ON THE FIRST SIGN CHANGE OF FOURIER COEFFICIENTS OF CUSP FORMS

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Abstract. We present a variant of the current widely-used method initiated by Choie and Kohnen in the study of the location of the first sign change of the Fourier coefficients of a holomorphic cusp form when all the coefficients are real. This version circumvents the use of Atkin-Lehner theory of newforms, instead utilizing the Eisenstein series, and it applies directly to cases including integral weight cusp forms on the congruence subgroup $\Gamma_0(N)$ of any level N as well as half-integral weight cusp forms.

1. Introduction and the main result

Within the last two decades there have been various vital studies on the first sign change in the sequence $\{a(n)\}_{n\in\mathbb{N}}$ of the Fourier coefficients of a holomorphic cusp form when all $a(n)$'s are real. Two scenarios have been developed respectively for the Hecke eigenforms and generic modular forms.

The former case was considered first and dated back to 2006 in Kohnen and Sengupta [13]. The intrinsic structure of Hecke eigenforms leads to the multiplicativity of Fourier coefficients, which provides a stage for multiplicative number theory to play. This was first invoked by Iwaniec, Kohnen and Sengupta (2007, [11]) and subsequently sharpened by Kowalski et al. (2010, [14]) and Matomäki (2012, [15]). A salient idea in [14] is the analogy between the sign of the Fourier coefficients (of newforms) and the values of quadratic Dirichlet characters. This first sign change problem then falls in the same context as Linnik's problem on the least quadratic non-residue. The work of Iwaniec et al. [11] is a significant breakthrough on the square root barrier in the Linnik problem. Readers are referred to Tao's post [19] on the Linnik problem and the square root barrier.

The study of the first sign change for generic modular forms was essentially pioneered in Choie and Kohnen (2009, [4]). As noted in [4], it follows from Siegel's work (1969, [21]) that a modular form g, not necessarily a cusp form, of weight $k \equiv 2 \bmod 4$ for the full modular group must exhibit a sign change within the first $d_k + 1$ Fourier coefficients, where $d_k := \dim M_k$ is the dimension of the space of modular forms. However Siegel did not specifically address the first change problem, and his method does not seems directly applicable to the general case.

Let $k \geqslant 2$ be an even integer, N be a positive integer and $S_k(N)$ be the space of holomorphic cusp forms of weight k and level N. When N is squarefree and $0 \neq f \in S_k(N)$ has real Fourier coefficients $a(n)$, we now have from the result of Choie and Kohnen (2009, [4]) that there exist $n_1, n_2 \in \mathbb{N}$ with $n_1, n_2 \ll k^5 N^{9/2}\Psi_k(N)$ such that $a(n_1)a(n_2) < 0$, where $\Psi_k(N) = (kN)^{o(1)}$ in general (when N does not have many prime factors). Their detection method of the first sign change lies in the comparison of the mean value $\sum_{n\leqslant x} a(n)\log^2(x/n)$ and the second moment $\sum_{n\leqslant x} a(n)^2\log^2(x/n)$ together with an upper bound on the size of $a(n)$. Explicit dependence on the weight k and level N is demanded in the evaluation process. To that end, Choie and Kohnen make use of Atkin-Lehner theory of newforms to express f in terms of a special orthogonal basis in $S_k(N)$ and to estimate with Rankin-Selberg L-functions for two newforms, keeping track of the contribution of their analytic conductors.

Two further research objectives were then identified for follow-up: (i) improve the upper bound for the location of the sign change and (ii) extend the result to a wider class of cusp forms, e.g., of non-squarefree levels or of other weights. Objective (ii) was attempted with a success in Choie and Kohnen (2013, [5]) for a nonzero cusp form h, with real Fourier coefficients, in the Kohnen plus subspace $S^{+\cdots+}_{k-1/2}(4m)$ of the space of cusp forms weight $k-\frac{1}{2}$ and level $4m$, where k is a positive even integer and m is an odd squarefree integer. They showed the occurrence of a sign change in the first $O((k+m)^{5+o(1)})$ coefficients. The proof method uses the theory of Jacobi forms to connect the Fourier coefficients of h and some cusp form of weight $\leqslant k+2m$ and level 1, and the sign change is detected with the method in [4]. The case of Siegel cusp forms was also studied in Choie, Gun and Kohnen (2015, [3]) using again the bound in [4].

Very recently, Cho, Jin and Lim (2023, [2]) extended the work of Choie and Kohnen to the case of a general level, using the same method but a more delicate treatment of an appropriate special orthogonal basis. However, their result is weaker than that of the squarefree level case, since it only ensures the occurrence of a sign change in the first $O(k^{6+\varepsilon}N^{9+\varepsilon})$ coefficients. A quantitative improvement on the upper bound, i.e. Objective (i), was worked out by He and Zhao (2018, [8]) with a substantial reduction to $(kN)^{2+o(1)}$ (from $k^5N^{9/2}\Psi_k(N)$) under the same conditions, i.e. $k\in 2\mathbb{N}$ and $N\geqslant 1$ squarefee. (One of the theorems in Jin (2019, [12]) improves the result of Choie and Kohnen in the prime-level case but does not supersede the bound in [8].) The framework of the method in [8] (as well as [12]) is the same as in Choie and Kohnen [4] – relying on Atkin-Lehner theory in the space of cusp forms, Rankin-Selberg L-function theory for (cuspidal) newforms, and Deligne's bound for the Ramanujan conjecture.

1.1. **The main result.** In this note, we approach the first sign-change detection with a variant of a key methodological component that appears in step two if we consider the method as a two-step process. In step one, we continue to utilize the first moment (or mean value) of the coefficients to detect the potential cancellations, applying the approach outlined in He and Zhao [8], and derive the mean value estimate in Proposition 3.2.

The second step is to rule out the possibility that the small first moment resulted from the small size of all coefficients. This is unveiled via the asymptotic formula for the second moment and hence Rankin-Selberg L-functions. Previous work has employed the Rankin-Selberg L-functions for a pair of newforms, expressing a generic form f as a linear combination $f=\sum\alpha_g g$ of newforms g, using the Atkin-Lehner theory of cuspidal newforms. However, in the case of non-squarefree levels, the structure of the coefficients α_g and the conductor of $L(s,g\times g')$ are more complex, leading to varying result quality between squarefree and non-squarefree levels.

The novelty here is the direct treatment of $L(s,f\times f)$ for a generic form f leveraging the connection among Eisenstein series. It relies on the link between the Eisenstein series for $\Gamma_0(N)$ at the cusp ∞ and the scaled Eisenstein series $E(dz,s)$ for the full modular group, where $d|N$. This provides a simpler method for evaluating the second moment, as detailed in Theorem 2 and its proof.

Our primary goal is to establish the first sign change result below, which follows from Proposition 3.2 and Theorem 2. This result applies to integral weight cusp forms of non-squarefree level and to half-integral weight cusp forms outside the Kohnen plus space.

Theorem 1. *Let $k \in \frac{1}{2}\mathbb{N}$, and $N \in \mathbb{N}$ or $4\mathbb{N}$ according as $k \in \mathbb{N}$ or not. Suppose $0 \neq f$ is a holomorphic cusp form of weight k, level N and nebentypus χ where χ is a Dirichlet character mod N. Assume all the Fourier coefficients $a(n)$ of f are real. Then for any $\varepsilon > 0$, there exist $n_1, n_2 \ll_\varepsilon (kN)^{2+\varepsilon}$ such that $a(n_1)a(n_2) < 0$.*

Remark 1. (i) We reach the best known bound to date of He and Zhao with more relaxed conditions on the level, weight and nebentypus. (ii) The approach here can easily be transferred to deal with the case of Maass cusp forms for $\Gamma_0(N)$.

Remark 2. Since Theorem 1 applies to integral weight cusp forms attached with a nebentypus, the case of half-integral weight can be deduced easily. This is because, for a half-integral weight form $\mathfrak{f}(z) = \sum_{n \geqslant 1} c(n)e(nz)$, the signs of $c(n)$ for $n \leqslant \ell$ cannot be all the same when $\mathfrak{f}^2(z) = \sum_{\ell \geqslant 2} \big(\sum_{m+n=\ell} c(m)c(n)\big)e(\ell z)$, which is an integral weight form, has a negative ℓth Fourier coefficient.

Organization. Section 2 summarizes and develops the key ingredients related to the congruence subgroups, Fourier expansions at cusps and Eisenstein series. Section 3 focuses on the mean value estimate with Voronoi formula. Section 4 is the core of the paper, presenting our new variant to establish the mean square estimate. Sections 5 and 6 contain the proofs of Theorem 1 and a lemma, respectively.

2. Preliminaries

In what follows, for any 2×2 matrix γ, we express its entries as $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$. We denote

$$n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \quad \text{and} \quad a(y) = \begin{pmatrix} y & \\ & 1/y \end{pmatrix}$$

for $x \in \mathbb{R}$ and $y \in \mathbb{R}^\times$.

In this section, Γ denotes finite-index discrete subgroup of the full modular group $SL_2(\mathbb{Z})$ unless specified in the context. We recap some basics from Diamond and Shurman [7], Iwaniec [9], etc., below.

2.1. **Cusps and right coset representatives.** Let Γ be a discrete subgroup of $SL_2(\mathbb{Z})$ with $[SL_2(\mathbb{Z}) : \Gamma] < \infty$. Fix a set of inequivalent cusps $\mathfrak{A}$ of Γ, which is finite. Every $s \in \mathbb{Q} \cup \{\infty\}$ is Γ-equivalent to some $\mathfrak{a} \in \mathfrak{A}$, written as $s \sim_\Gamma \mathfrak{a}$. Let $\Gamma_\mathfrak{a}$ be the stabilizer of $\mathfrak{a}$ in Γ, and $\sigma_\mathfrak{a} \in SL_2(\mathbb{Z})$ be a *non-standard* scaling matrix such that $\sigma_\mathfrak{a}\infty = \mathfrak{a}$. (A scaling matrix $\sigma_\mathfrak{a}$ is often required to satisfy $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} \cap SL_2(\mathbb{Z}) = \langle n(1)\rangle$.) This matrix $\sigma_\mathfrak{a}$ satisfies

$$\{\pm I\} \cdot \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = SL_2(\mathbb{Z})_\infty \cap \sigma_\mathfrak{a}^{-1}\{\pm I\} \cdot \Gamma\sigma_\mathfrak{a} = (\sigma_\mathfrak{a}^{-1}\{\pm I\} \cdot \Gamma\sigma_\mathfrak{a})_\infty = \langle \pm n(w_\mathfrak{a})\rangle.$$

The width $w_\mathfrak{a}$ of a cusp $\mathfrak{a}$ is defined as $w_\mathfrak{a} = w_\mathfrak{a}(\Gamma) := |SL_2(\mathbb{Z})_\infty/(\{\pm I\} \cdot \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a})|$. Clearly $\sigma_\mathfrak{a}a(\sqrt{w_\mathfrak{a}})$ is a (standard) scaling matrix. When $-I \notin \Gamma$, $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a}$ equals either $\langle n(w_\mathfrak{a})\rangle$ or $\langle -n(w_\mathfrak{a})\rangle$. We call $\mathfrak{a}$ a *regular cusp* or *irregular cusp* respectively. Both $\Gamma_\mathfrak{a}$ and

$w_{\mathfrak{a}}$ depend on $\mathfrak{a}$ only and are independent of the choice of $\sigma_{\mathfrak{a}}$. As $\Gamma_{\sigma\mathfrak{a}} = \sigma\Gamma_{\mathfrak{a}}\sigma^{-1}$, $\Gamma_{\mathfrak{b}}$ is conjugate to $\Gamma_{\mathfrak{a}}$ in Γ if $\mathfrak{b} \sim_\Gamma \mathfrak{a}$.

Let $\mathcal{R}$ be any complete set of inequivalent right coset representatives of $\{\pm I\}\cdot\Gamma$ in $SL_2(\mathbb{Z})$. Then $|\mathcal{R}| = \mu$ where μ is the index of $\{\pm I\}\cdot\Gamma$ in $SL_2(\mathbb{Z})$, which equals $\mu' := [SL_2(\mathbb{Z}) : \Gamma]$ or $\mu'/2$ according as $-I \in \Gamma$ or not. If the matrix $\sigma_{\mathfrak{a}}$ lies in the right coset $\{\pm I\}\cdot\Gamma\tau$ for some $\tau \in \mathcal{R}$, we shall have $\tau(\infty) \sim_\Gamma \mathfrak{a}$. Up to Γ-equivalence, $\mathfrak{A} = \{\tau(\infty) : \tau \in \mathcal{R}\}$ (as sets, not multi-sets). Besides, if $\tau(\infty) \sim_\Gamma \tau'(\infty)$ where $\tau, \tau' \in SL_2(\mathbb{Z})$, one checks that $\tau' \in \{\pm I\}\cdot\Gamma\tau n(r)$ for some $r \in \mathbb{Z}$. Let $h := |\mathfrak{A}|$ and fix h (non-standard) scaling matrices $\tau_{\mathfrak{a}}$ in $SL_2(\mathbb{Z})$. Then

$$\bigsqcup_{\mathfrak{a}\in\mathfrak{A}} \{\tau_{\mathfrak{a}} n(r) : r \bmod w_{\mathfrak{a}}\}$$

is a collection of right coset representatives of $\{\pm I\}\cdot\Gamma$. Hence we have $\mu = \sum_{\mathfrak{a}\in\mathfrak{A}} w_{\mathfrak{a}}$, and $\bigsqcup_{r \bmod w_{\mathfrak{a}}} \{\pm I\}\cdot\Gamma\tau_{\mathfrak{a}} n(r)$ collects all the matrices τ in $SL_2(\mathbb{Z})$ satisfying $\tau(\infty) \sim_\Gamma \mathfrak{a}$. Let $\mathfrak{a} \in \mathfrak{A}$ and $\tau \in SL_2(\mathbb{Z})$ such that $\tau(\infty) = \mathfrak{a}$. We have $\{\pm I\}\cdot\tau^{-1}\Gamma_{\mathfrak{a}}\tau = \langle \pm n(w_{\mathfrak{a}})\rangle$ and

$$\{\pm I\}\cdot\Gamma_{\mathfrak{a}} = \langle \pm \begin{pmatrix} 1 - caw_{\mathfrak{a}} & a^2 w_{\mathfrak{a}} \\ -c^2 w_{\mathfrak{a}} & 1 + caw_{\mathfrak{a}} \end{pmatrix}\rangle \quad \text{if } \tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1}$$

For our later use, we consider specifically the congruence subgroups $\Gamma_0(N)$ for $N \in \mathbb{N}$ and $\Gamma_0(N/\ell, \ell)$ for $1 \leqslant \ell | N$, which consists of matrices $\gamma \in SL_2(\mathbb{Z})$ with $c_\gamma \equiv 0 \bmod N/\ell$ and $b_\gamma \equiv 0 \bmod \ell$. It is easy to see that

$$\Gamma_0(N/\ell, \ell) = a(\sqrt{\ell})\Gamma_0(N)a(\sqrt{\ell})^{-1}.$$

2.1.1. *The subgroup* $\Gamma_0(N)$. In the following, $\mathfrak{A}$ will denote a specific cusp set (as chosen in (II) below). From [17, 3.4.1] and [7, Section 3.8], we get

(I) the set of right cosets $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$ has cardinality $N\prod_{p|N}(1+p^{-1})$ and

$$\Gamma_0(N)\backslash SL_2(\mathbb{Z}) = \left\{\Gamma_0(N)\tau : \ c_\tau = c|N,\ d_\tau = d \in \mathbb{Z}/\tfrac{N}{c}\mathbb{Z},\ (d, c, N/c) = 1\right\}.$$

(II) the orbit space of cusps $\{\Gamma_0(N)\mathfrak{a} : \mathfrak{a} \text{ cusp}\}$ is mapped bijectively to

$$\mathfrak{A} := \{\mathfrak{a}_{d/c} = a/c : c|N,\ d \in (\mathbb{Z}/(c, \frac{N}{c})\mathbb{Z})^\times\}$$

where a is an integer with $(a, c) = 1$ and $ad \equiv 1 \bmod (c, N/c)$. An immediate question is to check $a/c \sim_{\Gamma_0(N)} a'/c$ where $(a, c) = (a', c) = 1$ and $ad \equiv a'd \equiv 1 \bmod (c, N/c)$ for some $d \in (\mathbb{Z}/(c, N/c)\mathbb{Z})^\times$. This can be justified with [7, Proposition 3.8.3] if we find integers y, j that satisfy

$$y \equiv 1 \bmod N/c,\ \ (y, c) = 1 \ \text{ and } \ ya' \equiv a + jc \bmod N.$$

Since $(y, c) = 1$ follows from the last congruence, this is equivalent to solving the linear congruence

$$(1 + \ell\tfrac{N}{c})a' \equiv a + jc \bmod N$$

in the variables ℓ and j, for which a solution is guaranteed, as $(a'N/c, c, N) = (c, N/c)$ divides $a - a'$, cf. [18, Section 5.2].

