NEW CHARACTERIZATION OF PLURISUBHARMONIC FUNCTIONS AND POSITIVITY OF DIRECT IMAGE SHEAVES

FUSHENG DENG, ZHIWEI WANG, LIYOU ZHANG, AND XIANGYU ZHOU

Abstract. We discover a new characterizations of plurisubharmonic functions in terms of L^p extension from one point and Griffiths positivity of holomorphic vector bundles with singular Finsler metrics in terms of L^p extensions. As applications, we give stronger result or new proof of some well-known theorems on the Griffiths positivity of the holomorphic vector bundles and their direct image sheaves associated to certain holomorphic fibrations.

1. Introduction

In the famous and useful Ohsawa-Takegoshi L^2 extension theorem for holomorphic functions, plurisubharmonic (p.s.h) functions are used as weights to derive a conclusion about L^2 extension from complex submanifolds, in particular, from one point. In this paper, we show that the converse also holds, namely, if a function is given as the weight such that the conclusion about L^2 extension from one point in the Ohsawa-Takegoshi L^2 extension theorem holds, then the function must be plurisubharmonic. This result can be generalized to give a characterization of Griffiths positivity of holomorphic vector bundles with singular Finsler metrics. As applications, we give some stronger result and new proof of a couple of well-known results on the Griffiths positivity of the direct image sheaves associated to certain holomorphic fibrations, originally established by Berndtsson [3].

Our first main theorem is to discover the following surprising result.

Theorem 1.1. *Let $\varphi : D \to [-\infty, +\infty)$ be an upper semicontinuous function on a domain $D \subset \mathbb{C}^n$ that is not identically $-\infty$. Let $p > 0$ be a fixed constant. If for any $z_0 \in D$ with $\varphi(z_0) > -\infty$ and any $m > 0$, there is $f \in \mathcal{O}(D)$ such that $f(z_0) = 1$ and*

$$\int_D |f|^p e^{-m\varphi} \leq C_m e^{-m\varphi(z_0)},$$

where C_m are constants independent of z_0 and satisfying the growth condition $\lim\limits_{m\to\infty} \frac{1}{m} \log C_m = 0$, then φ is p.s.h.

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The proof of Theorem 1.1 is influenced by Demailly's regularization method of psh functions in [8]. Theorem 1.1 can be generalized to holomorphic vector bundles of higher rank.

We first introduce the notion of *multiple L^p-extension property* for holomorphic vector bundles with singular Finsler metrics.

Definition 1.1 (Multiple L^p-extension property)**.** Let (E,h) be a holomorphic vector bundle over a bounded domain $D\subset\mathbb{C}^n$ equipped with a singular Finsler metric h. Let $p>0$ be a fixed constant. Assume that for any $z\in D$, any nonzero element $a\in E_z$ with finite norm $|a|$, and any $m\geq 1$, there is a holomorphic section f_m of $E^{\otimes m}$ on D such that $f_m(z)=a^{\otimes m}$ and satisfies the following estimate:

$$\int_D |f_m|^p \leq C_m|a^{\otimes m}|^p = C_m|a|^{mp},$$

where C_m are constants independent of z and satisfying the growth condition $\frac{1}{m}\log C_m\to 0$ as $m\to\infty$. Then (E,h) is said to have the *multiple L^p-extension property.*

Theorem 1.2. *Let (E,h) be a holomorphic vector bundle over a bounded domain $D\subset\mathbb{C}^n$ equipped with a singular Finsler metric h, such that the norm of any local holomorphic section of E^* is upper semicontinuous. If (E,h) has multiple L^p-extension property for some $p>0$, then (E,h) is positively curved in the sense of Griffiths, namely* $\log|u|$ *is p.s.h for any local holomorphic section u of E^*.*

The reader is referred to §2 for the definitions of singular Finsler metrics and dual Finsler metrics, and §4.2 for the definition of Finsler metrics on tensor products of vector bundles.

Combing Theorem 1.1 with the Ohsawa-Takegoshi L^2 extension theorem, we give a new proof of the positivity of certain direct image sheaves associated to holomorphic fibrations.

A paper of the authors with the same title was posted on arXiv (arXiv:1809.10371) in 2018, and the present paper is a shorter and compressed version of the arXiv paper. After the first version of the present paper was posted on the arXiv, there have been several related papers motivated by and based on the present paper on the study of converse versions of L^2 theories for $\bar\partial$. This shows the usefulness of the present paper.

In [20], motivated by a question posed in the arXiv version of the present paper, Hosono-Inayama introduced the concept "twisted Hörmander condition" for hermitian holomorphic vector bundles and proved that this condition implies multiple L^2-extension property, a concept introduced in the present paper, thus get the Griffiths positivity by using Theorem 1.2 in the present paper.

Deng-Ning-Wang in [11] introduced another notion "optimal L^2-estimate condition" for plurisubharmonic functions (or metrics on line bundles). The essential difference between the optimal L^2-estimate condition and multiple L^2-extension property or twisted Hörmander condition is that the first involves an optimal constant but does not involve the tensor powers of the considered bundles. Deng-Ning-Wang shows in [11] that a $\mathcal{C}^2$ function is plurisubharmonic if it satisfies the optimal L^2-estimate condition.

In [12], Deng-Ning-Wang-Zhou generalized the concept "optimal L^2-estimate condition" from line bundles to vector bundles, and found that the optimal L^2-estimate condition implies the Nakano positivity of the Chern curvature. It is well-known that under the assumption of Nakano positivity one has an optimal L^2-estimates for solving $\bar{\partial}$. Therefore the main result in [12] establishes the equivalence between Nakano positivity and the optimal L^2-estimate condition for Hermitian holomorphic vector bundles.

Simply speaking, the main difference between the present paper and the above related papers is as follows. The present paper is to establish a converse version of the Ohsawa-Takegoshi L^2 extension theorem (Theorem 1.1), while the above papers are to establish certain converse versions of Hörmander's L^2-estimate for $\bar{\partial}$.

The structure of this paper is organized as follows. In §2, we introduce the singular Finsler metrics on coherent analytic sheaves and define the concept of curvature positivity. In §3, we show certain continuity of Hodge-type metrics on direct image sheaves. In §4, we give the proofs of Theorem 1.1 and Theorem 1.2. In §5, we apply the results in the previous sections to prove the Griffiths positivity of certain direct image sheaves associated to holomorphic fibrations.

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2. Singular Finsler metrics on coherent analytic sheaves

In this section, we recall the notions of singular Finsler metrics on holomorphic vector bundles and give a definition of positively curved singular Finsler metrics on coherent analytic sheaves.

