WEAKLY ADMISSIBLE LOCUS AND NEWTON STRATIFICATION IN p-ADIC HODGE THEORY

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Dedicated to Luc Illusie, with our utmost gratitude

ABSTRACT. The admissible locus $\mathcal{F}(G,\mu,b)^a$ inside the flag variety $\mathcal{F}(G,\mu)$, attached to a reductive group G with a minuscule cocharacter μ of G, is a p-adic analogue of the complex analytic period spaces. It has an algebraic approximation $\mathcal{F}(G,\mu,b)^{wa}$ inside the flag variety, called the weakly admissible locus. On the flag variety $\mathcal{F}(G,\mu)$, we have the Newton stratification which has the admissible locus as its unique open stratum. In this paper, we study the relation between the Newton strata and the weakly admissible locus. We show that $\mathcal{F}(G,\mu,b)^{wa}$ is maximal (in the sense that it's a union of Newton strata) is equivalent to (G,μ) weakly fully HN-decomposable, it's also equivalent to the condition that the Newton stratification is finer than the Harder-Narasimhan stratification. These equivalent conditions are generalizations of the fully HN-decomposable condition and the weakly accessible condition. Moreover, we give a criterion to determine whether a Newton stratum is completely contained in the weakly admissible locus involving G-bundles as extensions of M-bundles over the Fargues-Fontaine curve, where M is a Levi subgroup of G. When $G = \mathrm{GL}_n$, we also give a combinatorial inductive criterion to determine whether a vector bundle over the Fargues-Fontaine curve is an extension of two given vector bundles.

CONTENTS

INTRODUCTION

Let F be a finite extension of $\mathbb{Q}_p$, $\bar{F}$ an algebraic closure of F with $C = \widehat{\bar{F}}$ its p-adic completion. Let G be a connected reductive group over F, and $\{\mu\}$ the geometric conjugacy class of a minuscule cocharacter μ. Let $\breve{F}$ be the completion of the maximal unramified extension F^{un} of F inside $\bar{F}$, and we write σ the Frobenius on $\breve{F}$ relative to F. Fix $b \in G(\breve{F})$. Let $E = E(G,\{\mu\})$ be the reflex field, that is, the field of definition of the geometric conjugacy class $\{\mu\}$, which is a finite extension of F and is viewed as a subfield of $\bar{F}$. Attached to the pair $(G,\{\mu\})$, we have the flag variety $\mathcal{F}(G,\mu)$ which is a projective variety defined over the reflex field E. In the following, we shall consider its associated

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adic space, still denoted by $\mathcal{F}(G, \mu)$, defined over $\breve{E}$, the p-adic completion of the maximal unramified extension $E \cdot F^{un} \subset \bar{F}$ of E. For an element $b \in G(\breve{F})$ satisfying Kottwitz condition $[b] \in B(G, \mu)$ (cf. Section 1.1.4), Rapoport and Zink defined an open subset

$$\mathcal{F}(G, \mu, b)^{wa}$$

of the adic space $\mathcal{F}(G, \mu)$, called the *weakly admissible locus* (historically also called the *p-adic period domain*, see also [DOR10]), as a vast generalization of the Drinfeld upper half plane. For $K/\breve{E}$ a complete field extension, the points of $\mathcal{F}(G, \mu, b)^{wa}(K)$ correspond to the weakly admissible filtered isocrystals equipped with a G-structure, whose underlying isocrystals with a G-structure are induced by b. If $K/\breve{E}$ is finite, by a fundamental result of Colmez-Fontaine in p-adic Hodge theory ([CF00]), such filtered isocrystals are *admissible* in the sense that they come from crystalline Galois representations. If $K/\breve{E}$ is an arbitrary complete field extension, we cannot always get Galois representations attached to points in $\mathcal{F}(G, \mu, b)^{wa}(K)$. This phenomenon leads Rapoport and Zink to conjecture that there exists an open subspace, called *admissible locus*

$$\mathcal{F}(G, \mu, b)^{a}$$

inside $\mathcal{F}(G, \mu, b)^{wa}$ with the same classical points, i.e., the same points with values in finite extensions of $\breve{E}$ as $\mathcal{F}(G, \mu, b)^{wa}$, together with a p-adic étale local system with additional structures on $\mathcal{F}(G, \mu, b)^{a}$ interpolating the crystalline Galois representations attached to its classical points. By contrast to the subspace $\mathcal{F}(G, \mu, b)^{wa}$ of weakly admissible locus, the construction of $\mathcal{F}(G, \mu, b)^{a}$ is quite mysterious, and was previously known only for certain triples (G, μ, b) by the work of Hartl ([Har08] [Har13]) and Faltings ([Fal10]).

Recently, thanks to the new progress in p-adic Hodge theory, especially the discovery of Fargues-Fontaine curve, we can associate in a natural way to b a G-bundle

$$\mathcal{E}_b$$

on the Fargues-Fontaine curve X, whose isomorphism class only depends on the σ-conjugacy class of b. Furthermore we can use each C-point $x \in \mathcal{F}(G, \mu)(C)$ to modify $\mathcal{E}_b$ à la Beauville-Laszlo to get another G-bundle

$$\mathcal{E}_{b,x}$$

on X. We define $\mathcal{F}(G, \mu, b)^{a}$ as the (necessarily open) subspace of $\mathcal{F}(G, \mu)$ stable under generalization, whose set of C-points is given by

$$\mathcal{F}(G, \mu, b)^{a}(C) = \{x \in \mathcal{F}(G, \mu)(C) | \mathcal{E}_{b,x} \text{ is the trivial } G\text{-bundle}\}.$$

The existence of the étale local system on $\mathcal{F}(G, \mu, b)^{a}$ as in the conjecture of Rapoport-Zink is then a consequence of the work of Fargues-Fontaine ([FF18]), Kedlaya-Liu ([KL15]), and Scholze-Weinstein ([SW20]).

As a result, we have two open adic subspaces inside the flag varieties $\mathcal{F}(G, \mu)$:

$$\mathcal{F}(G, \mu, b)^{a} \subseteq \mathcal{F}(G, \mu, b)^{wa} \subseteq \mathcal{F}(G, \mu).$$

It is a natural question to understand the structure of these subspaces. Compared with the weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$, which has been intensively studied since the work of Rapoport-Zink, we don't know much about the admissible locus $\mathcal{F}(G, \mu, b)^{a}$. As the first step, we would like to see when $\mathcal{F}(G, \mu, b)^{a}$ coincides with $\mathcal{F}(G, \mu, b)^{wa}$ ([Rap18, Question A.20]). If $G = \mathrm{GL}_n$, Hartl gave a complete solution to this question. In particular, when $b \in \mathrm{GL}_n(\breve{F})$ is basic (cf. Section 1.1.1), up to twist by a central cocharacter, the equality

$$\mathcal{F}(\mathrm{GL}_n, \mu, b)^{a} = \mathcal{F}(\mathrm{GL}_n, \mu, b)^{wa}$$

holds if and only if

$$\mu \in \{(1, 0^{(n-1)}), (1^{(n-1)}, 0)\}, \quad \text{or} \quad n = 4 \text{ and } \mu = (1, 1, 0, 0).$$

For a general reductive group G, Fargues and Rapoport conjectured that, at least when b is basic, there exists a group-theoretic condition which expresses precisely the coincidence of $\mathcal{F}(G, \mu, b)^a$ and $\mathcal{F}(G, \mu, b)^{wa}$. By a recent joint work of the first-named author with Fargues and Shen, this conjecture is now a theorem.

Theorem (Chen-Fargues-Shen [CFS21]). *Suppose that $b \in G(\breve{F})$ is basic. Then $\mathcal{F}(G, \mu, b)^a = \mathcal{F}(G, \mu, b)^{wa}$ if and only if the pair (G, μ) is fully Hodge-Newton decomposable.*

Here the fully Hodge-Newton decomposability condition is purely group-theoretic. It was first introduced and systematically studied by Görtz, He and Nie in [GHN19]. Moreover, they also give equivalent conditions in terms of affine Deligne-Lusztig varieties for this condition. We refer to § 2.1 for the precise definition of the fully HN-decomposability condition. In her succeeding work, the first-named author proved the analogue of the above theorem in the non-basic case ([Che23]). Furthermore, Shen generalized the Fargues-Fontaine conjecture for non-minuscule cocharacters ([She23]).

On the other hand, Rapoport investigated in [Rap18] the question in which situations the subspace $\mathcal{F}(G, \mu, b)^{wa}$ (resp. $\mathcal{F}(G, \mu, b)^a$) is the whole flag variety $\mathcal{F}(G, \mu)$. If this is indeed the case, we call, following Rapoport, the triple (G, μ, b) *weakly accessible* (resp. *accessible*). Clearly, if (G, μ, b) is accessible, then it is weakly accessible. In [Rap18, A.4-A.5], Rapoport gave a full classification for such triples. Indeed, if (G, μ, b) is weakly accessible, then b is basic and hence determined by (G, μ). For instance, in the GL_n-case, up to a twist by a central cocharacter, weak accessibility is equivalent to the condition that $\mu = (1^{(r)}, 0^{(n-r)})$ with $\gcd(n, r) = 1$.

To go further, we consider the Newton stratification on the flag variety $\mathcal{F}(G, \mu)$:

$$\mathcal{F}(G, \mu) = \coprod_{[b'] \in B(G)} \mathcal{F}(G, \mu, b)^{[b']}.$$

Here $B(G)$ is the set of σ-conjugacy classes of $G(\breve{F})$ and $\mathcal{F}(G, \mu, b)^{[b']}$ is a locally closed subspace of $\mathcal{F}(G, \mu)$ such that its C-points are those $x \in \mathcal{F}(G, \mu)(C)$ with $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$. Rapoport determined in [Rap18] which Newton strata $\mathcal{F}(G, \mu, b)^{[b']}$ are non-empty when b is basic. For $[b] \in B(G, \mu)$, we know that

$$\mathcal{F}(G, \mu, b)^a = \mathcal{F}(G, \mu, b)^{[1]}$$

is the unique open stratum of this stratification. Naturally, we want to compare the Newton stratification with the weakly admissible locus. For example, we would like to classify the strata which have a non-empty intersection with $\mathcal{F}(G, \mu, b)^{wa}$ ([Che23, Conjecture 5.2]). By the work of Chen-Fargues-Shen mentioned above combined with a recent preprint of Viehmann ([Vie]), we have a complete answer to this question. Suppose a Newton stratum $\mathcal{F}(G, \mu, b)^{[b']}$ is non-empty, then

$$\mathcal{F}(G, \mu, b)^{[b']} \cap \mathcal{F}(G, \mu, b)^{wa} \neq \emptyset$$

if and only the triple $(G, \nu_b - \mu, b')$ is Hodge-Newton indecomposable (cf. Definition 2.1). In particular,

$$\mathcal{F}(G, \mu, b)^{wa} \subseteq \coprod_{\substack{[b'] \in B(G) \text{ s.t. } (G, \nu_b - \mu, b') \\ \text{is Hodge-Newton indecomposable}}} \mathcal{F}(G, \mu, b)^{[b']} \subseteq \mathcal{F}(G, \mu).$$

We say that the weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$ is *maximal* if the first inclusion above is an equality, or equivalently (by [Vie]), that $\mathcal{F}(G, \mu, b)^{wa}$ is a union of Newton strata. Note that the weakly admissible locus is maximal if $\mathcal{F}(G, \mu, b)^{wa} = \mathcal{F}(G, \mu, b)^a$ or $\mathcal{F}(G, \mu)$. Therefore the maximality condition of the weakly admissible locus can be considered as a uniform generalization of the above two extreme cases that we have discussed. Our first main result is a group theoretic characterization and a geometric characterization of the maximality of the weakly admissible locus when b is basic under the minuscule condition.

Theorem (Theorem 3.5). *Suppose b basic and μ minuscule. Then the following three assertions are equivalent:*

(a) *$\mathcal{F}(G, \mu, b)^{wa}$ is maximal;*
(b) *(G, μ) is weakly fully Hodge-Newton decomposable (cf. Definition 2.4);*
(c) *the Newton stratification is finer than the Harder-Narasimhan stratification in the sense that every Harder-Narasimhan stratum is a union of some Newton strata (see § 1.4.3 for a brief review of the Harder-Narasimhan stratification).*

The weakly fully Hodge-Newton decomposability condition in assertion (b) is a group theoretic condition. This condition is a common generalization of the fully Hodge-Newton decomposability and weak accessibility condition, and it includes also some new cases which are not covered by the last two conditions. For example, the following new cases arise in the GL_n-case:

n is even and $\mu \in \{(1^{(2)}, 0^{(n-2)}), (1^{(n-2)}, 0^{(2)})\}$, or $n = 6$ and $\mu = (1, 1, 1, 0, 0, 0)$.

In Theorem 2.15, we classify all the weakly fully HN-decomposable pairs when G is absolutely simple and adjoint.

We would also like to have a practical criterion to see if a single Newton stratum $\mathcal{F}(G, \mu, b)^{[b']}$ is entirely contained in the weakly admissible locus. Our second main result of this paper is such a criterion.

Theorem (Theorem 4.9). *Let μ be a minuscule cocharacter of G and $b \in G(\breve{F})$ be basic. Suppose $\mathcal{F}(G, \mu, b)^{[b']} \neq \emptyset$. Then*

$$\mathcal{F}(G, \mu, b)^{[b']} \nsubseteq \mathcal{F}(G, \mu, b)^{wa}$$

if and only if there exist some maximal proper standard Levi subgroup M of H, the quasi-split inner form of G over F, and an element w in the Weyl group of H, satisfying the following two conditions:

(1) *b has a reduction b_M to M and w is μ-negative for M (cf. Definition 4.1); and*
(2) *$[b']$ is an extension of some $[b'_M] \in B(M, \kappa(b_M) - \mu^{w,\#}, \nu_{b_M} - \mu^{w,\diamond})$, where $B(M, ...)$ denotes a generalized Kottwitz set (cf. §1.1.4 and definition 4.4).*

If $G = \mathrm{GL}_n$, the two conditions in the theorem can be reformulated in a more down-to-earth way as follows:

(i) the isocrystal $(\breve{F}^n, b \circ \sigma)$ has a decomposition by sub-isocrystals $(\breve{F}^n, b \circ \sigma) = D_1 \oplus D_2$; and
(ii) the vector bundle $\mathcal{E}_{b'}$ can be written as a extension of $\mathcal{E}_2$ by $\mathcal{E}_1$, such that the vector bundle $\mathcal{E}_1$ is of degree > 0 and that each $\mathcal{E}_i$ can be realized as a minuscule modification of $\mathcal{E}(D_i)$, the vector bundle attached to the isocrystal D_i.

If the Newton stratum $\mathcal{F}(\mathrm{GL}_n, \mu, b)^{[b']}$ contains a point which is not weakly admissible, such a point produces a decomposition of the isocrystal $(\breve{F}^n, b \circ \sigma)$ and an extension of vector bundles as required above. Conversely, once these two conditions are fulfilled, it is not hard to realize $\mathcal{E}_{b'}$ as a modification of $\mathcal{E}_b$ in the way compatible with the two modifications

given in (ii). But à priori the modification that we get randomly may be *not* minuscule. We need to adjust the initial modification properly to obtain a minuscule one, and this is done in Proposition 4.6 for general quasi-split groups. In this way, we obtain a point in $\mathcal{F}(\mathrm{GL}_n, \mu, b)^{[b']}$ violating the weak admissibility.

It is now obvious to the readers that, our problem of classifying the Newton strata contained in the weakly admissible locus is closely related to the difficult question to determine whether a G-bundle is an extension of an M-bundle on the Fargues-Fontaine curve, with M a Levi subgroup of the quasi-split inner form of G. When the M-bundle is semi-stable, this question is studied by [BFHHLWY] for GL_n and [Vie] for arbitrary G. In § 5, we study this question for GL_n for any M-bundle. Inspired by previous work of Schlesinger on the classification of extensions of vector bundles on the projective line ([Sch00]), we introduce in Definition 5.1, for two given vector bundles $\mathcal{E}_i$ ($i = 1, 2$), a combinatorial constraint on the slopes of a vector bundle $\mathcal{E}$ which can be realized as an extension of $\mathcal{E}_2$ by $\mathcal{E}_1$. We check that this constraint is indeed necessary, and shows that our combinatorial condition coincides with that of [BFHHLWY] when the two vector bundles $\mathcal{E}_i$ ($i = 1, 2$) are semistable. One might expect that this combinatorial condition is also sufficient to classify all the extensions, but unfortunately this is *not* the case (cf. Example 5.8). We can only give an inductive criterion to see if a vector bundle $\mathcal{E}$ verifying our combinatorial conditions indeed comes from an extension of vector bundles.

After we finished this work, we noticed that Hong has posted a new preprint [Hon22] on his webpage. In his new work, Hong has classified independently all the extensions of vector bundles over the Fargues-Fontaine curve in a similar way as we did in § 5.

We briefly describe the structure of this article. In §1, we collect some preliminaries about modifications of G-bundles on the Fargues-Fontaine curve that we will need in the sequel. In §2, we discuss the weakly fully HN-decomposable condition and give the minute criterion for the weakly fully HN-decomposability (Proposition 2.11). Using the minute criterion, we give the classification of the weakly fully HN-decomposable pairs (G, μ) when G is absolutely simple and adjoint in Theorem 2.15. In §3, we give several equivalent conditions when the weakly admissible locus is maximal in Theorem 3.5. In §4, we give a criterion when a single Newton stratum is contained in the weakly admissible locus in Theorem 4.9. In §5, we introduce a combinatorial condition which is necessary for a vector bundle over the Fargues-Fontaine curve to be realized as an extension of two given vector bundles (Proposition 5.3). Moreover, we give a combinatorial inductive criterion to classify all extensions of vector bundles in Proposition 5.9. At the end of this section, we apply this criterion to determine all the Newton strata contained in the weakly admissible locus for GL_n in an explicit example. In the appendix, we show that our combinatorial constraint above is also sufficient in some special cases, yielding hence classifications of extensions in some new cases not yet covered in the literature.

Notations

We use the following notations:

- F is a finite degree extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$ and a uniformizer π_F.
- $\bar{F}$ is an algebraic closure of F and $\Gamma = \mathrm{Gal}(\bar{F}|F)$.
- $\breve{F} = \widehat{F^{un}}$ is the completion of the maximal unramified extension $F^{un} \subset \bar{F}$, with Frobenius σ.
- G is a connected reductive group over F, and H is a quasi-split inner form of G.
- $A \subseteq T \subseteq B$, where A is a maximal split torus, $T = Z_H(A)$ is the centralizer of A in T, and B is a Borel subgroup in H.
- $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is the absolute root datum of H with positive roots Φ^+ and simple roots Δ with respect to the choice of B. Write $\Delta^\vee$ for the corresponding simple coroots.
- $W = N_H(T)/T$ is the absolute Weyl group of T in H, and w_0 is the longest element in W.
- $(X^*(A), \Phi_0, X_*(A), \Phi_0^\vee)$ is the relative root datum of H with positive roots Φ_0^+ and simple (reduced) roots Δ_0.
- If M is a standard Levi subgroup in H we denote by Φ_M the corresponding roots or coroots showing up in Lie M, and by W_M the Weyl group of M. If P is the standard parabolic subgroup of H with Levi component M, sometimes we also write W_P for W_M. For $\alpha \in \Delta_0$, let M_α be the standard Levi subgroup of H with $\Delta_{M_\alpha,0} = \Delta_0 \backslash \{\alpha\}$. Let P_α be the standard parabolic subgroup of H with Levi component M_α.

1. Preliminaries

In this section, we collect some basic definitions and results needed in the sequel.

1.1. **Generalized Kottwitz sets.** Let $B(G)$ be the set of σ-conjugacy classes of elements in $G(\breve{F})$. Kottwitz has defined two maps, the *Newton map* and the *Kottwitz map*, on $B(G)$ (cf. [Kot85], [Kot97]), which are of fundamental importance in this paper.

1.1.1. *The Newton map.* Let $\mathbb{D}$ be the pro-torus whose character group is $\mathbb{Q}$, and consider

$$\mathcal{N}(G) := \left(\mathrm{Hom}(\mathbb{D}_{\bar{F}}, G_{\bar{F}})/G(\bar{F})\right)^\Gamma,$$

with $\Gamma = \mathrm{Gal}(\bar{F}/F)$. The Newton map is a map

$$(1.1.1) \qquad\qquad\qquad \nu = \nu_G : B(G) \longrightarrow \mathcal{N}(G)$$

defined as follows. Let $b \in G(\breve{F})$. For (V, ρ) an object in $\mathrm{Rep}_F(G)$, write $V_{\breve{F}} := V \otimes_F \breve{F}$. The element b induces an isocrystal

$$(1.1.2) \qquad\qquad D_{b,\rho} = (D_{b,\rho}, \varphi_{b,\rho}) := (V_{\breve{F}}, \rho(b)(1 \otimes \sigma)),$$

whose slope decomposition gives rise to a $\mathbb{Q}$-graded vector space over $\breve{F}$. In this way, we obtain an exact tensor functor

$$\mathrm{Rep}_F(G) \longrightarrow \mathbb{Q} - \mathrm{Grad}_{\breve{F}},$$

$$(V, \rho) \longmapsto \text{slope decomposition of } D_{b,\rho},$$

from the category $\mathrm{Rep}_F(G)$ of rational algebraic representations of G over F to the category $\mathbb{Q} - \mathrm{Grad}_{\breve{F}}$ of $\mathbb{Q}$-graded $\breve{F}$-vector spaces. This tensor functor in turn, by Tannaka duality, gives a morphism of algbraic groups

$$\nu_b : \mathbb{D}_{\breve{F}} \longrightarrow G_{\breve{F}}.$$

As the slope decomposition of $D_{b,\rho}$ is defined over F^{un}, the morphism ν_b is naturally defined over F^{un}, thus can be viewed as an element of $X_*(G)_{\mathbb{Q}} = \mathrm{Hom}(\mathbb{D}_{\bar{F}}, G_{\bar{F}})$. Moreover, its $G(\bar{F})$-conjugacy class $[\nu_b]$ depends only on the σ-conjugacy of b, and is invariant under the action of Γ. The Newton map ν in (1.1.1) is then defined by setting $\nu([b]) = [\nu_b] \in \mathcal{N}(G)$.

Recall also that, an element $[b] \in B(G)$ is called *basic* if the rational cocharacter ν_b factors through the center of $G_{\breve{F}}$. In this case, we will also say that b is basic.

1.1.2. *The Kottwitz map.* Let $\pi_1(G)$ denote the algebraic fundamental group of G. In other words,

$$\pi_1(G) = X_*(T)/\langle \Phi^\vee \rangle,$$

where $\langle \Phi^\vee \rangle \subset X_*(T)$ is the subgroup generated by $\Phi^\vee$. Up to a canonical isomorphism, $\pi_1(G)$ does not depend on the choice of T, and is naturally equipped with an action of $\Gamma = \mathrm{Gal}(\bar{F}/F)$. When $G = \mathrm{GL}_{n,F}$, we have a canonical identification $\pi_1(G) \simeq \mathbb{Z}$.

The Kottwitz map (cf. [Kot97], [RR96]), written κ_G or even κ if there is no confusion, is a map

$$\kappa = \kappa_G : B(G) \longrightarrow \pi_1(G)_\Gamma$$

which is functorial on G as an F-reductive group, such that the following square is commutative:

$$
\begin{array}{ccc}
\breve{F}^* & \xrightarrow{\;\; v_{\pi_F}(-) \;\;} & \mathbb{Z} \\
{\scriptstyle \mathrm{can}} \downarrow & & \downarrow {\scriptstyle \mathrm{can}} \,\simeq \\
B(\mathbb{G}_{m,F}) & \xrightarrow{\;\; \kappa_{\mathbb{G}_{m,F}} \;\;} & \pi_1(\mathbb{G}_{m,F})_\Gamma,
\end{array}
$$

where $v_{\pi_F}(-)$ is the π_F-adic valuation on F. The Kottwitz map κ_G is uniquely determined by this property. The reader can find in [CFS21, § 1.5] a more direct construction of κ_G via the abelianization of Kottwitz set à la Borovoï.

Proposition 1.1 ([Kot97, 4.13])**.** *The map*

$$(\nu, \kappa) : B(G) \longrightarrow \mathcal{N}(G) \times \pi_1(G)_\Gamma, \quad [b] \mapsto ([\nu_b], \kappa([b]))$$

is injective.

Remark 1.2.   (1) *By a classical result of Steinberg, $H^1(\breve{F}, G)$ is trivial, from where one deduces a natural injective map ([Kot85, 1.8.3]):*

$$H^1(F, G) \to B(G)$$

whose image is the set of elements $[b] \in B(G)$ with trivial Newton vector ν_b.
     (2) *Composing the natural map in (1) with the Kottwitz map κ_G above, we get a map*

$$H^1(F, G) \longrightarrow \pi_1(G)_\Gamma.$$

It is known that this map is injective, with image the subgroup $\pi_1(G)_{\Gamma,\mathrm{tor}}$ of torsion elements in $\pi_1(G)_\Gamma$. If G is semisimple, the group $\pi_1(G)$ is of torsion. Consequently, the map above is bijective provided G semisimple.

1.1.3. *Poset structure on $\mathcal{N}(G)$ and on $B(G)$.* We have the following well-known bijective maps

$$X_*(A)^+_{\mathbb{Q}} \xrightarrow{\;\sim\;} (X_*(T)_{\mathbb{Q}}/W)^\Gamma \xrightarrow{\;\sim\;} (X_*(G)_{\mathbb{Q}}/G(\bar{F}))^\Gamma.$$

Here $X_*(A)^+_{\mathbb{Q}}$ is the closed Weyl chamber attached to the basis Δ_0

$$X_*(T)^+_{\mathbb{Q}} = \{\lambda \in X_*(A)_{\mathbb{Q}} | \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta_0\}.$$

Remark 1.3. *Up to a canonical bijection, the bijective map $X_*(T)_{\mathbb{Q}}/W \longrightarrow X_*(G)_{\mathbb{Q}}/G(\bar{F})$ does not depend on the choice of the maximal torus T. This allows us to equip a monoid structure on $X_*(G)_{\mathbb{Q}}/G(\bar{F})$, and the latter becomes a Γ-monoid in this way.*

We define a partial order $\preceq$ on $\mathcal{N}(G)$ (resp. on $B(G)$): for $v, v' \in \mathcal{N}(G)$ (resp. $[b], [b'] \in B(G)$), write $\tilde{v}$ and $\tilde{v}'$ the representatives of v and v' (resp. of $[\nu_b]$ and $[\nu_{b'}]$) in $X_*(A)^+$, and set

$$v \preceq v' \iff \tilde{v}' - \tilde{v} \in \left\{ \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee \mid n_\alpha \in \mathbb{Q}_{\geq 0} \right\} \subset X_*(A)_{\mathbb{Q}},$$

(resp. $[b] \preceq [b'] \iff \nu_b \preceq \nu_{b'}$ and $\kappa(b) = \kappa(b')$).

One can check that, the definition of the partial order $\preceq$ only depends on G. Using this partial order, we can equip $B(G)$ with the topology such that

$$[b'] \in \overline{\{[b]\}} \quad \iff \quad [b] \preceq [b'].$$

The set $\mathcal{N}(G)$ can be topologized in a similar way.

1.1.4. *The generalized Kottwitz sets.* Recall the Kottwitz set

$$B(G, \mu) := \{ [b] \in B(G) | [\nu_b] \preceq \mu^\diamond, \ \kappa([b]) = \mu^\# \},$$

where

- $\mu^\diamond$ denotes the Galois average of the cocharacter μ in $X_*(G)_{\mathbb{Q}}/G(\bar{F})$:

$$\mu^\diamond := \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\tau \in \Gamma/\Gamma_\mu} \mu^\tau \in X_*(G)_{\mathbb{Q}}/G(\bar{F}),$$

  with Γ_μ the stabilizer of the action Γ on $\mu \in X_*(G)$, and
- $\mu^\#$ is the image of μ in $\pi_1(G)_\Gamma$ via the canonical maps below

$$X_*(T) \longrightarrow \pi_1(G) = X_*(T)/\langle \Phi^\vee \rangle \longrightarrow \pi_1(G)_\Gamma.$$

One checks that both $\mu^\diamond$ and $\mu^\#$ depend only on the conjugacy class $\{\mu\}$ of μ. Set also

$$A(G, \mu) := \{ [b] \in B(G) | [\nu_b] \preceq \mu^\diamond \}.$$

It is known that $B(G, \mu) \subset A(G, \mu)$ are both finite sets equipped with the partial order $\preceq$ induced from $B(G)$. Later we will need a generalized version of the Kottwitz sets introduced in [CFS21, § 4.1].

Definition 1.4. *Let $\epsilon \in \pi_1(G)_\Gamma$ and $\delta \in \mathcal{N}(G)$. Set*

$$B(G, \epsilon, \delta) := \{ [b] \in B(G) | [\nu_b] \preceq \delta, \ \kappa([b]) = \epsilon \}.$$

If $\epsilon = \mu^\#$ and $\delta = \mu^\diamond$, we recover the Kottwitz set $B(G, \mu)$ above.