The set $\mathfrak{A}$ is partitioned into $\tau(N) := \sum_{c|N} 1$ classes $\mathfrak{A}_c$ of cusps $\mathfrak{a}_{d/c}$ of the same denominator c, i.e. $\mathfrak{A} = \bigsqcup_{c|N} \mathfrak{A}_c$. We can take the non-standard scaling $\tau_{\mathfrak{a}}$ to be a matrix $\tau \in SL_2(\mathbb{Z})$ with $a_\tau = a$ and $c_\tau = c$. For every $c|N$, $|\mathfrak{A}_c| = \phi((c, N/c))$, where ϕ is the Euler phi function, and all the cusps $\mathfrak{a}$ in $\mathfrak{A}_c$ have the same width $w_{\mathfrak{a}} = N/(N, c^2)$.

(III) the double coset space $\Gamma_0(N)\backslash SL_2(\mathbb{Z})/SL_2(\mathbb{Z})_\infty$ maps bijectively onto $\mathfrak{A}$ under the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \mathfrak{a}_{d/c}$, using (I) and (II). The number of double cosets is $\sum_{c|N} \phi((c, N/c))$, which is the number of inequivalent cusps.

2.1.2. *The subgroup $\Gamma_0(N/\ell, \ell)$.* Recall $\gamma \in \Gamma_0(N/\ell, \ell)$ if $\gamma \in SL_2(\mathbb{Z})$ with $c_\gamma \equiv 0 \bmod N/\ell$ and $b_\gamma \equiv 0 \bmod \ell$.

Lemma 2.1. *Let $\Gamma = \Gamma_0(N/\ell, \ell)$ and $\mathfrak{A} = \{\mathfrak{a}_{d/c}\}$ be the set of $\Gamma_0(N)$-inequivalent cusps in Section 2.1.1.*

(i) *The coset space $\Gamma\backslash SL_2(\mathbb{Z})$ has cardinality $N\prod_{p|N}(1 + p^{-1})$.*

(ii) *A set of inequivalent cusps of Γ is given by*

$$\mathfrak{A}(\Gamma) := \{\ell\mathfrak{a}_{d/c} : \ c|N, \ d \in (\mathbb{Z}/(c, N/c)\mathbb{Z})^\times\},$$

corresponding to $\mathfrak{A} = \{\mathfrak{a}_{d/c}\}$.

(iii) *The double coset space $\Gamma\backslash SL_2(\mathbb{Z})/SL_2(\mathbb{Z})_\infty$ admits the set of representatives $\mathcal{D} := \{\tau\}$ with*

$$a_\tau = \frac{\ell}{(c, \ell)} a \quad \text{and} \quad c_\tau = \frac{c}{(c, \ell)}$$

where a/c runs over all cusps $\mathfrak{a}_{d/c}$ in $\mathfrak{A} = \bigsqcup_{c|N} \mathfrak{A}_c$ (see (II)).

(iv) *The right coset space $SL_2(\mathbb{Z})_\infty\backslash SL_2(\mathbb{Z})$ admits the set of representatives*

$$\left\{\tau^{-1}\gamma : \tau \in \mathcal{D}, \gamma \in \Gamma_\mathfrak{b}\backslash\Gamma \text{ with } \mathfrak{b} = \ell\mathfrak{a}_{d/c}\right\},$$

where $\mathcal{D}$ and $\mathfrak{a}_{d/c}$ are defined as in (iii) (so $\mathfrak{b} = \ell\mathfrak{a}_{d/c} = a_\tau/c_\tau$ is determined by τ), and $\Gamma_\mathfrak{b} = (\tau SL_2(\mathbb{Z})_\infty\tau^{-1}) \cap \Gamma$.

(v) *The width of the cusp $\mathfrak{b} = \ell\mathfrak{a}_{d/c} \in \mathfrak{A}(\Gamma)$ is given by*

$$w_\mathfrak{b} = \frac{(c, \ell)^2}{\ell} w_{\mathfrak{a}_{d/c}} = \frac{(c, \ell)^2}{\ell} \frac{N}{(N, c^2)}.$$

Proof. (i) can be deduced with basic group theory. Write $G := SL_2(\mathbb{Z})$, $H := \Gamma_0(N)$ and $a := a(\sqrt{\ell})$. Operating in $SL_2(\mathbb{R})$ with $\Gamma = \Gamma_0(N/\ell, \ell) = a(\sqrt{\ell})\Gamma_0(N)a(\sqrt{\ell})^{-1}$, we get

$$[G : \Gamma] = [G : aHa^{-1}] = [G : aGa^{-1} \cap G][aGa^{-1} \cap G : aHa^{-1}].$$

As the conjugation is an automorphism, $[aGa^{-1} \cap G : aHa^{-1}] = [G \cap a^{-1}Ga : H]$. Also, G and a are invariant under transposes, i.e. $G^T = G$ and $a^T = a$. The transpose leads to an anti-automorphism mapping bijectively from the left coset space $(aGa^{-1} \cap G)\backslash G$ to the right coset space $G/(a^{-1}Ga \cap G)$. Thus, $[G : aGa^{-1} \cap G] = [G : G \cap a^{-1}Ga]$ and so

$$[G : \Gamma] = [G : G \cap a^{-1}Ga][G \cap a^{-1}Ga : H] = [G : H].$$

Section 2.1.1 (I) gives (i).

(ii) is clear as $a(\sqrt{\ell})\mathfrak{a} = \ell\mathfrak{a}$. This leads to an one-one correspondence between the cusps of $\Gamma_0(N)$ and Γ. Moreover, $\ell\mathfrak{a} \sim_\Gamma \ell\mathfrak{a}'$ if and only if $\mathfrak{a} \sim_{\Gamma_0(N)} \mathfrak{a}'$. (ii) follows from (II).

(iii) We first check that two different τ's in $\mathcal{D}$ give rise to different double cosets. (Two double cosets are either identical or disjoint.) Suppose $\Gamma\tau SL_2(\mathbb{Z})_\infty = \Gamma\tau' SL_2(\mathbb{Z})_\infty$. Acting on ∞, the cusp $\tau\infty$ is $\ell a/c = \ell\mathfrak{a}_{d/c}$, which is Γ-equivalent to $\tau'\infty = \ell\mathfrak{a}_{d'/c'}$ and hence $\mathfrak{a}_{d/c} \sim_{\Gamma_0(N)} \mathfrak{a}_{d'/c'}$, implying $c = c'$ and $d = d'$ by (II). Checking the cardinality, we see that $\left|\Gamma\backslash SL_2(\mathbb{Z})/SL_2(\mathbb{Z})_\infty\right| = |\mathfrak{A}(\Gamma)| = |\mathfrak{A}|$. The assertion follows.

(iv) By (iii) and taking the inverse, we have

$$\bigsqcup_{\tau\in\mathfrak{D}} SL_2(\mathbb{Z})_\infty\tau^{-1}\Gamma = SL_2(\mathbb{Z}) = \bigsqcup_{\sigma\in SL_2(\mathbb{Z})_\infty\backslash SL_2(\mathbb{Z})} SL_2(\mathbb{Z})_\infty\sigma.$$

Suppose σ lies in the double coset of τ^{-1}. Then $SL_2(\mathbb{Z})_\infty\sigma = SL_2(\mathbb{Z})_\infty\tau^{-1}\gamma$ for some $\gamma\in\Gamma$. If $SL_2(\mathbb{Z})_\infty\tau^{-1}\gamma' = SL_2(\mathbb{Z})_\infty\tau^{-1}\gamma$, then $\gamma'\in\Gamma_{\mathfrak{b}}\gamma$ where $\Gamma_{\mathfrak{b}} = \tau SL_2(Z)_\infty\tau^{-1}\cap\Gamma$. Clearly the converse is true. Thus the double coset $SL_2(\mathbb{Z})_\infty\tau^{-1}\Gamma$ decomposes into a disjoint union of $SL_2(\mathbb{Z})_\infty\tau^{-1}\gamma$ over $\gamma\in\Gamma_{\mathfrak{b}}\backslash\Gamma$.

(v) By definition, the cusp width of $\mathfrak{b}$ is $w_{\mathfrak{b}} = |SL_2(\mathbb{Z})_\infty/\tau^{-1}\Gamma_{\mathfrak{b}}\tau|$. Hence we are going to determine the smallest positive r for which $\tau n(r)\tau^{-1}\in\Gamma_0(N/\ell,\ell)$. Applying (iii) for a_τ and c_τ, it is equivalent to the minimal $r\in\mathbb{N}$ that fulfills

$$\left(\frac{\ell a}{(\ell,c)}\right)^2 r\equiv 0 \mod \ell,\ \text{ and }\ \left(\frac{c}{(\ell,c)}\right)^2 r\equiv 0 \mod N/\ell. \tag{2.2}$$

Verifying locally, we will show the minimal $r=(c,\ell)^2 w_{\mathfrak{a}}/\ell$. For every prime p, we write $\mathrm{n}=\mathrm{ord}_p(N)$, $\iota=\mathrm{ord}_p(\ell)$, $\mathrm{c}=\mathrm{ord}_p(c)$, $\mathrm{r}=\mathrm{ord}_p(r)$. Note that $\mathrm{c},\iota\leqslant\mathrm{n}$ for $c,\ell|N$.

- $2\mathrm{c}\leqslant\iota$: The first congruence in (2.2) can be voided and the second congruence implies $\mathrm{r}=\mathrm{n}-\iota$. Now, noting $2\mathrm{c}\leqslant\mathrm{n}$ from $\iota\leqslant\mathrm{n}$, we have
$$\mathrm{w} := \mathrm{ord}_p\left(\frac{(c,\ell)^2}{\ell}\frac{N}{(N,c^2)}\right) = 2\mathrm{c}-\iota+\mathrm{n}-2\mathrm{c} = \mathrm{n}-\iota = \mathrm{r}.$$
- $\mathrm{c}\leqslant\iota<2\mathrm{c}$: This case implies $p|c$ and so $p\nmid a$ from $(a,c)=1$. (2.2) implies r equals $\max(2\mathrm{c}-\iota,\mathrm{n}-\iota)$. Thus $\mathrm{r}=2\mathrm{c}-\iota$ occurs when $2\mathrm{c}\geqslant\mathrm{n}$, and in this case, $\mathrm{w}=2\mathrm{c}-\iota$. If $2\mathrm{c}<\mathrm{n}$, then $\mathrm{w}=2\mathrm{c}-\iota+\mathrm{n}-2\mathrm{c}=\mathrm{r}$ as well.
- $\iota<\mathrm{c}$: Now $p|\ell$ implies $p|c$, thus we can ignore a in the congruence in (2.2). So we have $\mathrm{r}=\max(\iota,\mathrm{n}-\iota-2(\mathrm{c}-\iota))=\max(\iota,\mathrm{n}+\iota-2\mathrm{c})$, which is also valid for $p\nmid\ell$ but $p|N$. Now, $\mathrm{w}=\iota+\mathrm{n}-\min(\mathrm{n},2\mathrm{c})=-\min(-\iota,2\mathrm{c}-\iota-\mathrm{n})=\mathrm{r}$.

In summary, $\mathrm{ord}_p(w_{\mathfrak{b}})=\mathrm{ord}_p(r)$ for all prime p. □

Remark 3. Lemma 2.1 yields that both $\Gamma_0(N/\ell,\ell)$ and $\Gamma_0(N)$ have the same index in $SL_2(\mathbb{Z})$ and the same number of inequivalent cusps but their cusp widths may differ. The sum of cusp widths in either group equals the index $N\prod_{p|N}(1+p^{-1})$. For cross-checking, the case of $\Gamma_0(N)$ is shown in [9, (2.30)]. We verify the case of $\Gamma_0(N/\ell,\ell)$ below:

$$\begin{aligned}\sum_{c|N}\sum_{d\in(\mathbb{Z}/(c,\frac{N}{c})\mathbb{Z})^\times}\frac{(c,\ell)^2}{\ell}\frac{N}{(N,c^2)} &= N\sum_{c|N}\frac{(c,\ell)^2}{c\ell}\prod_{p|(c,N/c)}(1-\frac{1}{p})\\ &= N\prod_{p|N}\left\{\frac{1}{p^\iota}+(1-p^{-1})\sum_{\mathrm{c}=1}^{\mathrm{n}-1}p^{\min(\mathrm{c}-\iota,\iota-\mathrm{c})}+p^{\iota-\mathrm{n}}\right\} = N\prod_{p|N}(1+p^{-1}).\end{aligned}$$

2.2. **Fundamental domains.** Let $D := SL_2(\mathbb{Z})\backslash\mathbb{H}$ be the fundamental domain for $SL_2(\mathbb{Z})$. A discrete subgroup Γ of $SL_2(\mathbb{Z})$ of finite index μ has a fundamental domain $\Gamma\backslash\mathbb{H}$ equal to the union of μ copies of D. Moreover for each cusp $\mathfrak{a}$, there are $w_{\mathfrak{a}}$ copies of D that meet at $\mathfrak{a}$. Wohlfahrt [23] defined the *general level* N for Γ by the least common multiple N of $w_{\mathfrak{a}}$ for all $\mathfrak{a}\in\mathfrak{A}$. For a congruence subgroup Γ of level N (defined by Klein), the two levels N and N are equal, cf. [22] or [23].

