} |u_\varepsilon - t| \le C(C_0^{-s} + \kappa). \end{aligned}$$

Recalling now (6.14) we obtain

$$\begin{aligned}\frac{2^n\omega_0}{|B_4|} &\le \min\left(\fint_{B_{4r_x}(z)}|u_\varepsilon-1|\,dx,\ \fint_{B_{4r_x}(z)}|u_\varepsilon+1|\,dx\right)\\ &\le \fint_{B_{4r_x}(z)}|u_\varepsilon-t|\,dx+|1-|t||\le C(\kappa+(C_0^{-s}+\kappa)^{1/2}),\end{aligned}$$

and this gives a lower bound for κ provided the C_0 is chosen sufficiently large. Thus, (6.16) is now proved.

We now produce a covering of B_1, by some of the balls $B_{r_x}(x)$, as follows. Given $k\le -4$, let $X_k:=\{x\in B_1\,:\,r_x\in(2^k,2^{k+1}]\}$ and let $\{x_j^k\}_{j\in\mathcal{J}_k}$ be a maximal subset of X_k with the property that the balls $B_{\frac14 r_{x_j^k}}(x_j^k)$ are disjoint. It then follows (using that all radii r_x belong to $(2^k,2^{k+1}]$ for $x\in X_k$) that

$$X_k\subset\bigcup_{j\in\mathcal{J}_k}B_{r_{x_j^k}}(x_j^k)$$

and that the family of quadruple balls

$$\{B_{4r_{x_j^k}}(x_j^k)\}_{j\in\mathcal{J}_k}$$

has (dimensional) finite overlapping.[6] Note also that, by construction, the union of the sets X_k when k runs on $\{\lfloor\log_2(R_0\varepsilon)\rfloor\le k\le -4\}$ covers all of B_1.

Now, on the one hand, the BV estimate $\int_{B_{3/2}}|\nabla u_\varepsilon|\,dx\le C$ (which follows from Theorem 2.1) yields, for all $k\le -4$,

$$\#\mathcal{J}_k\le C(2^k)^{1-n}. \tag{6.17}$$

Indeed, this follows using that the balls $B_{4r_{x_j^k}}(x_j^k)$ have finite overlapping and are contained in $B_{3/2}$: when $k<-4$ then $r_{x_j^k}<\frac18$ and hence all the balls satisfy (6.16) and are contained in $B_{3/2}$ by construction; while for $k=-4$ the radius of the balls is at least $\frac{1}{16}$ so their number must be bounded.

On the other hand, we claim that Lemma 4.4 yields

$$\int_{B_{r_x}(x)}\varepsilon^{-s}W(u_\varepsilon)\,dx\le\int_{B_{r_x}(x)}\varepsilon^{-s}(1-|u_\varepsilon|)^2\,dx\le C\varepsilon^{-s}\left(\frac{\varepsilon}{r_x}\right)^{\alpha}r_x^n$$

for any given $\alpha\in[0,2s]$. Indeed, note that if $r_x=C_0\varepsilon$ the previous estimate is trivial, while if $r_x>C_0\varepsilon$ then $r_x\le\frac12\mathrm{dist}(x,\{|u|\le\frac{9}{10}\})$ and hence we may apply Lemma 4.4 (recall that $r_x\ge C_0\varepsilon\ge\varepsilon$).

[6]That is, every point $x\in\mathbb{R}^n$ belongs to at most N of these balls, with N depending only on n. This is easy to check: if $x\in\mathbb{R}^n$ belonged to N of such balls, we would have the existence of N points x_j^k in $B_{4\cdot 2^{k+1}}(x)$ such that the balls $B_{\frac14 2^k}(x_j^k)$ are disjoint and contained in $B_{9\cdot 2^k}(x)$.

Therefore, choosing $\alpha := \min\left(\frac{1+s}{2}, 2s\right) \in (0,1)$ we obtain —using (6.17)—

$$
\begin{aligned}
\int_{B_1} \varepsilon^{-s} W(u_\varepsilon)\,dx &\le C \sum_{k=\lfloor \log_2(R_0\varepsilon)\rfloor}^{-4} \sum_{j\in\mathcal{J}_k} \int_{B_{r_{x_j^k}}(x_j^k)} \varepsilon^{-s} W(u_\varepsilon)\,dx \\
&\le C \sum_{k=\lfloor \log_2(R_0\varepsilon)\rfloor}^{-4} \sum_{j\in\mathcal{J}_k} \varepsilon^{-s}\left(\frac{\varepsilon}{r_{x_j}}\right)^{\alpha} r_{x_j}^n \\
&\le C \sum_{k=\lfloor \log_2(R_0\varepsilon)\rfloor}^{-4} \varepsilon^{-s}\left(\frac{\varepsilon}{2^k}\right)^{\alpha} (2^{k+1})^n\, \#\mathcal{J}_k \\
&\le C \sum_{k=\lfloor \log_2(R_0\varepsilon)\rfloor}^{-4} \varepsilon^{\alpha-s}(2^k)^{n-\alpha}(2^k)^{1-n} \le C\varepsilon^{\alpha-s}\sum_{k=-\infty}^{-4}(2^k)^{1-\alpha} \\
&\le C\varepsilon^{\beta},
\end{aligned}
$$

as we wanted to show. □

7. Proofs of the classification results

In this section we give the proof of our classification results. In order to prove Theorem 1.8, we will need some preliminary ingredients.

We start by recalling the main results in [37], which are a consequence of an improvement of flatness theory for phase transitions in the "genuinely nonlocal" regime (meaning that the order s of the operator is less than 1). The first one will be used to conclude one-dimensionality of solutions.

Theorem 7.1 (Theorem 1.2 in [37])**.** *Let $n \ge 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. Let $u : \mathbb{R}^n \to (-1,1)$ be a solution of $(-\Delta)^{s/2}u + W'(u) = 0$ in $\mathbb{R}^n$.*

Assume that there exists a function $a : (1,\infty) \to (0,1]$ such that $a(R) \downarrow 0$ as $R \uparrow \infty$ and such that, for all $R > 0$, we have

$$
\{e_R\cdot x \le -a(R)R\} \subset \left\{u \le -\tfrac{4}{5}\right\} \subset \left\{u \le \tfrac{4}{5}\right\} \subset \{e_R\cdot x \le a(R)R\} \quad \text{in } B_R \tag{7.1}
$$

for some $e_R \in S^{n-1}$ which may depend on R.

Then, $u(x) = \phi(e\cdot x)$ for some direction $e \in S^{n-1}$ and an increasing function $\phi : \mathbb{R} \to (-1,1)$.

We next prove a corollary of Theorem 1.1 in [37] which will be useful in the sequel. It is an"iterated version" of Theorem 1.1 in [37] in the particular case $L = (-\Delta)^s$ and $f(u) = -W'(u) = u - u^3$.

Proposition 7.2. *Let $s \in (0,1)$ and $n \ge 2$. There exist constants $\alpha_0 \in (0, s/2)$, $p_0 \in (2,\infty)$, and $a_0 \in (0,1/4)$, depending only on n and s, such that the following statement holds.*

Let $a \in (0, a_0]$ and let

$$
j_a := \left\lfloor \frac{\log a}{\log(2^{-\alpha_0})} \right\rfloor \in \mathbb{N}. \tag{7.2}
$$

Let $\varepsilon > 0$ and $k \in \mathbb{N}$ satisfy $2^{k-1}\varepsilon \le \big(2^{-\alpha_0(k-1)}a\big)^{p_0}$ and let $u_\varepsilon : \mathbb{R}^n \to (-1,1)$ be a solution of

$$(-\Delta)^s u_\varepsilon + \varepsilon^{-s} W'(u_\varepsilon) = 0 \quad \text{in } B_{2^{j_a}} \subset \mathbb{R}^n$$

satisfying $0 \in \big\{ -\frac{3}{4} \le u \le \frac{3}{4}\big\}$ and

$$\big\{\omega_j \cdot x \le -a2^{j(1+\alpha_0)}\big\} \subset \big\{u \le -\tfrac{3}{4}\big\} \subset \big\{u \le \tfrac{3}{4}\big\} \subset \big\{\omega_j \cdot x \le a2^{j(1+\alpha_0)}\big\} \quad \text{in } B_{2^j}, \tag{7.3}$$

for $0 \le j \le j_a$, for some $\omega_j \in S^{n-1}$.

Then, for all $i = 1, 2, \dots, k$ we have

$$\Big\{\omega_{-i} \cdot x \le -\frac{a}{2^{(1+\alpha_0)i}}\Big\} \subset \big\{u \le -\tfrac{3}{4}\big\} \subset \big\{u \le \tfrac{3}{4}\big\} \subset \Big\{\omega_{-i} \cdot x \le \frac{a}{2^{(1+\alpha_0)i}}\Big\} \quad \text{in } B_{2^{-i}}, \tag{7.4}$$

for certain $\omega_{-i} \in S^{n-1}$.

