Quadratic Weyl group multiple Dirichlet series of Type $D_4^{(1)}$

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In this paper and its sequel [27], we investigate the precise relationship between the quadratic affine Weyl group multiple Dirichlet series in the sense of [19, 13], and those defined axiomatically by Whitehead [49] and [48]. In particular, we show that the axiomatic quadratic Weyl group multiple Dirichlet series of type $D_4^{(1)}$ over rational function fields of odd characteristic admits meromorphic continuation to the interior of the corresponding complexified Tits cone. We shall also determine the polar divisor of this function, and compute the residue at each of its poles. As a consequence, we obtain an *exact* formula for a weighted 4-th moment of quadratic Dirichlet L-functions over rational function fields; we shall also derive an asymptotic formula for this weighted moment that is expected to generalize to any global field.

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1. Introduction

In this paper and its sequel [27], we investigate the precise relationship between the quadratic affine Weyl group multiple Dirichlet series in the sense of [19, 13], and those defined axiomatically by Whitehead [49] and [48]. In particular, we show that the axiomatic quadratic Weyl group multiple Dirichlet series of type $D_4^{(1)}$ over rational function fields of odd characteristic admits meromorphic continuation to the interior of the corresponding complexified Tits cone. We shall also determine the polar divisor of this function, and compute the residue at each of its poles. As a consequence, we obtain an *exact* formula for a weighted 4-th moment of quadratic Dirichlet L-functions over rational function fields; we shall also derive an asymptotic formula for this weighted moment that is expected to generalize to any global field.

Before discussing our results in more detail, let us first recall some basic facts about Weyl group multiple Dirichlet series associated to classical root systems.

A *Weyl group multiple Dirichlet series,* WMDS for short, is a Dirichlet series in several complex variables attached to a root system satisfying a group of functional equations isomorphic to the Weyl group of the root system; the number of variables is precisely the rank of the root system. These objects were initially introduced on a case-by-case basis in the 1980's, when the idea emerged that it could be useful to tie together a family of related L-functions in one variable (e.g., the family of quadratic Dirichlet L-functions) to create a *double Dirichlet series,* which could be used to study the average behavior of the original family of L-functions; see [35, 14, 33, 16, 29, 22]. Motivated by the problem of understanding the asymptotics of moments of L-functions, double Dirichlet series soon became multiple Dirichlet series (see [15] and [24]), and since approximately 2003, the idea of how to construct WMDS attached to classical root systems took shape; see [9, 10, 11, 12, 18, 19, 20]. It became clear that a *quadratic* WMDS over a global field, for example, associated to a (finite) reduced irreducible root system Φ_0 of rank r has the form

$$\sum_{n_1,\ldots,n_r} \frac{H(n_1,\ldots,n_r;m_1,\ldots,m_r)\Psi(n_1,\ldots,n_r)}{\prod |n_i|^{s_i}}$$

the sum being over r-tuples of representatives of non-zero S-integers modulo S-units in the number field setting (or r-tuples of effective divisors prime to S in the function field setting), for S a large enough finite set of places; see, e.g., [20, 29]. The r-tuple $(m_1,\ldots,m_r)$ of non-zero S-integers (resp. effective divisors prime to S) is a twisting parameter, and $|\cdot|$ denotes the corresponding norm of an element. The function H is the most important part, giving the structure of the series, and the function Ψ is just a technical device, varying over a certain finite-dimensional vector space of complex-valued functions, that makes the product $H\Psi$ well-defined.

The coefficients H satisfy a twisted multiplicativity that reduces their specification to the determination of the p-parts

$$\sum_{k_1,\ldots,k_r\geqslant 0} H\big(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r}\big)\,p^{-k_1s_1-\cdots-k_rs_r}$$

for odd primes p and tuples $(l_1,\ldots,l_r)\in\mathbb{N}^r$; see [18] and [20]. The p-parts are constructed so that the resulting WMDS satisfies a group of functional equations isomorphic to the Weyl group of Φ_0; the meromorphic continuation of this series to $\mathbb{C}^r$ is then essentially automatic. There are several equivalent methods of representing the correct p-parts, namely,

- Definition by the "averaging method", also known as the Chinta-Gunnells method [19, 18, 20].
- Definition as spherical p-adic Whittaker functions [11, 12].
- Definition as sums over crystal bases [11, 42].
- Definition as partition functions of statistical-mechanical lattice models [7, 8].

The Chinta-Gunnells method has been extended to the particular affine root system $D_4^{(1)}$ in [13], and to any root system associated with a symmetrizable Kac-Moody algebra in [38]. Moreover, a Casselman-Shalika type formula for Whittaker functions on metaplectic covers of Kac-Moody groups over non-archimedean local fields has been recently established by Patnaik and Puskás [45]. Despite these advances, we are quite far from a satisfactory theory of multiple Dirichlet series in the general setting of Kac-Moody Lie algebras and their Weyl groups. In a way or another, the main difficulties occurring in the infinite-dimensional case are caused by the existence of imaginary roots. To be more precise, let us consider, for example, the axiomatic multiple Dirichlet series associated to the fourth moment of quadratic Dirichlet L-functions over rational function fields $\mathbb{F}_q(x)$ of odd characteristics, see [25, 47, 49]; for simplicity, we shall assume throughout that $q\equiv 1 \pmod 4$. For $\Re(s_i)>1$, $i=1,\ldots,5$, this series can be expressed as

$$\sum_{\substack{d\text{ monic}\\ d=d_0d_1^2\\ d_0\text{ monic \& square-free}}} \frac{\prod_{i=1}^4 L\big(s_i+\frac12,\chi_{d_0}\big)\cdot P_d(\mathbf{s}';\chi_{d_0})}{|d|^{s_5}} \tag{1}$$

where $P_d(\mathbf{s}';\chi_{d_0})$, $\mathbf{s}'=(s_1,\ldots,s_4)$, are certain correction polynomials. Substitute $x_i=q^{-s_i-\frac12}$ ($i=1,\ldots,5$), and denote the resulting function by $Z(\mathbf{x};q)$, where $\mathbf{x}=(x_1,\ldots,x_5)$. Then the function $Z(q^{-1/2}\mathbf{x};q)$ and the Chinta-Gunnells average $Z_W^{\mathrm{CG}}(\mathbf{x};\sqrt{q})$ for $D_4^{(1)}$ satisfy the same group of functional equations. Consequently

$$Z(q^{-1/2}\mathbf{x};q)=F(\mathbf{x}^\delta)Z_W^{\mathrm{CG}}(\mathbf{x};\sqrt{q})$$

for some function F of one complex variable. In other words, the two functions differ by a power series whose coefficients (apart from the constant term which is 1) are supported on the positive affine imaginary roots $n\delta$, with $n\geqslant 1$. The appearance of such correction factors is a common feature when dealing with extensions of classical formulas to various infinite-dimensional settings, and pinning down these factors is usually quite a daunting task; see Macdonald's analogue [40] of Weyl's denominator identity for affine root systems, Kac's generalization [37] of the same identity to symmetrizable Kac-Moody algebras, the affine Gindikin-Karpelevich formula [4, 5], the affine Macdonald formula [41, 17, 6], and Whittaker functions on p-adic loop groups [44, 45].

One of our main results is the following (Theorem 8.1):

Theorem 1.1. — *The factor $F(z)$ is given by*

$$F(z)=\prod_{n\geqslant 1}\left(1-qz^{2n-1}\right)^{-2}.$$

In particular, the function $Z(q^{-1/2}\mathbf{x};q)$ has meromorphic continuation to $\Omega=\{\mathbf{x}\in\mathbb{C}^5:|\mathbf{x}^{\delta}|<1\}$, and in this domain it satisfies a group of functional equations.

Remark 1.1. The function $Z_W^{\mathrm{CG}}(\mathbf{x};\sqrt{q})$ cannot be meromorphically continued beyond the region Ω, and the fact that $Z(q^{-1/2}\mathbf{x};q)$ satisfies a group of functional equations was already known by the work of Whitehead [49]. The coefficients of the power series expansion of $Z_W^{\mathrm{CG}}(\mathbf{x};\sqrt{q})$ are polynomials in $u=\sqrt{q}$, and so $Z(u^{-1}\mathbf{x};u^2)$ can be considered as a function of the additional complex variable u. In particular, the function $Z(q^{1/2}\mathbf{x};q^{-1})$, obtained for $u=q^{-1/2}$, gives – after substituting $q\to|p|$ – the p-part of the global WMDS.

We have the following supplement to Theorem 1.1:

Theorem 1.2. — *Let Φ_{re}^{+} denote the set of positive affine real roots of a root system of type $D_4^{(1)}$. Then the singularities of $Z(q^{-1/2}\mathbf{x};q)$ occur along the hypersurfaces $\mathbf{x}^{2\alpha}=q^{-1}$, for $\alpha\in\Phi_{\mathrm{re}}^{+}$. In particular, this function has a simple pole at $x_5=q^{-1/2}$ with residue*

$$\operatorname*{Res}_{x_5\to q^{-\frac{1}{2}}} Z(q^{-1/2}\mathbf{x};q)=-\frac{q^{-\frac{1}{2}}}{\left(P^2;P^2\right)_\infty\left(qP^2;P^2\right)_\infty\prod_{i=1}^{4}\left(x_i^2;P^2\right)_\infty\left(qx_i^{-2}P^2;P^2\right)_\infty\cdot\ \prod_{1\leqslant i<j\leqslant 4}\left(x_ix_j;P\right)_\infty}$$

where $P=(x_1x_2x_3x_4)/q$, and $(a;b)_\infty=\prod_{k\geqslant 0}\left(1-ab^k\right)$ is the b-Pochhammer symbol.

Remark 1.2. The residue at any other pole of this function can be obtained from the residue at $x_5=q^{-1/2}$ by applying a specific functional equation.

As a first application, we obtain the precise relationship between the function $Z(q^{-1/2}\mathbf{x};q)$ and the Casselman-Shalika formula [45] for unramified Whittaker functions on metaplectic covers of Kac-Moody groups over non-archimedean local fields. With the notations and terminology of *loc. cit.*, the authors prove that, for each dominant coweight $\lambda^\vee$, the value $\mathscr{W}(\varpi^{\lambda^\vee})$ of the metaplectic Whittaker function on the toral element $\varpi^{\lambda^\vee}$, where ϖ is a chosen uniformizer, is the p-adic specialization of the expression

$$v^{\langle\lambda^\vee,\rho\rangle}\widetilde{\mathfrak{m}}\widetilde{\Delta}\cdot\sum_{w\in W}(-1)^{\ell(w)}\Big(\prod_{\tilde{a}^\vee\in\widetilde{\Phi}^\vee(w)}e^{-\tilde{a}^\vee}\Big)w\star e^{\lambda^\vee}. \tag{2}$$

Here $\widetilde{\Phi}^\vee$ is the dual root system constructed in [45] using a metaplectic structure (Q,n) on a root datum. The factor $\widetilde{\Delta}$ is defined by the formal infinite product

$$\widetilde{\Delta}:=\prod_{\tilde{a}\in\widetilde{\Phi}^+}\left(\frac{1-ve^{-\tilde{a}^\vee}}{1-e^{-\tilde{a}^\vee}}\right)^{m(\tilde{a})}$$

where $m(\tilde{a})$ is the root-multiplicity of $\tilde{a}$. When the root datum is associated with an untwisted affine root system Φ of ADE type, the correction factor $\widetilde{\mathfrak{m}}$ is

$$\widetilde{\mathfrak{m}}:=\mathrm{ct}(\widetilde{\Delta}^{-1})=\prod_{i=1}^{r}\prod_{j=1}^{\infty}\frac{1-v^{\widetilde{m}_i}e^{-j\tilde{\delta}}}{1-v^{\widetilde{m}_i+1}e^{-j\tilde{\delta}}}$$

where $\{\widetilde{m}_i\}_{i=1,\dots,r}$ is the set of *exponents* of the underlying *finite* root system to $\widetilde{\Phi}$, and $\tilde{\delta}$ is the minimal positive imaginary root of $\widetilde{\Phi}$. For example, if Φ is of $D_4^{(1)}$ type, then $\widetilde{\Phi}$ is again of $D_4^{(1)}$ type (see [45, Table 2.3.2]), and if $\{\alpha_i\}_{i=1,\dots,5} \subset \Phi_{\mathrm{re}}^+$ is the set of simple roots, $x_i := e^{-\alpha_i}$, then

$$\widetilde{\mathfrak{m}}(\mathbf{x}; v) = \prod_{i=1}^{4} \prod_{j=1}^{\infty} \frac{1 - v^{\widetilde{m}_i} \mathbf{x}^{2j\delta}}{1 - v^{\widetilde{m}_i + 1} \mathbf{x}^{2j\delta}}, \text{ the exponents being } 1, 3, 3, 5.$$

If we specialize (2) to the case when $\widetilde{G}$ is the metaplectic *double* cover of a simply connected affine Kac-Moody group G of type $D_4^{(1)}$ over a non-archimedean local field, we get:

Theorem 1.3. — *If we define*

$$\tilde{\mathfrak{c}}(\mathbf{x}; v) := \prod_{n=1}^{\infty} \left(1 - v\mathbf{x}^{n\delta}\right)^2 \left(1 - v\mathbf{x}^{2n\delta}\right)^2 \quad \textit{and} \quad D(\mathbf{x}; v) := \prod_{\alpha \in \Phi_{\mathrm{re}}^+} \left(1 - v\mathbf{x}^{2\alpha}\right)$$

then the value of the unramified Whittaker function on the group $\widetilde{G}$ at the identity element is

$$\mathscr{W}(1) = (\widetilde{\mathfrak{m}} \tilde{\mathfrak{c}} D)(\mathbf{x}; q^{-1}) Z(q^{1/2}\mathbf{x}; q^{-1})$$

where $q \equiv 1 \pmod 4$ is the size of the residue field.