For any subgroup G of $SL_2(\mathbb{Z})$, we let $c_G := \min\{c_\gamma>0:\gamma\in G\}$ if G contains a γ with $c_\gamma\neq 0$. When $\mathfrak{a}\not\sim_\Gamma\infty$, $c_{\mathfrak{a}} := \min\{c\geqslant 1 : \frac{a}{c}\sim_\Gamma\mathfrak{a}\}$ exists. We set $c_\infty := 1$ and

$c_{\Gamma_\infty} = w_\infty$ by convention (for consistency below). By (2.1), $c_{\Gamma_\mathfrak{a}} \geqslant c_\mathfrak{a}^2 w_\mathfrak{a}$ holds for $\mathfrak{a} \neq \infty$, because $c_{\Gamma_\mathfrak{a}} \in c_\tau^2 w_\mathfrak{a}\mathbb{Z}$ for $\tau \in SL_2(\mathbb{Z})$ with $\tau(\infty) = \mathfrak{a}$. Our convention maintains validity for the cusp ∞.

Geometrically, $\Gamma\backslash\mathbb{H}$ is identified with the glued domain of a (fundamental) polygon $\mathcal{F}$ in $\mathbb{H}$ for Γ; the interior of the polygon in $\mathbb{H}$ equals the intersection of exteriors to a set of isometric circles and the half-strip P_0 where

$$P_Y := \{z = x + \mathrm{i}y : |x| < \tfrac{1}{2}\mathrm{w}_\infty,\ y > Y\} \tag{2.3}$$

for any $Y \geqslant 0$, cf. [9, p.32]. The set P_0 is the ($\mathbb{H}$-)interior of the polygon for the fundamental domain $\Gamma_\infty\backslash\mathbb{H}$ of Γ_∞, and its closure $\overline{P_0}$ in $\mathbb{H}$ contains all $\gamma\mathcal{F}$ with $\gamma \in \Gamma_\infty\backslash\Gamma$.

We write $\mathrm{d}\mu = y^{-2}\,\mathrm{d}x\,\mathrm{d}y$ for the hyperbolic measure on $\mathbb{H}$ or its pushforward on $\Gamma\backslash\mathbb{H}$. The lemma below, basically following from [10], gives a quantitative version when $\Gamma\backslash\mathbb{H}$ and $\Gamma_\infty\backslash\mathbb{H}$ are identified with the fundamental polygons in $\overline{P_0}$.

Lemma 2.2. *Let $\pi : \Gamma_\infty\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H}$, $\Gamma_\infty z \mapsto \Gamma z$, be the natural projection, and $Y > 0$.*

(i) For any element $\mathrm{z} \in \Gamma\backslash\mathbb{H}$, the multiplicity of its preimages in P_Y, i.e. $|\pi^{-1}\{\mathrm{z}\} \cap P_Y|$, is at most $1 + O((c_\Gamma Y)^{-1})$.

(ii) For any nonnegative Γ-invariant function F on $\Gamma_\infty\backslash\mathbb{H}$, we have

$$\int_{P_Y} F\,\mathrm{d}\mu \leqslant (1 + O((c_\Gamma Y)^{-1})) \int_{\Gamma\backslash\mathbb{H}} F\,\mathrm{d}\mu.$$

The two implied O-constants are independent of Y and Γ.

Lemma 2.2 will be proved in Section 6.

2.3. **Multiplier systems.** Write $j(\gamma, z) := c_\gamma z + d_\gamma$ for $\gamma \in \Gamma$ and $z \in \mathbb{H}$. Let $k \in \frac{1}{2}\mathbb{Z}$ and χ be a multiplier system of weight k for Γ, which is by definition a function from Γ to $S^1 \subset \mathbb{C}$ and satisfies

(i) $\chi(\alpha\beta)/(\chi(\alpha)\chi(\beta)) = j(\alpha, \beta z)^k j(\beta, z)^k / j(\alpha\beta, z)^k$, $\forall\ \alpha, \beta \in \Gamma$, and

(ii) $\chi(-I) = e(-k/2)$ if $-I \in \Gamma$.

The multiplier system can actually be defined for any discrete subgroup Γ of $SL_2(\mathbb{R})$, not necessarily subgroup of $SL_2(\mathbb{Z})$. If $\Gamma_\mathfrak{a} = \langle\gamma_\mathfrak{a}\rangle$, then $\chi(\gamma_\mathfrak{a}) = e(\kappa_\mathfrak{a})$ for some $\kappa_\mathfrak{a} = \kappa_\mathfrak{a}(\chi) \in [0, 1)$. A cusp $\mathfrak{a} \in \mathfrak{A}$ is a *singular cusp* for the multiplier system χ if $\chi(\gamma_\mathfrak{a}) = 1$, i.e. $\kappa_\mathfrak{a} = 0$.

If $\mathfrak{a} \sim_\Gamma \mathfrak{a}'$, then $\kappa_\mathfrak{a} = \kappa_{\mathfrak{a}'}$: suppose $\mathfrak{a}' = \sigma\mathfrak{a}$ for some $\sigma \in \Gamma$. Then $\sigma_{\mathfrak{a}'} := \sigma\sigma_\mathfrak{a}$ is a nonstandard scaling matrix of $\mathfrak{a}'$ and so $\Gamma_{\mathfrak{a}'} = \sigma\Gamma_\mathfrak{a}\sigma^{-1}$. By [9, (2.57)], $\chi(\gamma_\mathfrak{a}) = \chi(\gamma_{\mathfrak{a}'})$.

2.3.1. *Example 1.* $k \in \mathbb{Z}$ and $\Gamma = \Gamma_0(N)$, $\chi(\gamma) := \chi_N(d)$, where $N \in \mathbb{N}$ and χ_N is an even Dirichlet character mod N and $d = d_\gamma$.

The cusps ∞ and $\frac{a}{c}$ with $(c, N/c) = 1$ (which includes 0) are necessarily singular and $S_k(\Gamma, \chi)$ is the space of modular forms of weight k, level N and nebentypus χ_N.

If $N \equiv 1 \bmod 4$ is squarefree and $\chi_N = \left(\frac{N}{\cdot}\right)$ is the Kronecker symbol, then all cusps are singular. If $N = 4N'$ where $N' \equiv 3 \bmod 4$ is squarefree and χ_N is the Kronecker symbol, then two thirds of the cusps are singular. See Theorem 1.1 of [1]. (We follow Iwaniec [9] to define a singular cusp, which is called regular cusp in [1], and a singular cusp in [1] means exactly a non-singular cusp in [9].)

For general $N \in \mathbb{N}$, by (2.1), $\chi(\gamma_\mathfrak{a}) = \chi(1 + caw_\mathfrak{a})$ for $\mathfrak{a} \in \mathfrak{A}_c$. As $w_\mathfrak{a} = N/(N, c^2)$, we infer that $\chi(\gamma_\mathfrak{a}^{(c,N/c)}) = 1$, which implies $(c, N/c)\kappa_\mathfrak{a} \in \mathbb{N} \cup \{0\}$, and therefore,

$$\kappa_\mathfrak{a} \geqslant 1/(c, N/c) \tag{2.4}$$

if $\kappa_a \neq 0$. Define $\tilde{\kappa}_{\mathfrak{a}} := \kappa_{\mathfrak{a}}$ if $\kappa_{\mathfrak{a}} \in (0,1)$ and $\tilde{\kappa}_{\mathfrak{a}} := 1$ if $\kappa_{\mathfrak{a}} = 0$, and

$$\mathcal{K}(\chi) := \max_{c|N} \frac{1}{|\mathfrak{A}_c|} \sum_{\mathfrak{a}\in\mathfrak{A}_c} \tilde{\kappa}_{\mathfrak{a}}^{-1}. \tag{2.5}$$

Lemma 2.3. $1 \leqslant \mathcal{K}(\chi) \ll (\log N)(\log\log N)$.

Proof. Given $c|N$ and any family $\mathcal{A}$ of $\phi((c,N/c))$ integers a's that are coprime to c and distinct modulo $(c,N/c)$. Then $\{a/c : a \in \mathcal{A}\}$ is $\Gamma_0(N)$-equivalent to $\mathfrak{A}_c$. Assume $(c,N/c) > 1$ and $\chi(\gamma_{\alpha/c}) \neq 1$ for some $\alpha \in \mathcal{A}$, because, otherwise, $\mathcal{K}(\chi) = 1$.

Observe that elements in $\{\alpha a : a \in \mathcal{A}\}$ are coprime to c and distinct modulo $(c,N/c)$. The set $\{\alpha a/c : a \in \mathcal{A}\}$ is equivalent to $\mathfrak{A}_c$ and $\{\kappa_{a/c} : a \in \mathcal{A}\} = \{\kappa_{\alpha a/c} : a \in \mathcal{A}\}$ as multisets. As $\chi(\gamma_{\alpha a/c}) = \chi(1+\alpha aN/(c,N/c))$ and

$$\chi(\gamma_{\alpha/c}^{a}) = \chi(\tau n(w_{\alpha/c})^a\tau^{-1}) = \chi(\tau n(aw_{\alpha/c})\tau^{-1}) = \chi(1+a\alpha N/(c,N/c)) = \chi(\gamma_{\alpha a/c})$$

where $\tau \in SL_2(\mathbb{Z})$ maps ∞ to α/c, it follows that $\kappa_{\alpha a/c} \equiv a\kappa_{\alpha/c} \bmod 1$. Write $\kappa_{\alpha/c} = \frac{u}{v}$ where $1 \leqslant u < v$ and $(u,v) = 1$. As $\chi(\gamma_{\alpha/c}) \neq 1$, we have $\kappa_{\alpha/c} \in \frac{1}{(c,N/c)}\mathbb{N}$, implying $v|(c,N/c)$, and $\kappa_{\alpha a/c} \neq 0$.

View $\{a \pmod{(c,N/c)} : a \in \mathcal{A}\} = \{1 \leqslant a \leqslant (c,N/c) : (a,(c,N/c)) = 1\}$. The family $\{au \pmod{v} : a \in \mathcal{A}\}$ is covered by $(c,N/c)/v$ sub-families of $\{1 \leqslant w < v : (w,v) = 1\}$. Thus the family $\{a\kappa_{\alpha/c} \pmod 1\}$ can be partitioned into, not necessarily disjoint, $(c,N/c)/v$ sub-families which are subsets of $\{w/v : 1 \leqslant w < v, (w,v) = 1\}$. Consequently,

$$\sum_{a\in\mathcal{A}} \tilde{\kappa}_{a/c}^{-1} = \sum_{a\in\mathcal{A}} \kappa_{\alpha a/c}^{-1} \leqslant \frac{(c,N/c)}{v} \sum_{1\leqslant w<v} \frac{v}{w} \leqslant (c,N/c)\log v.$$

As $|\mathfrak{A}_c| = \phi((c,N/c)) \gg (c,N/c)/(\log\log N)$, our assertion follows. □

2.3.2. *Example 2.* $k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$. Let Δ be a Fuchsian subgroup of the universal covering of $SL_2(\mathbb{R})$ in Shimura [20], in which an element $\xi \in \Delta$ is of the form $(\alpha,\varphi(z))$. Take $\chi(\alpha) := \varphi(z)^{2k}/j(\alpha,z)^k$, which is a multiplier system of $\Gamma := P(\Delta)$, the image of Δ under projection P. The space $S_k(\Delta)$ in [20] equals $S_k(\Gamma,\chi)$. (A transfer of viewpoints from the multiplier system to the metaplectic cover for half-integral weight modular forms is explained in [6, §1.1.6].)

A typical multiplier system of the half-integral weight k for $\Gamma = \Gamma_0(4N)$ is

$$\chi(\gamma) = \Big(\frac{-4}{d_\gamma}\Big)^{-k}\Big(\frac{c_\gamma}{d_\gamma}\Big)\chi_{4N}(\gamma)$$

where the nebentypus χ_{4N} is (induced by) an even Dirichlet character mod $4N$ and $\big(\frac{\cdot}{\cdot}\big)$ denotes the Kronecker symbol. By convention $\big(\frac{0}{\pm1}\big) = 1$, $-\frac{\pi}{2} < \arg(z^{1/2}) \leqslant \frac{\pi}{2}$ and $z^{\kappa/2} := (z^{1/2})^\kappa$ for $z \in \mathbb{C}$ and $\kappa \in \mathbb{Z}$, cf. [20]. Thus $\big(\frac{-4}{d}\big)^{1/2} = 1$ or i for $d \equiv 1$ or 3 mod 4. For $f \in S_k(\Gamma_0(4N),\chi_{4N})$, its square is an integral weight cusp form

$$f^2 \in S_{2k}(\Gamma_0(4N),\psi_{4N}) \text{ where } \psi_{4N}(\gamma) = \Big(\frac{-4}{d_\gamma}\Big)^{2k}\chi_{4N}^2(\gamma).$$

The factor $\kappa_{\mathfrak{a}}$ associated with χ at the cusp $\mathfrak{a}$ may be equal to or halved of the corresponding factor for ψ_{4N}, hence Lemma 2.3 remains valid.

2.4. **Fourier coefficients at cusps.** Let $k \in \frac{1}{2}\mathbb{N}$ and let $S_k(\Gamma,\chi)$ be the vector space of holomorphic cusp forms of weight k for Γ with respect to the multiplier system χ. The space $S_k(\Gamma,\chi)$ is a finite-dimensional Hilbert space With respect to the Petersson inner product $\langle f,g\rangle = \int_{\Gamma\backslash\mathbb{H}} y^k f(z)\overline{g(z)}\,\mathrm{d}\mu$. Denote $\|f\|_2^2 := \langle f,f\rangle$.

For any $\gamma \in GL_2^+(\mathbb{R})$, the slash operator on $f \in S_k(\Gamma,\chi)$ is defined as

$$f|_\gamma(z) := j(\gamma,z)^{-k} f(\gamma z).$$

Then for any cusp $\mathfrak{a} \in \mathfrak{A}$, $f\big|_{\sigma_\mathfrak{a}}(z) = e(\kappa_\mathfrak{a} z/\mathrm{w}_\mathfrak{a})\sum_{n\geqslant 0}\widehat{f_\mathfrak{a}}(n)e(nz/\mathrm{w}_\mathfrak{a})$, where $\widehat{f_\mathfrak{a}}(0) = 0$ if $\kappa_\mathfrak{a} = 0$, is of period $\mathrm{w}_\mathfrak{a}$ with $\mathrm{w}_\mathfrak{a} = w_\mathfrak{a}$ or $2w_\mathfrak{a}$ according as $\mathfrak{a}$ is regular or irregular.