Remark 1.5. *In this paper, for the generalized Kottwitz set $B(G, \epsilon, \delta)$, δ is written additively but not multiplicatively such as in [CFS21] and [Che23]. For example, the notation $B(G, 0, \nu_b \mu^{-1,\diamond})$ in [CFS21] and [Che23] is written $B(G, 0, \nu_b - \mu^\diamond)$ here.*

1.2. **Fargues-Fontaine curves and G-bundles.**

1.2.1. *Fargues-Fontaine curves.* We recall the construction and the basic properties of the Fargues-Fontaine curve (cf. [FF18]). Let K be a perfectoid field over $\mathbb{F}_q$, and $\omega_K \in K$ with $0 < |\omega_K| < 1$. Let $W_{\mathcal{O}_F}(\mathcal{O}_K) := W(\mathcal{O}_K) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_F$, and

$$\mathcal{Y}_K := \mathrm{Spa}(W_{\mathcal{O}_F}(\mathcal{O}_K)) \setminus V(\pi_F[\omega_K]).$$

The latter is an adic space over F equipped with an automorphism φ induced from the Frobenius on K relative to $\mathbb{F}_q$. The quotient

$$\mathcal{X} = \mathcal{X}_K := \mathcal{Y}_K / \varphi^{\mathbb{Z}}$$

is an adic curve which can be algebraized, and the resulting F-scheme, the *Fargues-Fontaine curve* associated with the perfectoid field K, is given by

$$X = X_K := \mathrm{Proj}\left(\bigoplus_i B_K^{\varphi=\pi_F^i}\right)$$

with $B_K := H^0(\mathcal{Y}_K, \mathcal{O}_{\mathcal{Y}_K})$. It is known that X is a one-dimensional Noetherian scheme. In the following, we shall take C a complete algebraically closed extension of $\bar{F}$, and $K = C^\flat$ the tilt of C. In particular, the curve X is equipped with a closed point $\infty \in X$ with residue field $k(\infty) = C$ and complete local ring $\hat{\mathcal{O}}_{X,\infty} = B_{dR}^+(C)$.

1.2.2. *G-bundles on Fargues-Fontaine curves.* In the following, we shall use intensively the notion of G-bundles on $X = X_{C^\flat}$ with C as above. Recall first that, a *G-bundle* on X is a (right) G-torsor on X for the étale topology. From a G-bundle $\mathcal{E}$ on X, one can construct an exact tensor functor

$$\mathrm{Rep}_F(G) \longrightarrow \mathrm{Bun}_X,$$

$$(V, \rho) \longmapsto \mathcal{E} \times^{G,\rho} (\mathcal{O}_X \otimes_F V).$$

Here Bun_X is the category of vector bundles on X and $\mathcal{E} \times^{G,\rho} (\mathcal{O}_X \otimes_F V)$ denotes the quotient of $\mathcal{E} \times (\mathcal{O}_X \otimes_F V)$ by the following action of G: for g (resp. x) a local section of G (resp. of $\mathcal{E}$), and $v \in V$,

$$g \cdot (x, 1 \otimes v) := (xg^{-1}, 1 \otimes \rho(g)(v)).$$

Conversely, every exact tensor functor

(1.2.1) $$\mathrm{Rep}_F(G) \longrightarrow \mathrm{Bun}_X$$

arises in this way (see [Far20, § 1]). Moreover, attached to a G-bundle $\mathcal{E}$, we have the Harder-Narasimhan polygon $\nu_{\mathcal{E}} \in \mathcal{N}(G)$ and the G-equivariant first Chern class $c_1^G(\mathcal{E}) \in \pi_1(G)_\Gamma$. See [CFS21, § 1.4] for more details.

Let $b \in G(\breve{F})$. For (V, ρ) an object in $\mathrm{Rep}_F(G)$, consider the isocrystal $D_{b,\rho}$ of (1.1.2). It gives rise the vector bundle $\mathcal{E}(D_{b,\rho})$ on X associated with the graded module

$$M(D_{b,\rho}) := \bigoplus_{i \geq 0} (B_K \otimes_{\breve{F}} D_{b,\rho})^{\varphi \otimes \varphi = \pi_F^i}$$

over the graded ring $P_K := \bigoplus_i B_K^{\varphi=\pi_F^i}$. In this way, we obtain an exact tensor functor of the form (1.2.1). The resulting G-bundle on X is denoted by

$$\mathcal{E}_b$$

in the following. The isomorphism class of the isocrystal $(D_{b,\rho}, \varphi_{b,\rho})$, and hence the isomorphism class of the G-bundle $\mathcal{E}_b$, depend only on the σ-conjugacy class of b. So we obtain a map

(1.2.2) $$B(G) \longrightarrow H^1(X, G), \quad [b] \mapsto [\mathcal{E}_b].$$

Here $B(G)$ is the set of σ-conjugacy classes $G(\breve{F})$.

Theorem 1.6 ([Far20] Théorème 5.1). *The map* (1.2.2) *is bijective. Moreover, for $[b] \in B(G)$, $\nu_{\mathcal{E}_b} = -w_0[\nu_b] \in \mathcal{N}(G)$ and $c_1^G(\mathcal{E}_b) = -\kappa_G([b])$.*

Remark 1.7. *For (V, ρ) an F-representation of G, the restriction of the vector bundle $\mathcal{E}(D_{b,\rho})$ to $\mathrm{Spec}(B_{dR}^+)$ via the map*

$$\mathrm{Spec}(B_{dR}^+) = \mathrm{Spec}(\hat{\mathcal{O}}_{X,\infty}) \longrightarrow X$$

is canonically *trivialized (see also* [FF18, p. 278]*). To show this, write* $\{\infty\} = V_+(t)$ *with* $t \in B_K^{\varphi = \pi}$*. Then* $\mathcal{O}_{X,\infty}$ *is the homogeneous localization of the graded ring* P_K *at the homogeneous prime ideal* $(t) \subset P_K$*. In other words, let*

$$S := \{b \in B_K \setminus t \cdot B_K \mid \varphi(b) = \pi^i b \text{ for some } i \in \mathbb{N}\},$$

then $\mathcal{O}_{X,\infty} = (S^{-1}B_K)^{\varphi = 1}$*. Similarly*

$$\mathcal{E}(D_{b,\rho})_\infty = \left(S^{-1}B_K \otimes_{\breve{F}} D_{b,\rho} \right)^{\varphi \otimes \varphi = 1}.$$

So it suffices to check that, for any isocrystal D *over* $\breve{F}/F$*, the natural morphism of* B_{dR}^+*-modules*

$$B_{dR}^+ \otimes_{\mathcal{O}_{X,\infty}} \left(S^{-1}B_K \otimes_{\breve{F}} D \right)^{\varphi \otimes \varphi = 1} \longrightarrow B_{dR}^+ \otimes_{\breve{F}} D$$

is an isomorphism: one reduces to the case where D *is simple of slope* d/h*, with* $d \in \mathbb{Z}, h \in \mathbb{Z}_{\geq 1}$ *and* $(h,d) = 1$*, and the result follows from the fact that the natural map below is an isomorphism:*

$$B_{dR}^+ \otimes_{\mathcal{O}_{X,\infty}} \left(S^{-1}B_K \right)^{\varphi^h = \pi^d} \longrightarrow B_{dR}^{+,h}, \quad 1 \otimes a \mapsto (a, \varphi(a), \dots, \varphi^{h-1}(a)).$$

As a corollary, the G*-bundle* $\mathcal{E}_b$ *is canonically trivialized over* $\mathrm{Spec}(B_{dR}^+)$*.*

1.3. **Flag varieties and the weakly admissible locus.** Let $\{\mu\}$ be a geometric conjugacy class of a cocharacter

$$\mu : \mathbb{G}_{m,\bar{F}} \longrightarrow G_{\bar{F}}.$$

Let $E = E(G, \{\mu\}) \subset \bar{F}$ be the field of definition of $\{\mu\}$: so E/F is a finite subextension of $\bar{F}/F$ such that $\mathrm{Gal}(\bar{F}/E)$ is the stabilizer of

$$\{\mu\} \in \mathrm{Hom}_{\bar{F}}(\mathbb{G}_{m,\bar{F}}, G_{\bar{F}})/G(\bar{F})$$

under the action of $\mathrm{Gal}(\bar{F}/F)$.

1.3.1. *Flag varieties.* There is a projective variety $\mathcal{P}$ over F parametrizing all parabolic subgroups of G, and the geometric connected component of $\mathcal{P}$ whose set of $\bar{F}$-points is given by the image of the map

$$\{\mu\} \longrightarrow \mathcal{P}(\bar{F}), \quad \lambda \mapsto P_\lambda$$

is defined over E. The corresponding open and closed subscheme of $\mathcal{P}_E$ will be written by

$$\mathcal{F}(G, \mu).$$

Here, for $\lambda : \mathbb{G}_{m,\bar{F}} \to G_{\bar{F}}$ a cocharacter, P_λ denotes its associated parabolic subgroup: for any field extension $L/\bar{F}$,

$$P_\lambda(L) = \left\{ g \in G(L) \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G_L \right\}.$$

Recall that, for $\lambda' : \mathbb{G}_{m,L} \to G_L$ a second cocharacter of G over L, $P_\lambda = P_{\lambda'}$ if and only if there exists $g \in P_\lambda(L)$ with $\lambda' = \mathrm{Int}(g) \circ \lambda$. Therefore, with a particular choice of $\mu \in \{\mu\}$, we have an identification

$$G_{\bar{F}}/P_\mu \longrightarrow \mathcal{F}(G, \mu)_{\bar{F}},$$
$$g P_\mu \longmapsto g P_\mu g^{-1}.$$

So $\mathcal{F}(G, \mu)$ is a twisted form of $G_{\bar{F}}/P_\mu$ over E.

1.3.2. *Weakly admissible locus in flag varieties.* In the following, we shall consider $\mathcal{F}(G, \mu)$ as an adic space over $\breve{E}$, the completion of the composite $E \cdot F^{un} \subset \bar{F}$. Let $b \in G(\breve{F})$. Rapoport and Zink defined a subspace, called weakly admissible locus

$$\mathcal{F}(G, \mu, b)^{wa} \subset \mathcal{F}(G, \mu)$$

attached to the triple $(G, \{\mu\}, b)$, as a vast generalization of Drinfeld upper half plane. Let us briefly recall its definition. Let $L/\breve{E}$ be a complete field extension. A point $x \in \mathcal{F}(G, \mu)(L)$ corresponds to a parabolic subgroup of G_L, and is thus of the form P_{μ_x} for some cocharacter $\mu_x : \mathbb{G}_{m,L} \to G_L$ over L. For any finite-dimensional F-representation (V, ρ) of G, we may consider the filtration on $V_L := V \otimes_F L$ induced by the cocharacter $\rho \circ \mu_x$, which does not depend on the auxiliary choice of μ_x. We shall denote this filtration by $\mathrm{Fil}_{x,\rho}^{\bullet}$. Let

$$\varphi - \mathrm{FilMod}_{L/\breve{F}}$$

be the category of filtered isocrystals over $L/\breve{F}$. Consider the functor

(1.3.1)
$$\mathcal{I}_{b,x} : \mathrm{Rep}_F(G) \longrightarrow \varphi - \mathrm{FilMod}_{L/\breve{F}},$$
$$(V, \rho) \longmapsto (V_{\breve{F}}, \rho(b)(1 \otimes \sigma), \mathrm{Fil}_{x,\rho}^{\bullet}).$$

Definition 1.8 ([RZ96, 1.18]). *Let $L/\breve{E}$ be a complete field extension. Let $b \in G(\breve{F})$ and $x \in \mathcal{F}(G, \mu)(L)$. We say that x is* weakly admissible *if the following equivalent conditions are satisfied:*

   (1) *for any object (V, ρ) in $\mathrm{Rep}_F(G)$, the filtered isocrystal $\mathcal{I}_{b,x}(V, \rho)$ over $L/\breve{F}$ is weakly admissible in the sense of Fontaine;*
   (2) *there is a faithful finite dimensional F-representation (V, ρ) of G such that $\mathcal{I}_{b,x}(V, \rho)$ is weakly admissible.*

It is a fundamental observation that the weakly admissible locus forms an open adic subspace of the flag variety:

Theorem 1.9 ([RZ96]). *There is a partially proper open subspace $\mathcal{F}(G, \mu, b)^{wa} \subset \mathcal{F}(G, \mu)$ such that for any complete field extension $L/\breve{E}$,*

$$\mathcal{F}(G, \mu, b)^{wa}(L) = \{x \in \mathcal{F}(G, \mu)(L) \mid x \text{ is weakly admissible}\}.$$

Moreover, $\mathcal{F}(G, \mu, b)^{wa} \neq \emptyset$ if and only if $[b] \in A(G, \mu)$.

Remark 1.10. *The space $\mathcal{F}(G, \mu, b)^{wa}$ is obtained from $\mathcal{F}(G, \mu)$ by removing a family of Zariski closed subspaces of $\mathcal{F}(G, \mu)$. To see this, consider the reductive group J_b over F, the σ-centralizer of b: so for any F-algebra R,*

$$J_b(R) = \{g \in G(\breve{F} \otimes_F R) \mid g b \sigma(g)^{-1} = b\}.$$

The group $J_b(F)$ acts on $\mathcal{F}(G, \mu)$ through the natural inclusion $J_b(F) \subset G(\breve{F}) \subset G(\breve{E})$. The weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$ is stable under this action of $J_b(F)$, and is of the form

$$\mathcal{F}(G, \mu) \setminus \bigcup_{i \in I} J_b(F) \cdot Z_i$$

where Z_i, $i \in I$, is a finite collection of Zariski closed Schubert varieties.

1.4. **Period domains and modifications of G-bundles.** Let $C/\bar{F}$ be a complete algebraically closed field extension, and consider the Fargues-Fontaine curve $X = X_{C^\flat}$, with $\infty \in X$ the associated closed point. So

$$k(\infty) = C \quad \text{and} \quad \hat{\mathcal{O}}_{X,\infty} = B_{dR}^+(C) =: B_{dR}^+.$$

Let also $\mu : \mathbb{G}_{m,\bar{F}} \to G_{\bar{F}}$ be a *minuscule* cocharacter of G. Let $b \in G(\breve{F})$. In this subsection, we shall construct modifications of the G-bundle $\mathcal{E}_b$ from C-points of the period domain. As an application, we relate the weak admissibility to certain stability condition of G-bundles on the Fargues-Fontaine curve.

Remark 1.11. *The natural ring homomorphism $\theta : B_{dR}^+ \longrightarrow C$ of Fontaine has a canonical section over $\bar{F}$. Thus B_{dR}^+ and its fraction field B_{dR} are naturally algebras over $\bar{F}$. Therefore the cocharacter $\mu : \mathbb{G}_m \to G$ yields a map $\mathbb{G}_m(B_{dR}) \to G(B_{dR})$, and for t a non-zero element of B_{dR}^+, we may consider its image $\mu(t)$ in $G(B_{dR})$.*

1.4.1. *Modifications of G-bundles.* Let $t \in B_{dR}^+$ be a uniformizer of the discrete valuation ring B_{dR}^+. Consider (the set of C-points of) the *affine Schubert cell* $\mathrm{Gr}_{G,\mu}^{B_{dR}}(C)$ associated with μ (which does not depend on the choice of the uniformizer t above) inside the B_{dR}^+-affine Grassmannian $\mathrm{Gr}_G^{B_{dR}}(C)$:

$$\mathrm{Gr}_{G,\mu}^{B_{dR}}(C) := G(B_{dR}^+)\mu(t)^{-1}G(B_{dR}^+)/G(B_{dR}^+) \subset \mathrm{Gr}_G^{B_{dR}}(C) := G(B_{dR})/G(B_{dR}^+),$$

and (the evaluation on C-points of) the *Bialynicki-Birula map* (cf. [CS17, 3.4.3])

$$\pi_{G,\mu} : \mathrm{Gr}_{G,\mu}^{B_{dR}}(C) \longrightarrow \mathcal{F}(G,\mu)(C),$$
$$g\mu(t)^{-1}G(B_{dR}^+) \longmapsto \theta(g)P_\mu\theta(g)^{-1}.$$

Proposition 1.12 ([CS17, 3.4.4]). *Assume that μ is minuscule. Then the map $\pi_{G,\mu}$ above is bijective.*

In particular, for any $x \in \mathcal{F}(G,\mu)(C)$, one can use it to modify the G-bundle $\mathcal{E}_b$ as follows. Recall that the pullback of $\mathcal{E}_b$ along the natural map

$$\mathrm{Spec}(B_{dR}^+) = \mathrm{Spec}(\hat{\mathcal{O}}_{X,\infty}) \longrightarrow X,$$

can be *canonically* trivialized (Remark 1.7). Let $\tilde{x} = gG(B_{dR}^+) \in \mathrm{Gr}_{G,\mu}^{B_{dR}}(C)$ be the unique element such that $\pi_{G,\mu}(\tilde{x}) = x$. We glue $\mathcal{E}|_{X \setminus \{\infty\}}$ and the trivial G-bundle $G \times_{\mathrm{Spec}(F)} \mathrm{Spec}(B_{dR}^+)$ over $\mathrm{Spec}(B_{dR}^+)$ through the isomorphism of G-bundles below:

$$\left(\mathcal{E}|_{X \setminus \{\infty\}}\right)|_{\mathrm{Spec}(B_{dR})} \xrightarrow{\simeq} G \times_{\mathrm{Spec}F} \mathrm{Spec}(B_{dR}) \xrightarrow{g^{-1}} G \times_{\mathrm{Spec}F} \mathrm{Spec}(B_{dR}).$$

The resulting G-bundle on X will be denoted by $\mathcal{E}_{b,x}$ in the sequel.

Remark 1.13. *The isomorphism class of the G-bundle $\mathcal{E}_b$ depends only on the class $[b] \in B(G)$ of b. However, the construction of $\mathcal{E}_{b,x}$ does depend on b as an element in $G(\breve{F})$. For $b_1, g \in G(\breve{F})$ with $b_1 = gb\sigma(g)^{-1}$, the element g induces an isomorphism $\mathcal{E}_{b,x} \xrightarrow{\sim} \mathcal{E}_{b_1,gx}$.*

1.4.2. *Weak admissibility and admissibility for G-bundles.* For $\mathcal{E}$ a G-bundle on X, and for $G' \subset G$ an algebraic subgroup of G, a *reduction of $\mathcal{E}$ to G'* is a G'-bundle $\mathcal{E}'$ on X, together with an isomorphism of G-bundles

$$\mathcal{E}' \times^{G'} G \xrightarrow{\sim} \mathcal{E}.$$

From the isomorphism above, we deduce a morphism $\mathcal{E}' \to \mathcal{E}$ of X-schemes, compatible with the action of G'. The latter yields a section

$$s : X \simeq \mathcal{E}'/G' \longrightarrow \mathcal{E}/G',$$

of the map $\mathcal{E}/G' \to X$ such that $\mathcal{E}' \xrightarrow{\sim} X \times_{s,\mathcal{E}/G'} \mathcal{E}$. Conversely, every section of $\mathcal{E}/G' \to X$ can be obtained in this way.

Proposition 1.14 ([CFS21], Lemma 2.4). *Let $P \subset G$ be a parabolic subgroup of G. Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be two G-bundles on X, equipped with an isomorphism $\mathcal{E}|_U \xrightarrow{\sim} \tilde{\mathcal{E}}|_U$ of G-bundles over some non-empty open subset U of X. Let $\mathcal{E}_P$ be a reduction of $\mathcal{E}$ to P. Then there is naturally a reduction $\tilde{\mathcal{E}}_P$ of $\tilde{\mathcal{E}}$ to P, together with an isomorphism $\mathcal{E}_P|_U \xrightarrow{\sim} \tilde{\mathcal{E}}_P|_U$ of P-bundles satisfying the obvious compatibility condition.*

One can reformulate the weak admissibility in terms of G-bundles on Fargues-Fontaine curves when G is quasi-split. To see this, we need the following key definition:

Definition 1.15 ([CFS21], Definition 2.5). *Let $b \in G(\breve{F})$ be an element. For a Levi subgroup M of G, a reduction of b to M is an element $b_M \in M(\breve{F})$ together with an element $g \in G(\breve{F})$ such that $b = g b_M \sigma(g)^{-1}$. By abuse of notation, such a reduction will simply be denoted by b_M in the following if there is no confusion.*

Proposition 1.16 ([CFS21], Proposition 2.7). *Assume G quasi-split, and μ minuscule. Let $b \in G(\breve{F})$ with $[b] \in A(G, \mu)$. Then $x \in \mathcal{F}(G, \mu)(C)$ is weakly admissible if and only if for any standard parabolic subgroup P with associated Levi M, any reduction b_M of b to M, and any $\chi \in X^*(P/Z_G)^+$, we have*

$$\deg(\chi_*(\mathcal{E}_{b,x})_P) \leq 0,$$

where $(\mathcal{E}_{b,x})|_P$ is the reduction to P of $\mathcal{E}_{b,x}$ induced by the reduction $\mathcal{E}_{b_M} \times^M P$ of $\mathcal{E}_b$ to P.

As mentioned in the introduction, Rapoport and Zink have conjectured the existence of an open subspace $\mathcal{F}(G, \mu, b)^a \subset \mathcal{F}(G, \mu, b)^{wa}$ together with an étale G-local system $\mathcal{L}$ on $\mathcal{F}(G, \mu, b)^a$ such that these two spaces have the same classical points, that is, points with values in finite field extensions of $\breve{E}$, and that $\mathcal{L}$ interpolates the crystalline representations arising from classical points of $\mathcal{F}(G, \mu, b)^{wa}$. With the above construction, for $[b] \in B(G, \mu)$, we define $\mathcal{F}(G, \mu, b)^a$, called admissible locus, the subspace of $\mathcal{F}(G, \mu)$ such that, for any complete algebraically closed field extension C of $\bar{F}$

$$\mathcal{F}(G, \mu, b)^a(C) = \{x \in \mathcal{F}(G, \mu)(C) | \mathcal{E}_{b,x} \text{ is trivial}\}$$

The space $\mathcal{F}(G, \mu, b)^a$ satisfies the properties required by Rapoport-Zink conjecture due to the work of Fargues-Fontaine ([FF18]), Kedlaya-Liu ([KL15]) and Scholze-Weinstein ([SW20]).

1.4.3. *Newton and Harder-Narasimhan stratifications.* In the following, we review the definition of the Newton and the Harder-Narasimhan stratification on the flag variety. Recall that $b \in G(\breve{F})$.

The Newton stratification on $\mathcal{F}(G, \mu)$ is indexed by elements in $B(G)$. More precisely, let C be a complete algebraically closed field extension of $\bar{F}$. For each $x \in \mathcal{F}(G, \mu)(C)$, the G-bundle $\mathcal{E}_{b,x}$ corresponds to a unique element $[b'_x] \in B(G)$ by Theorem 1.6. So we obtain a map $\mathcal{F}(G, \mu)(C) \to B(G)$. Letting C vary, we deduce a map

(1.4.1) $$\text{Newt}_b : |\mathcal{F}(G, \mu)| \longrightarrow B(G), \quad x \mapsto [b'_x].$$

For $[b'] \in B(G)$, the corresponding Newton stratum $\mathcal{F}(G, \mu, b)^{[b']}$ is defined to be the preimage of $[b'] \in B(G)$ via the map Newt_b: so for any complete algebraically closed field extension C of $\bar{F}$,

$$\mathcal{F}(G, \mu, b)^{[b']}(C) = \{x \in \mathcal{F}(G, \mu)(C) | \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}$$

In particular, we have a decomposition

$$(1.4.2) \qquad \mathcal{F}(G,\mu) = \coprod_{[b'] \in B(G)} \mathcal{F}(G,\mu,b)^{[b']}$$

of the flag variety $\mathcal{F}(G,\mu)$.

Remark 1.17. *Assume that μ is minuscule. Let $b, b' \in G(\breve{F})$.*

(1) *Each Newton stratum $\mathcal{F}(G,\mu,b)^{[b']}$ is locally closed in $\mathcal{F}(G,\mu)$ by the work of Kedlaya-Liu [KL15], and is stable under the action of $J_b(F) \subset G(\breve{F})$ on $\mathcal{F}(G,\mu)$. Moreover, if $[b] \in B(G)$ is basic, by [Rap18, A.10],*

$$\mathcal{F}(G,\mu,b)^{[b']} \neq \emptyset \iff [b'] \in B(G, \kappa([b]) - \mu^{\#}, \nu_b - \mu^{\diamond}).$$

However, when $[b] \in B(G)$ is non-basic, such a description is still unknown.

(2) *The decomposition (1.4.2) is a stratification by the work of Viehmann [Vie, Corollary 6.7].*

Now we turn to the Harder-Narasimhan stratification on the flag variety. The formalism of Harder-Narasimhan stratification on the flag varieties (or on the B_{dR}^+-Grassmannian) was studied by Dat-Orlik-Rapoport [DOR10], Cornut-Peche Irissarry[CPI19], Shen [She23] and Nguyen-Viehmann[NV23]. We briefly recall the definition and main properties that we will need in the sequel. For $D = (D, \varphi, \mathrm{Fil}^\bullet)$ a filtered isocrystal over $C/\breve{F}$. We call $\dim_{\breve{F}}(D)$ the *rank* of D, written $\mathrm{rank}(D)$, and

$$\deg(D) := \sum_i i \cdot \dim_C \mathrm{gr}_{\mathrm{Fil}^\bullet}^i(D_C) - v_\pi(\det(\varphi))$$

its *degree*. Using these two functions, one can develop a theory of Harder-Narasimhan filtration on the category $\varphi - \mathrm{FilMod}_{C/\breve{F}}$ of filtered isocrystals over $C/\breve{F}$ ([DOR10, Proposition 8.1.10]), and a remarkable property here is that the Harder-Narasimhan filtration for filtered isocrystals is compatible with tensor products ([DOR10, Theorem 8.1.9]).

Recall that, for $b \in G(\breve{F})$ and for $x \in \mathcal{F}(G,\mu)(C)$, we have the natural functor

$$\mathcal{I}_{b,x} : \mathrm{Rep}_F(G) \longrightarrow \varphi - \mathrm{FilMod}_{C/\breve{F}}.$$

of (1.3.1), or equivalently, a functor

$$\mathcal{I}_b : \mathrm{Rep}_F(G) \longrightarrow \varphi - \mathrm{Mod}_{\breve{F}}$$

together with a filtration $\mathrm{Fil}_x^\bullet$ on the fiber functor

$$\omega_C^G : \mathrm{Rep}_F(G) \longrightarrow \mathrm{Vect}_C, \quad (V, \rho) \mapsto V_C := V \otimes_F C.$$

Using the formalism of Harder-Narasimhan filtration, we deduce a unique $\mathbb{Q}$-filtration on $\mathcal{I}_b$, such that for any $V = (V, \rho) \in \mathrm{Rep}_F(G)$, the induced filtration on the isocrystal $\mathcal{I}_b(V)$ over $\breve{F}$ is the Harder-Narasimhan filtration of the filtered isocrystal $\mathcal{I}_{b,x}(V)$ as explained in the above paragraph. In particular, this defines an element $v_{b,x} \in X_*(G)_{\mathbb{Q}}/G$. It is known that $v_{b,x}$ is invariant under the action of $\Gamma = \mathrm{Gal}(\bar{F}/F)$. So

$$v_{b,x} \in \mathcal{N}(G) = (X_*(G)_{\mathbb{Q}}/G)^\Gamma.$$

We call it the *Harder-Narasimhan vector* of x.

Remark 1.18. *Suppose $b = 1 \in G(\breve{F})$. Recall that we are in the minuscule case, so we can identify the affine Schubert cell $\mathrm{Gr}_{G,\mu}^{B_{dR}}(C)$ with $\mathcal{F}(G,\mu)(C)$ via the Bialynicki-Birula map. Under this identification, the Harder-Narasimhan vector $v_{b,x} = v_{1,x}$ of $x \in \mathrm{Gr}_{G,\mu}^{B_{dR}}(C) = \mathcal{F}(G,\mu)(C)$ is denoted by $\mathrm{HN}(\mathcal{E}_1, x)$ by Nguyen-Viehmann in [NV23].*

Letting C vary, we get the following map on topological spaces

$$\mathrm{HN}_b : |\mathcal{F}(G,\mu)| \longrightarrow \mathcal{N}(G), \quad x \mapsto v_{b,x}^* := -w_0 v_{b,x}.$$

Remark 1.19. *We take the dual of the Harder-Narasimhan vector here is in order to compare it with* New_b *in (1.4.1).*

Proposition 1.20 ([She23] Theorem 3.5 and Theorem 4.4)**.** *The map* HN_b *above is upper-continuous. In other words, for any* $v \in \mathcal{N}(G)$*, the subset*

$$\mathcal{F}(G,\mu)^{\mathrm{HN}_b \geq v} := \{x \in \mathcal{F}(G,\mu) | v \preceq \mathrm{HN}_b(x)\}$$

is closed. Consequently, the subset

$$\mathcal{F}(G,\mu)^{\mathrm{HN}_b = v} := \{x \in \mathcal{F}(G,\mu) | \mathrm{HN}_b(x) = v\}$$

is locally closed, and we call it the Harder-Narasimhan stratum *associated with* $v \in \mathcal{N}(G)$*.*

Remark 1.21. *Recall that* μ *is minuscule. If* $[b] \in B(G,\mu)$*, then if* $[b'] \in B(G, 0, \nu_b - \mu^\diamond)$ *is basic, then*

$$\mathcal{F}(G,\mu)^{\mathrm{HN}_b = [\nu_{b'}]} = \mathcal{F}(G,\mu,b)^{wa}$$

is precisely the subspace of weakly admissible locus, thus is open and non-empty.

In [She23] and [NV23], Shen and Nguyen-Viehmann study the geometric properties of the Harder-Narasimhan stratification. For the convenience of the readers, we summarize below some of their results in the form that we will need in the sequel.