Proof. The proof will apply inductively Theorem 1.1 of [37] to $u^{(i)}(x) := u_\varepsilon(2^{i-1}x)$. Indeed, recall first that —see (4.10)— we have $-W''(t) \in [-2, -1/2]$ for $|t| \ge \frac{1}{\sqrt{2}}$. Since $\frac{3}{4} \ge \frac{1}{\sqrt{2}}$ we may take the constant κ from [37] equal to 1/4.

Notice that the case $i = 1$ of (7.4) follows directly from Theorem 1.1 in [37] since $2^{k-1}\varepsilon \le \big(2^{-\alpha_0(k-1)}a\big)^{p_0}$ and $k \ge 1$ guarantee $\varepsilon \le a^{p_0}$.

Assume now that (7.4) holds for $i = 1, 2, \dots, i_\circ - 1$ and that $i_\circ \le k$. Then $u^{(i_\circ)}(x)$ satisfies the assumptions of Theorem 1.1 in [37] with and a replaced by $2^{-\alpha_0(i-1)}a$ and ε replaced by $2^{i-1}\varepsilon$ (too see this it may be useful to notice that, by the definition of j_a in (7.2), we have $j_{2^{-\alpha_0(i-1)}a} = j_a + (i-1)$), since we have

$$2^{(i_\circ - 1)}\varepsilon \le 2^{k-1}\varepsilon \le \big(2^{-\alpha_0(k-1)}a\big)^{p_0} \le \big(2^{-\alpha_0(i_\circ-1)}a\big)^{p_0}.$$

Hence, $u^{(i_\circ)}$ satisfies the conclusion Theorem 1.1 in [37] so, after rescaling, we obtain that (7.4) also holds also for $i = i_\circ$. □

The second result is an easy consequence of Proposition 7.2: flatness implies a $C^{1,\alpha}$ type result.

Theorem 7.3. *Let $n \ge 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. Given $\tilde{a} > 0$ there exist positive constants σ_0, δ_0, α_0, and ϱ_0, depending only on $\tilde{a}$, n, and s, such that the following holds. Assume that $u_{\tilde{\varepsilon}}$ is a solution of $(-\Delta)^{s/2}u_{\tilde{\varepsilon}} + \tilde{\varepsilon}^{-s}W'(u_{\tilde{\varepsilon}}) = 0$ in B_1 satisfying*

$$\{x_n < -\sigma_0\} \subset \{u_{\tilde{\varepsilon}} < -\tfrac{3}{4}\} \subset \{u_{\tilde{\varepsilon}} < \tfrac{3}{4}\} \subset \{x_n < \sigma_0\} \quad \text{in } B_1. \tag{7.5}$$

Then, for all $z \in \{u_{\tilde{\varepsilon}} = 0\} \cap B_{3/4}$ and $k \ge 2$ satisfying $2^{-k} \ge \tilde{\varepsilon}^{\delta_0}$ we have

$$\begin{aligned} &\{\omega_{z,k} \cdot (x-z) < -\tilde{a}2^{-(1+\alpha_0)k}\varrho_0\} \subset \{u_{\tilde{\varepsilon}} < -\tfrac{3}{4}\} \subset \\ &\quad \subset \{u_{\tilde{\varepsilon}} < \tfrac{3}{4}\} \subset \{\omega_{z,k} \cdot (x-z) < \tilde{a}2^{-(1+\alpha_0)k}\varrho_0\} \end{aligned} \tag{7.6}$$

in $B_{2^{-k}\varrho_0}(z)$, for some $\omega_{z,k} \in S^{n-1}$.

Proof. Let a_0, α_0, p_0 be the constants from Proposition 7.2 (which depend only on n and s) and define $\delta_0 := \frac{1}{2+\alpha_0 p_0}$. It remains to choose $\sigma_0 > 0$ depending only on $\tilde a$, n, and s. Note that we may assume

$$\tilde a \tilde\varepsilon^{(1+\alpha_0)\delta_0} \le \sigma_0 \tag{7.7}$$

since otherwise (7.6) follows immediately from (7.5).

Choose

$$a := \min\{\tilde a, a_0\} \qquad \text{and} \qquad \varrho_0 := 2^{-j_a-2}, \tag{7.8}$$

where j_a was defined as in (7.2).

Now, for any $z \in \{u_{\tilde\varepsilon} = 0\} \cap B_{3/4}$, let

$$v^z := u_{\tilde\varepsilon}(z + \varrho_0\,\cdot\,).$$

Note that v^z satisfies $(-\Delta)^{s/2} v^z + (\tilde\varepsilon/\varrho_0)^{-s} W'(v^z) = 0$ in $B_{\frac{1}{4\varrho_0}} = B_{2^{j_a}}$.

Choose now $\sigma_0 > 0$ small so that $\sigma_0/\varrho_0 \le a$. Then it is immediate to verify that, thanks to (7.5), v^z satisfies the assumption (7.3) of Proposition 7.2 with $\omega_j = e_n$ for all $j = 0, \dots, j_a$.

Hence, defining $\varepsilon := \tilde\varepsilon/\varrho_0$, provided

$$2^{k-1}\varepsilon \le (2^{-\alpha_0(k-1)} a)^{p_0}, \tag{7.9}$$

Proposition 7.2 yields

$$\{\omega_{z,k}\cdot x < -\tilde a 2^{-(1+\alpha_0)k}\} \subset \{u_{\tilde\varepsilon} < -\tfrac34\} \subset \{u_{\tilde\varepsilon} < \tfrac34\} \subset \{\omega_{z,k}\cdot x < \tilde a 2^{-(1+\alpha_0)k}\} \tag{7.10}$$

in $B_{2^{-k}}$, for some $\omega_{z,k} \in S^{n-1}$. Note that (7.10) immediately yields (7.6) after scaling. Thus, it only remains to show that (7.9) is satisfied thanks to our assumption $2^{-k} \ge \tilde\varepsilon^{\delta_0}$ and our choice of δ_0.

Indeed,

$$(7.9) \quad \Leftrightarrow \quad \frac{\tilde\varepsilon}{\varrho_0} \le (2^{1-k})^{\alpha_0 p_0 + 1} a^{p_0} \quad \Leftrightarrow \quad 2^{-k} \ge \frac{(\varrho_0 a^{p_0})^{-\frac{1}{\alpha_0 p_0 + 1}}}{2} \tilde\varepsilon^{\frac{1}{1+\alpha_0 p_0}} = c_a \tilde\varepsilon^{\frac{1}{1+\alpha_0 p_0}}.$$

But since we choose $\delta_0 < \frac{1}{1+\alpha_0 p_0}$, recalling (7.7) we can absorb the multiplicative constant after possibly decreasing σ_0 in order to ensure that $\tilde\varepsilon$ is sufficiently small —recall that a was fixed in (7.8). □

Now, we introduce a class of "good" sets. By definition, the characteristic function of a "good" set must be the limit of stable solutions to the fractional Allen-Cahn equation with parameter ε, as a sequence of ε tends to 0. As we will show in Proposition 7.5 below, these sets are "good" in the sense that they inherit several good properties from the approximating sequence, such as BV and energy estimates, a monotonicity formula, density estimates, and the improvement of flatness. As an approximating sequence, we will take later the blow-downs of an entire stable solution to the fractional Allen-Cahn equation.

Definition 7.4. We say that a set $E \subset \mathbb{R}^n$ belongs to the class $\mathcal{A}$ when there exists a sequence of functions u_j, with $|u_j| < 1$, which are stable solutions of

$$(-\Delta)^{s/2} u_j + \varepsilon_j^{-s} W'(u_j) = 0 \quad \text{in } \mathbb{R}^n, \quad \text{with} \quad \varepsilon_j \downarrow 0 \ \text{ as } \ j \uparrow \infty, \tag{7.11}$$

where $W(u) = \frac{1}{4}(1-u^2)^2$, and such that

$$u_j \xrightarrow{L^1_{\text{loc}}} \chi_E - \chi_{E^c} \text{ as } j \uparrow \infty.$$

Proposition 7.5. *Any set $E \in \mathcal{A}$ satisfies the following properties.*

(1) ***BV and energy estimates.***

$$\operatorname{Per}(E, B_R) \le CR^{n-1} \quad \textit{and} \quad P_s(E, B_R) \le CR^{n-s}, \tag{7.12}$$

where C is a constant which depends only on n and s.