Definition 2.1. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. A (singular) Finsler metric h on E is a function $h : E \to [0, +\infty]$, such that $|cv|_h^2 = |c|^2 h(v)$, where $|v|_h^2 := h(v)$, for any $v \in E$ and $c \in \mathbb{C}$.

In the above definition, we do not assume the triangle inequality and any regularity property of a singular Finsler metric. Only when considering Griffiths positivity certain regularity is required, as shown in the following Definition 2.3.

Definition 2.2. For a singular Finsler metric h on E, its dual Finsler metric h^* on the dual bundle E^* of E is defined as follows. For $f \in E_x^*$, the fiber of E^* at $x \in X$, $|f|_{h^*}$ is defined to be 0 if $|v|_h = +\infty$ for all nonzero $v \in E_x$; otherwise,

$$|f|_{h^*} := \sup\{|f(v)|; v \in E_x, |v|_h \leq 1\} \leq +\infty.$$

Definition 2.3. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. A singular Finsler metric h on E is called negatively curved (in the sense of Griffiths) if for any local holomorphic section s of E the function

$\log |s|_h^2$ is p.s.h., and is called positively curved (in the sense of Griffiths) if its dual metric h^* on E^* is negatively curved.

As far as our knowledge, there has no natural definition of singular Finsler metric on a coherent analytic sheaf. In the present paper, we will propose a definition of positively curved Finsler metrics on coherent analytic sheaves. Let $\mathcal{F}$ be a coherent analytic sheaf on X, it is well known that $\mathcal{F}$ is locally free on some Zariski open subset U of X. On U, we will identify $\mathcal{F}$ with the vector bundle associated to it.

Definition 2.4. Let $\mathcal{F}$ be a coherent analytic sheaf on a complex manifold X. A positively curved singular Finsler metric h on $\mathcal{F}$ is a singular Finsler metric on the holomorphic vector bundle $\mathcal{F}|_{X\setminus Z}$, for some analytic subset $Z \subset X$ such that $\mathcal{F}|_{X\setminus Z}$ is locally free, satisfying the condition that for any local holomorphic section g of the dual sheaf $\mathcal{F}^*$ on an open set $U \subset X$, the function $\log |g|_{h^*}$ is p.s.h. on $U \setminus Z$, and can be extended to a p.s.h. function on U.

Remark 2.1.
(1) Suppose that $\log |g|_{h^*}$ is p.s.h. on $U \setminus Z$. It is well-known that if $\mathrm{codim}_{\mathbb{C}}(Z) \geq 2$ or $\log |g|_{h^*}$ is locally bounded above near Z, then $\log |g|_{h^*}$ extends across Z to U uniquely as a p.s.h function. In particular, when $\mathcal{F}$ is torsion free, then $\mathrm{codim}_{\mathbb{C}}(Z) \geq 2$.
(2) The above definition does not depend on the choice of Z, in the sense that, after extension, the norm functions $|g|_{h^*}$ for local sections g of $\mathcal{F}^*$, which are the original quantities that we are really concerning with, remain unchanged if we choose different Z. Also in this sense, Definition 2.4 matches Definition 2.1 and Definition 2.3 if $\mathcal{F}$ is a vector bundle.

3. Regularity of Hodge-type metrics

The aim of this section is to show certain continuity of Hodge-type metrics on direct image sheaves. One of the basic techniques is the following Ohsawa-Takegoshi type L^2 extension theorem:

Theorem 3.1 (c.f.[25][9][29]). *Let (X, ω) be a weakly pseudoconvex Kähler manifold and L be a holomorphic line bundle over X with a (singular) hermitian metric h. Let $s : X \to \mathbb{C}^r$ be a holomorphic map such that $0 \in \mathbb{C}^r$ is not a critical value of s. Assume that the curvature current of (L, h) is semi-positive and $|s(x)| \leq M$ for some constant M. Let $Y = s^{-1}(0)$ be the zero set of s. Then for every holomorphic section f of $K_X \otimes L$ over Y such that $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} dV_\omega < +\infty$, there exists a holomorphic section F of $K_X \otimes L$ over X such that $F|_Y = f$ and*

$$\int_X |F|_L^2 dV_{X,\omega} \leq C_{r,M} \int_Y \frac{|f|_L^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega}.$$

where $C_{r,M}$ is a constant depending only on r and M.

3.1. **For families of compact Kähler manifolds.** Let X, Y be Kähler manifolds of dimension $m+n$ and m respectively, let $p : X \to Y$ be a proper holomorphic submersion. Let L be a holomorphic line bundle over X, and h be a singular Hermitian metric on L, whose curvature current is semi-positive, i.e., L is pseudoeffective. Let $K_{X/Y}$ be the relative canonical bundle on X.

Let $\mathcal{E} = p_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h))$ be the direct image sheaf on Y, where $\mathcal{I}(h)$ is the multiplier ideal sheaf associated to (L, h). By Grauert's theorem, $\mathcal{E}$ is a coherent analytic sheaf on Y. We assume that $\mathcal{E}$ is locally free, then it is the sheaf of holomorphic sections of a holomorphic vector bundle, which will be denoted by E. For any $y \in Y$, we can identify the fiber E_y of E at y with $H^0(X_y, (K_{X/Y} \otimes L \otimes \mathcal{I}(h))|_{X_y}) \subset H^0(X_y, K_{X_y} \otimes L|_{X_y})$. For $u \in E_y$, the norm of u is defined to be

$$H_y(u) := \|u\| = \left(\int_{X_y} |u|_h^2 \right)^{1/2} \leq +\infty.$$

Note that here we view u as an element in $H^0(X_y, K_{X_y} \otimes L|_{X_y})$. Then H is a singular (Hermitian) metric on E. It is clear that H is locally bounded below by positive constants. The following proposition shows that H is lower semicontinuous.