Proposition 1.22 ([She23], [NV23])**.** *Suppose* $[b] \in B(G,\mu)$ *is basic and* $x \in \mathcal{F}(G,\mu,b)^{[b']}(C)$*. Then*

(1) $\mathrm{HN}_b(x) \in B(G, 0, \nu_b - \mu^\diamond)$;
(2) $\mathrm{HN}_b(x) \preceq [\nu_{b'}]$*. Moreover, if* $(G, b', \nu_b - \mu^\diamond)$ *is Hodge-Newton decomposable with respect to some standard Levi subgroup* M *inside the quasi-split inner form* H *of* G *(cf. definition 2.1 below), then*

$$\mathrm{HN}_b(x) \preceq_M [\nu_{b'}],$$

where $\preceq_M$ *means that* $[\nu_{b'}] - \mathrm{HN}_b(x)$ *is a non-negative combination of coroots in* $\Delta_{M,0}^\vee$*.*

Proof. (1) and the first assertion in (2) are proved in [She23, Proposition 4.3], and also in [NV23, Proposition 3.13 and Theorem 6.4] (via inner twisting). The last assertion of (2) is stated for the modification of the trivial bundle $\mathcal{E}_1$ in [NV23, Theorem 6.4]. Their results can be translated in the form mentioned above using the compatibility under inner twisting (cf. loc. cit. Section 8, or [She23, § 4.5]). $\qquad\square$

2. Weakly fully HN-decomposability

In this section, we introduce the notion of *weakly fully HN-decomposability* and discuss its minute criterion. We also give the complete classification of the weakly fully HN-decomposable pairs when the group G is absolutely simple and adjoint. In the next section we shall use this group theoretic condition to see when the flag variety $\mathcal{F}(G,\mu)$ has maximal weakly admissible locus $\mathcal{F}(G,\mu,b)^{wa}$.

2.1. **Weakly fully HN-decomposability.** Recall that the reductive group G is not necessarily quasi-split, and H is the quasi-split inner form of G over F. So we have natural identifications

$$X_*(G)/G(\bar{F}) = X_*(H)/H(\bar{F}), \quad \mathcal{N}(G) = \mathcal{N}(H) \quad \text{and} \quad \pi_1(G) = \pi_1(H).$$

Moreover, as in Notations, let

$$A \subset T \subset B$$

be subgroups of H, where A is a maximal split torus of H, $T = C_H(A)$ and B a Borel subgroup of H containing T. Let $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ (resp. $(X^*(A), \Phi_0, X_*(A), \Phi_0^\vee)$) be the root datum given respectively by the adjoint action of T (resp. of A), with Δ (resp. Δ_0) a fixed basis. In particular, the Galois group Γ acts on Φ and on Δ, inducing a natural bijection

$$\Phi/\Gamma \xrightarrow{\sim} \Phi_0, \quad \Gamma\text{-orbit of } \alpha \mapsto \alpha|_A.$$

Similarly, we have $\Delta/\Gamma \xrightarrow{\sim} \Delta_0$.

In order to discuss the concept of weakly fully Hodge-Newton decomposability for the non quasi-split reductive group G, we need to relate a generalized Kottwitz set for G to a certain Kottwitz set of H^{ad}, the adjoint quotient of H. This part is explained in [CFS21, 4.2]. For the sake of completeness, let us reproduce their argument here. Since H^{ad} is semisimple, we have natural identifications (see Remark 1.2)

$$H^1(F, H^{ad}) \xrightarrow{\sim} B(H^{ad})_{basic} \xrightarrow{\sim} \pi_1(H^{ad})_\Gamma = [\langle\Phi\rangle^\vee/\langle\Phi^\vee\rangle]_\Gamma.$$

The group G, being an inner form of H, gives a class $[b_G] \in B(H^{ad})_{basic}$ and a class $[\xi] \in [\langle\Phi\rangle^\vee/\langle\Phi^\vee\rangle]_\Gamma$, with $b_G \in H^{ad}(\breve{F})$ and $\xi \in \langle\Phi\rangle^\vee$. Moreover, $J_{b_G} = G^{ad}$ and there is a bijection

$$(2.1.1) \qquad\qquad B(G^{ad}) = B(J_{b_G}) \longrightarrow B(H^{ad})$$

that sends $[1]$ to $[b_G]$ ([Kot97, 3.4]), which can be inserted into the following commutative diagram

$$
\begin{array}{ccc}
B(G^{ad}) & \xrightarrow{\;(2.1.1)\;} & B(H^{ad}) \\
\kappa_{G^{ad}} \downarrow & & \downarrow \kappa_{H^{ad}} \\
\pi_1(G^{ad})_\Gamma = \pi_1(H^{ad})_\Gamma & \xrightarrow{\;\bullet + \xi\;} & \pi_1(H^{ad})_\Gamma
\end{array}
$$

As $[b_G] \in B(H^{ad})_{basic}$, its Newton vector is trivial. So the map (2.1.1) gives the commutative diagram below

$$
\begin{array}{ccc}
B(G^{ad}) & \xrightarrow{\;(2.1.1)\;} & B(H^{ad}) \\
\nu_{G^{ad}} \downarrow & & \downarrow \nu_{H^{ad}} \\
\mathcal{N}(G^{ad}) & = & \mathcal{N}(H^{ad})
\end{array}
$$

On the other hand, for $\epsilon \in \pi_1(G)_\Gamma$ and $\delta \in \mathcal{N}(G)$, via the natural map $B(G) \to B(G^{ad})$, the generalized Kottwitz set $B(G, \epsilon, \delta)$ can be identified with $B(G^{ad}, \epsilon^{ad}, \delta^{ad}) \subset B(G^{ad})$, where ϵ^{ad} (resp. δ^{ad}) denotes the natural image of ϵ in $\pi_1(G^{ad})_\Gamma$ (resp. in $\mathcal{N}(G^{ad})$): see [Kot97, 4.11]. Thus, by the bijection (2.1.1), $B(G, \epsilon, \delta)$ can be further identified with

$$B(H^{ad}, \epsilon^{ad} + \xi, \delta^{ad}) \subset B(H^{ad}).$$

Before giving the definition of the weakly fully HN-decomposability, let us review briefly the definition of the fully HN-decomposability and its minute criterion. The fully HN-decomposability condition was first introduced and systematically studied by ([GHN19]).

Definition 2.1 (fully HN-decomposablity [GHN19]). *Let $\{\mu\} \in X_*(G)/G(\bar{F}) = X_*(H)/H(\bar{F})$ with $\mu \in X_*(T)^+$, $\epsilon \in \pi_1(G)_\Gamma = \pi_1(H)_\Gamma$, and $\delta \in X_*(A)_\mathbb{Q}^+ = \mathcal{N}(H) = \mathcal{N}(G)$.*

(1) *Suppose $[b] \in A(G,\mu)$ (resp. $[b] \in B(G,\epsilon,\delta)$), and view its Newton vector $[\nu_b]$ as an element in $X(A)_\mathbb{Q}^+$. We say that the triple (G,μ,b) (resp. (G,δ,b)) is Hodge-Newton decomposable, or HN-decomposable for short, if there exists a standard proper Levi subgroup $M \subsetneq H$ such that:*
- *the centralizer of $[\nu_b]$ is contained in M; and*
- *$\mu^\diamond - [\nu_b] \in \langle \Phi_{0,M}^\vee \rangle_\mathbb{Q}$ (resp. $\delta - [\nu_b] \in \langle \Phi_{0,M}^\vee \rangle_\mathbb{Q}$).*

Moreover, for a HN-decomposable triple (G,μ,b) (resp. (G,δ,b), we say that the triple is HN-decomposable with respect to a standard Levi subgroup M in H if M is the unique minimal Levi subgroup satisfying the above conditions.

(2) *We say that the generalized Kottwitz set $B(G,\epsilon,\delta)$ is fully HN-decomposable if for any non-basic $[b] \in B(G,\epsilon,\delta)$, the triple (G,δ,b) is HN-decomposable. We say that the pair (G,μ) is fully HN-decomposable if so is $B(G,\mu)$.*

Remark 2.2.   (1) *The generalized Kottwitz set $B(G,\epsilon,\delta)$ is fully HN-decomposable if and only if the corresponding generalized Kottwitz set $B(H^{ad}, \epsilon^{ad} + \xi, \delta^{ad})$ for H^{ad} is fully HN-decomposable.*

(2) *Suppose G quasi-split. If (G,μ,b) (resp. (G,δ,b)) is HN-decomposable with respect to M. Let b_M be a reduction of b to M, then (M,μ,b) (resp. (M,δ,b)) is HN-indecomposable.*

For $\beta \in \Delta$, let $w_\beta \in \langle \Phi \rangle_\mathbb{Q}$ be the corresponding fundamental weight. For $\alpha \in \Delta_0$, let

$$\tilde{\omega}_\alpha = \sum_{\beta \in \Delta \ s.t. \ \beta|_A = \alpha} w_\beta \in X^*(T)_\mathbb{Q}^\Gamma = X^*(A)_\mathbb{Q}.$$

In particular, for $\gamma \in \Delta$ we have

$$\langle \gamma^{\vee,\diamond}, \tilde{\omega}_\alpha \rangle = \begin{cases} 0, & \text{if } \gamma|_A \neq \alpha; \\ 1, & \text{otherwise.} \end{cases}$$

Here $\gamma^{\vee,\diamond}$ denotes the Galois average of $\gamma^\vee$ in $X_*(T)$. In particular, for the element $\xi \in \langle \Phi \rangle^\vee$ above, the fractional part

$$\{\langle \xi^\diamond, \tilde{\omega}_\alpha \rangle\} \in [0,1[$$

of $\langle \xi^\diamond, \tilde{\omega}_\alpha \rangle$ only depends on its $\langle \Phi^\vee \rangle$-coset in $\pi_1(H^{ad}) = \langle \Phi \rangle^\vee / \langle \Phi^\vee \rangle$.

Proposition 2.3 (Minute criterion for the fully HN-decomposability, [GHN19, Theorem 3.3], [CFS21, Proposition 4.12]). *The set $B(G,\mu)$ is fully HN-decomposable if and only if for any $\alpha \in \Delta_0$, $\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle + \{\langle \xi^\diamond, \tilde{\omega}_\alpha \rangle\} \leq 1$.*

Now we are ready to give the definition of the weakly fully HN-decomposability.

Definition 2.4 (Weakly fully HN-decomposability).

(1) *Let $\epsilon \in \pi_1(G)_\Gamma$, and $\delta \in X_*(A)_\mathbb{Q}^+ = \mathcal{N}(G)$. We say that the generalized Kottwitz set $B(G,\epsilon,\delta)$ is weakly fully HN-decomposable if for every non-basic $[b'] \in B(G,\epsilon,\delta)$, either the triple (G,δ,b') is HN-decomposable or $[b]$ does not have reduction to $\text{Cent}_H(\nu_{b'})$, where $[b]$ is the basic element in $B(G,\epsilon,\delta)$.*

(2) *We say that the pair (G,μ) is weakly fully HN-decomposable if the Kottwitz set $B(G,\mu)$ is weakly fully HN-decomposable.*

Remark 2.5.   (1) *The generalized Kottwitz set $B(G,\epsilon,\delta)$ is weakly fully HN-decomposable if and only if so is $B(H^{ad}, \epsilon^{ad} + \xi, \delta^{ad})$.*

(2) *If the generalized Kottwitz set $B(G,\epsilon,\delta)$ is fully HN-decomposable, then it is weakly fully HN-decomposable.*

(3) *By the uniqueness of superbasic Levi subgroup (cf. [Nie22] Lemma 1.5), $[b]$ has a reduction to $\mathrm{Cent}_H(\nu_{b'})$ if and only if $[b]$ has a reduction to all the maximal proper standard Levi subgroups of H containing $\mathrm{Cent}_H(\nu_{b'})$.*

Suppose that $G = H$ is quasi-split. In the following, we will give a criterion when $[b]$ has reduction to a maximal proper standard Levi subgroup. Recall that for $\alpha \in \Delta_0$, M_α is the standard Levi subgroup of H with $\Delta_{M_\alpha,0} = \Delta_0 \backslash \{\alpha\}$.

Lemma 2.6. *Suppose that $G = H$ is quasi-split. Let $\mu \in X_*(T)^+$ and $[b] \in B(G)_{basic}$ such that $\kappa_G(b) = \mu^\#$. For any $\alpha \in \Delta_0$, the element $[b] \in B(G)$ has a reduction to M_α if and only if $\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$.*

Proof. We may assume that the identity component of the center Z of G is trivial. To see this, set $G' = G/Z^0$. As Z^0 is connected, we obtain from [Kot85, 1.9] an exact sequence of pointed sets

$$B(Z^0) \longrightarrow B(G) \longrightarrow B(G') \longrightarrow 0.$$

Let $M'_\alpha = M_\alpha/Z^0$ be the corresponding standard Levi subgroup of G'. We have a similar exact sequence of pointed sets

$$B(Z^0) \longrightarrow B(M_\alpha) \longrightarrow B(M'_\alpha) \longrightarrow 0.$$

As a result, one checks that the square below is cartesian

$$
\begin{array}{ccc}
B(G) & \longrightarrow & B(G') \\
\uparrow & & \uparrow \\
B(M_\alpha) & \longrightarrow & B(M'_\alpha)
\end{array}
$$

Therefore, $[b] \in B(G)$ has a reduction to M_α precisely when its image in $B(G')$ has a reduction to M'_α. Hence, for the remaining part of the proof, we assume that the center of G is finite. In particular, since $[b] \in B(G)_{basic}$, its Newton vector ν_b is trivial.

We next claim that $[b]$ has a reduction to M_α if and only if there exists a torsion element in $\pi_1(M_\alpha)_\Gamma$ which is mapped to $\kappa_G(b)$ via $\pi_1(M_\alpha)_\Gamma \to \pi_1(G)_\Gamma$. If $b_\alpha \in B(M_\alpha)$ is a reduction of b to M_α, the image of $[\nu_{b_\alpha}] \in \mathcal{N}(M_\alpha)$ in $\mathcal{N}(G)$ is $[\nu_b]$, hence is trivial. Therefore, ν_{b_α} itself is trivial, and $\kappa_{M_\alpha}(b_\alpha) \in \pi_1(M_\alpha)_\Gamma$ is a torsion element, which is sent to $\kappa_G(b) = \mu^\#$. Conversely, if there exists $a \in \pi_1(M_\alpha)_{\Gamma,tor}$ in the preimage of $\kappa_G(b) = \mu^\#$ via $\pi_1(M_\alpha)_\Gamma \to \pi_1(G)_\Gamma$, it corresponds to a unique element in $H^1(F, M_\alpha)$ via the composed map below (see Remark 1.2)

$$H^1(F, M_\alpha) \longrightarrow B(M_\alpha) \xrightarrow{\kappa_{M_\alpha}} \pi_1(M_\alpha)_\Gamma.$$

The latter gives rise to an element of $B(M_\alpha)$, which is a reduction of b to M_α, proving our claim.

Now, write $\mu = \sum_{\beta \in \Delta} c_\beta \beta^\vee \in X_*(T)$ with $c_\beta \in \mathbb{Q}$. The preimage of $\mu^\sharp$ via $\pi_1(M_\alpha)_\Gamma \to \pi_1(G)_\Gamma$ are the images in $\pi_1(M_\alpha)_\Gamma$ of the elements in $X_*(T)$ of the form

$$\sum_{\beta \in \Delta} c_\beta \beta^\vee + \sum_{\substack{\beta \in \Delta \\ \beta|_A = \alpha}} \lambda_\beta \beta^\vee,$$

with $\lambda_\beta \in \mathbb{Z}$ for any $\beta \in \Delta$ such that $\beta|_A = \alpha$. On the other hand, the image in $\pi_1(M_\alpha)_\Gamma$ of such an element is a torsion if and only if

$$\sum_{\substack{\beta \in \Delta \\ \beta|_A = \alpha}} (\lambda_\beta + c_\beta) = 0.$$

Therefore, there exists a torsion element in $\pi_1(M_\alpha)_\Gamma$ which is mapped to $\kappa_G(b) = \mu^\sharp$ via the natural map $\pi_1(M_\alpha)_\Gamma \to \pi_1(G)_\Gamma$ if and only if

$$\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle = \sum_{\substack{\beta \in \Delta \\ \beta|_A = \alpha}} c_\beta \in \mathbb{Z},$$

as required. $\qquad\square$

Lemma 2.7. *Suppose that G is quasi-split. Let $\mu \in X_*(T)^+$ and let $[b] \in B(G)_{basic}$ such that $\kappa_G(b) = \mu^\#$. Then for any $\alpha \in \Delta_0$, the element $[b]$ has a reduction to M_α if and only if $[b]$ has a reduction to M_{α^*}, where $\alpha^* = -w_0\alpha$.*

Proof. By Lemma 2.6, it suffices to show

$$\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle + \langle \mu^\diamond, \tilde{\omega}_{\alpha^*} \rangle \in \mathbb{Z}.$$

Indeed, the involution $*$ is equal to -1 on $\pi_1(G)_\Gamma$ since $(w_0\mu)^\sharp = \mu^\sharp$. As the fractional part of $\langle \mu^\diamond, \tilde{\omega}_{\alpha*} \rangle = \langle (-w_0\mu)^\diamond, \tilde{\omega}_\alpha \rangle$ only depends on the image $(-w_0\mu)^\sharp$, it follows that

$$\langle \mu^\diamond, \tilde{\omega}_\alpha \rangle + \langle \mu^\diamond, \tilde{\omega}_{\alpha^*} \rangle \in \langle \mu^\diamond, \tilde{\omega}_\alpha \rangle + \langle -\mu^\diamond, \tilde{\omega}_\alpha \rangle + \mathbb{Z} = \mathbb{Z}.$$

This completes the proof of our lemma. $\qquad\square$

For the weakly fully HN-decomposibility, we have a similar minute criterion as that for the fully HN-decomposability (Proposition 2.3). To see this, recall that, by Proposition 1.1, the generalized Kottwitz set $B(G, \epsilon, \delta)$ can be viewed as a subset of $\mathcal{N}(G)$ through the Newton map ν_G. Hence we can describe the elements in $B(G, \epsilon, \delta)$ in terms of their Newton vectors in $X_*(A)_\mathbb{Q}^+ = \mathcal{N}(G)$.

Proposition 2.8 ([CFS21], Proposition 4.6, Corollary 4.7 and Corollary 4.8). *Let $\epsilon \in \pi_1(G)_\Gamma$ and $\delta \in X_*(A)_\mathbb{Q}^+$. Suppose $\epsilon = \mu_1^\sharp$ with $\mu_1 \in X_*(T)^+$ not necessarily minuscule. Then as a subset of $\mathcal{N}(G)$, the generalized Kottwitz set $B(G, \epsilon, \delta)$ consists of the vectors $v \in X_*(A)_\mathbb{Q}^+$ such that*

(1) *$\delta - v \in \langle \Phi_0^\vee \rangle_\mathbb{Q}$;*
(2) *for all $\alpha \in \Delta_0$ with $\langle v, \alpha \rangle \neq 0$, one has $\langle \delta - v, \tilde{\omega}_\alpha \rangle \geq 0$, and $\langle \mu_1^\diamond + \xi^\diamond - v, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$.*

In particular, let $[b] \in B(G, \mu_1)$ be the basic element, then

(1) *the Kottwitz set $B(G, \mu_1)$ consists of the vectors $v \in X_*(A)_\mathbb{Q}^+$ such that*
   (a) *$\mu_1^\diamond - v \in \langle \Phi_0^\vee \rangle_\mathbb{Q}$;*
   (b) *for all $\alpha \in \Delta_0$ with $\langle v, \alpha \rangle \neq 0$, one has $\langle \mu_1^\diamond - v, \tilde{\omega}_\alpha \rangle \geq 0$, and $\langle \mu_1^\diamond + \xi^\diamond - v, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$.*
(2) *the Kottwitz set $B(G, 0, \nu_b - \mu_1^\diamond)$ consists of $v \in X_*(A)_\mathbb{Q}^+$ such that*
   (a) *$v \in \langle \Phi_0^\vee \rangle_\mathbb{Q}$;*
   (b) *for all $\alpha \in \Delta_0$ with $\langle v, \alpha \rangle \neq 0$, one has $\langle \nu_b - w_0\mu_1^\diamond - v, \tilde{\omega}_\alpha \rangle \geq 0$ and $\langle v - \xi^\diamond, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$.*

As a direct application of the previous proposition, we get the following corollary.

Corollary 2.9. *Suppose $G = H$ is quasi-split. Then the bijection*

$$X_*(A)_\mathbb{Q}^+ \simeq X_*(A)_\mathbb{Q}^+$$
$$v \mapsto v^* := -w_0 v$$

induces a bijection between generalized Kottwitz sets

$$B(G, \epsilon, \delta) \simeq B(G, -\epsilon, -w_0\delta).$$

Remark 2.10. *If G is not quasi-split, then in general, $B(G, \epsilon, \delta)$ is not in bijection with $B(G, -\epsilon, -w_0\delta)$. For example, let $H = \mathrm{PGL}_6$, $\mu = (1^{(2)}, 0^{(4)})$ and $\xi = \mu^{\sharp}$. Then*

$$B(G, \mu) \simeq B(H, \mu^{\sharp} + \xi^{\sharp}, \mu)$$

$$= \left\{ \left(\frac{2}{3}, \frac{4}{15}^{(5)}\right), \left(\frac{2}{3}, \frac{1}{3}^{(3)}, \frac{1}{6}^{(2)}\right), \left(\frac{2}{3}^{(2)}, \frac{1}{6}^{(4)}\right), \left(\frac{2}{3}^{(3)}, 0^{(3)}\right), \left(\frac{5}{12}^{(4)}, \frac{1}{6}^{(2)}\right), \left(\frac{1}{3}^{(6)}\right) \right\}$$

consists of 6 elements, while

$$B(G, -\mu) \simeq B(H, 0, -\mu)$$

$$= \left\{ \left(\frac{2}{3}^{(6)}\right), \left(1^{(3)}, \frac{1}{3}^{(3)}\right), \left(1^{(3)}, \frac{2}{3}, \frac{1}{6}^{(2)}\right), \left(\frac{11}{12}^{(4)}, \frac{1}{6}^{(2)}\right) \right\}$$

consists of 4 elements.

Proposition 2.11 (Minute criterion for weakly fully HN-decomposability). *Let $\mu \in X_*(T)^+$. The pair (G, μ) is weakly fully HN-decomposable if for any $\alpha \in \Delta_0$ with $\langle \mu^{\diamond} + \xi^{\diamond}, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$, we have*

$$(2.1.2) \qquad\qquad \langle \mu^{\diamond}, \tilde{\omega}_\alpha \rangle + \{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\} \le 1.$$

Proof. Without loss of generality, assume $G = G^{ad}$ and thus H are adjoint. Recall that there is a natural identification between the generalized Kottwitz set $B(G, \mu) = B(G, \mu^{\#}, \mu^{\diamond})$ for G, and the generalized Kottwitz set $B(H, \mu^{\#} + \xi^{\#}, \mu^{\diamond})$ for H.

Necessity. Suppose that $\langle \mu^{\diamond}, \tilde{\omega}_\alpha \rangle + \{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\} > 1$ is an integer for some $\alpha \in \Delta_0$. Let $v \in \langle \Phi_0^{\vee} \rangle_{\mathbb{Q}} = X_*(A)_{\mathbb{Q}}$ such that $\langle v, \alpha' \rangle = 0$ for all $\alpha' \in \Delta_0 \setminus \{\alpha\}$ and

$$\langle v, \tilde{\omega}_\alpha \rangle = \langle \mu^{\diamond}, \tilde{\omega}_\alpha \rangle + \{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\} - 1 \in \mathbb{Z}_{>0}.$$

As $\tilde{\omega}_\alpha \in \mathbb{Q}_{\ge 0} \Delta_0$, v is dominant. Moreover, $\langle \mu^{\diamond} - v, \omega_\alpha \rangle = 1 - \{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\} \ge 0$, and

$$\langle \mu^{\diamond} + \xi^{\diamond} - v, \tilde{\omega}_\alpha \rangle = 1 + \langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle - \{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\} \in \mathbb{Z}$$

So by Proposition 2.8, there exists $[b'] \in B(G, \mu)$ such that $\nu_{b'} = v$. Since $\langle v, \tilde{\omega}_\alpha \rangle = \langle \nu_{b'}, \tilde{\omega}_\alpha \rangle \ne 0$, $v \ne 0$ and the element $[b']$ is not basic. Moreover, M_α is the centralizer of $v = \nu_{b'}$ in H, and

$$\langle \mu^{\diamond} - v, \tilde{\omega}_\alpha \rangle = 1 - \{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\} > 0.$$

Hence $\mu^{\diamond} - v \notin \langle \Phi_{0,M_\alpha}^{\vee} \rangle_{\mathbb{Q}}$, and (G, μ, b') is not HN-decomposable. Moreover, as

$$\langle \mu^{\diamond} + \xi^{\diamond}, \tilde{\omega}_\alpha \rangle \in \mathbb{Z},$$

according to Lemma 2.6, the basic element $[b]$ in $B(G, \mu) = B(H, \mu^{\#} + \xi, \mu^{\diamond})$ has a reduction to $\mathrm{Cent}_H(\nu_{b'}) = M_\alpha$. This contradicts the weakly fully HN-decomposability condition.

Sufficiency. Suppose that $[b'] \in B(G, \mu) = B(H, \mu^{\#} + \xi, \mu^{\diamond})$ is a non-basic element which is HN-indecomposable with respect to μ. We want to show that $[b]$ does not have reduction to the maximal standard Levi $M_\alpha \supset \mathrm{Cent}_H([\nu_{b'}])$ of H for some $\alpha \in \Delta_0$, or equivalently, that for some $\langle \alpha, \nu_{b'} \rangle \ne 0$ we have $\langle \mu^{\diamond} + \xi^{\diamond}, \tilde{\omega}_\alpha \rangle \notin \mathbb{Z}$: see Lemma 2.6. Take $\alpha \in \Delta_0$ with $\langle \nu_{b'}, \alpha \rangle \ne 0$, such that $\langle \mu^{\diamond} + \xi^{\diamond}, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$. Then, M_α contains the centralizer of $\nu_{b'}$ and $\langle \nu_{b'}, \tilde{\omega}_\alpha \rangle > 0$. Moreover, the inequality (2.1.2) gives $\langle \mu^{\diamond}, \tilde{\omega}_\alpha \rangle + \{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\} \le 1$. As $[b'] \in B(G, \mu)$, again by Proposition 2.8, we find

$$\underbrace{\{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\}}_{\ge 0} + \underbrace{\langle \mu^{\diamond}, \tilde{\omega}_\alpha \rangle - \langle \nu_{b'}, \tilde{\omega}_\alpha \rangle}_{\ge 0} = \underbrace{\{\langle \xi^{\diamond}, \tilde{\omega}_\alpha \rangle\} + \langle \mu^{\diamond}, \tilde{\omega}_\alpha \rangle}_{\le 1} - \underbrace{\langle \nu_{b'}, \tilde{\omega}_\alpha \rangle}_{>0} \in \mathbb{Z}.$$

This implies that $\langle \mu^{\diamond} - \nu_{b'}, \tilde{\omega}_\alpha \rangle = 0$ since $\nu_{b'} \preceq \mu^{\diamond}$. In other words, $\mu^{\diamond} - \nu_{b'} \in \langle \Phi_{0,M_\alpha}^{\vee} \rangle_{\mathbb{Q}}$. This contradicts to the condition that (G, μ, b') is HN-indecomposable. $\qquad\square$

Remark 2.12. *Since $-w_0$ induces a bijection on Δ_0 and $w_0\xi - \xi \in \langle\Phi^\vee\rangle$, the inequality (2.1.2) given by the minute criterion (Proposition 2.11) is equivalent to the following: for $\alpha \in \Delta_0$ with $\langle -w_0\mu^\diamond, \tilde{\omega}_\alpha\rangle + \{\langle -\xi^\diamond, \tilde{\omega}_\alpha\rangle\} \in \mathbb{Z}$, we have*

$$\langle -w_0\mu^\diamond, \tilde{\omega}_\alpha\rangle + \{\langle -\xi^\diamond, \tilde{\omega}_\alpha\rangle\} \leq 1.$$

In particular, when G is quasi-split, (G, μ) is weakly fully HN-decomposable if and only if

$$\langle -w_0\mu^\diamond, \tilde{\omega}_\alpha\rangle \notin \mathbb{Z}_{>1}, \quad \forall\, \alpha \in \Delta_0.$$

So (G, μ) is weakly fully HN-decomposable if and only if this is the case for (G, μ^{-1}).

Remark 2.13. *Let $G = \mathrm{GL}_n$, and μ a minuscule cocharacter of G. Up to twist by a central cocharacter, we assume further that $\mu = (1^{(r)}, 0^{(n-r)})$. Then using the minute criterion, it is easy to see that, the pair (G, μ) is weakly fully HN-decomposable if it is fully HN-decomposable, or is one of the following forms:*

(1) *(weakly accessible case, cf. Definition 3.2 below), μ is central or $\mu = (1^{(r)}, 0^{(n-r)})$ with $\gcd(n, r) = 1$;*
(2) *n even and $\mu = (1, 1, 0, \cdots, 0)$;*
(3) *n even and $\mu = (1, \cdots, 1, 0, 0)$;*
(4) *$n = 6$ and $\mu = (1, 1, 1, 0, 0, 0)$.*

In the next subsection, we shall give the complete classification of weakly fully HN-decomposable pairs when the group G is absolutely simple and adjoint.

2.2. **Classification of weakly fully HN-decomposable pairs.**

Notations are the same as the last subsection. By definition, we know that the fully HN-decomposability condition (Definition 2.4) implies the weakly fully HN-decomposable case (Definition 2.1). Görtz, He and Nie classify in [GHN19] the fully HN-decomposable pairs. In this section, we classify the weakly fully HN-decomposable pairs (G, μ) when G is absolutely simple and adjoint.