(2) ***Monotonicity formula.*** *Let $\bar{u} := \chi_E - \chi_{E^c}$ and let $\bar{U}$ be its s-extension in $\mathbb{R}^{n+1}_+$ (see Definition 5.1). We set*

$$\Phi_E(R) = \frac{1}{R^{n-s}} \int_{\widetilde{B}^+_R} y^{1-s} |\nabla \bar{U}(x,y)|^2 \, dx \, dy.$$

Then, Φ_E is a nondecreasing function of R and $\Phi_E(R)$ is constant if and only if E is a cone (i.e., $\bar{U}$ is homogeneous of degree 0).

(3) ***Density estimate.*** *For some positive constant ω_0, which depends only on n and s, we have that if*

$$R^{-n}|E \cap B_R| \le \omega_0 \quad (\textit{respectively, } R^{-n}|E^c \cap B_R| \le \omega_0)$$

for some $R > 0$, then

$$|E \cap B_{R/2}| = 0 \quad (\textit{respectively, } |E^c \cap B_{R/2}| = 0).$$

(4) ***Improvement of flatness.*** *There exists $\sigma_0 > 0$, which depends only on n and s, such that*

$$\textit{if} \quad \partial E \cap B_2 \subset \{x \in \mathbb{R}^n : |x \cdot e_n| \le \tilde{\sigma}_0\},$$

then $\partial E \cap B_{1/2}$ is a $C^{1,\alpha}$ graph in the e_n direction.

(5) ***Blow-up.*** *Let $E_{r_i,x_0} := \frac{E - x_0}{r_i}$ with $r_i \downarrow 0$ as $i \uparrow \infty$. If $E_{r_i,x_0} \xrightarrow{L^1_{\text{loc}}} E_*$, then $E_* \in \mathcal{A}$.*

Remark 7.6. None of the properties (1)-(5) in Proposition 7.5 are known to hold within the class of all weakly stable sets for the fractional perimeter P_s. This is the reason that brings us to introduce the class $\mathcal{A}$.

The results established in our paper allow us to give the

Proof of Proposition 7.5. Recall that (see Definition 7.4) $E \in \mathcal{A}$ if there exists a sequence u_j of stable solutions to the fractional Allen-Cahn equation with parameter $\varepsilon_j \downarrow 0$ such that $u_j \to \chi_E - \chi_{E^c}$ as $j \uparrow \infty$.

(1) The first estimate in (7.12) follows easily passing to the limit the BV estimate u_j established in (2.1) (the total variation is lower-semicontinuous). We emphasize again —see Remark 2.2— that the BV estimate (2.1) is independent of the potential W in the statement of Theorem 2.1. We are strongly using this here since u_j satisfies $(-\Delta)^{s/2} u_j + \varepsilon_j^{-s} W'(u_j) = 0$ and hence the associated potential $\varepsilon_j^{-s} W(u)$ converges

to infinity (since $\varepsilon_j \downarrow 0$) except at $u = \pm 1$. Similarly, the estimate on the fractional perimeter follows, passing to the limit the estimate of Corollary 2.4.

(2) Denote $\bar u := \chi_E - \chi_{E^c}$. Let U_j be the s-extension of u_j (as defined in Section 5). By Proposition 5.3 (rescalled), we know that

$$\Phi_j(R) := \frac{1}{R^{n-s}}\Big\{\frac{d_s}{2}\int_{\tilde B_R^+} y^{1-s}|\nabla U_j(x,y)|^2\,dx\,dy + \int_{B_R} \varepsilon_j^{-s} W(u_j(x))\,dx\Big\}$$

is a nondecreasing function of R.

Hence, we have a sequence $\Phi_j(R)$ of nondecreasing functions which is uniformly bounded (in R and j) by the energy estimate (2.5) of Theorem 2.6. Moreover, arguing as in the proof of Proposition 6.1 —see (6.8)—, the convergence $u_j \to \bar u$ in $L^1_{\rm loc}(\mathbb{R}^n)$ can be upgraded to a strong convergence in $W^{s/2,2}_{\rm loc}(\mathbb{R}^n)$ and also $U_j \to \bar U$ in $W^{1,2}_{\rm loc}(\mathbb{R}^{n+1}_+, y^{1-s})$. Also, thanks to Proposition 6.2, the potential term in $\Phi_j(R)$ converges to zero (as $\varepsilon_j \downarrow 0$) for any fixed $R > 0$.

Hence, passing to the limit as $j \to \infty$, we deduce that this monotonicity property is satisfied by the limiting function $\Phi_E(R)$. Moreover similarly as in (6.4)-(6.5) we obtain

$$\Phi_j(R_2) - \Phi_j(R_1) \ge d_s \int_{B_{R_2}\setminus B_{R_1}} \frac{y^{1-s}}{(|x|^2+y^2)^{\frac{n-s}{2}}}(\partial_r U_j)^2\,dx\,dy. \tag{7.13}$$

Since $U_j \to \bar U$ strongly $W^{1,2}_{\rm loc}(\mathbb{R}^{n+1}_+, y^{1-s})$ we also obtain that if Φ_E is constant for $\bar U$ then E must be a cone.

(3) The density estimate follows easily by passing to the limit the corresponding density estimate (established in Proposition 2.7) for the approximating sequence u_j.

(4) It follows from Theorem 7.3 and the density estimates in Proposition 2.7. Indeed, fix $\tilde a > 0$ and assume that $\partial E \cap B_2 \subset \{x \in \mathbb{R}^n : |x\cdot e_n| \le \tilde\sigma_0\}$. Then the density estimates imply the convergence of $\{u_j > t\}$ in Hausdorff distance towards E for all $t \in (-1,1)$ (see the last part of the statement of Theorem 2.10) we have, for $\sigma_0 = 8\tilde\sigma$,

$$\big\{x_n < -\sigma_0\big\} \subset \{u_j < -\tfrac34\} \subset \{u_j < \tfrac34\} \subset \big\{x_n < \sigma_0\big\} \quad \text{in } B_1.$$

Then, Theorem 7.3 (rescaled) yields that for all $z \in \{u_j = 0\} \cap B_{\frac34}$ and $k \ge 1$ satisfying $2^{-k} \ge \varepsilon_j^{\delta_0}$ we have

$$\begin{aligned}\{\omega_{z,k}\cdot(x-z) < -\tilde a 2^{-(1+\alpha_0)k}\varrho_0\} \subset \{u_{\tilde\varepsilon_j} < -\tfrac34\} \subset \\ \subset \{u_{\tilde\varepsilon} < \tfrac34\} \subset \{\omega_{z,k}\cdot(x-z) < \tilde a 2^{-(1+\alpha_0)k}\varrho_0\}\end{aligned}$$

in $B_{2^{-k}\varrho_0}(z)$, for some $\omega_{z,k} \in S^{n-1}$. After passing this information to the limit (using again Hausdorff convergence), we deduce that for all $z \in \partial E \cap B_{3/4}$ we have

$$\{\omega_{z,k}\cdot(x-z) \le -\tilde a 2^{-(1+\alpha_0)k}\varrho_0\} \subset E \subset \{\omega_{z,k}\cdot(x-z) < \tilde a 2^{-(1+\alpha_0)k}\varrho_0\} \tag{7.14}$$

in $B_{2^{-k}\varrho_0}(z)$ for all $k \ge 0$. Similarly as for classical minimal surfaces, this implies that ∂E is a $C^{1,\alpha}$ graph inside $B_{1/2}$ provided $\tilde a$ is chosen small enough depending on α_0 (note that (7.14) yields $|\omega_{z,k} - \omega_{z,k+1}| \le C_0\tilde a 2^{-\alpha_0 k}$ and hence, by triangle inequality and summing a geometric series $|e_n - \omega_{z,k}| \le C_0 \tilde{a} \frac{1}{1-2^{-\alpha_0}} < \frac{1}{10}$ for all $k \ge 0$, provided $\tilde{a}$ is chosen small).

(5) Since $E^i := E_{r_i,x_0}$ belongs to $\mathcal{A}$ for every $i \in \mathbb{N}$, then it can be approximated by a sequence u^i_j as in (7.11). By assumption for each $m \in \mathbb{N}$ there exists i_m such that $|(E^{i_m} \setminus E_*) \cup (E_* \setminus E^{i_m})) \cap B_m| \le \frac{1}{m}$. Also given this i_m there exists j_m such that

$$\int_{B_m} \left| u^{i_m}_{j_m} - (\chi_{E^{i_m}} - \chi_{(E^{i_m})^c}) \right| \le \frac{1}{m}.$$

Hence, $u^{i_m}_{j_m} \xrightarrow{L^1_{\rm loc}} \chi_{E_*} - \chi_{E_*^c}$ and thus $E_* \in \mathcal{A}$. □

We also need the following classification theorem for cones in $\mathcal{A}$ which are translation invariant in all directions but two of them.