Proposition 3.2 ([21])**.** *Let s be a holomorphic section of E. The function $|s|(y) := \|s(y)\| : Y \to [0, +\infty]$ is lower semi-continuous.*

Using similar idea in [21] and combining Theorem 3.1, we can get the following

Proposition 3.3. *With the same notations and assumptions as in Proposition 3.2, for every $\xi \in H^0(Y, E^*)$, the function $|\xi|(y) := H^*(\xi(y)) : Y \to [0, +\infty]$ is upper semi-continuous.*

3.2. **For families of pseudoconvex domains.** Let $\Omega \subset \mathbb{C}^{m+n} = \mathbb{C}_t^m \times \mathbb{C}_z^n$ be a pseudo-convex domain. Let $p : \Omega \to \mathbb{C}^m$ be the natural projection. We denote $p(\Omega)$ by D and denote $p^{-1}(t)$ by Ω_t for $t \in D$. Let φ be a p.s.h. function on Ω. For an open subset U of D, we denote by $\mathcal{F}(U)$ the space of holomorphic functions F on $p^{-1}(U)$ such that $\int_{p^{-1}(K)} |F|^2 e^{-\varphi} < \infty$ for all compact subset K of U. For $t \in D$, let

$$E_t = \{F|_{\Omega_t} : F \in \mathcal{F}(U),\ U \subset D \text{ open and } t \in U\}.$$

E_t is a vector space and we define a norm on it as follows:

$$H(f) := \|f\| = \left(\int_{D_t} |f|^2 e^{-\varphi_t} \right)^{1/2} \leq \infty,$$

where $\varphi_t = \varphi|_{D_t}$. Let $E = \coprod_{t \in D} E_t$ be the disjoint union of all E_t. Then we have a natural projection $\pi : E \to D$ which maps elements in E_t to t. We view H as a singular Hermitian metric on E.

In general E is not a genuine holomorphic vector bundle over D. However, we can also talk about its holomorphic sections, which are the objects we are really interested in. By definition, a section $s : D \to E$ is a *holomorphic* *section* if it varies holomorphically with t, namely, the function $s(t,z)$: $\Omega \to \mathbb{C}$ is holomorphic with respect to the variable t. Note that $s(t,z)$ is automatically holomorphic on z for t fixed, by Hartogs theorem, $s(t,z)$ is holomorphic jointly on t and z and hence is a holomorphic function on Ω. In some sense, E can be viewed as an object similar to holomorphic vector fields studied in [22].

Let E_t^* be the dual space of E_t, namely the space of all complex linear functions on E_t. Let $E^* = \coprod_{t\in D} E_t^*$. The natural projection from E^* to D is denoted by π^*. Note that we do not define any topology on E_t^* and E^*. The only object we are interested in is holomorphic sections of E^* which we are going to define. Given a holomorphic section s of E on some open set U of D, s induces a function $|s| : U \to \mathbb{R}$ with $|s|(t)$ given by $\|s(t)\|$, which is lower semicontinuous and hence measurable, by the following Proposition 3.4.

Definition 3.1. A section ξ of E^* on D is holomorphic if:

(1) for any local holomorphic section s of E, $\langle \xi, s\rangle$ is a holomorphic function;
(2) for any sequence s_j of holomorphic sections of E on D such that $\int_D |s_j| \leq 1$, if $s_j(t,z)$ converges uniformly on compact subsets of Ω to $s(t,z)$ for some holomorphic section s of E, then $\langle \xi, s_j\rangle$ converges uniformly to $\langle \xi, s\rangle$ on compact subsets of D.

In the same way we can define holomorphic section of E^* on open subsets of D. The Finsler metric H on E induces a Finsler metric H^* on E^*, as defined in the Definition 2.2. We will show that H is lower semicontinuous and H^* is upper semicontinuous, as analogues of Proposition 3.2 and Proposition 3.3 in the case of families of pseudoconvex domains.

Proposition 3.4. *With the above notations and assumptions. Assume s is a holomorphic section of E, then the function $|s|(t) := H(s(t)) : D \to [0, +\infty]$ is lower semicontinuous.*

Proof. We assume $0 \in D$ and prove that $|s|$ is lower semicontinuous for a point 0. Let $K_1 \Subset K_2 \Subset \cdots \Subset K_j \Subset \cdots \Subset \Omega_0$ be an increasing sequence of compact subsets of Ω_0, such that $\cup_j K_j = \Omega_0$. Since the set valued function $t \to \Omega_t$ is lower semi-continuous, in the sense that if Ω_t contains a compact set K, then K is contained in all Ω_s for s sufficiently close to t. Thus for any j, there is a small disk $B_j \subset D$ centered at a, such that $B_j \times K_j \Subset \Omega$. Note that $e^{-\varphi}$ is lower semicontinuous, hence $\liminf_{t\to 0} |s|(t) \geq \left(\int_{K_j} |s(0,z)|^2 e^{-\varphi_t}\right)^{1/2}$ for all j. Let j go to ∞, we get $\liminf_{t\to 0} |s|(t) \geq |s|(0)$. □

The following lemma shows that $|\xi|(t)$ can not take value $+\infty$ anywhere.

Lemma 3.5. *Let ξ be a holomorphic section of E^*, then $|\xi|(t) < +\infty$ for all $t \in D$.*

Proof. We argue by contradiction. Assume $0 \in D$ and $|\xi(0)| = +\infty$. By definition, there is a sequence $\{u_j\} \subset E_0$ such that $|u_j| = 1$ and

$\lim_{j\to\infty}\langle\xi(0), u_j\rangle = +\infty$. By Theorem 3.1, there are holomorphic sections s_j of E such that $s_j(0) = u_j$ and $\int_D |s_j| \leq C$ for some constant C independent of j. By Montel's theorem there is a subsequence of $\{s_j\}$, may assumed to be $\{s_j\}$ itself, that converges uniformly on compact subsets of Ω to some holomorphic section s of E. By definition, $\langle\xi, s_j\rangle$ converges uniformly on compact sets of D to $\langle\xi, s\rangle$. In particular, $\langle\xi(0), u_j\rangle$ converges to $\langle\xi(0), s(0)\rangle \leq +\infty$, which is a contradiction. □

Proposition 3.6. *Let $\xi : D \to E^*$ be a holomorphic section of E^*. Then the function $|\xi|(t) := H^*(\xi(t)) : D \to [0, +\infty)$ is upper semicontinuous.*

Proof. We assume $0 \in D$ and prove that $|\xi|$ is upper semicontinuous at 0. We need to show that

$$\limsup_{j\to+\infty} |\xi|(t_j) \leq |\xi|(0).$$

for every sequence $t_1, t_2, \cdots \in D$ which converges to 0. We may assume that $|\xi|(t_j) \neq -\infty$ for all $j \in \mathbb{N}$, and that the sequence $|\xi|(t_j)$ actually has a limit. From the definition of the dual metric and Lemma 3.5, for each j, there exists $u_j \in E_{t_j}$, such that $|u_j| = 1$ and $|\xi|(t_j) < |\langle\xi(t_j), u_j\rangle| + \epsilon$, where $\epsilon > 0$ is an arbitrary constant.

By Theorem 3.1, there are holomorphic sections s_j of E such that

$$s_j(t_j) = u_j \quad \text{and} \quad \int_D |s_j(t)| \leq K$$

for some constant K independent of j.

By Montel's theorem, there is a subsequence of $\{s_j\}$, denoted by $\{s_j\}$ itself, that converges on compact subsets of Ω uniformly to some holomorphic section s of E.