The main tool for us to do the classification is the minute criterion for weakly fully HN-decomposability (Proposition 2.11): the pair (G, μ) is weakly fully HN-decomposable if and only if for any $\alpha \in \Delta_0$ with $\langle \mu^\diamond + \xi^\diamond, \tilde{\omega}_\alpha\rangle \in \mathbb{Z}$, we have

$$\langle \mu^\diamond, \tilde{\omega}_\alpha\rangle + \{\langle \xi^\diamond, \tilde{\omega}_\alpha\rangle\} \leq 1.$$

Suppose G and hence H are absolutely simple and adjoint. Note that in the minute criterion, whether (G, μ) is weakly fully HN-decomposable only depends on the quadruple $(H, |\mathrm{Im}\Gamma|, \mu, \xi)$, where $|\mathrm{Im}\Gamma|$ is the order of the image of the natural homomorphism $\Gamma \to \mathrm{Aut}(\Phi, \Delta)$ for H. So in order to classify the weakly fully HN-decomposable pairs (G, μ), it suffices to classify the possible quadruples $(H, |\mathrm{Im}\Gamma|, \mu, \xi)$.

We will write the type of Dynkin diagram to represent H. Here the labeling of the Dynkin diagram is as in [Bou81]. Moreover, as H is adjoint, there is a bijection

$$\langle\Phi\rangle^\vee / \langle\Phi^\vee\rangle \simeq \{\text{minuscule dominant cocharacters of } T\}.$$

For any $\xi \in (\langle\Phi\rangle^\vee / \langle\Phi^\vee\rangle)_\Gamma$, we will take a minuscule dominant cocharacter of T as a representative of ξ via this identification. By convention, $\omega_0^\vee := 0$ corresponds to $\xi = 0$.

Proposition 2.14. *Suppose G is absolutely simple and adjoint. Let μ be a minuscule cocharacter and $\mu \neq 0$. Then (G, μ) is weakly fully HN-decomposable if and only if (G, μ) is fully HN-decomposable or the associated quadruple $(H, |\mathrm{Im}\Gamma|, \mu, \xi)$ is one of the following up to isomorphism:*

H	$|\mathrm{Im}\Gamma|$	μ	ξ
A_n	1	$\omega_i^\vee$	$\omega_{i'}^\vee$, s.t. $\gcd(i+i', n+1) = 1$
A_n	1	$\omega_1^\vee$	arbitrary
A_n	1	$\omega_2^\vee$	$\omega_{i'}^\vee$, s.t. $\gcd(2+i', n+1) = 2$
A_5	1	$\omega_3^\vee$	0
A_3	2	$\omega_2^\vee$	$\omega_1^\vee$
C_3	1	$\omega_3^\vee$	0
D_n	1	$\omega_1^\vee$	$\omega_n^\vee$
D_5	1	$\omega_5^\vee$	$0, \omega_1^\vee$
D_n	2	$\omega_1^\vee$	$\omega_n^\vee$
D_4	2	$\omega_4^\vee$	0

Moreover, (G, μ) is weakly accessible if and only if the quadruple is the first row in the above table.

As a corollary of this proposition, we can list all the weakly fully HN-decomposable pairs using the notation of Tits' table ([Tit79]).

Theorem 2.15. *Suppose G is absolutely simple and adjoint. Let μ be a minusucle cocharacter and $\mu \neq 0$. Then (G, μ) is weakly fully HN-decomposable if and only if (G, μ) is fully HN-decomposable or weakly accessible or (G, μ) is one of the following up to isomorphism:*

$(^dA_n, \omega_1^\vee)$ with $d|n+1$	$(^dA_n, \omega_2^\vee)$ with n odd, $d|n+1$ (*)	$(A_5, \omega_3^\vee)$	$(^4A_3, \omega_2^\vee)$
$(^2C_n, \omega_1^\vee)$	$(^2C - B_n, \omega_1^\vee)$	$(C_3, \omega_2^\vee)$	
$(^2D_n'', \omega_1^\vee)$	$(D_5, \omega_5^\vee)$	$(^2D_5', \omega_5^\vee)$	$(^2D_4, \omega_4^\vee)$

where (*) after $(^dA_n, \omega_2^\vee)$ means that not all groups of type ^dA_n are allowed, but only the ones with Frobenius acting on the $n+1$-cycle by a clockwise rotation of i'-steps with $\gcd(2+i', n+1) = 2$ are allowed.

Remark 2.16. *In [GHN19], Görtz, He and Nie give characterizations of basic affine Deligne-Lusztig varieties associated to (G, μ) when it is fully HN-decomposable. For example, under this assumption, the affine Deligne-Lusztig variety is a union of Deligne-Lusztig varieties. It will also be an interesting question to investigate the basic affine Deligne-Lusztig varieties associated to a weakly fully HN-decomposable pair. For the case $(C_3, \omega_2^\vee)$, it is studied in [Ric98]. In a joint work with Viehmann in preparation, we study the case $(A_n, \omega_2^\vee)$.*

Proof of Proposition 2.14. Note that by the minute criterion (Proposition 2.11), the condition that (H, μ) is fully HN-decomposable implies that (G, μ) is weakly fully HN-decomposable for any inner form G of H. In the following, we only consider (H, μ) which is NOT fully HN-decomposable. We will discuss by the Dynkin diagram of H. Sometimes for simplicity, we write $\tilde{\omega}_i$ for $\tilde{\omega}_{\alpha_i} = \omega_i$ when H is split.

Case A_n: Note that for $1 \leq i, j \leq n$,

$$(2.2.1) \qquad \langle \omega_i^\vee, \omega_j \rangle = \min\{i, j\} - \frac{ij}{n+1}.$$

Consider

$$\theta_i(j) := \begin{cases} j - \frac{ij}{n+1}, & j \leq i \\ i - \frac{ij}{n+1}, & j > i, \end{cases}$$

as a function on $\mathbb{R} \cap [0, n+1]$. Then $\theta_i(j) = \langle \omega_i^\vee, \omega_j \rangle$ for $1 \leq i, j \leq n$. Moreover, as a function on j, $\theta_i(j)$ is strictly increasing for $j \leq i$ and strictly decreasing for $j \geq i$.

Subcase ^1A_n: $|\mathrm{Im}\Gamma| = 1$.

By (2.2.1),

$$\langle \omega_i^\vee, \tilde{\omega}_j \rangle + \{ \langle \omega_{i'}^\vee, \tilde{\omega}_j \rangle \} \equiv -\frac{(i+i')j}{n+1} \quad \mod \mathbb{Z}.$$

It follows that

$$\langle \omega_i^\vee, \tilde{\omega}_j \rangle + \{ \langle \omega_{i'}^\vee, \tilde{\omega}_j \rangle \} \in \mathbb{Z} \Longleftrightarrow n+1 | (i+i')j.$$

Suppose $(A_n, 1, \omega_i^\vee, \omega_{i'}^\vee)$ is weakly fully HN-decomposable. By the minute criterion (Proposition 2.11), it's equivalent to say

$$(2.2.2) \qquad \forall 1 \le j \le n, \frac{(i+i')j}{n+1} \in \mathbb{Z} \Longrightarrow \theta_i(j) = \langle \omega_i^\vee, \tilde{\omega}_j \rangle \le 1.$$

Obviously, this condition is satisfied if $\gcd(i+i', n+1) = 1$. Now assume $d := \gcd(i+i', n+1) > 1$. By symmetry, we may further assume $i \le \frac{n+1}{2}$. Moreover $(A_n, \omega_1^\vee)$ is fully HN-decomposable by [GHN19]. It remains to consider $2 \le i \le \frac{n+1}{2}$.

   Claim: $i = 3$ with $n=5$ or $i = 2$.

   If $i \ge 3$, then

$$\theta_i\left(\frac{n+1}{2}\right) = \frac{i}{2} > 1$$

$$\theta_i(2) = 2 - \frac{2i}{n+1} \ge 1$$

where the second inequality becomes an equality if and only if $i = \frac{n+1}{2}$. The weakly fully HN-decomposability condition (2.2.2) for $(A_n, 1, \omega_i^\vee, \omega_{i'}^\vee)$ is equivalent to

$$(2.2.3) \qquad \theta_i\left(\frac{n+1}{d}r\right) \le 1, \ \forall 1 \le r \le d-1.$$

If $i \ne \frac{n+1}{2}$, then $\theta_i(j) > 1$ for all $j \in [2, \frac{n+1}{2}]$. On the other hand, note that

$$\underbrace{\{\frac{n+1}{d}r | r \in \mathbb{N} \cap [1, d-1]\}}_{\text{value of } \theta_i \le 1 \text{ by } (2.2.3)} \cap \underbrace{[2, \frac{n+1}{2}]}_{\text{value of } \theta_i > 1} \ne \emptyset$$

which leads to a contradiction.

   If $i = \frac{n+1}{2}$, then $\theta_i(j) > 1$ for all $j \in]2, \frac{n+1}{2}]$. Again by (2.2.3),

$$\{\frac{n+1}{d}r | r \in \mathbb{N} \cap [1, d-1]\} \cap [2, \frac{n+1}{2}] = \{2\}$$

which implies that $\frac{n+1}{d} = 2$ and $4 > \frac{n+1}{2}$. Hence $n = 5$ and $i = 3$ as we ignore the fully HN-decomposable pairs. Now the Claim follows.

   For $i = 2$, $\theta_2(j) = 2 - \frac{2j}{n+1}$ for $j \ge 2$. By (2.2.3), it follows that

$$j \in \{\frac{n+1}{d}r | r \in \mathbb{N} \cap [1, d-1]\} \Longrightarrow \theta_2(j) \le 1 \Longleftrightarrow j \ge \frac{n+1}{2} \text{ or } j = 1.$$

This implies $d = 2$ or $n = 3$.

   For $i = 3$ with $n = 5$, $\theta_3(j) \le 1 \Leftrightarrow j \ne 3$. Hence (2.2.3) implies that $\frac{n+1}{d} \nmid 3$. It follows that $d = 3$.

   Subcase ^2A_n: $|\text{Im}\Gamma| = 2$. If n is even, then $\langle \mu, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$ for any μ and any $\alpha \in \Delta_0$. Hence (G, μ) is weakly fully HN-decomposable if and only if it's fully HN-decomposable.

   If $n = 2m - 1$ is odd. Note that

$$\langle \omega_i^\vee, \tilde{\omega}_{\alpha_j} \rangle \in \mathbb{Z}, \ \forall 1 \le j \le m-1, \forall 1 \le i \le m$$

where $\tilde{\omega}_{\alpha_j} = \omega_j + \omega_{n-j} = (1^{(j)}, 0^{(2m-2j)}, -1^{(j)})$. Suppose $(A_n, 2, \omega_i^\vee, \omega_{i'}^\vee)$ is weakly fully HN-decomposable. It follows that

$$\langle \omega_i^\vee, \tilde{\omega}_{\alpha_j} \rangle \le 1, \ \forall 1 \le j \le m-1.$$

It follows that $m = 2$ if $i = 2$. We can easily verify $(A_3, 2, \omega_2^\vee, \omega_1^\vee)$ is the only new case up to isomorphism.

Case B_n: $\mu = \omega_1^\vee$ is the only non-trivial minuscule cocharacter. We can check that the weakly fully HN-decomposable pairs are all fully HN-decomposable.

Case C_n: $\mu = \omega_n^\vee$ is the only non-trivial minuscule cocharacter. Note that

$$\langle \omega_n^\vee, \tilde{\omega}_j \rangle = \frac{j}{2}, \forall 1 \leq j \leq n.$$

So $(C_3, 1, \omega_3^\vee, 0)$ is the only new case.

Case D_n: $\omega_1^\vee$, $\omega_{n-1}^\vee$ and $\omega_n^\vee$ are the non-trivial minuscule cocharacters. Note that

$$\langle \omega_1^\vee, \tilde{\omega}_j \rangle = \begin{cases} 1, & j \leq n-2 \\ \frac{1}{2}, & j = n-1, n \end{cases}, \quad \langle \omega_n^\vee, \tilde{\omega}_j \rangle = \begin{cases} \frac{j}{2}, & j \leq n-2 \\ \frac{n-2}{4}, & j = n-1 \\ \frac{n}{4}, & j = n. \end{cases}$$

Subcase ^1D_n: $|\text{Im}(\Gamma)| = 1$. If $\mu = \omega_1^\vee$, then ξ can be taken arbitrarily. If $n = 5$, we can also take $\mu = \omega_n^\vee$ with $\xi = 0$ or $\omega_1^\vee$.

Subcase ^2D_n: $|\text{Im}(\Gamma)| = 2$. Note that

$$\langle \omega_1^\vee, \tilde{\omega}_{\alpha_j} \rangle = 1, \langle \omega_n^\vee, \tilde{\omega}_{\alpha_j} \rangle = \frac{j}{2}, \ \forall 1 \leq j \leq n-1.$$

It follows that if $\mu = \omega_1^\vee$, then ξ can be chosen arbitrarily. If $\mu = \omega_n^\vee$, then $n = 4$ and $\xi = 0$ or $\omega_1^\vee$ (which in fact have the same image in $(\langle \Phi \rangle^\vee / \langle \Phi^\vee \rangle)_\Gamma$)

Subcase $^3D_4$: $|\text{Im}(\Gamma)| = 3$.

$$\langle \omega_1^\vee, \tilde{\omega}_{\alpha_i} \rangle = \begin{cases} 2, & i = 1, \\ 1 & i = 2. \end{cases}$$

No new case is possible.

Case E_6, E_7: Same computations show there are NO weakly fully HN-decomposable but not fully HN-decomposable cases.

Case E_8, F_4, G_2: There is no non-trivial minuscule cocharacter.

The last assertion of the proposition follows directly from [Rap18, A.13].

$\square$

3. Maximal weakly admissible locus

In this section, we address the question when the weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$ is maximal (in the sense that it is a union of Newton strata, see Definition 3.2). We shall show in Theorem 3.5 that the previous condition is fulfilled if and only if the pair (G, μ) is weakly fully HN-decomposable, which is further equivalent to the fact that the Newton stratification is finer than the Harder-Narasimhan stratification.

Recall that there is a Newton decomposition for the flag variety $\mathcal{F}(G, \mu)$:

$$\mathcal{F}(G, \mu) = \coprod_{[b'] \in B(G, 0, \nu_b - \mu^\diamond)} \mathcal{F}(G, \mu, b)^{[b']}$$

where $\mathcal{F}(G, \mu, b)^{[b']}$ is a locally closed adic subspace of $\mathcal{F}(G, \mu)$, such that

$$\mathcal{F}(G, \mu, b)^{[b']}(C) := \{x \in \mathcal{F}(G, \mu)(C) | \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}$$

for any complete algebraically closed field extension $C/\bar{F}$. Moreover, $\mathcal{F}(G, \mu, b)^{[b']}$ defines a locally spatial subdiamond of $\mathcal{F}(G, \mu)^\diamond$.

Proposition 3.1. *Let $\{\mu\}$ be the geometric conjugacy class of a minuscule cocharacter μ, and $[b] \in A(G, \mu)$ which is basic. Suppose $[b'] \in B(G, 0, \nu_b - \mu^\diamond)$.*

(1) ([CFS21, Theorem 5.1]) *If $[b']$ is HN-decomposable with respect to $\nu_b - \mu^\diamond$, then*
$$\mathcal{F}(G,\mu,b)^{wa} \cap \mathcal{F}(G,\mu,b)^{[b']} = \emptyset;$$

(2) ([Vie, Theorem 1.3]) *If $[b']$ is HN-indecomposable with respect to $\nu_b - \mu^\diamond$, then*
$$\mathcal{F}(G,\mu,b)^{wa} \cap \mathcal{F}(G,\mu,b)^{[b']} \neq \emptyset.$$

As a corollary, we deduce

(3.0.1) $$\mathcal{F}(G,\mu,b)^{wa} \subseteq \coprod_{\substack{[b']\in B(G,0,\nu_b-\mu^\diamond) \text{ such that} \\ (b',\nu_b-\mu^\diamond) \text{ is HN-indecomposable}}} \mathcal{F}(G,\mu,b)^{[b']} \subseteq \mathcal{F}(G,\mu).$$

Definition 3.2. *Let $\{\mu\}$ a geometric conjugacy class of a cocharacter of G. Let $[b] \in A(G,\mu)$, so that the period domain $\mathcal{F}(G,\mu,b)^{wa}$ is non-empty.*

(1) ([Rap18, Definition A1]) *We say that the triple (G,b,μ) is* weakly accessible *if*
$$\mathcal{F}(G,\mu,b)^{wa} = \mathcal{F}(G,\mu).$$
We say that (G,μ) is weakly accessible *if this is the case for (G,b',μ), where $[b'] \in B(G,\mu)$ is the unique basic element.*

(2) *We say that the weakly admissible locus $\mathcal{F}(G,\mu,b)^{wa}$ is* maximal *if the first inclusion in (3.0.1) is an equality, or equivalently,*
$$\mathcal{F}(G,\mu,b)^{wa}(C) = \coprod_{\substack{[b']\in B(G,0,\nu_b-\mu^\diamond) \ s.t. \\ (b',\nu_b-\mu^\diamond) \ is \ HN\text{-}indecomposable}} \mathcal{F}(G,\mu,b)^{[b']}(C)$$
for any complete algebraically field extension $C/\bar{F}$.

If $G = \mathrm{GL}_n$, the weakly accessible pairs (G,μ) are given as in Remark 2.13 (1). In the general case, after some reduction steps ([Rap18, Lemma A.11]), the following proposition gives the complete classification of all weakly accessible triples (G,μ,b).

Proposition 3.3 ([Rap18] Proposition A.12)**.** *Suppose that (G,μ,b) defines a non-empty period domain where G is F-simple and μ is non-trivial. Then (G,μ,b) is weakly accessible if and only if the F-group J_b is anisotropic, in which case b is basic.*

Remark 3.4. *Clearly, fully HN-decomposability or weak accessibility implies the maximality of the weakly admissible locus.*

The following theorem gives several characterizations for the maximality of the weakly admissible locus $\mathcal{F}(G,\mu,b)^{wa}$, which is the first main result of our paper.

Theorem 3.5. *Suppose that $b \in G(\breve{F})$ is basic and that $\mu \in X_*(T)^+$ is minuscule. Then the following assertions are equivalent:*

(1) $\mathcal{F}(G,\mu,b)^{wa}$ *is maximal;*
(2) (G,μ) *is weakly fully HN-decomposable;*
(3) *the Newton stratification is finer than the Harder-Narasimhan stratification in the sense that every Harder-Narasimhan stratum is a union of some Newton strata.*

Proof. We may assume G and thus H adjoint. Indeed, this follows from the observation that, if we denote by
$$\pi : \mathcal{F}(G,\mu) \longrightarrow \mathcal{F}(G^{ad},\mu^{ad})$$
the natural map, then the preimage of a Newton stratum (resp. Harder-Narasimhan stratification) for the triple (G^{ad},μ^{ad},b^{ad}) is the corresponding stratum for the triple (G,μ,b). Therefore, for the rest of the proof, G is supposed to be adjoint. In particular, $\nu_b = 0$. Furthermore, we suppose $\mu \neq 0$: otherwise our theorem is trivially true.

Recall that we have the identification (2.1.1) between $B(G)$ and $B(H)$ induced by the class

$$\xi \in H^1(F, H) \simeq \pi_1(H)_{\Gamma, tor}$$

of G as an inner form of H, under which $B(G, \mu)$ is identified with $B(H, \mu^\# + \xi, \mu^\diamond)$. Moreover, using a fixed inner twisting

$$G_{\breve{F}} \xrightarrow{\sim} H_{\breve{F}},$$

we have an identification

$$\mathcal{F}(G, \mu) \xrightarrow{\sim} \mathcal{F}(H, \mu)$$

of flag varieties over $\breve{E}$, under which the Newton stratification (resp. the Harder-Narasimhan stratification) on $\mathcal{F}(G, \mu)$ for the triple (G, μ, b) is identified with the corresponding stratification on $\mathcal{F}(H, \mu)$ for the triple (H, μ, b^H), where $[b^H] \in B(H)$ is the image of $[b]$ via $B(G) \simeq B(H)$. In the following, we will deal with (H, μ, b^H) instead of (G, μ, b).

Moreover, we may assume that H is simple. Indeed, if $H = H_1 \times H_2$ with $\mu = (\mu_1, \mu_2)$ and $b^H = (b_1^H, b_2^H)$, we have a natural isomorphism

$$\mathcal{F}(H, \mu, b^H) \simeq \mathcal{F}(H_1, \mu_1, b_1^H) \times \mathcal{F}(H_2, \mu_2, b_2^H),$$

under which the Newton stratification and Harder-Narasimhan stratification on both sides are compatible.

$(1) \Rightarrow (2)$. Suppose that (G, μ) is not weakly fully HN-decomposable. By the minute criterion (Proposition 2.11 and Remark 2.12), there exists $\alpha \in \Delta_0$ such that

$$\langle \mu^\diamond, \tilde{\omega}_{-w_0\alpha} \rangle + \{\langle \xi^\diamond, \tilde{\omega}_{-w_0\alpha} \rangle\} = \langle -w_0\mu^\diamond, \tilde{\omega}_\alpha \rangle + \{\langle -w_0\xi^\diamond, \tilde{\omega}_\alpha \rangle\} \in \mathbb{Z}_{>1}.$$

We want to show that there exists a point in a HN-indecomposable Newton stratum that is not weakly admissible. Let $\alpha^* = -w_0\alpha$ and $M = M_{\alpha^*}$ the standard Levi such that $\Delta_{0,M} = \Delta_0 \setminus \{\alpha^*\}$. Let P be the standard parabolic subgroup of H corresponding to M. Since $\langle \mu^\diamond + \xi^\diamond, \tilde{\omega}_{\alpha^*} \rangle \in \mathbb{Z}$ and $[b^H] \in B(H, \mu^\# + \xi, \mu^\diamond)$, the element $[b^H]$ has a reduction $[b_M]$ to M (Lemma 2.6). By [Che23, Lemma 1.8],

$$\xi \in \mathrm{Im}(\pi_1(M)_{\Gamma, tor} \to \pi_1(G)_{\Gamma, tor}),$$

and there exists $\mu_1 \in W\mu \subset X_*(T)$ which is M-dominant, such that

$$[b_M] \in B(M, \mu_1^\# + \xi, \mu_1).$$

Here, ξ is view as an element in $\pi_1(M)_{\Gamma, tor}$. As $\langle \mu^\diamond, \tilde{\omega}_{\alpha^*} \rangle > 0$ while $\mu_1^\diamond = \mu_1^\diamond - \nu_{b_M} \in \langle \Phi_{M,0}^\vee \rangle_\mathbb{Q}$, it follows that $\mu \neq \mu_1$, and μ_1 is M-dominant but not H-dominant. So there exists $\beta \in \Delta$ with $\beta|_A = \alpha^*$, such that

$$\langle \mu_1, \beta \rangle < 0.$$

Therefore $\langle \mu_1, \beta \rangle = -1$ as μ and hence μ_1 are minuscule. Let $[b_M'] \in B(M)_{basic}$ such that

$$\kappa_M(b_M') = \text{image of } -\beta^\vee + \xi \text{ in } \pi_1(M)_\Gamma.$$

Then

$$\nu_{b_M'} = \mathrm{pr}_M(-\beta^\vee + \xi)^\diamond = \mathrm{pr}_M(-\beta^\vee)^\diamond,$$

with pr_M the projection of $X_*(T) = \langle \Phi_M^\vee \rangle_\mathbb{Q} \oplus \langle \Phi_M \rangle_\mathbb{Q}^\perp$ to the direct factor $\langle \Phi_M \rangle_\mathbb{Q}^\perp$. Let

$$\mu_2 = (s_\beta \mu_1)_{M\text{-dom}} = (\mu_1 + \beta^\vee)_{M\text{-dom}}.$$

We check easily that

$$[b_M'] \in B(M, (\mu_1 - \mu_2)^\# + \xi, \nu_{b_M} - \mu_2^\diamond).$$

Hence there exists a point $x_M \in \mathcal{F}(M, \mu_2)(C)$ such that $\mathcal{E}_{b'_M} \simeq \mathcal{E}_{b_M, x_M}$. Let $[b'^H]$ be the image of $[b'_M]$ in $B(H)$. It follows that

$$[b'^H] \in B(H, \xi, \nu_b - \mu^\diamond).$$

In particular, $\mathcal{F}(H, \mu, b^H)^{[b'^H]}$ is a non-empty Newton stratum of the flag variety. Let $x \in \mathcal{F}(H, \mu)(C)$ the image of x_M via the natural map $\mathcal{F}(M, \mu_2) \to \mathcal{F}(H, \mu)$. In order to show the weakly admissible locus is not maximal, it suffices to prove the following Claim.

   Claim: $(H, b'^H, -w_0\mu)$ is HN-indecomposable and $x \in \mathcal{F}(H, \mu, b^H)^{[b'^H]}(C)$ is not weakly admissible.

   As $\nu_{b'_M}$ is H-antidominant, $\nu_{b'^H} = (\nu_{b'_M})_{H\text{-dom}} = w_0\nu_{b'_M}$. It follows that (H, b'^H, μ^{-1}) is HN-indecomposable as

$$
\begin{aligned}
\langle -w_0\mu^\diamond - w_0\nu_{b'_M}, \tilde{\omega}_\alpha \rangle &= \langle -w_0\mu^\diamond, \tilde{\omega}_\alpha \rangle + \langle \nu_{b'_M}, \tilde{\omega}_{\alpha^*} \rangle \\
&= \langle -w_0\mu^\diamond, \tilde{\omega}_\alpha \rangle - 1 > 0.
\end{aligned}
$$

since H is simple. Moreover, $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$, and x is not weakly admissible by definition: for the dominant root $\chi = mw_\beta \in X^*(P/Z_H)^+$ with $m > 0$, we have

$$\deg\chi_*((\mathcal{E}_{b,x})_P) = \deg(\chi_*(\mathcal{E}_{b'_M})) = \langle -w_0\nu_{b'_M}, mw_\beta \rangle > 0,$$

since $\nu_{b'_M}$ is H-anti-dominant and non trivial.

   $\underline{(2) \Rightarrow (1)}$. Suppose that (G, μ) is weakly fully HN-decomposable. Let

$$x \in \mathcal{F}(H, \mu, b^H)^{[b'^H]}(C)$$

with $(H, b'^H, -w_0\mu)$ HN-indecomposable. So $\mathcal{E}_{b^H, x} = \mathcal{E}_{b'^H}$. We want to show that x is weakly admissible. Suppose that x is not weakly admissible, then there exist a maximal standard Levi subgroup M_α, a reduction of b^H to M_α, and thus a reduction $(\mathcal{E}_{b^H, x})_{P_\alpha}$ of $\mathcal{E}_{b^H, x} = \mathcal{E}_{b'^H}$ to the corresponding standard parabolic P_α, such that for some $\chi \in X^*(P_\alpha/Z_H)^+ = X^*(P_\alpha)^+$

$$(3.0.2) \qquad\qquad \deg\chi_*((\mathcal{E}_{b,x})_{P_\alpha}) > 0.$$

In particular, $\langle \mu^\diamond + \xi^\diamond, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}$ by Lemma 2.6. Let $v \in X_*(A)_{\mathbb{Q}}$ be the slope vector for the reduction $(\mathcal{E}_{b,x})_{P_\alpha}$ of $\mathcal{E}_{b,x} = \mathcal{E}_{b'}$ to P_α. Indeed,

$$v \in \mathrm{Hom}(X^*(P_\alpha), \mathbb{Z})^\Gamma = \mathrm{Hom}(X^*(M_\alpha^{ab}), \mathbb{Z})^\Gamma \xrightarrow{\sim} X_*(M_\alpha^{ab})^\Gamma \subseteq X_*(A_{M_\alpha})_{\mathbb{Q}} \subset X_*(A)_{\mathbb{Q}}$$

where M_α^{ab} is the cocenter of M_α and $A_{M_\alpha} \subseteq A$ is a maximal split central torus of M_α.

   In other words, let $\mathcal{E}_{b'_{M_\alpha}} := (\mathcal{E}_{b,x})_{P_\alpha} \times^{P_\alpha} M_\alpha$ with $[b'_{M_\alpha}] \in B(M_\alpha)$, then

$$[b'_{M_\alpha}] \in B(M_\alpha, \kappa_{M_\alpha}(b'_{M_\alpha}), \nu_{b'_{M_\alpha}}).$$

Moreover, as

$$v = \mathrm{pr}_{M_\alpha}(\nu_{\mathcal{E}_{b'_{M_\alpha}}})^\diamond = -\mathrm{pr}_{M_\alpha}(\nu_{b'_{M_\alpha}})^\diamond,$$

with pr_{M_α} the projection of $X_*(T) = \langle \Phi_{M_\alpha}^\vee \rangle_{\mathbb{Q}} \oplus \langle \Phi_{M_\alpha} \rangle_{\mathbb{Q}}^\perp$ to the direct factor $\langle \Phi_{M_\alpha} \rangle_{\mathbb{Q}}^\perp$, It follows that

$$-v \in B(M_\alpha, \kappa_{M_\alpha}(b'_{M_\alpha}), \nu_{b'_{M_\alpha}}).$$

   The fact that $M_\alpha = \mathrm{Cent}_G(v)$ and the inequality (3.0.2) implies

$$(3.0.3) \qquad\qquad \langle v, \tilde{\omega}_\alpha \rangle = \langle \nu_{\mathcal{E}_{b'_{M_\alpha}}}, \tilde{\omega}_\alpha \rangle > 0.$$

Hence v is H-dominant. Furthermore, as

$$\kappa_H(b'_{M_\alpha}) = -c_1^H(\mathcal{E}_{b'_{M_\alpha}} \times^{M_\alpha} H) = -c_1^H(\mathcal{E}_{b,x}) = \kappa_H(b') = \xi \in \pi_1(H)_\Gamma$$

we deduce that $-w_0 v \in B(H, \xi, (\nu_{b'_{M_\alpha}})_{H-dom})$. In particular, $-w_0 v \in B(H, \xi, -w_0 v)$. On the other hand, $v \in X_*(A)_{\mathbb{Q}}^+$ and $v \preceq \nu_{\mathcal{E}_{b,x}} = -w_0 \nu_{b'}$ by [CFS21, Theorem 1.8]. Hence,

$$-w_0 v \preceq \nu_{b'} \preceq -w_0 \mu^\diamond,$$

and then

$$-w_0 v \in B(H, \xi, -w_0 \mu^\diamond).$$

By proposition 2.8, it follows that

(3.0.4) $$\langle -w_0 v, \tilde{\omega}_{\alpha^*} \rangle + \{ \langle -\xi^\diamond, \tilde{\omega}_{\alpha^*} \rangle \} \in \mathbb{Z}_{\geq 0}$$

As (G, μ) is weakly fully HN-decomposable, by Remark 2.12, and the fact that b^H has reduction to M_{α^*}, we have

(3.0.5) $$\langle -w_0 \mu^\diamond, \tilde{\omega}_{\alpha^*} \rangle + \{ \langle -\xi^\diamond, \tilde{\omega}_{\alpha^\vee} \rangle \} = 0 \text{ or } 1$$

Moreover,

(3.0.6) $$\langle -w_0 \mu^\diamond - (-w_0 v), \tilde{\omega}_{\alpha^*} \rangle \geq \langle -w_0 \mu - \nu_{b'}, \tilde{\omega}_{\alpha^*} \rangle > 0.$$

Here we have the last inequality because the centralizer of $\nu_{b'} = -w_0 \nu_{\mathcal{E}_{b'}}$ is contained in M_{α^*} and $\nu_{b'}$ is HN-indecomposable relative to $-w_0 \mu$. Combined with (3.0.4), (3.0.5), (3.0.6), it follows that $\langle -w_0 v, \tilde{\omega}_{\alpha^*} \rangle = 0$ which contradicts with (3.0.3).

