THE SEMICLASSICAL STRUCTURE OF THE SCATTERING MATRIX FOR A MANIFOLD WITH INFINITE CYLINDRICAL END

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Abstract. We study the microlocal properties of the scattering matrix associated to the semiclassical Schrödinger operator $P = h^2\Delta_X + V$ on a Riemannian manifold with an infinite cylindrical end. The scattering matrix at $E = 1$ is a linear operator $S = S_h$ defined on a Hilbert subspace of $L^2(Y)$ that parameterizes the continuous spectrum of P at energy 1. Here Y is the cross section of the end of X, which is not necessarily connected. We show that, under certain assumptions, microlocally S is a Fourier integral operator associated to the graph of the scattering map $\kappa : \mathcal{D}_\kappa \to T^*Y$, with $\mathcal{D}_\kappa \subset T^*Y$. The scattering map κ and its domain $\mathcal{D}_\kappa$ are determined by the Hamilton flow of the principal symbol of P. As an application we prove that, under additional hypotheses on the scattering map, the eigenvalues of the associated unitary scattering matrix are equidistributed on the unit circle.

1. Introduction

For certain Euclidean or asymptotically conic scattering problems it is known that the scattering matrix quantizes the scattering relation, a mapping determined by the bicharacteristic flow of the principal symbol of the operator in question, e.g. [1, 2, 4, 29, 28]. Here we consider this problem for a class of manifolds with infinite cylindrical ends with an application to the equidistribution of phase shifts of the unitary scattering matrix. Our results are related to results of [41], but are quite different in methodology and technically apply to different classes of manifolds.

Throughout this paper, (X, g) will denote a smooth connected Riemannanian manifold with *infinite cylindrical end*. That is, X has a decomposition as $X = X_C \cup X_\infty$, where X_C is a smooth compact manifold with boundary $\partial X_C = Y$, and $X_\infty \cong (-4, \infty) \times Y$. More precisely, if we denote by g_Y the restriction of g to $TY = T\partial X_C$ (this is a metric on Y), we assume that X_∞ is isometric to $(-4, \infty) \times Y$ with the product metric $(dr)^2 + g_Y$ where r is the natural coordinate on $(-4, \infty)$. We do not necessarily assume that Y is connected. For convenience, we extend r to a smooth function on X, so that $r \leq -4$ on X_C. The (non-negative) Laplacians on X and Y are denoted by Δ_X, Δ_Y respectively.

The purpose of this paper is to study the microlocal properties of the scattering matrix associated to the semiclassical Schrödinger operator

$$P = h^2\Delta_X + V.$$

Here $V = V(h, x) = V_0(x) + h^2V_2(x)$, with $V_0,\ V_2 \in C_c^\infty(X_C)$. The scattering matrix is a linear operator $S = S(h) : \mathbb{1}_{[0,1]}(h^2\Delta_Y)L^2(Y) \to \mathbb{1}_{[0,1]}(h^2\Delta_Y)L^2(Y)$ (where $\mathbb{1}_I$ denotes the characteristic function of the interval I), whose definition we recall in Section 1.2. The space

$$\mathcal{H}_Y := \mathbb{1}_{[0,1]}(h^2\Delta_Y)L^2(Y) = \{f \in L^2(Y) \mid \mathbb{1}_{[0,1]}(h^2\Delta_Y)f = f\} \tag{1}$$

parameterizes the continuous spectrum of P at energy 1, see [12, Section 2.4]. When $1 \notin \operatorname{spec}(h^2\Delta_Y)$, the spectrum of $h^2\Delta_Y$, $\mathcal{H}_Y$ parameterizes the space of bounded, but not L^2, elements of the null space of $P - I$.

We fix once and for all a semiclassical quantization scheme denoted Op_h, associating to compactly supported smooth functions ψ on T^*Y semiclassical pseudodifferential operators $\operatorname{Op}_h(\psi)$ on $L^2(Y)$. Our main Theorems, 1.4 and 1.5, state that, under certain assumptions on the resolvent $(P - 1 - i0)^{-1}$, for suitable functions $\psi \in C_c^\infty(T^*Y)$ the compositon $S \circ \operatorname{Op}_h(\psi)$ is a Fourier integral operator associated to the graph of the scattering map κ. We define κ in Section 1.1. Under some additional hypotheses, including that the set of fixed points of κ^m has measure zero for all $m = 1, 2 \ldots$, we use these results to prove in Theorem 8.4 that the eigenvalues of the associated unitary scattering matrix S_U (unitary on $\mathcal{H}_Y$), are equidistributed on $\mathbb{S}^1$.

The main results are precisely stated in Section 1.3.

1.1. **The scattering map.** The *scattering map* κ is defined on an open subset $\mathcal{D}_\kappa$ of the open unit tangent ball bundle of Y,

$$\mathcal{B} := \{\overline{y} = (y, \eta) \in T^*Y \mid |\eta| < 1\}.$$

The map κ is analogous to the scattering map of [29], and related to the scattering relation of [1, 2] and others.

The definition involves the Hamilton flow Φ_t of $p(x, \xi) = |\xi|^2 + V_0(x)$, the principal symbol of P, on T^*X. Note that since $p_{\restriction T^*X_\infty} = |\xi|^2$, the projections of the trajectories of Φ_t in $T^*X_\infty \subset T^*X$ to X_∞ are geodesics on the product manifold $(-4, \infty) \times Y$.

We will use the following notation: If $\overline{y} = (y, \eta) \in \mathcal{B}$, we let

$$\nu_\pm(\overline{y}) := (0, y, \pm\sqrt{1 - |\eta|^2}, \eta) \in T^*_{(0,y)}X_\infty \subset T^*X. \tag{2}$$

That is, $\nu_\pm(\overline{y})$ are unit covectors in $T^*_{(0,y)}X$ whose restrictions to $\{0\} \times T_yY$ is η, and $\nu_-(\overline{y})$ (resp. $\nu_+(\overline{y})$) has a non-trivial ∂_r component in the direction of X_C (resp. away from X_C).

Definition 1.1. A point $\overline{y}_- = (y_-, \eta_-) \in \mathcal{B}$ is in the domain $\mathcal{D}_\kappa$ of the scattering map κ if and only if the trajectory of $\nu_-(\overline{y})$ under the Hamilton flow Φ_t of p is not forward trapped, that is, if and only if

$$\exists\, T > 0 \text{ such that } \forall\, t > T \qquad \Phi_t(\nu_-(\overline{y})) \in X_\infty.$$

For such $\overline{y}_-$, there is a $t_+ = t_+(\overline{y}_-) > 0$ and a $\overline{y}_+ \in \mathcal{B}$ such that

$$\Phi_{t_+}(\nu(\overline{y}_-)) = \nu_+(\overline{y}_+),$$

and we define

$$\kappa(\overline{y}_-) := \overline{y}_+.$$

Thus $\kappa : \mathcal{D}_\kappa \to \mathcal{B} \subset T^*Y$.

Remarks 1.2. Some remarks may be in order.

(1) Under the hypotheses of the definition, let $(x(t), \xi(t)) = \Phi_t(\nu_-(\overline{y}_-))$. Since $x(0) = (r(0), y(0)) = (0, y_-) \in X_\infty$ and $\dot{r}(0) < 0$, $x(t) \in X_C$ for some $t > 0$. The non-trapping condition means that at some later time the trajectory $(x(t), \xi(t))$ will exit T^*X_C and lie over X_∞.
(2) If $V \equiv 0$, the map κ is the billiard map of $\{r \leq 0\}$, a Riemannian manifold with boundary.
(3) The scattering map does depend on the choice of decomposition of X as $X = X_C \cup X_\infty$, since this choice determines the location of the set $\{r = 0\} \subset X_\infty$. We will see in Remark 6.5

that a different choice of origin for the r coordinate results in a scattering map $\kappa' : \mathcal{D}_{\kappa'} \to \mathcal{B}$ which is of the form

$$\kappa' = \vartheta \circ \kappa \circ \vartheta \quad \text{and} \quad \mathcal{D}_{\kappa'} = \vartheta^{-1}(\mathcal{D}_\kappa), \tag{3}$$

for a certain canonical transformation $\vartheta : \mathcal{B} \to \mathcal{B}$. (Note that κ and κ' are *not* conjugate.)

(4) For all $\overline{y} = (y, \eta) \in T^*Y$ let $\overline{y}' = (y, -\eta)$. Then, using the time-reversibility of the flow Φ_t, it is not hard to see that $\kappa(\kappa(\overline{y})') = \overline{y}'$. Therefore κ is one-to-one.

(5) Examples show that $\mathcal{D}_\kappa$ can be a proper subset of $\mathcal{B}$.

1.2. **The scattering matrix.** For a manifold with an infinite cylindrical end, the scattering matrix for the operator $P = h^2\Delta_X + V$ is a linear operator from $\mathcal{H}_Y$ to itself, where $\mathcal{H}_Y$ is defined in (1). Thus the scattering matrix acts on a finite-dimensional space whose dimension increases as $h > 0$ decreases, and thus can in fact be identified with a matrix, albeit one whose dimension changes with h. In [31, 8, 34] the scattering matrix is defined via its entries in a particular basis. It is more convenient here to take an approach like that is used in the Euclidean or cylindrical end case in [32, Sections 2.7, 7.3], defining the scattering matrix by its action on any element of $\mathcal{H}_Y$. That the two approaches yield the same operator is well-known, easy to check, and is a consequence of our proof of Lemma 1.3.

We also note that there are several conventions in the literature as to exactly which operator is referred to as the scattering matrix. One, which we shall denote S_U, is normalized to be unitary on $\mathcal{H}_Y$; this is found in [8, 34], for example. We shall work primarily with the unnormalized scattering matrix that we denote S, found in [31]. The two are related by $S_U = (I - h^2\Delta_Y)_+^{1/4} S (I - h^2\Delta_Y)_+^{-1/4}$, where $(\bullet)_+$ is the Heaviside function. We shall refer to S_U as the unitary scattering matrix.

The (unitary) scattering matrix can also be defined in a time-dependent way, via the scattering operator. This is done in [35, Section 5]; see also [8, Section 2].

Let $(I - h^2\Delta_Y)^{1/2}$ be the operator on $L^2(Y)$ defined by the spectral theorem, with non-negative real and imaginary parts. Suppose $F \in \langle r \rangle^{1/2+\epsilon} H^2(X)$ for all $\epsilon > 0$ and F is in the null space of $P - 1$. Suppose in addition that h^2 is not the reciprocal of an eigenvalue of Δ_Y. Then on X_∞ a separation of variables argument shows that we can write

$$F \restriction_{X_\infty} (r, y) = e^{-ir(I - h^2\Delta_Y)^{1/2}/h} \mathbb{1}_{[0,1]}(h^2\Delta_Y) f_- + e^{ir(I - h^2\Delta_Y)^{1/2}/h} f_+ \tag{4}$$

for some functions $f_\mp \in L^2(Y)$. We shall refer to f_- as the incoming data, and f_+ as the outgoing data. If $1 \notin \operatorname{spec}(h^2\Delta_Y)$, then the (unnormalized) scattering matrix $S = S(h)$ is such that:

$$S\left(\mathbb{1}_{[0,1]}(h^2\Delta_Y) f_-\right) := \mathbb{1}_{[0,1]}(h^2\Delta_Y) f_+. \tag{5}$$

More precisely:

Lemma 1.3. *If $1 \notin \operatorname{spec}(h^2\Delta_Y)$, for every $f \in \mathcal{H}_Y$ there exists $F \in \langle r \rangle^{1/2+\epsilon} H^2(X)$ in the null space of $P - 1$ such that (4) holds with $\mathbb{1}_{[0,1]}(h^2\Delta_Y) f_- = f$, and the relation $S(f) = \mathbb{1}_{[0,1]}(h^2\Delta_Y) f_+$ defines an operator $S : \mathcal{H}_Y \to \mathcal{H}_Y$. Moreover, if $1 \in \operatorname{spec}(h_0^2\Delta_Y)$, then $\lim_{h' \uparrow h_0} S(h')$ exists as a bounded operator.*

If $1 \in \operatorname{spec}(h_0^2\Delta_Y)$, then we define $S(h_0) = \lim_{h' \uparrow h_0} S(h')$.

Although the results of Lemma 1.3 are known (e.g. [31, 8, 34, 32]), for the convenience of the reader we give a proof in Section 3. Additionally, the proof shows the operator S is (up to sign conventions) consistent with the non-unitary scattering matrices of [31, 8, 34].

Like the scattering map, the scattering matrix depends on the choice of coordinate r on the end, which corresponds to fixing the decomposition $X = X_C \cup X_\infty$. For example, if for $c_0 > -4$

we instead write $X = X'_C \cup X'_\infty$, with $X'_C = X_C \cup \{x = (r, y) \in X_\infty \mid -4 < r \leq c_0\}$ and $X'_\infty = X_\infty \setminus \{x = (r, y) \in X_\infty \mid -4 < r \leq c_0\}$, then the coordinate in the new decomposition is $r' = r - c_0 - 4$. With S' denoting the scattering matrix for the decomposition $X'_C \cup X'_\infty$, $S' = e^{i(c_0+4)(I-h^2\Delta_Y)_+^{1/2}/h} S e^{i(c_0+4)(I-h^2\Delta_Y)_+^{1/2}/h}$. Compare this with the corresponding change in the scattering map, (3).

1.3. **Main results.** In our main theorem we assume that an appropriate cut-off resolvent is bounded at high energy–this is hypothesis (6) of Theorem 1.4. Section 2 contains examples of manifolds and potentials for which this hypothesis holds, and [11, Theorem 3.1] gives a technique for constructing such manifolds. Section 2 also contains examples for which the weaker resolvent bound (7) and the other hypotheses of Theorem 1.5 hold.

Throughout the paper, we use the notation $(P - 1 \pm i0)^{-1} = \lim_{\delta \downarrow 0} (P - 1 \pm i\delta)^{-1}$.

Theorem 1.4. *Suppose there are constants $C_0,\ N_0,\ h_0 > 0$ so that*

$$\|\mathbb{1}_{[0,1]}(h^2\Delta_Y)\mathbb{1}_{[0,1]}(r)(P - 1 - i0)^{-1}\mathbb{1}_{(-\infty,0]}(r)\| \leq C_0 h^{-N_0} \text{ for } 0 < h \leq h_0. \tag{6}$$

Let $\psi \in C_c^\infty(T^*Y)$ have its support in the domain of the scattering map. Then for $0 < h < h_0$ $S \operatorname{Op}_h(\psi)$ and $S_U \operatorname{Op}_h(\psi)$ are semi-classical Fourier integral operators associated with the graph of the scattering map κ.

Proposition 7.4 gives a more explicit expression for the scattering matrix using the Schrödinger propagator and some operators which map between $L^2(Y)$ and $L^2(X_\infty)$. This explicit expression shows how the scattering matrix is a quantum analog of the scattering map defined in Section 1.1; see also Section 1.4.

We remark here that there is some flexibility in choosing the exact cut-offs in (6): we could replace $\mathbb{1}_{[0,1]}(r)$ by $\mathbb{1}_{[b,c]}(r)$ and $\mathbb{1}_{(-\infty,0]}(r)$ by $\mathbb{1}_{(-\infty,a]}(r)$ if $-4 < a < b < c < \infty$. Although we do not prove this, Section 5 proves some results in this direction.

A more restrictive assumption on the manifold and operator than in Theorem 1.4 allows us to make a weaker assumption on the resolvent bound. In this next theorem we assume that X is diffeomorphic to $\mathbf{R} \times Y_0$, but we do not assume that the metric is globally a product metric. In Section 2 we give two families of examples for which the metrics on X have a warped product structure and the resolvent for $P = h^2\Delta_X$ satisfies the estimate (7), but which have quite different trapping properties and quite different quantitative behavior of the eigenvalues of Δ_X.

Theorem 1.5. *Let (Y_0, g_{Y_0}) be a smooth compact Riemannian manifold, and let g be a metric on $X = \mathbf{R} \times Y_0$ which is the product metric $(dr)^2 + g_{Y_0}$ outside of a compact set. Let $P = h^2\Delta_X + V$ satisfy $[P, \Delta_{Y_0}] = 0$. Suppose for any $\epsilon > 0$ there are constants $C_0 = C_0(\epsilon),\ N_0 = N_0(\epsilon),\ h_0 = h_0(\epsilon) > 0$ so that*

$$\|\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)\mathbb{1}_{[0,1]}(r)(P - 1 - i0)^{-1}\mathbb{1}_{(-\infty,0]}(r)\| \leq C_0 h^{-N_0} \text{ for } 0 < h \leq h_0. \tag{7}$$

Let $\psi \in C_c^\infty(T^*Y)$ have its support in the domain of the scattering map. Then for $0 < h < h_0$ $S \operatorname{Op}_h(\psi)$ and $S_U \operatorname{Op}_h(\psi)$ are semi-classical Fourier integral operators associated with the graph of the scattering map κ.

These two theorems are proved by combining the results of Propositions 6.9 and 7.4.

In Section 8 we use these theorems to prove Theorem 8.4. This shows that under some additional hypotheses in the semiclassical limit the eigenvalues of the unitary scattering matrix S_U are equidistributed.

We now comment on the resolvent estimates, (6) and (7). In Euclidean or hyperbolic scattering settings bounds on a cut-off resolvent of a semiclassical operator are well known under non-trapping assumptions on the bicharacteristic flow of the associated Hamiltonian. Moreover, some estimates are known under assumptions that the trapping is relatively mild; see, for example, [42, Section 3] for a recent survey. All of the operators we consider here have nontrivial trapping, as each geodesic in Y corresponds to a trapped bicharacteristic of P in $\{p = 1\} \cap \{r = c\} \subset T^*X_\infty$ for any $c > -4$.

For manifolds with infinite cylindrical ends, $(P - 1 - i0)^{-1} = (h^2\Delta_X + V - 1 - i0)^{-1}$ can have poles for a sequence of $h_j \downarrow 0$. For example, let (Y_0, g_0) be a compact Riemannian manifold, and consider the simplest case $X = \mathbf{R} \times Y_0$ with the product metric. Then for any nontrivial $\chi \in C_c^\infty(X)$, $\chi(h^2\Delta_X - 1 - i0)^{-1}\chi$ has a pole whenever $1/h^2$ is an eigenvalue of Δ_{Y_0}– though in this case including a spectral projection in Δ_Y as is done in (6), as well as a spatial cut-off, is enough to ensure a bound which is polynomial in h. Theorem 3.1 of [11] gives a technique of constructing manifolds (X, g) and operators $P = h^2\Delta_X + V$ so that for any $\chi \in C_c^\infty(X)$, $\|\chi(P - 1 - i0)^{-1}\chi\|$ is polynomially bounded in h. In Section 2 below we give some examples, most using results from [11], for which (6) or (7) holds.

In an effort to simplify the exposition, our results are for the scattering matrix at fixed energy 1, with corresponding hypotheses (6) and (7) on the resolvent at energy 1. However, as is well known a rescaling can be used to prove corresponding results at other positive energies. Let $E > 0$, and write $P - E = E(\frac{1}{E}P - 1) = E(\frac{1}{E}(\Delta_X + V) - 1)$. Setting $(X', g') = (X, Eg)$, we have $\Delta_{X'} = \frac{1}{E}\Delta_X$. By defining $r' = E^{-1/2}(r + 4) - 4$, we see that we can decompose $X' = X'_C \cup X'_\infty$ so that $g' \restriction_{X'_\infty} = (dr')^2 + Eg_Y$, as required in our definition of a manifold with infinite cylindrical end. Then results for the scattering matrix of $P' = \Delta_{X'} + \frac{1}{E}V$ at energy 1 then imply results for the scattering matrix of P at energy E.