Define the regularized Fourier coefficient of $f\big|_{\sigma_\mathfrak{a}}$ by

$$\lambda_\mathfrak{a}(n) := r_k^{-1}\mathrm{w}_\mathfrak{a}^{k/2}\widehat{f_\mathfrak{a}}(n)(n+\kappa_\mathfrak{a})^{(1-k)/2} \tag{2.6}$$

where $r_k := \sqrt{k/\Gamma(k)}(4\pi)^{(k-1)/2}$. By definition, $\lambda_\mathfrak{a}(0) = 0$ if $\kappa_\mathfrak{a} = 0$. Then,

$$f\big|_{\sigma_\mathfrak{a}}(z) = r_k\mathrm{w}_\mathfrak{a}^{-k/2}e(\kappa_\mathfrak{a} z/\mathrm{w}_\mathfrak{a})\sum_{n\geqslant 0}\lambda_\mathfrak{a}(n)(n+\kappa_\mathfrak{a})^{(k-1)/2}e(nz/\mathrm{w}_\mathfrak{a}). \tag{2.7}$$

Proposition 2.4. *Let $f \in S_k(\Gamma,\chi)$ have $\|f\|_2 = 1$, and let $c_\Gamma = \min\{c_\tau > 0 : \tau \in \Gamma\}$. For $n \geqslant 0$,*

$$|\lambda_\infty(n)|^2 \ll 1 + \frac{n}{c_\Gamma \mathrm{w}_\infty}.$$

Proof. The method is based on [9, Section 5.1]. Let $Y > 0$ and P_Y be the strip defined as in (2.3). We suppress the subscript ∞ in $\widehat{f_\infty}(n)$, κ_∞ and w_∞. As in [9, p.70], from $f(z) = \sum_{n\geqslant 1}\widehat{f}(n)e((n+\kappa)z/\mathrm{w})$, we infer that for any $n \geqslant 1$,

$$\mathrm{w}|\widehat{f}(n)|^2\int_Y^\infty y^{k-2}e^{-4\pi(n+\kappa)y/\mathrm{w}}\,\mathrm{d}y \leqslant \int_{P_Y} y^k\big|f(z)\big|^2\,\mathrm{d}\mu,$$

where $\mathrm{d}\mu = y^{-2}\,\mathrm{d}x\,\mathrm{d}y$. Take $Y = \mathrm{w}/(4\pi(n+\kappa))$. The integral $\int_Y^\infty$ is

$$\gg \Big(\frac{\mathrm{w}}{4\pi(n+\kappa)}\Big)^{k-1}\int_0^\infty e^{-u}\big(u+1\big)^{k-2}\,\mathrm{d}u \gg \Big(\frac{\mathrm{w}}{4\pi(n+\kappa)}\Big)^{k-1}\frac{\Gamma(k)}{k}.$$

Applying Lemma 2.2 (ii), we obtain

$$\begin{aligned} |\widehat{f}(n)|^2(n+\kappa)^{1-k} &\ll \frac{(4\pi)^{k-1}\mathrm{w}^{-k}}{\Gamma(k)/k}\big(1+O((c_\Gamma Y)^{-1})\big)\int_{\Gamma\backslash\mathbb{H}} y^k\big|f(z)\big|^2\,\mathrm{d}\mu \\ &\ll \frac{(4\pi)^{k-1}\mathrm{w}^{-k}}{\Gamma(k)/k}\big(1+O((c_\Gamma\mathrm{w})^{-1}n)\big)\|f\|_2^2, \end{aligned}$$

as $c_\Gamma\mathrm{w} \geqslant 1$. This gives the upper bound to the regularized coefficient $\lambda_\infty(n)$. □

Corollary 2.5. *Let $\alpha \in SL_2(\mathbb{Z})$ and $f \in S_k(\Gamma,\chi)$ have $\|f\|_2 = 1$. At the cusp $\mathfrak{a} := \alpha\infty$, f has the Fourier expansion as in (2.7). The regularized Fourier coefficient $\lambda_\mathfrak{a}(n)$, $n \geqslant 0$, satisfies*

$$|\lambda_\mathfrak{a}(n)|^2 \ll 1 + \frac{n}{c_{\Gamma'}\mathrm{w}_\mathfrak{a}}$$

where $\Gamma' := \alpha^{-1}\Gamma\alpha$ and $c_{\Gamma'} = \min\{c_\tau > 0 : \tau \in \Gamma'\}$, which is a positive integer.

Proof. For $\Gamma' = \alpha^{-1}\Gamma\alpha$, which is also a subgroup of $SL_2(\mathbb{Z})$, the map $\mathfrak{b} \mapsto \alpha\mathfrak{b}$ gives a bijection between $\mathfrak{A}(\Gamma')$ and $\mathfrak{A}(\Gamma)$, and $\sigma_{\alpha\mathfrak{b}} \in \alpha\sigma_{\mathfrak{b}}SL_2(\mathbb{Z})_\infty$ for the cusp $\alpha\mathfrak{b} \in \mathfrak{A}(\Gamma)$. Thus we have $w_{\mathfrak{b}} = w_{\alpha\mathfrak{b}}$ and $\kappa_{\mathfrak{b}} = \kappa_{\alpha\mathfrak{b}}$ since $\Gamma'_{\mathfrak{b}} = \alpha^{-1}\Gamma_{\alpha\mathfrak{b}}\alpha$.

Let $\chi'(\gamma) = \chi(\alpha\gamma\alpha^{-1})$ for $\gamma \in \Gamma'$. Then, $f\big|_\alpha \in S_k(\Gamma', \chi')$ and $\widehat{f}_{\mathfrak{a}}(n)$ is the Fourier coefficients of $f\big|_\alpha$ at $\mathfrak{b} = \infty$. The corollary follows readily from Proposition 2.4. □

2.5. **Eisenstein series.** The Eisenstein series $E_{\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}(\mathrm{w}_{\mathfrak{a}}^{-1}\Im\sigma_{\mathfrak{a}}^{-1}\gamma z)^s$ at any cusp $\mathfrak{a}$ converges absolutely for $\Re s > 1$, where $\sigma_{\mathfrak{a}}$ is the matrix defined in Section 2.1. By the general theory of Eisenstein series, we have

(1) $E_{\mathfrak{a}}(z,s)$ extends meromorphically in s to the whole complex plane.
(2) $E_{\mathfrak{a}}(z,s)$ has a simple pole at $s = 1$ with the residue

$$\lim_{s\to 1}(s-1)E_{\mathfrak{a}}(z,s) = \mathrm{vol}(\Gamma\backslash\mathbb{H})^{-1}.$$

(3) $E_{\mathfrak{a}}(z,s)$ has no poles in $\Re s \geqslant \frac{1}{2}$ except for a finite number of simple poles in $(\frac{1}{2}, 1]$.
(4) Let $\mathcal{E}(z,s)$ be a column vector consisting of $E_{\mathfrak{a}}(z,s)$ where $\mathfrak{a}$ runs over all inequivalent cusps (in any ordering). Then $\mathcal{E}(z,s) = \Phi(s)\mathcal{E}(z,1-s)$ where the scattering matrix $\Phi(s)$, of order $|\mathfrak{A}|$, satisfies $\Phi(s)\Phi(1-s) = I$ (the identity matrix).

The functional equation in (4) connects $E_{\mathfrak{a}}(z,s)$ to $E_{\mathfrak{b}}(z,1-s)$'s with $\mathfrak{b} \in \mathfrak{A}$, but the complexity of $\Phi(s)$ in the general case prevents us from using it to solve our problem. Fortunately the case of $\Gamma_0(N)$ is much more transparent. When $N = 1$, the Eisenstein series $E(z,s)$ for $SL_2(\mathbb{Z})$ is, for $\Re s > 1$,

$$\begin{aligned} E(z,s) &:= \sum_{\gamma\in SL_2(\mathbb{Z})_\infty\backslash SL_2(\mathbb{Z})} (\Im\gamma z)^s \\ &= \frac{1}{2}\sum_{\substack{(m,n)\in\mathbb{Z}\times\mathbb{Z}\\(m,n)=1}} \Big(\frac{y}{|mz+n|^2}\Big)^s. \end{aligned} \tag{2.8}$$

Lemma 2.6. *Let $N \in \mathbb{N}$ and $\Gamma = \Gamma_0(N)$. The Eisenstein series $E_\infty(z,s)$ associated to the cusp ∞ of Γ decomposes into*

$$E_\infty(z,s) = N^{-s}\frac{\zeta(2s)}{L(2s,\chi_{0,N})}\sum_{q|N}\frac{\mu(q)}{q^s}E\Big(\frac{N}{q}z, s\Big) \tag{2.9}$$

where $\mu(\cdot)$ is the Möbius function, $\zeta(s)$ and $L(s,\chi_{0,N})$ are the Riemann zeta-function and the Dirichlet L-function associated to the principal character $\chi_{0,N}$ mod N, and $E(z,s)$ is defined in (2.8)*.*

Proof. It follows from the feature of Dirichlet convolution and the explicit formula

$$E_\infty(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(\Im\gamma z)^s = \frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\(cN,d)=1}}\Big(\frac{y}{|cNz+d|^2}\Big)^s,$$

likewise (2.8). Multiplying by $2L(2s,\chi_{0,N})$ on both sides of (2.9), the left-hand side gives

$$2L(2s,\chi_{0,N})E_\infty(z,s) = \sum_{(m,N)=1}\frac{1}{m^{2s}}\sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\(cN,d)=1}}\Big(\frac{y}{|cNz+d|^2}\Big)^s$$
$$= \sum_{\substack{(c,d)\neq(0,0)\\(N,d)=1}}\Big(\frac{y}{|cNz+d|^2}\Big)^s,$$

while the right-hand side is

$$N^{-s}\sum_{N=q\ell}\frac{\mu(q)}{q^s}\zeta(2s)E\big(\ell z,s\big) = N^{-s}\sum_{N=q\ell}\frac{\mu(q)}{q^s}\sum_{(c,d)\neq(0,0)}\Big(\frac{\ell y}{|c\ell z+d|^2}\Big)^s,$$
$$= N^{-s}\sum_{N=quv}\frac{\mu(q)}{q^s}\frac{v^s}{u^s}\sum_{\substack{(c,f)\neq(0,0)\\(v,f)=1}}\Big(\frac{y}{|cvz+f|^2}\Big)^s$$

by expressing $\ell = uv$, $d = uf$ where $u = (\ell,d)$ (and so $(v,f) = 1$). A rewrite leads to

$$\sum_{v|N}\Big(\sum_{q|(N/v)}\mu(q)\Big)\Big(\frac{v}{N}\Big)^{2s}\sum_{\substack{(c,f)\neq(0,0)\\(v,f)=1}}\Big(\frac{y}{|cvz+f|^2}\Big)^s = \sum_{\substack{(c,f)\neq(0,0)\\(N,f)=1}}\Big(\frac{y}{|cNz+f|^2}\Big)^s,$$

as $\sum_{d|n}\mu(d) = 1$ or 0 according to $n = 1$ or $n > 1$. The formula (2.9) follows plainly. □

We shall use the following properties of $E(z,s)$.

Lemma 2.7. *In the half-plane $\Re s \geqslant \frac{1}{2}$, $E(z,s)$ is holomorphic except for a simple pole at $s = 1$ with the residue $3/\pi$. Moreover, it satisfies the functional equation*

$$(E^*(z,s) :=)\, \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = \pi^{-(1-s)}\Gamma(1-s)\zeta(2(1-s))E(z,1-s).$$

Moreover, letting $s = \sigma+it$ and $z = x+iy$, for some small constant $c > 0$ and any $\varepsilon > 0$, we have

$$E(z,s) - y^s - \frac{\pi^{-(1-s)}\Gamma(1-s)\zeta(2-2s)}{\pi^{-s}\Gamma(s)\zeta(2s)}y^{1-s} \ll_{\sigma,\varepsilon} y^{-\sigma}(1+|t|)^{1+\varepsilon} \tag{2.10}$$

for $\sigma \geqslant \frac{1}{2} - \frac{c}{\log(2+|t|)}$, $t\in\mathbb{R}$ and $y > 0$.

Proof. These follow from [9, p.237] and [24, Lemma 3.2]. □

Remark 4. The completed Eisenstein series $E^*(z,s)$ is holomorphic except for simple poles at $s = 0$ and 1, because, from $E^*(z,s) = E^*(z,1-s)$ and the holomorphicity of $E(z,s)$ at $s = 1/2$, the simple pole of the factor $\zeta(2s)$ does not yield a pole.

3. Voronoi formula and a mean value estimate

To compute the mean value estimate, we shall apply the Voronoi formula which is deduced from the (Hecke) functional equation of the L-function $L(s,f)$ associated to the cusp form f. Although the Voronoi formula is derived from a complex integral of $L(s,f)$, interestingly it seems not easy to obtain the same quality in the weight aspect of the mean value result calculated by the Voronoi formula if one estimates directly with the complex integral.

Let $f \in S_k(\Gamma, \chi)$ satisfy $\|f\|_2 = 1$, and write $g(z) := f\Big|_W(z)$ with $W = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. By (2.7),

$$\begin{aligned} f(z) &= r_k \mathrm{w}_\infty^{-k/2} \sum_{n\geqslant 0} \lambda_\infty(n)(n+\kappa_\infty)^{(k-1)/2} e((n+\kappa_\infty)z/\mathrm{w}_\infty), \\ g(z) &= r_k \mathrm{w}_0^{-k/2} \sum_{n\geqslant 0} \lambda_0(n)(n+\kappa_0)^{(k-1)/2} e((n+\kappa_0)z/\mathrm{w}_0). \end{aligned}$$

Note that $\lambda_\infty(0) = 0$ (resp. $\lambda_0(0) = 0$) if $\kappa_\infty = 0$ (resp. $\kappa_0 = 0$). Then $g \in S_k(\Gamma^W, \chi^W)$ where $\Gamma^W := W^{-1}\Gamma W$ and $\chi^W(\alpha) := \chi(W\alpha W^{-1})$, and $\|g\|_2 = 1$. Set $c_k := (2\pi)^{k/2}/r_k$. The L-functions

$$\begin{aligned} L(s,f) &:= \sum_{n\geqslant 0} \lambda_\infty(n)(n+\kappa_\infty)^{-s} = c_k \frac{(2\pi/\mathrm{w}_\infty)^{s-1/2}}{\Gamma(s+\frac{k-1}{2})} \int_0^\infty f(iy) y^{s+\frac{k-1}{2}} \frac{\mathrm{d}y}{y}, \\ L(s,g) &:= \sum_{n\geqslant 0} \lambda_0(n)(n+\kappa_0)^{-s} = c_k \frac{(2\pi/\mathrm{w}_0)^{s-1/2}}{\Gamma(s+\frac{k-1}{2})} \int_0^\infty g(iy) y^{s+\frac{k-1}{2}} \frac{\mathrm{d}y}{y} \end{aligned}$$

are absolutely convergent for $\Re s > \frac{3}{2}$ (by Proposition 2.4), and satisfy the functional equation

$$\Big(\frac{\mathrm{w}_\infty}{2\pi}\Big)^s \Gamma(s+\tfrac{k-1}{2}) L(s,f) = \mathrm{i}^k \Big(\frac{\mathrm{w}_\infty}{\mathrm{w}_0}\Big)^{1/2} \Big(\frac{\mathrm{w}_0}{2\pi}\Big)^{1-s} \Gamma(1-s+\tfrac{k-1}{2}) L(1-s,g).$$

3.1. **Voronoi formula.** Let $J_\nu(\cdot)$ denote the J-Bessel function of order $\nu \geqslant -1/2$. We have the following.