By definition, $\langle\xi, s_j\rangle$, as a sequence of the holomorphic functions on D, converges uniformly on compact subsets of D to $\langle\xi, s\rangle$. In particular

$$\limsup_{j\to\infty} |\xi|(t_j) \leq \limsup_{j\to\infty}(|\langle\xi(t_j), u_j\rangle| + \epsilon) = |\langle\xi(0), s(0)\rangle| + \epsilon.$$

If $s(0) = 0$, we are done. We assume $s(0) \neq 0$. Then it suffices to prove that $|s(0)| \leq 1$. But this is true since s_j converges to s uniformly on compacts sets, $|u_j| = 1$, and $e^{-\varphi}$ is lower semicontinuous. □

Remark 3.1. Let ξ be a holomorphic section of E^*. By Lemma 3.5 and Theorem 3.6, $|\xi|(t)$ is locally bounded above by positive constants. On the other hand, it is not difficult to show that a section of E^* is holomorphic if it satisfies condition (1) in Definition 3.1 and its norm is locally bounded above.

4. New characterizations of psh functions and positively curved bundles

4.1. **Characterization of plurisubharmonic functions.** We give the proof of Theorem 1.1 in this subsection. We need some preparations.

Let D, φ, and p be as in Theorem 1.1. Let

$$H^p(D,\varphi)=\{f\in\mathcal{O}(D); \|f\|_{\varphi,p}:=\int_D |f|^p e^{-\varphi}<\infty\}.$$

For $z\in D$, define

$$K_{\varphi,p}=\sup\{|f(z)|^p; f\in H^p(D,\varphi), \|f\|_{\varphi,p}=1\}.$$

It is also easy to see that

$$K_{\varphi,p}(z)=\left(\inf\{\|f\|_{\varphi,p}; f\in H^p(D,\varphi), |f(z)|=1\}\right)^{-1}$$

if there exists an $f\in H^p(D,\varphi)$ with $f(z)\neq 0$, and otherwise $K_{\varphi,p}(z)$ is 0.

We have the following

Lemma 4.1. *With the above notations, $K_{\varphi,p}$ is a continuous function on D.*

Proof. This is proved by a normal family argument.

By definition, it is clear that $K_{\varphi,p}$ is lower semicontinuous. We now show it is also upper semicontinuous.

Assume $a\in D$ and $z_j\in D$ which converges to a as $j\to\infty$. Let $\epsilon>0$ be arbitrary. There exists $f_j\in H^p(D,\varphi)$ such that $\|f_j\|_{\varphi,p}=1$ and $|f_j(z_j)|^p> K_{\varphi,p}(z_j)-\epsilon$. Since φ is upper semicontinuous and hence bounded above locally, $\{f_j\}$ is a normal family on D and hence has a subsequence, which is denoted by $\{f_j\}$ itself, that converges uniformly to some $f\in\mathcal{O}(D)$ on compact subsets of D. By Fatou's lemma, we have $f\in H^p(D,\varphi)$ and $\|f\|_{\varphi,p}\leq 1$. So

$$K_{\varphi,p}(a)\geq |f(a)|^p=\lim_{j\to\infty}|f_j(z_j)|^p\geq \limsup_{j\to\infty} K_{\varphi,p}(z_j)-\epsilon.$$

Letting ϵ go to 0, we see $K_{\varphi,p}$ is upper semicontinuous. □

Using the same normal family argument as in the proof of Lemma 4.1, one can show that the supremum in the definition of $K_{\varphi,p}$ can be attained.

Lemma 4.2. *Let D,φ and p be as above. Then for any $z\in D$ there exits $f\in H^p(D,\varphi)$ such that $\|f\|_{\varphi,p}=1$ and $K_{\varphi,p}(z)=|f(z)|^p$.*

Lemma 4.3. *$\log K_{\varphi,p}$ is a p.s.h. function on D.*

Proof. Note that $\log K_{\varphi,p}=\sup\{p\log|f|; f\in H^p(D,\varphi), \|f\|_{\varphi,p}=1\}$ and $\log K_{\varphi,p}$ is upper semicontinuous by Lemma 4.1, $\log K_{\varphi,p}$ is plurisubharmonic. □

Now we're ready to prove Theorem 1.1.

Proof of Theorem 1.1. We will use the above notations and definitions. We denote $\frac{1}{m}\log K_{m\varphi,p}$ by φ_m. By Lemma 4.3, φ_m is p.s.h on D. We want to show that φ_m converges to φ as $m\to\infty$.

By assumption, for any $z\in D$ with $\varphi(z)>-\infty$, there exists $f\in D$ such that $f(z)=1$ and $\|f\|_{m\varphi,p}\leq C_m e^{-m\varphi}$. Then we have

$$-\varphi_m(z)\leq \frac{1}{m}\log\|f\|_{m\varphi,p}\leq \frac{1}{m}\log\left(e^{-m\varphi(z)}\right)+\frac{\log C_m}{m}=-\varphi(z)+\frac{\log C_m}{m}.$$

This is

$$\varphi_m(z) \geq \varphi(z) - \frac{\log C_m}{m}.$$

Let $z \in D$ and $r > 0$ such that $d(z, \partial D) > r$. We have

$$\varphi|_{B(z,r)} \leq \sup_{\zeta \in B(z,r)} \varphi(\zeta).$$

By Lemma 4.2, we can choose $f \in H^p(D, m\varphi)$ such that $\|f\|_{m\varphi,p} = 1$ and $K_{m\varphi,p}(z) = |f(z)|^p$. Note that $|f|^p$ is a p.s.h function, we get

$$\int_{B(z,r)} |f(\zeta)|^p \geq \frac{\pi^n r^{2n}}{n!} |f(z)|^p,$$

by the mean value inequality. Therefore, we get

$$1 = \|f\|_{m\varphi,p} \geq \int_{B(z,r)} |f(\zeta)|^p e^{-m\varphi(\zeta)} \geq \frac{\pi^n r^{2n}}{n!} |f(z)|^p e^{-m \sup_{\zeta \in B(z,r)} \varphi(\zeta)},$$

which implies

$$\varphi_m(z) = \frac{1}{m} \log |f(z)|^p \leq \sup_{\zeta \in B(z,r)} \varphi(\zeta) - \frac{1}{m} \log(\frac{\pi^n r^{2n}}{n!}).$$

In summary, we have

$$\varphi(z) - \frac{\log C_m}{m} \leq \varphi_m(z) \leq \sup_{\zeta \in B(z,r)} \varphi(\zeta) - \frac{1}{m} \log(\frac{\pi^n r^{2n}}{n!}).$$