1.4. **Idea of the proof.** In order to prove the theorem, we construct the *Poisson operator* $\mathbb{P}$, or, more precisely, the Poisson operator multiplied on the right by $\operatorname{Op}_h(\psi)$, $\mathbb{P}\operatorname{Op}_h(\psi)$. We define the Poisson operator below, and show in Section 3 that it is in fact well-defined.

Definition 1.6. Suppose $1 \notin \operatorname{spec}(h^2\Delta_Y)$. The Poisson operator is a linear operator $\mathbb{P} : L^2(Y) \to \langle r\rangle^{1/2+\delta}H^2(X)$ for any $\delta > 0$ so that for all $f \in L^2(Y)$, $(P-1)\mathbb{P}f = 0$ and $\mathbb{P}f$ has specified incoming data, namely

$$(\mathbb{P}f) \restriction_{X_\infty} = e^{-ir(I-h^2\Delta_Y)^{1/2}/h}\mathbb{1}_{[0,1]}(h^2\Delta_Y)f + e^{ir(I-h^2\Delta_Y)^{1/2}/h}f_+ \tag{8}$$

for some $f_+ \in L^2(Y)$. Moreover, we require that $\langle \mathbb{P}f, g\rangle = \langle \langle r\rangle^{-1/2-\delta}\mathbb{P}f, \langle r\rangle^{1/2+\delta}g\rangle = 0$ for any L^2 eigenfunction g of P with eigenvalue 1.

By the definition of the scattering matrix,

$$S\mathbb{1}_{[0,1]}(h^2\Delta_Y)f = \mathbb{1}_{[0,1]}(h^2\Delta_Y)f_+,$$

where f_+ is as in (8).

We note that a separation of variables on the end X_∞ shows that any L^2 eigenfunction of P must be exponentially decreasing on X_∞, so that its product with an element of $\langle r\rangle^{1/2+\delta}L^2(X)$ is integrable. Thus the pairing $\langle \mathbb{P}f, g\rangle := \langle \langle r\rangle^{-1/2-\delta}\mathbb{P}f, \langle r\rangle^{1/2+\delta}g\rangle$ makes sense. Without the restriction involving the eigenfunctions with eigenvalue 1, $\mathbb{P}$ is not uniquely determined at values of h for which 1 is an eigenvalue of $P = P(h)$.

We now outline the ideas behind the microlocal construction of $\mathbb{P}\operatorname{Op}_h(\psi)$, omitting details here for clarity. Let $\psi \in C_c^\infty(\mathcal{B})$ and choose $\psi_{sp} \in C_c^\infty([0, 1))$ so that $\|\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)\| = O(h^\infty)$.

First, we construct $\mathbb{P}_{\rm appr}$, an approximation of $\mathbb{P}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)$, satisfying the following properties:

- If $f_- = \{f_-(h)\}_{0<h<h_0} \subset L^2(Y)$ is a tempered family, then $\mathbb{P}_{\rm appr} f_-$ has an expansion of the form (8), with f replaced by $\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)f_-$.
- $\|(P-1)\mathbb{P}_{\rm appr}\| = O(h^\infty)$, and $(P-1)\mathbb{P}_{\rm appr} f_-$ is compactly supported.

If we can construct such an approximation $\mathbb{P}_{\rm appr}$, then we can find $\mathbb{P}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)$ by using the outgoing resolvent on X to solve away the error, giving

$$\mathbb{P}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi) = \mathbb{P}_{\rm appr} - (P-1-i0)^{-1}(P-1)\mathbb{P}_{\rm appr}.$$

This is the point at which the hypothesis (6) or (7) is used. We note that $\mathbb{P}$ does depend on ψ, but we omit this in our notation.

We construct $\mathbb{P}_{\rm appr}$ using cut-off functions to piece together three terms: on the end X_∞ we use both the incoming and the outgoing resolvents on the product

$$\tilde{X} = \mathbf{R}_s \times Y, \text{ metric } (ds)^2 + g\restriction_Y,$$

and on a compact subset of X we use (roughly) $\int_0^{t_\psi} e^{it/h}e^{-itP/h}dt$, where $t_\psi > 0$ is appropriately chosen. In studying the outgoing and incoming resolvents on the product manifold $\tilde{X} = \mathbf{R}\times Y$, the operators

$$T_\pm f = \int_{\mathbf{R}} e^{\mp ir'((I-h^2\Delta_Y)^{1/2})/h} f(r',\bullet)dr',\ T_\pm : L^2_c(\mathbf{R}\times Y) \to L^2(Y) \tag{9}$$

arise naturally, see Lemma 4.2. In order to ensure $\mathbb{P}_{\rm appr}$ has the desired incoming data, we shall need a right inverse of T_-. Let $\chi \in C_c^\infty((-1/4,0);\mathbf{R}_+)$ satisfy $\int \chi(r)dr = 1$, and set, for $g \in C^\infty(Y)$,

$$R_\pm g = \chi(r)e^{\pm ir(I-h^2\Delta_Y)_+^{1/2}/h}g, \tag{10}$$

so that $T_\pm R_\pm = I$.

Thus we find $\mathbb{P}_{\rm appr}$ as a sum $\mathbb{P}_{\rm appr} = A_1 + A_2 + A_3$ (see (45) for details). The operator A_1 is determined by $(h^2\Delta_{\tilde{X}} - 1 + i0)^{-1}R_-$ and is chosen so that

$$A_1 \operatorname{Op}_h(\psi)f_-\restriction_{X_\infty} = e^{-ir(I-h^2\Delta_Y)^{1/2}/h}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)f_-$$

–that is, A_1 gives the desired incoming data; compare (8) and Lemma 18. Moreover, the way in which A_1 is chosen ensures the error $(P-1)A_1f_-$ is incoming in the sense that

$$\operatorname{WF}_h((P-1)A_1f_-) \subset \{(r,y,-\sqrt{1-|\eta|^2},\eta) \in T^*X_\infty :\ (y,\eta)\in T^*Y :\ |\eta|<1\},$$

that is, the wavefront set is contained in bicharacteristics pointing towards X_C. The operator A_2 is supported on a compact subset of the manifold X, and it is built using $\int_0^{t_\psi} e^{it/h}e^{-itP/h}dt$ for suitably chosen $t_\psi \in (0,\infty)$. Using the identity $(P-1)\int_0^T e^{it/h}e^{-itP/h}dt = ih(e^{iT/h}e^{-itP/h} - I)$, the operator A_2 is chosen to solve away the incoming error $(P-1)A_1$, but at the cost of a new error. The new error, $(P-1)(A_1+A_2)$, is $O(h^\infty)$ away from the end X_∞, and on the end X_∞ is outgoing in the sense that

$$\operatorname{WF}_h((P-1)(A_1+A_2)f_-) \subset \{(r,y,\sqrt{1-|\eta|^2},\eta) \in T^*X_\infty :\ (y,\eta)\in T^*Y,\ |\eta|<1\}.$$

The operator A_3 solves away this new error, by using the outgoing resolvent on $\tilde{X}$. This third operator, A_3, contributes an outgoing term to the expansion of $\mathbb{P}_{\rm appr}$ on the end X_∞.

Proposition 7.4 gives, up to small error, an explicit expression for the (cut-off) scattering matrix involving R_-, $e^{-it_\psi P/h}$, and T_+. We can read off an approximation of $S\operatorname{Op}_h(\psi)f_-$ from the outgoing term in $\mathbb{P}_{\rm appr}f_-$, and this comes from the term A_3 described above. Since A_3 solves away

the error $(P-1)(A_1+A_2)$, we can see the operator R_- in the expression for the scattering matrix as originating from A_1, $e^{-it_\psi P/h}$ as originating from the use of the propagator in the construction of A_2, and T_+ as coming from the use of the outgoing resolvent on $\tilde{X}$ in A_3, see Lemma 4.2. In conclusion, if $1 \notin \operatorname{spec}(h^2\Delta_Y)$ and (6) holds, then we have that

$$S \operatorname{Op}_h(\psi) = B_1 T_+ B_2 e^{-it_\psi P/h} B_3 R_- B_4 \operatorname{Op}_h(\psi) + O(h^\infty),$$

where B_j, $j = 1, ..., 4$ are semiclassical pseudodifferential operators needed to make the equality hold. (They are explicitly given in the statement of Proposition 7.4, but we omit their definitions here to focus on the bigger picture.)

The operators $e^{-it_\psi P/h}$, T_+, and R_- (the latter two pre-composed with suitable cutoffs that appear in the expression above) are semi-classical Fourier integral operators whose canonical relations roughly parallel the construction of the scattering map κ, as we now explain. (For the precise descriptions of these relations see Section 6.) Recall that for every $\overline{y} \in \mathcal{B}$ there is a corresponding incoming bicharacteristic, namely

$$\gamma_t^{\overline{y}} := \Phi_t(\nu_-(\overline{y})).$$

The relation Θ_-^T of R_- relates points $\overline{y}$ in the unit cotangent bundle of Y with the points of a certain segment $\widetilde{\gamma}^{\overline{y}}$ of $\gamma^{\overline{y}}$,

$$\widetilde{\gamma}^{\overline{y}} = \left\{ \gamma_t^{\overline{y}} \mid t \in (0, t(\overline{y})) \right\}.$$

The pseudodifferential factors to the right of R_- restrict the domain of this relation to a subset of $\mathcal{B}$ such that, for a certain time t_ψ, the image under Φ_{t_ψ} of $\widetilde{\gamma}^{\overline{y}}$ is over $(0,\infty) \times Y \subset X_\infty$ for $\overline{y} \in \operatorname{supp}\psi$. Then $\Phi_{t_\psi}(\widetilde{\gamma}^{\overline{y}})$ is the image of $\widetilde{\gamma}^{\overline{y}}$ under the canonical relation of $e^{-it_\psi P/h}$. Finally, the relation Θ_+ of T_+ relates points in $\Phi_{t_\psi}(\gamma^{\overline{y}})$ with the image $\kappa(\overline{y})$ of $\overline{y}$ under the scattering map. See Figure 1.

Comparing with the steps in the construction of κ, the canonical relation of R_- is analogous to the map $\mathcal{B} \ni \overline{y} \mapsto \nu_-(\overline{y})$, the relation of $e^{-it_\psi P/h}$ is analogous to flowing $\Phi_{t(\overline{y})}(\nu_-(\overline{y}))$, and the relation of T_+ is analogous to projecting $\Phi_{t(\overline{y})}(\nu_-(\overline{y})) \to \kappa(\overline{y})$.

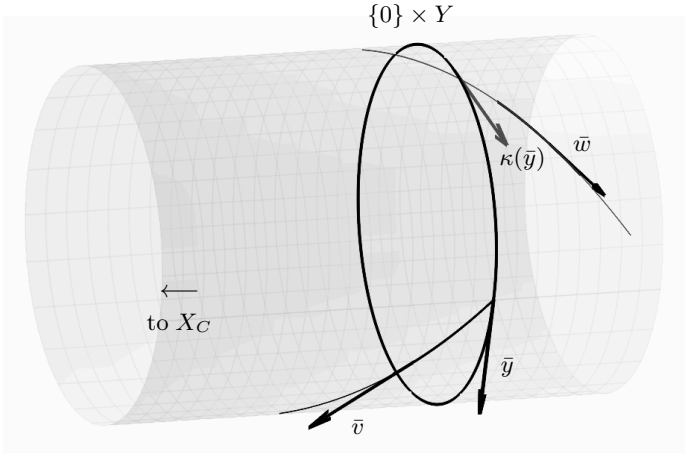

FIGURE 1. $\overline{y} \in \mathcal{D}_\kappa$ and two segments of the corresponding bicharacteristic $\gamma^{\overline{y}}$. The pair $(\overline{v}, \overline{y})$ is in the canonical relation of R_-, $\overline{w} = \Phi_{t_\psi}(\overline{v})$, and $(\kappa(\overline{y}), \overline{w})$ is in the canonical relation of T_+.

Part of the proof of the theorems is to carefully check the various compositions of FIOs which occur, not only in the expression for the scattering matrix, but also in the construction of $\mathbb{P}_{\text{appr}}$. This is done in Section 6. The constructions of $\mathbb{P}\operatorname{Op}_h(\psi)$ and $S\operatorname{Op}_h(\psi)$ are carried out rigorously in Section 7.

1.5. **Background and related work.** An introduction to the spectral and scattering theory of manifolds with infinite cylindrical ends can be found in [24, 27, 31], with further results for the scattering matrix in [8, 34]. A relatively short self-contained introduction may also be found in [12, Section 2]. The papers [9, 13] use a detailed microlocal analysis of the scattering matrix applied to a specific function in an inverse problem.

In [41] Zelditch and Zworski consider a family of surfaces of revolution having a single connected asymptotically cylindrical end, proving a result for the pair correlation measure of the phase shifts of the (unitary) scattering matrix. This is stronger than our equidistribution result, Theorem 8.4, but is for a particular class of surfaces of revolution. Additionally, Proposition 3 of [41] shows that for the surfaces under consideration the truncated scattering matrix (in their setting, $S\mathbb{1}_{[\epsilon,1-\epsilon]}(h^2\Delta_Y)$) is a semiclassical quantum map associated to the scattering map κ. The paper [41] uses the warped product structure of the surface and a separation of variables argument to reduce the problem to a study of a family of one-dimensional problems.

There are many papers which use microlocal analysis to study the properties of the scattering matrix in Euclidean scattering. Alexandrova [1, 2] shows that under suitable assumptions the scattering amplitude for a compactly supported perturbation of the semiclassical Euclidean Laplacian quantizes the scattering relation (see also [3, 4]). A related result for the asymptotically conic setting is [28]. Ingremeau [29] studies mapping properties of the scattering matrix on Gaussian coherent states for (non-trapping) semiclassical Schrödinger operators on $\mathbf{R}^n$. Our proofs of Theorems 1.4 and 1.5 have been influenced by both [1] and [29], as well as by [18, Section 3.11]. There are many other results which use microlocal techniques to find asymptotics of the scattering matrix in Euclidean settings. We mention just a few, [39, 25, 37, 38, 23, 33] and refer the reader to the cited papers for further references.

The distribution of phase shifts has been studied in a number of Euclidean settings, e.g. [6, 16, 20, 30, 21, 19]. Some of these papers use the results of Alexandrova or Ingremeau on the microlocal structure of the scattering matrix. Our Theorem 8.4 is an application of Theorem 1.4 or 1.5 to prove an equidistribution result in the cylindrical end setting.

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2. Examples for which one of the resolvent estimates holds

In this section we give some examples of manifolds for which the estimates (6) or (7) on the cut-off resolvent for $P = h^2\Delta_X$ or $P = h^2\Delta_X + V$ for certain potentials $V = V_0 + h^2V_2$ holds.

2.1. **An example with a single connected end.** For $n \geq 2$ we can give $X = \mathbf{R}^n$ a warped product structure that makes it a manifold with an infinite cylindrical end. Let ρ be the radial

coordinate on $\mathbf{R}^n$, and let $g_0 = d\rho^2 + f(\rho)g_{\mathbb{S}^{n-1}}$, where $g_{\mathbb{S}^{n-1}}$ is the usual metric on the unit sphere $\mathbb{S}^{n-1}$. We assume $f \in C^\infty([0,\infty))$, $f(\rho) = \rho^2$ in a neighborhood of $\rho = 0$, the support of f' is $[0,\rho_0]$, with $f'(\rho) > 0$ for $\rho \in (0,\rho_0)$. The $d = 2$ case is illustrated by Figure 2. Then the only trapped geodesics are those which lie in a hypersurface $\{\rho = c\}$ for any $c \geq \rho_0$.

FIGURE 2. A cigar-shaped, two-dimensional warped product.

Let g be any metric on X so that $g - g_0$ is supported in $\{(\rho, y) \mid \rho < \rho_0\}$, and so that g has the same trapped geodesics as g_0 does. A discussion of constructing such metrics can be found in [11, Example 1]. We remark that there are metrics satisfying these conditions which are not rotationally symmetric.

For such manifolds (X, g), $Y = \mathbb{S}^{n-1}$ and all of $\mathcal{B}$ is in the domain of the scattering map.

By [11, Theorem 1.1], for any $\chi \in C_c^\infty(X)$, $\|(\chi h^2\Delta_X - 1 - i0)^{-1}\chi\| = O(h^{-2})$ when $h > 0$ is sufficiently small. Hence the estimate (6) holds for $P = h^2\Delta_X$ on (X, g), with $N_0 = 2$. Moreover, by [11, Theorem 3.1], (6) holds for $P = h^2\Delta_X + V$ for a class of potentials $V \in C_c^\infty(X; \mathbf{R})$.

We remark that in the case of a rotationally symmetric surface these manifolds are very similar to, but not the same as, the surfaces considered in [41].

2.2. **Examples modifying hyperbolic surfaces.** Starting with a convex cocompact hyperbolic surface (X, g_H), one can modify the metric on the ends of the manifold X in such a way as to obtain a manifold with cylindrical ends so that the cut-off resolvent is polynomially bounded.

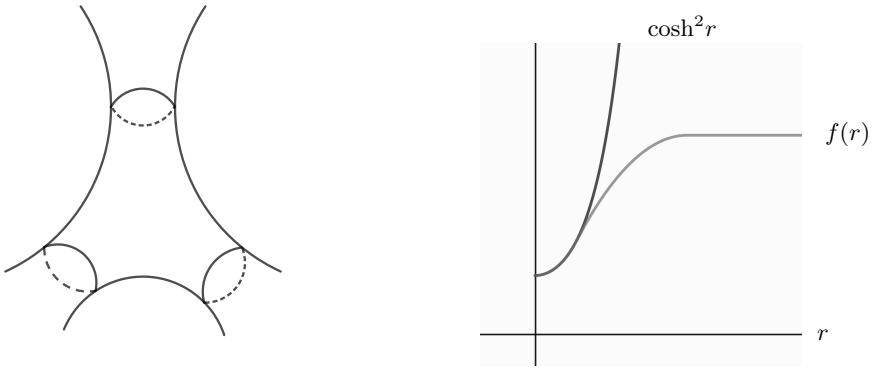

FIGURE 3. A hyperbolic surface (X, g_H) with three funnels, and an example of a function f satisfying the conditions on the warping function in Section 2.2.

There is a compact set $N \subset X$ so that $X \backslash N = (-4, \infty)_r \times Y_y$, and $g_H \restriction_{X\backslash N} = dr^2 + \cosh^2(r+4) g_Y$. Here we have modified slightly the usual convention to fit with our convention of using the coordinate $r \in (-4, \infty)$ on the ends of our manifold X. The set N is called the convex core of X, and the manifold Y is the disjoint union of k circles that might not have the same length. Let $f \in C^\infty(\mathbf{R}; (0,\infty))$ be equal to $\cosh^2 r$ near $r = 0$, with f' compactly supported and $f' > 0$ on the interior of the convex hull of its support. Let g be the smooth metric on X defined by $g \restriction_N = g_H \restriction_N$, and $g \restriction_{X\backslash N} = dr^2 + f(r+4) g_Y$.

Under these hypotheses, by [11, Theorem 1.1] using results of [5, 17] the Laplacian Δ_X on (X, g) satisfies $\|\chi(h^2\Delta_X - 1 - i0)^{-1}\chi\| = O(h^{-2})$ for any $\chi \in C_c^\infty(X)$, implying (6) with $N_0 = 2$. In fact the result is a bit stronger; see [11, Section 3.3] for further discussion and references.

In higher dimensions it is possible, but more complicated, to do a similar construction to modify the metrics on (certain) hyperbolic manifolds to give manifolds with infinite cylindrical ends so that the resolvent of the semiclassical Laplacian satisfies (6); see [11, Section 3.3].

2.3. **Right circular cylinder.** Set $X = \mathbf{R}_s \times \mathbb{S}^1_y$, where $\mathbb{S}^1$ is the unit circle, and consider the product metric on X. Let $W \in C_c^\infty(X; \mathbf{R})$ satisfy $W_0(s) := \int_0^{2\pi} W(s,y)dy \geq 0$, with $W_0 \not\equiv 0$. Then by [10, Proposition 4.4 and Lemma 4.5] the operator $P = h^2\Delta_X + h^2 W$ satisfies (6) with $N_0 = 2$. Thus our results can be interpreted to give results for the nonsemiclassical Schrödinger operator at high energy.