Proposition 3.1. *Let $\varphi \in \mathcal{C}_0^\infty(0,\infty)$ and $x \geqslant 1$ be a parameter. Then*

$$\begin{aligned} &\sum_{n\geqslant 0} \lambda_\infty(n) \varphi\Big(\frac{n+\kappa_\infty}{x}\Big) \\ &= 2\pi \mathrm{i}^k \frac{x}{\sqrt{\mathrm{w}_\infty \mathrm{w}_0}} \sum_{n\geqslant 1} \lambda_0(n) \int_0^\infty \varphi(u) J_{k-1}\Big(\frac{4\pi\sqrt{(n+\kappa_0)xu}}{\sqrt{\mathrm{w}_0\mathrm{w}_\infty}}\Big) \mathrm{d}u. \end{aligned} \tag{3.1}$$

Proof. Let $\widehat{\varphi}(s) := \int_0^\infty \varphi(u) u^{s-1} \,\mathrm{d}u$ be the Mellin transform of φ. Then $\widehat{\varphi}(s)$ is entire and satisfies

$$\widehat{\varphi}(s) \ll_{r,\varphi} (1+|s|)^{-r} \quad \text{for all } r \geqslant 0. \tag{3.2}$$

Let $\varepsilon > 0$ be arbitrarily small. A shift of path with (3.2) leads to

$$\sum := \sum_{n\geqslant 0} \lambda_\infty(n) \varphi\Big(\frac{n+\kappa_\infty}{x}\Big) = \frac{1}{2\pi \mathrm{i}} \int_{(\varepsilon)} L(s,f) \widehat{\varphi}(s) x^s \,\mathrm{d}s.$$

Inserting the functional equation, it follows with the standard argument that

$$\begin{aligned} \sum &= \mathrm{i}^k \Big(\frac{\mathrm{w}_\infty}{\mathrm{w}_0}\Big)^{1/2} \frac{2\pi}{\mathrm{w}_\infty} \frac{1}{2\pi\mathrm{i}} \int_{(\varepsilon)} \Big(\frac{\mathrm{w}_0\mathrm{w}_\infty}{4\pi^2}\Big)^{1-s} \frac{\Gamma(1-s+\frac{k-1}{2})}{\Gamma(s+\frac{k-1}{2})} L(1-s,g) \widehat{\varphi}(s) x^s \,\mathrm{d}s \\ &= \mathrm{i}^k \Big(\frac{\mathrm{w}_\infty}{\mathrm{w}_0}\Big)^{1/2} \frac{\mathrm{w}_0}{2\pi} \frac{1}{2\pi\mathrm{i}} \int_{(\frac{3}{2}+\varepsilon)} \Big(\frac{4\pi^2}{\mathrm{w}_0\mathrm{w}_\infty}\Big)^{1-s} \frac{\Gamma(s+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})} L(s,g) \widehat{\varphi}(1-s) x^{1-s} \,\mathrm{d}s \\ &= \frac{\mathrm{i}^k}{2\pi} \sqrt{\mathrm{w}_\infty \mathrm{w}_0} \sum_{n\geqslant 0} \frac{\lambda_0(n)}{n+\kappa_0} I\Big(\frac{4\pi^2(n+\kappa_0)x}{\mathrm{w}_0\mathrm{w}_\infty}\Big), \end{aligned}$$

where with (3.2)

$$I(y)=\frac{1}{2\pi \mathrm{i}}\int_{(\frac13)}\frac{\Gamma(s+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})}\widehat{\varphi}(1-s)y^{1-s}\,\mathrm{d}s.$$

Replacing $\widehat{\varphi}$ with its Mellin transformation formula, and applying the formula

$$J_\nu(x)=\frac{1}{2\pi \mathrm{i}}\int_{(c)}\frac{\Gamma(s+\frac{\nu}{2})}{\Gamma(1-s+\frac{\nu}{2})}\Big(\frac{x}{2}\Big)^{-2s}\mathrm{d}s \quad \text{where } \nu\geqslant -\tfrac12 \text{ and } c\in(\tfrac14,\tfrac12),$$

one sees immediately that for $k\geqslant 1/2$,

$$\begin{aligned} I(y) &= \int_0^\infty \varphi(u)\frac{1}{2\pi \mathrm{i}}\int_{(\frac13)}\frac{\Gamma(s+\frac{k-1}{2})}{\Gamma(1-s+\frac{k-1}{2})}(uy)^{1-s}\,\mathrm{d}s\frac{\mathrm{d}u}{u}\\ &= y\int_0^\infty \varphi(u)J_{k-1}(2\sqrt{yu})\,\mathrm{d}u. \end{aligned}$$

Our assertion follows after a little calculation. □

3.2. **Mean value estimate.** Next we show the rapid decay of the smoothed sum of length longer than the "analytic conductor" for $\mathrm{w}_\infty^{(k-1)/2}L(s,f)$, following the calculation in [8, Lemma 6].

Proposition 3.2. *Let $\varepsilon>0$ be arbitrarily small and $\varphi\in\mathcal{C}_0^\infty(0,\infty)$. Suppose $f\in S_k(\Gamma,\chi)$ satisfies $\|f\|_2=1$. Let $\tilde\kappa_0=1$ or κ_0 according as $\kappa_0=0$ or not (corresponding to that 0 is singular cusp or not). For any $x\geqslant(k^2\mathrm{w}_\infty\mathrm{w}_0/\tilde\kappa_0)^{1+\varepsilon}$ and any constant $A\geqslant 1$,*

$$\sum_{n\geqslant 0}\lambda_\infty(n)\varphi\Big(\frac{n+\kappa_\infty}{x}\Big)\ll_{A,\varepsilon}x^{-A}.$$

Proof. By Corollary 2.5 with $\alpha=W$, we see that

$$|\lambda_0(n)|^2\ll(n+\kappa_0).$$

Applying it to the right side of (3.1), the left-hand side of (3.1) is bounded by

$$\ll x\cdot\sum_{n\geqslant0}\sqrt{n+\kappa_0}\bigg|\int_0^\infty\varphi(u)J_{k-1}\Big(\frac{4\pi\sqrt{(n+\kappa_0)xu}}{\sqrt{\mathrm{w}_0\mathrm{w}_\infty}}\Big)\,\mathrm{d}u\bigg|. \tag{3.3}$$

It remains to handle the integral. We invoke the following facts for the Bessel function $J_\nu(x)$ with $x>0$: (1) $(x^\nu J_\nu(x))'=x^\nu J_{\nu-1}(x)$ for all real ν, (2) $J_\nu(x)\ll1$ for $\nu\geqslant0$ or $J_\nu(x)\ll x^\nu+1$ for $\nu\in[-\frac12,0]$, where the implied constant is absolute, i.e. independent of ν. These facts can be justified by the power series of $J_\nu(x)$, together with

$$J_\nu(x)=\frac1\pi\int_0^\pi\cos(x\sin\theta-\nu\theta)\,\mathrm{d}\theta-\frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-x\sinh\alpha-\nu\alpha}\,\mathrm{d}\alpha.$$

For any given $\varphi_0\in\mathcal{C}_0^\infty(0,\infty)$, if we define $\varphi_r(x):=x^{-1}\varphi_{r-1}'(x)$ for $r=1,2,\cdots$, then we get from (1) and (2) that

$$\int\varphi_0(x)\cdot x^kJ_{k-1}(x)\,\mathrm{d}x=(-1)^r\int\varphi_r(x)\cdot x^{k+r}J_{k+r-1}(x)\,\mathrm{d}x.$$

Hence, with $B:=4\pi\sqrt{(n+\kappa_0)x/(\mathrm{w}_0\mathrm{w}_\infty)}$, we express the integral as

$$\frac{2}{B^2}\int v^{1-k}\varphi\Big(\frac{v^2}{B^2}\Big)\cdot v^kJ_{k-1}(v)\,\mathrm{d}v=\frac{2}{B^2}\int\Big(\frac1v\frac{d}{dv}\Big)^r\Big(v^{1-k}\varphi\Big(\frac{v^2}{B^2}\Big)\Big)v^{k+r}J_{k+r-1}(v)\,\mathrm{d}v.$$

Due to the support of φ, the integration is over an interval of the form $[a,b]$, where $a,b \asymp B$. The operator $v^{-1}\frac{d}{dv}$ acts on v^K as a multiplication by $Kv^{-2} \asymp KB^{-2}$ and acts on $\varphi(v^2/B^2)$ as a multiplication by B^{-2}. Thus, for $v \in [a,b]$,

$$\Big(\frac{1}{v}\frac{d}{dv}\Big)^r \Big(v^{1-k}\varphi\Big(\frac{v^2}{B^2}\Big)\Big) \ll_r k^r B^{-2r} v^{1-k}.$$

With (2), the integral is $\ll_r B^{-2}\int_a^b k^r B^{-2r} v^{1+r}\,\mathrm{d}v \ll (k\sqrt{\mathrm{w}_0\mathrm{w}_\infty/((n+\kappa_0)x)})^r$. When $x \geqslant (k^2\mathrm{w}_0\mathrm{w}_\infty/\tilde{\kappa}_0)^{1+\varepsilon}$, we can bound (3.3), in both cases $\kappa_0 = 0$ or not, by

$$\ll_r x \cdot x^{-\frac{r\varepsilon}{2+2\varepsilon}} \sum_{n\geqslant 1} n^{(1-r)/2} \ll x^{-A},$$

implying our result by taking sufficiently large $r \geqslant (2+2\varepsilon)(A+1)/\varepsilon$. □

4. Mean square estimate

In this section we derive the following result. A noteworthy point is the different treatment here from the typical approach with the functional equation of the column $(E_{\mathfrak{a}}(z,s))^t$ of Eisenstein series. A typical approach hurdle is the calculation of the scattering matrix. Instead, we exploit the Eisenstein series of the full modular group, which is much simpler, and correspondingly, we form a L-function $\mathcal{L}_\ell(s)$, see (4.2), by combining the L-function $L(s, f_{\mathfrak{a}} \otimes f_{\mathfrak{a}})$. The function $\mathcal{L}_\ell(s)$ satisfies a simple functional equation, see Lemma 4.2 (b), and the L-function $L(s, f\otimes f)$ applied in the proof of theorem below is composed of $\mathcal{L}_\ell(s)$'s.

Theorem 2. *Let $k \in \frac{1}{2}\mathbb{N}$, and $N \in \mathbb{N}$ or $4\mathbb{N}$ according as $k \in \mathbb{N}$ or not. Suppose $f \in S_k(\Gamma_0(N),\chi)$ has $\|f\|_2 = 1$, where χ is a Dirichlet character mod N. Given any $\varepsilon > 0$ and any weight function $\varphi \in \mathcal{C}_0^\infty(0,\infty)$, we have, for all $x \geqslant (kN)^{2+\varepsilon}$,*

$$\sum_{\substack{l,m\geqslant 1\\(l,N)=1}} |\lambda(m)|^2 \varphi\Big(\frac{l^2 m}{x}\Big) = x\widehat{\varphi}(1)\frac{2\pi^2}{kN}\prod_{p|N}(1-p^{-1}) + O_{\varepsilon,\varphi}\big((kNx)^\varepsilon kN\big)$$

where $f(z) = \sqrt{\frac{(4\pi)^{k-1}k}{\Gamma(k)}} \cdot \sum_{n\geqslant 1}\lambda(n)n^{(k-1)/2}e(nz)$ at the cusp ∞.

4.1. **Some Preparation.** We set up some notation and prove two lemmas.

Let $\mathfrak{a} \in \mathfrak{A}$ (see Section 2.1.1). Denote $f_{\mathfrak{a}} := f\big|_{\sigma_{\mathfrak{a}}}$ and define, for $\Re s > 2$,

$$L(s, f_{\mathfrak{a}} \otimes f_{\mathfrak{a}}) := \sum_{n\geqslant 0} |\lambda_{\mathfrak{a}}(n)|^2 (n+\kappa_{\mathfrak{a}})^{-s} \tag{4.1}$$

where $\lambda_{\mathfrak{a}}(n)$ is the regularized Fourier coefficient of $f_{\mathfrak{a}}$ defined in (2.6). When $\mathfrak{a} = \infty$, we write $\lambda(n) = \lambda_\infty(n)$ and $L(s, f\otimes f) = L(s, f_\infty \otimes f_\infty)$. We also define, for $\Re s > 2$,

$$\mathcal{L}_\ell(s) := \sum_{c|N}\Big(\frac{(c,\ell)^2}{(N,c^2)}\Big)^s \sum_{\mathfrak{a}\in\mathfrak{A}_c} \zeta(2s) L(s, f_{\mathfrak{a}}\otimes f_{\mathfrak{a}}). \tag{4.2}$$

Lemma 4.1. *Let $N \in \mathbb{N}$, $\ell | N$ and $E(z,s)$ be the Eisenstein series for $SL_2(\mathbb{Z})$ in (2.8). Define*

$$I_\ell(s) := \int_{\Gamma_0(N)\backslash\mathbb{H}} y^k |f(z)|^2 E(\ell z, s)\,\mathrm{d}\mu. \tag{4.3}$$

(i) For $\Re s > 2$,

$$\zeta(2s)I_\ell(s) = \frac{k}{\Gamma(k)}(4\pi)^{-s}\Gamma(s+k-1)\Big(\frac{N}{\ell}\Big)^s\mathcal{L}_\ell(s).$$

(ii) The function

$$\Lambda_\ell(s) := \pi^{-s}\Gamma(s)\zeta(2s)I_\ell(s)$$

is meromorphic on $\mathbb{C}$, *having simple poles at* $s = 0, 1$, *and being holomorphic elsewhere. It satisfies the functional equation*

$$\Lambda_\ell(s) = \Lambda_\ell(1-s). \tag{4.4}$$

Moreover, for $|t| \geqslant 1$,

$$\Lambda_\ell(\sigma+it) \ll_{f,\sigma} |t|^{\max(\sigma,1-\sigma)+1/2+\varepsilon}e^{-\pi|t|/2}. \tag{4.5}$$

Proof. (i) Let $\Gamma := \Gamma_0(N/\ell, \ell) = a(\sqrt{\ell})\Gamma_0(N)a(1/\sqrt{\ell})$. With the notation in Section 2, for $f \in S_k(\Gamma_0(N), \chi)$, we set

$$f_\ell(z) := f\big|_{a(1/\sqrt{\ell})}(z) = \ell^{-k/2}f(z/\ell) \in S_k(\Gamma, \chi).$$