$\underline{(3) \Rightarrow (1)}$. It follows directly from a result of Viehmann (cf. Proposition 3.1 (2)).

$\underline{(2) \Rightarrow (3)}$. Suppose (G, μ) is weakly fully HN-decomposable. As we have proved $(2) \Rightarrow (1)$, for any $[b'] \in B(G, 0, -w_0 \mu)$ which is HN-indecomposable with respect to $-w_0 \mu$, the Newton stratum $\mathcal{F}(G, \mu, b)^{[b']}$ is contained in the weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$. Therefore it suffices to consider $[b'] \in B(G, 0, -w_0 \mu)$ which are HN-decomposable. Let $[b'^H] \in B(H)$ be the image of $[b']$ via inner twist. Suppose that $[b'^H] \in B(H, \xi^\sharp, -w_0 \mu)$ is HN-decomposable with respect to a standard Levi subgroup M of H (i.e. $\nu_{b'^H} \preceq_M -w_0 \mu^\diamond$ and M is the smallest standard Levi subgroup with this property). For any $x \in \mathcal{F}(H, \mu, b^H)^{[b'^H]}(C)$, we want to show that the Harder-Narasimhan vector $\mathrm{HN}_{b^H}(x)$ does not depend on x. By Proposition 1.22 (2), $\mathrm{HN}_{b^H}(x) \preceq_M \nu_{b'^H}$. We claim that

$$\mathrm{HN}_{b^H}(x) = \mathrm{pr}_M \nu_{b'^H},$$

or equivalently, that $\mathrm{HN}_{b^H}(x)$ is central in M. Write $v := \mathrm{HN}_{b^H}(x)$.

Suppose v that is NOT central in M. Then there exists $\alpha \in \Delta_{M,0}$ such that $\langle v, \alpha \rangle > 0$. As $-w_0 \mu^\diamond - v$ is a nonnegative combination of coroots in $\Delta_{M,0}^\vee$ and $\alpha \in \Delta_{M,0}$, we have

(3.0.7) $$\langle -w_0 \mu^\diamond - v, \tilde{\omega}_\alpha \rangle > 0.$$

Moreover, $\langle \mathrm{HN}_{b^H}(x), \alpha \rangle = \langle v, \alpha \rangle \neq 0$ implies that b_H has reduction to M_α. Therefore, by Lemma 2.6 and Lemma 2.7, we have

$$\langle \mu^\diamond + \xi^\diamond, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}.$$

Then by the minute criterion for the weakly fully HN-decomposability (Proposition 2.11 and Remark 2.12),

(3.0.8) $$\langle -w_0 \mu^\diamond, \tilde{\omega}_\alpha \rangle + \{ \langle -\xi^\diamond, \tilde{\omega}_\alpha \rangle \} = 0 \text{ or } 1.$$

On the other hand, by Proposition 1.22 (1) and Corollary 2.9, $v \in B(H, \xi^\sharp, -w_0 \mu^\diamond)$. In particular, by Proposition 2.8,

(3.0.9) $$\langle \xi^\diamond - v, \tilde{\omega}_\alpha \rangle \in \mathbb{Z}.$$

Combined with (3.0.8) and (3.0.9), we deduce

(3.0.10) $$\langle -w_0 \mu^\diamond, \tilde{\omega}_\alpha \rangle + \{ \langle -v, \tilde{\omega}_\alpha \rangle \} = 0 \text{ or } 1.$$

As v is dominant, $\langle v, \tilde{\omega}_\alpha \rangle \geq 0$. It follows that

$$\underbrace{\langle -w_0\mu^\diamond, \tilde{\omega}_\alpha \rangle + \{\langle -v, \tilde{\omega}_\alpha \rangle\}}_{=0 \text{ or } 1} - \underbrace{\langle -w_0\mu^\diamond - v, \tilde{\omega}_\alpha \rangle}_{>0 \text{ by } (3.0.7)} \in \mathbb{N}.$$

Therefore the left hand side is 0 and it follows that $\langle v, \tilde{\omega}_\alpha \rangle = 0$ which implies $v = 0$ since H is simple. This contradicts the fact that v is not central in M. $\qquad\square$

4. Newton strata completely contained in the weakly admissible locus

In this section, we study the question when a single Newton stratum is contained in the weakly admissible locus of the flag variety. We shall start with some general observation which works for a quasi-split reductive group, and give an answer to the above question by establishing a criterion for a Newton stratum to be completely contained in the weakly admissible locus. At the end of this section, we illustrate our criterion by an explicit example in the GL_n-case.

4.1. **Extensions and weakly admissible locus.** Recall that H is the quasi-split inner form of G over F, and T is a maximal F-torus of H, with

$$W = (N_H(T)/T)(\bar{F})$$

its Weyl group. Let $\mu \in X_*(T)$ be a minuscule cocharacter. Let M be a standard Levi subgroup of H, with corresponding parabolic subgroup P. We have the decomposition in Schubert cells of $\mathcal{F}(H, \mu)_{\bar{F}}$ according to the $P_{\bar{F}}$-orbits given as follows. For $w \in W$, let $\mathcal{F}(H, \mu)_P^w$ be the schematic image of the map

$$P_{\bar{F}} \longrightarrow \mathcal{F}(H, \mu)_{\bar{F}}, \quad g \mapsto gP_{\mu^w}g^{-1}.$$

So $\mathcal{F}(H, \mu)_P^w$ is the $P_{\bar{F}}$-orbit of $P_{\mu^w} \in \mathcal{F}(H, \mu)(\bar{F})$, and

$$\mathcal{F}(H, \mu)_P^w = P_{\bar{F}}/P_{\bar{F}} \cap P_{\mu^w}.$$

Moreover, every $P_{\bar{F}}$-orbit of $\mathcal{F}(H, \mu)_{\bar{F}}$ is obtained in this way: this follows from the relative Bruhat decomposition

$$H_{\bar{F}} = \coprod_{[w] \in W_P \backslash W / W_{P_\mu}} P_{\bar{F}} w P_\mu$$

Here W_P denotes the stabilizer of the standard parabolic subgroup $P_{\bar{F}} \subset H$ under the action of Weyl group W (on the set of standard parabolic subgroups of $H_{\bar{F}}$), and $[w]$ is the double coset in $W_P \backslash W / W_{P_\mu}$ of an element w in the Weyl group W. By taking the projection to the Levi quotient M, we obtain a map

(4.1.1) $$\mathrm{pr}_{P,w} : \mathcal{F}(H, \mu)_P^w \to \mathcal{F}(M, \mu^w),$$

It is known that the above map $\mathrm{pr}_{P,w}$ above is an affine fibration.

Definition 4.1. *Let μ be a minuscule cocharacter of H, $[b] \in A(H, \mu)_{basic}$. Let M be a standard Levi subgroup of H. An element $w \in W$ is called μ-negative for M if $\langle \nu_b - w\mu, \chi \rangle < 0$ for some $\chi \in X^*(P/Z_H)^+$.*

Proposition 4.2. *Let μ be a minuscule cocharacter of H, $[b] \in A(H, \mu)_{basic}$.*

(1) *Let M be a standard Levi subgroup of H, with P its corresponding standard parabolic subgroup. Assume that $w \in W$ is μ-negative and that b has a reduction (b_M, h) to M with $b_M \in M(\breve{F})$ and $h \in H(\breve{F})$, then*

$$h \cdot \mathcal{F}(H, b)_P^w \subset \mathcal{F}(H, \mu)_{\bar{F}} \setminus \mathcal{F}(H, \mu, b)_{\bar{F}}^{wa}.$$

(2) *The complement in $\mathcal{F}(H,\mu)$ of the weakly admissible locus can be described as follows: for $C/\bar{F}$ a complete algebraically closed field extension of $\bar{F}$,*

$$(\mathcal{F}(H,\mu)\backslash\mathcal{F}(H,\mu,b)^{wa})\,(C) = \bigcup_{\alpha\in\Delta_0}\left(\bigcup_{\substack{(b_M,h)\ is\ a\ reduction\\ of\ b\ to\ M_\alpha}}\left(\bigcup_{\substack{w\in W\ is\ \mu\text{-}negative\\ for\ M_\alpha}} h\cdot\mathcal{F}(H,\mu)^w_{P_\alpha}(C)\right)\right).$$

Remark 4.3. *The part (2) of this proposition can be considered as a reformulation of [NV23, Corollary 4.5]. Indeed, to describe the complement of the weakly admissible locus, [NV23] takes the union over all the parabolic subgroups that contradict the weakly admissibility condition while our union is taken over all the* maximal *parabolic subgroups. These two description coincide since in order to verify the weakly admissibility, it suffices to test for all the maximal parabolic subgroups instead of all the parabolic subgroups.*

Proof. (1) Since $b = hb_M\sigma(h)^{-1}$, the element $h\in H(\check{F})$ induces an isomorphism of H-bundles

$$\iota = \iota_h : \mathcal{E}_{b_M}\times^M H \xrightarrow{\sim} \mathcal{E}_b,$$

such that, under the natural identification in Remark 1.7, the restriction of ι to $\mathrm{Spec}(\hat{\mathcal{O}}_{X,\infty}) = \mathrm{Spec}(B_{dR}^+)$ is the right-multiplication by h. So for C a complete algebraically closed field extension of $\bar{F}$ and for $x\in\mathcal{F}(H,\mu)(C)$, the isomorphism ι above induces a compatible isomorphism between the modifications of H-bundles (see also Remark 1.13):

$$\left(\mathcal{E}_{b_M}\times^M H\right)_{h^{-1}x}\xrightarrow{\sim}\mathcal{E}_{b,x}.$$

If furthermore $x\in h\cdot\mathcal{F}(H,\mu)^w_P$, and by [CFS21, Lemma 2.6] there is an isomorphism

$$(\mathcal{E}_{b,x})_P\times^P M \simeq \mathcal{E}_{b_M,\mathrm{pr}_{P,w}(h^{-1}x)}.$$

On the other hand, w is μ-negative for M, so $\langle w\mu - \nu_b,\chi\rangle > 0$ for some $\chi\in X(P/Z_G)^+$. Thus

$$\deg(\chi_*(\mathcal{E}_{b,x})_P) = \deg(\chi_*(\mathcal{E}_{b_M,\mathrm{pr}_{P,w}(h^{-1}x)})) = \langle w\mu - \nu_{b_M},\chi\rangle > 0.$$

Therefore x is not weakly admissible by Proposition 1.16.

(2) In view of (1) and the criterion for the weak admissibility in the quasi-split case (Proposition 1.16), this follows from the the following two facts: once an element b has a reduction to a proper standard Levi M, it has a reduction to every maximal standard Levi containing M; moreover, if $\langle\nu_b - w\mu,\chi\rangle < 0$ for some $\chi\in X^*(P/Z_H)^+$, then the same holds with $\chi = \tilde{\omega}_\alpha$ for a certain $\alpha\notin\Delta_{0,M}$. $\qquad\square$

Suppose $[b]\in A(H,\mu)_{basic}$ with b_M a reduction of b to a standard Levi M of H. Recall that we have Newton stratifications

$$\mathcal{F}(H,\mu) = \coprod_{[b']\in B(H,\kappa(b)-\mu^\#,\nu_b-\mu^\diamond)}\mathcal{F}(H,\mu,b)^{[b']},\quad\text{and}$$

$$\mathcal{F}(M,\mu^w) = \coprod_{[b'_M]\in B(M,\kappa_M(b_M)-\mu^{w,\#},\nu_{b_M}-\mu^{w,\diamond})}\mathcal{F}(M,\mu^w,b_M)^{[b'_M]}.$$

Let P be the standard parabolic subgroup corresponding to M. We would like to compare these two Newton stratifications via the affine fibration $\mathrm{pr}_{P,w}$ in (4.1.1) above.

Definition 4.4. *Let M be a standard Levi subgroup of H, with P the corresponding standard parabolic subgroup. Let $[b'^H]\in B(H)$ and $[b'_M]\in B(M)$. We say that $[b'^H]$ is an extension of $[b'_M]$ if $\mathcal{E}_{b'^H}$ has a reduction $(\mathcal{E}_{b'^H})_P$ to P such that $(\mathcal{E}_{b'^H})_P\times^P M \simeq \mathcal{E}_{b'_M}$. Let $[b']\in B(G)$ be the preimage of $[b'^H]$ via the identification $B(G)\simeq B(H)$. We also say that $[b']$ is an extension of $[b'_M]$ if $[b'^H]$ is an extension of $[b'_M]$.*

Remark 4.5.   (1) *Suppose $H = \mathrm{GL}_n$ and $M = \mathrm{GL}_r \times \mathrm{GL}_{n-r}$ a standard Levi of H. An element $[b'] \in B(H)$ is an extension of*

$$[b'_M] = ([b''_1], [b''_2]) \in B(M) = B(\mathrm{GL}_r) \times B(\mathrm{GL}_{n-r})$$

if and only if the rank n vector bundle $\mathcal{E}_{b'}$ on the Fargues-Fontaine curve X is an extension of the vector bundle $\mathcal{E}_{b''_2}$ of rank $n - r$ by the vector bundle $\mathcal{E}_{b''_1}$ of rank r.

(2) *When $[b'_M] \in B(M)$ is basic with $\nu_{b'_M}$ anti-H-dominant, then the extensions $[b'] \in B(G)$ of $[b'_M]$ are classified by* [BFHHLWY] *for GL_n and by* [Vie] *for arbitrary G.*

Suppose that $[b] \in A(H, \mu)_{basic}$ has a reduction (b_M, h) to a proper standard Levi subgroup M of H. Let P be the standard parabolic subgroup corresponding to M. Then, for every $w \in W$ and for every $[b'] \in B(H, \kappa(b) - \mu^\sharp, \nu_b - w_0 \mu^\diamond)$, we have

(4.1.2)
$$\mathrm{pr}_{P,w}(\mathcal{F}(H, \mu)^w_P \cap h^{-1} \mathcal{F}(H, \mu, b)^{[b']}) \subseteq \coprod_{\substack{[b'_M] \in B(M, \kappa(b_M) - \mu^{w,\#}, \nu_{b_M} - \mu^{w,\diamond}) \\ \text{such that } [b'] \text{ is an extension of } [b'_M]}} \mathcal{F}(M, \mu^w, b_M)^{[b'_M]}.$$

Proposition 4.6. *The inclusion in* (4.1.2) *is an equality.*

Proof. Since (b_M, h) is a reduction of b to M, we have $b = h b_M \sigma(h)^{-1}$. So by Remark 1.13, we have

$$h^{-1} \cdot \mathcal{F}(H, \mu, b)^{[b']} = \mathcal{F}(H, \mu, b_M)^{[b']} \subset \mathcal{F}(H, \mu).$$

Here for the second term above, we view $b_M \in M(\breve{F})$ as an element of $G(\breve{F})$. Therefore, replacing $b \in H(\breve{F})$ by $b_M \in M(\breve{F}) \subset H(\breve{F})$, we shall assume $h = 1$ for the remaining part of the proof.

Let $[b'_M] \in B(M, \kappa(b_M) - \mu^{w,\#}, \nu_{b_M} - \mu^{w,\diamond})$ such that $[b']$ is an extension of $[b'_M]$, and let $x' \in \mathcal{F}(M, \mu^w, b_M)^{[b'_M]}$. We need to find $x \in \mathcal{F}(H, \mu)^w_P \cap \mathcal{F}(H, \mu, b)^{[b']}$ such that $\mathrm{pr}_{P,w}(x) = x'$. Write $U = X \setminus \{\infty\}$. Recall that the M-bundle $\mathcal{E}_{b'_M} = \mathcal{E}_{b_M, x'}$ is a modification of $\mathcal{E}_{b_M}$: so there is an isomorphism

(4.1.3)
$$\mathcal{E}_{b_M}|_U \xrightarrow{\sim} \mathcal{E}_{b_M, x'}|_U = \mathcal{E}_{b'_M}|_U \simeq \mathcal{E}_{b', P}|_U \times^P M$$

of M-bundles over U, where $\mathcal{E}_{b', P}$ is defined in Definition 4.4 as $[b']$ is an extension of $[b'_M]$. We claim that the isomorphism (4.1.3) can be extended to a modification of P-bundles over U

(4.1.4)
$$\mathcal{E}_{b_M}|_U \times^M P \xrightarrow{\sim} \mathcal{E}_{b', P}|_U.$$

Indeed, by a result of Anschütz ([Ans19, Theorem 4]), the M-bundle $\mathcal{E}_{b_M}$ is trivial over U. So the isomorphism (4.1.3) corresponds to a section s of $\mathcal{E}_{b', P} \times^P M$ over U. We want to lift the section s to a section of $\mathcal{E}_{b', P}$ over U through the canonical map

$$\mathcal{E}_{b', P} \longrightarrow \mathcal{E}_{b', P} \times^P M.$$

The latter makes $\mathcal{E}_{b', P}$ a $R_u(P)$-bundle over $\mathcal{E}_{b', P} \times^P M$, where $R_u(P)$ denotes the unipotent radical of P. Consider the cartesian diagram below

$$
\begin{array}{ccc}
\mathcal{U} & \longrightarrow & \mathcal{E}_{b', P} \\
\downarrow & & \downarrow \\
U & \xrightarrow{\ s\ } & \mathcal{E}_{b', P} \times^P M
\end{array}
\qquad .
$$

In particular, $\mathcal{U}$ is a $R_u(P)$-bundle over U. Since U is affine and $R_u(P)$ is unipotent over F (thus a successive extension of $\mathbb{G}_a$), every $R_u(P)$-bundle must be trivial. So $\mathcal{U}$ has a section over U, giving a section of $\mathcal{E}_{b',P}$ over U lying above s, as claimed.

On the other hand, the restriction $\mathcal{E}_{b_M}|_{\mathrm{Spec}(\hat{\mathcal{O}}_{X,\infty})}$ is canonically trivialized (Remark 1.7), so the modification (4.1.4) corresponds to a coset

$$x_P P(B_{dR}^+) \in \mathrm{Gr}_P^{B_{dR}}(C) = P(B_{dR})/P(B_{dR}^+),$$

whose image in $\mathrm{Gr}_M^{B_{dR}}(C) = M(B_{dR})/M(B_{dR}^+)$ is

$$\pi_{M,\mu^w}^{-1}(x') = m\mu^{w,-1}(t)M(B_{dR}^+) \in M(B_{dR}^+)\mu^{w,-1}(t)M(B_{dR}^+)/M(B_{dR}^+) = \mathrm{Gr}_{M,\mu^w}^{B_{dR}}(C).$$

Here

$$\pi_{M,\mu} : \mathrm{Gr}_{M,\mu^w}^{B_{dR}}(C) \xrightarrow{\sim} \mathcal{F}(M,\mu^w)(C)$$

is the Bialynicki-Birula map (for M) recalled in § 1.4.1. Hence the element x_P is of the form

$$um\mu^{w,-1}(t)m' \in P(B_{dR}),$$

with $u \in R_u(P)(B_{dR})$, $m, m' \in M(B_{dR}^+)$. In order to obtain the desired minuscule modification, recall that since b is basic we have an identification (cf. [FS21, Proposition III.4.7])

$$J_b \times X \xrightarrow{\sim} \underline{\mathrm{Aut}}(\mathcal{E}_b),$$

so that the following composed map

(4.1.5) $$J_b(B_{dR}) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{E}_b|_{\mathrm{Spec}(B_{dR})}) \xrightarrow{\sim} G(B_{dR})$$

is given by the natural isomorphism

$$\iota : J_{b,\breve{F}} \xrightarrow{\sim} G_{\breve{F}}.$$

Here the second identification in (4.1.5) is induced by the canonical trivialization of the G-bundle $\mathcal{E}_b|_{\mathrm{Spec}(B_{dR})}$. Let $\tilde{u} \in J_b(B_{dR})$ be the element in $J_b(B_{dR})$ corresponding to $u \in R_u(P)(B_{dR}) \subset G(B_{dR})$. As $b = b_M \in M(\breve{F}) \subset P(\breve{F})$, the preimages via the isomorphism ι of the $\breve{F}$-subgroups $M_{\breve{F}} \subset P_{\breve{F}} \subset G_{\breve{F}}$ descend to F-subgroups $\tilde{M} \subset \tilde{P} \subset J_b$, and $\tilde{U} \in R_u(\tilde{P})(B_{dR})$. Applying the lemma below to the F-unipotent group $N = R_u(\tilde{P})$, we can further write $\tilde{u}$ as

$$\tilde{u} = \tilde{u}'\tilde{u}''$$

with $\tilde{u}' \in R_u(\tilde{P})(U)$ and $\tilde{u}'' \in \tilde{P}(B_{dR}^+)$. Now, composing (4.1.4) with the inverse of $\tilde{u}' \in J_b(U) \simeq \mathrm{Aut}(\mathcal{E}_b|_U)$, we get a new modification $\mathcal{E}_{b_M}|_U \times^M P \xrightarrow{\sim} \mathcal{E}_{b',P}|_U$ compatible with (4.1.3), which is now minuscule of type μ^w. From this new modification, we deduce a minuscule modification of type μ

$$\mathcal{E}_b|_U \xrightarrow{\sim} \mathcal{E}_{b'},$$

or equivalently, a point $x \in \mathcal{F}(H,\mu,b)_P^{[b']}$ lying above x'. This completes the proof of our proposition. $\qquad \square$

Lemma 4.7. *Let $U = X \backslash \{\infty\}$. Let N be a unipotent group over F. Then the map*

$$N(U) \oplus N(B_{dR+}) \longrightarrow N(B_{dR})$$
$$(n', n'') \longmapsto n'n''$$

is surjective and $H^1(U, N) = 0$.

Proof. We prove by induction on dimension of N. Write $B_e = H^0(U, \mathcal{O}_X)$.

For $\dim N = 1$, then $N \simeq \mathbb{G}_a$. The map

$$B_e \oplus B_{dR}^+ \to B_{dR}, \quad (a, b) \mapsto a + b$$

is surjective, since $0 = H^1(X, \mathcal{O}_X) = B_{dR}/(B_e + B_{dR}^+)$ by [FF18, Prop. 8.2.3]. Moreover, as $U = \mathrm{Spec}(B_e)$ is affine, $H^1(U, \mathcal{O}_X) = H^1(U, \mathbb{G}_a) = 0$. Thus our lemma holds in this case.

For $\dim N > 1$, N has a filtration by F-subgroups

$$0 = N_0 \subset N_1 \subset \cdots \subset N_r = N,$$

such that $N_{i+1}/N_i \simeq \mathbb{G}_a$. It follows by induction that $H^1(*, N) = 0$ for $* = \mathrm{Spec}(B_{dR}), \mathrm{Spec}(B_{dR}^+)$ or U and we have short exact sequences:

$$1 \longrightarrow N_{r-1}(B_{dR}) \longrightarrow N(B_{dR}) \xrightarrow{\alpha} \mathbb{G}_a(B_{dR}) \longrightarrow 1,$$
$$1 \longrightarrow N_{r-1}(B_{dR}^+) \longrightarrow N(B_{dR}^+) \xrightarrow{\alpha} \mathbb{G}_a(B_{dR}^+) \longrightarrow 1,$$
$$1 \longrightarrow N_{r-1}(U) \longrightarrow N(U) \xrightarrow{\alpha} \mathbb{G}_a(U) \longrightarrow 1.$$

Let $n \in N(B_{dR})$, then there exists $n' \in N(U)$ and $n'' \in N(B_{dR}^+)$ with $\alpha(n) = \alpha(n')\alpha(n'')$ (the group law is written multiplicatively), or equivalently,

$$n'^{-1}nn''^{-1} \in \mathrm{Ker}(\alpha) = N_{r-1}(B_{dR}).$$

By induction hypothesis, $n'^{-1}nn''^{-1} = \tilde{n}'\tilde{n}''$ with $\tilde{n}' \in N_{r-1}(U)$ and $\tilde{n}'' \in N_{r-1}(B_{dR}^+)$. It follows that

$$n = (\underbrace{n'\tilde{n}'}_{\in N(U)}) \cdot (\underbrace{\tilde{n}''n''}_{\in N(B_{dR}^+)}).$$

$\square$

Corollary 4.8. *Let μ be a minuscule cocharacter of H, and $[b] \in A(H, \mu)_{basic}$. Assume that $[b]$ has a reduction (b_M, h) to some standard Levi subgroup M of H, with $b_M \in M(\breve{F})$ and $h \in H(\breve{F})$. Let $[b'] \in B(H, \kappa(b) - \mu^{\#}, \nu_b - \mu^{\diamond})$ and $w \in W$. Then*

$$(h \cdot \mathcal{F}(H, \mu)_P^w) \cap \mathcal{F}(H, \mu, b)^{[b']} \neq \emptyset.$$

if and only if $[b']$ is an extension of a certain $[b'_M] \in B(M, \kappa(b_M) - \mu^{w,\#}, \nu_{b_M} - \mu^{w,\diamond})$.

Proof. This follows directly from Proposition 4.6. $\square$

Corollary 4.8 allows us to have a criterion about whether a single Newton stratum is completely contained in the weakly admissible locus for a general connected reductive group.

Theorem 4.9. *Let μ be a minuscule cocharacter of the (not necessarily quasi-split) connected reductive group G, and $b \in G(\breve{F})$ such that $[b] \in B(G, \mu)_{basic}$. Let $[b'] \in B(G, 0, \nu_b - \mu)$. Then*

$$\mathcal{F}(G, \mu, b)^{[b']} \not\subseteq \mathcal{F}(G, \mu, b)^{wa}$$

if and only if there exists some maximal proper standard Levi subgroup M of H, the quasi-split inner form of G over F, and $w \in W$ satisfying the following two properties:

(1) *b has a reduction b_M to M and w is μ-negative for M; and*
(2) *$[b']$ is an extension of some $[b'_M] \in B(M, \kappa(b_M) - \mu^{w,\#}, \nu_{b_M} - \mu^{w,\diamond})$.*

Proof. Recall the identification (2.1.1) between $B(G)$ and $B(H)$ induced by the class

$$\xi \in H^1(F, H) \simeq \pi_1(H)_{\Gamma, tor}$$

of G as an inner form of H, under which $B(G, \mu)$ is identified with $B(H, \mu^{\#} + \xi, \mu^{\diamond})$. Moreover, using a fixed inner twisting $G_{\breve{F}} \xrightarrow{\sim} H_{\breve{F}}$, we have an identification

$$\mathcal{F}(G, \mu) \xrightarrow{\sim} \mathcal{F}(H, \mu)$$

of flag varieties over $\breve{E}$, under which the Newton stratification (resp. the Harder-Narasimhan stratification) on $\mathcal{F}(G, \mu)$ for the triple (G, μ, b) is identified with the corresponding stratification on $\mathcal{F}(H, \mu)$ for the triple (H, μ, b^H), where $[b^H] \in B(H)$ is the image of $[b]$ via $B(G) \simeq B(H)$. Therefore the theorem follows immediately from Corollary 4.8 and Proposition 4.2 (2). $\qquad\square$

4.2. **An example in the GL_n-case.** In this §, we illustrate an application of Theorem 4.9, and we refer to the next section for some similar but more complicated applications of the same result.