Lemma 7.7. *Assume that some nontrivial cone $\Sigma \subset \mathbb{R}^n$ belongs to $\mathcal{A}$ and is of the form*

$$\widetilde{\Sigma} \times \mathbb{R}^{n-2}$$

for some cone $\widetilde{\Sigma} \subset \mathbb{R}^2$. Then, Σ is a half-space.

Proof. If Σ (which is nontrivial) is not a half-space, then $\tilde{\Sigma}$ is not a half-plane (and is also nontrivial). Hence, $\partial\widetilde{\Sigma}$ must contain at least two non-aligned rays. Recall that, thanks to the density estimates for the class $\mathcal{A}$, we can chose a representative among sets that differ from Σ for a set of measure zero such that every point of the topological boundary of Σ has positive density for both Σ and Σ^c. Also, thanks to the improvement of flatness property for the class $\mathcal{A}$, the outwards normal vectors (in $\mathbb{R}^2$) ν_1 and ν_2 to these two non-aligned rays need to form some positive angle, that is, $|\nu_1 - \nu_2|^2 \ge c > 0$, for some positive constant c depending only on n and s. Let us denote by $\widetilde{H}_1$ and $\widetilde{H}_2$ these two non aligned rays, that is for $i = 1, 2$, we set

$$\widetilde{H}_i := \{\widetilde{x} \in \mathbb{R}^2 \,|\, \widetilde{x} = t\,\omega_i,\ t > 0\},$$

for some vectors ω_1, $\omega_2 \in \mathbb{R}^2$ such that $\omega_i \cdot \nu_i = 0$ for $i = 1, 2$. Moreover, we set

$$H_i = \widetilde{H}_i \times \mathbb{R}^{n-2}, \quad i = 1,\, 2.$$

With these notations, we have that

$$H_1 \times H_2 \subset \partial\Sigma \times \partial\Sigma.$$

We fix a non-increasing cutoff function $\xi \in C^\infty_c([0,3))$ such that $\xi \equiv 1$ in $[0,2]$, and $\xi \le 1$, and we define

$$\psi(x) = \xi\left(\sqrt{x_1^2 + x_2^2}\right)\xi(|x_3|) \cdots \xi(|x_n|). \tag{7.15}$$

In what follows, we change notation and we denote points in $\mathbb{R}^n \times \mathbb{R}^n$ by (x, y) —instead of the usual $(x, \bar{x})$.

We claim that, for every $0<r<1$, the following estimate holds:

$$I_1 := \iint_{\partial\Sigma\times\partial\Sigma} \frac{\big|\nu_\Sigma(x)-\nu_\Sigma(y)\big|^2}{(r^2+|x-y|^2)^{\frac{n+s}{2}}}\,\psi^2(x)\psi^2(y)d\mathcal{H}^{n-1}(x)d\mathcal{H}^{n-1}(y) \geq C(n,s)r^{-s}, \tag{7.16}$$

for some positive constant $C(n,s)$ depending on n and s.

To prove the claim, we use the notation $x=(\widetilde{x},x')\in\mathbb{R}^2\times\mathbb{R}^{n-2}$. We have that

$$\begin{aligned} I_1 &\geq c\int_{H_1}\int_{H_2} \frac{\psi^2(x)\psi^2(y)}{(r^2+|x-y|^2)^{\frac{n+s}{2}}} d\mathcal{H}^{n-1}(x)d\mathcal{H}^{n-1}(y)\\ &\geq c\iint_{\mathbb{R}^{n-2}\times\mathbb{R}^{n-2}} dx'\,dy' \int_{\widetilde{H}_1} d\mathcal{H}^1(\widetilde{x})\int_{\widetilde{H}_2} d\mathcal{H}^1(\widetilde{y}) \frac{\psi^2(x)\psi^2(y)}{(r^2+|x-y|^2)^{\frac{n+s}{2}}}. \end{aligned}$$

Using the change of variables $X=(0,|\widetilde{x}|)$, $Y=(0,-|\widetilde{y}|)$ and the triangle inequality $|\widetilde{x}-\widetilde{y}|\leq|\widetilde{x}|+|\widetilde{y}|=|X-Y|$, we get

$$\begin{aligned} I_1 &\geq c\iint_{\mathbb{R}^{n-2}\times\mathbb{R}^{n-2}} dx'\,dy' \int_{\{0\}\times\mathbb{R}^+} dX\int_{\{0\}\times\mathbb{R}^-} dY \frac{\psi^2(X,x')\psi^2(Y,y')}{(r^2+|X-Y|^2+|x'-y'|^2)^{\frac{n+s}{s}}}\\ &\geq c\int_{B_1^{n-2}} dx' \int_{\{y'\in\mathbb{R}^{n-2}\,|\,|x'-y'|<1\}} dy'\int_0^1 dX\int_{-1}^0 dY \frac{1}{(r^2+|X-Y|^2+|x'-y'|^2)^{\frac{n+s}{s}}}\\ &\geq Cc|B_1^{n-2}|\int_0^1 d\rho\int_0^1 dX\int_{-1}^0 dY\frac{\rho^{n-3}}{(r^2+|X-Y|^2+\rho^2)^{\frac{n+s}{s}}}, \end{aligned}$$

where we are identifying a point $(0,X)\in\{0\}\times\mathbb{R}^+$ with the real number $X\in\mathbb{R}^+$ and the integration over $\{0\}\times\mathbb{R}^+$ with integration over $\mathbb{R}^+$ (analogously for $Y\in\mathbb{R}^-$), and in the last inequality we have used polar coordinates in $\mathbb{R}^{n-2}$.

Finally, using the change of variables $\bar{X}=X/r$, $\bar{Y}=Y/r$, $\bar{\rho}=\rho/r$, and recalling that $0<r<1$, we deduce

$$\begin{aligned} I_1 &\geq C(n,s)\frac{1}{r^{n+s}}\int_0^1 d\rho\int_0^1 dX\int_{-1}^0 dY\frac{\rho^{n-3}}{\left(1+\frac{|X-Y|^2}{r^2}+\frac{\rho^2}{r^2}\right)^{\frac{n+s}{2}}}\\ &= C(n,s)\frac{1}{r^{n+s}}\int_0^{1/r} d\bar{\rho}\int_0^{1/r} d\bar{X}\int_{-1/r}^0 d\bar{Y}\frac{r^n\cdot\bar{\rho}^{n-3}}{(1+|\bar{X}-\bar{Y}|^2+\bar{\rho}^2)^{\frac{n+s}{2}}}\\ &\geq C(n,s)r^{-s}\int_0^1 d\bar{\rho}\int_0^1 d\bar{X}\int_{-1}^0 d\bar{Y}\frac{\bar{\rho}^{n-3}}{(1+|\bar{X}-\bar{Y}|^2+\bar{\rho}^2)^{\frac{n+s}{2}}} = C(n,s)r^{-s}, \end{aligned}$$

which concludes the proof of (7.16).

Recall now that since Σ belongs to the class $\mathcal{A}$, there exists, by definition, a sequence of functions $u_j:\mathbb{R}^n\to(-1,1)$ which are stable solutions of $(-\Delta)^{s/2}u_j+\varepsilon_j^{-s}W'(u_j)=0$ and such that

$$u_j \xrightarrow{L^1_{\rm loc}} \chi_\Sigma-\chi_{\Sigma^c} \quad \text{as } j\uparrow\infty.$$

We claim now that the following inequality holds:

$$
\begin{aligned}
I_2 &:= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big|n_j(x)-n_j(y)\big|^2}{|x-y|^{n+s}}\psi(x)^2|\nabla u_j|(x)dx\,|\nabla u_j|(y)dy \\
&\qquad\qquad \le \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+s}}|\nabla u_j|(x)dx\,|\nabla u_j|(y)dy =: I_3,
\end{aligned}
\tag{7.17}
$$

where

$$
n_j(x) := \begin{cases} \frac{\nabla u_j}{|\nabla u_j|} & \text{where } \nabla u \neq 0 \\ 0 & \text{where } \nabla u = 0. \end{cases}
\tag{7.18}
$$

Let us prove (7.17). We start by observing that the stability condition (1.12) (with $\Omega = \mathbb{R}^n$), written for the functions u_j, is equivalent to requiring that

$$
\int_{\mathbb{R}^n} \xi(x)(-\Delta)^{s/2}\xi(x)\,dx + \int_{\mathbb{R}^n} \varepsilon_j^{-s}W''(u_j)\xi^2(x)\,dx \ge 0
\tag{7.19}
$$

for any Lipschitz function ξ which is compactly supported in $\mathbb{R}^n$.