Remark 1.3. A similar comparison result for metaplectic Whittaker functions on simply laced affine Kac-Moody groups will appear in [27]. In the finite-dimensional case, the equivalence between the Chinta-Gunnells method and the representation of the p-parts as spherical p-adic Whittaker functions was established by McNamara [43].

Our main reason for singling out the study of WMDS of type $D_4^{(1)}$ concerns the fourth moment of quadratic Dirichlet L-functions. Traditionally the moment problem for this family of L-functions is asking for an asymptotic formula for $\sum_{\deg d = D} L\left(\frac{1}{2}, \chi_d\right)^r$ $(r \geqslant 1)$ as $D \to \infty$, the sum being over monic square-free polynomials $d \in \mathbb{F}_q[x]$. A conjectural asymptotic formula, for all r, was first given by Andrade and Keating [2] (see also [46]), and a refined version, exhibiting additional lower order terms in the asymptotic formula, was recently proposed in [28]. This conjecture is known only for $r \leqslant 3$, see [30, 31, 23], and when $r = 4$, a weaker form of the asymptotic formula is known by the work of Florea [32].

As an immediate consequence of Theorem 1.1, we have:

Theorem 1.4. — *Assuming $q \equiv 1 \pmod 4$ and $D \geqslant 1$, we have the exact formula:*

$$\sum_{\substack{\deg d = D \\ d \text{ monic}}} L\left(\tfrac{1}{2}, \chi_{d_0}\right)^4 P_d(\chi_{d_0}) = \mathrm{Coeff}_{\xi^D} \left[\prod_{n \geqslant 1} \left(1 - q\xi^{4n-2}\right)^{-2} \cdot Z_W^{\mathrm{CG}}(\underline{1}, \xi; \sqrt{q}) \right]$$

where $P_d(\chi_{d_0}) = P_d(0, \dots, 0; \chi_{d_0})$ and $\underline{1} := (1,1,1,1)$. Here, for a monic polynomial $d \in \mathbb{F}_q[x]$, we write $d = d_0 d_1^2$ with d_0 monic and square-free.

While this result is probably special to rational function fields, the following asymptotic formula is expected to generalize to any global field.

Theorem 1.5. — *For $D, N \geqslant 1$ and $(N+1)^{-1} < \Theta < N^{-1}$, we have*

$$\sum_{\substack{\deg d = D \\ d \text{ monic}}} L\left(\tfrac{1}{2}, \chi_{d_0}\right)^4 P_d(\chi_{d_0}) = \sum_{n \leqslant N} Q_n(D, q) q^{\frac{D}{2n}} + O_{\Theta, q}\left(q^{\frac{D\Theta}{2}}\right) \tag{3}$$

where $Q_n(D, q)$ is a polynomial in D of degree 10 if n is odd and of degree 7 if n is even. Furthermore, if we let $\varrho = q^{-1/n} \in \mathbb{R}$, then the leading coefficient of $Q_n(D, q)$ is given by an expression of the form

$$(-1)^{\lfloor n/2 \rfloor} \varrho^{3\lfloor n/2 \rfloor \cdot \lfloor (n+1)/2 \rfloor} g_{n,D}(\sqrt{\varrho})$$

where $g_{n,D} \neq 0$ is a power series in $\varrho^{1/2}$ with non-negative coefficients.

According to [28], a similar asymptotic formula should hold for $\sum_{\deg d = D} L\left(\frac{1}{2}, \chi_d\right)^4$, summed only over square-free monics; it should be a consequence of the analytic properties of $Z(q^{-1/2}\mathbf{x}; q)$ (i.e., Theorem 1.1 and Theorem 1.2 above) and its twists, which shall be addressed in a future work. However, from the analytic point of view, the asymptotic formula (3) should be sufficient. This is so since it has the correct order of magnitude, and the coefficients $P_d(\chi_{d_0})$ are, as we shall see, non-negative; when $d = d_0$ is square-free, we have $P_d(\chi_{d_0}) = |d|^{-1/2}$, which also explains the discrepancy in (3) by a factor of $q^{D/2}$ (cf. [28, Conjecture 1.2]).

Remark 1.4. The study of the analytic properties of twisted versions of $Z(q^{-1/2}\mathbf{x}; q)$ mentioned in the previous paragraph is also relevant to the problem of extending our results to higher genus function fields. The main difficulty in dealing with both these general situations comes from the failure of the so-called *local-to-global* property whose complexity grows as the degrees of the twisting parameters increase. Recently, Friedlander [34] obtained an interesting explicit formula that can be interpreted as measuring the *degree of failure* of the local-to-global property of a (finite-dimensional) twisted Weyl group multiple Dirichlet series over rational function fields. We expect that an analogous formula will hold for affine Weyl group multiple Dirichlet series.

Using the p-parts $Z(p^{1/2}\mathbf{x}; p^{-1})$, with p a rational odd prime, one constructs (cf. [28, Remark 1]) the analogue of (1) over the rationals. This series has the form

$$Z(s_1, \ldots, s_5; \chi_{a_2}, \chi_{a_1}) = \sum_{\substack{d = d_0 d_1^2 \geqslant 1 \\ (d,2)=1}} \frac{\prod_{i=1}^4 L\left(s_i + \frac{1}{2}, \chi_{a_1 d_0}\right) \cdot \chi_{a_2}(d_0) P_d(\mathbf{s}'; \chi_{a_1 d_0})}{d^{s_5}} \tag{4}$$

where $a_1, a_2 \in \{\pm 1, \pm 2\}$ and χ_d, for $d \in \mathbb{Z}$ non-zero and square-free, is the usual quadratic character; as in the function field case, this expression is certainly valid when $\Re(s_i) > 1$ ($i = 1, \ldots, 5$). It has some initial continuation, and in that region it satisfies a group W of functional equations isomorphic to a Weyl group of type $D_4^{(1)}$. If $a_2 = 1$, this function has a simple pole at $s_5 = \frac{1}{2}$, and its residue can now be explicitly computed as the *infinite* product of zeta functions

$$\prod_{p \neq 2} R\left(p^{-\frac{1}{2} - s_1}, p^{-\frac{1}{2} - s_2}, p^{-\frac{1}{2} - s_3}, p^{-\frac{1}{2} - s_4}; p^{-\frac{1}{2}}\right)$$

with $R(\underline{x}; u)$ ($\underline{x} := (x_1, \ldots, x_4)$) given in Theorem 6.1; the residues at the other simple poles (under the action of W) can be computed using the functional equations.

Conjecturally $Z(\mathbf{s}; \chi_{a_2}, \chi_{a_1})$ ($\mathbf{s} := (s_1, \ldots, s_5)$) admits meromorphic continuation to $\Re(\delta(\mathbf{s})) > 0$, with all its singularities contained in the set $\left\{w\left(s_5 = \frac{1}{2}\right)\right\}_{w \in W}$.

Let us say some words about the proofs of our main results; the proofs of Theorems 1.3-1.5 are (essentially) straightforward applications of Theorems 1.1 and 1.2. The key result in establishing these two theorems is a *new* functional equation satisfied by the Chinta-Gunnells average for affine root systems. This functional equation has no finite-dimensional analogue. To state this in the $D_4^{(1)}$ case, consider the average $Z_W(\mathbf{x}; u) = \sum_{w \in W} 1|w(\mathbf{x}; u)$, where $f|w$ is the Chinta-Gunnells action. We will show in Proposition 3.1 that, as a function of the complex variables $x_1, \ldots, x_5, u$, $Z_W(\mathbf{x}; u)$ is holomorphic for $|\mathbf{x}^\delta| < 1, u \in \mathbb{C}$ except for the set of points for which $u\mathbf{x}^\alpha \pm 1 = 0$ for some $\alpha \in \Phi_{\mathrm{re}}^+$. For $a, b \in \{\mathscr{e}, \mathscr{o}\}$, where $\mathscr{e}$ (resp. $\mathscr{o}$) stands for even (resp. odd), let $Z_W^{a,b}$ denote the part of Z_W which has parity a with respect to the involution $\varepsilon_5(\underline{x}, x_5) = (-\underline{x}, x_5)$ and parity b with respect to the involution $\varepsilon_1(\underline{x}, x_5) = (\underline{x}, -x_5)$. If we let $\mathbf{Z} = {}^t(Z_W^{\mathscr{e},\mathscr{e}}, Z_W^{\mathscr{e},\mathscr{o}}, Z_W^{\mathscr{o},\mathscr{e}})$, then this vector function satisfies the functional equation

$$\mathbf{Z}(\mathbf{x}; u) = B(\mathbf{x}; u)\mathbf{Z}\big(\mathbf{x}; u\mathbf{x}^\delta\big) \tag{5}$$

for a 3 by 3 matrix $B(\mathbf{x}; u)$ with *rational* entries. In fact, the matrix

$$\prod_{\substack{\alpha \in \Phi_{\mathrm{re}}^+ \\ \alpha < \delta}} \big(1 - u^2\mathbf{x}^{2\alpha}\big) \cdot B(\mathbf{x}; u)$$

has polynomial entries in $\mathbf{x}$ and u, and each entry of $B(\mathbf{x}; u)$ is divisible by $\big(1 - u^2\mathbf{x}^\delta\big)^2$, see Theorem 4.3. (An explicit formula for $B(\mathbf{x}; u)$ is obtained via the identity $B(\mathbf{x}; u) = A^{-1}(\mathbf{x}; u\mathbf{x}^\delta)$, with A given in [1].)

The strategy to deduce Theorem 1.1 and Theorem 1.2 from the functional equation (5) goes as follows. The divisibility of the matrix $B(\mathbf{x}; u)$ by $\big(1 - u^2\mathbf{x}^\delta\big)^2$ implies that the renormalization $\tilde{Z}_W$ of the Chinta-Gunnells average, defined by (26), and the function Z_W have the same singularities in Ω. We then compute the residue of the function $\tilde{Z}_W(\mathbf{x}; u)$ at $x_5 = u^{-1}$ by using (5) and the invariance of the average $Z_W(\mathbf{x}; u)$ under the Weyl group, which reduce the calculation to a simple application of the classical Macdonald's identity [40]. The structure of this residue (given by the formula in Theorem 1.2, with q replaced by u^2) allows us to construct a WMDS of type (1), with $\tilde{Z}_W(\mathbf{x}; |p|^{-1/2})$ as the p-parts, which after substituting $x_i = q^{-s_i}$, turns out to have the same residue as $\tilde{Z}_W(\mathbf{x}; \sqrt{q})$ at the simple pole $x_5 = 1/\sqrt{q}$. Thus, by letting $\mathscr{Z}(\mathbf{x})$ denote this WMDS in the variables x_i, we deduce that $\mathscr{Z}(\mathbf{x}) = \tilde{Z}_W(\mathbf{x}; \sqrt{q})$. Finally, we note that the matrix $B(\mathbf{x}; u)/\big(1 - u^2\mathbf{x}^\delta\big)^2$ determines $\tilde{Z}_W(\mathbf{x}; u)$ recursively, see Lemma 5.3; this implies that the coefficients of $\tilde{Z}_W(u\mathbf{x}; u)$ satisfy the dominance axiom [25, 49], hence $\tilde{Z}_W(\mathbf{x}; \sqrt{q}) = Z(q^{-1/2}\mathbf{x}; q)$, and that the coefficients $P_d(\chi_{d_0})$ in Theorem 1.4 are non-negative.

Concluding remark. By the analogy between number fields and function fields of curves over finite fields, it is conceivable that the WMDS (4) satisfies some analogue of the functional equation (32). It is also quite probable that the mechanism behind this additional symmetry will provide a natural approach to prove the meromorphic continuation of (4) to the half-space $\Re(\delta(\mathbf{s})) > 0$.

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2. Notation

We denote by $\mathbb{N}$ the set of non-negative integers, by $\mathbb{Z}$ the ring of (rational) integers, and by $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ the fields of rational numbers, real numbers and complex numbers, respectively.

By $X \ll Y$ or $X = O(Y)$ we denote an inequality of the form $|X| \leqslant CY$, for some constant C. We refer to C as the ***implied*** constant, and its dependence on parameters that we wish to keep track of (e.g., ϵ, q) will be indicated by appropriate subscripts (e.g., $X \ll_{\epsilon,q} Y$ or $X = O_{\epsilon,q}(Y)$).

3. Preliminaries

We begin by recalling some basic facts about affine root systems and their Weyl groups, following closely the standard reference [39].

Let V be a finite-dimensional real vector space, equipped with a positive definite symmetric scalar product $\langle x, y\rangle$; we shall identify V with its dual space V^* via the scalar product. Consider the vector space $\mathscr{F}$ of affine-linear functions on V (i.e., the functions of the form $f = f_0 + c\delta$, where $f_0(x) = \langle Df, x\rangle$ is a linear functional on V, δ is the constant function 1, and $c \in \mathbb{R}$). We define the positive semidefinite scalar product on $\mathscr{F}$ by

$$\langle f, g\rangle = \langle Df, Dg\rangle.$$

The radical of this form is the one-dimensional subspace $\mathbb{R}\delta$ of constant functions. For non-zero $x \in V$, let $x^\vee = 2x/\langle x, x\rangle$, and for each non-constant $f \in \mathscr{F}$, let $f^\vee = 2f/\langle f, f\rangle$.

Let $f \in \mathscr{F}$ be non-constant. The orthogonal reflection σ_f in the affine hyperplane on which f vanishes is given by

$$\sigma_f(x) = x - f^\vee(x)Df = x - f(x)Df^\vee.$$

The reflection σ_f acts on $\mathscr{F}$ by $\sigma_f(g) = g \circ \sigma_f$. For notational convenience, we shall write from now on $\sigma_f g$ instead of $\sigma_f(g)$.

For each $v \in V$, the translation $x \to x + v$ will be denoted by $\tau(v)$.