Consider the case where $G = \mathrm{GL}_{10}$ and $\mu = (1^{(4)}, 0^{(6)})$. Let $b \in B(G)_{basic}$ with $\kappa(b) = \mu^{\#}$. In particular,

$$\nu_b = \left(\frac{2}{5}^{(10)}\right) = \left(\underbrace{\frac{2}{5}, \frac{2}{5}, \cdots, \frac{2}{5}}_{10}\right) \in \mathcal{N}(\mathrm{GL}_{10}).$$

The element b has a reduction to only one proper maximal standard Levi

$$M = \mathrm{GL}_5 \times \mathrm{GL}_5 \hookrightarrow G = \mathrm{GL}_{10},$$

and if $w \in W$ is such that $\langle \nu_b - w\mu, \chi \rangle < 0$ for some $\chi \in X^*(P/Z_G)^+$, then

$$w\mu \in \left\{ \left((1^{(3)}, 0^{(2)}), (1, 0^{(4)})\right), \left((1^{(4)}, 0), (0^{(5)})\right) \right\} \in \mathcal{N}(M) = \mathcal{N}(\mathrm{GL}_5) \times \mathcal{N}(\mathrm{GL}_5).$$

So

$$\nu_{b_M} - \mu^w \in \left\{ \left(\left(\frac{2}{5}^{(2)}, -\frac{3}{5}^{(3)}\right), \left(\frac{2}{5}^{(4)}, -\frac{3}{5}\right)\right), \left(\left(\frac{2}{5}, -\frac{3}{5}^{(4)}\right), \left(\frac{2}{5}^{(5)}\right)\right) \right\} \in \mathcal{N}(M).$$

Therefore, by Theorem 4.9, for $[b'] \in B(G, 0, \nu_b\mu^{-1})$, the Newton stratum $\mathcal{F}(G, \mu, b)^{[b']}$ contains a point that is not weakly admissible if and only if $\mathcal{E}_{b'}$ is an extension of some element in

$$[b'_M] \in B(M, \kappa(b_M) - \mu^{w,\#}, \nu_{b_M} - \mu^{w,\diamond}),$$

or equivalently, by explicit computation, $\mathcal{E}_{b'}$ is an extension of $\mathcal{E}'$ by $\mathcal{E}''$, where the pair $(\mathcal{E}', \mathcal{E}'')$ is either contained in

$$\left\{ \mathcal{O}\left(-\frac{1}{3}\right) \oplus \mathcal{O}^2, \mathcal{O}\left(-\frac{1}{4}\right) \oplus \mathcal{O}, \mathcal{O}\left(-\frac{1}{5}\right) \right\} \times \left\{ \mathcal{O}\left(\frac{1}{5}\right), \mathcal{O} \oplus \left(\frac{1}{4}\right), \mathcal{O}^2 \oplus \mathcal{O}\left(\frac{1}{3}\right), \mathcal{O}^3 \oplus \mathcal{O}\left(\frac{1}{2}\right) \right\},$$

or contained in

$$\left\{ \mathcal{O}\left(-\frac{2}{5}\right) \right\} \times \left\{ \mathcal{O}\left(\frac{2}{5}\right), \mathcal{O} \oplus \mathcal{O}\left(\frac{1}{2}\right)^2, \mathcal{O}\left(\frac{1}{3}\right) \oplus \mathcal{O}\left(\frac{1}{2}\right) \right\}.$$

Observe that, for each choice of $(\mathcal{E}', \mathcal{E}'')$ above, the slopes of $\mathcal{E}'$ are less or equal to those of $\mathcal{E}''$. In particular, every extension of $\mathcal{E}'$ by $\mathcal{E}''$ splits. On the other hand,

$$\nu_b\mu^{-1} = \left(\frac{2}{5}^{(6)}, \frac{-3}{5}^{(4)}\right),$$

so there are 26 non-empty Newton strata, listed as follows:

- $\nu_{b'} = \left(\frac{2}{5}^{(5)}, 0^{(i)}, -\frac{2}{5-i}^{(5-i)}\right)$ for $i = 0, 1$; or $\left(\frac{2}{5}^{(5)}, -\frac{1}{3}^{(3)}, -\frac{1}{2}^{(2)}\right)$. We have 3 possibilities in this case, and the polygons are HN-decomposable, in the sense that they all touch the polygon for $\nu_b \mu^{-1}$. As a result, the corresponding Newton strata do not contain any weakly admissible point.
- $\nu_{b'} = \left(\frac{1}{3}^{(6)}, -\frac{1}{2}^{(4)}\right)$. So $\mathcal{E}_{b'}$ is not a direct sum of $\mathcal{E}' \oplus \mathcal{E}''$ for all possible choices $(\mathcal{E}', \mathcal{E}'')$, and the corresponding Newton stratum is completely contained in $\mathcal{F}(G, \mu, b)^{wa}$.
- $\nu_{b'} = \left(\frac{1}{i}^{(i)}, 0^{(j)}, -\frac{1}{10-i-j}^{(10-i-j)}\right)$ for $3 \leq i \leq 8$ and $0 \leq j \leq 8 - i$. We have 21 possibilities for this case. It is easy to see that, $\mathcal{E}_{b'} = \mathcal{E}' \oplus \mathcal{E}''$ for some choice $(\mathcal{E}', \mathcal{E}'')$ if and only if $i \leq 5$ and $i + j \geq 5$. So in this case, we get 9 Newton strata that are completely contained in the weakly admissible locus.
- $\nu_{b'} = \left(0^{(6)}\right)$, which corresponds to the admissible locus $\mathcal{F}(G, \mu, b)^a$ thus is completely contained in $\mathcal{F}(G, \mu, b)^{wa}$.

In summary, there are 26 non-empty Newton strata in the Newton stratification of $\mathcal{F}(G, \mu)$, and 11 of them are completely contained in the weakly admissible locus.

Remark 4.10. *The reason why we can determine all the Newton strata completely contained in the weakly admissible locus in this example is that all the involved extensions of M-bundles to G-bundles are trivial. In general, it's a difficult question to determine whether a G-bundle is an extension of a M-bundle. In the next section, we will study this question for GL_n.*

5. Extensions of Vector bundles over Fargues-Fontaine curve

As seen in the previous section, to determine if a single Newton stratum $\mathcal{F}(G, \mu, b)^{[b']}$ is completely contained in the weakly admissible locus $\mathcal{F}(G, \mu, b)^{wa}$, we need to have a classification of extensions of G-bundles over the Fargues-Fontaine curve. In this section, we give such a classification for GL_n in an inductive way: see also the appendix below for a direct classification in some special cases. As an application, this gives an algorithm to determine which Newton strata are completed contained in the weakly addmissible locus for $G = \mathrm{GL}_n$.

5.1. **Inductive classification of extensions of vectors bundles.** When $G = \mathrm{GL}_n$, the Kottwitz map is determined by the Newton map, hence by Proposition 1.1, the Newton map

$$\nu : B(\mathrm{GL}_n) \to \mathcal{N}(\mathrm{GL}_n)$$

is injective. Let $\mathcal{N}(n)$ be its image. Then it's in bijection with $B(\mathrm{GL}_n)$. The set $\mathcal{N}(n)$ has an explicit description as follows:

$$\mathcal{N}(n) = \Big\{ (\underbrace{\lambda_1, \ldots, \lambda_1}_{m_1}, \cdots, \underbrace{\lambda_r, \ldots, \lambda_r}_{m_r}) \in \mathbb{Q}^n \mid \begin{array}{l} r \in \mathbb{N}_{\geq 1}, \ \lambda_1 > \lambda_2 > \cdots > \lambda_r \\ \sum_{i=1}^r m_i = n, \ m_i \lambda_i \in \mathbb{Z}, \ \forall 1 \leq i \leq r \end{array} \Big\} \subset X_*(T)^+_{\mathbb{Q}}.$$

For $\underline{a} \in \mathcal{N}(n)$, let $\mathcal{O}(\underline{a})$ be the corresponding vector bundle of rank n over the Fargues-Fontaine curve X. We say that $\underline{a}$ is semistable (resp. stable) if the corresponding vector bundle $\mathcal{O}(\underline{a})$ is semistable (resp. stable). In general, the stable (resp. semi-stable) blocks in $\mathcal{O}(\underline{a})$ are also called the stable (resp. semistable) blocks in $\underline{a}$. For $\underline{a} \in \mathcal{N}(n)$ and $\underline{b} \in \mathcal{N}(m)$, their direct sum

$$\underline{a} \oplus \underline{b} \in \mathcal{N}(n + m)$$

is defined in such a way that $\mathcal{O}(\underline{a} \oplus \underline{b}) = \mathcal{O}(\underline{a}) \oplus \mathcal{O}(\underline{b})$. For $\underline{a} = (a_1, \ldots, a_n) \in \mathcal{N}(n)$, we define its dual $\underline{a}^\vee$ by

$$\underline{a}^\vee := (-a_n, \ldots, -a_1) \in \mathcal{N}(n).$$

In other words, $\underline{a}^\vee = -w_0\underline{a}$. Clearly $\mathcal{O}(\underline{a}^\vee) = \mathcal{O}(\underline{a})^\vee$. Finally, for $\underline{c} \in \mathbb{Q}^n$, set

$$S_n\underline{c} := \{(c_{\sigma(1)}, \ldots, c_{\sigma(n)}) \in \mathbb{Q}^n | \sigma \in S_n\}.$$

Definition 5.1. *Let $r, s \in \mathbb{N}$, $\underline{c} \in \mathcal{N}(r)$ and $\underline{d} \in \mathcal{N}(s)$. Set $n = r + s$ and define the following two subsets $\mathrm{Ext}^1(\underline{c}, \underline{d})$ and $\widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$ of $\mathcal{N}(n)$:*

(1) $\mathrm{Ext}^1(\underline{c}, \underline{d})$ *is the set of $\underline{a} \in \mathcal{N}(n)$ such that there exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}(\underline{d}) \longrightarrow \mathcal{O}(\underline{a}) \longrightarrow \mathcal{O}(\underline{c}) \longrightarrow 0,$$

or equivalently, that $\mathcal{O}(\underline{a})$ is an extension of $\mathcal{O}(\underline{c})$ by $\mathcal{O}(\underline{d})$;

(2) $\widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$ *is the set of $\underline{a} \in \mathcal{N}(n)$ satisfying the following condition: there exists a partition*

$$\{1, \ldots, n\} = H \coprod K$$

of $\{1, \ldots, n\}$ with

$$H = \{h_1 < h_2 < \ldots < h_r\} \quad and \quad K = \{k_1 < k_2 \ldots < k_s\},$$

and $\underline{b} \in \mathbb{Q}^n$ with

$$(b_{h_1}, \ldots, b_{h_r}) \in S_r\underline{c} \quad and \quad (b_{k_1}, \ldots, b_{k_s}) \in S_s\underline{d},$$

such that the following properties hold:

  (a) *for any $i \in H$, $b_i \geq a_i$;*

  (b) *for any $i \in K$, $b_i \leq a_i$;*

  (c) *for any $1 \leq l \leq n$, $\sum_{i=1}^l b_i \geq \sum_{i=1}^l a_i$, with equality if $l = n$.*

Remark 5.2.   (1) *The combinatorial definition of $\widetilde{\mathrm{Ext}}^1$ is motivated by [Sch00] in which the extensions of vector bundles on $\mathbb{P}^1$ over an algebraically closed field are classified in a similar way. After we finish our work, we noticed that similar condition is also considered independently by Hong in [Hon22].*

(2) *In the definition of $\widetilde{\mathrm{Ext}}^1$, if an element $\underline{b} \in \mathbb{Q}^n$ satisfies the conditions (2.a)-(2.c) above and if we take $\tilde{\underline{b}} \in \mathbb{Q}^n$ such that*

$$(\tilde{\underline{b}}_{h_1}, \ldots, \tilde{\underline{b}}_{h_r}) = (c_1, \ldots, c_r), \quad and \quad (\tilde{\underline{b}}_{k_1}, \ldots, \tilde{\underline{b}}_{k_s}) = (d_1, \ldots, d_s),$$

then $\tilde{\underline{b}}$ satisfies equally the conditions (2.a)-(2.c) above.

These two subsets of $\mathcal{N}(n)$ are closely related by the following proposition:

Proposition 5.3. *For $r, s \in \mathbb{N}$, $\underline{c} \in \mathcal{N}(r)$ and $\underline{d} \in \mathcal{N}(s)$, we have $\mathrm{Ext}^1(\underline{c}, \underline{d}) \subset \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$.*

Proof. Take $\underline{a} \in \mathrm{Ext}^1(\underline{c}, \underline{d})$. Let $N \in \mathbb{N}$ such that all the components of $N\underline{a}, N\underline{c}, N\underline{d}$ are integers. Consider the cyclic covering $f : X_N \longrightarrow X$ of order N between Fargues-Fontaine curves. It follows that $f^*\mathcal{O}(\underline{a}), f^*\mathcal{O}(\underline{c})$ and $f^*\mathcal{O}(\underline{d})$ are sum of line bundles: for example, if $\underline{a} = (a_1, \ldots, a_n) \in \mathcal{N}(n) \subset \mathbb{Q}^n$, $f^*\mathcal{O}(\underline{a})$ is the direct sum of line bundles $\mathcal{O}(Na_i)$, $1 \leq i \leq n$. Then the same argument as in the proof of [Sch00, Proposition 3.1] shows that $N\underline{a} \in \widetilde{\mathrm{Ext}}^1(N\underline{c}, N\underline{d})$, which implies $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. $\qquad\square$

Let $\underline{c} \in \mathcal{N}(r)$, $\underline{d} \in \mathcal{N}(s)$. Proposition 5.3 gives a necessary condition for the property that $\mathcal{O}(\underline{a})$ can be realized as an extension of $\mathcal{O}(\underline{c})$ by $\mathcal{O}(\underline{d})$. One could find some other conditions in the literature for the last property. For example, if $\underline{c}$ and $\underline{d}$ are semistable, one can reformulate the conditions in Definition 5.1 (2) in a more direct way, which allows us to relate the set $\widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$ with the description of extensions in [BFHHLWY].

Lemma 5.4. *Let $\underline{c} \in \mathcal{N}(r)$, $\underline{d} \in \mathcal{N}(s)$ and $\underline{a} \in \mathcal{N}(n)$. Assume $n = r + s$, and that $\underline{c}$ and $\underline{d}$ are both semistable. Set $c := c_1 = \cdots = c_r$ and $d := d_1 = \ldots = d_s$. Then $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$ if and only if one of the following two conditions is verified:*

(1) *$c \leq d$ and $\underline{a} = \underline{c} \oplus \underline{d}$; or*
(2) *$c > d$, and $\underline{a} \leq \underline{c} \oplus \underline{d}$.*

Proof. If one of the two conditions is verified, it is clear that $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$: we can take $H = \{1 < \ldots < r\}$ and $K = \{r + 1 < \ldots < r + s = n\}$ in both cases.

Conversely, suppose that $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$, with $H = \{h_1 < \ldots < h_r\}$ and $K = \{k_1 < \ldots < k_s\}$ a partition of $\{1, \ldots, n\}$ given in Definition 5.1. We shall distinguish two different cases:

- Assume first $c \leq d$. If $h_1 < k_s$, by the conditions (2.a) and (2.b) of Definition 5.1 we have
$$c = b_{h_1} \geq a_{h_1} \geq a_{k_s} \geq b_{k_s} = d.$$
So we get $c = d$. Combing (2.c) in Definition 5.1, we deduce moreover $a_1 = \ldots = a_n$, and thus $\underline{a} = \underline{c} \oplus \underline{d}$ as asserted in (1). If $h_1 > k_s$, then $H = \{s + 1 < \ldots < n\}$ and $K = \{1 < \ldots < s\}$. By (2.c) of Definition 5.1, for all $1 \leq l \leq s$, we have
$$\sum_{i=1}^{l} d_i = \sum_{i=1}^{l} b_i \geq \sum_{i=1}^{l} a_i$$
while $d_i = b_i \leq a_i$ for all $1 \leq i \leq s$ by (2.b) of Definition 5.1, therefore $a_1 = \ldots = a_s = d$. On the other hand, as
$$\sum_{i=1}^{s} d_i + \sum_{i=s+1}^{n} c_i = \sum_{i=1}^{n} b_i \geq \sum_{i=1}^{l} a_i = \sum_{i=1}^{s} a_i + \sum_{i=s+1}^{n} a_i,$$
we find
$$\sum_{i=s+1}^{n} c_i \geq \sum_{i=s+1}^{n} a_i.$$
But $c_i = b_i \geq a_i$ for every $s + 1 \leq i \leq n$ by (2.a) of Definition 5.1. Thus $a_i = c_i = c$ for all $s + 1 \leq i \leq n$. Consequently we still obtain $\underline{a} = \underline{c} \oplus \underline{d}$ as asserted in (1).
- It remains to consider the case where $c > d$. Necessarily $a_1 \leq c$: otherwise $1 \in K$ and we obtain $d = b_1 \leq a_1 \leq c$ which is absurd. Similarly $a_n \geq d$. Since $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ by (2.c) of Definition 5.1, it follows that $\underline{c} \oplus \underline{d} \geq \underline{a}$, as claimed by (b).

This completes the proof of our lemma. $\qquad\square$

Corollary 5.5. *Let $\underline{c} \in \mathcal{N}(r)$ and $\underline{d} \in \mathcal{N}(s)$. Assume that $\underline{c}$ and $\underline{d}$ are both semistable. Then $\mathrm{Ext}^1(\underline{c}, \underline{d}) = \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$.*

Proof. Let $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. We must check that $\mathcal{O}(\underline{a})$ is an extension of $\mathcal{O}(\underline{c})$ by $\mathcal{O}(\underline{d})$. Set $c := c_1 = \ldots = c_r$ and $d := d_1 = \ldots = d_s$. According to the lemma above, we only need to consider the following two cases.

- If $c \leq d$ and $\underline{a} = \underline{c} \oplus \underline{d}$, then $\mathcal{O}(\underline{a}) = \mathcal{O}(\underline{c}) \oplus \mathcal{O}(\underline{d})$, which is the split extension of $\mathcal{O}(\underline{c})$ by $\mathcal{O}(\underline{d})$.
- If $c > d$ and $\underline{a} \leq \underline{c} \oplus \underline{d}$, then the vector bundles $\mathcal{F}_1 := \mathcal{O}(\underline{d})$, $\mathcal{F}_2 := \mathcal{O}(\underline{c})$, and $\mathcal{E} := \mathcal{O}(\underline{a})$ satisfy the condition of [BFHHLWY, Theorem 1.1.2]. So according to loc. cit., $\mathcal{O}(\underline{a})$ is an extension of $\mathcal{O}(\underline{c})$ by $\mathcal{O}(\underline{d})$, as required.

$\qquad\square$

We have the following property of $\widetilde{\mathrm{Ext}}^1$:

Lemma 5.6. *Let $\underline{c} \in \mathcal{N}(r)$, $\underline{d} \in \mathcal{N}(s)$ and $\underline{a} \in \mathcal{N}(n)$. Assume $n = r+s$, and $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. Then the following assertions hold.*

(i) *$\underline{a}$ strongly slopewise dominates $\underline{d}$: for every $\mu \in \mathbb{Q}$,*

$$n_\mu := \#\{a_i | a_i \geq \mu\} \geq \#\{d_i | d_i \geq \mu\},$$

with equality if and only if $(a_1, \ldots, a_{n_\mu}) = (d_1, \ldots, d_{n_\mu})$;

(ii) *$\underline{a}^\vee$ strongly slopewise dominates $\underline{c}^\vee$, or equivalently, for every $\mu \in \mathbb{Q}$,*

$$m_\mu := \#\{a_i | a_i \leq \mu\} \geq \#\{c_i | c_i \leq \mu\},$$

with equality if and only if $(a_1, \ldots, a_{m_\mu}) = (c_1, \ldots, c_{m_\mu})$; and

(iii) *$\underline{a} \leq \underline{c} \oplus \underline{d}$.*

Proof. Suppose for some $\mu \in \mathbb{Q}$,

$$n_\mu = \#\{a_i | a_i \geq \mu\} \leq \#\{d_i | d_i \geq \mu\}.$$

We may assume without lossing generality that $\mu = a_i$ or d_i for some i. By (2.b) of Definition 5.1, the inequality above must be an equality and $\{1, \ldots, n_\mu\} \subset K$. Therefore $(a_1, \ldots, a_{n_\mu}) = (d_1, \ldots, d_{n_\mu})$, showing (i) above. For (ii), note that

$$\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d}) \Longleftrightarrow \underline{a}^\vee \in \widetilde{\mathrm{Ext}}^1(\underline{d}^\vee, \underline{c}^\vee).$$

So (ii) is just the dual version of (i).

It remains to check (iii). Write $\underline{c} \oplus \underline{d} = (b'_1, \ldots, b'_n) \in \mathcal{N}(n)$. So the $b'_1 \geq \ldots \geq b'_n$ is just the permutation by order of $b_1, \ldots, b_n$. So, combing (2.c) of Definition 5.1, we obtain

$$\sum_{i=1}^l b'_i \geq \sum_{i=1}^l b_i \geq \sum_{i=1}^l a_i, \quad \text{for all } 1 \leq l \leq n.$$

In other words, $\underline{a} \leq \underline{c} \oplus \underline{d}$, as claimed by (iii). $\qquad\square$

Remark 5.7. *We keep the notations of Lemma 5.6.*

(1) *According to [Hon23], the condition that $\underline{a}$ strongly slopewise dominates $\underline{d}$ is equivalent to the fact that $\mathcal{O}(\underline{d})$ is a subbundle (i.e. locally direct factor) of $\mathcal{O}(\underline{a})$ and the condition that $\underline{a}^\vee$ strongly slopewise dominates $\underline{c}^\vee$ is equivalent to the fact that $\mathcal{O}(\underline{c})$ is a quotient of the vector bundle $\mathcal{O}(\underline{a})$.*

(2) *In general, as shown by the following example, the combination of the above conditions (i)-(iii) is weaker than the conditions defining $\widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$ even when one of $\underline{a}$ and $\underline{b}$ is semistable. Consider*

$$\underline{a} = (6, 5, 2, 1) \in \mathcal{N}(4), \quad \underline{c} = (10, 4) \in \mathcal{N}(2), \quad \text{and} \quad \underline{d} = (0, 0) \in \mathcal{N}(2).$$

Then the triple $(\underline{a}, \underline{c}, \underline{d})$ does verify the conditions (i)-(iii) above. But $\underline{a} \notin \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. Otherwise, let $\{1, 2, 3, 4\} = H \coprod K$ be the partition given in Definition 5.1. Since $0 < 6$, we must have $1 \in H$ and thus $b_1 = c_1 = 10$. Then as

$$a_1 + a_2 = 11 > b_1 + 0,$$

by (2.c) of Definition 5.1, $2 \in H$ and thus $b_2 = c_2 = 4$. But this contradicts to the fact that $b_2 \geq c_2$. In particular, by Proposition 5.3, $\mathcal{O}(\underline{a}) = \mathcal{O}(6) \oplus \mathcal{O}(5) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)$ is not an extension of $\mathcal{O}(\underline{c}) = \mathcal{O}(10) \oplus \mathcal{O}(4)$ by $\mathcal{O}(\underline{d}) = \mathcal{O} \oplus \mathcal{O}$.

One would probably expect that

$$\mathrm{Ext}^1(\underline{c}, \underline{d}) = \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d}),$$

or equivalently, for $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$, there exists a short exact sequence of the form

$$0 \longrightarrow \mathcal{O}(\underline{d}) \longrightarrow \mathcal{O}(\underline{a}) \longrightarrow \mathcal{O}(\underline{c}) \longrightarrow 0.$$

However this fails in general by the following example.

Example 5.8. *Let*

$$\underline{a} = \left(1, \frac{5}{7}^{(7)}, \frac{4}{7}^{(7)}, 0\right), \quad \underline{c} = \left(3, \frac{3}{5}^{(5)}\right), \quad \text{and} \quad \underline{d} = \left(\frac{5}{9}^{(9)}, -1\right).$$

In particular, $\underline{a} \in \mathcal{N}(16)$, $\underline{c} \in \mathcal{N}(6)$ and $\underline{d} \in \mathcal{N}(10)$. Then we have $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$, with

$$H = \{1, 9, 10, 11, 12, 13\} \subset \{1, 2, \ldots, 16\}.$$

However, $\mathcal{O}(\underline{a})$ is not an extension of $\mathcal{O}(\underline{c})$ by $\mathcal{O}(\underline{d})$, or equivalently, there does not exist a short exact sequence as follows

$$0 \longrightarrow \mathcal{O}\left(\frac{5}{9}\right) \oplus \mathcal{O}(-1) \xrightarrow{\phi} \mathcal{O}(1) \oplus \mathcal{O}\left(\frac{5}{7}\right) \oplus \mathcal{O}\left(\frac{4}{7}\right) \oplus \mathcal{O} \xrightarrow{\psi} \mathcal{O}(3) \oplus \mathcal{O}\left(\frac{3}{5}\right) \longrightarrow 0$$

To prove this, suppose that such an extension exists. Consider the subbundle $\mathcal{O}(\underline{a}') \subset \mathcal{O}(\underline{a})$, with

$$\underline{a}' = \left(1, \frac{5}{7}^{(7)}, \frac{4}{7}^{(7)}\right) \in \mathcal{N}(15).$$

Write $\mathcal{O}(\underline{c}') := \psi(\mathcal{O}(\underline{a}')) \subset \mathcal{O}(\underline{a})$ and $\mathcal{O}(\underline{d}') = \phi^{-1}(\mathcal{O}(\underline{a}'))$. So we have the following commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Coker}(\alpha') & \longrightarrow & \mathcal{O} & \longrightarrow & \mathrm{Coker}(\alpha'') & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}\left(\frac{5}{9}\right) \oplus \mathcal{O}(-1) & \xrightarrow{\phi} & \mathcal{O}(1) \oplus \mathcal{O}\left(\frac{5}{7}\right) \oplus \mathcal{O}\left(\frac{4}{7}\right) \oplus \mathcal{O} & \xrightarrow{\psi} & \mathcal{O}(3) \oplus \mathcal{O}\left(\frac{3}{5}\right) & \longrightarrow & 0 \\
& & \alpha' \uparrow & & \alpha \uparrow & & \alpha'' \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(\underline{d}') & \longrightarrow & \mathcal{O}(\underline{a}') & \longrightarrow & \mathcal{O}(\underline{c}') & \longrightarrow & 0
\end{array}
$$

We claim that $\mathcal{O}(\underline{d}') = \mathcal{O}(\frac{5}{9})$. To see this, observe first that α' is not surjective: otherwise $\mathcal{O} \xrightarrow{\sim} \mathrm{Coker}(\alpha'')$ and thus $\mathcal{O}$ would be a direct factor of $\mathcal{O}(3) \oplus \mathcal{O}(\frac{3}{5})$, which is absurd. So $\mathrm{Coker}(\alpha')$ is a line bundle contained in $\mathcal{O}$, and its degree ≤ 0. In particular, there is no non-zero morphism $\mathcal{O}(\frac{5}{9}) \to \mathrm{Coker}(\alpha')$. As a result, being a quotient of $\mathcal{O}(\frac{5}{9}) \oplus \mathcal{O}(-1)$, $\mathrm{Coker}(\alpha') \simeq \mathcal{O}(-1)$, and thus $\mathcal{O}(\underline{d}') = \mathcal{O}(\frac{5}{9}) \subset \mathcal{O}(\underline{d})$. Furthermore, we claim that $\mathcal{O}(\underline{c}') = \mathcal{O}(2) \oplus \mathcal{O}(\frac{3}{5})$. As $\mathrm{Coker}(\alpha'')$ is torsion, α'' is generically an isomorphism. It follows that the induced map

$$\mathcal{O}(\underline{a}') \xrightarrow{\psi} \mathcal{O}(3) \oplus \mathcal{O}\left(\frac{3}{5}\right) \longrightarrow \mathcal{O}\left(\frac{3}{5}\right)$$

is generically an epimorphism, hence must be an epimorphism since $\frac{2}{5} < \frac{4}{7}$. In particular, the composition

$$\beta : \mathcal{O}(\underline{c}') \xrightarrow{\alpha''} \mathcal{O}(3) \oplus \mathcal{O}\left(\frac{3}{5}\right) \longrightarrow \mathcal{O}\left(\frac{3}{5}\right)$$

must be surjective. On the other hand, $\mathcal{O}(\underline{c}')$ is a vector bundle of rank 6 and of degree 5, so $\ker(\beta) \simeq \mathcal{O}(2)$. As $2 > \frac{3}{5}$, we get $\mathcal{O}(\underline{c}') = \mathcal{O}(2) \oplus \mathcal{O}(\frac{3}{5})$. So

$$\underline{c}' = \left(2, \frac{3}{5}^{(5)}\right), \quad \text{and} \quad \underline{d}' = \left(\frac{5}{9}^{(9)}\right).$$

Moreover, $\mathcal{O}(\underline{a}')$ is an extension of $\mathcal{O}(\underline{c}')$ by $\mathcal{O}(\underline{d}')$, so $\underline{a}' \in \widetilde{\mathrm{Ext}}^1(\underline{c}', \underline{d}')$. But one could check directly that this is not the case. In fact, since

$$2 > 1 > \frac{5}{9} > \frac{5}{7} > \frac{3}{5} > \frac{4}{7}$$

if $\underline{a}' \in \widetilde{\mathrm{Ext}}^1(\underline{c}', \underline{d}')$, the corresponding partition $\{1, \ldots, 15\} = H' \coprod K'$ would satisfy $1 \in H'$ and $\{2, 3, \ldots, 8\} \subset K'$. But this is impossible as

$$2 + \frac{5}{9} \times 7 < 1 + \frac{5}{7} \times 7 = 6.$$

Next we want to give an inductive criterion for an element $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$ to be contained in $\mathrm{Ext}^1(\underline{c}, \underline{d})$.