We now take $r = e^{-\sqrt{m}/2n}$, then the above inequality becomes

$$\varphi(z) - \frac{\log C_m}{m} \leq \varphi_m(z) \leq \sup_{\zeta \in B(z, e^{-\sqrt{m}/n})} \varphi(\zeta) - \frac{1}{m} \log \frac{\pi^n}{n!} + \frac{1}{\sqrt{m}}.$$

Note that φ is u.s.c, the above inequality implies that $\limsup_{\zeta \to z} \varphi(\zeta) = \varphi(z)$ and hence φ_m converges to φ pointwise as $m \to \infty$. Let $\psi_m = \sup_{j \geq m} \varphi_j$ and let ψ_m^* be the upper semicontinuous regularization of ψ_m. We have

$$\varphi(z) \leq \psi_m(z) \leq \sup_{\zeta \in B(z, e^{-\sqrt{m}/n})} \varphi(\zeta) + \frac{2}{\sqrt{m}}$$

for $m >> 1$. Since the last term of the above inequality is u.s.c, ψ_m^* also satisfies the same inequality. So we also have ψ_m^* converges to φ pointwise as $m \to \infty$, thus φ is plurisubharmonic.

□

Remark 4.1. It is worth mentioning that Berndtsson proved in [1] that a continuous function φ on a planar domain is subharmonic if $e^{-m\varphi}$ can be used as a weight for Hörmander's L^2-estimate for $\bar{\partial}$. For the arguments in [1], dimension-one condition and continuity for φ seem to be necessary assumptions. It seems interesting to generalize Berndtsson's result to higher dimensions and upper semi-continuous functions.

Remark 4.2. After the present paper appeared on the arXiv, there are several groups who made contributions to the question raised in Remark 4.1, for instance, Hosono-Inayama [20], Deng-Ning-Wang [11], and Deng-Ning-Wang-Zhou [12]. For a discription on these papers, the readers are referred to a few paragraphes at the end of Introduction (§1), where we explain the progress on the aforementioned question, as well as their relations with the present paper.

4.2. **Characterization of positive vector bundles.** The aim of this subsection is to prove Theorem 1.2.

We start from some basic linear algebra. Let V be a vector space of finite dimension. Recall that a Finsler metric on V is defined to be a map $h : V \to [0, +\infty]$ such that $h(cv) = |c|^2 h(v)$ for all $v \in V$ and $c \in \mathbb{C}$. Given a Finsler metric on V, the dual metric h^* on the dual space V^* is defined as in Definition 2.2. For a positive integer m, the m-th tensor power of V is denoted by $V^{\otimes m}$. Then h and h^* induces naturally Finsler metrics h^m and h^{*m} on $V^{\otimes m}$ and $(V^*)^{\otimes m}$ as follows. Recall that a vector $\xi \in (V^*)^{\otimes m}$ can be viewed as a map $\xi : V^m \to \mathbb{C}$ which is multilinear, namely linear on each component.

Definition 4.1. The metric $h^{*m} : (V^*)^{\otimes m} \to [0, +\infty]$ on $(V^*)^{\otimes m}$ is defined as:

$$h^{*m}(\xi) := \sup\{|\xi(u_1, \cdots, u_m)|; u_i \in V, h(u_i) \leq 1, 1 \leq i \leq m\}$$

if $h(u) < +\infty$ for some $u \in V$; otherwise, $h^{*m}(\xi)$ is defined to be 0. The metric h^m on $V^{\otimes m}$ is defined in the same way by identifying V and $(V^*)^*$, the dual space of V^*.

According to this definition, for $\xi_1, \cdots, \xi_m \in V^*$, we have the product formula $h^{*m}(\xi_1 \cdots \xi_m) = h^*(\xi_1) \cdots h^*(\xi_m)$. Definition 4.1 can be applied to holomorphic vector bundles. If E is a holomorphic vector bundle over a complex manifold X and h is a Finsler metric on E. The induced metrics h^m and h^{*m} on $E^{\otimes m}$ and $(E^*)^{\otimes m}$ is defined pointwise.

We now give the proof of Theorem 1.2.

Proof. Let u be a local holomorphic section of E^* on $U \subset D$. We need to show that the function $\varphi := \log |u|$ is p.s.h. on U. Our strategy is to prove that φ satisfies the condition in Theorem 1.1.

Without loss of generality, we assume $U = D$. Let z be a fixed point in D. We assume that $|u(z)| \neq 0$. Let $a \in E_z$ such that $|a| = 1$ and $\langle u(z), a\rangle = |u(z)|$. By assumption, there is a holomorphic section f of $E^{\otimes m}$ over D such that $f(z) = a^{\otimes m}$ and $\int_D |f|^p \leq C_m |a^{\otimes m}|^p = C_m |a|^{mp} = C_m$. We view $u^{\otimes m}$ as a holomorphic section of $(E^*)^{\otimes m}$. It is obvious that $|u^{\otimes m}| = |u|^m$ and $|u^{\otimes m}(z)| = \langle u^{\otimes m}(z), a^{\otimes m}\rangle$. By definition,

$$|u(\zeta)|^m \geq |\langle u^{\otimes m}(\zeta), f(\zeta)\rangle|/|f(\zeta)|$$

for $\zeta \in D$, which is

$$e^{-m\varphi(\zeta)} \leq e^{-\log|\langle u^{\otimes m}(\zeta), f(\zeta)\rangle|}|f(\zeta)|. \qquad (1)$$

Since $u^{\otimes m}, f$ are holomorphic section of $(E^*)^{\otimes m}$ and $E^{\otimes m}$ respectively, $\langle u^{\otimes m}, f\rangle$ is a holomorphic function on D. By the Ohsawa-Takegoshi extension theorem, there is a holomorphic function h on D such that $h(z) = 1$ and

$$\int_D |h|^2 e^{-p\log|\langle u^{\otimes m}(\zeta), f(\zeta)\rangle|} \leq Ce^{-p\log|\langle u^{\otimes m}(z), f(z)\rangle|} = Ce^{-pm\varphi(z)},$$

where C is a constant independent m and z. By the above inequality, we have

$$\begin{aligned}\int_D |h|e^{-\frac{p}{2}m\varphi} &\leq \int_D |h|e^{-\frac{p}{2}\log|\langle u^{\otimes m}(\zeta), f(\zeta)\rangle|}|f|^{\frac{p}{2}} \\ &\leq \left(\int_D |h|^2 e^{-p\log|\langle u^{\otimes m}(\zeta), f(\zeta)\rangle|}\int_D |f|^p\right)^{1/2} \\ &\leq \left(Ce^{-pm\varphi(z)}C_m\right)^{1/2} \\ &= \sqrt{CC_m}e^{-\frac{p}{2}m\varphi(z)}.\end{aligned}$$

By Theorem 1.1, $\varphi = \log|u|$ is p.s.h on D. □

Remark 4.3. Although this theorem is stated and proved for vector bundles of finite rank, the same argument also works for holomorphic vector bundles of infinite rank.