In this case, the scattering map has as its domain all of $\mathcal{B}$ and we can find the scattering map explicitly. Here Y is the disjoint union of two circles, which we write $Y = \mathbb{S}^1_L \sqcup \mathbb{S}^1_R$ for the cross sections of the connected ends of X on which s is bounded above (for $\mathbb{S}^1_L$; the "left" end) or s is bounded below (for $\mathbb{S}^1_R$; the "right" end). We use global coordinates $(s, y) \in \mathbf{R} \times [0, 2\pi)$ on $\mathbf{R} \times \mathbb{S}^1$, and use these same coordinates y on $\mathbb{S}^1_L$ and $\mathbb{S}^1_R$. Thus we can see in a particularly simple example how our choice of function r giving a coordinate on X_∞ (or equivalently the decomposition $X = X_C \cup X_\infty$) affects the scattering map.

Suppose $\operatorname{supp}(W) \subset [-a, a] \times \mathbb{S}^1$, and set $X_C = [-a, a] \times \mathbb{S}^1$. Then the sets $\{\pm s = a + 4\}$ correspond to the set $\{r = 0\} \subset X$. Recalling that $P = h^2\Delta_X + h^2 W$ here, a simple computation finds that if $(y_-, \eta_-) \in T^*\mathbb{S}^1_R \subset T^*Y$ with $|\eta_-| < 1$ then $\kappa(y_-, \eta_-) = (y_+, \eta_-)$, where $y_+ \in \mathbb{S}^1_L$ and, modulo 2π, $y_+ = y_- + 2(a+4)\eta_- / \sqrt{1 - \eta_-^2}$. A similar computation works for points in $T^*\mathbb{S}^1_L$.

2.4. **Warped products.** Set $X = \mathbf{R}_s \times (Y_0)_y$ and $g = ds^2 + (f(s))^{4/(n-1)} dg_{Y_0}$, where (Y_0, g_{Y_0}) is a smooth compact Riemannian manifold and $f \in C^\infty(\mathbf{R}; \mathbf{R}_+)$, with $f(s) = 1$ if $|s| > a$. We consider two special classes of functions f, which give rise to manifolds with qualitatively different behavior both in terms of the trapped geodesics and in terms of the number of embedded eigenvalues of $h^2\Delta_X$. For the first one (6) (and hence also (7)) holds for $P = h^2\Delta_X$ (and $P = h^2\Delta_X + V$ for some V), and for the second we show that (7) holds for $P = h^2\Delta_X$.

Here Y is the disjoint union of two copies of Y_0. We write $Y = Y_{0L} \sqcup Y_{0R}$, where Y_{0L} and Y_{0R} are copies of Y_0 identified with the cross section of the "left" and "right" ends of X, respectively.

2.4.1. *Hourglass-type warped products.* In addition to the assumptions made on f above, assume that f has a single critical point in $(-a, a)$, and it is a nondegenerate minimum. The surface on the left in Figure 4 provides an example. Then by [11, Theorem 3.1], see [11, Section 3.4], for any $\chi \in C_c^\infty(X)$,

$$\|\chi(h^2\Delta_X - 1 - i0)^{-1}\chi\| = O(h^{-2}) \text{ for } h > 0 \text{ sufficiently small.} \tag{11}$$

Thus the estimate (6) holds with $N_0 = 2$ for $P = h^2\Delta_X$. We note that the estimate (11) implies that Δ_X has only finitely many eigenvalues. For this manifold, each geodesic which is both forward and backward trapped lies in a set $\{s = c\}$ for some $c \in \mathbf{R}$ with $f'(c) = 0$.

For Schrödinger operators $P = h^2\Delta_X + V$, where $V \in C_c^\infty(X; \mathbf{R})$ satisfies certain conditions the estimate (11) holds, see [11, Theorem 3.1]. For example, if $V_2 \in C_c^\infty(X; \mathbf{R})$, and $V(x) = V(x, h) = h^2 V_2(X)$, then (11) holds. For this example, because we use the results of [11] to prove the estimate (6), the potentials V need not be functions of s alone.

We now return to the case $P = h^2\Delta_X$. Let f_m be the minimum value of f, and set $|\eta|_c = f_m^{2/(n-1)}$. Then using properties of geodesics on warped products, the domain of the scattering map is $\{(y, \eta) \in T^*Y \mid |\eta| < 1 \text{ and } |\eta| \neq |\eta|_c\}$. Suppose $(y_-, \eta_-) \in T^*Y_{0L}$. If $|\eta_-| < |\eta|_c$, then $\kappa(y_-, \eta_-) \in T^*Y_{0R}$, while if $|\eta|_c < |\eta_-| < 1$, $\kappa(y_-, \eta_-) \in T^*Y_{0L}$.

We introduce some notation to describe one consequence of this for the scattering matrix. Let $\pi_L : L^2(Y_{0L} \sqcup Y_{0R}) \to L^2(Y_{0L})$ and $\pi_R : L^2(Y_{0L} \sqcup Y_{0R}) \to L^2(Y_{0R})$ be the natural orthogonal projections. Then if $\psi_s \in C_c^\infty(\mathbf{R})$ is supported in $(-\infty, |\eta|_c^2)$, it follows from the mapping properties of κ and Theorem 1.4 that $\|\pi_L S \psi_s(h^2\Delta_Y)\pi_L\| = O(h^\infty)$. Likewise, if $\psi_l \in C_c^\infty(\mathbf{R})$ is supported in $(|\eta|_c^2, 1)$, then $\|\pi_R S \psi_l(h^2\Delta_Y)\pi_L\| = O(h^\infty)$.

Of course, there are similar results focusing on right multiplication by π_R rather than π_L.

2.4.2. *Warped products with bulges.* Now consider what is in some sense the opposite situation to that of Section 2.4.1: in addition to the general assumptions on f in Section 2.4, assume that f has a single critical point in $(-a, a)$, and it is a maximum. In Figure 4, the figure on the left illustrates the hourglass-type warped products of Section 2.4.1, while that on the right illustrates the warped products with bulges discussed in this section.

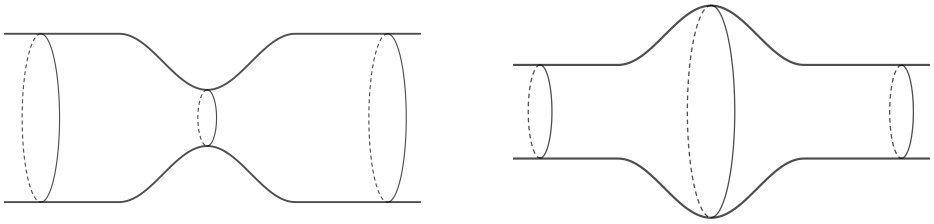

FIGURE 4. An hourglass-shaped warped product (left), and a warped product with a bulge (right).

With these assumptions on f, X has infinitely many trapped geodesics that lie entirely in the region with $s \in (-a, a)$, and it is straightforward to show via a separation of variables and results from semiclassical analysis that Δ_X has infinitely many eigenvalues accumulating at infinity, [14, 34]. Hence if $\chi \in C_c^\infty(X)$ is nontrivial, then there is a sequence $\{h_j\}$ tending to 0 so that $\lim_{\epsilon \downarrow 0} \|\chi(h_j^2\Delta_X - 1 - i\epsilon)^{-1}\chi\| = \infty$. Nonetheless, we show in Lemma A.1 that (7) holds with $N_0 = 1$. In comparison with the example of Section 2.4.1, the estimate is improved: $N_0 = 1$ here, compared to $N_0 = 2$ in the hourglass-type example. This may be surprising, since the trapping in the warped products with bulges is stronger than that in the hourglass-type warped products. This difference might be attributed to the fact that our microlocal cutoff in the cross-section, $\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)$ has the effect of cutting off away from trapped bicharacteristics in $\{p = 1\}$ in the

examples of this section, but not the examples of Section 2.4.1. Alternatively, the difference may be an artifact of the proof.

In this setting, the domain of the scattering map κ is all of $\mathcal{B}$. Using the notation of Section 2.4.1, if $(y_-,\eta_-)\in\mathcal{B}\cap T^*Y_{0L}$, then $\kappa(y_-,\eta_-)\in T^*Y_{0R}$. Then by Theorem 1.5 for any $\psi\in C_c^\infty([-1,1))$, $\|\pi_{0L}S\psi(h^2\Delta_Y)\pi_{0L}\|=O(h^\infty)$.

For the special case of a surface of revolution with a bulge we compute the scattering map in Section A.2.

3. Existence of the Poisson operator and the scattering matrix

In this section we discuss the Poisson operator, introduced in Section 1.4, and prove some consequences for the scattering matrix, Lemmas 1.3 and 3.1. The construction we give of the Poisson operator in this section is different from the more microlocal construction that we will give in Section 7. Much of the content of this section is known, see e.g. [31, 8, 34, 32, 12], but we include it for the reader's convenience.

We begin by checking that there is an operator satisfying the conditions given in Definition 1.6 to define the Poisson operator, and that this uniquely determines the operator. Recall that we assume $1\notin\operatorname{spec}(h^2\Delta_Y)$. Let $\mathcal{PR}_1$ denote orthogonal projection onto the eigenfunctions of P with eigenvalue 1, with $\mathcal{PR}_1=0$ if 1 is not an eigenvalue of P. Then it follows from [31, Section 6.8] or [12, Lemmas 2.2 and 2.3] that $\lim_{\epsilon\downarrow 0}\langle r\rangle^{-(1/2+\delta)}(P-1\pm i\epsilon)^{-1}\left(I-\mathcal{PR}_1\right)\tilde\chi$ is a bounded operator on $L^2(X)$ for any $\tilde\chi\in L_c^\infty(X)$ and $\delta>0$.

Let $\varphi\in C^\infty(\mathbb{R};[0,1])$ satisfy $\varphi(r)=1$ for $r\geq 0$, and $\varphi(r)=0$ for $r\leq -1/2$. Given $f\in L^2(Y)$, set

$$F_{\tilde X}(r,y)=e^{-ir(I-h^2\Delta_Y)_+^{1/2}/h}\mathbb{1}_{[0,1]}(h^2\Delta_Y)f\in\langle r\rangle^{1/2+\delta}H^2(\mathbf{R}\times Y) \tag{12}$$

and

$$F_{X_\infty}(r,y)=\varphi(r)F_{\tilde X}\in\langle r\rangle^{1/2+\delta}H^2(X). \tag{13}$$

Then

$$(P-1)F_{X_\infty}=(h^2\Delta_X-1)F_{X_\infty}=\left(-h^2\varphi''(r)-2h^2\varphi'(r)\partial_r\right)F_{\tilde X}$$

has compact support on $X_\infty\subset X$. Moreover, this function is orthogonal to any eigenfunction of P with eigenvalue 1. This is because a separation of variables argument shows that if $g\in L^2(X)$ satisfies $(P-1)g=0$, then $\mathbb{1}_{[0,1]}(h^2\Delta_Y)(g\restriction_{X_\infty})=0$.

We now set

$$\mathbb{P}f=F_{X_\infty}-(P-1-i0)^{-1}(h^2\Delta_X-1)F_{X_\infty}=F_{X_\infty}-\lim_{\epsilon\downarrow 0}(P-1-i\epsilon)^{-1}(h^2\Delta_X-1)F_{X_\infty} \tag{14}$$

and check that it satisfies the requirements on $\mathbb{P}f$ made in the definition of the Poisson operator. Note that by construction, $(P-1)\mathbb{P}f=0$ and if $g\in L^2(X)$ satisfies $(P-1)g=0$, then $\langle\mathbb{P}f,g\rangle=0$. Since

$$\left((P-1-i0)^{-1}(h^2\Delta_X-1)F_{X_\infty}\right)\restriction_{r>0}=e^{ir(I-h^2\Delta_Y)^{1/2}/h}f_+$$

for some function $f_+\in L^2(X)$, we have shown that $\mathbb{P}f\in\langle r\rangle^{1/2+\delta}H^2(X)$, and thus have shown that there is an operator satisfying the conditions of Definition 1.6.

Next we consider uniqueness. Suppose there are two such Poisson operators, $\mathbb{P}$ and $\tilde{\mathbb{P}}$. For incoming data $f\in L^2(Y)$, denote the corresponding outgoing data by f_+ and $\tilde f_+$, respectively. Let $\{\phi_j\}$ be a complete set of orthonormal eigenfunctions of Δ_Y, with

$$\Delta_Y\phi_j=\sigma_j^2\phi_j,\ \text{and}\ 0=\sigma_1^2\leq\sigma_2^2\leq\cdots. \tag{15}$$

We have $(P-1)(\mathbb{P}-\tilde{\mathbb{P}})f = 0$ and

$$\Big((\mathbb{P}-\tilde{\mathbb{P}})f\Big)\restriction_{X_\infty} = e^{ir(I-h^2\Delta_Y)^{1/2}/h}(f_+ - \tilde{f}_+) = \sum_j c_j e^{ir(1-h^2\sigma_j^2)^{1/2}/h}\phi_j \tag{16}$$

for some $\{c_j\}$, $c_j \in \mathbb{C}$. Here we use the convention that $(1-h^2\sigma_j^2)^{1/2}$ has nonnegative real and imaginary parts, as in the definition of $(I-h^2\Delta_Y)^{1/2}$. Applying a Stokes' identity gives

$$0 = \lim_{R\to\infty}\int_{X:r<R}\Big((P-1)(\mathbb{P}-\tilde{\mathbb{P}})f\Big)\overline{(\mathbb{P}-\tilde{\mathbb{P}})f}\,dx = -\frac{2i}{h}\sum_{h^2\sigma_j^2<1}\sqrt{1-h^2\sigma_j^2}|c_j|^2$$

implying that $c_j = 0$ if $h^2\sigma_j^2 < 1$, so that $(\mathbb{P}-\tilde{\mathbb{P}})f \in L^2(X)$. Thus $(\mathbb{P}-\tilde{\mathbb{P}})f$ is an L^2 element of the null space of $P-1$, and hence is either 0 or an eigenfunction. But $\langle(\mathbb{P}-\tilde{\mathbb{P}})f, g\rangle = 0$ for any eigenfunction g of P with eigenvalue 1, so $(\mathbb{P}-\tilde{\mathbb{P}})f \equiv 0$ and the Poisson operator is uniquely defined.

We remark that our argument above shows that if we omit the requirement $\langle \mathbb{P}f, g\rangle = 0$ for every $g \in L^2(X)$ in the null space of $P-1$, then $\mathbb{P}f$ is determined up to addition of an eigenfunction of P with eigenvalue 1. We shall use this below.

Proof of Lemma 1.3. For $f \in \mathcal{H}_Y$, the function $F = \mathbb{P}f$ satisfies the conditions of the first part of the Lemma. Our argument above shows that if $F,\ \tilde{F} \in \langle r\rangle^{1/2+\delta}H^2(X)$ with $(P-1)F = 0 = (P-1)\tilde{F}$, and both F and $\tilde{F}$ have an expansion as in (4) with the same incoming data $f = f_- = \tilde{f}_-$, then $F-\tilde{F}$ is in the L^2 null space of $P-1$. Hence $F-\tilde{F}$ is either 0 or an L^2 eigenfunction. Since for any L^2 eigenfunction g with eigenvalue 1, $\mathbb{1}_{[0,1]}(h^2\Delta_Y)(g\restriction_{X_\infty}) = 0$, this ensures $\mathbb{1}_{[0,1]}(h^2\Delta_Y)(f_+ - \tilde{f}_+) = 0$, where f_+, $\tilde{f}_+$ are the outgoing data as in (4) for F and $\tilde{F}$, respectively. Thus the scattering matrix is well-defined when $1 \notin \operatorname{spec}(h^2\Delta_Y)$.

We next relate the scattering matrix S defined above to those of [8, Section 1.3] and [34]; see also [31, Section 6.10], again under the assumption that $1 \notin \operatorname{spec}(h^2\Delta_Y)$. Suppose $h^2\sigma_k^2 < 1$. Then we can write the expansion as in (8) for $\mathbb{P}\phi_k$ as

$$(\mathbb{P}\phi_k)\restriction_{X_\infty}(r,y) = e^{-ir(1-h^2\sigma_k^2)^{1/2}/h}\phi_k + \sum_j e^{ir(1-h^2\sigma_j^2)^{1/2}/h}S^{\#}_{jk}\phi_j$$

where $S^{\#}_{jk} = S^{\#}_{jk}(h)$ are some scalars. This uniquely determines the $S^{\#}_{jk}$ if $h^2\sigma_j^2 < 1$. Comparing our definition of S, we find $S\phi_k = \sum_{h^2\sigma_j^2\le 1} S^{\#}_{jk}\phi_j$. Moreover, this shows that the scattering matrix of [8, Definition 1.3] is $S_U = (I-h^2\Delta_Y)_+^{1/4}S(I-h^2\Delta_Y)_+^{-1/4}$; this operator is unitary on $\mathcal{H}_Y$.

Combining this with results of [8, Section 1.3] shows that if $h^2\sigma_j^2 = 1$ for some j, then $\lim_{h'\uparrow h} S(h')$ exists as a bounded operator. □

We shall later need a bound on $\|S\|$. We recall that the operator $(1-h^2\Delta_Y)_+^{1/4}S(1-h^2\Delta_Y)_+^{-1/4}$ is unitary on $\mathcal{H}_Y = \mathbb{1}_{[0,1]}(h^2\Delta_Y)L^2(Y)$. This does not immediately give a good bound on $\|S\|$ itself, since $\|(I-h^2\Delta_Y)_+^{-1/2}\|$ is large when $1/h^2$ is near an eigenvalue of Δ_Y. However, under the assumptions of Theorem 1.5, $h^2\Delta_Y$ commutes with S, and in this setting $\|S\| = 1$. In general we have the following lemma. Recall $\mathcal{PR}_1$ is orthogonal projection onto the eigenfunctions of P with eigenvalue 1.