3.1. Affine root systems

Let Φ_0 be a rank r reduced irreducible root system, spanning a real vector space V, and let $W_0 = W(\Phi_0)$ denote the Weyl group of Φ_0. The dual root system of Φ_0 with respect to a W_0-invariant positive definite scalar product $\langle x, y\rangle$ on V is $\Phi_0^\vee = \{\alpha^\vee : \alpha \in \Phi_0\}$. Regarding each $\alpha \in \Phi_0$ as a linear function on V (i.e., $\alpha(x) = \langle \alpha, x\rangle$ for $x \in V$), one gets the ***affine root system***

$$\Phi = \{\alpha + n\delta : \alpha \in \Phi_0, n \in \mathbb{Z}\} \cup \{m\delta\}_{m \in \mathbb{Z} \smallsetminus \{0\}}$$

associated with Φ_0. The elements of the subset $\Phi_{\mathrm{re}} = \{\alpha + n\delta : \alpha \in \Phi_0, n \in \mathbb{Z}\}$ are called ***affine real roots,*** and the elements $m\delta$ ($m \in \mathbb{Z} \smallsetminus \{0\}$) are called ***affine imaginary roots.***

For each $\beta \in \Phi_{\mathrm{re}}$, let σ_β denote the orthogonal reflection in the affine hyperplane on which β vanishes. Explicitly, if $\beta = \alpha + n\delta$ then

$$H_\beta = \beta^{-1}(0) = \{x \in V : \langle \alpha, x\rangle = -n\} \text{ and } \sigma_\beta(x) = x - (\langle \alpha, x\rangle + n)\alpha^\vee.$$

The reflection σ_β acts on Φ_{re} by

$$\sigma_\beta \beta' = \beta' \circ \sigma_\beta = \beta' - \langle \alpha', \alpha^\vee \rangle \beta, \quad \beta' = \alpha' + n'\delta \in \Phi_{\mathrm{re}}.$$

The ***affine Weyl group*** W of Φ is defined to be the group of affine isometries of V generated by all reflections σ_β, $\beta \in \Phi_{\mathrm{re}}$. Note that, for each $\alpha \in \Phi_0$, the composition $\tau(\alpha^\vee) := \sigma_\alpha \circ \sigma_{\alpha+\delta}$ sends an element $x \in V$ to $x + \alpha^\vee$, that is, $\tau(\alpha^\vee)$ is translation by $\alpha^\vee$. Thus W contains a subgroup of translations $\tau(Q^\vee)$ isomorphic to the root lattice $Q^\vee$ of $\Phi_0^\vee$, and we have $W = W_0 \ltimes \tau(Q^\vee)$.

Let $\alpha_1, \ldots, \alpha_r$ be a set of simple roots of Φ_0, and let Φ_0^+ (resp. Φ_0^-) be the set of positive (resp. negative) roots determined by $\alpha_1, \ldots, \alpha_r$. If we let $\theta \in \Phi_0^+$ denote the highest root, then the affine roots α_i ($1 \leqslant i \leqslant r$) together with $\alpha_0 = -\theta + \delta$ form a set of simple roots for Φ.

An affine real root β is positive (resp. negative) relative to the open r-simplex

$$C = \{x \in V : \alpha_i(x) > 0 \ (1 \leqslant i \leqslant r) \text{ and } (-\theta + \delta)(x) > 0\}$$

if $\beta(x) > 0$ (resp. $\beta(x) < 0$) for all $x \in C$. If we let Φ_{re}^+ (resp. Φ_{re}^-) denote the set of positive (resp. negative) affine real roots, then $\Phi_{\mathrm{re}}^- = -\Phi_{\mathrm{re}}^+$, and $\Phi_{\mathrm{re}} = \Phi_{\mathrm{re}}^+ \cup \Phi_{\mathrm{re}}^-$. The set of positive affine real roots is

$$\Phi_{\mathrm{re}}^+ = \{\alpha + (n + \chi(\alpha))\delta : \alpha \in \Phi_0, n \in \mathbb{N}\}$$

where χ is the characteristic function of Φ_0^-. An affine imaginary root $m\delta$ is positive or negative according as $m > 0$ or $m < 0$. If α and β are distinct affine roots then $\alpha < \beta$ will signify that $\beta - \alpha \in \sum_i \mathbb{N}\alpha_i$.

The affine Weyl group W is also a Coxeter group on the generators $\sigma_i = \sigma_{\alpha_i}$ ($1 \leqslant i \leqslant r$), and $\sigma_0 = \sigma_{-\theta+\delta}$, subject to the relations $\sigma_i^2 = 1$ for all $0 \leqslant i \leqslant r$, and $(\sigma_i\sigma_j)^{m_{ij}} = 1$ if $i \neq j$ and $m_{ij} < \infty$.

Note that

$$w\delta = \delta \circ w^{-1} = \delta \text{ for all } w \in W. \tag{6}$$

3.2. Extended Affine Weyl Group

With notations as before, let $P^\vee$ denote the *coweight* lattice of Φ_0, that is,

$$P^\vee = \{\lambda \in V : \langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi_0\}.$$

The ***extended affine Weyl*** group is defined to be $\widetilde{W} := W_0 \ltimes \tau(P^\vee)$, where $\tau(P^\vee)$ is the group of translations by elements of $P^\vee$. The extended affine Weyl group contains W as a normal subgroup, and the quotient $\widetilde{W}/W \cong P^\vee/Q^\vee$ is a finite (abelian) group.

The extended affine Weyl group acts on the set of affine roots. To see this, take an element w of $\widetilde{W}$, and express it as $w = w_0\tau(\lambda)$, with $w_0 \in W_0$ and $\lambda \in P^\vee$. If $\beta = \alpha + n\delta \in \Phi$, then

$$(w\beta)(x) = \beta(w^{-1}x) = \langle w_0\alpha, x \rangle + n - \langle \lambda, \alpha \rangle \quad (\text{for } x \in V).$$

Thus $w\beta = w_0\beta - \langle \lambda, \alpha \rangle \delta \in \Phi$ because $\langle \lambda, \alpha \rangle \in \mathbb{Z}$. It follows that $\widetilde{W}$ permutes the affine real roots and fixes the imaginary roots.

For an element $w \in W$, we shall denote its *length* (with respect to the generators $\sigma_i, 0 \leqslant i \leqslant r$) by $\ell(w)$. If we let $\Phi(w) = \{\beta \in \Phi^+ : w\beta \in \Phi^-\}$, then $\ell(w) = |\Phi(w)|$. The length function is extended to $\widetilde{W}$ by the same formula. If $w_0 \in W_0$ and $\lambda \in P^\vee$, then

$$\ell(w_0\tau(\lambda)) = \sum_{\alpha\in\Phi_0^+} |\langle\lambda,\alpha\rangle + \chi(w_0\alpha)| \tag{7}$$

where, as before, χ is the characteristic function of Φ_0^-.

Let $O = \{w \in \widetilde{W} : \ell(w) = 0\}$. The elements of O permute the simple affine roots, and we have

$$\widetilde{W} = W \rtimes O$$

so $O \cong \widetilde{W}/W \cong P^\vee/Q^\vee$. In particular, O is a finite abelian group.

3.3. The Chinta-Gunnells action

From now on we will assume that the affine root system Φ is simply laced. Let $\alpha_0, \dots, \alpha_r$ be the set of affine simple roots, where $\alpha_0 = -\theta + \delta$. If two α_i and α_j are connected in the Dynkin diagram of Φ, we shall write $i \sim j$. Define an action of W on monomials $\mathbf{x}^\beta := \prod_{i=0}^r x_i^{k_i}$, for $\beta = \sum_{i=0}^r k_i\alpha_i$ in the root lattice of Φ, by $w\mathbf{x}^\beta = \mathbf{x}^{w^{-1}\beta}$, which corresponds to the contragredient action on roots. We also have an action of W on variables $\mathbf{x} = (x_0, \dots, x_r)$ by $(w\mathbf{x})_j = \mathbf{x}^{w^{-1}\alpha_j}$, that is (using that Φ is simply laced):

$$(\sigma_i\mathbf{x})_j = \begin{cases} 1/x_j & \text{if } j = i \\ x_i x_j & \text{if } j \sim i \\ x_j & \text{otherwise.} \end{cases}$$

Let $\varepsilon_i\mathbf{x}$ be the involution defined by

$$(\varepsilon_i\mathbf{x})_j = \begin{cases} -x_j & \text{if } j \sim i \\ x_j & \text{otherwise.} \end{cases}$$

Let $\mathbb{C}(\mathbf{x}, u)$ be the field of rational functions in $x_1, \dots, x_r, u$, for an additional variable u. With this notation, we define the action of a simple reflection σ_i on $f \in \mathbb{C}(\mathbf{x}, u)$ by

$$f|\sigma_i(\mathbf{x}; u) = f(\sigma_i\mathbf{x}; u)J(x_i, 0) + f(\varepsilon_i\sigma_i\mathbf{x}; u)J(x_i, 1),$$

where, for $\varepsilon \in \{0, 1\}$,

$$J(x, \varepsilon) = J(x, u, \varepsilon) = \frac{x}{2}\left(\frac{u - x}{1 - ux} - (-1)^\varepsilon\right). \tag{8}$$

One verifies as in [19, Lemma 3.2] that this action extends to a well-defined W-action on $\mathbb{C}(\mathbf{x}, u)$.

To construct the analogue of the Chinta-Gunnells (average) function in our context, let $\Delta(\mathbf{x})$ be defined by

$$\Delta(\mathbf{x}) = \prod_{n\geqslant 1}\left(1 - \mathbf{x}^{2n\delta}\right)^r \cdot \prod_{\beta\in\Phi_{\mathrm{re}}^+}\left(1 - \mathbf{x}^{2\beta}\right). \tag{9}$$

The product (9) is absolutely convergent in the region $|\mathbf{x}^\delta| < 1$ of $\mathbb{C}^{r+1}$, and by (6), it satisfies the transformation formulas

$$\Delta(\mathbf{x}) = -x_i^2\Delta(\sigma_i\mathbf{x}) \qquad (\text{for } i = 0, \dots, r). \tag{10}$$

The Chinta-Gunnells function can now be defined by $Z_W^{\mathrm{CG}}(\mathbf{x};u) = Z_W(\mathbf{x};u)/\Delta(\mathbf{x})$, where

$$Z_W(\mathbf{x};u) := \sum_{w\in W} 1|w(\mathbf{x};u). \tag{11}$$

For affine root systems, however, there is a highly non-trivial correction of $Z_W^{\mathrm{CG}}(\mathbf{x};u)$ (corresponding to the affine imaginary roots) that one should take into account. This issue will be addressed in Section 5 when the root system is $D_4^{(1)}$, but for now let us concentrate on the function $Z_W(\mathbf{x};u)$ that we just defined.

The following proposition (cf. [49, Proposition 3.1.2]) gives the largest possible region of convergence for the series (11), as a function of the complex variables $x_0,\ldots,x_r,u$.

Proposition 3.1. — *For $x_0,\ldots,x_r,u\in\mathbb{C}$ such that $|\mathbf{x}^\delta|<1$, the series defining $Z_W(\mathbf{x};u)$ converges absolutely and uniformly on every compact subset away from the points for which $u\mathbf{x}^\beta\pm1=0$ for some $\beta\in\Phi_{\mathrm{re}}^+$. In addition, the function $Z_W(\mathbf{x};u)$ is $W-$invariant under the Chinta-Gunnells action.*

Proof. We shall follow closely [49]. Let w be a fixed Weyl group element, and write $w=\sigma_{i_\ell}\cdots\sigma_{i_1}$ in reduced form. It is well-known that the set $\Phi(w)=\{\beta\in\Phi^+ : w\beta\in\Phi^-\}$ is explicitly given by the ℓ distinct positive roots

$$\Phi(w)=\{\beta_1=\alpha_{i_1},\ \beta_2=\sigma_{i_1}\alpha_{i_2},\ \beta_3=\sigma_{i_1}\sigma_{i_2}\alpha_{i_3},\ldots,\ \beta_\ell=\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_{\ell-1}}\alpha_{i_\ell}\}.$$

Using the fact that $f|w_1|w_2=f|w_1w_2$, one can verify by induction on ℓ the formula:

$$f|w(\mathbf{x};u) \;=\; \sum_{\delta_1,\ldots,\delta_\ell\in\{0,1\}} f\Big(w\varepsilon^{\sum_{i=1}^{\ell}\delta_i\beta_i}\mathbf{x};u\Big)\prod_{k=1}^{\ell} J\Big((-1)^{\langle\beta_k,\,\sum_{i<k}\delta_i\beta_i\rangle}\mathbf{x}^{\beta_k},\delta_k\Big) \tag{12}$$

where $\varepsilon^\alpha:=\prod_i\varepsilon_i^{k_i}$, for $\alpha=\sum_i k_i\alpha_i$, and J is defined by (8); we are interpreting the factor corresponding to $k=1$ to be $J(\mathbf{x}^{\beta_1},\delta_1)$.