5. Positivity of direct images of twisted relative canonical bundles

In this section, we apply Theorem 1.1 to show that the direct image of relative canonical bundles twisted by pseudoeffective line bundles associated to certain families of pseudoconvex domains or compact Kähler manifolds is semi-positive in the sense of Griffiths. The proof is given by combing Theorem 1.1 and Theorem 3.1.

5.1. **For families of pseudoconvex domains.** Let U, D be bounded pseudoconvex domains in $\mathbb{C}^r$ and $\mathbb{C}^n$ respectively, and let $\Omega = U \times D \subset \mathbb{C}^r \times \mathbb{C}^n$. Let φ be a p.s.h function on Ω, which is for simplicity assumed to be bounded. For $t \in U$, let $D_t = \{t\} \times D$ and $\varphi_t(z) = \varphi(t, z)$. Let $E_t = H^2(D_t, e^{-\varphi_t})$ be the space of L^2 holomorphic functions on D_t with respect to the weight $e^{-\varphi_t}$. Then E_t are Hilbert spaces with the natural inner product. Since φ is assumed to be bounded on Ω, all E_t for $t \in U$ are equal as vector spaces, however, the inner products on them depend on t if $\varphi(t, z)$ is not constant with t. So, under the natural projection, $E = \coprod_{t\in U} E_t$ is a trivial holomorphic vector bundle (of infinite rank) over U with varying Hermitian metric.

In [3] , Berndtsson proved that E is semipositive in the sense of Griffths, namely, for any local holomorphic section ξ of the dual bundle E^* of E, the function $\log|\xi|$ is p.s.h. (indeed Berndtsson proved a stronger result which says that E is semipositive in the sense of Nakano). The aim here is to provide a new proof of the positivity of E, based on our new charaterization of p.s.h. functions (Theorem 1.1).

Theorem 5.1. *The vector bundle E is semipositive in the sense of Griffths.*

Before giving the proof of Theorem 5.1, we first recall the notion of *Hilbert tensor product* of Hilbert spaces and prove a related lemma. Let V and W be two Hilbert spaces. For $v \in V, w \in W$, the norm of $v \otimes w$ is defined to be $\|v\|\|w\|$. If $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ are orthonormal bases of V and W respectively, then the Hilbert tensor product $V\hat{\otimes}W$ of V and W is defined to be the Hilbert space with $\{v_i \otimes w_j\}_{i\in I, j\in J}$ as an orthonormal basis. It is easy to show that the definition of $V\hat{\otimes}W$ is independent of the choices of the orthonormal bases of V and W. By definition one can check that $(V\hat{\otimes}W)^* = V^*\hat{\otimes}W^*$.

The definition can be naturally generalized to the tensor product of several Hibert spaces. In particular, we can define the tensor powers $V^{\hat{\otimes}k} := V\hat{\otimes}\cdots\hat{\otimes}V$ $((k \geq 1))$ of a Hilbert space V. Let V be a Hilbert space. For $v \in V$, it is obvious that the norm of $v^{\otimes k} := v \otimes \cdots \otimes v \in V^{\hat{\otimes}k}$ is $\|v\|^k$ for all $k \geq 1$.

Lemma 5.2. *Let D_1, D_2 be bounded domains in $\mathbb{C}^n_z$ and $\mathbb{C}^m_w$ respectively. Let φ_1 and φ_2 be p.s.h. functions on D_1 and D_2. Then*

$$H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)}) = H^2(D_1, e^{-\varphi_1})\hat{\otimes}H^2(D_2, e^{-\varphi_2}).$$

Proof. Let $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ be orthonormal bases of $H^2(D_1, e^{-\varphi_1})$ and $H^2(D_2, e^{-\varphi_2})$ respectively. Then $K_1(z) = \sum_i |f_i(z)|^2$ is the Bergman kernel of $H^2(D_1, e^{-\varphi_1})$ and $K_2(w) = \sum_j |g_j(w)|^2$ is the Bergman kernel of $H^2(D_2, e^{-\varphi_2})$. Let $K(z, w) = \sum_{i,j} |f_i(z)g_j(w)|^2$. It is clear that $K(z, w) = K_1(z)K_2(w)$. By Fubini theorem, $\{f_i(z)g_j(w)\}_{i,j=1}^\infty$ is an orthonormal set of $H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)})$. By the product property of the Bergman kernel, the Bergman kernel of $H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)})$ equals to $K_1(z)K_2(w)$. So $\{f_i(z)g_j(w)\}_{i,j=1}^\infty$ is an orthonormal basis of $H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)})$ and hence

$$H^2(D_1 \times D_2, e^{-(\varphi_1+\varphi_2)}) = H^2(D_1, e^{-\varphi_1})\hat{\otimes}H^2(D_2, e^{-\varphi_2}).$$

□

It is clear that Lemma 5.2 can be generalized to product of several domains. We now give the proof of Theorem 5.1.

Proof. Let u be a local holomorphic section of the dual bundle E^* of E. We need to prove that $\log|u(t)|$ is a p.s.h. function. Without loss of generality, we assume that u is a global holomorphic section, namely a holomorphic section of E^* on U. The upper semi-continuity of $\log|u(t)|$ follows from Proposition 3.6. We now prove that $\log|u(t)|$ satisfies the condition in Theorem 1.1 for some $p > 0$.

For $m \geq 1$, let $\Omega_m = U \times D^m$ and $\varphi_m(t, z_1, \cdots, z_m) = \varphi(t, z_1) + \cdots + \varphi(t, z_m)$. For $t \in U$, we denote $t \times D^m$ by D^m_t. Let $E_t^{\hat{\otimes}m} = H^2(D^m_t, e^{-\varphi_m})$, and $E^{\hat{\otimes}m} = \coprod_{t\in U} E_t^{\hat{\otimes}m}$. By Lemma 5.2, $E^{\hat{\otimes}m}$ is the m-th tensor power of E in the Hilbert space sense.