Lemma 3.1. *There is a $C > 0$ independent of h so that*

$$\|S\| \le Ch\|\mathbb{1}_{[0,1]}(h^2\Delta_Y)\mathbb{1}_{[0,1]}(r)(P-1-i0)^{-1}(I-\mathcal{PR}_1)\mathbb{1}_{[-1,0]}(r)\|.$$

Proof. First suppose $1 \notin \operatorname{spec}(h^2\Delta_Y)$. We use the Poisson operator $\mathbb{P}$ as constructed in (14). Let $f \in \mathcal{H}_Y$, and denote the outgoing data in $\mathbb{P}f$ by f_+, so that $Sf = f_+$. We note from our construction of $\mathbb{P}$ in (14) that

$$\begin{aligned}
&\mathbb{1}_{[0,1]}(h^2\Delta_Y)\left(\mathbb{P}f\restriction_{X_\infty|r>0} - e^{-ir(I-h^2\Delta_Y)^{1/2}/h}f\right)\\
&= \left(e^{ir(I-h^2\Delta_Y)^{1/2}/h}\mathbb{1}_{[0,1]}(h^2\Delta_Y)f_+\right)\restriction_{X_\infty|r>0}\\
&= \mathbb{1}_{[0,1]}(h^2\Delta_Y)\left((P-1-i0)^{-1}[h^2\partial_r^2,\varphi]F_{\tilde X}\right)\restriction_{X_\infty|r>0}\\
&= \mathbb{1}_{[0,1]}(h^2\Delta_Y)\left((P-1-i0)^{-1}(I-\mathcal{PR}_1)[h^2\partial_r^2,\varphi]F_{\tilde X}\right)\restriction_{X_\infty|r>0}.
\end{aligned}$$

where $F_{\tilde X}$ is defined in (12) and φ is as in (13). Thus using that $[h^2\partial_r^2,\varphi] = \mathbb{1}_{[-1,0]}(r)[h^2\partial_r^2,\varphi]$

$$\begin{aligned}
&\|\mathbb{1}_{[0,1]}(h^2\Delta_Y)f_+\|_{L^2(Y)}\\
&\leq \|\mathbb{1}_{[0,1]}(h^2\Delta_Y)\mathbb{1}_{[0,1]}(r)(P-1-i0)^{-1}(I-\mathcal{PR}_1)\mathbb{1}_{[-1,0]}(r)\|_{L^2(X)\to L^2(X)}\|[h^2\partial_r^2,\varphi]F_{\tilde X}\|_{L^2(X)}\\
&\leq Ch\|\mathbb{1}_{[0,1]}(h^2\Delta_Y)\mathbb{1}_{[0,1]}(r)(P-1-i0)^{-1}(I-\mathcal{PR}_1)\mathbb{1}_{[-1,0]}(r)\|_{L^2(X)\to L^2(X)}\|f\|_{L^2(Y)}
\end{aligned}$$

for some constant C, since

$$\|[h^2\partial_r^2,\varphi]F_{\tilde X}\|_{L^2(X)} = \|[h^2\partial_r^2,\varphi]e^{-ir(I-h^2\Delta_Y)_+^{1/2}/h}\mathbb{1}_{[0,1]}(h^2\Delta_Y)f\| \leq C\|f\|_{L^2(Y)}.$$

To handle the case of $1 \in \operatorname{spec}(h^2\Delta_Y)$, take the limit as $h' \uparrow h$ to obtain the desired bound. □

4. The resolvent on $\tilde X = \mathbf{R}\times Y$

We shall need some facts about the behavior of the resolvent of the Laplacian on the manifold $\tilde X = \mathbb{R}\times Y$ with the product metric, and the operators that arise when studying it. We begin with a simple lemma about the resolvent of $-h^2\partial_r^2$ on $\mathbb{R}$.

Lemma 4.1. *Let $\tau > 0$ and $f \in L^2_c(\mathbb{R})$ be supported in the interval $[a,b]$. Then, if $r > b$*

$$\left((-h^2\partial_r^2 - \tau^2 \mp i0)^{-1}f\right)(r) = \pm\frac{i}{2\tau h}e^{\pm i\tau r/h}\int_a^b e^{\mp i\tau r'/h}f(r')dr'$$

and

$$\left((-h^2\partial_r^2 + \tau^2)^{-1}f\right)(r) = \frac{1}{2\tau h}e^{-\tau r/h}\int_a^b e^{\tau r'/h}f(r')dr'.$$

If $r < a$, then

$$\left((-h^2\partial_r^2 - \tau^2 \mp i0)^{-1}f\right)(r) = \pm\frac{i}{2\tau h}e^{\mp i\tau r/h}\int_a^b e^{\pm i\tau r'/h}f(r')dr'$$

and

$$\left((-h^2\partial_r^2 + \tau^2)^{-1}f\right)(r) = \frac{1}{2\tau h}e^{\tau r/h}\int_a^b e^{-\tau r'/h}f(r')dr'.$$

Proof. We use that, for $\lambda \in \mathbb{C}$ with $\Im\lambda > 0$,

$$\left((-h^2\partial_r^2 - \lambda^2)^{-1}f\right)(r) = \frac{i}{2\lambda h}\int_{-\infty}^{\infty} e^{i\lambda|r-r'|/h}f(r')dr'.$$

Then the lemma follows directly using the support properties of f. □

Recall the operators $T_\pm : L^2_c(\tilde{X}) \to L^2(Y)$ are defined by

$$(T_\pm f)(y) = \int_{\mathbf{R}} e^{\mp ir'((I-h^2\Delta_Y)^{1/2})/h} f(r',y)dr'.$$

Lemma 4.2. *Let $f \in L^2_c(\tilde{X})$ be supported in $[a,b]\times Y$, and let $\psi_{sp} \in C^\infty_c([0,1))$. Then*

(17)
$$\left(\psi_{sp}(h^2\Delta_Y)(I-h^2\Delta_Y)^{1/2}(h^2\Delta_{\tilde{X}} - (1\pm i0))^{-1} f\right)\restriction_{r>b} = \frac{\pm i}{2h}\psi_{sp}(h^2\Delta_Y)e^{\pm ir(I-h^2\Delta_Y)^{1/2}/h}T_\pm f$$

Moreover,

(18)
$$\left(\psi_{sp}(h^2\Delta_Y)(I-h^2\Delta_Y)^{1/2}(h^2\Delta_{\tilde{X}} - 1 - i0)^{-1} f\right)\restriction_{r<a} = \frac{i}{2h}\psi_{sp}(h^2\Delta_Y)e^{-ir(I-h^2\Delta_Y)^{1/2}/h}T_- f.$$

Proof. The proof uses separation of variables and the spectral theorem.

Let $\{\sigma_j^2\}$, $\{\phi_j\}$ be the eigenvalues and eigenfunctions of Δ_Y as in (15). Define

$$\tau_j = \tau_j(h) := (1-h^2\sigma_j^2)^{1/2}, \tag{19}$$

where our convention is the square root has nonnegative real and imaginary parts. Then writing $f(r,y) = \sum_{j=1}^\infty f_j(r)\phi_j(y)$, we have

$$\left((I-h^2\Delta_Y)^{1/2}(h^2\Delta_{\tilde{X}} - 1 - i0)^{-1}) f\right)(r,y) = \sum_{j=1}^\infty \tau_j \left((-h^2\partial_r^2 - \tau_j^2 - i0)^{-1} f_j\right)(r)\phi_j(y).$$

Now we assume that $r > b$. Then from Lemma 4.1,

$$\left((I-h^2\Delta_Y)^{1/2}(h^2\Delta_{\tilde{X}} - 1 - i0)^{-1} f\right)(r,y) = \sum_{j=1}^\infty \frac{i}{2h} e^{i\tau_j r/h}\left(\int_a^b e^{-i\tau_j r'/h} f_j(r')dr'\right)\phi_j(y)$$

so that

$$\begin{aligned}\left(\psi_{sp}(h^2\Delta_Y)(I-h^2\Delta_Y)^{1/2}(h^2\Delta_{\tilde{X}} - 1 - i0)^{-1} f\right)(r,y) \\ = \sum_{j=1}^\infty \frac{i}{2h}\psi_{sp}(h^2\sigma_j^2)e^{i\tau_j r/h}\left(\int_a^b e^{-i\tau_j r'/h} f_j(r')dr'\right)\phi_j(y).\end{aligned} \tag{20}$$

But then, for the top choice of sign, this is the representation of the operator on the right hand of (17) given by the spectral theorem.

The proofs of the remaining equalities are similar. □

5. Resolvent estimates on X

This section contains two lemmas that we use later to allow some flexibility in exactly how we cut-off the resolvent on the end X_∞. We recall that $r > -4$ only on the end X_∞. These estimates do not require the bounds (6) or (7).

Lemma 5.1. *Let $c > -4$, $M > c$ and $z \in \mathbb{C}\setminus \operatorname{spec}(P)$. Then*

$$\|\mathbb{1}_{[M,M+1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,c]}(r)\| \le \|\mathbb{1}_{[c,c+1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,c]}(r)\|$$

and for any $\epsilon > 0$

$$\|\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)\mathbb{1}_{[M,M+1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,c]}(r)\| \\ \leq \|\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)\mathbb{1}_{[c,c+1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,c]}(r)\|.$$

Proof. Recall that on the end X_∞, $P = -h^2\partial_r^2 + h^2\Delta_Y$. We use here the notation

$$\tilde{\tau}_j(z) = \tilde{\tau}_j(z,h) = (z - h^2\sigma_j^2)^{1/2}, \tag{21}$$

where the square root has positive imaginary part, which is possible since $z \notin [0,\infty)$. Notice $\lim_{z\to 1,\, \Im z>0} \tilde{\tau}_j(z) = \tau_j$. Then for any $f \in L^2(X)$ there are $c_j \in \mathbb{C}$ so that

$$(P-z)^{-1}(\mathbb{1}_{(-\infty,c]}(r)f)\,\mathbb{1}_{r>c}\,(r,y) = \sum_{j=1}^\infty c_j e^{i\tilde{\tau}_j(z)r/h}\phi_j(y).$$

Since for each j, $|e^{i\tilde{\tau}_j(z)r/h}|$ is monotonically decreasing on $(-4,\infty)$,

$$\left\|\mathbb{1}_{[M,M+1]}(r)\sum_{j=1}^\infty c_j e^{i\tilde{\tau}_j(z)r/h}\phi_j(y)\right\|^2 \leq \left\|\mathbb{1}_{[c,c+1]}(r)\sum_{j=1}^\infty c_j e^{i\tau_j(z)r/h}\phi_j(y)\right\|^2$$

proving the first statement of the lemma.

The proof of the second statement is very similar. □

Lemma 5.2. *For* $M > 0$ *there is a* $C = C(M) > 0$ *so that for all* $z \in \mathbb{C}\setminus \operatorname{spec}(P)$, $h \in (0,1]$

$$\|\mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,M]}(r)\| \leq C\left(h^{-2} + \|\mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,0]}(r)\|\right).$$

Moreover, for every M, $\epsilon > 0$ *there is a* $C = C(M,\epsilon)$ *so that*

$$\|\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)\mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,M]}(r)\| \\ \leq C\left(h^{-2} + \|\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)\mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,0]}(r)\|\right).$$

Proof. We can write

$$\mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,M]}(r) = \mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{(-\infty,0]}(r) + \mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{[0,M]}(r). \tag{22}$$

It is immediate that the first term on the right is bounded as desired, so we need only bound $\|\mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{[0,M]}(r)\|$.

Let P_{0D} denote the operator $-h^2\partial_r^2 + h^2\Delta_Y$ on the product manifold $((-4,\infty)\times Y, (dr)^2 + g_Y)$ with Dirichlet boundary conditions at $\{r=-4\}$. Using $\tilde{\tau}_j(z)$ as in (21), if $f \in L^2((-4,\infty)\times Y)$ and we write $f(r,y) = \sum_{j=1}^\infty f_j(r)\phi_j(y)$, then for $z \in \mathbb{C}\setminus[0,\infty)$

$$((P_{0D}-z)^{-1}f)(r,y) = \sum_{j=1}^\infty \frac{i}{2\tilde{\tau}_j(z)h}\left(\int_{-4}^\infty (e^{i|r-r'|\tilde{\tau}_j(z)/h} - e^{i(r+r'+8)\tilde{\tau}_j(z)/h})f_j(r')dr'\right)\phi_j(y).$$

Note that the choice of Dirichlet boundary condition ensures that for any $M' \in \mathbf{R}$,

$$\|\mathbb{1}_{[-4,M']}(r)(P_{0D}-z)^{-1}\mathbb{1}_{[-4,M']}(r)\| + \|\mathbb{1}_{[-4,M']}(r)h\partial_r(P_{0D}-z)^{-1}\mathbb{1}_{[-4,M']}(r)\| \leq \frac{C'}{h^2} \tag{23}$$

for some $C' = C'(M')$, independent of $z \in \mathbb{C}\setminus[0,\infty)$ and $h \in (0,1]$.

Now choose $\chi \in C_c^\infty(\mathbf{R})$ so that $\chi(r) = 1$ for $r \leq -1$ and $\chi(r) = 0$ for $r \geq 0$. Then, with $\chi = \chi(r)$,

$$(P-z)(1-\chi)(P_{0D}-z)^{-1}\mathbb{1}_{[0,M]}(r) = \mathbb{1}_{[0,M]}(r) - [P,\chi](P_{0D}-z)^{-1}\mathbb{1}_{[0,M]}(r)$$

so that

(24)
$$(P-z)^{-1}\mathbb{1}_{[0,M]}(r) = (1-\chi)(P_{0D}-z)^{-1}\mathbb{1}_{[0,M]}(r) + (P-z)^{-1}(h^2\chi'' + 2h^2\chi'\partial_r)(P_{0D}-z)^{-1}\mathbb{1}_{[0,M]}(r).$$

Since χ' is supported in $[-1,0]$, using (23) and (24) proves $\|\mathbb{1}_{[0,1]}(r)(P-z)^{-1}\mathbb{1}_{[0,M]}(r)\|$ is bounded as desired, completing the proof of the first part of the lemma.

To prove the second statement, we left multiply both sides of (22) and (24) by $\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)$, and proceed as before. □

We remark that although Lemma 5.1 and 5.2 are stated for the resolvent $(P-z)^{-1}$ for $z \in \mathbb{C}\setminus \operatorname{spec}(P)$, by a limiting argument they also hold for $(P - z \pm i0)^{-1}$, when $z \in [0,\infty)$. Of course, if $z \in [0,\infty)$ the estimates are only meaningful if the right hand side is finite.

6. Microlocal properties of components of the scattering matrix

In this section we analyze the operators that go into the approximation to the scattering matrix, proving that they are Fourier integral operators (FIOs). In dealing with canonical relations between cotangent bundles, it will be convenient to use the following notational principles:

- We will identify the cotangent bundle of a Cartesian product with the product of the cotangent bundles, and separate points in cotangent bundle factors by a semi-colon. For example, $(r,\rho\,;\,y,\eta) \in T^*(\mathbb{R}\times Y)$ denotes the generic point in $T^*(\mathbb{R}\times Y)$ with $(r,\rho)\in T^*\mathbb{R}$ and $(y,\eta)\in T^*Y$. This differs from some notation in the introduction.
- On occasion we will use the notation $\overline{x} = (x,\xi) \in T^*X$, $\overline{y} = (y,\eta)$, $\overline{w} = (w,\theta) \in T^*Y$, and $\overline{r} = (r,\rho)$. Also, the "prime" operation is defined to be $\overline{y}' := (y,-\eta)$.
- A canonical relation from a symplectic manifold $\mathcal{M}_1$ to $\mathcal{M}_2$ will be a Lagrangian submanifold of $\mathcal{M}_2 \times \mathcal{M}_1^-$ (the domain of the relation is a subset of the second factor).
- If an FIO e.g. from $C_c^\infty(Y)$ to $C^\infty(X)$ has a Schwartz kernel $\mathcal{K} \in C^{-\infty}(X\times Y)$, its canonical relation is
$$\{(\overline{x},\overline{y}) \mid (\overline{x},\overline{y}') \in \mathrm{WF}(\mathcal{K})\}.$$

6.1. The operators $T_\pm\psi_{sp}(h^2\Delta_Y)$ and $R_\pm\psi_{sp}(h^2\Delta_Y)$.

In this section we prove that $T_+\psi_{sp}(h^2\Delta_Y)$ and $R_-\psi_{sp}(h^2\Delta_Y)$ are semi-classical Fourier integral operators for any $\psi_{sp}\in C_c^\infty([0,1))$. We recall from (19) that $\tau_j = (1-h^2\sigma_j^2)^{1/2}$.

Proposition 6.1. *Let*

$$W_\pm(r,w,y) = \sum_j \psi_{sp}(h^2\sigma_j^2) e^{\mp i r h^{-1}\tau_j}\phi_j(w)\overline{\phi_j(y)}, \tag{25}$$

where $\psi_{sp}\in C_c^\infty([0,1))$. Then $W_\pm$ is a Lagrangian semi-classical function on $\mathbb{R}\times Y\times Y$, associated with the Lagrangian submanifold $\Gamma_\pm \subset T^*(\mathbb{R}\times Y\times Y)$ given by

$$\Gamma_\pm = \{(\overline{r};\overline{w};\overline{y}') \mid \overline{y}\in\mathcal{B},\ \overline{w} = \Psi_{\pm r}(\overline{y}),\ \rho = \mp H(\overline{y})\}, \tag{26}$$

where $\mathcal{B}$ is the open unit tangent ball bundle of Y,

$$H(y,\eta) = \sqrt{1-|\eta|^2}$$

and Ψ is the Hamilton flow of H.

Proof. Microlocally in $\mathcal{B}$, the operator $\sqrt{I-h^2\Delta_Y}$ is a semi-classical self-adjoint pseudodifferential operator of order zero. The function $W_\pm$ is the Schwartz kernel of the composition

$$\psi_{sp}(h^2\Delta)e^{\mp irh^{-1}\sqrt{I-h^2\Delta_Y}},$$

regarded as an operator $L^2(Y)\to L^2(\tilde{X})$. It is well-known that if A is a self adjoint semi-classical pseudodifferential operator of order zero, the exponential $e^{-irh^{-1}A}$ is a semi-classical Fourier integral operator [26, Theorem 11.5.1], [36, Section IV.6] associated with a Lagrangian strictly analogous to Γ_+. The presence of the factor $\psi_{sp}(h^2\Delta)$ in U microlocalizes $\sqrt{I-h^2\Delta_Y}$ to where it is a pseudodifferential operator, so the same construction can be applied verbatim to the $W_\pm$. □

Note that the operator $T_\pm\psi_{sp}(h^2\Delta_Y):L^2_c(\tilde{X})\to L^2(Y)$ has $W_\pm$ as its Schwartz kernel, except for a trivial permuation of the variables:

$$T_\pm\psi_{sp}(h^2\Delta_Y)(f)(w)=\int W_\pm(r,w,y)\,f(r,y)\,dr\,dy.$$

Similarly, $R_\pm\psi_{sp}(h^2\Delta_Y)$, where $R_\pm$ is defined by (10), has for Schwartz kernel $W_\mp$, this time in the standard manner:

$$R_\pm\psi_{sp}(h^2\Delta_Y)(g)(r,w)=\chi(r)\int W_\mp(r,w,y)g(y)\,dy.$$

Therefore:

Corollary 6.2. *The operator $T_\pm\psi_{sp}(h^2\Delta_Y):L^2_c(\tilde{X})\to L^2(Y)$ is a Fourier integral operator associated with the canonical relation*

$$\Theta_\pm=\left\{\left(\overline{w}\,;\,(\overline{r};\overline{y})\right)\mid \overline{y}\in\mathcal{B},\,\overline{w}=\Psi_{\pm r}(\overline{y}),\,\rho=\pm H(\overline{y})\right\}. \tag{27}$$

Moreover, $R_\pm\psi_{sp}(h^2\Delta_Y):L^2_c(Y)\to L^2(\tilde{X})$ is a Fourier integral operator associated with the canonical relation

$$\Theta_\pm^T=\left\{\left((\overline{r};\overline{w})\,;\,\overline{y}\right)\mid \overline{y}\in\mathcal{B},\,\overline{w}=\Psi_{\mp r}(\overline{y}),\,\rho=\pm H(\overline{w}),\,r\in supp(\chi)\right\}, \tag{28}$$

which is the transpose of (27) (except for the restriction on r).

In what follows we will work with the compositions $e^{-it_\psi P/h}R_-$ and $T_+\tilde{\chi}(r)e^{-it_\psi P/h}R_-$, where $\tilde{\chi}$ is compactly supported on X_∞. We will use the composition theorem for FIOs to prove that each of these operators is an FIO, [26, Theorem 18.13.1]. We will show that the clean-intersection hypothesis of that theorem is satisfied in each case.

6.2. **Geometric considerations.** To better understand the previous canonical relations, introduce the co-isotropic submanifold of T^*X

$$\mathcal{C}=p^{-1}(1)=\{\overline{x}=(x,\xi)\in T^*X\mid p(\overline{x})=|\xi|^2+V_0(x)=1\}.$$

(We are working microlocally in a region of T^*X where $p^{-1}(1)$ is a submanifold, and therefore without loss of generality for simplicity we will assume it is a submanifold everywhere). The null leaves of $\mathcal{C}$ are the (unparametrized) Hamilton trajectories of p. Let

$$\mathcal{C}_\pm:=\{(\overline{r};\,\overline{y})\in T^*X_\infty\mid \overline{y}\in\mathcal{B},\ \rho=\pm H(\overline{y})\} \tag{29}$$

where $\mathcal{B}\subset T^*Y$ is the open unit cotangent bundle. Note that $\mathcal{C}_\pm\subset\mathcal{C}$ and that $\rho\neq 0$ on $\mathcal{C}_\pm$. Also introduce the embeddings

$$\nu_\pm:\mathcal{B}\to\mathcal{C}_\pm,\qquad \nu_\pm(\overline{y})=(0,\pm H(\overline{y})\,;\,\overline{y}). \tag{30}$$

In view of our notation conventions, this notation is consistent with (2).