To estimate $1|w(\mathbf{x};u)$, express each root $\beta\in\Phi(w)$ as $\beta=\alpha+(n+\chi(\alpha))\delta$ with $\alpha\in\Phi_0$, $n\in\mathbb{N}$, and χ the characteristic function of Φ_0^-. By assuming that these roots are as small as possible, one finds that

$$\sum_{\alpha+(n+\chi(\alpha))\delta\in\Phi(w)} n+\chi(\alpha)\;\geqslant\;\frac{|\Phi_0|\left\lfloor\frac{\ell}{|\Phi_0|}\right\rfloor\left(\left\lfloor\frac{\ell}{|\Phi_0|}\right\rfloor-1\right)}{2}$$

where $\lfloor x\rfloor$ denotes the integer part of x, and $|\Phi_0|$ is the cardinality of Φ_0. One can also see that, for a positive affine real root $\beta=\alpha+(n+\chi(\alpha))\delta$,

$$\left|J\big(\mathbf{x}^\beta,\varepsilon\big)\right|\leqslant|\mathbf{x}^{(n+\chi(\alpha))\delta}|\,K(\mathbf{x},u)/2$$

where we set

$$K(\mathbf{x},u):=(1+|u|)\max_{\alpha\in\Phi_0}\{|\mathbf{x}^\alpha|\,(1+|\mathbf{x}^\alpha|)\}\cdot\sup_{\beta\in\Phi_{\mathrm{re}}^+}\left|1-u\mathbf{x}^\beta\right|^{-1};$$

the supremum is finite since $|\mathbf{x}^{m\delta}|\to0$ as $m\to\infty$, and $u\mathbf{x}^\beta\neq\pm1$ for all $\beta\in\Phi_{\mathrm{re}}^+$. It follows that

$$|1|w(\mathbf{x};u)|\leqslant|\mathbf{x}^\delta|^{\frac{\ell^2}{2|\Phi_0|}-O(\ell)}K(\mathbf{x},u)^\ell \tag{13}$$

and thus

$$\sum_{w\in W}|1|w(\mathbf{x};u)| \leqslant \sum_{\ell\geqslant 0}\sum_{\substack{w\in W\\ \ell(w)=\ell}} |\mathbf{x}^\delta|^{\frac{\ell^2}{2|\Phi_0|}-O(\ell)}K(\mathbf{x},u)^\ell$$
$$< \sum_{\ell\geqslant 0}(r+1)^\ell|\mathbf{x}^\delta|^{\frac{\ell^2}{2|\Phi_0|}-O(\ell)}K(\mathbf{x},u)^\ell$$

where, for the last inequality, we applied the trivial bound $\#\{w\in W : \ell(w)=\ell\}\leqslant r^{\ell-1}(r+1)<(r+1)^\ell$. The last series converges since $|\mathbf{x}^\delta|<1$ and $u\mathbf{x}^\beta\neq\pm1$ for all $\beta\in\Phi^+_{\mathrm{re}}$. This implies our first assertion.

Finally, for $x_0,\dots,x_r,u$ in the region of absolute convergence, we have

$$Z_W|w'(\mathbf{x};u) = \sum_{w\in W}1|ww'(\mathbf{x};u) = Z_W(\mathbf{x};u)$$

which completes the proof. □

Remark 3.1. The singularities $u\mathbf{x}^\beta\pm1=0$ ($\beta\in\Phi^+_{\mathrm{re}}$) of the function $Z_W(\mathbf{x};u)$ are at most simple poles, and the residues at these poles of a closely related function will be evaluated in [27] by generalizing the method introduced in the present paper to arbitrary simply laced affine root systems.

We conclude this subsection by extending the Chinta-Gunnells action to the group $\widetilde{W}$, which is done as follows. An element $w\in\widetilde{W}$ acts on the multivariable $\mathbf{x}$ by $(w\mathbf{x})_j=\mathbf{x}^{w^{-1}\alpha_j}$, and for $\eta\in O$ and a function $f(\mathbf{x};u)$, we define

$$f|\eta(\mathbf{x};u)=f(\eta\mathbf{x};u).$$

This defines an action of $\widetilde{W}=W\rtimes O$, and to check that it is well-defined, it suffices to check its compatibility with the commutation relations between the elements of O and the simple reflections σ_i. Since W is a normal subgroup of $\widetilde{W}$, it follows at once from [39, (2.2.5) and (2.2.6)] that, for $\eta\in O$, we have: $\eta\alpha_i=\alpha_j$ if and only if $\eta\sigma_i=\sigma_j\eta$. Moreover, one checks that

$$\varepsilon^\alpha w = w\varepsilon^{w^{-1}\alpha} \text{ for all } \alpha \text{ in the root lattice and } w\in\widetilde{W}$$

with ε_i the sign action defined at the beginning of this subsection. It is easy to check that $f|\eta|\sigma_i=f|\sigma_j|\eta$, for $\eta\in O$ and i,j such that $\eta\alpha_i=\alpha_j$, so the action is, indeed, well-defined.

Remark 3.2. One could also define the function $Z_{\widetilde{W}}=\sum_{w\in\widetilde{W}}1|w$, but it follows from the definition of the extended action that $Z_{\widetilde{W}}=n\cdot Z_W$, with n the cardinality of O. For this reason, the extension of the Chinta-Gunnells action to $\widetilde{W}$ will in fact not be needed. However, with no additional effort required, some of the relations used in the proofs of our results will be stated (for completeness) in the extended Weyl group.

3.4. Chinta-Gunnells averages of polynomials

We will also need the convergence and properties of the Chinta-Gunnells average

$$Z_{W,g} := \sum_{w\in W} g|w$$

for any Laurent polynomial $g(\mathbf{x})$ with coefficients in $\mathbb{C}(u)$. Note that $Z_{W,1}=Z_W$.

Proposition 3.2. — *For any Laurent polynomial $g(\mathbf{x})$, there is an integer $N = N_g \geqslant 0$ such that the series defining $\mathbf{x}^{N\delta} Z_{W,g}(\mathbf{x})$ converges absolutely and uniformly on compacta in the same region as $Z_W(\mathbf{x})$, and it is W-invariant under the Chinta-Gunnells action.*

Proof. By linearity, it is enough to consider the case $g(\mathbf{x}) = \mathbf{x}^\alpha$ for $\alpha = a_0\alpha_0 + \cdots + a_r\alpha_r$ an element in the root lattice of Φ. Using (12) and (13), it also suffices to show that $|\mathbf{x}^{w^{-1}\alpha}| \ll |\mathbf{x}^\delta|^{-O(\ell)}$, for the length ℓ of w sufficiently large, say $\ell \geqslant \ell(w_0)$ for all $w_0 \in W_0$; here we are taking $\mathbf{x} = (x_0, \ldots, x_r)$ in a compact set with $x_i \neq 0$ for all i.

We decompose $w = w_0 t$, with $w_0 \in W_0$ and $t = \tau(\lambda)$ a translation with $\lambda \in P^\vee$. Letting $\mu_i \in P^\vee$ be the fundamental coweights ($i = 1, \ldots, r$), we can write $t^{-1} = \prod_{i=1}^r \tau(\mu_i)^{n_i}$ for some $n_i \in \mathbb{Z}$; we have

$$\tau(\mu_i)\alpha_i = \alpha_i - \delta, \quad \tau(\mu_i)\alpha_0 = \alpha_0 + m_i\delta \quad \text{and} \quad \tau(\mu_i)\alpha_j = \alpha_j \ (j \neq 0, i)$$

where $\theta = \sum_{i=1}^r m_i\alpha_i$ is the highest root in Φ_0. It follows from (7) that

$$\sum_{i=1}^r |n_i| \leqslant \ell(t) = \sum_{\alpha \in \Phi_0^+} |\langle \lambda, \alpha \rangle| \leqslant \ell(w_0) + \ell \leqslant 2\ell$$

where the lower bound of $\ell(t)$ was obtained by retaining only the terms corresponding to $\alpha = \alpha_i \in \Phi_0^+$.

On the other hand, if we write $w_0^{-1}\alpha = \sum_{j=0}^r k_j\alpha_j$, then $w^{-1}\alpha = t^{-1}w_0^{-1}\alpha = w_0^{-1}\alpha + n\delta$, with n given by

$$n = \langle \lambda, w_0^{-1}\alpha \rangle = \sum_{i=1}^r (k_0 m_i - k_i) n_i.$$

Thus

$$|n| \leqslant \sum_{i=1}^r |k_0 m_i - k_i||n_i| \leqslant C \sum_{i=1}^r |n_i| \leqslant 2C\ell$$

for a positive constant C depending only upon Φ_0 and α. It follows that

$$|\mathbf{x}^{w^{-1}\alpha}| \leqslant |\mathbf{x}^{w_0^{-1}\alpha}||\mathbf{x}^\delta|^{-|n|} \ll |\mathbf{x}^\delta|^{-2C\ell}$$

where the implied constant can be taken to be the maximum over $\mathbf{x}$ in the compact set and $w_0 \in W_0$ of $|\mathbf{x}^{w_0^{-1}\alpha}|$. Accordingly $|g|w|$ satisfies an estimate of type (13). It is also clear that there are only finitely many $w \in W$ such that $g|w$ has poles when $x_i = 0$, which completes the proof. □

For brevity, we state the following result only for the case $D_4^{(1)}$, the general case being considered in [27].

Proposition 3.3. — *Assume the root system Φ is affine of type $D_4^{(1)}$. Then for every monomial g, the function $Z_{W,g}$ satisfies*

$$Z_{W,g}(\mathbf{x}) = C_g(\mathbf{x}^\delta) Z_W(\mathbf{x})$$

where $C_g(x)$ is a Laurent polynomial with coefficients in $\mathbb{Z}[u]$, which can be determined recursively in terms of g.

Proof. We use the labelling of simple roots for $D_4^{(1)}$ given in the beginning of the next section, so that $\delta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5$. We abbreviate $Z_g = Z_{W,g}$ in this proof.

Let $g(\mathbf{x}) = \mathbf{x}^\alpha$ be a monomial with $\alpha = a_1\alpha_1 + \cdots + a_5\alpha_5$, with $a_i \in \mathbb{Z}$. Then $g(\mathbf{x})$ is even (resp. odd) for the involution ε_i if and only if $v_i(g) := \sum_{j\sim i} a_j$ is even (resp. odd). Note that in this case we have only two sign functions, $\varepsilon_1 = \varepsilon_i$ for $i = 1, \ldots, 4$, and ε_5.

We use the following properties of the Chinta-Gunnells action. If $g(\mathbf{x})$ is odd for the involution ε_i, then

$$g|\sigma_i = -x_i\sigma_i g$$

and if $g(\mathbf{x})$ is even for ε_i, then

$$[(x_i - u)g]|\sigma_i = (u - x_i)\sigma_i g.$$

By the previous lemma, we can act with $w \in W$ and sum these relations over W to obtain (replacing first g by g/x_i in the even case):

$$Z_g = \begin{cases} -Z_{x_i\sigma_i g} & \text{for } v_i(g) \text{ odd} \\ uZ_{x_i\sigma_i g} + uZ_{g/x_i} - Z_{x_i^2\sigma_i g} & \text{for } v_i(g) \text{ even.} \end{cases} \tag{14}$$

We now introduce an order on the monomials g. Let $\mathbf{d}(g)$ be the 8-tuple consisting of the differences:

$$\pm(a_1 - a_2),\ \pm(a_5 - a_1 - a_2),\ \pm(a_3 + a_4 - a_5),\ \pm(a_3 - a_4) \tag{15}$$

ordered decreasingly. We order the monomials g by the lexicographic order of the tuples $\mathbf{d}(g)$. Note that $g = \mathbf{x}^{n\delta}$ if and only if $\mathbf{d}(g) = (0, \ldots, 0)$, and that these are the smallest elements for the order just defined. Using the relations above, we show that Z_g with $g \neq \mathbf{x}^{n\delta}$ can be expressed in terms of $Z_{g'}$ with $\mathbf{d}(g') < \mathbf{d}(g)$, which shows recursively that Z_g can be expressed in terms of $Z_{\mathbf{x}^{n\delta}} = \mathbf{x}^{n\delta}Z_1$, with coefficients in $\mathbb{Z}[u]$.

The differences in (15) containing a_i are of type $\pm(a_i - b_i)$, $\pm(a_i - c_i)$, with $b_i + c_i = v_i(g)$. For each index i, we define two differences:

$$d_i^{(1)} = a_i - b_i \quad \text{and} \quad d_i^{(2)} = c_i - a_i$$

(fixing throughout a choice of b_i, c_i, e.g., $b_1 = a_2$, $c_1 = a_5 - a_2$, etc.). Since $\sigma_i g = g x_i^{v_i(g)-2a_i}$, and $b_i + c_i = v_i(g)$, the differences $\pm d_i^{(1)}$ and $\pm d_i^{(2)}$ are switched for $\sigma_i g$, and the other four differences are the same for g and $\sigma_i g$, so $\mathbf{d}(\sigma_i g) = \mathbf{d}(g)$.

If $g \neq \mathbf{x}^{n\delta}$, then there exists an index i with $d_i^{(1)} > d_i^{(2)}$; indeed, if $d_i^{(1)} \leqslant d_i^{(2)}$, that is $2a_i \leqslant v_i(g)$ for all i, then one must have equality for all i, and so $g = \mathbf{x}^{n\delta}$ for some n, a contradiction. We claim that the relations (14) for this choice of i, express Z_g in terms of $Z_{g'}$, with $\mathbf{d}(g') < \mathbf{d}(g)$, hence finishing the proof. We have $-d_i^{(2)} > -d_i^{(1)}$ as well, and the pairs $(d_i^{(1)}, d_i^{(2)})$, $(-d_i^{(2)}, -d_i^{(1)})$ for $x_i\sigma_i g$ (resp. $x_i^2\sigma_i g$) are

$$(d_i^{(2)} + 1, d_i^{(1)} - 1),\ (-d_i^{(1)} + 1, -d_i^{(2)} - 1) \quad (\text{resp. } (d_i^{(2)} + 2, d_i^{(1)} - 2),\ (-d_i^{(1)} + 2, -d_i^{(2)} - 2)).$$

It follows that $\mathbf{d}(g/x_i) = \mathbf{d}(x_i\sigma_i g) < \mathbf{d}(g)$, unless $d_i^{(1)} = d_i^{(2)} + 1$, in which case $g = x_i\sigma_i g$ and $Z_g = 0$. Similarly, if $v_i(g)$ is even, we have $\mathbf{d}(x_i^2\sigma_i g) < \mathbf{d}(g)$, unless $d_i^{(1)} = d_i^{(2)} + 2$, in which case $g = x_i^2\sigma_i g$ and $Z_g = uZ_{g/x_i}$ with $\mathbf{d}(g/x_i) < \mathbf{d}(g)$.