Let us now choose ξ of the form $\xi = \eta\cdot\psi$, where η is a Lipschitz function and $\psi \in C_0^\infty(\mathbb{R}^n)$. A simple computation gives that

$$
(-\Delta)^{s/2}\xi(x) = \psi(x)(-\Delta)^{s/2}\eta(x) + \int_{\mathbb{R}^n} \eta(y)\frac{\psi(x)-\psi(y)}{|x-y|^{n+s}}\,dy,
$$

which implies

$$
\begin{aligned}
\int_{\mathbb{R}^n} \xi(x)(-\Delta)^{s/2}\xi(x)\,dx &= \int_{\mathbb{R}^n} \psi^2(x)\eta(x)(-\Delta)^{s/2}\eta(x)\,dx \\
&\quad + \iint_{\mathbb{R}^n\times\mathbb{R}^n} \eta(x)\eta(y)\psi(x)\frac{\psi(x)-\psi(y)}{|x-y|^{n+s}}\,dx\,dy \\
&= \int_{\mathbb{R}^n} \psi^2(x)\eta(x)(-\Delta)^{s/2}\eta(x)\,dx \\
&\quad + \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \eta(x)\eta(y)\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+s}}\,dx\,dy.
\end{aligned}
$$

Hence, the stability condition (7.19) becomes

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \psi^2(x)\eta(x)(-\Delta)^{s/2}\eta(x)\,dx + \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n} \eta(x)\eta(y)\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+s}}\,dx\,dy \\
&\qquad + \int_{\mathbb{R}^n} \varepsilon_j^{-s}W''(u_j)\eta^2(x)\psi^2(x)\,dx \ge 0.
\end{aligned}
\tag{7.20}
$$

We use now the fact that, for any $i = 1,\dots,n$, $\partial_{x_i}u_j$ satisfies the linearized equation $(-\Delta)^{s/2}v + \varepsilon_j^{-s}W''(u)v = 0$. By multiplying the (vectorial) equation satisfied by ∇u_j by ∇u_j itself, we deduce that

$$
\nabla u_j\cdot(-\Delta)^{s/2}\nabla u_j + \varepsilon_j^{-s}W''(u)|\nabla u_j|^2 = 0.
\tag{7.21}
$$

Let us now choose $\eta = |\nabla u_j|$ in the stability inequality (7.20) (this is an admissible choice by the regularity results of Appendix C) and use (7.21), to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} &\psi^2(x)|\nabla u_j|(x)(-\Delta)^{s/2}|\nabla u_j|(x)\,dx\\ &+\frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}^n}|\nabla u_j(x)||\nabla u_j(y)|\frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+s}}\,dx\,dy\\ &-\int_{\mathbb{R}^n}\psi^2(x)\nabla u_j(x)\cdot(-\Delta)^{s/2}\nabla u_j(x)\,dx\geq 0.\end{aligned}$$

To conclude the proof of (7.17), it is enough to observe that

$$\begin{aligned}&\int_{\mathbb{R}^n}\psi^2(x)|\nabla u_j|(x)(-\Delta)^{s/2}|\nabla u_j|(x)\,dx-\int_{\mathbb{R}^n}\psi^2(x)\nabla u_j(x)\cdot(-\Delta)^{s/2}\nabla u_j(x)dx\\ &=\iint_{\mathbb{R}^n\times\mathbb{R}^n}\psi^2(x)\left(|\nabla u_j|(x)\frac{|\nabla u_j|(x)-|\nabla u_j|(y)}{|x-y|^{n+s}}-\nabla u_j(x)\cdot\frac{\nabla u_j(x)-\nabla u_j(y)}{|x-y|^{n+s}}\right)dxdy\\ &=\iint_{\mathbb{R}^n\times\mathbb{R}^n}\psi^2(x)\frac{\nabla u_j(x)\cdot\nabla u_j(y)-|\nabla u_j|(x)|\nabla u_j|(y)}{|x-y|^{n+s}}dxdy.\end{aligned}$$

This, together with the definition (7.18) of n_j, proves (7.17).

By our uniform BV estimates, we have that $\nabla u_j \to -2D\chi_\Sigma$ weakly* as Radon measures (see, e.g., Proposition 3.13 and formula (3.11) in [5]). Here and in the following, we denote by $D\chi_\Sigma$ the perimeter measure and by $|D\chi_\Sigma|$ its total variation.

Let us show that

$$I_1 \leq I_2$$

for j large enough, where these quantities were defined in (7.16) and (7.17).

Indeed, we start by observing that

$$\begin{aligned}I_2 &= 2\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\big(|\nabla u_j|(x)|\nabla u_j|(y)-\nabla u_j(x)\cdot\nabla u_j(y)\big)}{|x-y|^{n+s}}\psi^2(x)\,dx\,dy\\ &\geq 2\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\big(|\nabla u_j|(x)|\nabla u_j|(y)-\nabla u_j(x)\cdot\nabla u_j(y)\big)}{(r^2+|x-y|^2)^{\frac{n+s}{2}}}\psi^2(x)\psi^2(y)\,dx\,dy.\end{aligned}$$

Now, we claim that:

$$\begin{aligned}\lim_{j\to\infty}&\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\nabla u_j(x)\cdot\nabla u_j(y)}{(r^2+|x-y|^2)^{\frac{n+s}{2}}}\psi^2(x)\psi^2(y)\,dx\,dy\\ &=\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{2D\chi_\Sigma(dx)\cdot 2D\chi_\Sigma(dy)}{(r^2+|x-y|^2)^{\frac{n+s}{2}}}\psi^2(x)\psi^2(y)\end{aligned}\tag{7.22}$$

and

$$\begin{aligned}\liminf_{j\to\infty}&\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|\nabla u_j|(x)|\nabla u_j|(y)}{(r^2+|x-y|^2)^{\frac{n+s}{2}}}\psi^2(x)\psi^2(y)\,dx\,dy\\ &\geq\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{2|D\chi_\Sigma|(dx)2|D\chi_\Sigma|(dy)}{(r^2+|x-y|^2)^{\frac{n+s}{2}}}\psi^2(x)\psi^2(y).\end{aligned}\tag{7.23}$$

We first prove (7.22). For every $y \in \mathbb{R}^n$, we define

$$G_j(y) := \int_{\mathbb{R}^n} \nabla u_j(x) \frac{\psi^2(x)\psi^2(y)}{(r^2 + |x-y|^2)^{\frac{n+s}{2}}}\, dx = \int_{\mathbb{R}^n} \nabla u_j(x) \Psi(x,y)\, dx.$$

Using the BV estimate for u_j (which is uniform in j) and that, for y fixed, the function Ψ is smooth and compactly supported (as function of x), we deduce that the family of functions $\{G_j\}_{j\in\mathbb{N}}$ is equibounded and equicontinuous. This, combined with the weak*-convergence of ∇u_j to $-2D\chi_\Sigma$, gives that, as $j \to \infty$, G_j converges uniformly to G, where

$$G(y) := -2 \int_{\mathbb{R}^n} D\chi_\Sigma(dx)\Psi(x,y).$$

Hence, we have that

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \nabla u_j(x)\cdot\nabla u_j(y)\Psi(x,y)\, dx\, dy \\
&\qquad - \iint_{\mathbb{R}^n\times\mathbb{R}^n} 2D\chi_\Sigma(dx)\cdot 2D\chi_\Sigma(y)\Psi(x,y)dy \\
&\quad = \int_{\mathbb{R}^n} \nabla u_j(y)\cdot G_j(y)\, dy + 2\int_{R^n} D\chi_\Sigma(dy)\cdot G(y) \\
&\quad = \int_{\mathbb{R}^n} \nabla u_j(y)\left(G_j(y) - G(y)\right)\, dy + \int_{\mathbb{R}^n} G(y)\left(\nabla u_j(y) + 2D\chi_\Sigma(dy)\right) \to 0,
\end{aligned}$$

where the first term tends to zero thanks to the uniform convergence of G_j to G and the uniform BV estimate (on compact sets) for u_j, and the second term also vanishes in the limit since ∇u_j weak*-converge to $-2D\chi_\Sigma$ and G is smooth and compactly supported.

To get (7.23), we reason similarly as before and use the lower-semicontinuity of the total variation.

Hence, from (7.22) and (7.23), we get, as $j \to \infty$,

$$I_2 \geq 4I_1 - o(1) \geq I_1.$$

Finally, let us show that

$$I_3 \leq C_1(n,s),$$

where I_3 was defined in (7.17).