Proposition 6.3. *The images $\mathcal{T}_\pm := \nu_\pm(\mathcal{B}) = \mathcal{C}_\pm \cap \{r = 0\}$ are symplectic submanifolds, and $\nu_\pm : \mathcal{B} \cong \mathcal{T}_\pm$ is a symplectomorphism. Moreover, $\mathcal{T}_\pm$ are Poincaré cross sections of $\mathcal{C}_\pm$. Explicitly, the null leaf of $\mathcal{C}_\pm$ through $(\overline{r};\, \overline{y})$ intersects the transversal $\mathcal{T}_\pm$ at $\nu_\pm(\Psi_{\pm r}(\overline{y}))$.*

Proof. Let G_t denote the Hamilton flow on T^*Y of the square of the Riemannian norm function, $\gamma_Y(\overline{y}) = |\eta|^2$. Then on T^*X_∞

$$\Phi_t(r, \rho\,;\, \overline{y}) = (r + 2t\rho, \rho\,;\, G_t(\overline{y}))$$

as long as $r + 2t\rho > -4$. On the other hand, $dH = -\frac{1}{2H} d\gamma_Y$, and therefore a similar relation holds among the corresponding Hamilton fields of H and γ_Y. It follows that $G_t = \Psi_{-2tH}$ and on T^*X_∞

$$\Phi_t(r, \rho\,;\, \overline{y}) = (r + 2t\rho, \rho\,;\, \Psi_{-2tH(\overline{y})}(\overline{y})). \tag{31}$$

Therefore

$$\forall\ (r, \rho\,;\, \overline{y}) \in \mathcal{C}_\pm \quad \Phi_{-r/2\rho}(r, \rho\,;\, \overline{y}) = \nu_\pm(\Psi_{\pm r}(\overline{y})). \tag{32}$$

□

Note that (32) implies that for all $(r, \rho\,;\, \overline{y}) \in \mathcal{C}_\pm$, $(r, \rho\,;\, \overline{y}) = \Phi_{r/2\rho}\left(\nu_\pm(\Psi_{\pm r}(\overline{y}))\right)$. Replacing $\overline{y}$ by $\Psi_{\mp r}(\overline{y})$ yields

$$(r, \rho\,;\, \Psi_{\mp r}(\overline{y})) = \Phi_{r/2\rho}\left(\nu_\pm(\overline{y})\right). \tag{33}$$

Therefore, we can re-state Corollary 6.2 as follows:

Corollary 6.4. *The canonical relation $\Theta^T_\pm$ of $R_\pm\psi_{sp}(h^2\Delta_Y)$ is*

$$\Theta^T_\pm = \left\{ \left(\Phi_t(\nu_\pm(\overline{y}))\,,\, \overline{y}\right) \mid \overline{y} \in \mathcal{B},\ t = \pm\frac{r}{2H(\overline{y})},\ r \in (-1/4, 0) \right\}. \tag{34}$$

Remark 6.5. We can use (31) to see how the scattering map changes if we change the origin of the r coordinate. If we replace r by $r - c$ for some constant $c > 0$, we obtain a scattering map κ' with domain $\mathcal{D}_{\kappa'}$. A point $\overline{y} \in \mathcal{B}$ is in the domain $\mathcal{D}_{\kappa'}$ if and only if the trajectory $t \mapsto \Phi_t(c, -H(\overline{y}); \overline{y})$ is not forward-trapping. By (31),

$$\Phi_t(c, -H(\overline{y}); \overline{y}) = (c - 2tH(\overline{y}), -H(\overline{y})\,;\, \Psi_{-2tH(\overline{y})}(\overline{y})). \tag{35}$$

Therefore this trajectory traverses the hypersurface $\{r = 0\}$ at time $t = \frac{c}{2H(\overline{y})}$ and at the point $(0, -H(\overline{y}); \Psi_{-c}(\overline{y}))$. It follows that if we let

$$\vartheta := \Psi_{-c} : \mathcal{B} \to \mathcal{B},$$

then ϑ maps $\mathcal{D}_{\kappa'}$ into $\mathcal{D}_\kappa$. The converse is analogous, that is, ϑ maps $\mathcal{D}_{\kappa'}$ bijectively onto $\mathcal{D}_\kappa$.

To find $\kappa'(\overline{y})$, we are to follow the trajectory described above until the time $t'_+(\overline{y}) > 0$ where it intersects $\{r = c\}$, and then take the T^*Y component of the point of intersection. By the previous discussion,

$$t'_+(\overline{y}) = \frac{c}{2H(\overline{y})} + t_+(\vartheta(\overline{y})) + s$$

where s is such that

$$\Phi_s(0, H(\kappa(\vartheta(\overline{y}))); \kappa(\vartheta(\overline{y}))) = (c, H(\kappa(\vartheta(\overline{y}))); \kappa'(\overline{y})).$$

Applying (35) again to the left-hand side of this identity, we obtain

$$\Phi_s(0, H(\kappa(\vartheta(\overline{y}))); \kappa(\vartheta(\overline{y}))) = (2sH(\kappa(\vartheta(\overline{y}))), H(\kappa(\vartheta(\overline{y}))); \Psi_{-2sH(\kappa(\vartheta(\overline{y})))}(\kappa(\vartheta(\overline{y})))),$$

that is, $s = c/2H(\kappa(\vartheta(\overline{y})))$. Substituting, we have

$$(c, H(\kappa(\vartheta(\overline{y}))); \Psi_{-c}(\kappa(\vartheta(\overline{y})))) = (c, H(\kappa(\vartheta(\overline{y}))); \kappa'(\overline{y})),$$

which shows that $\kappa' = \vartheta \circ \kappa \circ \vartheta$.

6.3. **The operator $T_+\chi(r)e^{-it_\psi P/h}R_-\psi_{sp}(h^2\Delta_Y)\mathbf{Op}_h(\psi)$.** Let us fix $\delta > 0$ and use $\mathfrak{Y}$ (the fraktur letter Y) for

$$\mathfrak{Y} = \{\overline{y} = (y, \eta) \in T^*Y \mid |\eta| < 1 - \delta\} \cap \mathcal{D}_\kappa,$$

where $\delta > 0$ is small enough that $\operatorname{supp}(\psi_{sp}) \subset [0, 1-\delta)$. Let $\psi \in C_c^\infty(\mathfrak{Y})$, and let $t_\psi > 0$ be chosen sufficiently large so that if $(y, \eta) \in \operatorname{supp}\psi$ then $\Phi_t(0, -\sqrt{1-|\eta|^2}; y, \eta) \in \{(x, \xi) \in T^*X_\infty \mid \ r \geq 0\}$ for all $t \geq t_\psi$. The existence of such a t_ψ follows from the assumption that the support of ψ is compact and is contained in the domain of the scattering map.

We first consider $e^{-it_\psi P/h}R_-\psi_{sp}(h^2\Delta_Y)\mathrm{Op}_h(\psi)$.

Proposition 6.6. *For any $\tilde{\chi} \in C_c^\infty((-4, \infty))$ the composition $\tilde{\chi}(r)e^{-it_\psi P/h}R_-\psi_{sp}(h^2\Delta_Y)Op_h(\psi)$ is a Fourier integral operator whose canonical relation $\Sigma \subset T^*X \times T^*Y$ is*

$$\Sigma = \left\{ \left(\Phi_{t_\psi + t}(\nu_-(\overline{y}))\,;\, \overline{y}\right) \in T^*X_\infty \times T^*Y,\ \mid \overline{y} \in \mathfrak{Y},\ t \in \left(0, \frac{1}{8H(\overline{y})}\right) \right\}. \tag{36}$$

Remark 6.7. Clearly (36) is parametrized by

$$\mathcal{D} := \left\{ (\overline{y}, t) \in \mathfrak{Y} \times \mathbb{R} \mid t \in \left(0, \frac{1}{8H(\overline{y})}\right) \right\}.$$

The condition on t ensures that (36) is over the portion of the cylinder defined by $r \in (0, \infty)$.

Remark 6.8. It is important to note that by the assumption on t_ψ and $\mathfrak{Y}$, in (36)

$$t \mapsto \Phi_{t_\psi + t}(\nu_-(\overline{y}_0))$$

is an outward-going geodesic on $T^*X_\infty \cap \mathcal{C}$. It is therefore of the form

$$\Phi_{t_\psi + t}(\nu_-(\overline{y}_0)) = (r(t), H(\overline{y}(t))\,;\, \overline{y}(t)) \tag{37}$$

with $\frac{dr}{dt} > 0$.

Proof of Proposition 6.6. Again by [26, Theorem 11.5.1], the factor $e^{-it_\psi P/h}$ is a Fourier integral operator associated to the graph of Φ_{t_ψ}. It is known ([26, §4.3]) that left-composition by an FIO associated with a canonical transformation is always clean (in fact, transverse), and therefore $\chi(r)e^{-it_\psi P/h}R_-$ is a Fourier integral operator whose canonical relation is the composition of the graph of Φ_{t_ψ} with Θ_-^T. The result follows directly from Corollary 6.4.

□.

For $\tilde{\chi} \in C_c^\infty((-4, \infty))$ and $\psi_{sp} \in C_c^\infty(\mathcal{B})$, we analyze next the composition

$$\psi_{sp}(h^2\Delta_Y)T_+ \circ \left(\tilde{\chi}(r)e^{-it_\psi P/h}R_-\psi_{sp}(h^2\Delta_Y)\mathrm{Op}_h(\psi)\right), \tag{38}$$

which is a bit more complicated.

Proposition 6.9. *The operator (38) is an FIO, associated to the graph of the scattering map κ restricted to $\mathfrak{Y}$.*

Proof. Introduce the manifolds:

$$T^*X \overset{\Delta}{\times} T^*X := \{(\overline{x};\overline{x}) \mid \overline{x} \in T^*X)\}, \quad \mathfrak{A} = T^*Y \times \left(T^*X \overset{\Delta}{\times} T^*X\right) \times T^*Y,$$

and $\mathfrak{B} = \Theta_+ \times \Sigma$. Recall that Θ_+ and Σ are the canonical relations associated to the factors of the composition (38), and note that $\mathfrak{A}$ and $\mathfrak{B}$ are submanifolds of $T^*Y \times T^*X \times T^*X \times T^*Y$.

We first prove that the manifolds $\mathfrak{A}$ and $\mathfrak{B}$ intersect cleanly. We claim that the intersection is the set

$$\mathfrak{A} \cap \mathfrak{B} = \left\{ \left(\kappa(\overline{y})\,;\, \Phi_{t_\psi+t}(\nu_-(\overline{y}))\,;\, \Phi_{t_\psi+t}(\nu_-(\overline{y}))\,;\, \overline{y}\right) \mid (\overline{y},t) \in \mathcal{D} \right\}, \tag{39}$$

where κ is the scattering map. To see this, let $\zeta \in \mathfrak{A} \cap \mathfrak{B}$, and let us write

$$\zeta = (\overline{w}\,;\, \Phi_s(\nu_+(\overline{w}))\,;\, \Phi_{t_\psi+t}(\nu_-(\overline{y}))\,;\, \overline{y}),$$

where $\Phi_s(\nu_+(\overline{w})) = \Phi_{t_\psi+t}(\nu_-(\overline{y}))$. Therefore

$$\nu_+(\overline{w}) = \Phi_{t_\psi+t-s}(\nu_-(\overline{y})),$$

which is a relation that characterizes the scattering map, namely

$$\nu_+(\kappa(\overline{y})) = \Phi_{t_+(\overline{y})}(\nu_-(\overline{y})), \tag{40}$$

where $\overline{y} \to t_+(\overline{y})$ is a smooth function by the implicit function theorem. Therefore $\overline{w} = \kappa(\overline{y})$, which yields (39). We also obtain the relation $t_+(\overline{y}) = t_\psi + t - s$.

The set $\mathfrak{A} \cap \mathfrak{B}$ is clearly a submanifold parametrized by $(\overline{y},t) \in \mathcal{D}$, and elements in $T_\zeta(\mathfrak{A} \cap \mathfrak{B})$ are of the form

$$\left(d\kappa(\delta\overline{y})\,;\, d\Phi_{t_\psi+t} \circ d\nu_-(\delta\overline{y}) + \delta t \Xi\,;\, d\Phi_{t_\psi+t} \circ d\nu_-(\delta\overline{y}) + \delta t \Xi\,;\, \delta\overline{y}\right) \tag{41}$$

where $\delta\overline{y} \in T_{\overline{y}}T^*Y$, $\delta t \in T_t\mathbb{R} \cong \mathbb{R}$, and Ξ is the Hamilton field of p (the generator of Φ) evaluated at the appropriate point.

To prove that the intersection is clean, we need to show that

$$T_\zeta\left(\mathfrak{A} \cap \mathfrak{B}\right) = T_\zeta\mathfrak{A} \cap T_\zeta\mathfrak{B}.$$

The inclusion $T_\zeta\left(\mathfrak{A} \cap \mathfrak{B}\right) \subset T_\zeta\mathfrak{A} \cap T_\zeta\mathfrak{B}$ is automatic, so let $v \in T_\zeta\mathfrak{A} \cap T_\zeta\mathfrak{B}$. Since $v \in T_\zeta\mathfrak{B}$, it is of the form

$$v = \left(\delta\overline{w}\,;\, d(\Phi_s) \circ d\nu_+(\delta\overline{w}) + \delta s \Xi\,;\, d(\Phi_{t_\psi+t}) \circ d\nu_-(\delta\overline{y}) + \delta t \Xi\,;\, \delta\overline{y}\right)$$

where $\delta\overline{w} \in T_{\overline{w}}\mathcal{B}$, $\delta s \in T_s\mathbb{R} \cong \mathbb{R}$, etc. The condition $v \in T_\zeta\mathfrak{A}$ means that the middle entries in v are equal, that is

$$d(\Phi_s) \circ d\nu_+(\delta\overline{w}) + \delta s \Xi \;= d(\Phi_{t_\psi+t}) \circ d\nu_-(\delta\overline{y}) + \delta t \Xi. \tag{42}$$

Comparing with (41), in order to conclude that $v \in T_\zeta\left(\mathfrak{A} \cap \mathfrak{B}\right)$ all we need to show is that $\delta\overline{w} = d\kappa(\delta\overline{y})$. To see this, let us rewrite (42) as

$$d(\Phi_{t_\psi+t-s}) \circ d\nu_-(\delta\overline{y}) = d\nu_+(\delta\overline{w}) + (\delta s - \delta t)\Xi.$$

But, by (40), this also equals

$$d(\Phi_{t_+(\overline{y})}) \circ d\nu_-(\delta\overline{y}) = d\nu_+(d\kappa(\delta\overline{y})) - dt(\delta\overline{y})\Xi.$$

Now the summands on the right-hand sides of these expressions correspond to the direct sum decomposition $T\mathcal{C}_+ = T\mathcal{T}_+ \oplus \mathbb{R}\Xi$. Therefore, corresponding summands must equal each other, that is

$$d\nu_+(\delta\overline{w}) = d\nu_+(d\kappa(\delta\overline{y})) \quad \text{and} \quad (\delta s - \delta t)\Xi = -dt(\delta\overline{y})\Xi.$$

Since $d\nu_+$ is injective, the first of these relations yields $\delta\overline{w} = d\kappa(\delta\overline{y})$, and the proof that $\mathfrak{A}$ and $\mathfrak{B}$ intersect cleanly is complete.

By the composition theorem for semi-classical FIOs [26, Theorem 18.13.1], the operator (38) is a semi-classical FIO associated to the relation which is the image of $\mathfrak{A}\cap\mathfrak{B}$ under the projection onto $T^*Y\times T^*Y$. By (39), this is precisely the graph of κ restricted to $\mathfrak{Y}$. □

7. A microlocal approximation of the Poisson operator and the scattering matrix

In this section we give a microlocal construction of $\mathbb{P}\operatorname{Op}_h(\psi)$, the Poisson operator composed with $\operatorname{Op}_h(\psi)$. Recall $\psi\in C_c^\infty(T^*Y)$ has support contained in the domain of the scattering map κ. A consequence of our construction is an expression for the scattering matrix in terms of R_-, T_+, and the Schrödinger propagator, see Proposition 7.4. Propositions 6.9 and 7.4 combine to prove our theorems.

Recall $t_\psi>0$ is chosen sufficiently large so that if $(y,\eta)\in\operatorname{supp}\psi$, then $\Phi_t(0,-\sqrt{1-|\eta|^2};y,\eta)\in\{(x,\xi)\in T^*X_\infty:\ r\geq 0\}$ for all $t\geq t_\psi$. Here we continue to use the notation for the cotangent variables on T^*X_∞ introduced in Section 6. Choose $M\in\mathbb{N}$ so that if $(y,\eta)\in\operatorname{supp}\psi$ and $-1/4\leq s\leq 0$ then

$$\forall\, t\in[0,t_\psi]\qquad \Phi_t(s,-\sqrt{1-|\eta|^2};\Psi_s(y,\eta))\in\{(x,\xi)\in T^*X_\infty:\ r\leq M-2\}. \tag{43}$$

In particular, this implies $M\geq 2$. Let $b_\psi<1$ be chosen so that if $(y,\eta)\in\operatorname{supp}(\psi)$ or $(r,\sqrt{1-|\eta|^2};y,\eta)=\Phi_{t_\psi}(r_-,-\sqrt{1-|\eta_-|^2};y_-,\eta_-)$ for some $(y_-,\eta_-)\in\operatorname{supp}\psi$ and $-1/4\leq r_-\leq 0$, then $|\eta|\leq b_\psi$. Choose $\psi_{sp}\in C_c^\infty([0,1))$ so that ψ_{sp} is 1 on $[0,b_\psi]$.

Let $\chi_j\in C^\infty(\mathbf{R};[0,1])$ satisfy $\chi_j(r)=1$ if $r<-2+j$ and $\chi_j(r)=0$ if $r>-3/2+j$, ensuring $\chi_j\chi_{j+1}=\chi_j$.

Recall R_- is defined in (10), and $\operatorname{supp}(R_-f_-)\subset\{(r,y)\in X_\infty\mid -1/4\leq r\leq 0\}$. Set

$$\begin{aligned} U_- &= (I-h^2\Delta_Y)^{1/2}\psi_{sp}(h^2\Delta_Y)R_-\operatorname{Op}_h(\psi):L^2(Y)\to H_c^\infty(X_\infty)\subset H^\infty(X),\\ U_+ &= (\chi_M(r)-\chi_1(r))e^{-it_\psi P/h}U_-:L^2(Y)\to L^2(X_\infty)\subset L^2(X) \end{aligned}\tag{44}$$

and

$$\begin{aligned}\mathbb{P}_{\text{appr}} = 2ih\Big((1-\chi_0(r))(h^2\Delta_{\tilde X}-1+i0)^{-1}U_- + \frac{1}{ih}\chi_M(r)\int_0^{t_\psi}e^{it/h}e^{-itP/h}U_-dt\\ -e^{it_\psi/h}(1-\chi_0(r))(h^2\Delta_{\tilde X}-1-i0)^{-1}\left(\psi_{sp}(h^2\Delta_Y)U_+\right)\Big).\end{aligned}\tag{45}$$

We shall see that the operator $\mathbb{P}_{\text{appr}}$ is an approximation of $\mathbb{P}\operatorname{Op}_h(\psi)$. The mapping properties of $(h^2\Delta_{\tilde X}-1\pm i0)^{-1}$ ensure that if $f_-\in L^2(Y)$, then $\mathbb{P}_{\text{appr}}f_-\in\langle r\rangle^{1/2+\delta}H^\infty(X)$ for any $\delta>0$. Note that our definition of $\mathbb{P}_{\text{appr}}$ involves $\operatorname{Op}_h(\psi)$, and so depends on choice of ψ, even though our notation does not indicate this.