Note that $a(\sqrt{\ell})(\Gamma_0(N)\backslash\mathbb{H}) = (a(\sqrt{\ell})\Gamma_0(N)a(\sqrt{\ell})^{-1})\backslash\mathbb{H}$ and $\tau^{-1}(\Gamma_\mathfrak{b}\backslash\mathbb{H}) = \tau^{-1}\Gamma_\mathfrak{b}\tau\backslash\mathbb{H}$. By changing variables, unfolding and using Lemma 2.1 (iv), we have

$$\begin{aligned}
I_\ell(s) &= \int_{a(\sqrt{\ell})(\Gamma_0(N)\backslash\mathbb{H})} y^k|f_\ell(z)|^2E(z,s)\,\mathrm{d}\mu \\
&= \sum_{\sigma\in SL_2(\mathbb{Z})_\infty\backslash SL_2(\mathbb{Z})}\int_{\Gamma\backslash\mathbb{H}} y^k|f_\ell(z)|^2(\Im\sigma z)^s\,\mathrm{d}\mu \\
&= \sum_{\tau\in\mathfrak{D}}\sum_{\gamma\in\Gamma_\mathfrak{b}\backslash\Gamma}\int_{\Gamma\backslash\mathbb{H}} y^k|f_\ell(z)|^2(\Im\tau^{-1}\gamma z)^s\,\mathrm{d}\mu \\
&= \sum_{\tau\in\mathfrak{D}}\int_{\Gamma_\mathfrak{b}\backslash\mathbb{H}} y^k|f_\ell(z)|^2(\Im\tau^{-1}z)^s\,\mathrm{d}\mu \\
&= \sum_{\tau\in\mathfrak{D}}\int_{\tau^{-1}\Gamma_\mathfrak{b}\tau\backslash\mathbb{H}} y^{k+s}|f_\ell|_\tau(z)|^2\,\mathrm{d}\mu. \qquad (4.6)
\end{aligned}$$

By Lemma 2.1 (iv), for $\mathfrak{b} = \ell\mathfrak{a}$ with $\mathfrak{a} = a/c$, we have $a_\tau = \ell a/(c,\ell)$ and $c_\tau = c/(c,\ell)$ (see Section 2 for the notation a_τ, c_τ). Observing that the matrix $u := \sigma_\mathfrak{a}^{-1}a(1/\sqrt{\ell})\tau$ is

$$u = \begin{pmatrix} d' & -b' \\ -c & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{\ell}} & \\ & \sqrt{\ell}\end{pmatrix}\begin{pmatrix}\ell a/(\ell,c) & b \\ c/(\ell,c) & d\end{pmatrix} = \begin{pmatrix}\sqrt{\ell}/(\ell,c) & D/\sqrt{\ell} \\ & (\ell,c)/\sqrt{\ell}\end{pmatrix}$$

for some $b, b', d, d' \in \mathbb{Z}$ and $D = bd' - \ell b'd$, we get $a(1/\sqrt{\ell})\tau = \sigma_\mathfrak{a}u$ and hence, with (2.7),

$$\begin{aligned}
f_\ell|_\tau(z) &= f|_{a(1/\sqrt{\ell})}|_\tau(z) \\
&= \Big(\tfrac{\sqrt{\ell}}{(\ell,c)}\Big)^k f|_{\sigma_\mathfrak{a}}\big(\tfrac{\ell}{(\ell,c)^2}z + \tfrac{D}{(\ell,c)}\big) \\
&= \varpi\Big(\tfrac{\sqrt{\ell}}{(\ell,c)}\Big)^k r_k w_\mathfrak{a}^{-k/2}e\Big(\frac{\kappa_\mathfrak{a}\ell z}{w_\mathfrak{a}(\ell,c)^2}\Big)\sum_{n\geqslant 0}\vartheta^n\lambda_\mathfrak{a}(n)(n+\kappa_\mathfrak{a})^{(k-1)/2}e\Big(\frac{n\ell z}{w_\mathfrak{a}(\ell,c)^2}\Big)
\end{aligned}$$

where $|\varpi| = 1$, but ϖ may not be 1 when k is half-integral, r_k is defined as in the line before (2.7) and $\vartheta := e\big(\frac{D/(\ell,c)}{w_{\mathfrak{a}}}\big)$. Consequently, as $w_{\mathfrak{b}} = w_{\mathfrak{a}}(\ell,c)^2/\ell$ and $w_{\mathfrak{a}} = N/(N,c^2)$ by Lemma 2.1 (v),

$$\begin{aligned}
&\int_{\tau^{-1}\Gamma_{\mathfrak{b}}\tau\backslash\mathbb{H}} y^{k+s}|f_\ell|_\tau(z)|^2\,\mathrm{d}\mu \\
={}& \Big(\frac{\ell}{(\ell,c)^2}\Big)^k \int_{\langle n(w_{\mathfrak{b}})\rangle\backslash\mathbb{H}} y^{k+s}\big|f|_{\sigma_{\mathfrak{a}}}\big(\tfrac{\ell}{(\ell,c)^2}z+\tfrac{D}{(\ell,c)}\big)\big|^2\,\mathrm{d}\mu \\
={}& r_k^2\Big(\frac{\ell}{w_{\mathfrak{a}}(\ell,c)^2}\Big)^k \sum_{n\geqslant 0}\frac{|\lambda_{\mathfrak{a}}(n)|^2}{(n+\kappa_{\mathfrak{a}})^{1-k}}\cdot w_{\mathfrak{b}}\int_0^\infty y^{k+s-1}\exp\Big(-\frac{4\pi\ell(n+\kappa_{\mathfrak{a}})}{w_{\mathfrak{a}}(\ell,c)^2}y\Big)\frac{\mathrm{d}y}{y} \\
={}& \frac{k}{\Gamma(k)}(4\pi)^{-s}\Gamma(s+k-1)\Big(\frac{(c,\ell)^2}{\ell}\cdot\frac{N}{(N,c^2)}\Big)^s L(s,f_{\mathfrak{a}}\otimes f_{\mathfrak{a}}).
\end{aligned}$$

Combining with (4.6) and the formation of $\mathcal{D}$ in Lemma 2.1 (iii) and (v), we obtain that

$$I_\ell(s) = \frac{k}{\Gamma(k)}(4\pi)^{-s}\Gamma(s+k-1)\Big(\frac{N}{\ell}\Big)^s\sum_{c|N}\Big(\frac{(c,\ell)^2}{(N,c^2)}\Big)^s\sum_{\mathfrak{a}\in\mathfrak{A}_c}L(s,f_{\mathfrak{a}}\otimes f_{\mathfrak{a}}).$$

This completes the proof of Part (i).

(ii) We apply the properties of $E(z,s)$ in Lemma 2.7. By Remark 4, in the half plane $\Re s \geqslant \frac12$, $\Lambda_\ell(s) = \pi^{-s}\Gamma(s)\zeta(2s)I_\ell(s)$ has a simple pole at $s=1$, coming from $E^*(z,s)$. Clearly the functional equation for $E(z,s)$ leads to $\Lambda_\ell(s) = \Lambda_\ell(1-s)$. Therefore, in the plane $\Re s < \frac12$, $\Lambda_\ell(s)$ has a simple pole at $s=0$.

For $|t|\geqslant 1$, we know that

$$\Gamma(\sigma+it) \asymp_\sigma |t|^{\sigma-1/2}e^{-\pi|t|/2},\quad \zeta(\sigma+it) \ll_\sigma \begin{cases} |t|^{\frac12(1-\sigma)+\varepsilon} & \text{if } 0\leqslant\sigma\leqslant 1,\\ |t|^{\frac12-\sigma+\varepsilon} & \text{if } \sigma\leqslant 0,\end{cases}$$

and for all $t\in\mathbb{R}$,

$$\zeta(\sigma+it)^{-1} \ll \log(|t|+3) \qquad \text{for } \sigma \geqslant 1-\tfrac{C}{\log(|t|+3)}.$$

It follows from (2.10) that $I_\ell(s) \ll_{f,\delta} (1+|t|)^{1+\varepsilon}$ in (an open region containing) $\Re s\geqslant\frac12$ and $|s-1|>\delta$. Thus, for $\sigma\geqslant\frac12$ and $|t|\geqslant 1$,

$$\Lambda_\ell(\sigma+it) \ll_f |\Gamma(\sigma+it)||t|^{1+\varepsilon} \ll_{f,\sigma} |t|^{\sigma+1/2+\varepsilon}e^{-\pi|t|/2}.$$

The upper bound in (4.5) follows with the functional equation $\Lambda_\ell(s) = \Lambda_\ell(1-s)$. □

Lemma 4.2. *The Dirichlet series $\mathcal{L}_\ell(s)$ defined in* (4.2) *extends to a meromorphic function on $\mathbb{C}$, having only one pole, which is simple, at $s=1$. Moreover,*

(a) $$L(2s,\chi_{0,N})L(s,f\otimes f) = N^{-s}\sum_{q\ell=N}\mu(q)\mathcal{L}_\ell(s),\quad \forall\ s\in\mathbb{C}$$

(b) $$\mathcal{L}_\ell(s) = \frac{\Gamma(1-s)\Gamma(k-s)}{\Gamma(s)\Gamma(s+k-1)}\Big(\frac{N}{4\pi^2\ell}\Big)^{1-2s}\mathcal{L}_\ell(1-s),\quad \forall\ s\in\mathbb{C}$$

(c) $$\mathcal{L}_\ell(\sigma+it) \ll_\varepsilon \big(|t|(|t|+k)N\big)^{2-\sigma}\ell^{\sigma+1}(kN)^\varepsilon,\quad \forall\ s\in S_{\varepsilon,1}$$

where $L(s,\chi_{0,N})$ is the Dirichlet L-function associated to the principal character $\chi_{0,N}$ mod N and $S_{\varepsilon,1} := \{\sigma+it:\ -1+\varepsilon\leqslant\sigma\leqslant 2+\varepsilon,\ |t|\geqslant 1\}$.

Remark 5. (i) By Lemma 4.2 (a) and (4.2), we have the formula: for $\Re s > 2$,

$$L(2s,\chi_{0,N})L(s,f\otimes f) = N^{-s}\sum_{q\ell=N}\mu(q)\sum_{c\mid N}\Big(\frac{(c,\ell)^2}{(N,c^2)}\Big)^s\sum_{\mathfrak{a}\in\mathfrak{A}_c}\zeta(2s)L(s,f_{\mathfrak{a}}\otimes f_{\mathfrak{a}}).$$

(ii) By (a), the L-function $L(2s,\chi_{0,N})L(s,f\otimes f)$ has exactly one simple pole at $s=1$, and by (4.8) below, the residue is

$$\operatorname*{res}_{s=1} L(2s,\chi_{0,N})L(s,f\otimes f) = \frac{4\pi}{k}\frac{L(2,\chi_{0,N})}{\mathrm{vol}(\Gamma_0(N)\backslash\mathbb{H})}\|f\|^2 = \frac{2\pi^2}{kN}\prod_{p\mid N}(1-p^{-1})\|f\|^2. \tag{4.7}$$

Proof. The analytic properties of $\mathcal{L}_\ell(s)$ follows from Lemma 4.1, as $(\Gamma(k)/k)\Lambda_\ell(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s+k-1)(N/\ell)^s\mathcal{L}_\ell(s)$. The simple pole of $\Lambda_\ell(s)$ at $s=0$ is accounted for by $\Gamma(s)$, hence $\mathcal{L}_\ell(s)$ remains holomorphic at $s=0$. Since $\Lambda_\ell(s)=\Lambda_\ell(1-s)$ by (4.4), we get the functional equation in (b).

Unfolding with the Eisenstein series for $\Gamma_0(N)$ at ∞, we get, for $\Re s > 1$,

$$\begin{aligned}\int_{\Gamma_0(N)\backslash\mathbb{H}} y^k|f(z)|^2E_\infty(z,s)\,\mathrm{d}\mu &= \int_{\Gamma_0(N)_\infty\backslash\mathbb{H}} y^{k+s}|f(z)|^2\,\mathrm{d}\mu\\ &= r_k^2\sum_{n\geqslant 1}|\lambda(n)|^2n^{k-1}\int_0^\infty y^{k+s-1}e^{-4\pi ny}\frac{\mathrm{d}y}{y}\\ &= \frac{k}{\Gamma(k)}(4\pi)^{-s}\Gamma(s+k-1)L(s,f\otimes f).\end{aligned}\tag{4.8}$$

Applying Lemma 2.6 to (4.8), we deduce with (4.3) that

$$\frac{k}{\Gamma(k)}(4\pi)^{-s}\Gamma(s+k-1)L(s,f\otimes f) = N^{-s}\frac{\zeta(2s)}{L(2s,\chi_{0,N})}\sum_{q\mid N}\frac{\mu(q)}{q^s}I_{N/q}(s).$$

Invoking (i) in Lemma 4.1, it leads to

$$L(2s,\chi_{0,N})L(s,f\otimes f) = N^{-s}\sum_{q\ell=N}\frac{\mu(q)}{q^s}\Big(\frac{N}{\ell}\Big)^s\mathcal{L}_\ell(s),$$

which implies (a) for $\Re s > 2$, and for all $s\in\mathbb{C}$ by analytic continuation.