The proof clearly gives an algorithm for computing recursively the polynomial $C_g(x)$ in terms of g. $\square$

Remark 3.3. For $D_4 = \langle \sigma_2, \ldots, \sigma_5 \rangle$, one can define the same 8-tuple $\mathbf{d}(g)$ for a monomial g in $x_2, \ldots, x_5$, by setting $a_1 = 0$ in (15). The above proof then shows that for all monomials g, we have $Z_{W,g} = p(u) Z_{W,h}$, for a polynomial p and a monomial $h = \prod x_i^{a_i}$ such that $2a_i \leqslant v_i(h)$ for $i = 2, \ldots, 5$. However, there are infinitely many such h, for example $h = x_2^b x_3^a x_4^a x_5^{2a}$ for $2a \leqslant b \leqslant a$. Therefore the previous proposition fails for finite Weyl groups.

Remark 3.4. When specializing $u = -1$, the quotient of the averages appearing in Proposition 3.3, for an arbitrary simply laced affine root system Φ, becomes

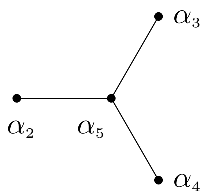
$$\frac{Z_{W,g}}{Z_W}(\mathbf{x}; -1) = \chi_\lambda(\mathbf{x}) := \frac{\sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{w(\lambda - \rho) + \rho}}{\sum_{w \in W} (-1)^{\ell(w)} \mathbf{x}^{\rho - w\rho}}$$

where $g = \mathbf{x}^\lambda$, and $\rho = \omega_0 + \cdots + \omega_r$, with ω_i the affine fundamental weights. If λ were an anti-dominant affine weight, then χ_λ would be the character of the infinite-dimensional representation of the associated affine Kac-Moody Lie algebra with lowest weight λ. However, in our situation λ is an element of the affine root lattice, and letting $w_0 \in W$ be such that $w_0(\lambda - \rho) = \mu$ is the unique anti-dominant affine weight in the W-orbit of $\lambda - \rho$, there are two possibilities. Either μ is singular, i.e., it is fixed by a simple reflection, and then $\chi_\lambda = 0$; or μ is regular, in which case one checks easily that it must be of the form $\mu = n\delta - \rho$ for some $n \in \mathbb{Z}$, and $\chi_\lambda(\mathbf{x}) = (-1)^{\ell(w_0)} \mathbf{x}^{n\delta}$. We conclude that the specialization at $u = -1$ of the function $C_g(x)$ in Proposition 3.3 is either 0 or $\pm x^n$, for some $n \in \mathbb{Z}$. For example, when Φ is of type $D_4^{(1)}$ and $g = x_1^2 x_2^2 x_3^2$, we have

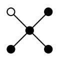
$$C_g(x) = -u^4 x^{-1} + (u^2 - 1)^3, \text{ and } C_g(x)|_{u=-1} = -x^{-1}.$$

4. An extra functional equation

From now on we take the affine root system Φ to be of type $D_4^{(1)}$, see [37, p. 54, TABLE Aff 1]. The Dynkin diagram of the finite root system $\Phi_0 = D_4$ is



Dynkin diagram of D_4

and the Dynkin diagram of $D_4^{(1)}$ is  with the additional simple root denoted by α_1; note the shift in the labeling of simple roots compared to the previous section. The set of positive roots Φ_0^+ is given explicitly by

$$\Phi_0^+ = \{\alpha_i,\ \alpha_5,\ \alpha_i + \alpha_5,\ \alpha_i + \alpha_j + \alpha_5,\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,\ \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5\}_{2 \leqslant i \neq j \leqslant 4};$$

thus the highest root is $\theta = \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5$, and $\delta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5$. The Weyl group of Φ is $W = \langle \sigma_i \rangle_{i=1,\ldots,5}$, where $\sigma_i = \sigma_{\alpha_i}$.

4.1. A cocycle associated with the Chinta-Gunnells action

For any function $f(\mathbf{x})$, and $a, b \in \{\mathscr{e}, \mathscr{o}\}$, where $\mathscr{e}$ (resp. $\mathscr{o}$) stands for even (resp. odd), let $f^{a,b}$ denote the part of f which has parity a with respect to the involution $\varepsilon_5(\underline{x}, x_5) = (-\underline{x}, x_5)$ and parity b with respect to the involution $\varepsilon_1(\underline{x}, x_5) = (\underline{x}, -x_5)$, where $\underline{x} = (x_1, \ldots, x_4)$. For the following lemma, it is convenient to write the Chinta-Gunnells action as

$$f|\sigma_i(\mathbf{x}) = x_i \frac{u - x_i}{1 - ux_i} f_i^+(\sigma_i \mathbf{x}) - x_i f_i^-(\sigma_i \mathbf{x}) \tag{16}$$

where $f_i^{\pm}(\mathbf{x}) = (f(\mathbf{x}) \pm f(\varepsilon_i \mathbf{x}))/2$ are the even and odd components of $f(\mathbf{x}) = f(\mathbf{x}; u)$ with respect to ε_i.

Lemma 4.1. — *The Chinta-Gunnells action of $w \in \widetilde{W}$ preserves the subspace*

$$\mathbb{C}(\mathbf{x}, u)_0 := \{f \in \mathbb{C}(\mathbf{x}, u) \mid f^{\mathscr{o},\mathscr{o}} = 0\}.$$

Proof. Assume $f^{\mathscr{o},\mathscr{o}} = 0$. Then for all i and $j \sim i$, we have

$$(f|\sigma_i)^{\mathscr{o},\mathscr{o}}(\mathbf{x}) = [[-x_i f_i^-(\sigma_i \mathbf{x})]_i^-]_j^- = -x_i f^{\mathscr{o},\mathscr{o}}(\sigma_i \mathbf{x}) = 0.$$

Similarly, $(f|\eta)^{\mathscr{o},\mathscr{o}} = 0$ for all $\eta \in O$. □

In particular, $(1|w)^{\mathscr{o},\mathscr{o}} = 0$ for all $w \in \widetilde{W}$, so $Z_W^{\mathscr{o},\mathscr{o}} \equiv 0$. We restrict henceforth the Chinta-Gunnells action to the invariant subspace $\mathbb{C}(\mathbf{x}, u)_0$.

Letting $\bar{f}$ be the column vector

$$\bar{f} := {}^t(f^{\mathscr{e},\mathscr{e}}, f^{\mathscr{e},\mathscr{o}}, f^{\mathscr{o},\mathscr{e}})$$

we define a 3×3 matrix $\Lambda_w(\mathbf{x})$ such that

$$\overline{f|w}(\mathbf{x}) = \Lambda_w(\mathbf{x}) \bar{f}(w\mathbf{x}) \qquad (\text{for } w \in \widetilde{W}) \tag{17}$$

with Λ_w satisfying the 1-cocycle relation $\Lambda_{ww'}(\mathbf{x}) = \Lambda_{w'}(\mathbf{x}) \Lambda_w(w'\mathbf{x})$ for all $w, w' \in \widetilde{W}$. On the generators σ_i, this cocycle is given by

$$\Lambda_{\sigma_i}(\mathbf{x}) = \Lambda_1(x_i, u) = -x_i \begin{pmatrix} \frac{(1-u^2)x_i}{1-u^2x_i^2} & 0 & -\frac{u(1-x_i^2)}{1-u^2x_i^2} \\ 0 & 1 & 0 \\ -\frac{u(1-x_i^2)}{1-u^2x_i^2} & 0 & \frac{(1-u^2)x_i}{1-u^2x_i^2} \end{pmatrix}$$

for $i = 1, \ldots, 4$, and

$$\Lambda_{\sigma_5}(\mathbf{x}) = \Lambda_2(x_5, u) = -x_5 \begin{pmatrix} \frac{(1-u^2)x_5}{1-u^2x_5^2} & -\frac{u(1-x_5^2)}{1-u^2x_5^2} & 0 \\ -\frac{u(1-x_5^2)}{1-u^2x_5^2} & \frac{(1-u^2)x_5}{1-u^2x_5^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The group $O \subset \widetilde{W}$ has order 4, and it is generated by elements η of order 2 with $\eta\alpha_i = \alpha_j$ and $\eta\alpha_k = \alpha_l$ for $\{i, j, k, l\}$ a permutation of $\{1, 2, 3, 4\}$. From (17) we have that Λ_η is the identity matrix for $\eta \in O$.

We denote by $\mathbf{Z}$ the vector $\overline{Z_W}$. The functional equation $Z_W = Z_W|w$ becomes:

$$\mathbf{Z}(\mathbf{x}; u) = \Lambda_w(\mathbf{x}) \mathbf{Z}(w\mathbf{x}; u) \tag{18}$$

for all $w \in \widetilde{W}$, which follows from the cocycle relation of Λ_w.

4.2. An extension of the Chinta-Gunnells action

Define the transformation τ acting on $f \in \mathbb{C}(\mathbf{x}, u)$ by

$$\tau f(\mathbf{x}; u) = f\big(\mathbf{x}; u/\mathbf{x}^{\delta}\big).$$

This transformation extends the natural action of $w \in \widetilde{W}$ given by $wf(\mathbf{x}; u) = f(w^{-1}\mathbf{x}; u)$, and we denote by $\widetilde{W} \oplus \mathbb{Z}$ the group generated by $\widetilde{W}$ and τ^a, $a \in \mathbb{Z}$, which can be seen as a subgroup of $\mathrm{End}(\mathbb{C}(\mathbf{x}, u))$. We show that there is an extension of the Chinta-Gunnells action to this larger group $\widetilde{W} \oplus \mathbb{Z}$, from which we will derive an extra functional equation for the vector function $\mathbf{Z}$. As in the previous subsection, we will restrict the action to the space $\mathbb{C}(\mathbf{x}, u)_0$ of functions f with $f^{o,o} = 0$.

Similarly to the way the Chinta-Gunnells action is defined in (16), we look for an action of the form

$$f|\tau(\mathbf{x}; u) = A_{e,e}(\mathbf{x}; u)\tau f^{e,e}(\mathbf{x}; u) + A_{e,o}(\mathbf{x}; u)\tau f^{e,o}(\mathbf{x}; u) + A_{o,e}(\mathbf{x}; u)\tau f^{o,e}(\mathbf{x}; u) \tag{19}$$

defined for $f \in \mathbb{C}(\mathbf{x}, u)_0$, for three unknown functions $A_{e,e}$, $A_{e,o}$, $A_{o,e} \in \mathbb{C}(\mathbf{x}, u)$. In order for this action to preserve the space $\mathbb{C}(\mathbf{x}, u)_0$, we require that

$$A_{e,e}^{o,o} = 0, \quad A_{e,o}^{o,e} = 0, \quad A_{o,e}^{e,o} = 0.$$

We also require this action to be compatible with the Chinta-Gunnells action, namely

$$f|\tau|w = f|w|\tau \qquad (\text{for } w \in \widetilde{W}). \tag{20}$$

Formula (19) can be written:

$$\overline{f|\tau}(\mathbf{x}; u) = A(\mathbf{x}; u)\bar{f}\big(\mathbf{x}; u/\mathbf{x}^{\delta}\big) \tag{21}$$

for the 3×3 matrix

$$A(\mathbf{x}; u) = \Big(\overline{A_{e,e}} \quad \tfrac{1}{x_5}\overline{x_5 A_{e,o}} \quad \tfrac{1}{x_1}\overline{x_1 A_{o,e}}\Big). \tag{22}$$

Defining $\Lambda_\tau := A$, condition (20) is then equivalent to the fact that the cocycle Λ has a well-defined extension to the monoid $\widetilde{W} \times \{\tau^n\}_{n \geqslant 0}$ by the 1-cocycle relation, namely it satisfies $\Lambda_{\tau w} = \Lambda_{w\tau}$ for all $w \in \widetilde{W}$, that is,

$$\Lambda_w(\mathbf{x}; u)A(w\mathbf{x}; u) = A(\mathbf{x}; u)\Lambda_w\big(\mathbf{x}; u/\mathbf{x}^{\delta}\big). \tag{23}$$

We also require that A is invertible, so that $\Lambda_{\tau^{-1}}$ is well-defined by the cocycle relation. Therefore $f|\tau^{-1}$ is well-defined as well, by the analogue of (21) written for τ^{-1}, with A replaced by $\Lambda_{\tau^{-1}}$.

The main theorem of this section shows that there exists an extension of the Chinta-Gunnells action to $\widetilde{W} \oplus \mathbb{Z}$ as above, and that $\mathbf{Z}$ satisfies an additional functional equation under the transformation τ.

Theorem 4.2. — *There exists an invertible matrix $A(\mathbf{x}; u)$ of the form (22) satisfying the following conditions:*

- *The cocycle relation (23) is satisfied.*
- *There exists an element $a \in \mathbb{C}\big(u, \mathbf{x}^{\delta}\big)$ such that the matrix $a \cdot A$ has polynomial entries in $\mathbf{x}$ and u.*
- *The vector function $\mathbf{Z}(\mathbf{x}; u)$ satisfies the functional equation*

$$\mathbf{Z}(\mathbf{x}; u) = A(\mathbf{x}; u)\mathbf{Z}\big(\mathbf{x}; u/\mathbf{x}^{\delta}\big). \tag{24}$$

In [27], we will use results in [26] to show that a similar result holds for arbitrary affine irreducible reduced root systems.

Remark 4.1. The matrix A in the theorem is essentially unique; more precisely, any invertible matrix A of the form (22) that satisfies the cocycle relation is unique, up to multiplication by elements in $\mathbb{C}(u, \mathbf{x}^\delta)$. We omit the proof of this fact since it is quite technical, and we will not need it in the sequel.