We begin with a preliminary lemma.

Lemma 7.1. *Let $\tilde\chi\in C_c^\infty(X_\infty)$ have support in the region with $r\geq M-2\geq 0$. Then $\|\tilde\chi e^{-itP/h}U_-\|=O(h^\infty)$ uniformly for t satisfying $0\leq t\leq t_\psi$.*

Proof. First observe that, by (10),

$$\forall\, g\in L^2(Y)\qquad U_-g=\chi(r)(I-h^2\Delta_Y)^{1/2}\psi_{sp}(h^2\Delta_Y)e^{-ir(I-h^2\Delta_Y)_+^{1/2}/h}g,$$

where χ is supported in $(-1/4,0)$.

For each value of h and each t, the operator $e^{-itP/h}U_-$ is a smoothing operator of finite rank. By Corollary 6.2 and the composition theorem for FIOs, it is also a semi-classical FIO whose canonical relation is

$$\left\{ \left(\Phi_t(r,\rho;\overline{w})\,,\,\overline{y}\right) |\; \overline{w} = \Psi_r(\overline{y}),\, \overline{y} \in \mathrm{supp}(\psi),\, \rho = -H(\overline{w}),\, r \in \mathrm{supp}(\chi) \right\}. \tag{46}$$

More precisely, let $\chi^\sharp \in C_c^\infty(\mathbb{R})$ be identically equal to one in a neighborhood of $1 \in \mathbb{R}$, and note that

$$\chi^\sharp(h^2\Delta_X)U_- = U_- + O(h^\infty)$$

because the image of the canonical relation of U_- (which is the same as that of R_-) is contained in $p^{-1}(1)$. We now use the well-known approximation of $e^{-itP/h}\chi^\sharp(h^2\Delta_X)$ by oscillatory integrals, uniformly for t in a bounded interval, see e.g. [36, Theorem IV-30] or [4, Lemma 3.2]. That is, one can write $e^{-itP/h}\chi^\sharp(h^2\Delta_X) = \mathcal{F}_1 + \mathcal{R}_1$, where the Schwartz kernel of $\mathcal{F}_1$ is a finite sum of oscillatory integrals of the form $\int e^{ih^{-1}\varpi(t,x,x',p)}a(t,x,x',p,h)dp$ where ϖ are generating functions for portions of the canonical relation of the graph of the Hamilton flow of p, the amplitudes a are smooth and have an asymptotic expansion in powers of h, and the Schwartz kernel of $\mathcal{R}_1$ is $O(h^\infty)$ uniformly for t in a bounded interval. Similarly, one can write $U_- = \mathcal{F}_2 + \mathcal{R}_2$, where the Schwartz kernel of $\mathcal{F}_2$ is a finite sum of oscillatory integrals of the form $\int e^{ih^{-1}\varpi(y,x,p)}a(y,x,p,h)dp$ where ϖ are generating functions for the canonical relation of R_-, and the Schwartz kernel of $\mathcal{R}_2$ is $O(h^\infty)$. It follows that

$$e^{-itP/h}\chi^\sharp(h^2\Delta_X)U_- = \mathcal{F}_1\mathcal{F}_2 + \mathcal{S}, \qquad \mathcal{S} = \mathcal{F}_1\mathcal{R}_2 + \mathcal{R}_1\mathcal{F}_2 + \mathcal{R}_1\mathcal{R}_2.$$

Note that the Schwartz kernel of $\mathcal{S}$ is $O(h^\infty)$ uniformly for t in a compact interval.

Now recall how $M \in \mathbb{N}$ is chosen, (43), and also recall that $\mathrm{supp}(\chi) \subset (-1/4, 0)$. It follows that the Schwartz kernel of $\tilde{\chi}\mathcal{F}_1\mathcal{F}_2$ is a finite sum of oscillatory integrals whose phase functions do not have critical points in the support of their amplitudes. Therefore the Hilbert-Schmidt norm of $\tilde{\chi}\mathcal{F}_1\mathcal{F}_2$ can be estimated as $h \to 0$ by a finite sum of absolute values of oscillatory integrals without critical points. By smoothness of the integrands, the estimate is uniform in $t \in [0, t_\psi]$.

In combination with the rapid decrease of the Schwartz kernel of $\mathcal{S}$, we can conclude that the Hilbert-Schmidt norm of $\tilde{\chi}e^{-itP/h}U_-$ is $O(h^\infty)$ uniformly in $t \in [0, t_\psi]$. □

Lemma 7.2. *Set $(P-1)\mathbb{P}_{\mathrm{appr}} = 2ihE$. Then for any $f_- \in L^2(Y)$, Ef_- is compactly supported with support in $X_c \cup \{(r,y) \in X_\infty : r \le M-1\}$, and $\|E\|_{L^2(Y)\to H^2(X)} = O(h^\infty)$.*

Proof. Using that

$$(P-1)\int_0^{t_\psi} e^{it/h}e^{-itP/h}dt = ih\left(e^{it_\psi/h}e^{-it_\psi P/h} - I\right)$$

and $(1-\chi_0)U_- = U_- = \chi_M U_-$ gives $E = \sum_{j=1}^4 E_j$, where

$$\begin{aligned}
E_1 &= [h^2\partial_r^2, \chi_0](h^2\Delta_{\tilde{X}} - 1 + i0)^{-1}U_- \\
E_2 &= -\frac{1}{ih}[h^2\partial_r^2, \chi_M]\int_0^{t_\psi} e^{it/h}e^{-itP/h}U_-dt \\
E_3 &= e^{it_\psi/h}\left(\chi_M e^{-it_\psi P/h}U_- - \psi_{sp}(h^2\Delta_Y)U_+\right) \\
E_4 &= -e^{it_\psi/h}[h^2\partial_r^2, \chi_0](h^2\Delta_{\tilde{X}} - 1 - i0)^{-1}\psi_{sp}(h^2\Delta_Y)U_+.
\end{aligned}$$

The claim about the support of E is immediate from our expression for E.

We begin with bounding E_2. Since $[h^2\partial_r^2, \chi_M]$ is supported in $\{x \in X_\infty \mid x = (r,y), M-2 \leq r \leq -3/2 + M\}$, as a corollary of Lemma 7.1 we obtain that $\|E_2\| = O(h^\infty)$.

For E_3, we use

$$\begin{aligned} \|E_3\| &= \|\chi_M e^{-it_\psi P/h} U_- - \psi_{sp}(h^2\Delta_Y)(\chi_M - \chi_0)e^{-it_\psi P/h}U_-\| \\ &\leq \|(1-\psi_{sp}(h^2\Delta_Y))(\chi_M-\chi_0)e^{-it_\psi P/h}U_-\| + \|\chi_0 e^{-it_\psi P/h}U_-\|. \end{aligned} \tag{47}$$

That $\|(1-\psi_{sp}(h^2\Delta_Y))(\chi_M-\chi_0)e^{-it_\psi P/h}U_-\| = O(h^\infty)$ follows from Proposition 6.6 and our choice of ψ_{sp}, and that $\|\chi_0 e^{-it_\psi P/h}U_-\| = O(h^\infty)$ follows from Proposition 6.6, the support properties of χ_0, and our choice of t_ψ.

Now consider E_4. The support properties of $[h^2\partial_r^2, \chi_0]$ and $(\chi_M - \chi_1)$ mean that by Lemma 4.2

$$\|E_4\| = (2h)^{-1}\|(h^2\chi_0''(1-h^2\Delta_Y)^{-1/2} - 2i\chi_0')\psi_{sp}(h^2\Delta_Y)T_-U_+\|.$$

But by Corollary 6.2 and Proposition 6.6, the composition of the canonical relations of T_- and U_+ is empty. Therefore $\|T_-U_+\|_{L^2(Y)\to H^m(Y)} = O(h^\infty)$, and hence $\|E_4\| = O(h^\infty)$.

The term E_1 is handled in a way similar to E_4, using that

$$(h^2\Delta_{\bar X} - 1 + i0)^{-1} f = \overline{(h^2\Delta_{\bar X} - 1 - i0)^{-1}\overline{f}}.$$

□

For the next lemma, we continue to use the functions χ_j introduced above.

Lemma 7.3. *Suppose $f \in L^2_c(X)$ has support in $X_C \cup \{(r,y) \in X_\infty \mid -4 < r \leq M'\}$, with $-4 < M' < \infty$. Then for $r \geq M'$,*

$$\begin{aligned} &\| \left(\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)(1-\chi_0)(P-1-i0)^{-1}f\right)(r,\bullet)\|_{L^2(Y)} \\ &\qquad \leq \|\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)\mathbb{1}_{[M',M'+1]}(r)(P-1-i0)^{-1}\mathbb{1}_{(-\infty,M']}(r)\|\|f\| \end{aligned} \tag{48}$$

for any $\epsilon > 0$. Moreover,

$$\begin{aligned} &\| \left(\mathbb{1}_{[0,1)}(h^2\Delta_Y)(1-\chi_0)(P-1-i0)^{-1}f\right)(r,\bullet)\|_{L^2(Y)} \\ &\qquad \leq \|\mathbb{1}_{[0,1)}(h^2\Delta_Y)\mathbb{1}_{[M',M'+1]}(r)(P-1-i0)^{-1}\mathbb{1}_{(-\infty,M']}(r)\|\|f\|. \end{aligned} \tag{49}$$

Although these two are almost the same, and have essentially identical proofs, the operator norm on the right side of (48) may be smaller than that in (49); see, for example, Section 2.4.2.

Proof. From the support properties of f, there are $c_j = c_j(h,f) \in \mathbb{C}$ so that for $r > M'$

$$(\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)(1-\chi_0)(P-1-i0)^{-1}f)(r,y) = \sum_{h^2\sigma_j^2 \leq 1-\epsilon} c_j e^{i\tau_j r/h}\phi_j(y)$$

where $\tau_j = (1-\sigma_j^2h^2)^{1/2} \geq 0$ for $h^2\sigma_j^2 \leq 1$. Then for $r > M'$

$$\begin{aligned} &\| \left(\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)(1-\chi_0)(P-1-i0)^{-1}f\right)(r,\bullet)\|^2_{L^2(Y)} \\ &= \sum_{h^2\sigma_j^2\leq 1-\epsilon} |c_j|^2 \\ &= \int_{X_\infty \mid M'\leq r\leq M'+1} |\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)(1-\chi_0)(P-1-i0)^{-1}f|^2 \\ &= \|\mathbb{1}_{[0,1-\epsilon]}\mathbb{1}_{[M',M'+1]}(r)(P-1-i0)^{-1}\mathbb{1}_{(-\infty,M']}(r)f\|^2 \end{aligned}$$

proving (48). The proof of (49) is essentially identical. □

Recall that $\mathbb{P}_{\text{appr}}$ as defined in (45) depends on $\psi \in C^\infty(T^*Y)$. Set

$$Q = \mathbb{P}_{\text{appr}} - (P - 1 - i0)^{-1}(P-1)\mathbb{P}_{\text{appr}}. \tag{50}$$

Then $(P-1)Q = 0$, and for any $f_- \in L^2(Y)$, $\delta > 0$, $\langle r\rangle^{-1/2-\delta} Q f_- \in L^2(X)$. We shall show that Q is actually $\mathbb{P}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)$, and that under the hypotheses of Theorem 1.4 or 1.5 we can use this to find an expression for the cut-off scattering matrix, up to a small error.

Proposition 7.4. *If $1 \notin \operatorname{spec}(h^2\Delta_Y)$, then the operator Q defined in (50) is $Q = \mathbb{P}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)$. Moreover, if (6) holds, then*

$$\begin{aligned} &S\operatorname{Op}_h(\psi) \\ &= e^{it_\psi/h}(I-h^2\Delta_Y)^{-1/2}\psi_{sp}(h^2\Delta_Y)T_+(\chi_M-\chi_1)e^{-it_\psi P/h}(I-h^2\Delta_Y)^{1/2}R_-\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)+O(h^\infty), \end{aligned} \tag{51}$$

and if the hypotheses of Theorem 1.5 hold,

$$\begin{aligned} &S\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)\operatorname{Op}_h(\psi) \\ &= e^{it_\psi/h}(I-h^2\Delta_Y)^{-1/2}\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_Y)T_+(\chi_M-\chi_1)e^{-it_\psi P/h}(I-h^2\Delta_Y)^{1/2}R_-\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)+O(h^\infty) \end{aligned} \tag{52}$$

for any $\epsilon > 0$. If $1 \in \operatorname{spec}(h^2\Delta_Y)$ the equations (51) and (52) hold as limits as $h' \uparrow h$.

Proof. As noted already, $(P-1)Q = 0$. Thus to show that $Q = \mathbb{P}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)$ it remains to study the expansion of Qf_- on X_∞ for $f_- \in L^2(Y)$. We begin by studying the behavior of $\mathbb{P}_{\text{appr}} f_-$ for $r > M$. By Lemma 4.2,

$$\begin{aligned} &(1-\chi_0(r))(h^2\Delta_{\tilde X} - 1 + i0)^{-1}U_-f_-\,\mathbb{1}_{r>M} \\ &= \frac{-i}{2h}e^{-ir(I-h^2\Delta_Y)^{1/2}/h}(I-h^2\Delta_Y)^{-1/2}T_-\psi_{sp}(h^2\Delta_Y)(I-h^2\Delta_Y)^{1/2}R_-\operatorname{Op}_h(\psi)f_-\,\mathbb{1}_{r>M} \\ &= \frac{-i}{2h}e^{-ir(I-h^2\Delta_Y)^{1/2}/h}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)f_-\,\mathbb{1}_{r>M} \end{aligned} \tag{53}$$

using that $T_\pm$ commutes with functions of $h^2\Delta_Y$ and $T_\pm R_\pm = I$. Also by Lemma 4.2,

$$\begin{aligned} &(1-\chi_0(r))(h^2\Delta_{\tilde X} - 1 - i0)^{-1}\left(\psi_{sp}(h^2\Delta_Y)U_+f_-\right)\mathbb{1}_{r>M} \\ &\qquad = \frac{i}{2h}e^{ir(I-h^2\Delta_Y)^{1/2}/h}(1-h^2\Delta_Y)^{-1/2}T_+\left(\psi_{sp}(h^2\Delta_Y)U_+f_-\right)\mathbb{1}_{r>M}\,. \end{aligned} \tag{54}$$

The term $\chi_M(r)\int_0^{t_\psi} e^{it/h}e^{-itP/h}U_-\,dt$ in (45) vanishes if $r > M$.

If $1 \notin \operatorname{spec}(h^2\Delta_Y)$,

$$\left((1-\chi_0)(P-1-i0)^{-1}Ef_-\right)(r,y)\,\mathbb{1}_{r>M} = \sum_{0\le h^2\sigma_j^2} c_j e^{i\tau_j r/h}\phi_j(y) \tag{55}$$

for some $c_j \in \mathbb{C}$. Recall τ_j defined in (19) has $\Re\tau_j \ge 0$, $\Im\tau_j \ge 0$. Combining these four observations, we see that if $1 \notin \operatorname{spec}(h^2\Delta_Y)$,

$$\begin{aligned} (Qf_-)\,\mathbb{1}_{r>M} = {}& e^{-ir(I-h^2\Delta_Y)^{1/2}/h}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)f_- \\ &+ e^{ir(I-h^2\Delta_Y)^{1/2}/h}e^{it_\psi/h}(I-h^2\Delta_Y)^{-1/2}T_+\left(\psi_{sp}(h^2\Delta_Y)U_+f_-\right) \\ &- 2ih\sum_{0\le h^2\sigma_j^2} c_j e^{i\tau_j r/h}\phi_j(y). \end{aligned} \tag{56}$$

This shows that $Q = \mathbb{P}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)$.

We now turn to proving (51), so suppose (6) holds. Then using in addition Lemmas 5.1 and 5.2, for any $M' > 0$ there is a constant C so that

$$\|\mathbb{1}_{[M',M'+1]}(r)(P-1-i0)^{-1}\mathbb{1}_{(-\infty,M']}(r)\| \leq C(h^{-2}+h^{-N_0}) \text{ for } 0 < h \leq h_0.$$

Then by (55) and Lemmas 7.2 and 7.3, $\sum_{0\leq h^2\sigma_j^2\leq 1}|c_j|^2 = O(h^\infty\|f_-\|)$ where the c_j are defined via (56). Thus by our definition of the scattering matrix and (56)

$$S\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)f_- = e^{it_\psi/h}(1-h^2\Delta_Y)^{-1/2}T_+\left(\psi_{sp}(h^2\Delta_Y)U_+f_-\right) + O(h^\infty\|f_-\|).$$

Using $\|(1-\psi_{sp}(h^2\Delta_Y))\operatorname{Op}_h(\psi)\| = O(h^\infty)$ and the fact that Lemma 3.1 and (6) imply $\|S\| = O(h^{1-\max(2,N_0)})$ finishes the proof when (6) holds, if $1 \notin \operatorname{spec}(h^2\Delta_Y)$. If $1 \in \operatorname{spec}(h^2\Delta_Y)$, then the equality holds by taking the limits as $h' \uparrow h$.

Now suppose the hypotheses of Theorem 1.5 hold. Since P commutes with Δ_{Y_0}, the scattering matrix S commutes with Δ_{Y_0} and, as mentioned earlier, this implies $\|S\| = 1$. Thus

$$\|S(1-\psi_{sp}(h^2\Delta_Y))\operatorname{Op}_h(\psi)\| = \|(1-\psi_{sp}(h^2\Delta_Y))\operatorname{Op}_h(\psi)\| = O(h^\infty).$$

Applying Lemmas 5.1, 5.2, 7.2, and 7.3 as before gives $\sum_{0\leq h^2\sigma_j^2\leq 1-\epsilon}|c_j|^2 = O(h^\infty)$, where the c_j are as in (56). Thus (52) holds. □

Proof of Theorems 1.4 and 1.5. Combining Propositions 6.9 and 7.4 proves Theorems 1.4 and 1.5 for $S\operatorname{Op}_h(\psi)$.

Turning to the proof for $S_U\operatorname{Op}_h(\psi)$, choose $\psi_{sp} \in C_c^\infty([0,1))$ so that

$$(I-\psi_{sp}(h^2\Delta_Y))\operatorname{Op}_h(\psi_0) = O(h^\infty).$$

Then the unitarity of S_U implies

$$S_U\operatorname{Op}_h(\psi) = (I-h^2\Delta_Y)_+^{1/4}S(I-h^2\Delta_Y)_+^{-1/4}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi) + O(h^\infty).$$

Since $(I-h^2\Delta_Y)_+^{-1/4}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)$ is a pseudodifferential operator with symbol supported in the support of ψ, using the result for S we see there is a $\psi_1 \in C_c^\infty([0,1))$ so that

$$\|(I-\psi_1(h^2\Delta_Y))S\psi_{sp}(h^2\Delta_Y)(I-h^2\Delta_Y)_+^{-1/4}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi)\| = O(h^\infty).$$

Thus

$$S_U\operatorname{Op}_h(\psi) = (I-h^2\Delta_Y)_+^{1/4}\psi_1(h^2\Delta_Y)S(I-h^2\Delta_Y)_+^{-1/4}\psi_{sp}(h^2\Delta_Y)\operatorname{Op}_h(\psi) + O(h^\infty),$$

and the result for S_U follows from the result for S and composition properties of Fourier integral and pseudodiffierential operators. □

8. Equidistribution of phase shifts

As an application of our theorems on the microlocal structure of the unitary scattering matrix S_U, in this section we prove Theorem 8.4, a result about the distribution of its phase shifts. This requires some additional hypotheses, for which we need some background.