By (4.1) and Corollary 2.5, we see that for any $\sigma = \Re s > 2$,

$$\zeta(2s)L(s,f_{\mathfrak{a}}\otimes f_{\mathfrak{a}}) \ll \kappa_{\mathfrak{a}}^{-\sigma}\delta_{\kappa_{\mathfrak{a}}\neq 0} + \sum_{n\geqslant 1}(n+\kappa_{\mathfrak{a}})^{1-\sigma} \ll (c,N/c)^\sigma,$$

by (2.4), where $\delta_* = 1$ if $*$ is true. Thus, for $\Re s = 2+\varepsilon$, we obtain

$$\begin{aligned}\mathcal{L}_\ell(s) &\ll_\varepsilon (kN)^\varepsilon\sum_{c\mid N}\Big(\frac{(c,\ell)^2}{(N,c^2)}\Big)^2\sum_{\mathfrak{a}\in\mathfrak{A}_c}(c,N/c)^2\\ &\ll (kN)^\varepsilon\sum_{c\mid N}\frac{(c,\ell)^4}{(N,c^2)^2}(c,N/c)^3\\ &\ll (kN)^\varepsilon\ell^3.\end{aligned}\tag{4.9}$$

Stirling's formula gives the following estimates: for $\sigma\ll 1$ and any $|t|\geqslant 1$,

$$\frac{\Gamma(1-s)}{\Gamma(s)}\ll |t|^{1-2\sigma}\quad\text{and}\quad\frac{\Gamma(k-s)}{\Gamma(s+k-1)}\ll(|t|+k)^{1-2\sigma}. \tag{4.10}$$

With (4.9), we infer by the functional equation in (b) that on $\Re s = -1-\varepsilon$,

$$\mathcal{L}_\ell(s) \ll_\varepsilon \Big(\frac{|t|(|t|+k)N}{\ell}\Big)^3 (kN)^\varepsilon \ell^3.$$

Note that by Lemma 4.1, $\mathcal{L}_\ell(s)$ is a meromorphic function of finite order with two simple poles. Using the convexity principle, we obtain for $-1+\varepsilon \leqslant \sigma \leqslant 2+\varepsilon$ and $|t| \geqslant 1$,

$$\mathcal{L}_\ell(\sigma+it) \ll_\varepsilon \big(|t|(|t|+k)N\big)^{2-\sigma} \ell^{\sigma+1} (kN)^\varepsilon.$$

□

4.2. **Proof of Theorem 2.** Let $\varepsilon > 0$ be arbitrarily small. Consider the contour integral along the vertical line $\Re s = 2+\varepsilon$,

$$J \ := \ \frac{1}{2\pi i}\int_{(2+\varepsilon)} L(2s, \chi_{0,N}) L(s, f\otimes f)\widehat{\varphi}(s) x^s \, \mathrm{d}s.$$

We shift the line of integration to $\Re s = -1-\varepsilon$ and hence get, with Lemma 4.2 (a),

$$\begin{aligned} J &= \mathrm{res}_{s=1} + \sum_{q\ell=N} \mu(q) \frac{1}{2\pi i}\int_{(-1-\varepsilon)} N^{-s}\mathcal{L}_\ell(s)\widehat{\varphi}(s) x^s \, \mathrm{d}s \\ &= M + \sum_{q\ell=N} \mu(q) J_\ell, \quad \text{say.} \end{aligned} \tag{4.11}$$

In view of (4.8), M is contributed from the pole of $E_\infty(z,s)$, which is an Eisenstein series for $\Gamma_0(N)$, at $s=1$, thus with (4.7) and $\|f\| = 1$, we have

$$M = x\widehat{\varphi}(1)\frac{2\pi^2}{kN}\prod_{p|N}(1-p^{-1}). \tag{4.12}$$

The integral J_ℓ needs to be treated carefully with the functional equation and the estimate for $\lambda_{\mathfrak{a}}(n)$. We apply Lemma 4.2 (b), a change of the variable of s into $1-s$ and then (4.2) to deduce that

$$\begin{aligned} J_\ell &= \frac{1}{2\pi i}\int_{(-1-\varepsilon)} N^{-s}\frac{\Gamma(1-s)\Gamma(k-s)}{\Gamma(s)\Gamma(s+k-1)}\Big(\frac{N}{4\pi^2\ell}\Big)^{1-2s}\mathcal{L}_\ell(1-s)\widehat{\varphi}(s)x^s \, \mathrm{d}s \\ &= \frac{4\pi^2\ell}{N^2}\frac{1}{2\pi i}\int_{(2+\varepsilon)} \frac{\Gamma(s)\Gamma(s+k-1)}{\Gamma(1-s)\Gamma(k-s)}\Big(\frac{N^3}{16\pi^4\ell^2}\Big)^s \mathcal{L}_\ell(s)\widehat{\varphi}(1-s)x^{1-s}\, \mathrm{d}s \\ &= \frac{4\pi^2\ell}{N^2}\sum_{c|N}\sum_{\mathfrak{a}\in\mathfrak{A}_c} J_{\ell,c,\mathfrak{a}} \end{aligned} \tag{4.13}$$

where

$$J_{\ell,c,\mathfrak{a}} = \frac{1}{2\pi i}\int_{(2+\varepsilon)} G_k(s)\Big(\frac{N^3}{16\pi^4\ell^2}\cdot\frac{(c,\ell)^2}{(N,c^2)}\Big)^s \zeta(2s)L(s, f_{\mathfrak{a}}\otimes f_{\mathfrak{a}})\widehat{\varphi}(1-s)x^{1-s}\, \mathrm{d}s$$

with

$$G_k(s) := \frac{\Gamma(s)\Gamma(s+k-1)}{\Gamma(1-s)\Gamma(k-s)}.$$

By the definition of $L(s, f_{\mathfrak{a}} \otimes f_{\mathfrak{a}})$ in (4.1), we expand $J_{\ell,c,\mathfrak{a}}$ into

$$\begin{aligned} J_{\ell,c,\mathfrak{a}} &= \sum_{n\geqslant 0} \frac{1}{2\pi i} \int_{(2+\varepsilon)} \frac{|\lambda_{\mathfrak{a}}(n)|^2}{(n+\kappa_{\mathfrak{a}})^s} \Big(\frac{N^3(c,\ell)^2}{16\pi^4\ell^2(N,c^2)}\Big)^s G_k(s)\zeta(2s)\widehat{\varphi}(1-s)x^{1-s}\,\mathrm{d}s \\ &= \sum_{0\leqslant n\leqslant w_{\mathfrak{a}}} + \sum_{n>w_{\mathfrak{a}}} = J_{\ell,c,\mathfrak{a}}^{\leqslant w_{\mathfrak{a}}} + J_{\ell,c,\mathfrak{a}}^{>w_{\mathfrak{a}}}, \quad \text{say.} \end{aligned} \tag{4.14}$$

From Corollary 2.5 and the fact that $c_\Gamma \geqslant 1$, we have

$$|\lambda_{\mathfrak{a}}(n)|^2 \ll 1 + \frac{n}{w_{\mathfrak{a}}} \ll 1 \;\text{ or }\; \frac{(N,c^2)}{N}n,$$

according as $n \leqslant w_{\mathfrak{a}}$ or $n > w_{\mathfrak{a}}$. For $J_{\ell,c,\mathfrak{a}}^{\leqslant w_{\mathfrak{a}}}$, we shift the line of integration from $2+\varepsilon$ to 1 and bound each summand trivially. Thus, bounding $G_k(s)$ with (4.10), we obtain

$$J_{\ell,c,\mathfrak{a}}^{\leqslant w_{\mathfrak{a}}} \ll \frac{N^3(c,\ell)^2}{\ell^2(N,c^2)} \sum_{0\leqslant n\leqslant w_{\mathfrak{a}}} (n+\kappa_{\mathfrak{a}})^{-1} \int_{-\infty}^{\infty} (|t|+1)(|t|+k)\big|\widehat{\varphi}(-it)\big|\,\mathrm{d}t.$$

The integral $\int_{-\infty}^{\infty}$ is clearly $\ll k$. Therefore, we obtain

$$J_{\ell,c,\mathfrak{a}}^{\leqslant w_{\mathfrak{a}}} \ll k\frac{N^3(c,\ell)^2}{\ell^2(N,c^2)}(\tilde{\kappa}_{\mathfrak{a}}^{-1} + N^{\varepsilon}) \tag{4.15}$$

where $\tilde{\kappa}_{\mathfrak{a}} = \kappa_{\mathfrak{a}}$ if $\kappa_{\mathfrak{a}} \in (0,1)$ and $\tilde{\kappa}_{\mathfrak{a}} = 1$ if $\kappa_{\mathfrak{a}} = 0$.

For $J_{\ell,c,\mathfrak{a}}^{>w_{\mathfrak{a}}}$ in (4.11), without shifting the line of integration, we bound each summand trivially and use the fast decay $\widehat{\varphi}(-1-\varepsilon-it) \ll (1+|t|)^{-8}$ to obtain

$$J_{\ell,c,\mathfrak{a}}^{>w_{\mathfrak{a}}} \ll N^{\varepsilon}\Big(\frac{N^3(c,\ell)^2}{\ell^2(N,c^2)}\Big)^2 \frac{(N,c^2)}{N} \sum_{n>w_{\mathfrak{a}}} \frac{1}{n^{1+\varepsilon}} \int_{-\infty}^{\infty} \big((1+|t|)(|t|+k)\big)^{3+\varepsilon} \frac{x^{-1-\varepsilon}}{(1+|t|)^8}\,\mathrm{d}t.$$

Plainly the integral $\int_{-\infty}^{\infty}$ is $\ll k^{3+\varepsilon}$. Consequently,

$$J_{\ell,c,\mathfrak{a}}^{>w_{\mathfrak{a}}} \ll (kNx)^{\varepsilon}\frac{N^5(c,\ell)^4}{\ell^4(N,c^2)}k^3x^{-1}. \tag{4.16}$$

Inserting (4.15) and (4.16) into (4.14), we deduce with (4.13) that

$$\begin{aligned} \sum_{q\ell=N} \mu(q)J_\ell &\ll k\sum_{\ell\mid N}\frac{\ell}{N^2}\sum_{c\mid N}\frac{N^3(c,\ell)^2}{\ell^2(N,c^2)}\sum_{\mathfrak{a}\in\mathfrak{A}_c}\tilde{\kappa}_{\mathfrak{a}}^{-1} \\ &\quad + N^{\varepsilon}k\sum_{\ell\mid N}\frac{\ell}{N^2}\sum_{c\mid N}\frac{N^3(c,\ell)^2}{\ell^2(N,c^2)}|\mathfrak{A}_c| \\ &\quad + (kNx)^{\varepsilon}k^3x^{-1}\sum_{\ell\mid N}\frac{\ell}{N^2}\sum_{c\mid N}\frac{N^5(c,\ell)^4}{\ell^4(N,c^2)}|\mathfrak{A}_c|. \end{aligned}$$

Using $|\mathfrak{A}_c| = \phi(c, N/c) \leqslant (c, N/c)$, the three sums are respectively evaluated as:

$$\begin{aligned}
\sum_{\ell|N} \frac{\ell}{N^2} \sum_{c|N} \frac{N^5(c,\ell)^4}{\ell^4(N,c^2)} |\mathfrak{A}_c| &\leqslant N^3 \sum_{\ell|N}\sum_{c|N} \frac{(c,\ell)^4}{\ell^3 c(N/c,c)}(c,N/c) \ll N^{3+\varepsilon} \\
\sum_{\ell|N} \frac{\ell}{N^2} \sum_{c|N} \frac{N^3(c,\ell)^2}{\ell^2(N,c^2)} |\mathfrak{A}_c| &\leqslant N \sum_{\ell|N}\sum_{c|N} \frac{(c,\ell)^2}{\ell c(N/c,c)}(c,N/c) \ll N^{1+\varepsilon} \\
\sum_{\ell|N} \frac{\ell}{N^2} \sum_{c|N} \frac{N^3(c,\ell)^2}{\ell^2(N,c^2)} \sum_{\mathfrak{a}\in\mathfrak{A}_c} \tilde{\kappa}_{\mathfrak{a}}^{-1} &\leqslant \mathcal{K}(\chi) N \sum_{\ell|N}\sum_{c|N} \frac{(c,\ell)^2}{\ell(N,c^2)}(c,N/c) \ll N^{1+\varepsilon}
\end{aligned}$$

where $\mathcal{K}(\chi) = \max_{c|N} |\mathfrak{A}_c|^{-1} \sum_{\mathfrak{a}\in\mathfrak{A}_c} \tilde{\kappa}_{\mathfrak{a}}^{-1} \ll N^{\varepsilon}$, by Lemma 2.3.

In summary,

$$\sum_{q\ell=N} \mu(q) J_\ell \ll (kNx)^{\varepsilon}\big(kN + k^3N^3x^{-1}\big),$$

and together with (4.11) and (4.12), our theorem follows because

$$\sum_{\substack{l,m\geqslant 1\\(l,N)=1}} |\lambda(m)|^2 \varphi\Big(\frac{l^2m}{x}\Big) \quad = \quad J.$$

5. Proof of Theorem 1

We first demonstrate that the terms $|\lambda(m)|$ in Theorem 2 with large m are more dominating in the following lemma, the idea of which was exploited in [8].

Lemma 5.1. *Under the same setting of Theorem 2, we choose* $\varphi \in \mathcal{C}_0^\infty(0,\infty)$ *to be compactly supported on* $[\frac{1}{2}, 2]$, $0 \leqslant \varphi(u) \leqslant 1$ *for all* u *and* $\varphi(u) = 1$ *for* $u \in [1, \frac{3}{2}]$. *For any* $\varepsilon > 0$, *there exists* $N_0(\varepsilon) \in \mathbb{N}$ *such that for all* $N \geqslant N_0(\varepsilon)$ *and any* $N^{2\varepsilon} \leqslant z \leqslant x^{1/2}$,

$$\sum_{\substack{l,m\geqslant 1\\(l,N)=1,\, l\geqslant z}} |\lambda(m)|^2 \varphi\Big(\frac{l^2m}{x}\Big) \ll N^{\varepsilon} z \sum_{\substack{l,m\geqslant 1\\(l,N)=1}} |\lambda(m)|^2 \varphi\Big(\frac{l^2m}{cN^{2\varepsilon}x/z^2}\Big)$$

holds for some $c \in [1, 6]$.

Proof. Using $\phi(N) = N\sum_{d|N} \mu(d)/d$ (or [18, Section 8.3, Problem 8]), we have

$$\sum_{\substack{1\leqslant l\leqslant X\\(l,N)=1}} 1 = X\frac{\phi(N)}{N} + O(\tau(N)).$$

Thus for any $\varepsilon > 0$, there exists $N_0(\varepsilon)$ such that if $N \geqslant N_0(\varepsilon)$, then

$$\sum_{\substack{X\leqslant l\leqslant 2X\\(l,N)=1}} 1 \geqslant \frac{X}{2}\frac{\phi(N)}{N}, \qquad \forall\ X \geqslant N^{\varepsilon}. \tag{5.1}$$

Under this situation, for any $x \geqslant N^{4\varepsilon}$ and any $N^{2\varepsilon} \leqslant z \leqslant x^{1/2}$,

$$\begin{aligned}
\sum_{\substack{l,m\geqslant 1\\(l,N)=1,\, l\geqslant z}} |\lambda(m)|^2 \varphi\Big(\frac{l^2m}{x}\Big) &\leqslant \sum_{m\leqslant 2x/z^2} |\lambda(m)|^2 \sqrt{2x/m} \\
&= z \sum_{m\leqslant 2x/z^2} |\lambda(m)|^2 \sqrt{(2x/z^2)/m},
\end{aligned}$$

which is, by (5.1),

$$\begin{aligned} &\leqslant z \sum_{m\leqslant 2x/z^2} |\lambda(m)|^2 N^{-\varepsilon}\frac{2N}{\phi(N)} \sum_{\substack{N^{\varepsilon}\sqrt{(2x/z^2)/m}\leqslant l\leqslant 2N^{\varepsilon}\sqrt{(2x/z^2)/m} \\ (l,N)=1}} 1 \\ &\leqslant z \sum_{\substack{2N^{2\varepsilon}x/z^2\leqslant l^2m\leqslant 8N^{2\varepsilon}x/z^2 \\ (l,N)=1}} |\lambda(m)|^2 \\ &\leqslant 4z\cdot \max_{c\in[1,6]} \sum_{\substack{l,m\geqslant 1 \\ (l,N)=1}} |\lambda(m)|^2\varphi\Big(\frac{l^2m}{cN^{2\varepsilon}x/z^2}\Big). \end{aligned}$$

□

Now we prove our main result Theorem 1.