Proof. The relations (23), for w running through the generators σ_i of W, reduce to a linear system of equations over the field $\mathbb{Q}(u)$, with unknowns being the coefficients of the entries of A (assuming that these entries are ***polynomials*** of bounded degree in $\mathbf{x}$). Using MAGMA [3], we solved this sparse system assuming that the degree of each entry is at most 16 in the variables $\mathbf{x}$ (in which case there are about 100,000 unknowns and about four times as many equations). We found an explicit non-singular matrix $A_0(\mathbf{x}; u)$ of the form (22) with polynomial entries in $\mathbf{x}, u$ satisfying (23), and the interested reader can find it in [1]. Thus we have the corresponding extensions of the cocycle Λ, and of the Chinta-Gunnells action, to all of $\widetilde{W} \oplus \mathbb{Z}$. By (21) it follows that

$$A_0(\mathbf{x}; u)\mathbf{Z}(\mathbf{x}; u/\mathbf{x}^\delta) = A_0(\mathbf{x}; u) \cdot \sum_{w \in W} \overline{1|w}(\mathbf{x}; u/\mathbf{x}^\delta) = \sum_{w \in W} \overline{1|\tau w}(\mathbf{x}; u) = \overline{Z_{W,F}}(\mathbf{x}; u)$$

where $F = 1|\tau$, and $Z_{W,F} = \sum_{w \in W} F|w$. The function F is the sum of the entries in the first column of A_0, so it is a polynomial in $\mathbf{x}$. By Proposition 3.3, it follows that $Z_{W,F} = aZ_W$, with $a \in \mathbb{C}(u, \mathbf{x}^\delta)$, and thus the matrix $A = A_0/a$ satisfies all the conditions in the theorem. Given A_0, the constant a can be determined explicitly using the algorithm in the proof of Proposition 3.3. More explicitly, the leading term among all the entries of A_0 is $\mathbf{x}^{16\delta}$ (it occurs in the $(2, 2)$ entry), and the value of a is given by

$$a = u^8 \mathbf{x}^{5\delta}(\mathbf{x}^\delta - u^2)^3(\mathbf{x}^\delta - u^4)(\mathbf{x}^{3\delta} - u^4). \qquad \square$$

We shall also need the following information about $\Lambda_{\tau^{-1}}$.

Theorem 4.3. — *The vector function* $\mathbf{Z}(\mathbf{x}; u)$ *satisfies the functional equation*

$$\mathbf{Z}(\mathbf{x}; u) = B(\mathbf{x}; u)\mathbf{Z}(\mathbf{x}; u\mathbf{x}^\delta)$$

for a 3 by 3 matrix $B(\mathbf{x}; u)$ *satisfying, in addition, the following two conditions:*

- *The matrix*
$$\prod_{\substack{\alpha \in \Phi^+_{\mathrm{re}} \\ \alpha < \delta}} (1 - u^2\mathbf{x}^{2\alpha}) \cdot B(\mathbf{x}; u)$$
has polynomial entries in $\mathbf{x}$ *and* u.
- *Each entry of* $B(\mathbf{x}; u)$ *is divisible by* $(1 - u^2\mathbf{x}^\delta)^2$.

Proof. Take $B(\mathbf{x}; u) = \Lambda_{\tau^{-1}}(\mathbf{x}; u) = A^{-1}(\mathbf{x}; u\mathbf{x}^\delta)$, with A from Theorem 4.2. The functional equation follows at once from the previous theorem, and the other conditions follow from the explicit formula of A, see [1]. $\square$

We shall also need:

Lemma 4.4. — *The specialization of the vector function* $\mathbf{Z}$ *to* $u = 0$ *is given by* $\mathbf{Z}(\mathbf{x}; 0) = {}^t(\Delta(\mathbf{x}), 0, 0)$.

Proof. By induction on length, for $w \in W$, we have $1|w(\mathbf{x}; 0) = (-1)^{\ell(w)} \prod_{\beta \in \Phi(w)} \mathbf{x}^{2\beta}$, where we recall that $\Phi(w) = \Phi^+ \cap w^{-1}(\Phi^-)$. Thus our assertion is just Macdonald's identity [40] in type $D_4^{(1)}$. □

Note that, by combining the functional equations (18) and (24), we get

$$\mathbf{Z}(\mathbf{x}; u) = \Lambda_{w\tau^k}(\mathbf{x}; u)\mathbf{Z}\big(w\mathbf{x}; u/\mathbf{x}^{k\delta}\big) \tag{25}$$

for all $w \in \widetilde{W}$ and $k \in \mathbb{Z}$.

5. Renormalization

The correction of the Chinta-Gunnells average for the affine root system $D_4^{(1)}$ is given by

$$\begin{aligned}\tilde{Z}_W(\mathbf{x}; u) &= \prod_{n \geqslant 1} \Big(1 - u^2\mathbf{x}^{(2n-1)\delta}\Big)^{-2} \cdot Z_W^{\mathrm{CG}}(\mathbf{x}; u) \\ &= \frac{1}{\Delta(\mathbf{x}) \prod_{n \geqslant 1}\big(1 - u^2\mathbf{x}^{(2n-1)\delta}\big)^2} \cdot \sum_{w \in W} 1|w(\mathbf{x}; u).\end{aligned} \tag{26}$$

As we shall see in Section 8, this function is directly connected to a Weyl group multiple Dirichlet series associated with the 4-th moment of quadratic Dirichlet L-functions.

Letting $D(\mathbf{x}; u)$, the denominator of $\tilde{Z}_W(\mathbf{x}; u)$, be defined by

$$D(\mathbf{x}; u) := \prod_{\alpha \in \Phi_{\mathrm{re}}^+} \big(1 - u^2\mathbf{x}^{2\alpha}\big) \tag{27}$$

we can now show the following:

Theorem 5.1. — *The function* $D\tilde{Z}_W(\mathbf{x}; u)$ *is holomorphic in the region* $|\mathbf{x}^\delta| < 1$.

Proof. For $|\mathbf{x}^\delta| < 1$, the absolutely convergent product $\Delta^{\mathrm{im}}(\mathbf{x}) := \prod_{n \geqslant 1}\big(1 - \mathbf{x}^{2n\delta}\big)^4$ is non-vanishing, and the divisibility (in the obvious sense) of Z_W by $\Delta^{\mathrm{re}} = \Delta/\Delta^{\mathrm{im}}$ in this region has already been discussed in [13, Section 4].

On the other hand, by (12) and (8), the function $Z_W(\mathbf{x}; u)$ is a sum of rational functions whose denominators are products of *distinct* factors of the form $1 - u^2\mathbf{x}^{2\alpha}$ ($\alpha \in \Phi_{\mathrm{re}}^+$). It follows that DZ_W/Δ is holomorphic when $|\mathbf{x}^\delta| < 1$, and by Theorem 4.3, that $(D/\Delta)\mathbf{Z}(\mathbf{x}; u)$ is divisible by $\big(1 - u^2\mathbf{x}^\delta\big)^2$. Accordingly, the vector function $(D/\Delta)\mathbf{Z}\big(\mathbf{x}; u\mathbf{x}^\delta\big)$ is divisible by $\big(1 - u^2\mathbf{x}^{3\delta}\big)^2$, and by the functional equation

$$\Big(\frac{D}{\Delta}\mathbf{Z}\Big)(\mathbf{x}; u) = \prod_{\substack{\alpha \in \Phi_{\mathrm{re}}^+ \\ \alpha < \delta}} \big(1 - u^2\mathbf{x}^{2\alpha}\big) \cdot B(\mathbf{x}; u)\Big(\frac{D}{\Delta}\mathbf{Z}\Big)\big(\mathbf{x}; u\mathbf{x}^\delta\big)$$

so does $(D/\Delta)\mathbf{Z}(\mathbf{x}; u)$. Proceeding by induction on n, we see at once that $(D/\Delta)\mathbf{Z}(\mathbf{x}; u)$ is divisible by $\big(1 - u^2\mathbf{x}^{(2n-1)\delta}\big)^2$ for all $n \geqslant 1$. Thus DZ_W/Δ is also divisible by the product $\prod_{n \geqslant 1}\big(1 - u^2\mathbf{x}^{(2n-1)\delta}\big)^2$, which completes the proof. □

Since Z_W is W-invariant under the Chinta-Gunnells action, it follows from (10) and the W-invariance of the product over imaginary roots that $\tilde{Z}_W$ itself satisfies a functional equation with respect to each $w \in W$. More precisely, we have:

$$\tilde{Z}_W = \tilde{Z}_W \| w \tag{28}$$

where we write $\|$ for the action defined on generators by

$$f\|\sigma_i(\mathbf{x}) := -\frac{1}{x_i^2} f|\sigma_i(\mathbf{x}) = \frac{1}{x_i} f_i^-(\sigma_i\mathbf{x}) + \frac{1-u/x_i}{1-ux_i} f_i^+(\sigma_i\mathbf{x}).$$

It is clear that the subspace $\mathbb{C}(\mathbf{x},u)_0$ defined in Lemma 4.1 is invariant under this action. The functional equations satisfied by the vector $\tilde{\mathbf{Z}} := \overline{\tilde{Z}_W}$ are:

$$\tilde{\mathbf{Z}}(\mathbf{x};u) = \tilde{\Lambda}_w(\mathbf{x})\tilde{\mathbf{Z}}(w\mathbf{x};u) \tag{29}$$

where $\tilde{\Lambda}_w(\mathbf{x}) = \tilde{\Lambda}_w(\mathbf{x};u)$ is the 3 by 3 matrix-cocycle such that

$$\overline{f\|w}(\mathbf{x};u) = \tilde{\Lambda}_w(\mathbf{x})\bar{f}(w\mathbf{x};u) \tag{30}$$

for $w \in W$ and $f \in \mathbb{C}(\mathbf{x},u)_0$. On generators, $\tilde{\Lambda}_w(\mathbf{x})$ is given by $\tilde{\Lambda}_{\sigma_i}(\mathbf{x}) = -\frac{1}{x_i^2}\Lambda_{\sigma_i}(\mathbf{x})$.

The following lemma provides some structural properties of the function $\tilde{Z}_W(\mathbf{x};u)$ that will be used to get some analytic information about the Weyl group multiple Dirichlet series we will introduce in Section 7.

Lemma 5.2. — *Set $\underline{x} = (x_1,\ldots,x_4)$, $\underline{k} = (k_1,\ldots,k_4)$ and $l = k_5$. Then we have:*

1. *The function $\tilde{Z}_W(\mathbf{x};u)$ can be written as*

$$\begin{aligned}\tilde{Z}_W(\mathbf{x};u) &= \frac{\sum_{l-\text{even}} P_l(\underline{x};u)x_5^l}{\prod_{j=1}^4(1-ux_j)} + \sum_{l-\text{odd}} P_l(\underline{x};u)x_5^l \\ &= \frac{\sum_{|\underline{k}|-\text{even}} Q_{\underline{k}}(x_5;u)\underline{x}^{\underline{k}}}{1-ux_5} + \sum_{|\underline{k}|-\text{odd}} Q_{\underline{k}}(x_5;u)\underline{x}^{\underline{k}}\end{aligned}$$

   where $P_l(\underline{x};u)$ and $Q_{\underline{k}}(x_5;u)$ are polynomials in $x_1,\ldots,x_4,u$ and x_5,u, respectively. Here we set $|\underline{k}| = k_1+\cdots+k_4$.

2. *The power series obtained by expanding*

$$\sum_{l\geqslant 0} P_l(\underline{x};u)x_5^l$$

   is absolutely convergent for arbitrary $\underline{x} \in \mathbb{C}^4$, provided $|x_5|$ is sufficiently small, and the power series obtained by expanding

$$\sum_{|\underline{k}|\geqslant 0} Q_{\underline{k}}(x_5;u)\underline{x}^{\underline{k}}$$

   is absolutely convergent for any $x_5 \in \mathbb{C}$, provided all $|x_1|,\ldots,|x_4|$ are sufficiently small.

3. *We have*

$$P_0(\underline{x};u) = Q_{\underline{0}}(x_5;u) \equiv 1$$

   where $\underline{0} = (0,\ldots,0)$.

4. *The polynomials* $P_l(\underline{x};u)$ *are symmetric in* $\underline{x}$*, and if* l *is odd then* $P_l(\underline{x};u)$ *is even, i.e.,* $P_l(\underline{x};u) = P_l(-\underline{x};u)$.
5. *We have the functional equations*

$$P_l(x_1,x_2,x_3,x_4;u) = x_1^{l-\delta_l} P_l\left(\frac{1}{x_1},x_2,x_3,x_4;u\right) \quad \text{and} \quad Q_{\underline{k}}(x_5;u) = x_5^{|\underline{k}|-\delta_{|\underline{k}|}} Q_{\underline{k}}\left(\frac{1}{x_5};u\right) \tag{31}$$

with $\delta_n = 0$ *or* 1 *according as* n *is even or odd.*

Proof. For notational simplicity, we denote $\tilde{Z}(\mathbf{x}) \coloneqq \tilde{Z}_W(\mathbf{x};u)$ in this proof (the variable u being fixed). The functional equation of $\tilde{Z}(\mathbf{x})$ can be broken into its even and odd parts, according to ε_i, as:

$$\tilde{Z}_i^{+}(\mathbf{x}) = \frac{1-u/x_i}{1-ux_i}\tilde{Z}_i^{+}(\sigma_i\mathbf{x}), \quad \tilde{Z}_i^{-}(\mathbf{x}) = \frac{1}{x_i}\tilde{Z}_i^{-}(\sigma_i\mathbf{x})$$

for $i = 1,\dots,5$. Note that $\tilde{Z}_i^{\pm} = \tilde{Z}_1^{\pm}$ for $i = 1,\dots,4$. The even functional equations can be expressed in terms of the functions

$$G_1(\mathbf{x}) = \prod_{i=1}^{4}(1-ux_i)\tilde{Z}_1^{+}(\mathbf{x}), \quad G_5(\mathbf{x}) = (1-ux_5)\tilde{Z}_5^{+}(\mathbf{x})$$

as $G_1(\mathbf{x}) = G_1(\sigma_i\mathbf{x})$ for $i = 1,\dots,4$, and $G_5(\mathbf{x}) = G_5(\sigma_5\mathbf{x})$.