8.1. **Distance on T^*Y and Minkowski content.** Fix any smooth Riemannian metric on T^*Y. This induces a distance on each connected component of T^*Y. If $\overline{y}$, $\overline{w} \in T^*Y$ belong to different connected components of T^*Y, we shall say the (generalized) distance (in T^*Y) between them is infinite. We will denote this (generalized) distance by dist_{T^*Y}; $\mathrm{dist}_{T^*Y} : T^*Y \times T^*Y \to [0, \infty]$. We use this to define the $(2n-2)$-dimensional Minkowski content of a bounded set $A \subset T^*Y$, where $2n-2 = \dim(T^*Y)$. The $(2n-2)$-dimensional upper Minkowski content of A is

$$\mathcal{M}^{*2n-2}(A) = \limsup_{\delta \downarrow 0} \mu(\{\overline{y} \in T^*Y \mid \mathrm{dist}_{T^*Y}(\overline{y}, A) < \delta\})$$

where for $B \subset T^*Y$, $\mu(B)$ is the Liouville measure of B. Similarly, the $(2n-2)$-dimensional lower Minkowski content is

$$\mathcal{M}_*^{2n-2}(A) = \liminf_{\delta \downarrow 0} \mu(\{\overline{y} \in T^*Y \mid \mathrm{dist}_{T^*Y}(\overline{y}, A) < \delta\}).$$

If $\mathcal{M}^{*2n-2}(A) = \mathcal{M}_*^{2n-2}(A)$, then the $(2n-2)$-dimensional Minkowski content of A is $\mathcal{M}^{2n-2}(A) = \mathcal{M}_*^{2n-2}(A)$.

For general A, the Minkowski content may depend on the choice of the metric on T^*Y via the induced distance or the chosen measure. However, we shall only apply this for bounded sets A that have zero $(2n-2)$-dimensional Minkowski content. For such sets, the property of having zero Minkowski content is independent of the choice of smooth metric on T^*Y. Moreover, this is also true of the choice of measure, as long as the measures are mutually absolutely continuous.

Remark 8.1. A set in a d-dimensional manifold that has zero d-dimensional Minkowski content has measure zero, but the converse is not true. For example, let $\mathcal{Q}$ be the intersection of the unit cube in $\mathbb{R}^d$ with $\mathbb{Q}^d$. Then $\mathcal{Q}$ has measure zero but d-dimensional Minkowski content one.

8.2. **Hypotheses and Theorem 8.4.** Throughout Sections 8.2 and 8.3, we assume:

(1) The assumptions of at least one of Theorems 1.4 or 1.5 hold.
(2) For $m \in \mathbf{Z}$, let $\mathcal{D}_{\kappa^m} \subset \mathcal{B}$ be the domain of κ^m, where we recall κ is the scattering map. We assume that for each $m \in \mathbf{N}$ the $(2n-2)$-dimensional Minkowski content of $\mathcal{B} \setminus \mathcal{D}_{\kappa^m}$ is 0.
(3) For each $m \in \mathbf{Z} \setminus \{0\}$, the set of fixed points of κ^m has measure 0.

In reference [20], where the authors studied the equidistribution property for semiclassical Schrödinger operators on $\mathbf{R}^n$, the analogs of the first and second assumptions are implied by a non-trapping assumption, while the analog of the third assumption is made explicitly. The proof we give here follows in outline much of the strategy of [20]. Some differences include not having knowledge of the microlocal structure of S_U near $\partial\mathcal{B}_Y$, and allowing for the possibility that the domain of the scattering map may not be all of $\mathcal{B}$.

Remark 8.2. Recall that for $\overline{y} = (y, \eta) \in \mathcal{B}$ we write $\overline{y}' = (y, -\eta)$. We shall use that since $\kappa(\kappa(\overline{y})') = \overline{y}'$, $\overline{y} \in \mathcal{D}_\kappa$ if and only if $\overline{y}' \in \mathcal{D}_{\kappa^{-1}}$, and similarly for iterates of κ. Hence the condition we made on the Minkowski content in assumption (2) is equivalent to making the assumption for all $m \in \mathbf{Z} \setminus \{0\}$.

Remark 8.3. The examples described in Sections 2.1 and 2.4 satisfy conditions (1) and (2). We show in Section A.2 that a surface of revolution with a bulge, as introduced in Section 2.4.2, satisfies condition (3) as well.

Let $S_U = S_U(h) = (I - h^2\Delta_Y)_+^{1/4} S (I - h^2\Delta_Y)_+^{-1/4}$ if $1 \notin \mathrm{spec}(h^2\Delta_Y)$, and $S_U(h) = \lim_{h' \uparrow h} S_U(h')$ if $1 \in \mathrm{spec}(h^2\Delta_Y)$. It will be helpful to recall here that S, $S_U : \mathcal{H}_Y \to \mathcal{H}_Y$, where $\mathcal{H}_Y = \mathbb{1}_{[0,1]}(h^2\Delta_Y) L^2(Y)$. The operator S_U is the unitary (on $\mathcal{H}_Y$) scattering matrix. We note that

both the scattering matrix and the hypothesis 3 depend on the choice of coordinate r on the the cylindrical end.

Theorem 8.4. *Suppose (X, g) is an n-dimensional manifold with infinite cylindrical end, and (X, g) and the associated scattering map κ satisfy all the conditions listed above. Let $f \in C(\mathbb{S}^1)$. Then*

$$\lim_{h\downarrow 0} \left(h^{n-1}\, Tr_{\mathcal{H}_Y}(f(S_U))\right) = \frac{c_{n-1}\,\mathrm{vol}(Y)}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta$$

where c_{n-1} is the usual Weyl constant in dimension $n-1$.

Subscripts on the trace in this section and the next indicate the space in which the trace is taken.

An immediate corollary of this Theorem is the following equidistribution result.

Corollary 8.5. *Let $0 \leq \theta_1 < \theta_2 < 2\pi$. Then*

$$\lim_{h\downarrow 0} \left(h^{n-1} N(\theta_1, \theta_2, h)\right) = \frac{c_{n-1}\,\mathrm{vol}(Y)}{2\pi}(\theta_2 - \theta_1)$$

where $N(\theta_1, \theta_2, h)$ is the number of eigenvalues of S_U with argument between θ_1 and θ_2.

8.3. **Proof of Theorem 8.4.** We begin with a result on the structure of the iterates of the unitary scattering matrix.

Lemma 8.6. *Let $m \in \mathbf{Z}\backslash\{0\}$ and let $\psi_0 \in C_c^\infty(\mathcal{B})$ be supported in $\mathcal{D}_{\kappa^m}$. Then under the hypotheses of Theorem 8.4, $(S_U)^m \operatorname{Op}_h(\psi_0)$ is a semiclassical Fourier integral operator associated to the graph of κ^m.*

Proof. Theorem 1.4 or 1.5 implies the result for $m = 1$.

Now suppose the lemma has been proved for $1 \leq m \leq m'$. We shall show that it holds for $m = m' + 1$, proving the lemma for positive m by induction. Recall now we assume that $\operatorname{supp}\psi_0 \subset \mathcal{D}_{\kappa^{m'+1}}$, and use $\mathcal{D}_{\kappa^{m'+1}} \subset \mathcal{D}_{\kappa^{m'}}$. Choose $\psi_{m'} \in C_c^\infty(\mathcal{B})$ to be supported on the domain of κ and to be 1 on $\{\kappa^{m'}(y,\eta) \mid (y,\eta) \in \operatorname{supp}\psi_0\}$. Then choose $\psi_{sp,m'} \in C_c^\infty([0,1))$ so that $(I - \psi_{sp,m'}(h^2\Delta_Y))\operatorname{Op}_h(\psi_{m'}) = O(h^\infty)$. We write

$$\begin{aligned}
&(S_U)^{m'+1} \operatorname{Op}_h(\psi_0) \\
&= S_U(\psi_{sp,m'}(h^2\Delta_Y) + I - \psi_{sp,m'}(h^2\Delta_Y))\operatorname{Op}_h(\psi_{m'})(S_U)^{m'} \operatorname{Op}_h(\psi_0) \\
&\quad + S_U(I - \operatorname{Op}_h(\psi_{m'}))(S_U)^{m'} \operatorname{Op}_h(\psi_0) \\
&= S_U\psi_{sp,m'}(h^2\Delta_Y)\operatorname{Op}_h(\psi_{m'})(S_U)^{m'} \operatorname{Op}_h(\psi_0) + S_U(I - \operatorname{Op}_h(\psi_{m'}))(S_U)^{m'} \operatorname{Op}_h(\psi_0) + O(h^\infty).
\end{aligned}$$

That this is a semiclassical FIO associated to $\kappa^{m'+1}$ follows from the inductive hypothesis, an application of Theorem 1.4 or 1.5, and the composition properties of Fourier integral operators. Thus concludes the proof for positive m.

We now turn to the result for S_U^{-1}. We shall use that since $\kappa((\kappa(\overline{y}))') = \overline{y}'$, using the notation $(\mathcal{D}_{\kappa^{-1}})' = \{\overline{y} \mid \overline{y}' \in \mathcal{D}_{\kappa^{-1}}\}$, gives $(\mathcal{D}_{\kappa^{-1}})' = \mathcal{D}_\kappa$.

Lemma 3.1 of [34] implies that $S_U^T = S_U$, where S_U^T denotes the transpose of S_U. Then for any $\psi \in C_c^\infty(\mathcal{D}_\kappa)$, $S_U^T \operatorname{Op}_h(\psi) = S_U \operatorname{Op}_h(\psi)$ is a semiclassical FIO associated to the scattering map κ. Denote complex conjugation by $\mathcal{C}$, and let $\psi_0 \in C_c^\infty(\mathcal{D}_{\kappa^{-1}})$. As an operator on $\mathcal{H}_Y$, $S_U^{-1} = S_U^*$ and $S_U^* \operatorname{Op}_h(\psi_0) = \mathcal{C}\, S_U^T \mathcal{C} \operatorname{Op}_h(\psi_0)$. But $\mathcal{C} \operatorname{Op}_h(\psi_0) = \operatorname{Op}_h(\psi)\mathcal{C}$ for some $\psi \in C_c^\infty((\mathcal{D}_{\kappa^{-1}})') = C_c^\infty(\mathcal{D}_\kappa)$, so that $S_U^* \operatorname{Op}_h(\psi_0) = \mathcal{C}\, S_U \operatorname{Op}_h(\psi)\mathcal{C}$. Now using that we know that $S_U \operatorname{Op}_h(\psi)$ is a semiclassical FIO, the properties of FIOs under conjugation by the action of the complex conjugate $\mathcal{C}$, and the

equality of sets $\{((\kappa(\overline{y}))', \overline{y}') \mid \overline{y} \in \mathcal{D}_\kappa\} = \{(\kappa^{-1}(\overline{y}), \overline{y}) \mid \overline{y} \in \mathcal{D}_{\kappa^{-1}}\}$ we prove the second assertion in the special case $m = 1$.

The general case of negative values of m can be proved by induction, in much the same manner as for positive m. □

Lemma 8.7. *Under the hypotheses of Theorem 8.4, for any $m \in \mathbf{N}$, $\epsilon > 0$ there is a $\psi \in C_c^\infty(\mathcal{D}_{\kappa^m} \cap \mathcal{D}_{\kappa^{-m}})$ so that for $h > 0$ sufficiently small, $\|(I - \mathrm{Op}_h(\psi))\mathbb{1}_{[0,1]}(h^2\Delta_Y)\|_{Tr_{L^2(Y)}} \leq \epsilon h^{-n+1}$.*

Proof. For m fixed and $\delta > 0$, set

$$U_\delta := \{\overline{y} \in \mathcal{B} \mid \mathrm{dist}_{T^*Y}(\overline{y}, T^*Y \setminus (\mathcal{D}_{\kappa^m} \cap \mathcal{D}_{\kappa^{-m}})) > \delta\}$$

and $V_\delta := \mathcal{B} \setminus \overline{U_\delta}$. Note that U_δ is open, and $U_\delta \subset U_{\delta/2}$. Let $\psi_\delta \in C_c^\infty(U_{\delta/2}) \subset C_c^\infty(\mathcal{B})$ satisfy $0 \leq \psi_\delta \leq 1$ and $1 - \psi_\delta = 0$ on U_δ.

Let $\chi_\delta \in C_c^\infty([0, 1+\delta); [0,1])$ with $\chi_\delta(t) = 1$ for $t \in [0,1]$, and note

$$\begin{aligned}\|(I - \mathrm{Op}_h(\psi_\delta))\mathbb{1}_{[0,1]}(h^2\Delta_Y)\|_{\mathrm{Tr}_{L^2(Y)}} &= \|(I - \mathrm{Op}_h(\psi_\delta))\chi_\delta(h^2\Delta_Y)\mathbb{1}_{[0,1]}(h^2\Delta_Y)\|_{\mathrm{Tr}_{L^2(Y)}} \\ &\leq \|(I - \mathrm{Op}_h(\psi_\delta))\chi_\delta(h^2\Delta_Y)\|_{HS_{L^2(Y)}}\|\mathbb{1}_{[0,1]}(h^2\Delta_Y)\|_{HS_{L^2(Y)}}\end{aligned}$$

where $\| \bullet \|_{HS}$ denotes the Hilbert-Schmidt norm. Now

$$\begin{aligned}&\|(I - \mathrm{Op}_h(\psi_\delta))\chi_\delta(h^2\Delta_Y)\|^2_{HS_{L^2(Y)}} \\ &= \mathrm{Tr}_{L^2(Y)}\left(\left((I - \mathrm{Op}_h(\psi_\delta))\chi_\delta(h^2\Delta_Y)\right)^*(I - \mathrm{Op}_h(\psi_\delta))\chi_\delta(h^2\Delta_Y)\right) \\ &\leq C(2\pi h)^{1-n}\int_{T^*Y} |(1 - \psi_\delta(y,\eta))\chi_\delta(|\eta|)|^2\, d\mu + O(h^{2-n})\end{aligned}$$

for some $C > 0$ independent of δ and h. Here μ is the Liouville measure. By the Weyl law, $\|\mathbb{1}_{[0,1]}(h^2\Delta_Y)\|^2_{HS_{L^2(Y)}} = c_{n-1}h^{1-n}\,\mathrm{vol}(Y) + O(h^{2-n})$. Thus there is a constant C_0 independent of δ and h so that

(57)

$$\|(I - \mathrm{Op}_h(\psi))\mathbb{1}_{[0,1]}(h^2\Delta_Y)\|_{\mathrm{Tr}_{L^2(Y)}} \leq C_0 h^{1-n}\left(\int_{T^*Y} |(1 - \psi_\delta(y,\eta))\chi_\delta(|\eta|)|^2\right)^{1/2} d\mu + O(h^{2-n}).$$

The integrand on the right in (57) takes values in $[0,1]$ and is supported in $V_\delta \cup \{\overline{y} \in T^*Y \mid 1 \leq |\eta| \leq 1 + \delta\}$. Let $W_\delta = \{\overline{y} = (y,\eta) \in T^*Y \mid 1 - \delta < |\eta| < 1 + \delta\}$, and note

$$\begin{aligned}V_\delta \setminus \overline{W_\delta} &\subset \{\overline{y} \in \mathcal{B} \mid \mathrm{dist}_{T^*Y}(\overline{y}, \mathcal{B} \setminus (\mathcal{D}_{\kappa^m} \cap \mathcal{D}_{\kappa^{-m}})) < \delta\} \\ &\subset \{\overline{y} \in T^*Y \mid \mathrm{dist}_{T^*Y}((\overline{y}, \mathcal{B} \setminus (\mathcal{D}_{\kappa^m} \cap \mathcal{D}_{\kappa^{-m}})) < \delta\}.\end{aligned} \tag{58}$$

Since by hypothesis (2) both $\mathcal{B} \setminus \mathcal{D}_{\kappa^m}$ and $\mathcal{B} \setminus \mathcal{D}_{\kappa^{-m}}$ have zero $(2n-2)$-dimensional Minkowski content, so does $\mathcal{B} \setminus (\mathcal{D}_{\kappa^m} \cap \mathcal{D}_{\kappa^{-m}}) = (\mathcal{B} \setminus \mathcal{D}_{\kappa^m}) \cup (\mathcal{B} \setminus \mathcal{D}_{\kappa^{-m}})$. Thus (58) implies $\int_{V_\delta \setminus \overline{W_\delta}} 1 d\mu \to 0$ as $\delta \downarrow 0$. Of course $\int_{\overline{W_\delta}} 1 d\mu \to 0$ as $\delta \downarrow 0$. Hence, since

$$\int_{T^*Y} |(1 - \psi_\delta(y,\eta))\chi_\delta(|\eta|)|^2\, d\mu \leq \int_{V_\delta \cup W_\delta} 1 d\mu \tag{59}$$

we may choose $\delta_0 > 0$ small enough so that

$$C_0\left(\int_{T^*Y} |((1 - \psi_{\delta_0}(y,\eta))\chi_{\delta_0}(|\eta|)|^2\right)^{1/2} d\mu < \epsilon/2.$$

Then set $\psi = \psi_{\delta_0}$, and we have chosen ψ so that

$$\|(I - \mathrm{Op}_h(\psi))\mathbb{1}_{[0,1]}(h^2\Delta_Y)\|_{\mathrm{Tr}_{L^2(Y)}} \leq (\epsilon/2)h^{-n+1} + O(h^{2-n}).$$

When $h > 0$ is sufficiently small, we have the desired estimate. □

Corollary 8.8. *Under the hypotheses of Theorem 8.4, for any $m \in \mathbf{Z}$ and $\epsilon > 0$ there is a $\psi \in C_c^\infty(\mathcal{D}_{\kappa^m} \cap \mathcal{D}_{\kappa^{-m}})$ so that for h sufficiently small $|Tr_{\mathcal{H}_Y}(f(S_U)(I - \mathrm{Op}_h(\psi))| \leq \epsilon h^{-n+1} \sup|f|$.*

Proof. Let ψ be as guaranteed by Lemma 8.7. Then

$$\begin{aligned}|\mathrm{Tr}_{\mathcal{H}_Y}(f(S_U)(I - \mathrm{Op}_h(\psi))| &= |\mathrm{Tr}_{L^2(Y)}(f(S_U)(I - \mathrm{Op}_h(\psi))\mathbb{1}_{[0,1]}(h^2\Delta_Y)| \\ &\leq \|f(S_U)\|\|(I - \mathrm{Op}_h(\psi))\mathbb{1}_{[0,1]}(h^2\Delta_Y)\|_{\mathrm{Tr}_{L^2(Y)}} \leq \sup|f|\epsilon h^{-n+1}\end{aligned}$$

for h sufficiently small. □

Lemma 8.9. *Let $m \in \mathbf{Z} \setminus \{0\}$, and $\psi \in C_c^\infty(\mathcal{D}_{\kappa^m})$. Then under the hypotheses of Theorem 8.4, $Tr_{\mathcal{H}_Y}(S_U^m \mathrm{Op}_h(\psi)) = o(h^{1-n})$.*

Proof. This follows from Lemma 8.6, our hypothesis (3) on the fixed point set of κ^m, and [20, Proposition 7.1]. □

Proof of Theorem 8.4. Given $\epsilon > 0$ and $f \in C(\mathbb{S}^1)$, use the density of the polynomials in $e^{i\theta}$ and $e^{-i\theta}$ in the continuous functions on $\mathbb{S}^1$ to choose $q \in \mathbb{C}^\infty(\mathbb{S}^1)$ with $\sup|f(\theta) - q(\theta)| < \epsilon$ and so that $q(e^{i\theta}) = \sum_{j=-J}^J a_j e^{ij\theta}$ for some $J \in \mathbb{N}$, $a_j \in \mathbb{C}$. Choose $\psi \in C_c^\infty(\mathcal{D}_{\kappa^J} \cap \mathcal{D}_{\kappa^{-J}})$ as guaranteed by Corollary 8.8, applied with $m = J$.