Proposition 5.9. *Let $\underline{c} \in \mathcal{N}(r)$, $\underline{d} \in \mathcal{N}(s)$. Let $n := r + s$, and $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. Write $c_r = \frac{q}{p}$ with $p \in \mathbb{Z}_{\geq 1}$ and $q \in \mathbb{Z}$ such that $(p, q) = 1$, so $\underline{c} = \left(c_1, \ldots, c_{r-p}, \frac{q}{p}, \ldots, \frac{q}{p}\right)$. Set*

$$\underline{c}' := (c_1, \ldots, c_{r-p}) \in \mathcal{N}(r - p), \quad \text{and} \quad \underline{c}'' = c_r^{(p)} := \left(\frac{q}{p}, \ldots, \frac{q}{p}\right) \in \mathcal{N}(p).$$

Then $\underline{a} \in \mathrm{Ext}^1(\underline{c}, \underline{d})$ if and only if there exists some $\underline{e} \in \mathcal{N}(n - p)$ such that

- *$\underline{e} \in \mathrm{Ext}^1(\underline{c}', \underline{d})$; and*
- *$\underline{a} \in \mathrm{Ext}^1(\underline{c}'', \underline{e})$, or equivalently, $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}'', \underline{e})$: see Proposition A.8 below or [Hon22, Theorem 1.1].*

Remark 5.10. *Proposition 5.9 is also proved independently by Hong in [Hon22, Theorem 1.2].*

Proof. Assume first that $\underline{a} \in \mathrm{Ext}^1(\underline{c}, \underline{d})$. So we have a short exact sequence of vector bundles over the Fargues-Fontaine curve X:

$$0 \longrightarrow \mathcal{O}(\underline{d}) \longrightarrow \mathcal{O}(\underline{a}) \longrightarrow \mathcal{O}(\underline{c}) \longrightarrow 0.$$

Let $\mathcal{E} \subset \mathcal{O}(\underline{a})$ be the inverse image of the subbundle $\mathcal{O}(\underline{c}') \subset \mathcal{O}(\underline{c})$ by the surjective morphism $\mathcal{O}(\underline{a}) \to \mathcal{O}(\underline{c})$, and write $\mathcal{E} = \mathcal{O}(\underline{e})$. Then $\mathcal{E} = \mathcal{O}(\underline{e})$ is an extension of $\mathcal{O}(\underline{c}')$ by $\mathcal{O}(\underline{d})$, or equivalently, $\underline{e} \in \mathrm{Ext}^1(\underline{c}', \underline{d})$. Furthermore, by Snake Lemma,

$$\mathcal{O}(\underline{a})/\mathcal{E} \xrightarrow{\simeq} \mathcal{O}(\underline{c})/\mathcal{O}(\underline{c}') \simeq \mathcal{O}(\underline{c}'').$$

So $\mathcal{O}(\underline{a})$ is an extension of $\mathcal{O}(\underline{c}'')$ by $\mathcal{E} = \mathcal{O}(\underline{e})$. In other words, $\underline{a} \in \mathrm{Ext}^1(\underline{c}'', \underline{e})$.
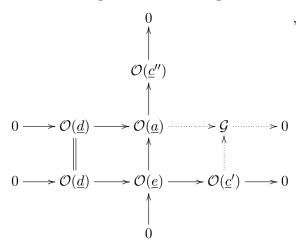
Conversely, suppose that there exists $\underline{e} \in \mathcal{N}(n - p)$ such that $\underline{e} \in \mathrm{Ext}^1(\underline{c}', \underline{d})$, and that $\underline{a} \in \mathrm{Ext}^1(\underline{c}'', \underline{e})$. Then we get two short exact sequences of vector bundles

$$0 \longrightarrow \mathcal{O}(\underline{d}) \longrightarrow \mathcal{O}(\underline{e}) \longrightarrow \mathcal{O}(\underline{c}') \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}(\underline{e}) \longrightarrow \mathcal{O}(\underline{a}) \longrightarrow \mathcal{O}(\underline{c}'') \longrightarrow 0.$$

We can insert them into the following commutative diagram

$$
\begin{array}{ccccccccc}
& & & & 0 & & & & \\
& & & & \uparrow & & & & \\
& & & & \mathcal{O}(\underline{c}'') & & & & \\
& & & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}(\underline{d}) & \longrightarrow & \mathcal{O}(\underline{a}) & \dashrightarrow & \mathcal{G} & \dashrightarrow & 0 \\
& & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}(\underline{d}) & \longrightarrow & \mathcal{O}(\underline{e}) & \longrightarrow & \mathcal{O}(\underline{c}') & \longrightarrow & 0 \\
& & & & \uparrow & & & & \\
& & & & 0 & & & &
\end{array}
\quad ,
$$

where $\mathcal{G}$ is the cokernel of the injective morphism $\mathcal{O}(\underline{d}) \to \mathcal{O}(\underline{a})$. By the Snake Lemma, the coherent sheaf $\mathcal{G}$ on the Fargues-Fontaine curve X can be put into the following short exact sequence

$$0 \longrightarrow \mathcal{O}(\underline{c}') \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}(\underline{c}'') \longrightarrow 0.$$

As $c_{r-p} \geq c_r$, the extension of $\mathcal{O}(\underline{c}'')$ by $\mathcal{O}(\underline{c}')$ splits. In particular, $\mathcal{G}$ is again a vector bundle, and $\mathcal{G} = \mathcal{O}(\underline{c}') \oplus \mathcal{O}(\underline{c}'') = \mathcal{O}(\underline{c})$. It follows that $\underline{a} \in \mathrm{Ext}^1(\underline{c}, \underline{d})$. $\square$

Proposition 5.9 can be used as an algorithm to compute $\mathrm{Ext}^1(\underline{c}, \underline{d})$ inductively on the number of stable blocks in $\underline{c} \oplus \underline{d}$ using duality combined with Corollary 5.5.

Example 5.11. *We have*

$$
\mathrm{Ext}^1\left(\left(0, -\frac{1}{6}^{(6)}\right), \left(-\frac{1}{3}^{(3)}\right)\right) = \left\{\begin{array}{c} \left(-\frac{1}{5}^{(10)}\right), \left(-\frac{1}{6}^{(6)}, -\frac{1}{4}^{(4)}\right), \left(0, -\frac{2}{9}^{(9)}\right) \\ \left(0, -\frac{1}{5}^{(5)}, -\frac{1}{4}^{(4)}\right), \left(0, -\frac{1}{6}^{(6)}, -\frac{1}{3}^{(3)}\right) \end{array}\right\}.
$$

Indeed, by Proposition 5.9, we have

$$
\mathrm{Ext}^1\left(\left(0, -\frac{1}{6}^{(6)}\right), \left(-\frac{1}{3}^{(3)}\right)\right) = \bigcup_{\underline{e} \in \mathrm{Ext}^1\left((0), \left(-\frac{1}{3}^{(3)}\right)\right)} \mathrm{Ext}^1\left(\left(-\frac{1}{6}^{(6)}\right), \underline{e}\right).
$$

Using the main result of [BFHHLWY] *or Corollary 5.5 above, we have*

$$
(5.1.1) \qquad \mathrm{Ext}^1\left((0), \left(-\frac{1}{3}^{(3)}\right)\right) = \left\{\left(-\frac{1}{4}^{(4)}\right), \left(0, -\frac{1}{3}^{(3)}\right)\right\}.
$$

So it remains to compute, for $\underline{e}$ one of the two elements in (5.1.1), the set $\mathrm{Ext}^1\left(\left(-\frac{1}{6}^{(6)}\right), \underline{e}\right)$:

- *if $\underline{e} = \left(-\frac{1}{4}^{(4)}\right)$, again by the main result of* [BFHHLWY]*,*

$$
\mathrm{Ext}^1\left(\left(-\frac{1}{6}^{(6)}\right), \underline{e}\right) = \left\{\left(-\frac{1}{5}^{(10)}\right), \left(-\frac{1}{6}^{(6)}, -\frac{1}{4}^{(4)}\right)\right\};
$$

- *if $\underline{e} = \left(0, -\frac{1}{3}^{(3)}\right)$, then as $0 > -1/6$ any extension of $\mathcal{O}(-1/6)$ by $\mathcal{O}$ splits, so we reduce to computing $\mathrm{Ext}^1\left(\left(-\frac{1}{6}^{(6)}\right), \left(-\frac{1}{3}^{(3)}\right)\right)$, and finally we get*

$$
\mathrm{Ext}^1\left(\left(-\frac{1}{6}^{(6)}\right), \underline{e}\right) = \left\{\left(0, -\frac{2}{9}^{(9)}\right), \left(0, -\frac{1}{5}^{(5)}, -\frac{1}{4}^{(4)}\right), \left(0, -\frac{1}{6}^{(6)}, -\frac{1}{3}^{(3)}\right)\right\}.
$$

Remark 5.12. *As the vector bundle $\mathcal{O}(-1/3)$ is (semi-)stable, by Proposition A.8 we have*

$$\mathrm{Ext}^1\left(\left(0,-\frac{1}{6}^{(6)}\right),\left(-\frac{1}{3}^{(3)}\right)\right) = \widetilde{\mathrm{Ext}}^1\left(\left(0,-\frac{1}{6}^{(6)}\right),\left(-\frac{1}{3}^{(3)}\right)\right).$$

So we can also compute the extension set in Example 5.11 by exploring directly the combinatorial condition in Definition 5.1.

5.2. **Applications.** As we see at the end of last section, the key for verifying if a single Newton stratum is not contained in the weakly admissible locus is the existence of certain particular extensions of vector bundles on the Fargues-Fontaine curve. In this §, we shall give further applications of Theorem 4.9 for the general linear group $G = \mathrm{GL}_n$. The new input here is our discussions in the previous subsection, which allows us to handle more complicated extensions of vector bundles. In the following, to simplify the notation, for an element

$$v = (\lambda_1^{(n_1)}, \ldots, \lambda_s^{(n_s)}) \in \mathcal{N}(n), \quad \text{with} \quad \lambda_1 > \lambda_2 > \ldots > \lambda_s, \text{ and } n_i \in \mathbb{Z}_{>0},$$

if the integer $\lambda_i n_i$ is coprime to n_i, then we omit the exponent (n_i) from the notation.

5.2.1. We take $G = \mathrm{GL}_{14}$, $\mu = (1^{(6)}, 0^{(8)})$, and $b \in B(G)_{basic}$ with $\nu_b = \left(\frac{3}{7}^{(14)}\right) \in \mathcal{N}(14)$. Then b has a reduction b_M to only one proper maximal standard Levi subgroup

$$M = \mathrm{GL}_7 \times \mathrm{GL}_7 \hookrightarrow \mathrm{GL}_{14}.$$

The cocharacters $w\mu$, with w an element in the Weyl group, such that $\langle \nu_b - w\mu, \chi \rangle < 0$ for some $\chi \in X^*(P_1/Z_G)^+$ are

$$(5.2.1) \qquad w\mu \in \left\{ \begin{array}{c} ((1^{(4)},0^{(3)}),(1^{(2)},0^{(5)})), \ ((1^{(5)},0^{(2)}),(1,0^{(6)})) \\ ((1^{(6)},0),(0^{(7)})) \end{array} \right\} \subset \mathcal{N}(M).$$

Thus

$$\nu_{b_M} - w\mu \in \left\{ \begin{array}{c} \left(\left(\frac{3}{7}^{(3)},-\frac{4}{7}^{(4)}\right),\left(\frac{3}{7}^{(5)},-\frac{4}{7}^{(2)}\right)\right), \ \left(\left(\frac{3}{7}^{(2)},-\frac{4}{7}^{(5)}\right),\left(\frac{3}{7}^{(6)},-\frac{4}{7}\right)\right) \\ \left(\left(\frac{3}{7},-\frac{4}{7}^{(6)}\right),\left(\frac{3}{7}^{(7)}\right)\right) \end{array} \right\} \subset \mathcal{N}(M).$$

We want to describe explicitly the generalized Kottwitz set

$$B(M,(k_1,k_2),\nu_{b_M} - w\mu) = B(\mathrm{GL}_7,k_1,v_1) \times B(\mathrm{GL}_7,k_2,v_2)$$

for the rational cocharacter $\nu_b - \mu^w = v_1 \times v_2 \in \mathcal{N}(M) = \mathcal{N}(7) \times \mathcal{N}(7)$ as above (here $k_i := |v_i|$). It suffices to do this for $B(\mathrm{GL}_7,k_i,v_i)$ $(i=1,2)$ respectively.

(i) $\underline{w\mu = ((1^{(4)},0^{(3)}),(1^{(2)},0^{(5)}))}$. So $v_1 = ((3/7)^{(3)},(-4/7)^{(4)})$ and $v_2 = ((3/7)^{(5)},(-4/7)^{(2)})$. The generalized Kottwitz set $B\left(\mathrm{GL}_7,-1,((3/7)^{(3)},(-4/7)^{(4)})\right) \subset \mathcal{N}(7)$ has 7 elements:

$$\left(\frac{1}{3}^{(3)},-\frac{1}{2}^{(4)}\right); \quad \left(0^{(i)},-\frac{1}{7-i}\right), \ 0 \le i \le 5,$$

and $B(\mathrm{GL}_7,1,((3/7)^{(5)},(-4/7)^{(2)})) \subset \mathcal{N}(7)$ consists of the following 6 elements:

$$\left(\frac{2}{5}^{(5)},-\frac{1}{2}^{(2)}\right); \quad \left(\frac{1}{7-i}^{(7-i)},0^{(i)}\right), \ 0 \le i \le 4.$$

(ii) $\underline{w\mu = ((1^{(5)},0^{(2)}),(1,0^{(6)}))}$. So $v_1 = ((3/7)^{(2)},(-4/7)^{(5)})$ and $v_2 = ((3/7)^{(6)},-4/7)$. The generalized Kottwitz set $B\left(\mathrm{GL}_7,-2,((3/7)^{(2)},(-4/7)^{(5)})\right) \subset \mathcal{N}(7)$ has 8 elements:

$$\left(0^{(i)},-\frac{2}{7-i}^{(7-i)}\right), \ 0 \le i \le 3; \quad \left(0^{(j)},-\frac{1}{5-j}^{(5-j)},-\frac{1}{2}^{(2)}\right), \ 0 \le j \le 2; \quad \left(-\frac{1}{4}^{(4)},-\frac{1}{3}^{(3)}\right),$$

and $B(\mathrm{GL}_7, 2, ((3/7)^{(6)}, -4/7)) \subset \mathcal{N}(7)$ consists of the following 6 elements:

$$\left(\frac{2}{5}^{(5)}, 0^{(2)}\right); \quad \left(\frac{1}{3}^{(6)}, 0\right); \quad \left(\frac{1}{3}^{(3)}, \frac{1}{4}^{(4)}\right); \quad \left(\frac{2}{7}^{(7)}\right).$$

(iii) $\underline{w\mu = ((1^{(6)}, 0), (0^{(7)}))}$. So $v_1 = (3/7, (-4/7)^{(6)})$, and $v_2 = ((3/7)^{(7)})$. The generalized Kottwitz set $B\left(\mathrm{GL}_7, -3, (3/7, (-4/7)^{(6)})\right) \subset \mathcal{N}(7)$ has 4 elements:

$$\left(0, -\frac{1}{2}^{(6)}\right), \quad \left(-\frac{1}{3}, -\frac{1}{2}^{(4)}\right); \quad \left(-\frac{2}{5}^{(5)}, -\frac{1}{2}^{(2)}\right); \quad \left(-\frac{3}{7}^{(7)}\right),$$

and $B(\mathrm{GL}_7, 3, ((3/7)^{(7)})) \subset \mathcal{N}(7)$ consists of one single element $((3/7)^{(7)})$.

We would like to give a complete list of the non-empty Newton strata

$$\mathcal{F}(\mathrm{GL}_{15}, \mu, b)^{[b']}, \quad [b'] \in B(G, 0, \nu_b - w_0\mu)$$

that are contained in the weakly admissible locus $\mathcal{F}(\mathrm{GL}_{14}, \mu, b)^{wa}$. As before, it is enough to consider those $[b'] \in B(G, 0, \nu_b - w_0\mu)$ which are HN-indecomposable relative to $\nu_b - w_0\mu$. By Theorem 4.9, the Newton stratum $\mathcal{F}(G, \mu, b)^{[b']} \not\subset \mathcal{F}(G, \mu, b)^{wa}$ if and only if the vector bundle $\mathcal{E}_{b'}$ can be written as an extension of the form

(5.2.2) $$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_{b'} \longrightarrow \mathcal{E}_2 \longrightarrow 0,$$

with $\mathcal{E}_1$ and $\mathcal{E}_2$ two vector bundles, or equivalently, if

$$-\nu_{b'} \in \mathrm{Ext}^1(v_{\mathcal{E}_2}, v_{\mathcal{E}_1}) \subset \mathcal{N}(14),$$

such that the pair $(-v_{\mathcal{E}_1}, -v_{\mathcal{E}_2}) \in B(M, (k_1, k_2), \nu_{b_M} - w\mu)$ for the cocharacters $w\mu$ in (5.2.1). Write

$$\mathsf{E} = \bigcup_{\substack{(-v_{\mathcal{E}_1}, -v_{\mathcal{E}_2}) \in B(M, (k_1, k_2), \nu_{b_M} - w\mu) \\ \text{for a cocharacter } w\mu \text{ in (5.2.1).}}} \mathrm{Ext}^1(v_{\mathcal{E}_2}, v_{\mathcal{E}_1}) \subset \mathcal{N}(14)$$

Proposition 5.13. *Let $v \in \mathcal{N}(14)$. Then $v \in \mathsf{E}$ if and only if one of the following holds:*

(1) *$v = v_{\mathcal{E}_1 \oplus \mathcal{E}_2}$, with $(-v_{\mathcal{E}_1}, -v_{\mathcal{E}_2}) \in B(M, (k_1, k_2), \nu_{b_M} - w\mu)$ for a certain cocharacter $w\mu$ in (5.2.1); or*

(2) *$v \in \mathcal{N}(14)$ is one of the following 10 elements:*

$$\left(\frac{1}{2}^{(4)}, -\frac{1}{5}^{(10)}\right), \quad \left(\frac{1}{2}^{(4)}, -\frac{1}{6}, -\frac{1}{4}\right), \quad \left(\frac{1}{2}^{(4)}, 0, -\frac{2}{9}\right), \quad \left(\frac{1}{2}^{(4)}, 0, -\frac{1}{5}, -\frac{1}{4}\right), \quad \left(\frac{1}{2}^{(4)}, 0^{(2)}, -\frac{1}{4}^{(8)}\right)$$
$$\left(\frac{1}{3}, \frac{1}{6}, -\frac{2}{5}\right), \quad \left(\frac{1}{4}, \frac{1}{5}, -\frac{2}{5}\right), \quad \left(\frac{1}{3}, \frac{1}{5}, 0, -\frac{2}{5}\right), \quad \left(\frac{2}{9}, -\frac{2}{5}\right), \quad \left(\frac{2}{5}, 0, -\frac{2}{5}\right).$$

Moreover, the last 5 elements are all contained in $\mathrm{Ext}^1\left(\left(\frac{1}{2}, -\frac{2}{5}\right), \left(\frac{1}{6}, 0\right)\right)$.

Proof. Observe that, for most of the pairs $(\mathcal{E}_1, \mathcal{E}_2)$ as above, the slopes of $\mathcal{E}_2$ are less or equal to those of $\mathcal{E}_1$, so the extension (5.2.2) is trivial and hence

$$\mathrm{Ext}^1(v_{\mathcal{E}_2}, v_{\mathcal{E}_1}) = \{v_{\mathcal{E}_1 \oplus \mathcal{E}_2}\} \subset \mathcal{N}(14).$$

It remains for us to consider the pairs $(\mathcal{E}_1, \mathcal{E}_2)$ for which there exist non-trivial extensions. In other words, we only need to compute the following sets

$$\mathrm{Ext}^1\left(\left(\frac{1}{2}, -\frac{2}{5}\right), \left(\frac{1}{2}^{(4)}, -\frac{1}{3}\right)\right), \quad \mathrm{Ext}^1\left(\left(0^{(i)}, -\frac{1}{7-i}\right), \left(\frac{1}{2}^{(4)}, -\frac{1}{3}\right)\right), \quad 0 \leq i \leq 4,$$

and

$$\mathrm{Ext}^1\left(\left(\frac{1}{2}, -\frac{2}{5}\right), \left(\frac{1}{7-j}, 0^{(j)}\right)\right), \quad 0 \leq j \leq 5.$$

Using the inductive criterion in Proposition 5.9, we can determine explicitly these 12 sets. For example, we have

$$\mathrm{Ext}^1\left(\left(0,-\frac{1}{6}\right),\left(\frac{1}{2}^{(4)},-\frac{1}{3}\right)\right)=\left\{\begin{array}{c}\left(\frac{1}{2}^{(4)},-\frac{1}{5}^{(10)}\right),\left(\frac{1}{2}^{(4)},-\frac{1}{6},-\frac{1}{4}\right),\left(\frac{1}{2}^{(4)},0,-\frac{2}{9}\right)\\ \left(\frac{1}{2}^{(4)},0,-\frac{1}{5},-\frac{1}{4}\right),\left(\frac{1}{2}^{(4)},0,-\frac{1}{6},-\frac{1}{3}\right)\end{array}\right\}.$$

Indeed, as $1/2$ is bigger than 0 and $-1/6$, there is no non-trivial extension of $\mathcal{O}\oplus\mathcal{O}(-1/6)$ by $\mathcal{O}(1/2)^2$. So, if a vector bundle $\mathcal{E}$ is an extension of $\mathcal{O}\oplus\mathcal{O}(-1/6)$ by $\mathcal{O}(1/2)^2\oplus\mathcal{O}(-1/3)$, then $\mathcal{E}=\mathcal{O}(1/2)^2\oplus\mathcal{E}'$ with $\mathcal{E}'$ an extension of $\mathcal{O}\oplus\mathcal{O}(-1/6)$ by $\mathcal{O}(-1/3)$. So it suffices to compute

$$\mathrm{Ext}^1\left(\left(0,-\frac{1}{6}\right),\left(-\frac{1}{3}\right)\right),$$

which is done in Example 5.11. The other extension sets can be computed in a similar way. To list all the non-trivial extensions, one just keeps in mind that a vector bundle $\mathcal{E}$ may be realized as an extension of $\mathcal{E}_2$ by $\mathcal{E}_1$ for different pairs of vector bundles $(\mathcal{E}_1,\mathcal{E}_2)$ as above. □

Remark 5.14. *As in Example 5.11 above (cf. Remark 5.12), we can also prove Proposition 5.13 by using Proposition A.8.*

With the help of Proposition 5.13, we can now easily list all the non-empty Newton strata that are contained in the admissible locus. In the sequel, we shall list all the HN-indecomposable Newton strata according to the explicit form of $\nu_{b'}$.

(1) $\nu_{b'}=(\frac{3}{8},\dots)$. So $\nu_{b'}=(\frac{3}{8},-\frac{1}{2}^{(6)})$, and the only Newton stratum here is contained in the weakly admissible locus.

(2) $\nu_{b'}=(\frac{2}{5},\dots)$. We have 16 non-empty Newton strata in this case, and none of them is contained in the weakly admissible locus.

(3) $\nu_{b'}=(\frac{1}{3}^{(6)},\dots)$. We have 11 non-empty Newton strata in this case, and 3 of them are contained in the weakly admissible locus: $(\frac{1}{3}^{(6)},-\frac{1}{4}^{(8)})$, $\left(\frac{1}{3}^{(6)},-\frac{1}{5},-\frac{1}{3}\right)$ and $\left(\frac{1}{3}^{(6)},-\frac{1}{6},-\frac{1}{2}\right)$.

(4) $\nu_{b'}=(\frac{1}{3},\frac{1}{4},\dots)$. We find 8 non-empty Newton strata in this case, and none of them is contained in the weakly admissible locus.

(5) $\nu_{b'}=(\frac{1}{3},\frac{1}{5},\dots)$. We find 5 non-empty Newton strata in this case, and 4 of them are contained in the weakly admissible locus: $(\frac{1}{3},\frac{1}{5},-\frac{1}{3}^{(6)})$, $(\frac{1}{3},\frac{1}{5},-\frac{1}{4},-\frac{1}{2})$, $(\frac{1}{3},\frac{1}{5},0,-\frac{2}{5})$, $\left(\frac{1}{3},\frac{1}{5},0,-\frac{1}{3},-\frac{1}{2}\right)$.

(6) $\nu_{b'}=(\frac{1}{3},\frac{1}{6},\dots)$. We have 3 non-empty Newton strata in this case, and 2 of them are in the wa locus: $\nu_{b'}=\left(\frac{1}{3},\frac{1}{6},-\frac{2}{5}\right)$, $\left(\frac{1}{3},\frac{1}{6},-\frac{1}{3},-\frac{1}{2}\right)$.

(7) $\nu_{b'}=(\frac{1}{3},\frac{1}{7},\dots)$. So $\nu_{b'}=(\frac{1}{3},\frac{1}{7},-\frac{1}{2}^{(4)})$ and this Newton stratum is not contained in the weakly admissible locus.

(8) $\nu_{b'}=(\frac{2}{7},\dots)$. We have 8 non-empty Newton strata in this case, and none of them is contained in the weakly admissible locus.

(9) $\nu_{b'}=(\frac{1}{4}^{(8)},\dots)$. We have 5 non-empty Newton strata in this case, and all of them are contained in the weakly admissible locus.

(10) $\nu_{b'}=(\frac{1}{4},\frac{1}{5},\dots)$. We have 3 non-empty Newton strata in this case, and 2 of them are contained in the weakly admissible locus: $(\frac{1}{4},\frac{1}{5},-\frac{2}{5})$ or $(\frac{1}{4},\frac{1}{5},-\frac{1}{3},-\frac{1}{2})$.

(11) $\nu_{b'}=(\frac{1}{4},\frac{1}{6},\dots)$. So $\nu_{b'}=(\frac{1}{4},\frac{1}{6},-\frac{1}{2}^{(4)})$, and the corresponding stratum is not contained in the weakly admissible locus.

(12) $\nu_{b'} = (\frac{2}{9}, \dots)$. We have 3 non-empty Newton strata, and 2 of them are contained in the weakly admissible locus: $(\frac{2}{9}, -\frac{2}{5})$ or $(\frac{2}{9}, -\frac{1}{3}, -\frac{1}{2})$.

(13) $\nu_{b'} = (\frac{1}{5}^{(2)}, \dots)$. So that $\nu_{b'} = (\frac{1}{5}^{(2)}, -\frac{1}{2}^{(2)})$, and the only one Newton stratum here is not contained in the weakly admissible locus.

(14) $\nu_{b'} = (\frac{1}{i}, 0^{(j)}, -\frac{1}{14-i-j})$, $3 \leq i \leq 12$, $0 \leq j \leq 12 - i$. Such a stratum is contained in the weakly admissible locus if and only if $i \geq 8$, or $i \leq 7$ and $0 \leq j \leq 6 - i$.

To summarize, we have 121 non-empty HN-indecomposable Newton strata, and 44 of them are contained in the weakly admissible locus.

5.2.2. We take $G = \mathrm{GL}_{21}$, $\mu = (1^{(9)}, 0^{(12)})$ and thus $b \in B(G)_{basic}$ with $\nu_b = \left(\frac{3}{7}^{(21)}\right) \in \mathcal{N}(21)$. Then b has a reduction to two proper maximal standard Levi subgroups

$$M_1 = \mathrm{GL}_7 \times \mathrm{GL}_{14}, \ M_2 = \mathrm{GL}_{14} \times \mathrm{GL}_7 \hookrightarrow G.$$

For the first Levi subgroup, the cocharacters $w\mu$ such that $\langle \nu_b - w\mu, \chi \rangle < 0$ for some $\chi \in X^*(P_1/Z_G)^+$ are

$$w\mu \in \left\{ \begin{array}{c} \left((1^{(4)}, 0^{(3)}), (1^{(5)}, 0^{(9)})\right), \ \left((1^{(5)}, 0^{(2)}), (1^{(4)}, 0^{(10)})\right) \\ \left((1^{(6)}, 0), (1^{(3)}, 0^{(11)})\right), \ \left((1^{(7)}), (1^{(2)}, 0^{(12)})\right) \end{array} \right\} \subset \mathcal{N}(M_1).$$

Thus

$$\nu_{b_{M_1}} \mu^{w,-1} \in \left\{ \begin{array}{c} \left((\frac{3}{7}^{(3)}, -\frac{4}{7}^{(4)}), (\frac{3}{7}^{(9)}, -\frac{4}{7}^{(5)})\right), \ \left((\frac{3}{7}^{(2)}, -\frac{4}{7}^{(5)}), (\frac{3}{7}^{(10)}, -\frac{4}{7}^{(4)})\right) \\ \left((\frac{3}{7}, -\frac{4}{7}^{(6)}), (\frac{3}{7}^{(11)}, -\frac{4}{7}^{(3)})\right), \ \left((-\frac{4}{7}^{(7)}), (\frac{3}{7}^{(12)}, -\frac{4}{7}^{(2)})\right) \end{array} \right\} \subset \mathcal{N}(M_1).$$

Similarly, for the Levi subgroup M_2, , the cocharacters $w\mu$ such that $\langle \nu_b - w\mu, \chi \rangle < 0$ for some $\chi \in X^*(P_1/Z_G)^+$ are

$$w\mu \in \left\{ \begin{array}{c} \left((1^{(7)}, 0^{(7)}), (1^{(2)}, 0^{(5)})\right), \ \left((1^{(8)}, 0^{(6)}), (1, 0^{(6)})\right) \\ \left((1^{(9)}, 0^{(5)}), (0^{(7)})\right) \end{array} \right\} \subset \mathcal{N}(M_2),$$

and hence

$$\nu_{b_{M_2}} \mu^{w,-1} \in \left\{ \begin{array}{c} \left((\frac{3}{7}^{(7)}, -\frac{4}{7}^{(7)}), (\frac{3}{7}^{(5)}, -\frac{4}{7}^{(2)})\right), \ \left((\frac{3}{7}^{(6)}, -\frac{4}{7}^{(8)}), (\frac{3}{7}^{(6)}, -\frac{4}{7})\right) \\ \left((\frac{3}{7}^{(5)}, -\frac{4}{7}^{(9)}), (\frac{3}{7}^{(7)})\right) \end{array} \right\} \subset \mathcal{N}(M_2).$$

Let $b' \in G(\breve{F})$ with slope vector $\nu_{b'} = \left(\frac{5}{12}, -\frac{5}{9}\right)$. So $\mathcal{E}_{b'} = \mathcal{O}(\frac{5}{9}) \oplus \mathcal{O}(-\frac{5}{12})$. But for each $w\mu \in \mathcal{N}(M_i)$ as above, and for each possible pair $(\mathcal{E}', \mathcal{E}'')$ with

$$-w_{0,M_i} \nu_{\mathcal{E}' \oplus \mathcal{E}''} \preceq \nu_{b_{M_i}} \mu^{w,-1}$$

one checks (by a simple drawing for example) that the maximal slope of $\mathcal{E}'$ is less or equal to $\frac{1}{2} < \frac{5}{9}$. But on the other hand, none of the slope of $\mathcal{E}''$ is $\frac{5}{9}$. Therefore, $\mathcal{E}_{b'}$ can not be an extension of $\mathcal{E}'$ by $\mathcal{E}''$ by our discussions in the previous §: see for example Lemma 5.3 and the combinatorial condition in Definition 5.1. Consequently, by Theorem 4.9, $\mathcal{F}(G, \mu, b)^{[b']}$ is entirely contained in the weakly admissible locus.