Indeed, by Theorem 2.1 we have $\int_{B_\varrho(x)} |\nabla u_j(y)|\, dy \leq C(n,s)\varrho^{n-1}$ for all $x \in \tilde{B}_3 \times [-3,3]^{n-2}$ and $\rho > 0$ (here $\tilde{B}_3$ denote the ball in $\mathbb{R}^2$ centered at 0 and with radius 3). Hence for all $x \in \tilde{B}_3 \times [-3,3]^{n-2}$, defining $A_j(x) = B_{2^{j+1}}(x) \setminus B_{2^j}(x)$ and using that $|\psi(x) - \psi(y)|^2 \leq C(n)(|x-y|^2 \wedge 1)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{(|x-y|^2 \wedge 1)}{|x-y|^{n+s}} |\nabla u_j(y)| dy &\le C \sum_{j\in\mathbb{Z}} \frac{2^{2j}\wedge 1}{2^{j(n+s)}} 2^{j(n-1)} \\ &= C\left(\sum_{j<0} 2^{(1-s)j} + \sum_{j\ge 0} 2^{-(1+s)j} \right) \le C_1(n,s).\end{aligned}$$

Hence we have shown that, for j large enough,

$$C(n,s)r^{-s} \le I_1 \le I_2 \le I_3 \le C_1(n,s).$$

Choosing $r > 0$ small we obtain a contradiction. □

We now have all the ingredients to prove our main theorem.

Proof of Theorem 1.8. By our convergence result Theorem 2.10 for any given blow-down sequence $u_{R_j}(x) = u(R_j x)$ with $R_j \uparrow \infty$, there is a subsequence R_{j_ℓ} such that

$$u_{R_{j_\ell}} \to \chi_\Sigma - \chi_{\Sigma^c} \quad \text{in } L^1(B_1),$$

where Σ is a cone which is a weakly stable set in $\mathbb{R}^n$ for the s-perimeter, which belongs to the class $\mathcal{A}$ (by definition of $\mathcal{A}$), and which is nontrivial. We next prove that under our assumption, i.e., that the half-spaces are the only smooth (away from 0) stable nonlocal s-minimal cones in $\mathbb{R}^m \setminus \{0\}$ for any $3 \le m \le n$, Σ must be a half-space. The proof follows Federer's dimension reduction argument and is done by contradiction. Indeed, assume that Σ is not a half-space, then, by the just mentioned assumption, there exists at least one point $p_1 \in \partial\Sigma \cap S^{n-1}$ at which $\partial\Sigma$ is not smooth in any neighborhood of p_1.

Let us consider the blow-up of Σ at p_1,

$$\Sigma_{p_1,r} := \frac{\Sigma - p_1}{r}.$$

Using the energy estimate and the monotonicity formula (points (1) and (2) in Proposition 7.5) we have that, up to a subsequence,

$$\Sigma_{p_1,r} \xrightarrow{L^1_{\rm loc}} \Sigma_1,$$

where Σ_1 belongs to $\mathcal{A}$ by point (5) in Proposition 7.5. Moreover, by the density estimate (point (3) in Proposition 7.5), the convergence of blow-ups in the $L^1_{\rm loc}$-sense can be upgraded to a local uniform convergence (i.e., locally in Hausdorff distance). Now, if Σ_1 were a half-space, then by the improvement of flatness property (point (4) of Proposition 7.5) we would deduce that $\partial\Sigma$ is smooth in some neighborhood of p_1, reaching a contradiction. Hence, Σ_1 is not a half-space. Now, Σ_1 being the blow-up of a cone at a point $p_1 \neq 0$, we find that it must be translation invariant in the direction p_1.[7] Hence, up to a rotation, Σ_1 it must be of the form

$$\Sigma_1 = \widetilde{\Sigma}_1 \times \mathbb{R},$$

where $\widetilde{\Sigma}_1 \subset \mathbb{R}^{n-1}$ is a nontrivial cone different from a half-space (since we proved that Σ_1 cannot be a half-space).

We can now iterate the same argument: if $\tilde{\Sigma}_1$ were smooth, then it should be flat by our assumption. Hence $\widetilde{\Sigma}_1$ is not smooth, and thus, there exists $p_2 = (\tilde{p}_2, 0)$, with $\tilde{p}_2 \in \widetilde{\Sigma}_1 \cap S^{n-2}$, such that the blow-up Σ_2 of Σ_1 at the point p_2 belongs to $\mathcal{A}$ is nontrivial, is not a half-space, and is translation invariant in the direction x_1 and x_2. In other words, up to a rotation this second blow-up must be some set Σ_2 in $\mathcal{A}$ of the form

$$\Sigma_2 = \widetilde{\Sigma}_2 \times \mathbb{R}^2,$$

where $\widetilde{\Sigma}_2 \subset \mathbb{R}^{n-2}$ is a nontrivial cone different from a half-space.

After $n-2$ iterations we arrive at a blow-up Σ_{n-2} which belongs to $\mathcal{A}$ and must be of the form

$$\Sigma_{n-2} = \widetilde{\Sigma}_{n-2} \times \mathbb{R}^{n-2},$$

where $\widetilde{\Sigma}_{n-2} \subset \mathbb{R}^2$ is a nontrivial cone different from a half-space. Hence, using Lemma 7.7, we reach a contradiction, proving that the initial cone Σ must be a half-space.

Having proved that Σ must be a half-space, we now recall that (by Theorem 2.10) the convergence of sub-level sets of $u_{R_{j_\ell}}$ to the half-space Σ (in B_1) also holds in the sense of the Hausdorff distance. As a consequence u satisfies the asymptotic flatness assumption of Theorem 7.1 and hence it follows that $u(x) = \phi(e \cdot x)$ for some direction $e \in S^{n-1}$ and some increasing function $\phi : \mathbb{R} \to (-1, 1)$. $\square$

Remark 7.8. The following will be used in Appendix A to deal with global stable s-minimal sets. Notice that, reasoning exactly as in the proof of Theorem 1.8, one can prove that under the same assumption, i.e., that, for some pair (n, s) with $n \geq 3$ and $s \in (0, 1)$, hyperplanes are the only stable s-minimal cones in $\mathbb{R}^n \setminus \{0\}$, then any set $E \subset \mathbb{R}^n$ belonging to the class $\mathcal{A}$ (and which is not a.e. equal to $\mathbb{R}^n$ or $\varnothing$) is necessarily a half-space. Indeed, after doing a blow-down of E, we reduce to a cone Σ, which is stable and belongs to $\mathcal{A}$. Hence, by the exact same argument as before, one gets that Σ is a half-space, and finally, by the improvement of flatness property of $\mathcal{A}$, that so is E. More generally, an analog abstract result that reduces the classification of a stable set E to the classification of stable cones (smooth away from the origin), holds whenever E belongs to a class for which the properties listed in Proposition 7.5 are satisfied (BV and energy estimates, monotonicity formula, density estimates, improvement of flatness, and blow-up).

[7]This is done exactly as in the case of minimal surfaces:

$$x \in \Sigma_{p_1,r} \;\Leftrightarrow\; (p_1 + rx) \in \Sigma \;\Leftrightarrow\; \frac{\lambda}{r}(p_1 + rx) \in \Sigma \;\Leftrightarrow\; \left(\tfrac{\lambda-1}{r} p_1 + \lambda x\right) \in \Sigma_{p_1,r}$$

and hence, taking $\lambda = 1 + tr$ and sending $r \to 0$, we obtain $x \in \Sigma_1 \;\Leftrightarrow\; x + tp_1 \in \Sigma_1$.

Notice that here there is again a strong similarity with the classical theory of area minimizing minimal surfaces but a strong divergence from the classical theory of stable minimal surfaces. Genuinely nonlocal effects (namely our BV and density estimates) are what allow us to exploit conical blow-downs and Federer dimension reduction arguments even when dealing with merely stable critical points.

We can now easily deduce our rigidity result in $\mathbb{R}^3$.

Proof of Corollary 1.10. Thanks to Theorem 1.8 we just need to show that for s sufficiently close to 1, any weakly stable s-minimal cone in $\mathbb{R}^3 \setminus \{0\}$ whose boundary is smooth away from 0 must be either trivial or a half-space. This result was obtained in our previous paper [14]. $\square$

Instead, to prove Corollary 1.11 on monotone solutions in $\mathbb{R}^4$ we first need to establish the following result, which holds in any dimension.

Proposition 7.9. *Let $n \geq 2$, $s \in (0,1)$, and $W(u) = \frac{1}{4}(1-u^2)^2$. Assume that $u : \mathbb{R}^n \to (-1,1)$ is a solution of (1.4) satisfying $\partial_{x_n} u > 0$. Define $u^{\pm} := \lim_{x_n \to \pm\infty} u$.*

If each u^+ and u^- is either a increasing 1D solution or is identically ± 1, then u is a minimizer in $\mathbb{R}^n$.

The result for $s = 2$ was proven in [39, Theorem 1.1]. Here we show that the same type of argument works also for $s \in (0,2)$.