By Theorem 5.1, the function $D\tilde{Z}$ is holomorphic in $\Omega \coloneqq \{\mathbf{x} \in \mathbb{C}^5 : |\mathbf{x}^{\delta}| < 1\}$, and in this domain, it satisfies the functional equations

$$D\tilde{Z}_i^{+}(\mathbf{x}) = \frac{x_i(1+ux_i)}{u+x_i}D\tilde{Z}_i^{+}(\sigma_i\mathbf{x}), \quad D\tilde{Z}_i^{-}(\mathbf{x}) = -\frac{x_i(1-u^2x_i^2)}{u^2-x_i^2}D\tilde{Z}_i^{-}(\sigma_i\mathbf{x})$$

for $i = 1,\dots,5$, which imply that $D\tilde{Z}_i^{+}(\mathbf{x})$ and $D\tilde{Z}_i^{-}(\mathbf{x})$ are divisible by $1+ux_i$ and $1-u^2x_i^2$, respectively. Thus the functions

$$F_1^{+}(\mathbf{x}) \coloneqq \prod_{\substack{\alpha\in\Phi_{\mathrm{re}}^{+}\\ \alpha\neq\alpha_1,\dots,\alpha_4}} \left(1-u^2\mathbf{x}^{2\alpha}\right)\cdot G_1(\mathbf{x}), \quad F_1^{-}(\mathbf{x}) \coloneqq \prod_{\substack{\alpha\in\Phi_{\mathrm{re}}^{+}\\ \alpha\neq\alpha_1,\dots,\alpha_4}} \left(1-u^2\mathbf{x}^{2\alpha}\right)\cdot \tilde{Z}_1^{-}(\mathbf{x})$$

and

$$H_5^{+}(\mathbf{x}) \coloneqq \prod_{\substack{\alpha\in\Phi_{\mathrm{re}}^{+}\\ \alpha\neq\alpha_5}} \left(1-u^2\mathbf{x}^{2\alpha}\right)\cdot G_5(\mathbf{x}), \quad H_5^{-}(\mathbf{x}) \coloneqq \prod_{\substack{\alpha\in\Phi_{\mathrm{re}}^{+}\\ \alpha\neq\alpha_5}} \left(1-u^2\mathbf{x}^{2\alpha}\right)\cdot \tilde{Z}_5^{-}(\mathbf{x})$$

are still holomorphic in Ω. Notice that, for $i = 1,\dots,4$, $F_1^{+}(\mathbf{x})$ is σ_i-invariant, $F_1^{-}(\mathbf{x}) = x_i^{-1}F_1^{-}(\sigma_i\mathbf{x})$, and since σ_i is just permuting the roots in $\Phi_{\mathrm{re}}^{+} \smallsetminus \{\alpha_1,\dots,\alpha_4\}$, the product in the definition of $F_1^{\pm}$ is also σ_i-invariant. By expanding the inverse of this product, and $F_1^{\pm}(\mathbf{x})$ in power series, we can write

$$G_1(\mathbf{x}) = \sum_{l-\text{even}} P_l(\underline{x};u)x_5^l, \quad \tilde{Z}_1^{-}(\mathbf{x}) = \sum_{l-\text{odd}} P_l(\underline{x};u)x_5^l;$$

these expansions hold as long as $|\mathbf{x}^{\alpha}| < |u|^{-1}$ for all $\alpha \in \Phi_{\mathrm{re}}^{+} \smallsetminus \{\alpha_1,\dots,\alpha_4\}$ (e.g., $\underline{x} \in \mathbb{C}^4$ is arbitrary, and $|x_5|$ is sufficiently small), which justifies the first part (i.e., the P-part) of *2.* The functional equation (31) of the coefficients $P_l(\underline{x};u)$ now follows from the σ_1-invariance of G_1 and the functional equation $\tilde{Z}_1^{-}(\mathbf{x}) =$

$x_1^{-1}\tilde{Z}_1^{-}(\sigma_1\mathbf{x})$. It follows that $P_l(\underline{x};u)$ are polynomials in $x_1,\dots,x_4,u$, and our assertions *4* follow from the fact that $\tilde{Z}_1^{-}(\mathbf{x}) = \tilde{Z}_1^{-}(\varepsilon_5\mathbf{x})$ (as $\tilde{Z}^{\circ,\circ} = 0$), and from the symmetry of $\tilde{Z}_1^{+}(\mathbf{x})$ and $\tilde{Z}_1^{-}(\mathbf{x})$ with respect to $x_1,\dots,x_4$.

The Q-parts of *1, 2* and *5* follow from the same argument, applied to H_5^+ and H_5^-.

Finally, notice that

$$\tilde{Z}(\underline{x},0) = \Delta(\underline{x},0)^{-1}\cdot\sum_{w\in\langle\sigma_i\rangle_{1\leqslant i\leqslant 4}} 1|w(\underline{x},0) = \prod_{j=1}^{4}(1-ux_j)^{-1}$$

and

$$\tilde{Z}(\underline{0},x_5) = \Delta(\underline{0},x_5)^{-1}(1+1|\sigma_5(x_5)) = (1-ux_5)^{-1}$$

which give *3*. This completes the proof. □

The extra functional equation satisfied by $\tilde{\mathbf{Z}}(\mathbf{x};u)$ is

$$\tilde{\mathbf{Z}}(\mathbf{x};u) = \tilde{B}(\mathbf{x};u)\tilde{\mathbf{Z}}\big(\mathbf{x};u\mathbf{x}^{\delta}\big) \tag{32}$$

where $\tilde{B}(\mathbf{x};u) = B(\mathbf{x};u)/\big(1-u^2x^{\delta}\big)^2$, with the matrix $B(\mathbf{x};u)$ from Theorem 4.3. Using this functional equation, we now show that $\tilde{\mathbf{Z}}(\mathbf{x};u)$ is completely determined by $\tilde{\mathbf{Z}}(\mathbf{x};0)$ and $\tilde{B}(\mathbf{x};u)$. By contrast, the functional equations (29) alone determine $\tilde{\mathbf{Z}}(\mathbf{x};u)$ only up to a power series in u and $\mathbf{x}^{\delta}$ (see [13, Theorem 3.7]).

Lemma 5.3. — *Expand the vector function $\tilde{\mathbf{Z}}(\mathbf{x};u)$ as*

$$\tilde{\mathbf{Z}}(\mathbf{x};u) = \tilde{\mathbf{Z}}_0(\mathbf{x}) + u\tilde{\mathbf{Z}}_1(\mathbf{x}) + u^2\tilde{\mathbf{Z}}_2(\mathbf{x}) + \cdots$$

and let $\tilde{B}(\mathbf{x};u) = \tilde{B}_0(\mathbf{x}) + u\tilde{B}_1(\mathbf{x}) + \cdots$. Then we have

$$\tilde{\mathbf{Z}}_0(\mathbf{x}) = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \tilde{B}_0(\mathbf{x}) = \tilde{B}_0 := \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

and the matrix $\tilde{B}(\mathbf{x};u)$ determines $\tilde{\mathbf{Z}}(\mathbf{x};u)$ recursively by

$$\tilde{\mathbf{Z}}_n(\mathbf{x}) = (I-\mathbf{x}^{n\delta}\tilde{B}_0)^{-1}\cdot\sum_{i=0}^{n-1}\mathbf{x}^{i\delta}\tilde{B}_{n-i}(\mathbf{x})\tilde{\mathbf{Z}}_i(\mathbf{x}) \tag{33}$$

for $n\geqslant 1$, where I is the identity matrix.

Proof. The expression for $\tilde{\mathbf{Z}}_0(\mathbf{x})$ follows at once from Lemma 4.4, and the explicit formula of the matrix $B(\mathbf{x};u)$ yields $\tilde{B}_0$. The recursion of $\tilde{\mathbf{Z}}_n(\mathbf{x})$ follows at once from (32). □

The following corollary will be needed in the proof of the main theorem in Section 8.

Corollary 5.4. — *We have that*

$$\tilde{Z}_W(\mathbf{x};u) \pmod{u^2} = 1 + u(x_1+x_2+x_3+x_4+x_5).$$

Proof. From the explicit formula of the matrix $\tilde{B}(\mathbf{x};u)$, one finds that

$$\tilde{B}_1(\mathbf{x}) = \begin{pmatrix} 0 & \mathbf{x}^\delta x_5^{-1} & \mathbf{x}^\delta(x_1^{-1}+x_2^{-1}+x_3^{-1}+x_4^{-1}) \\ x_5 & 0 & 0 \\ x_1+x_2+x_3+x_4 & 0 & 0 \end{pmatrix}.$$

The formula now follows from (33) with $n=1$, after multiplying by the row vector $(1,1,1)$. □

We shall also need a positivity result about the specializations $P_l(\underline{1};u)$ of the polynomials in Lemma 5.2, with $\underline{1}=(1,1,1,1)$.

Corollary 5.5. — *The polynomial $P_l(\underline{1};u)$ for l odd, and the power series $P_l(\underline{1};u)/(1-u)^4$ for l even have non-negative coefficients.*

Proof. By Proposition A.2, the entries of the matrix $\tilde{B}(\underline{1},x_5;u)$, as power series in x_5 and u, have non-negative coefficients. Thus the matrices $\tilde{B}_n(\underline{1},x_5)$ have entries with non-negative coefficients, and by induction using (33), the same is true for the vectors $\tilde{\mathbf{Z}}_n(\underline{1},x_5)$. Accordingly, the function $\tilde{Z}_W(\underline{1},x_5;u)$ has non-negative coefficients when expanded as a power series, and now Lemma 5.2 finishes the proof. □

6. Residues

By Theorem 5.1, the singularities of the function $\tilde{Z}_W(\mathbf{x};u)$ can only occur at the zeros of the denominator $D(\mathbf{x};u)$, and thus $\tilde{Z}_W(\mathbf{x};u)$ can only have simple poles. Our goal is now to compute the residue of this function at each of these poles.

The following theorem provides the key calculation:

Theorem 6.1. — *The residue of $\tilde{Z}_W(\mathbf{x};u)$ at $x_5=u^{-1}$ is given by the formula*

$$\begin{aligned} R(\underline{x};u) &:= \lim_{x_5\to 1/u}(1-ux_5)\tilde{Z}_W(\mathbf{x};u) \\ &= \frac{1}{(P^2;P^2)_\infty (u^2P^2;P^2)_\infty \prod_{i=1}^4 (x_i^2;P^2)_\infty (u^2x_i^{-2}P^2;P^2)_\infty \cdot \prod_{1\leqslant i<j\leqslant 4}(x_ix_j;P)_\infty} \end{aligned}$$

where $P=(x_1x_2x_3x_4)/u^2$ and $(a;b)_\infty=\prod_{k\geqslant 0}(1-ab^k)$ is the b-Pochhammer symbol.

The proof of this theorem will be given in the rest of this section. Using results in [27], similar ideas can be used to prove explicit formulas for multiple residues of the average zeta function of arbitrary simply laced reduced irreducible affine root systems.

To compute the residue at any other pole of $\tilde{Z}_W(\mathbf{x};u)$, let α be an affine positive real root, and write it as $\alpha=\sum_i n_i\alpha_i$ in terms of simple roots. Let w_α be an element of the Weyl group sending α to α_5, and if $\alpha\neq\alpha_i$, hence $n_5\geqslant 1$, let ζ be a $2n_5$-th root of 1 in $\mathbb{C}$. Let $C_{\alpha,\zeta}(\underline{x};u)$ denote the limit

$$C_{\alpha,\zeta}(\underline{x};u) := \lim_{x_5\to\zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n_5}} \left(1-\zeta u^{1/n_5}\mathbf{x}^{\alpha/n_5}\right)\tilde{Z}_W(\mathbf{x};u)$$

where $\alpha' := \alpha - n_5\alpha_5$. Here, we chose the principal branch of the complex logarithm, and so $C_{\alpha,\zeta}(\underline{x};u)$ is well-defined and analytic at least when $x_1,\ldots,x_4$ and u are away from the non-positive real axis.

Lemma 6.2. — *With notation as above, we have*

$$C_{\alpha,\zeta}(\underline{x};u) = \frac{1}{2n_5}\{R(w_\alpha \underline{x};u)\cdot f_{w_\alpha}(\mathbf{x};u)\}|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n_5}} \tag{34}$$

where $f_w = (1+ux_5)\|w$ and $w_\alpha \underline{x}$ represents the first four components of $w_\alpha \mathbf{x}$.

Proof. By applying the functional equation (29) of the vector function $\tilde{\mathbf{Z}} = \overline{\tilde{Z}_W}$ corresponding to $w = \sigma_5$, one finds that

$$\lim_{x_5\to e/u}(1-eux_5)\tilde{\mathbf{Z}}(\mathbf{x};u) = \frac{1}{2}R(\underline{x};u)\cdot {}^t(1,e,0)$$

where $e \in \{-1,1\}$. Applying (29) again, for $w = w_\alpha$, we obtain

$$\lim_{x_5\to\zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n_5}}\left(1-\zeta u^{1/n_5}\mathbf{x}^{\alpha/n_5}\right)\tilde{\mathbf{Z}}(\mathbf{x};u) = \frac{1}{2n_5}\left\{R(w_\alpha\underline{x};u)\tilde{\Lambda}_{w_\alpha}(\mathbf{x})\right\}\Big|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n_5}}\cdot {}^t(1,\zeta^{n_5},0).$$

The formula in the lemma now follows using (30), after multiplying on the left by the vector $(1,1,1)$. □

If $\alpha = \alpha_i$, for some $1 \leqslant i \leqslant 4$, the residue of $\tilde{Z}_W(\mathbf{x};u)$ at $x_i = 1/u$ can be computed similarly from the functional equation (29) with $w_\alpha = \sigma_i\sigma_5$, which sends the simple root α_i to α_5.

6.1. Proof of Theorem 6.1

We start by recalling a result from [13], which identifies the residue $R(\underline{x};u)$ up to a function depending only upon $P = (x_1x_2x_3x_4)/u^2$ and u.