Now

$$\mathrm{Tr}_{\mathcal{H}_Y} f(S_U) = \mathrm{Tr}_{\mathcal{H}_Y}(f(S_U) - q(S_U)) + \mathrm{Tr}_{\mathcal{H}_Y}(q(S_U)(I - \mathrm{Op}_h(\psi))) + \mathrm{Tr}_{\mathcal{H}_Y}(q(S_U)\,\mathrm{Op}_h(\psi)). \tag{60}$$

Since by the Weyl law $\mathcal{H}_Y$ is of dimension $h^{1-n}c_{n-1}\,\mathrm{vol}(Y) + O(h^{2-n})$ and $\|f(S_U) - q(S_U)\| < \epsilon$,

$$|\mathrm{Tr}_{\mathcal{H}_Y}(f(S_U) - q(S_U))| < \epsilon h^{1-n}c_{n-1}\,\mathrm{vol}(Y) + O(h^{2-n}). \tag{61}$$

By our choice of ψ as in Corollary 8.8, for $h > 0$ sufficiently small

$$|\mathrm{Tr}_{\mathcal{H}_Y}(q(S_U)(I - \mathrm{Op}_h(\psi))| \leq \epsilon h^{1-n}\sup|q| \leq \epsilon h^{1-n}(\epsilon + \sup|f|). \tag{62}$$

Using $a_0 = \frac{1}{2\pi}\int_0^{2\pi} q(e^{i\theta})d\theta$ and Lemma 8.9,

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_Y}(q(S_U)\,\mathrm{Op}_h(\psi)) &= \sum_{j=-J}^J a_j \mathrm{Tr}_{\mathcal{H}_Y}(S_U^j\,\mathrm{Op}_h(\psi)) \\ &= \frac{1}{2\pi}\int_0^{2\pi} q(e^{i\theta})d\theta\ \mathrm{Tr}_{\mathcal{H}_Y}\,\mathrm{Op}_h(\psi) + o(h^{1-n}). \qquad (63)\end{aligned}$$

But by our choice of ψ as in Corollary 8.8, for h sufficiently small

$$|\mathrm{Tr}_{\mathcal{H}_Y}(\mathrm{Op}_h(\psi) - I_{\mathcal{H}_Y})| < \epsilon h^{1-n},$$

and since the dimension of $\mathcal{H}_Y$ is $c_{n-1}\,\mathrm{vol}(Y)h^{1-n} + O(h^{2-n})$ by the Weyl law,

$$|\mathrm{Tr}_{\mathcal{H}_Y}\,\mathrm{Op}_h(\psi) - c_{n-1}\,\mathrm{vol}(Y)h^{1-n}| < \epsilon h^{1-n} + O(h^{2-n}). \tag{64}$$

Using (61- 64) in (60), we find for h sufficiently small

$$\left|\mathrm{Tr}_{\mathcal{H}_Y} f(S_U) - \frac{c_{n-1}}{2\pi}\,\mathrm{vol}(Y)h^{1-n}\int_0^{2\pi} f(e^{i\theta})d\theta\right| \leq 2\epsilon h^{1-n}\,(c_{n-1}\,\mathrm{vol}(Y) + \epsilon + \sup|f| + 1) + o(h^{1-n})$$

implying

$$\lim_{h\downarrow 0}\left|h^{n-1}\mathrm{Tr}_{\mathcal{H}_Y} f(S_U) - \frac{c_{n-1}}{2\pi}\,\mathrm{vol}(Y)\int_0^{2\pi} f(e^{i\theta})d\theta\right| \leq 2\epsilon\left(c_{n-1}\,\mathrm{vol}(Y) + \epsilon + \sup|f| + 1\right).$$

Since $\epsilon > 0$ is arbitrary, this proves the theorem. □

Appendix A. Warped products with a bulge

This section collects two results for warped products with bulges, as introduced in Section 2.4.2. These results are a resolvent estimate and a computation of the scattering map for the special case in which the manifold is a surface of revolution.

We recall the setting. Let $f \in C^\infty(\mathbf{R};(0,\infty))$ satisfy $f(s) = 1$ if $|s| > a$ and suppose f has a single nondegenerate critical point in $(-a,a)$, and this point is a maximum of f. Let (Y_0, g_{Y_0}) be a smooth compact Riemannian manifold, and set $(X,g) = (\mathbf{R}\times Y_0, ds^2 + f^{4/(n-1)}g_{Y_0})$.

A.1. **The Resolvent estimate for the warped product with a bulge.** Here we bound the microlocally cut-off resolvent on a warped product with a bulge. We give a result that is stronger than we need in terms of the spatial cut-off (a weight in $|s|$, rather than a compactly supported function in s). Our presentation uses a commutator argument and is inspired by [40, 15] and references therein; see also [11, Section 2].

Lemma A.1. *Let f, Y_0, and X be as described above, and let $P = h^2\Delta_X$. Then for any $\epsilon,\ \alpha > 0$ there are $C_0 = C_0(\epsilon,\alpha),\ h_0 = h_0(\epsilon,\alpha) > 0$ so that*

$$\|\mathbb{1}_{[0,1-\epsilon]}(h^2\Delta_{Y_0})(1+|s|)^{-(1+\alpha)/2}(P-1-i0)^{-1}(1+|s|)^{-(1+\alpha)/2}\| \leq C_0h^{-1} \text{ for } 0<h\leq h_0. \tag{65}$$

We emphasize that while $\epsilon > 0$ is small, it is fixed here.

Proof. In this case

$$h^2\Delta_X = f(s)^{-1}\left(-h^2\partial_s^2 + h^2f''(s)/f(s) + h^2\Delta_{Y_0}f(s)^{-4/(n-1)}\right)f(s). \tag{66}$$

Since f is bounded, and is bounded below away from 0, it suffices to study the resolvent of the operator in parentheses on the right hand side of (66). To do so, we will separate variables. Set $\varphi = f(s)^{-4/(n-1)}$. We will show that for any $\alpha > 0$ and $\epsilon > 0$ there is a $h_0 = h_0(\epsilon,\alpha)$, $C_0 = C_0(\epsilon,\alpha)$ so that

$$\|(1+|s|)^{-(1+\alpha)/2}(-h^2\partial_s^2 + \tau\varphi - 1 - i\delta)^{-1}(1+|s|)^{-(1+\alpha)/2}\|_{L^2(\mathbf{R})\to L^2(\mathbf{R})} \leq C_0h^{-1} \quad \text{for } 0<h<h_0,\ 0<\delta<1,\ 0\leq\tau\leq 1-\epsilon. \tag{67}$$

Then using that this implies $\|h^2(f''/f)(-h^2\partial_s^2 + \tau\varphi - 1 - i\delta)^{-1}(1+|s|)^{-(1+\alpha)/2}\| = O(h)$, the estimate (67) together with a separation of variables using (66) proves the lemma.

We give a proof of (67) that is valid uniformly for all $\tau \in [0, 1-\epsilon]$. Without loss of generality we can assume that the maximum of f, and hence the minimum of φ, occurs at $s = 0$ so that $s\varphi'(s) \geq 0$. We also remark that $0 < \varphi \leq 1$. In order to simplify notation, we introduce $Q_\tau := -h^2\partial_s^2 + \tau\varphi - 1$, local to this proof.

Let $u \in H^2(\mathbf{R})$ satisfy $u(s),\ u'(s) \to 0$ as $s \to \pm\infty$ and $(1+|s|)^{(1+\alpha)/2}(Q_\tau - i\delta)u \in L^2(\mathbf{R})$. Let $w \in C^1(\mathbf{R};\mathbf{R})$ be bounded, along with its first derivative. Now using inner products on $L^2(\mathbf{R})$, add

the equalities

$$\langle w'u, u\rangle = -2\Re\langle wu, u'\rangle$$
$$h^2\langle w'u', u'\rangle = -2\Re\langle wh^2u'', u'\rangle$$

and

$$-\tau\langle (w\varphi)'u, u\rangle = 2\Re\langle \tau w\varphi u, u'\rangle$$

to get

$$\langle w'u, u\rangle + h^2\langle w'u', u'\rangle - \tau\langle (w\varphi)'u, u\rangle = -2\Re\langle w(Q_\tau u - i\delta u), u'\rangle + 2\delta\Im\langle wu, u'\rangle. \tag{68}$$

We wish to choose w so that both w' and $w' - \tau(w\varphi)'$ are nonnegative, with $w' > 0$. To do so, set $w(s) = w_1(s)\varphi^\beta$, where $w_1(s)$ is the odd function that is given for $s > 0$ by $w_1(s) = 1 - (1+s)^{-\alpha}$ and $\beta > 0$ is a constant to be chosen below. The restriction $\alpha, \beta > 0$ ensures $w' > 0$, since $w_1' > 0$ and $w_1(s)\varphi'(s) \geq 0$. We compute

$$w' - \tau(w\varphi)' = \varphi^{\beta-1}\left(w_1'\varphi(1-\tau\varphi) + w_1\varphi'(\beta(1-\tau\varphi) - \tau\varphi)\right).$$

Choosing $\beta = 2/\epsilon$ and using $\tau \leq 1 - \epsilon$, $0 < \varphi \leq 1$, yields

$$\begin{aligned} w' - \tau(w\varphi)' &\geq \varphi^{\beta-1}\left(w_1'\varphi\epsilon + w_1\varphi'(\beta\epsilon - 1 + \epsilon)\right) \\ &\geq \varphi^\beta w_1'\epsilon. \end{aligned} \tag{69}$$

Since $\epsilon > 0$ is fixed and the minimum of φ is strictly positive, there is a $c_0 > 0$, independent of $\tau \in [0, 1-\epsilon]$ so that

$$w' - \tau(w\varphi)' \geq c_0 w_1' = c_0\alpha(1+|s|)^{-(1+\alpha)}. \tag{70}$$

Using these in (68) and estimating the right hand side of (68) using the Cauchy-Schwarz inequality yields, for some constant C independent of h, $\tau \in [0, 1-\epsilon]$ and $\delta > 0$, and any $\gamma > 0$

$$\|\sqrt{w_1'}u\|^2 + h^2\|\sqrt{w'}u'\|^2 \leq \frac{C}{\gamma h^2}\|(w/\sqrt{w'})(Q_\tau u - i\delta u)\|^2 + C\gamma h^2\|\sqrt{w'}u'\|^2 + C\delta\|u\|\|u'\|. \tag{71}$$

We will use below that we can simplify this somewhat, by using that w is bounded and that $w' \geq c_1 w_1'$ for some $c_1 > 0$. Now

$$\begin{aligned} \|u'\|^2 = \frac{1}{h^2}\langle -h^2u'', u\rangle = \Re\frac{1}{h^2}\langle (Q_\tau - i\delta)u, u\rangle + \frac{1}{h^2}\langle (1-\tau\varphi)u, u\rangle \\ \leq \frac{1}{h^2}\left\|\frac{1}{\sqrt{w_1'}}(Q_\tau - i\delta)u\right\|\left\|\sqrt{w_1'}u\right\| + \frac{1}{h^2}\|u\|^2 \end{aligned} \tag{72}$$

and

$$\delta\|u\|^2 = \Im\langle (Q_\tau - i\delta)u, u\rangle \leq \left\|\frac{1}{\sqrt{w_1'}}(Q_\tau - i\delta)u\right\|\left\|\sqrt{w_1'}u\right\|$$

giving, if $0 < \delta \leq 1$ and $\gamma > 0$

$$\delta\|u\|\|u'\| \leq \frac{1}{h}\left\|\frac{1}{\sqrt{w_1'}}(Q_\tau - i\delta)u\right\|\left\|\sqrt{w_1'}u\right\|(1+\delta)^{1/2} \leq \frac{1}{\gamma h^2}\left\|\frac{1}{\sqrt{w_1'}}(Q_\tau - i\delta)u\right\|^2 + \gamma\left\|\sqrt{w_1'}u\right\|^2. \tag{73}$$

Using this in (71) and simplifying as indicated above yields, for some constant C independent of $\tau \in [0, 1-\epsilon]$ and $\delta \in (0,1]$,

$$\|\sqrt{w_1'}u\|^2 + h^2\|\sqrt{w'}u'\|^2 \leq \frac{C}{\gamma h^2}\left\|\frac{1}{\sqrt{w_1'}}(Q_\tau - i\delta)u\right\|^2 + C\gamma h^2\|\sqrt{w'}u'\|^2 + C\gamma\|\sqrt{w_1'}u\|^2.$$

Choosing γ sufficiently small, we can absorb the second and third terms on the right into the corresponding terms on the left, yielding, on using estimates for w', w_1' and with a new constant C

$$\|(1+|s|)^{-(1+\alpha)/2}u\|^2 + h^2\|(1+|s|)^{-(1+\alpha)/2}u'\|^2 \leq \frac{C}{h^2}\left\|(1+|s|)^{(1+\alpha)/2}(Q_\tau - i\delta)u\right\|^2.$$

Dropping the second term on the left and applying the resulting inequality with $u = (Q_\tau - i\delta)^{-1}(1+|s|)^{-(1+\alpha)/2}v$ for $v \in L^2(\mathbf{R})$ proves (67). □

We remark that the estimate (67) holds for any fixed $\tau \in [0, 1-\epsilon]$ from well-known non-trapping results, e.g. [37, 22, 7]. In fact, a rescaling and these known non-trapping results prove the estimate uniformly for $\tau \in [\epsilon', 1-\epsilon]$ for any fixed $\epsilon,\ \epsilon' > 0$. We are unaware, however, of a result that directly implies (67) uniformly for all $\tau \in [0, 1-\epsilon]$, so we have chosen to give a direct proof here, valid for all values of τ in this interval.

A.2. **Scattering map for a surface of revolution with a bulge.** In this section we compute the scattering map for a surface of revolution with a bulge, as in Section 2.4.2. We use the function f and manifold X introduced above (Section A or 2.4.2), but specialize to the case $Y_0 = \mathbb{S}^1$ and $X = \mathbf{R} \times \mathbb{S}^1$.

It will be convenient to use a coordinate $\theta \in \mathbf{R}$ on $\mathbb{S}^1$, identifying points which differ by an integral multiple of 2π. The manifold X has two connected ends and $Y = \mathbb{S}^1_L \sqcup \mathbb{S}^1_R$, where $\mathbb{S}^1_L$ corresponds to $s \to -\infty$, the "left" end. On $\mathbb{S}^1_L$ and $\mathbb{S}^1_R$ we use the coordinate θ which is inherited from the factor of $\mathbb{S}^1$ in X.

Lemma A.2. *Let $X = \mathbf{R}_s \times \mathbb{S}^1_\theta$ be a surface of revolution with a bulge as described above, and let $\{r = 0\} = \{s = -a-4\} \sqcup \{s = a+4\}$. Then if $(\theta_-, \eta_-) \in T^*\mathbb{S}^1_R$ with $|\eta_-| < 1$,*

$$\kappa(\theta_-, \eta_-) = \left(\theta_- + \eta_- \int_{-a-4}^{a+4} \frac{1}{f^2(\tilde{s})\sqrt{(f(\tilde{s}))^4 - \eta_-^2}} d\tilde{s}, \eta_-\right) \in T^*\mathbb{S}^1_L.$$

On the other hand, if $(\theta_-, \eta_-) \in T^*\mathbb{S}^1_L$ and $|\eta_-| < 1$, then $\kappa(\theta_-, \eta_-) \in T^*\mathbb{S}^1_R$ and it is given by the same expression.

Proof. We use the coordinates (s, θ, ρ, η) on T^*X. (We are going back to the notation used at the beginning of the paper where the spatial variables come first, followed by the corresponding fiber variables.) The principal symbol of the Laplacian is $p = \rho^2 + (f(s))^{-4}\eta^2$. Thus the equations for the Hamiltonian flow are

$$\begin{array}{ll} \dot{s} = 2\rho & \dot{\theta} = 2(f(s))^{-4}\eta \\ \dot{\rho} = 4(f(s))^{-5}f'(s)\eta^2 & \dot{\eta} = 0. \end{array} \tag{74}$$

Denote the initial conditions by $(s_0, \theta_0, \rho_0, \eta_0)$, and note that η is constant under the Hamiltonian flow, while s and ρ are independent of θ_0. Thus, denoting the Hamiltonian flow by Φ_t, we have

$$\Phi_t(s_0, \theta_0, \rho_0, \eta_0) = (s(t, s_0, \rho_0, \eta_0), \theta(t, s_0, \theta_0, \rho_0, \eta_0), \rho(t, s_0, \rho_0, \eta_0), \eta_0).$$

We shall prove the first equality of the lemma; the second can be derived from the first. Thus we wish to consider initial data

$$(s_0, \theta_0, \rho_0, \eta_0) = (a+4, \theta_0, -\sqrt{1-\eta_0^2}, \eta_0), \text{ where } |\eta_0| < 1. \tag{75}$$

Since p is constant under the Hamilton flow, $\rho^2 + f^{-4}(s)\eta^2 = \rho_0^2 + \eta_0^2 = 1$ using that the initial data are as in (75). Thus since $f(s) \geq 1$ and $\rho_0 < 0$, $\rho = -(1 - f^{-4}(s)\eta^2)^{1/2}$. Using (74) shows that s is a strictly decreasing function of t for such initial data. Thus $\kappa(\theta_0, \eta_0) \in T^*\mathbb{S}^1_L$, and we wish to find $\theta(t_{-a-4}, s_0, \theta_0, \rho_0, \eta_0)$ where t_{-a-4} is the value of t for which $s(t, s_0, \rho_0, \eta_0) = -a-4$. This value of t depends on η_0, but we suppress this in our notation. Using (74),

$$\theta(t_{-a-4}, s_0, \theta_0, \rho_0, \eta_0) - \theta_0 = \eta_0 \int_0^{t_{-a-4}} 2f^{-4}(s(t, s_0, \rho_0, \eta_0))dt. \tag{76}$$

To evaluate the integral in (76) we shall think of s, rather than t, as the independent variable, which works since s is a strictly decreasing function of t. Using (74) to find the derivative of t with respect to s gives

$$\theta(t_{-a-4}, s_0, \theta_0, \rho_0, \eta_0) - \theta_0 = 2\eta_0 \int_{a+4}^{-a-4} f^{-4}(\tilde{s}) \frac{1}{2\rho(\tilde{s})} d\tilde{s} = \eta_0 \int_{-a-4}^{a+4} f^{-4}(\tilde{s}) \frac{1}{\sqrt{1 - f^{-4}(\tilde{s})\eta_0^2}} d\tilde{s}$$

where we use $\tilde{s}$ as a variable to emphasize it is not a function of t here.

□

A similar, but more complicated, computation can be made for an hourglass-type surface of revolution.

Using Lemma A.2 we can see that for a surface of revolution with a bulge the scattering map κ satisfies Hypothesis 3 of Section 8. Indeed, it is clear that for any $m \in \mathbf{Z}$, κ^{2m+1} has no fixed points. Moreover, for each fixed value of η_-, the θ component of $\kappa^{2m}(\bullet, \eta_-)$ is a rotation by $m\delta_\theta(\eta_-)$, where $\delta_\theta(\eta_-) := 2\eta_- \int_{-a-4}^{a+4} \frac{1}{f^2(\tilde{s})\sqrt{(f(\tilde{s}))^4 - \eta_-^2}} d\tilde{s}$. Thus fixed points of κ^{2m} correspond to values of η_- so that $m\delta_\theta(\eta_-)$ is an integral multiple of 2π. But since δ_θ is a smooth, strictly increasing function of $\eta_- \in (-1, 1)$, for $m \neq 0$ this can happen only for isolated vales of η_-, with accumulation points only at $\eta_- = \pm 1$.

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