Appendix A. Direct description of Ext^1 for GL_n in some cases

In § 5, we describe Ext^1 in an inductive way. In this appendix, we want to prove $\mathrm{Ext}^1 = \widetilde{\mathrm{Ext}}^1$ in some cases. We first need some combinatorial lemmas.

Definition A.1. *Let $n \in \mathbb{N}$ and $\epsilon \in \mathbb{Q} \cap [0, 1)$. Recall that*

$$\mathbb{Q}_+^n = \{\underline{a} = (a_1, \cdots, a_n) \in \mathbb{Q}^n | a_1 \geq a_2 \geq \cdots \geq a_n\}.$$

(1) *An element $\underline{a} = (a_1, \cdots, a_n) \in \mathbb{Q}^n$ is said to have ϵ-breakpoints if the following two conditions are verified:*

- $|\underline{a}| := a_1 + \cdots + a_n \in \mathbb{Z} + \epsilon$; and
- for any $0 < i < n$ with $a_i \neq a_{i+1}$, we have $\sum_{j=1}^{i} a_j \in \mathbb{Z} + \epsilon$.

We say that $\underline{a}$ has integral breakpoints if it has 0-breakpoints.

(2) Set

$$\mathcal{N}(n, \epsilon) := \{\underline{a} = (a_1, \cdots, a_n) \in \mathbb{Q}_+^n \mid \underline{a} \text{ has } \epsilon - \text{breakpoints}\}.$$

(3) For $\underline{a} \in \mathbb{Q}^n$, let $P_{\underline{a}} : [0, n] \to \mathbb{R}$ be the piecewise linear function such that
- $P_{\underline{a}}(0) = 0$;
- $P_{\underline{a}}(i) = a_1 + a_2 + \cdots + a_i$ for $i = 1, \cdots, n$; and
- $P_{\underline{a}}$ is linear on the segment $[i - 1, i]$ for $i = 1, \cdots, n$.

Remark A.2.   (1) $\mathcal{N}(n, 0) = \mathcal{N}(n)$, where $\mathcal{N}(n)$ is the image of the Newton map $\nu :$ $B(\mathrm{GL}_n) \to \mathcal{N}(\mathrm{GL}_n)$ defined in Section 5.1.

(2) Let $0 < d < n$. For $\underline{a} \in \mathcal{N}(n)$, let

$$\tau_{>d}(\underline{a}) := (a_{d+1}, \cdots, a_n).$$

Then $\tau_{>d}(\underline{a}) \in \mathcal{N}(n - d, \epsilon)$ for ϵ to be the fractional part of $-\sum_{j=1}^{d} a_j$ in $[0, 1)$.

Lemma A.3. Let $\epsilon \in [0, 1)$. Let $\underline{a} \in \mathcal{N}(n, \epsilon)$ and $\underline{c} \in \mathcal{N}(n)$, such that $a_i \geq c_i$ for all $1 \leq i \leq n$. Then for any $m \in \mathbb{Z}$ with $|\underline{c}| \leq m \leq |\underline{a}|$, there exists $\underline{b} \in \mathcal{N}(n)$ such that $c_i \leq b_i \leq a_i$ for all $1 \leq i \leq n$ and $|\underline{b}| = m$.

Proof. We prove by induction on $|\underline{a}| - |\underline{c}| \in \mathbb{N} + \epsilon$. If $|\underline{a}| - |\underline{c}| \leq 1$, there is nothing to prove. So we may assume $|\underline{a}| - |\underline{c}| > 1$. Let

$$m_0 := \max\{0 \leq m \leq n | m \in \mathbb{Z}, P_{\underline{a}}(m) \in \mathbb{Z} + \epsilon, P_{\underline{a}}(m) - P_{\underline{c}}(m) < \delta + \epsilon\}$$
$$m_1 := \min\{0 \leq m \leq n | m \in \mathbb{Z}, P_{\underline{a}}(m) \in \mathbb{Z} + \epsilon, P_{\underline{a}}(m) - P_{\underline{c}}(m) \geq \delta + \epsilon\}$$
$$n_0 := \max\{0 \leq m \leq n | m \in \mathbb{Z}, P_{\underline{c}}(m) \in \mathbb{Z}, P_{\underline{a}}(m) - P_{\underline{c}}(m) < \delta + \epsilon\}$$
$$n_1 := \min\{0 \leq m \leq n | m \in \mathbb{Z}, P_{\underline{c}}(m) \in \mathbb{Z}, P_{\underline{a}}(m) - P_{\underline{c}}(m) \geq \delta + \epsilon\}$$

where

$$\delta = \begin{cases} 0, & \text{if } \epsilon \neq 0, \\ 1, & \text{if } \epsilon = 0. \end{cases}$$

Then a_i is constant for $m_0 < i \leq m_1$ and c_i is constant for $n_0 < i \leq n_1$. Moreover,

$$\max(m_0, n_0) < \min(m_1, n_1)$$

as $P_{\underline{a}} - P_{\underline{c}}$ is an increasing function. It suffices to find $\underline{b} \in \mathcal{N}(n)$ such that $|\underline{c}| < |\underline{b}| < |\underline{a}|$.

Let

$$\tilde{b}_i := \begin{cases} c_i, & i \leq n_0 \\ \frac{P_{\underline{a}}(m_1) - \delta - \epsilon - P_{\underline{c}}(n_0)}{m_1 - n_0}, & n_0 < i \leq m_1 \\ a_i, & i > m_1 \end{cases}$$

By definition, $|\underline{\tilde{b}}| = |\underline{a}| - \delta - \epsilon$ and therefore $|\underline{c}| < \tilde{\underline{b}} < |\underline{a}|$.

Claim: $c_i \leq \tilde{b}_i \leq a_i$ for any $n_0 < i \leq m_1$.

Then the existence of $\underline{b}$ follows from Lemma A.4. Now it remains to prove the Claim. For any $n_0 < i \leq m_1$, it suffices to show

$$\text{(A.0.1)} \qquad c_{n_1} = \frac{P_{\underline{c}}(n_1) - P_{\underline{c}}(n_0)}{n_1 - n_0} \leq \tilde{b}_i \leq a_{m_1} = \frac{P_{\underline{a}}(m_1) - P_{\underline{a}}(m_0)}{m_1 - m_0}.$$

We first prove the first inequality. If $m_1 = n_1$, then it holds by the definition of m_1 and $\tilde{b}_i$.

If $m_1 > n_1$, the first inequality is equivalent to $c_{n_1} \leq \frac{P_{\underline{a}}(m_1) - \delta - \epsilon - P_{\underline{c}}(n_1)}{m_1 - n_1}$. This follows from

$$c_{n_1} \leq a_{n_1} = \frac{P_{\underline{a}}(m_1) - P_{\underline{a}}(n_1)}{m_1 - n_1} \leq \frac{P_{\underline{a}}(m_1) - \delta - \epsilon - P_{\underline{c}}(n_1)}{m_1 - n_1}$$

by the definition of n_1.

If $m_1 < n_1$, the first inequality is equivalent to

$$\frac{P_{\underline{c}}(n_1) + \delta + \epsilon - P_{\underline{a}}(m_1)}{n_1 - m_1} \leq \frac{P_{\underline{a}}(m_1) - \delta - \epsilon - P_{\underline{c}}(n_0)}{m_1 - n_0}.$$

This follows from

$$\frac{P_{\underline{c}}(n_1) + \delta + \epsilon - P_{\underline{a}}(m_1)}{n_1 - m_1} \leq \frac{P_{\underline{c}}(n_1) - P_{\underline{c}}(m_1)}{n_1 - m_1} \leq \frac{P_{\underline{c}}(m_1) - P_{\underline{c}}(n_0)}{m_1 - n_0} \leq \frac{P_{\underline{a}}(m_1) - \delta - \epsilon - P_{\underline{c}}(n_0)}{m_1 - n_0},$$

where the inequality in the middle holds because $\underline{c}$ is decreasing.

For the second inequality in (A.0.1), we again distinguish three subcases.

If $m_0 = n_0$, then it's obvious by the definition of m_0.

If $m_0 > n_0$, then the second inequality is equivalent to

$$\frac{P_{\underline{a}}(m_0) - \delta - \epsilon - P_{\underline{c}}(n_0)}{m_0 - n_0} \leq \frac{P_{\underline{a}}(m_1) - P_{\underline{a}}(m_0)}{m_1 - m_0} = a_{n_1}.$$

This holds because the left hand side is bounded by c_{n_1} by the defintion of m_0.

If $m_0 < n_0$, then the second inequality is equivalent to

$$\frac{P_{\underline{a}}(m_1) - \delta - \epsilon - P_{\underline{c}}(n_0)}{m_1 - n_0} \leq \frac{P_{\underline{c}}(n_0) + \delta + \epsilon - P_{\underline{a}}(m_0)}{n_0 - m_0}.$$

This follows from

$$\frac{P_{\underline{a}}(m_1) - \delta - \epsilon - P_{\underline{c}}(n_0)}{m_1 - n_0} \leq \frac{P_{\underline{a}}(m_1) - P_{\underline{a}}(n_0)}{m_1 - n_0} \leq \frac{P_{\underline{a}}(n_0) - P_{\underline{a}}(m_0)}{n_0 - m_0} \leq \frac{P_{\underline{c}}(n_0) + \delta + \epsilon - P_{\underline{a}}(m_0)}{n_0 - m_0}.$$

$\square$

Lemma A.4. *Suppose $\underline{a} \in \mathcal{N}(n, \epsilon)$, $\underline{c} \in \mathcal{N}(n)$. Let $\tilde{\underline{b}} = (\tilde{b}_1, \cdots, \tilde{b}_n) \in \mathbb{Q}^n$ has integral breakpoints such that $a_i \geq \tilde{b}_i \geq c_i$ for all $1 \leq i \leq n$. Then there exists a permutation $\underline{b} := (b_1, \cdots, b_n)$ of $\tilde{\underline{b}}$ such that $\underline{b} \in \mathcal{N}(n)$ and $a_i \geq b_i \geq c_i$ for all $1 \leq i \leq n$.*

Proof. This can be checked directly. $\square$

Lemma A.5. *Suppose $\epsilon_1, \epsilon_2 \in [0, 1)$. Let $\underline{a} \in \mathcal{N}(n, \epsilon_1)$ and $\underline{c} \in \mathcal{N}(n, \epsilon_2)$ such that $a_i \geq c_i$ for all $1 \leq i \leq n$. Assume that $c_1 = \cdots = c_n$. Then for any $m \in \mathbb{Z}$ such that $|\underline{c}| \leq m \leq |\underline{a}|$, there exists $\underline{b} \in \mathcal{N}(n)$ such that $c_i \leq b_i \leq a_i$ for all $1 \leq i \leq n$ and $|\underline{b}| = m$.*

Remark A.6. *If we do not require all the coordinates in $\underline{c}$ are equal, then Lemma A.5 does not hold in general. For example, take $n = 3$, $\underline{a} = (\frac{4}{7}, \frac{4}{7}, 0)$ and $\underline{c} = (\frac{5}{9}, \frac{5}{9}, -1)$. Then $|\underline{a}| = \frac{8}{7} > 1 > |\underline{c}| = \frac{1}{9}$, but there doesn't exist $\underline{b} \in \mathcal{N}(3)$ such that $a_i \geq b_i \geq c_i$ for $i = 1, 2, 3$ and $|\underline{b}| = 1$.*

Proof of Lemma A.5. We prove by induction on n. If $n = 1$, it's obvious. Now we deal with general n. Suppose $|\underline{a}| \geq \lceil |\underline{c}| \rceil =: m_0$. According to Lemma A.3, it suffices to find $\underline{b} \in \mathcal{N}(n)$ such that $a_i \geq b_i \geq c_i$ for all i and $|\underline{b}| = m_0$. Let

$$\underline{c}' = (a_n, \cdots, a_n) \in \mathcal{N}(n, \epsilon_3)$$

where $\epsilon_3 = \{na_n\}$. If $na_n \geq m_0$, then $\underline{b} = (\frac{m_0}{n}, \cdots, \frac{m_0}{n}) \in \mathcal{N}(n)$ is the desired element. If $|\underline{c}'| = na_n < m_0$. Then $|\underline{c}'| < m_0 \leq |\underline{a}|$. Write $a_n = \frac{s}{r}$ with r and s coprime and $r \geq 1$. Let

$$\tau_{\leq n-r}(\underline{a}) := (a_1, \cdots a_{n-r}) \in \mathcal{N}(n - r, \epsilon_1),$$

and $\tau_{\leq n-r}(\underline{c}') \in \mathcal{N}(n-r, \epsilon_3)$ is defined in the same way. Then

$$|\tau_{\leq n-r}(\underline{c}')| = |\underline{c}'| - s < m_0 - s \leq |\tau_{\leq n-r}(\underline{a})| = |\underline{a}| - s.$$

By the induction hypothesis for $n - r$, we may find $(b_1, \cdots, b_{n-r}) \in \mathcal{N}(n - r)$ such that $a_n \leq b_i \leq a_i$ for $1 \leq i \leq n - r$ and $\sum_{i=1}^{n-r} b_i = m_0 - s$. Then

$$b = (b_1, \cdots, b_{n-r}, \underbrace{a_n, \cdots, a_n}_{r}) \in \mathcal{N}(n)$$

satisfies the desired properties. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma A.7. *Let $0 < d < n$. Suppose $\epsilon_1, \epsilon_2 \in [0, 1)$. Let $\underline{a} \in \mathcal{N}(n, \epsilon_1)$ and $\underline{c} \in \mathcal{N}(n - d, \epsilon_2)$ such that $a_{i+d} \geq c_i$ for all $1 \leq i \leq n - d$. Assume that either $c_1 = \cdots = c_{n-d}$ or $\underline{c} \in \mathbb{Z}^{n-d}$. Then for any $m \in \mathbb{Z}$ such that $|\underline{c}| + dc_1 \leq m \leq |\underline{a}|$, there exists $\underline{b} \in \mathcal{N}(n)$ such that $|\underline{b}| = m$, $b_i \leq a_i$ for all $1 \leq i \leq n$, $c_i \leq b_{i+d}$ and for $1 \leq i \leq n - d$.*

Proof. Let

$$\underline{c}' := \begin{cases} (\underbrace{c_1, \cdots, c_1}_{n}), & \text{if } c_1 = \cdots = c_{n-d} \\ (\underbrace{c_1, \cdots, c_1}_{d}, \underline{c}), & \text{if } \underline{c} \in \mathbb{Z}^{n-d} \end{cases} .$$

In the first case, we apply Lemma A.5 to the pair $(\underline{a}, \underline{c}')$ and in the second case, we apply Lemma A.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition A.8. *Suppose $0 < r < n$. Let $\underline{c} \in \mathcal{N}(r)$, $\underline{d} \in \mathcal{N}(n-r)$ and $\underline{a} \in \mathcal{N}(n)$. Suppose $\underline{c}$ or $\underline{d}$ is semistable, then $\mathcal{O}(\underline{a})$ is an extension of $\mathcal{O}(\underline{c})$ by $\mathcal{O}(\underline{d})$ if and only if $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. Equivalently, $\mathrm{Ext}^1(\underline{c}, \underline{d}) = \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$.*

Remark A.9. *Proposition A.8 is also obtained independently by Hong in [Hon22, Theorem 1.1].*

Proof of Proposition A.8. By Proposition 5.3, it suffices to prove the if part. By duality, we may assume that $\underline{d} = d_1^{(n-r)}$ is semistable.

   Claim: We may assume that $d_1 \leq c_r$.

   Indeed, if $d_1 > c_r$, then there exists $1 \leq m < r$ such that $c_{m+1} < d_1 \leq c_m$. There exists natural bijections:

$$\mathrm{Ext}^1(\underline{c}, \underline{d}) \simeq \mathrm{Ext}^1(\tau_{\leq m}\underline{c}, \underline{d}) \text{ and } \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d}) \simeq \widetilde{\mathrm{Ext}}^1(\tau_{\leq m}\underline{c}, \underline{d}).$$

We may replace $\underline{c}$ by $\tau_{\leq m}\underline{c}$. The Claim follows.

   We prove by induction on n. If $n = 1$, then it's trivial. Now assume that the proposition holds for $< n$. We also use induction on $c_1 - d_1 \in \frac{\mathbb{N}}{(n!)^2}$. If $c_1 = d_1$, then $c_1 = \ldots = c_r = d_1$, and $\underline{c}$ is also semistable, then the result follows from Corollary 5.5. Now we consider the general case $c_1 > d_1$. Moreover, we assume $\underline{c}$ non semi-stable: otherwise it follows again from Corollary 5.5.

   Take $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. Without loss of generality, we assume that the slope d_1 of $\underline{d}$ is not bigger than a_n. Then there exists a partition of $\{1, 2, \cdots, n\}$ into two disjoint subsets

$$H = \{h_1 < \cdots < h_r\}, \quad K = \{k_1 < \cdots < k_{n-r}\},$$

and $b \in \mathbb{Q}^n$ such that

- $(b_{h_1}, \cdots, b_{h_r}) \in S_r\underline{c}$, and $(b_{k_1}, \cdots, b_{k_{n-r}}) \in S_{n-r}\underline{d}$;
- $b_i \geq a_i$ if $i \in H$, and $b_i \leq a_i$ if $i \in K$; and
- $\underline{b} \geq \underline{a}$.

Let
$$m := \min\{0 \le s \le n | a_{s+1} \le c_r\}.$$
In particular, $P_{\underline{a}}(m) \in \mathbb{Z}$. Moreover, as $a_{m+1} \le c_r$ and $d_1 \le a_n$, we may assume that

- $H \cap (m, n] = \{i \in \mathbb{N} | m+1 \le i \le h_r\}$;
- $b_{m+1} = \cdots = b_{h_r-l} = c_r$ for some $0 \le l < h_r - m$ and $b_i > c_r$ for $h_r - l < i \le h_r$.

As $a_m > c_r$, we have $c_r \cdot (h_r - l - m) \in \mathbb{Z}$. Let

$$\underline{a}' := \tau_{>h_r-l}(\underline{a}) \in \mathcal{N}(n - h_r + l, \epsilon_1) \quad \text{and} \quad \underline{c}' := \tau_{>h_r}(\underline{b}) \in \mathcal{N}(n - h_r, \epsilon_2)$$

for some $\epsilon_1, \epsilon_2 \in \mathbb{Q} \cap [0, 1)$. So $\underline{c}' = d_1^{(n-h_r)}$. By Lemma A.7, there exists $\underline{e}' \in \mathcal{N}(n - h_r + l)$ such that

$$c_i' \le e_{i+l}' \le a_{i+l}'$$

for all $1 \le i \le n - h_r$, and

$$\begin{aligned}
|\underline{e}'| &= |\underline{a}'| + (a_{m+1} + \ldots + a_{h_r-l}) - c_r(h_r - l - m) \\
&= P_{\underline{a}}(n) - P_{\underline{a}}(m) - c_r(h_r - l - m) \in \mathbb{Z}.
\end{aligned}$$

Note that

$$|\underline{c}'| + l d_1 \le |\underline{e}'| = |\underline{a}'| + (a_{m+1} + \ldots + a_{h_r-l}) - c_r(h_r - l - m) \le |\underline{a}'|.$$

Here the first inequality holds because

$$\begin{aligned}
a_{m+1} + \ldots + a_n &\ge b_{m+1} + \ldots + b_n \\
&\ge (b_{m+1} + \ldots + b_{h_r-l}) + (b_{h_r-l+1} + \ldots + b_n) \\
&\ge (h_r - l - m)c_r + (n - h_r + l)d_1.
\end{aligned}$$

In particular,

$$|\underline{a}'| - |\underline{e}'| = P_{\underline{b}}(h_r - l) - P_{\underline{b}}(m) - P_{\underline{a}}(h_r - l) + P_{\underline{a}}(m).$$

Let
$$e := (a_1, \cdots, a_m, e_1', \cdots, e_{n-h_r+l}') \in \mathcal{N}(n - h_r + l + m).$$

Note that
$$\underline{c} = (\tau_{\le r-h_r+l+m}(\underline{c}), c_r^{(h_r-l-m)}).$$

By Proposition 5.9, to complete the proof, it suffices to verify

$$\underline{a} \in \text{Ext}^1(c_r^{(h_r-l-m)}, \underline{e}) \quad \text{and} \quad \underline{e} \in \text{Ext}^1(\tau_{\le r-h_r+l+m}(\underline{c}), \underline{d}).$$

But we can check

$$\underline{e} \in \widetilde{\text{Ext}}^1(\tau_{\le r-h_r+l+m}(\underline{c}), \underline{d}) = \text{Ext}^1(\tau_{\le r-h_r+l+m}(\underline{c}), \underline{d})$$

where the equality follows from the induction hypothesis, and

$$\underline{a} \in \widetilde{\text{Ext}}^1(c_r^{(h_r-l-m)}, \underline{e}) \simeq \widetilde{\text{Ext}}^1(c_r^{(h_r-l-m)}, \underline{e}') = \text{Ext}^1(c_r^{(h_r-l-m)}, \underline{e}') \simeq \text{Ext}^1(c_r^{(h_r-l-m)}, \underline{e})$$

where the first and third bijection follows from the proof of the Claim and the equality in the middle follows from induction on n if $m > 0$ or the fact that $c_r - e_{n-h_r+l}' < c_1 - d_1$ as $\underline{c}$ is non semi-stable and the induction hypothesis on $c_1 - d_1$ if $m = 0$. $\qquad\square$

Proposition A.10. *Suppose $0 < r < n$. Let $\underline{c} \in \mathcal{N}(r)$, $\underline{d} \in \mathcal{N}(n - r)$ and $\underline{a} \in \mathcal{N}(n)$. Suppose that two elements among $\underline{a}$, $\underline{b}$ and $\underline{c}$ are with all coordinates in $\mathbb{Z}$, then $\mathcal{O}(\underline{a})$ is an extension of $\mathcal{O}(\underline{c})$ by $\mathcal{O}(\underline{d})$ if and only if $\underline{a} \in \widetilde{\text{Ext}}^1(\underline{c}, \underline{d})$. In particular, $\text{Ext}^1(\underline{c}, \underline{d}) = \widetilde{\text{Ext}}^1(\underline{c}, \underline{d})$ if $\underline{d} \in \mathbb{Z}^{n-r}$ and $\underline{c} \in \mathbb{Z}^r$.*

Proof. The proof is similar to that of Proposition A.8. By Proposition 5.3, it suffices to prove the if part.

Assume first $\underline{d} \in \mathbb{Z}^{n-r}$ and $\underline{c} \in \mathbb{Z}^r$. We prove by induction on r. When $r = 1$, it's proved in Proposition A.8. Next, suppose $r \geq 2$ and take $\underline{a} \in \widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. Then there exist a partition

$$\{1, 2, \cdots, n\} = H \coprod K$$

with $H = \{h_1 < \cdots < h_r\}$, and $K = \{k_1 < \cdots < k_{n-r}\}$, and $\underline{b} \in \mathbb{Q}^n$ such that

- $b_i \geq a_i$ if $i \in H$, $b_i \leq a_i$ if $i \in K$;
- $\underline{b} \geq \underline{a}$;
- $(b_{h_1}, \cdots, b_{h_r}) = (c_1, \ldots, c_r)$, and $(b_{k_1}, \cdots, b_{k_{n-r}}) = (d_1, \ldots, d_{n-r})$.

Let

$$m := \min\{0 \leq s \leq n | a_{s+1} = a_{h_r}\}.$$

In particular $P_{\underline{a}}(m) \in \mathbb{Z}$. Without loss of generality, assume $H \cap (m, n] = \{i \in \mathbb{N} | m + 1 \leq i \leq h_r\}$. Let

$$\underline{a}' := \tau_{>m+1}(\underline{a}) \in \mathcal{N}(n - m - 1, \epsilon), \quad \text{and} \quad \underline{b}' := \tau_{>h_r}(\underline{b}) \in \mathcal{N}(n - h_r)$$

for some ϵ. By Lemma A.5, there exists $\underline{e}' \in \mathcal{N}(n - m - 1)$ such that $e_i' \leq a_i'$ for all i, $b_j \leq e_{h_r-m-1+j}'$ for all $1 \leq j \leq n - h_r$, and

$$P_{\underline{a}}(n) - P_{\underline{a}}(m) = |\underline{e}'| + b_{h_r}.$$

Let $\underline{e} = (a_1, \cdots, a_m, \underline{e}') \in \mathcal{N}(n-1)$. Note that $\underline{c} = (\tau_{\leq r-1}(\underline{c}), c_r)$. We can check

$$\underline{a} \in \widetilde{\mathrm{Ext}}^1((c_r), \underline{e}) = \mathrm{Ext}^1((c_r), \underline{e})$$

and

$$\underline{e} \in \widetilde{\mathrm{Ext}}^1(\tau_{\leq r-1}(\underline{c}), \underline{d}) = \mathrm{Ext}^1(\tau_{\leq r-1}(\underline{c}), \underline{d})$$

Indeed, the first equality follows from Proposition A.8, while the second equality follows from induction hypothesis. Then we conclude with Proposition 5.9.

Finally suppose that all the coordinates of $\underline{a}$ and $\underline{d}$ are integers. As above, we prove by induction on r. When $r = 1$, it's proved in Proposition A.8. Next, suppose $r \geq 2$. Then there exist a partition $\{1, 2, \cdots, n\} = H \coprod K$ with

$$H = \{h_1 < \cdots < h_r\}, \quad \text{and} \quad K = \{k_1 < \cdots < k_{n-r}\},$$

and $\underline{b} \in \mathbb{Q}^n$ satisfying

- $(b_{h_1}, \ldots, b_{h_r}) = (c_1, \ldots, c_r)$; and $(b_{k_1}, \ldots, b_{k_{n-r}}) = (d_1, \ldots, d_{n-r})$;
- $b_i \geq a_i$ if $i \in H$, $b_i \leq a_i$ if $i \in K$;
- $\underline{b} \geq \underline{a}$.

Let

$$m := \min\{0 \leq s \leq r | c_{s+1} = c_r\}.$$

In particular $(r - m)c_r \in \mathbb{Z}$, and $b_{h_{m+1}} = b_{h_{m+2}} = \ldots = b_{h_r} = c_r$. Let $\underline{a}' \in \mathcal{N}(n - h_{m+1} + 1 - r + m)$ be the element obtained from

$$\tau_{\geq h_{m+1}} \underline{a} \in \mathcal{N}(n - h_{m+1} + 1)$$

by removing the coordinates $a_{h_{m+1}}, a_{h_{m+2}}, \ldots, a_{h_r}$, and $\underline{b}' \in \mathcal{N}(n - h_{m+1} + 1 - r + m)$ defined from $\underline{b}$ in a similar way. Then $b_i' \leq a_i'$ and all the coordinates of $\underline{b}'$ are integers. As $\underline{b} \geq \underline{a}$,

$$|\underline{a}'| + (a_{h_{m+1}} + a_{h_{m+2}} + \ldots + a_{h_r}) \geq |\underline{b}'| + (r - m)c_r.$$

So it is easy to see that there exists $\underline{e}' \in \mathcal{N}(n - h_{m+1} + 1 - r + m)$ with all the coordinates in $\mathbb{Z}$, such that $b_i' \leq e_i' \leq a_i'$ and that

$$|\underline{a}'| - |\underline{e}'| = (r - m)c_r - (a_{h_{m+1}} + a_{h_{m+2}} + \ldots + a_{h_r}) \in \mathbb{Z}.$$

Let $\underline{e} = (a_1, \cdots, a_{h_{m+1}-1}, \underline{e}') \in \mathcal{N}(n - r + m)$. Note that $\underline{c} = (\tau_{\leq m}(\underline{c}), c_r^{(r-m)})$. We can check that

$$\underline{a} \in \widetilde{\mathrm{Ext}}^1(c_r^{(r-m)}, \underline{e}) = \mathrm{Ext}^1(c_r^{(r-m)}, \underline{e})$$

and

$$\underline{e} \in \widetilde{\mathrm{Ext}}^1(\tau_{\leq m}(\underline{c}), \underline{d}) \cap \mathbb{Z}^{n-r+m} = \mathrm{Ext}^1(\tau_{\leq m}(\underline{c}), \underline{d}) \cap \mathbb{Z}^{n-r+m}.$$

Indeed, the first equality follows from Proposition A.8, while the second equality follows from induction hypothesis. This concludes the proof by Proposition 5.9. $\qquad\square$

Remark A.11. *In Proposition A.10, if all the coordinates of $\underline{a}, \underline{c}$ and $\underline{d}$ are integers, then the same proof as that of the main result of* [Sch00] *shows that $\underline{a} \in \mathrm{Ext}^1(\underline{c}, \underline{d})$ if and only if $\widetilde{\mathrm{Ext}}^1(\underline{c}, \underline{d})$. The argument adopted above is inspired from the proof of Schlesinger in* [Sch00]. *This result is also mentioned in the remark after Example 4.5 in* [Hon22].

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