Let $f(z) = \sum_{n\geqslant 1} a(n)e(nz)$ and assume $\|f\|_2 = 1$. We note that $\mathrm{w}_\infty = w_\infty = 1$, $\mathrm{w}_0 = w_0 = N$ and $\tilde{\kappa}_0 = 1$. Thus, $\lambda(n) = r_k^{-1}a(n)n^{(1-k)/2}$. Let $0 < \varepsilon < \frac{1}{100}$ be arbitrarily small and φ be chosen as in Lemma 5.1.

By Proposition 3.2, for $x \geqslant (k^2N)^{1+\varepsilon}$ and any $A \geqslant 1$,

$$\sum_{n\geqslant 0}\lambda(n)\varphi(n/x) \ll_A x^{-A}. \tag{5.2}$$

Let $\eta > 0$ be chosen later and $X := (kN)^{2+\eta\varepsilon}$. Without loss of generality, we may only consider sufficiently large kN, meaning that $kN \geqslant B^{2/\varepsilon}$ for some constant $B \geqslant 2$ specified later. Note that $\prod_{p|N}(1-p^{-1}) \gg 1/\log N$. By Theorem 2 and Lemma 5.1, we have

$$\sum_{\substack{l,m\geqslant 1 \\ (l,N)=1}} \lambda(m)^2\varphi(l^2m/X) \gg \frac{X}{kN^{1+\varepsilon}} - O((kN)^{1+\varepsilon}X^{\varepsilon}) \gg \frac{X}{kN^{1+\varepsilon}}$$

when $\frac{1}{2\varepsilon} \geqslant \eta \geqslant 5$ and B is larger than the implied O-constant, and for $z = N^{5\varepsilon}$,

$$\sum_{\substack{l,m\geqslant 1 \\ (l,N)=1,\, l\geqslant z}} \lambda(m)^2\varphi\Big(\frac{l^2m}{X}\Big) \ll N^{\varepsilon}z\frac{N^{2\varepsilon}X/z^2}{kN} \leqslant \frac{X}{kN^{1+2\varepsilon}}.$$

Hence,

$$\sum_{\substack{l,m\geqslant 1 \\ (l,N)=1,\, l\leqslant N^{5\varepsilon}}} \lambda(m)^2\varphi\Big(\frac{l^2m}{X}\Big) \gg \frac{X}{kN^{1+\varepsilon}}.$$

Set $\eta = 11$. Then $(kN)^{2+\varepsilon} \leqslant X/l^2 \leqslant (kN)^{2+11\varepsilon}$. As φ is supported in $[\frac{1}{2}, 2]$ and $\|\varphi\|_\infty \leqslant 1$, it follows that

$$\lambda(m)^2 \gg \frac{1}{kN^{1+\varepsilon}} \quad \text{for some } \tfrac{1}{2}(kN)^{2+\varepsilon} \leqslant m \leqslant 2(kN)^{2+11\varepsilon}. \tag{5.3}$$

Assume all $\lambda(n)$'s with $1 \leqslant n \leqslant 2(kN)^{2+11\varepsilon}$ are of the same sign. Under this assumption, applying (5.2) with $\frac{1}{2}(kN)^{2+\varepsilon} \leqslant x \leqslant 2(kN)^{2+11\varepsilon}$ and $A = 1$, we infer with $\varphi(1) = 1$ that

$$\lambda(n) \ll (kN)^{-2}, \quad \forall\ \tfrac{1}{2}(kN)^{2+\varepsilon} \leqslant n \leqslant 2(kN)^{2+11\varepsilon}.$$

This contradicts (5.3), and hence we conclude the occurrence of a sign change among $\lambda(n)$'s with $n \leqslant 2(kN)^{2+11\varepsilon}$. This completes the proof.

6. Proof of Lemma 2.2

Plainly we have $\Gamma_\infty\backslash\mathbb{H} = \bigsqcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma(\Gamma\backslash\mathbb{H})$ (a disjoint union). The projection $\pi : \Gamma_\infty\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is continuous, and the restriction of π on $\gamma(\Gamma\backslash\mathbb{H})$ is a homeomorphism whose inverse map is γ, for every $\gamma \in \Gamma_\infty\backslash\Gamma$. A fundamental polygon for $\Gamma_\infty\backslash\mathbb{H}$, also denoted by $\Gamma_\infty\backslash\mathbb{H}$, is given by $\bigcup_{\gamma\in\Gamma_\infty\backslash\Gamma}\gamma\mathcal{F}$ which is contained in $\overline{P_0}$, where $\mathcal{F}$ is the fundamental polynomial for $\Gamma\backslash\mathbb{H}$ and P_0 is defined by (2.3) with $Y = 0$.

(i) Let $\mathrm{z} \in \Gamma_\infty\backslash\mathbb{H}$. The multiplicity $|\pi^{-1}\{\mathrm{z}\} \cap P_Y|$ equals $|\{\gamma \in \Gamma_\infty\backslash\Gamma : \Im\gamma z > Y\}|$, where $z \in \mathbb{H}$ is any representative of z. Let $s := a(\sqrt{w_\infty})$ and consider $\Gamma' := s^{-1}\Gamma s$. It is easy to see that $\Gamma'_\infty = \langle n(1)\rangle$ and $c_{\Gamma'} = w_\infty c_\Gamma$. Then for any $z \in \mathbb{H}$, we have

$$\begin{aligned}|\{\gamma \in \Gamma_\infty\backslash\Gamma : \Im\gamma z > Y\}| &= \left|\{\gamma' \in \Gamma'_\infty\backslash\Gamma' : \Im s\gamma' z' > Y\}\right| \\ &= \left|\{\gamma' \in \Gamma'_\infty\backslash\Gamma' : \Im\gamma' z' > w_\infty^{-1}Y\}\right|,\end{aligned}$$

where $\gamma' = s^{-1}\gamma s$ and $z' = s^{-1}z \in \mathbb{H}$. By Lemma 2.10 of Iwaniec [10], we know that the above formula is less than $1 + 10w_\infty/(c_{\Gamma'}Y) = 1 + 10/(c_\Gamma Y)$.

(ii) It is intuitively plausible as (i) implies that a point $\mathrm{z} \in \Gamma\backslash\mathbb{H}$ is locally covered by at most $|\pi^{-1}\{\mathrm{z}\} \cap P_Y|$ neighborhoods in P_Y. In fact, we shall divide $\Gamma\backslash\mathbb{H}$ into connected subsets V_i $(i \in I)$ so that P_Y can be covered by the family $\{\gamma V_i : \gamma \in \mathcal{M}_i, i \in I\}$ together with a neighborhood at ∞, where every $\mathcal{M}_i$ $(i \in I)$ is a finite subset of $\Gamma_\infty\backslash\Gamma$ with a manageable cardinality.

For a sufficiently large number X, the strip P_X lies in $\mathcal{F}$ and the closure $\overline{P}$ of the box $P := P_Y - P_{X+1}$ (the complement of P_{X+1} in P_Y) in $\mathbb{H}$ is a compact subset of $\mathbb{H}$. Then $\overline{P}$ is away from any cusp of Γ and is contained in the union of a finite subfamily of $\{\gamma\mathcal{F} : \gamma \in \Gamma\}$ (see [16, Theorem 1.6.2]).

Transferring P, P_Y and $\mathcal{F}$ to the manifold $\Gamma_\infty\backslash\mathbb{H}$, we consider $\Gamma\backslash\mathbb{H}$ (i.e. the transferred $\mathcal{F}$) to be a connected subset of $\Gamma_\infty\backslash\mathbb{H}$ and the transferred $\overline{P}$ to be a subset of $\bigcup_{\gamma\in\mathcal{J}}\gamma(\Gamma\backslash\mathbb{H})$ for some finite family $\mathcal{J}$ of $\Gamma_\infty\backslash\Gamma$. Denote the transferred $\overline{P}$, which is in $\Gamma_\infty\backslash\mathbb{H}$, by $\overline{P}$ as well. Then $\overline{P}$ is compact and connected in $\Gamma_\infty\backslash\mathbb{H}$ as is $\overline{P}$ in $\mathbb{H}$.

Inside $\mathbb{H}$, the set $\overline{\gamma(\Gamma\backslash\mathbb{H})}$ is the polygon $\overline{\gamma\mathcal{F}}$ whose boundary is composed of a finite number of geodesic segments. Its intersection with the rectangle $\overline{P}$ consists of a finite number of connected components. Likewise, in $\Gamma_\infty\backslash\mathbb{H}$, the intersection $\overline{P} \cap \gamma(\Gamma\backslash\mathbb{H})$ can be written into a disjoint union of finitely many connected components, i.e.

$$\overline{P} \cap \gamma(\Gamma\backslash\mathbb{H}) = \bigsqcup_{t\in\mathcal{K}_\gamma} U_{\gamma,t}$$

where $\mathcal{K}_\gamma$ is finite and $U_{\gamma,t}$ is connected. Denote the boundary of a set S by ∂S. Observe that $\partial U_{\gamma,t}$ is of measure zero. For any $\gamma \neq \tau$ in $\Gamma_\infty\backslash\Gamma$, a $U_{\gamma,t}$ may intersect with $U_{\tau,s}$ at the boundary only, i.e. $U_{\gamma,t} \cap U_{\tau,s} \subset (\partial U_{\gamma,t}) \cap (\partial U_{\tau,s})$ for any $t \in \mathcal{K}_\gamma$ and $s \in \mathcal{K}_\tau$. In summary, we express

$$\overline{P} = \bigcup_{(\gamma,t)\in T} U_{\gamma,t}$$

(in $\Gamma_\infty\backslash\mathbb{H}$), where $T = \{(\gamma, t) : \gamma \in \mathcal{J}, t \in \mathcal{K}_\gamma\}$ is finite, every $U_{\gamma,t}$ is connected and $U_{\gamma,t} \cap U_{\tau,s} \subset \partial U_{\gamma,t} \cap \partial U_{\tau,s}$ for any $(\gamma, t), (\tau, s) \in T$.

Now $\pi(\overline{P})$ is a compact subset of $\Gamma\backslash\mathbb{H}$ and for $(\gamma, t) \in T$, $\pi(U_{\gamma,t}) \subset \pi(\overline{P})$ and $\partial\pi(U_{\gamma,t})$ is a curve lying in $\pi(\overline{P})$. If $(\gamma, t) \neq (\tau, s) \in T$, then $\partial\pi(U_{\tau,s})$ is another curve on $\pi(\overline{P})$ and it may intersect with $\partial\pi(U_{\gamma,t})$. Consequently the curves $\partial\pi(U_{\gamma,t})$, for $(\gamma,t)\in T$, altogether weave a mesh on $\pi(\overline{P})$ of the surface $\Gamma\backslash\mathbb{H}$. That is,

$$\pi(\overline{P}) = \bigcup_{i\in I} V_i$$

where V_i is a connected compact subset of $\pi(\overline{P})$ and $\partial V_i \subset \bigcup_{(\gamma,t)\in T}\partial\pi(U_{\gamma,t})$.

Recall that γ is the inverse of the restricted map $\pi\big|_{\gamma(\Gamma\backslash\mathbb{H})}$. Suppose $\gamma\in\mathcal{J}$ maps some $z\in V_i^\circ$ to P where V_i° denotes the interior of V_i. Then $\gamma V_i^\circ \cap U_{\gamma,r}\neq\emptyset$ for some r, and consequently $\gamma V_i^\circ\subset U_{\gamma,r}$ because γV_i° is an open connected subset of $\overline{P}\cap\gamma(\Gamma\backslash\mathbb{H}) = \bigsqcup_{t\in\mathcal{K}_\gamma} U_{\gamma,t}$. Thus, for any $i\in I$ and any $\gamma\in\mathcal{J}$, we have

(*) $\qquad$ either $\gamma V_i^\circ\subset U_{\gamma,r}$ for some $r\in\mathcal{J}_\gamma$ or $\gamma V_i^\circ\cap P=\emptyset$.

Moreover, $U_{\gamma,r}\subset\bigcup_{i\in I,\,\gamma V_i^\circ\subset U_{\gamma,r}}\gamma V_i$ for any γ and r.

Therefore, we infer that $\int_{P_Y} = \int_{P_{X+1}} + \int_P$ and that

$$\int_P F = \sum_{\gamma\in\mathcal{J}}\sum_{r\in\mathcal{K}_\gamma}\int_{U_{\gamma,r}} F \leqslant \sum_{i\in I}\sum_{\gamma\in\mathcal{J}}\sum_{\substack{r\in\mathcal{K}_\gamma\\ \gamma V_i^\circ\subset U_{\gamma,r}}}\int_{\gamma V_i} F \leqslant \sum_{i\in I}\sum_{\gamma\in\mathcal{M}_i}\int_{V_i} F,$$

as $\partial U_{\gamma,t}$ and ∂V_i are of measure zero, where $\mathcal{M}_i := \{\gamma\in\mathcal{J} : \gamma V_i^\circ\subset\overline{P}\}$. Here we have used the Γ-invariance of the integrand F in the last inequality.

Note $\gamma V_i^\circ\cap P=\emptyset$ for $\gamma\notin\mathcal{M}_i$ by our construction. Let $\partial\overline{P}$ be the boundary of $\overline{P}$ in $\mathbb{H}$. The projection $\pi(\partial\overline{P})$ of its transfer in $\Gamma_\infty\backslash\mathbb{H}$ is a closed set of measure zero. If $V_i^\circ\neq\emptyset$, then as $V_i^\circ\subset\pi(\overline{P})$ has positive measure, we obtain $V_i^\circ\cap\pi(\overline{P}-\partial\overline{P})\neq\emptyset$. This implies $V_i^\circ\cap\pi(P)\neq\emptyset$, so by (*), we can find $\mathrm{z}\in V_i^\circ$ such that

$$|\mathcal{M}_i| = |\{\gamma\in\mathcal{J} : \gamma\mathrm{z}\in P\}| \leqslant |\pi^{-1}(\mathrm{z})\cap P_Y|.$$

Since $\pi^{-1}(\mathrm{z}) = \{\gamma\mathrm{z} : \gamma\in\Gamma_\infty\backslash\Gamma\}$ and $\int_{V_i}=0$ if $V_i^\circ=\emptyset$, we deduce from Part (i) that

$$\int_P \leqslant (1+O((c_\Gamma Y)^{-1}))\sum_{i\in I}\int_{V_i} \leqslant (1+O((c_\Gamma Y)^{-1}))\int_{\pi(P)}.$$

Therefore, (ii) follows readily from $\int_{\pi(P_{X+1})} = \int_{P_{X+1}}$ and $\int_{\pi(P_{X+1})} + \int_{\pi(P)} = \int_{\Gamma\backslash\mathbb{H}}$.

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