Let $t_0 \in U$ be an arbitrary point such that $u(t_0) \neq 0$. By the definition of tensor powers of Hilbert spaces given as above, $u^{\otimes m}$ is a nonvanishing holomorphic section of $(E^{\hat{\otimes} m})^* = (E^*)^{\hat{\otimes} m}$, and $|u^{\otimes m}(t)| = |u(t)|^m$. Let $f \in E_{t_0}^{\hat{\otimes} m}$ such that

$$\int_{D_{t_0}^m} |f|^2 e^{-\varphi_m(t_0, z_1, \cdots z_m)} = 1$$

and $\langle u^{\otimes m}(t_0), f\rangle = |u(t_0)|^m$.

By Theorem 3.1, there eixsts $F \in \mathcal{O}(\Omega_m)$ such that $F|_{D_{t_0}^m} = f$ and

$$\int_{\Omega_m} |F(t, z_1, \cdots, z_m)|^2 e^{-\varphi_m(t, z_1, \cdots z_m)} \leq C, \tag{2}$$

where C is a constant independent of t_0 and m. Let $F_t(z_1, \cdots, z_m) = F(t, z_1, \cdots, z_m)$ and

$$||F_t||_t^2 = \int_{D_t^m} |F_t|^2 e^{-\varphi_m(t, z_1, \cdots, z_m)}.$$

Since φ is bounded, by the mean value inequality, $\|F_t\|_t < +\infty$. This implies F_t lies in $E_t^{\hat{\otimes} m}$ for all $t \in U$ and hence F can be seen as a holomorphic section of $E^{\hat{\otimes} m}$.

From the definition of $|u^{\otimes m}(t)|$, it is clear that

$$\|F_t\|_t |u(t)|^m \geq |\langle u^{\otimes m}(t), F_t\rangle|,$$

and hence

$$e^{-m \log |u(t)|} \leq e^{-\log |\langle u^{\otimes m}(t), F_t\rangle|} ||F_t||_t.$$

Note that $\langle u^{\otimes m}(t), F_t\rangle$ is a holomorphic function on U. By Theorem 3.1, there is a holomorphic function h on U such that $h(t_0) = 1$ and

$$\int_U |h(t)|^2 e^{-2\log |\langle u^{\otimes m}(t), F_t\rangle|} \leq C' e^{-2\log |\langle u^{\otimes m}(t_0), F_{t_0}\rangle|} = C' e^{-2m \log |u(t_0)|}, \tag{3}$$

where C' is a constant independent of m and t_0. So we have the estimate

$$\begin{aligned} &\int_U |h(t)| e^{-m \log |u(t)|} \\ \leq &\int_U |h(t)| e^{-\log |\langle u^{\otimes m}(t), F_t\rangle|} ||F_t||_t \\ \leq &\left(\int_U |h(t)|^2 |e^{-2\log |\langle u^{\otimes m}(t), F_t\rangle|} \int_U ||F_t||_t^2 \right)^{1/2} \\ \leq &\sqrt{CC'} e^{-m \log |u(t_0)|}, \end{aligned} \tag{4}$$

where the last inequality follows from (2), (3) and Fubini theorem. By Theorem 1.1, $\log |u(t)|$ is subharmonic. □

5.2. **For families of compact Kähler manifolds.** In this subsection, we study the positivity of the direct image sheaf of the twisted relative canonical bundle associated to a family of compact Kähler manifolds.

Let X, Y be Kähler manifolds of dimension $r + n$ and r respectively, and let $p : X \to Y$ be a proper holomorphic map. For $y \in Y$ let $X_y = p^{-1}(y)$, which is a compact submanifold of X of dimension n if y is a regular value of p. Let L be a holomorphic line bundle over X, and h be a singular

Hermitian metric on L, whose curvature current is semi-positive. Let $K_{X/Y}$ be the relative canonical bundle on X.

Let $\mathcal{E} = p_*(K_{X/Y} \otimes L \otimes \mathcal{I}(h))$, and $\tilde{\mathcal{E}} = p_*(K_{X/Y} \otimes L)$ be the direct image sheaves on Y, where $\mathcal{I}(h)$ is the multiplier ideal sheaf associated to (L, h). We can choose a proper analytic subset $A \subset Y$ such that:

(1) p is submersive over $Y \backslash A$,
(2) both $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are locally free on $Y \backslash A$,
(3) for $y \in Y \backslash A$, the fibers E_y and $\tilde{E}_y$ are naturally identified with $H^0(X_y, K_{X_y} \otimes L|_{X_y} \otimes \mathcal{I}(h)|_{X_y})$ and $H^0(X_y, K_{X_y} \otimes L|_{X_y})$ respectively,

where E and $\tilde{E}$ are the vector bundles on $Y \backslash A$ associated to $\mathcal{E}$ and $\tilde{\mathcal{E}}$ respectively. For $u \in \tilde{E}_y$, the norm of u is defined to be

$$H(u) := \|u\| = \left(\int_{X_y} |u|_h^2 \right)^{1/2} \leq +\infty.$$

Then H is a Finsler metric on $\tilde{E}$, whose restriction on E gives a singular Hermitian metric on E, which will be also denoted by H. The following theorem says that H is positively curved as a singular Finsler metric on the coherent sheaf $\mathcal{E}$ (see Definition 2.4 for definition).

Theorem 5.3. *With the above assumptions and notations, H is a positively curved singular metric on $\mathcal{E}$.*

Proof. The proof splits into three steps.

Step 1. We prove that H is a positively curved singular Finsler metric on $\tilde{E} \to U := Y \backslash A$. The argument is similar to that in the proof of Theorem 5.1. Let u be a local holomorphic section of $\tilde{E}^*$. By definition, we need to show that $\log |u|$ is a p.s.h. function. Without loss of generality, in this step we can assume that $U = \mathbb{B}^r$ is the unit ball and u is a holomorphic section of $\tilde{E}^*$ on U.

For $m \geq 1$, let $X_m = \{(y, z_1, \cdots z_m); y \in U, z_1, \cdots, z_m \in X_y\}$ be the m-th fiber-product power of X. The is a natural proper holomorphic submersion from X_m to U, which is denoted by $p_m : X_m \to U$. Let $X_y^m = p_m^{-1}(y)$ be the fiber over $y \in U$.

For $1 \leq i \leq m$, we have a projection $\pi_i : X_m \to X$ which sends $(y, z_1, \cdots, z_m)$ to (y, z_i). Let $L_m = \pi_1^* L \otimes \cdots \otimes \pi_m^* L$ and let h_m be the singular Hermitian metric on L_m induced from the metric h on L. Then the curvature current of h_m is nonnegative.