Proof of Proposition 7.9. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let v be a minimizer of $\mathcal{E}_\Omega$ with exterior datum u outside of Ω. Let us first show that $u^- \leq v \leq u^+$ in $\mathbb{R}^n$. Indeed, in the complement of Ω the inequality holds since $v = u$ there and $u^- \leq u \leq u^+$ in $\mathbb{R}^n$. To show that $u^- \leq v \leq u^+$ in Ω, let us consider $\overline{w}^{\pm} = \max(v, u^{\pm})$ and $\underline{w}^{\pm} = \min(v, u^{\pm})$.

Assume by contradiction that $v > u^+$ at some point in Ω. Then, arguing similarly as in (3.7), we would have

$$\mathcal{E}_\Omega(\overline{w}^+) + \mathcal{E}_\Omega(\underline{w}^+) < \mathcal{E}_\Omega(v) + \mathcal{E}_\Omega(u^+). \tag{7.24}$$

But now since on the one hand v is a minimizer with exterior datum equal to u in the complement of Ω we have $\mathcal{E}_\Omega(v) \leq \mathcal{E}_\Omega(\underline{w}^+)$. On the other hand, since u^+ is either $+1$ or $1D$ and increasing, it follows[8] that u^+ is also a minimizer. Hence, noticing that $\overline{w}^+$ and u^+ coincide outside of Ω, we obtain $\mathcal{E}_\Omega(u^+) \leq \mathcal{E}_\Omega(\overline{w}^+)$. We therefore reach a contradiction with (7.24).

A similar argument using $\overline{w}^-$ and $\underline{w}^-$ shows $u^- \leq v$.

Finally, using the standard "foliation" $\big\{u(x', x_n + t),\ t \in \mathbb{R}\big\}$, unless $v \equiv u$ we may find a translation of the graph of u touching by above (or by below) the graph of v as some point interior point in Ω, and this contradicts the strong maximum principle. Hence the only possibility is that $v \equiv u$ and thus u is a minimizer in Ω. Since Ω is an arbitrary bounded domain, we conclude that u is a minimizer in $\mathbb{R}^n$. $\square$

[8]A simple way of proving this consists of using the standard argument which involves the foliation of $\mathbb{R}^n \times (-1,1)$ given by the horizontal translations of the graph of u^+; see [13, Proof of Proposition 6.2].

We can finally give the

Proof of Corollary 1.11. By Corollary 1.10 the limits $u^{\pm} := \lim_{x_4\to\pm\infty} u$ (which are stable solutions in $\mathbb{R}^3$) must be either ± 1 or increasing $1D$ solutions. Thus, by Proposition 7.9, u is a minimizer in $\mathbb{R}^4$. Now, since s is sufficiently close to 1, the corollary follows from Theorem 1.5 in [37]. $\square$

Appendix A. Smooth stable s-minimal surfaces in $\mathbb{R}^3$ are flat when $s \sim 1$

We give next the details of the proof of Theorem 2.11 and, as a consequence, of Corollary 2.12.

To prove Theorem 2.11, we need to introduce the following definition, which is analogous to the one of the class $\mathcal{A}$ introduced in Section 7.

Definition A.1. We say that a set $E \subset \mathbb{R}^n$ belongs to the class $\mathcal{A}'$ when there exist a sequence of sets $E_j \subset \mathbb{R}^n$ with $E_j \to E$ in $L^1_{\rm loc}(\mathbb{R}^n)$ such that:

- the boundaries ∂E_j are $(n-1)$-dimensional manifolds of class C^2;
- E_j are weakly stable sets for the s-perimeter in $\mathbb{R}^n$.

Proposition A.2. *Any set of the class $\mathcal{A}'$ satisfies the five properties* (1)-(5) *in Proposition* 7.5*.*

Proof. (1) **BV and energy estimates**. Since E_j have smooth boundaries, the fact that these sets are weakly stable easily gives that they are also stable in the sense of Definition 1.6 in [29]. Hence, by Corollary 1.8 in [29], we obtain that $\mathrm{Per}(E_j, B_R) \le CR^{n-1}$. Since the perimeter is lower semicontinuous, the same estimate holds by approximation for sets in $\mathcal{A}'$. Now, this BV estimate leads to the corresponding energy estimate by the exact same argument that we have given for solutions of the fractional Allen-Cahn equation.

(2) **Monotonicity formula**. We claim that if $F \subset \mathbb{R}^n$ is a weakly stable set with C^2 boundary, then the quantity

$$\Phi_F(R) = \frac{1}{R^{n-s}} \int_{\widetilde{B}_R^+} y^{1-s} |\nabla \bar{V}(x,y)|^2 \, dx\, dy$$

is nondecreasing in R, where $\bar{V}$ is the s-extension of $\bar{v} := \chi_F - \chi_{F^c}$.

This fact follows from the proof of Theorem 8.1 in [19]. Indeed, although Theorem 8.1 in [19] is stated for minimizers, its proof only needs that $\bar{V}$ is a minimizer with respect to sufficiently small Lipschitz perturbations (the size of the perturbations used actually converges to zero). If ∂E is smooth and E is weakly stable, then it is a standard fact that any compact subset of E is strictly stable (since the first eigenvalue of the Jacobi operator is strictly monotone with respect to the inclusion of domains). In particular, for any given ball, smooth perturbations of ∂E supported in this ball and with a small enough size (depending on the ball) lead to an increased nonlocal perimeter. Since ∂E is smooth, it is easy to see that the same property also holds for Lipchitz perturbations of sufficiently small size. Consequently, the argument in the proof of Theorem 8.1 in [19] —without any substantial modification— also applies to the case of weakly stable sets with smooth boundaries. (More generally, the monotonicity formula also holds for stationary critical points of the fractional perimeter with respect to inner variations, as established in [25]. Here we need only the case of stable critical points —for which the less technical argument in [19] can be used.)

Now, with an analogous argument as in the proof of Proposition 7.5, if E belongs to $\mathcal{A}'$, then we can take a sequence of approximating sets E_j with smooth boundary. Since the convergence of $E_j \to E$ in L^1_{loc} gives that $\bar{U}_j \to \bar{V}$ strongly in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, y^{1-s})$, where $\bar{U}_j$ are the s-extensions of $\bar{u}_j := \chi_{E_j} - \chi_{E_j^c}$, we obtain that Φ_E must be monotone since Φ_{E_j} are.

(3) **Density estimate**. The density estimate for sets in the class $\mathcal{A}'$ follows from the BV estimate with the exact same arguments as those in Section 5.

(4) **Improvement of flatness**. If $F \subset \mathbb{R}^n$ is any stable (or even stationary) set with C^2 boundary, the improvement of flatness result in [19, Theorem 6.8] applies to F (without any change in its proof). This is true because (unlike in the case "$s = 1$" of classical minimal surfaces) the proof of [19, Theorem 6.8] applies to any viscosity solution of the nonlocal minimal surface equation (in the sense given in [19, Theorem 5.1]). In [19] the minimality assumption is only used to show that the considered surfaces are viscosity solutions of the nonlocal minimal surface equation (this is done in [19, Theorem 5.1]). If one assumes that the boundary of F is smooth then it is easy to see using the computation of the first variation (see for instance [40]) that F must be a viscosity solution of the nonlocal minimal surface equation. Hence, [19, Theorem 6.8] applies to F.

As a consequence (with a similar approximation argument as in the proof of Proposition 7.5), we obtain that the improvement of flatness property holds true for sets E in the class $\mathcal{A}'$.

(5) **Blow-up**. The closedness of the class $\mathcal{A}'$ under blow-up follows by the same argument as in the proof of Proposition 7.5. □

Proof of Theorem 2.11. Thanks to Proposition A.2, by the same argument as in the proof of Theorem 1.8 (see Remark 7.8), we may reduce the classification in $\mathbb{R}^n$ of stable s-minimal sets in the class $\mathcal{A}'$ to the classification of stable s-minimal cones (smooth away from 0) in dimensions $3 \le m \le n$. □

Proof of Corollary 2.12. It follows from Theorem 2.11 analogously as in the proof Proof of Corollary 1.10. □

Appendix B. On the control of the potential energy by the Sobolev energy

In this section we give a short proof of a weaker version of the estimate in Proposition 2.5. It is a weaker estimate since it has an additional additive term on its right hand side.