Note that $H^0(X_y^m, K_{X_y^m} \otimes L_m|_{X_y^m}) = H^0(X_y, K_{X_y} \otimes L_{X_y})^{\otimes m}$ for $y \in U$. Indeed, this follows the same proof of Lemma 5.2 by putting a smooth Hermitian metric on L. In particular, the dimension of $H^0(X_y^m, K_{X_y^m} \otimes L_m|_{X_y^m})$ is independent of $y \in U$. Since U is assumed to be the unit ball, we can identify $K_{p_m^{-1}(U)/U}$ with $K_{p_m^{-1}(U)}$. So $p_{m*}(K_{X_m/U} \otimes L_m)|_U$ is locally free and corresponds to a holomorphic vector bundle, say $\tilde{E}^m$, on U, and we have $\tilde{E}^m = \tilde{E}^{\otimes m}$. In the same way as defining H, we can define a Finsler metric, say H^m, on $\tilde{E}^m$. For $y \in U$, let $\tilde{E}_{y,b}$ and $\tilde{E}_{y,b}^m$ be subspaces of

$\tilde{E}_y$ and $\tilde{E}_y^m$ consisting of vectors of finite norm. By Lemma 5.2, we have $\tilde{E}_{y,b}^m = (\tilde{E}_{y,b})^{\otimes m}$.

Recall that u is a holomorphic section of $\tilde{E}^*$ on U, and we need to prove that $\log|u|$ is a p.s.h. function on U. Note that the restriction $u|_E$ of u on E is a holomorphic section of E^*. The point is that, by definition of the dual norm in Definition 2.2, the norm of $u|_E$ and u are equal. Therefore, by Proposition 3.3, $\log|u|$ is upper semicontinuous. Now it suffices to prove that $\log|u|$ satisfies the condition in Theorem 1.1.

Let $y_0 \in U$ be any given point such that $|u(y_0)| \neq 0$. $u^{\otimes m}$ is a holomorphic section of $(\tilde{E}^*)^{\otimes m} = \tilde{E}^{m*}$. Note that the definition of the norm of $u^{\otimes m}$ only involves vectors in $\tilde{E}^m$ of finite norm, by Lemma 5.2, we have $|u^{\otimes m}(y)| = |u(y)|^m$. There exists $f_{y_0} \in \tilde{E}^m_{y_0}$ such that $\|f_{y_0}\| := H^m(f_{y_0}) = 1$ and $|\langle u^{\otimes m}(y_0), f_{y_0}\rangle| = |u(y_0)|^m$. By Theorem 3.1, there is $F \in H^0(p_m^{-1}(U), (K_{X_m} \otimes L_m)|_{p_m^{-1}(U)})$ such that $F|_{X^m_{y_0}} = f_{y_0}$ and

$$\int_{X_m} |F(y, z_1, \cdots, z_m)|^2 e^{-\varphi_m(z,z_1,\cdots,z_m)} \leq C,$$

where φ_m is the weight of h_m and C is an absolute constant independent of y_0 and m. For $y \in U$, let $F_y(z_1, \cdots, z_m) = F(y, z_1, \cdots, z_m)$, then $F_y \in \tilde{E}^m_y$ and

$$\|F_y\|^2 = \int_{X^m_y} |F_y|^2 e^{-\varphi_m(y,z_1,\cdots,z_m)}.$$

From the definition of $|u^{\otimes m}(y)|$, it is clear that

$$\|F_y\||u(y)|^m \geq |\langle u^{\otimes m}(y), F_y\rangle|,$$

and hence

$$e^{-m\log|u(y)|} \leq e^{-\log|\langle u^{\otimes m}(y),F_y\rangle|}||F_y||.$$

Note that F can be seen as a holomorphic section of $\tilde{E}^m$ on U, so $< u^{\otimes m}(y), F_y >$ is a holomorphic function on U. By Ohsawa-Takegoshi L^2 extension theorem (Theorem 3.1), there is a holomorphic function h on U such that $h(y_0) = 1$ and

$$\int_U |h(y)|^2 e^{-2\log|\langle u^{\otimes m}(y),F_y\rangle|} \leq C' e^{-2m\log|u(y_0)|},$$

where C' is an absolute constant independent of m and y_0. So we have the estimate

$$\begin{aligned}
&\int_U |h(y)| e^{-m\log|u(y)|} \\
\leq &\int_U |h(y)| e^{-\log|\langle u^{\otimes m}(y),F_y\rangle|}||F_y|| \\
\leq &\left(\int_U |h(y)|^2 |e^{-2\log|\langle u^{\otimes m}(y),F_y\rangle|} \int_U ||F_y||^2\right)^{1/2} \\
\leq &\sqrt{CC'} e^{-m\log|u(y_0)|}.
\end{aligned} \tag{5}$$

By Theorem 1.1, $\log|u(t)|$ is plurisubharmonic.

Step 2. We prove that H is a positively curved singular Finsler metric on $E \to U := Y \backslash A$. We also assume that $U = \mathbb{B}^r$ be the unit ball. Note that E is a holomorphic subbundle of $\tilde{E}$. Since U is a Stein manifold, there is a holomorphic subbundle E' of $\tilde{E}$ such that $\tilde{E}$ splits as $E \oplus E'$.

So any holomorphic section u of E^* on U can be extended to a holomorphic section $\tilde{u}$ of $\tilde{E}^*$ by setting $u(a) = 0$ for all $a \in E'$. Note that the norm of any vector in $\tilde{E} \backslash E$ is $+\infty$ (by Theorem 3.1), by definition, the norm of u and $\tilde{u}$ are equal. By the result in Step 1, $\log |\tilde{u}|$ is plurisubharmonic, so $\log |u|$ is plurisubharmonic.

Step 3. We will complete the proof of Theorem 5.3 in this final step. Let u be a holomorphic section of the dual sheaf $\mathcal{E}^*$ of $\mathcal{E}$ on some open set V in Y. We want to show that $|u|_{V \backslash A}$ is bounded above on all compact subsets of V. Once this is established, $\log |u|$ can be extended uniquely to a p.s.h. function on V and we are done.

The boundeddness of $\log |u|$ follows from the idea in the proof of Proposition 23.3 in [21]. Since the proof is quite standard, we omit it here.

□

Remark 5.1. Theorem 5.3 was proved by Berndtsson in [3] in the case that the metric on L is smooth and p is a submersion, and is similar to results obtained in [4] and [26]. Our method to Theorem 5.1 and Theorem 5.3 is different from the previous ones. As showed in the proofs, our arguments are based on Theorem 1.1 and a fiber-product technique.

Remark 5.2. The same method to Theorem 5.1 and Theorem 5.3 can be used to prove the plurisubharmonic variation of k-Bergman kernel metrics for all integers $k \geq 1$, and the positivity of NS metrics (see [4] for definitions, and [2, 3, 18, 4, 26, 21, 30] for related results).

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