Proposition B.1. *Given $s_0 > 0$, let $n \geq 2$, $s \in (s_0, 2]$, $W(u) = \frac{1}{4}(1-u^2)^2$, and K satisfy* (1.8) *and* (1.9). *Let $u : \mathbb{R}^n \to (-1,1)$ be a stable solution of $L_K u + W'(u) = 0$ in $\mathbb{R}^n$ (meaning $-a_{ij}\partial_{ij}u + W'(u) = 0$ when $s = 2$).*

Then,

$$\mathcal{E}^{\mathrm{Pot}}_{B_R}(u) \leq \begin{cases} C\big(\mathcal{E}^{\mathrm{Sob}}_{B_{R+1}}(u) + R^{n-s}\big) & \text{if } s \in (0,1) \\ C\big(\mathcal{E}^{\mathrm{Sob}}_{B_{R+1}}(u) + R^{n-1}\log R\big) & \text{if } s = 1 \\ C\big(\mathcal{E}^{\mathrm{Sob}}_{B_{R+1}}(u) + R^{n-1}\big) & \text{if } s \in (1,2], \end{cases} \tag{B.1}$$

for all $R \geq 1$, where C is a constant which depends only on n, λ, Λ, and s_0.

For brevity we give the proof only for the archetypal case $-\Delta u + W'(u) = 0$, where $W(u) = \frac{1}{4}(1-u^2)^2$. The same proof can be modified, with not too much effort, to cover also the operators of fractional order, as well as more general double well potentials as in Remark 1.9.

Proof of Proposition B.1 in the particular case $-\Delta u - (u - u^3) = 0$. Integrating by parts and using that $|\nabla u| \leq C$ in $\mathbb{R}^n$ for some dimensional constant C, we obtain

$$\begin{aligned} I_R &:= \int_{B_R} u^2(1-u^2)^2\,dx \leq \int_{B_R} u^2(1-u^2)\,dx = \int_{B_R} u(-\Delta u)\,dx \\ &\leq \int_{B_R} |\nabla u|^2\,dx + CR^{n-1}. \end{aligned}$$

Also, letting $\eta_R = 1 - (|x| - R)_+$ (note that $\eta_R = 0$ on ∂B_{R+1}) and testing the stability inequality $\int (1-3u^2)\xi^2\,dx \leq \int |\nabla \xi|^2\,dx$ in B_{R+1} with the function $\xi = (1-u^2)\eta_R$, we obtain

$$\begin{aligned} J_R :&= \int_{B_R} (1-u^2)^3\,dx \leq \int_{B_{R+1}} (1 - 3u^2 + 2u^2)\big((1-u^2)\eta_R\big)^2\,dx \\ &\leq \int_{B_{R+1}} \big|\nabla\big((1-u^2)\eta_R\big)\big|^2\,dx + 2\int_{B_{R+1}} u^2(1-u^2)^2\eta_R^2\,dx \\ &\leq \int_{B_{R+1}} 4u^2|\nabla u|^2\,dx + C|B_{R+1}\setminus B_R| + 2I_{R+1} \leq 6\int_{B_{R+1}} |\nabla u|^2\,dx + CR^{n-1}. \end{aligned}$$

Therefore,

$$\int_{B_R} (1-u^2)^2\,dx = \int_{B_R} (1-u^2)^2(u^2+1-u^2)\,dx = I_R + J_R \leq 7\int_{B_{R+1}} |\nabla u|^2\,dx + CR^{n-1},$$

as claimed. □

Appendix C. Regularity of solutions

In this appendix, for the reader's convenience, we prove a local smoothness result for solutions of semilinear equations involving the fractional Laplacian. Such result is well-know to experts but it is not easy to find in a clean form in the existing literature. Our goal is to show that every bounded *distributional solution*[9] of the fractional semilinear equation (1.6), for kernels in the class $\mathcal{L}_2$, satisfies interior $C^{2,\alpha}$ estimates in compact subsets of Ω. As we see next, this is a consequence of known regularity results for linear equations. When $\Omega = \mathbb{R}^n$, the situation is simpler than in the following arguments and one can conclude via a bootstrap argument that $u \in C^2(\mathbb{R}^n)$ even for kernels K in the class $\mathcal{L}_0$, i.e., kernels satisfying only (1.8). This is because the equation is posed in all space and one can differentiate it without introducing errors that come from rough exterior data (in particular the truncation arguments given in the proof of Propostion C.1 are not needed).

We will prove the following.

Proposition C.1. *Let $s_0 \in (0,1)$. Assume that $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function which solves $L_K u = f(u)$ in B_1 in the sense of distributions, with $f \in C^2$ and K satisfying* (1.8) *and* (1.9) *for some positive constants λ and Λ and for some $s \in (s_0, 1)$.*

Then,

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \le C(n, s_0, \lambda, \Lambda, f, \|u\|_{L^\infty(\mathbb{R}^n)}), \tag{C.1}$$

where $\alpha = \alpha(n, s_0, \lambda, \Lambda)$ is a positive constant.

Proof. Let us show first that if v is a distributional solution of $L_K v = g$ in $B_{2r}(x_0)$ belonging to $L^\infty(\mathbb{R}^n)$, then it satisfies

$$r^\alpha [v]_{C^\alpha(B_r(x_0))} \le C(n, s_0, \lambda, \Lambda)\big(\|v\|_{L^\infty(\mathbb{R}^n)} + r^s \|g\|_{L^\infty(B_{2r}(x_0))}\big). \tag{C.2}$$

Indeed, since L_K is translation invariant we can apply the C^α estimate for solutions to integro-differential equations in [21] to $v * \phi^\varepsilon$ and $g * \phi^\varepsilon$, where ϕ^ε is a smooth mollifier and then send $\varepsilon \to 0$.

Second, if $0 < r_1 < r_2 < r_3$, $B_{r_3}(x_0) \subset B_1$, and $\eta \in C_c^\infty(B_{r_3}(x_0))$, $0 \le \eta \le 1$ is some radial cutoff satisfying $\eta \equiv 1$ in $B_{r_2}(x_0)$, then thanks to the smoothness of the tails of the kernels assumed in (1.9) we have —see the proof of Corollary 1.2 in [61] for more details—

$$\begin{aligned} \|L_K(v\eta)\|_{C^\beta(B_{r_1}(x_0))} &\le C\big(\|L_K v\|_{C^\beta(B_{r_3}(x_0))} + \|v\|_{L^\infty(\mathbb{R}^n)}\big) \quad \text{and} \\ \|v\eta\|_{C^\beta(\mathbb{R}^n)} &\le C\|v\|_{C^\beta(B_{r_3}(x_0))} \end{aligned} \tag{C.3}$$

for all $\beta \le 2$, where C depends only on n, Λ, and r_i.

Finally, we show that using (C.2)-(C.3) we can adapt the standard local bootstrap argument for semilinear equations to make it work on our nonlocal equation $L_K u = f(u)$ in B_1. Observe first that since $u \in L^\infty(\mathbb{R}^n)$, applying (C.2) to $v = u$ we obtain (up to a scaling and covering argument) $\|u\|_{C^\alpha(B_{1-\varrho})} \le C_1$, where $\varrho > 0$. Our next goal will be to show that, whenever $k\alpha \le 2$, the following implication holds

$$\|u\|_{C^{k\alpha}(B_{1-k\varrho})} \le C_k \quad \Rightarrow \quad \|u\|_{C^{(k+1)\alpha}(B_{1-(k+1)\varrho})} \le C_{k+1}. \tag{C.4}$$

Here, the constants C_k depend only on n, s_0, λ, Λ, f, $\|u\|_{L^\infty(\mathbb{R}^n)}$, and $\varrho > 0$.

---

[9] We say that a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ is a distributional of $L_K u = g$ in a domain $\Omega \subset \mathbb{R}^n$ if $\int (u L_K \xi - g\xi)dx = 0$ for all $\xi \in C_c^\infty(\Omega)$.

Indeed, let $r_1 := 1-(k+1/2)\varrho$, $r_2 := 1-(k+1/4)\varrho$, and $r_3 := 1-k\varrho$ and choose the cut-off η as above. Define $\bar u := u\eta$. Thanks to the assumption in (C.4), and using $f \in C^2$, $Lu = f(u)$, and $u \in C^{k\alpha}(B_{r_3})$ we obtain $f(u) \in C^{k\alpha}(B_{r_3})$. Hence using (C.3) we find we obtain

$$\|L_K \bar u\|_{C^\beta(B_{r_1})} + \|\bar u\|_{C^\beta(\mathbb{R}^n)} \le CC_k, \quad \text{where } \beta := \alpha k \le 2.$$

By the C^α estimate in (C.2), used with v replaced by the incremental quotients (or incremental quotients of derivatives) of order $\beta = \alpha k \le 2$ of $\bar u$, we obtain $\|u\|_{C^{(k+1)\alpha}(B_{1-(k+1)\varrho})} = \|\bar u\|_{C^{(k+1)\alpha}(B_{1-(k+1)\varrho})} \le C_{k+1}$, proving (C.4). Hence after $N := 1/\alpha + 1$ iterations (taking $\varrho = \frac{1}{2N}$) we obtain (C.1). □

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