Proposition 6.3. — *We have*

$$R(\underline{x};u) = f(P,u)\cdot\frac{1}{\prod_{i=1}^4\left(x_i^2;P^2\right)_\infty\left(u^2x_i^{-2}P^2;P^2\right)_\infty\cdot\ \prod_{1\leqslant i<j\leqslant 4}(x_ix_j;P)_\infty}$$

where the function $f(P,u)$ is meromorphic in the region $|P|<1$, with possible poles only when $P^n = \pm u^{-2}$ for some $n \geqslant 1$. In addition, if we set $\mathrm{z} := x_1x_2x_3x_4$, then for every $\epsilon > 0$, the function $f(u^2\mathrm{z},u)$ is holomorphic in the polydisc $|\mathrm{z}| < (1+\epsilon)^{-4}$, $|u| < 1+\epsilon$.

Proof. For $|P|$ sufficiently small, the decomposition follows from [13, Lemma 4.30]. From Theorem 5.1, it follows that the function $[D(\mathbf{x};u)/(1-u^2x_5^2)]|_{x_5=1/u}\cdot R(\underline{x};u)$ is holomorphic in the region $|P|<1$, $u \neq 0$; when $|P|$ is small, we can express this function as

$$f(P,u)\left(P^2;P^2\right)_\infty\left(u^4P^2;P^2\right)_\infty\prod_{i=1}^4\left(u^2x_i^2;P^2\right)_\infty\left(u^4x_i^{-2}P^2;P^2\right)_\infty\cdot\prod_{1\leqslant i<j\leqslant 4}(-x_ix_j;P)_\infty\,.$$

However, the product of the Pochhammer symbols is holomorphic when $|P|<1$, and if in addition we take $P^n \neq \pm u^{-2}$ for all $n \geqslant 1$, we can choose $x_i \in \mathbb{C}$ $(i = 1,\dots,4)$ such that this product is non-zero. It follows that indeed $f(P,u)$ is holomorphic for $|P|<1$ except for possible poles when $P^n = \pm u^{-2}$ for some $n \geqslant 1$, and that the decomposition of $R(\underline{x};u)$ extends by analytic continuation.

On the other hand, by Lemma 5.2, part *1*, and the functional equation (31) of the polynomials $Q_{\underline{k}}(x_5;u)$, we have

$$R(u\underline{x};u) \;=\; \sum_{|\underline{k}|\equiv 0 \pmod 2} Q_{\underline{k}}(u;u)\underline{x}^{\underline{k}} \tag{35}$$

which implies easily that $f(u^2\mathrm{z},u)$ (as a function of the variables $\mathrm{z} = x_1x_2x_3x_4$ and u) is holomorphic in the polydisc $|\mathrm{z}| < (1+\epsilon)^{-4}$, $|u| < 1+\epsilon$. This completes the proof. □

Thus it remains to show that

$$f(P,u) = \frac{1}{\left(P^2;P^2\right)_\infty\left(u^2P^2;P^2\right)_\infty}. \tag{36}$$

The proof of this formula involves the following two steps:

(1) Formula (36) holds for $u=-1$. This is an immediate consequence of Macdonald's formula.

(2) The function

$$g(P,u) = f(P,u)\left(P^2;P^2\right)_\infty\left(u^2P^2;P^2\right)_\infty$$

is invariant under the transformation $(P,u)\mapsto(P,u/P)$. (It is here where we crucially use the extra functional equation from 4.2.)

The proofs of (1) and (2) will be given in the next two subsections, but for now let us show how they imply (36).

By Proposition 6.3, the function $g(u^2\mathrm{z},u)$ is holomorphic in the polydisc $|\mathrm{z}| < (1+\epsilon)^{-4}$, $|u| < 1+\epsilon$, hence a normally convergent power series in z and u. Substituting $\mathrm{z}\to P/u^2$, we can then write

$$g(P,u) \;=\; \sum_{m,n\in\mathbb{Z}} a_{m,n}u^mP^n$$

the series being normally convergent for, say $\epsilon \leqslant |u| < 1+\epsilon$ and $|P| < |u|^2(1+\epsilon)^{-4}$. Notice that $a_{m,n}=0$ if $n<0$. Since $g(P,u) = g(P,uP)$, it follows at once (after taking u and P such that $\epsilon \leqslant |uP| < 1+\epsilon$) that $a_{m,n} = a_{m,n-m}$ for all $m,n\in\mathbb{Z}$. Iterating, one finds that $a_{m,n} = a_{m,n+km}$ for $k\in\mathbb{Z}$. Since $a_{m,n}=0$ for $n<0$, it follows that $a_{m,n}=0$ if $m\neq 0$. Therefore $g(P,u)$ is independent of u, and by (1), $g\equiv 1$.

6.2. The residue for $u=-1$

From the relation between $Z_W(\mathbf{x};u)$ and $\tilde{Z}_W(\mathbf{x};u)$, we can write

$$R(\underline{x};u) = \frac{1}{\left(u^2P\,;P^2\right)_\infty^2\cdot\Delta_5(\underline{x},1/u)}\cdot\lim_{x_5\to 1/u}\frac{1-ux_5}{1-x_5^2}Z_W(\mathbf{x};u)$$

where we put $\Delta_5(\mathbf{x}) := \Delta(\mathbf{x})/(1-x_5^2)$. When $u=-1$, we see by induction on $\ell(w)$ that

$$1|w(\mathbf{x};-1) = (-1)^{\ell(w)}\prod_{\beta\,\in\,\Phi(w)}\mathbf{x}^\beta$$

and so $Z_W(\mathbf{x};-1) = F_{MD}(\mathbf{x})$, where $F_{MD}(\mathbf{x}) = \sum_{w\in W}(-1)^{\ell(w)}\prod_{\beta\,\in\,\Phi(w)}\mathbf{x}^\beta$ is the function studied by Macdonald [40]. Then

$$\lim_{u\to-1}\lim_{x_5\to1/u}\frac{1-ux_5}{1-x_5^2}Z_W(\mathbf{x};u) = \frac{1}{2}Z_W(\underline{x},-1;-1) = \frac{1}{2}F_{MD}(\underline{x},-1)$$

where in the first equality we interchanged the two limits; this is justified since $\Delta(\mathbf{x})$, and in particular $1-x_5^2$, divides $Z_W(\mathbf{x};u)$ (see [13, Section 4]), and so the function inside the double limit is continuous at $(1,-1)$ as a function of $z = ux_5$ and u. Consequently,

$$R(\underline{x};-1) = \frac{\frac{1}{2}F_{MD}(\underline{x},-1)}{\left(P\,;P^2\right)_\infty^2 \cdot \Delta_5(\underline{x},-1)}. \tag{37}$$

Using the list of roots of D_4 at the beginning of Section 4, we have by Macdonald's formula [40]

$$\begin{aligned} F_{MD}(\mathbf{x}) = &\prod_{i=1}^{4}\left(x_i;\mathbf{x}^\delta\right)_\infty \left(x_i^{-1}\mathbf{x}^\delta;\mathbf{x}^\delta\right)_\infty \left(x_ix_5;\mathbf{x}^\delta\right)_\infty \left(x_i^{-1}x_5^{-1}\mathbf{x}^\delta;\mathbf{x}^\delta\right)_\infty \prod_{1\leqslant i<j\leqslant 4}\left(x_ix_jx_5;\mathbf{x}^\delta\right)_\infty \\ &\cdot \left(\mathbf{x}^\delta;\mathbf{x}^\delta\right)_\infty^4 \left(x_5;\mathbf{x}^\delta\right)_\infty \left(x_5^{-1}\mathbf{x}^\delta;\mathbf{x}^\delta\right)_\infty . \end{aligned}$$

Also, $\Delta(\mathbf{x}) = F_{MD}(x_1^2,\ldots,x_5^2)$, and one can easily check that (37) matches the formula in Theorem 6.1 for $u=-1$. This completes the proof of step (1) in 6.1.

6.3. Functional equations of the residue function

From the formula of the residue function in Proposition 6.3, it is clear that $R(\underline{x};u) = R(-\underline{x};u)$. Then by Lemma 5.2, part *1*, it follows at once that

$$\lim_{x_5\to 1/u}(1-ux_5)\tilde{\mathbf{Z}}(\mathbf{x};u) = \frac{1}{2}R(\underline{x};u)\cdot v_0 \tag{38}$$

where we set $v_0 = {}^t(1,1,0)$.

We need the extension of the cocycle $\tilde{\Lambda}(\mathbf{x};u)$ to $W\oplus\mathbb{Z}$, as in Section 4.2, by $\tilde{\Lambda}_{\tau^{-1}}(\mathbf{x};u) = \tilde{B}(\mathbf{x};u)$, with $\tilde{B}(\mathbf{x};u)$ defined as in the line following (32), and

$$\tilde{\Lambda}_\tau(\mathbf{x};u) = \tilde{A}(\mathbf{x};u) := A(\mathbf{x};u)\cdot\left(1-u^2/\mathbf{x}^\delta\right)^2$$

with $A(\mathbf{x};u)$ from Theorem 4.2. Then the functional equation (25) holds for $\tilde{\mathbf{Z}}(\mathbf{x};u)$, with $\Lambda(\mathbf{x};u)$ replaced by $\tilde{\Lambda}(\mathbf{x};u)$. As in Section 4.2, we have an action of τ extending the action $\|$ such that

$$\overline{f\|\tau}(\mathbf{x};u) = \tilde{A}(\mathbf{x};u)\cdot\bar{f}(\mathbf{x};u/\mathbf{x}^\delta) \tag{39}$$

for all $f\in\mathbb{C}(\mathbf{x},u)_0$.

Proposition 6.4. — *Let $w\in W$ be such that $w\alpha_5 = \alpha_5-\delta$. Then*

$$R(\underline{x};u) = \lambda_w(\underline{x};u)\cdot R(w\underline{x};u/\mathbf{x}^\delta)\big|_{x_5=1/u}$$

where $\lambda_w(\underline{x};u) = \left[(1+ux_5)\|\tau w\right]^{e,e}(\mathbf{x};u)|_{x_5=1/u}$.

Notice from (39) that $[(1+ux_5)\|\tau](\mathbf{x};u) = \tilde{A}_{e,e}(\mathbf{x};u)+(ux_5/\mathbf{x}^\delta)\tilde{A}_{e,o}(\mathbf{x};u)$, where $\tilde{A}_{e,e}$ (resp. $\tilde{A}_{e,o}$) is the sum of the entries in the first (resp. second) column of the matrix $\tilde{A}(\mathbf{x};u)$.

Proof. Since $w\mathbf{x} = (w\underline{x}, x_5\mathbf{x}^\delta)$, the analogue of the functional equation (25) for $\tilde{\mathbf{Z}}(\mathbf{x}; u)$, applied for $w\tau$, together with (38) gives

$$R(\underline{x}; u)v_0 = \left\{R(w\underline{x}; u/\mathbf{x}^\delta)\tilde{\Lambda}_{w\tau}(\mathbf{x}; u)\right\}\Big|_{x_5=1/u} \cdot v_0.$$

Note that $\tilde{\Lambda}_{w\tau}(\mathbf{x}; u) = \tilde{\Lambda}_\tau(\mathbf{x}; u)\tilde{\Lambda}_w(\mathbf{x}; u/\mathbf{x}^\delta)$ does not have a pole at $x_5 = 1/u$; by Theorem 4.2, the matrix $\tilde{\Lambda}_\tau(\mathbf{x}; u)$ does not have a pole there, and since $\alpha_5 + \delta \notin \Phi(w)$, neither does $\tilde{\Lambda}_w(\mathbf{x}; u/\mathbf{x}^\delta)$. We thus obtain the functional equation of $R(\underline{x}; u)$, with $\lambda_w(\underline{x}; u)$ such that $\tilde{\Lambda}_{w\tau}(\mathbf{x}; u)\big|_{x_5=1/u} \cdot v_0 = \lambda_w(\underline{x}; u)v_0$. Combining (39) and (30), we get the formula for $\lambda_w(\underline{x}; u)$, and completes the proof. □

By applying this proposition to the element $t = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$, we obtain that

$$R(\underline{x}; u) = \lambda_t(\underline{x}; u)R\left(u/x_1, \ldots, u/x_4; u/P\right)$$

with $\lambda_t(\underline{x}; u)$ computed explicitly from the matrix $A(\mathbf{x}; u)$ in [1] as

$$\lambda_t(\underline{x}; u) = (1 - u^2)\prod_{i=1}^{4}(1 - u^2/x_i^2)\prod_{1\leqslant i<j\leqslant 4}(1 - u^2/x_ix_j).$$

One can check easily that this factor matches the one from the same functional equation applied to the right-hand side of the formula in Theorem 6.1, and thus completing the proof of step (2) in 6.1.

7. WMDS associated to moments of L-series

By Theorem 5.1 and the definition (27) of $D(\mathbf{x}; u)$, when $0 < u < 1$, the function $\tilde{Z}_W(\mathbf{x}; u)$ is holomorphic for $|x_i| < 1$ ($i = 1, \ldots, 5$), and thus, in this polydisc,

$$\tilde{Z}_W(\mathbf{x}; u) = 1 + \sum_{\mathrm{k}\neq\mathbf{0}} a(\mathrm{k}; u)\mathbf{x}^{\mathrm{k}} \tag{40}$$

the sum being over tuples $\mathrm{k} = (k_1, \ldots, k_5) \in \mathbb{N}^5 \smallsetminus \{\mathbf{0}\}$. The coefficients $a(\mathrm{k}; u)$ are polynomials in u, and by Cauchy's inequalities, for every $\epsilon > 0$, we have the estimate $|a(\mathrm{k}; u)| \ll_\epsilon u^{-\epsilon|\mathrm{k}|}$.

Before making the transition to multiple Dirichlet series, let us briefly recall some facts about quadratic Dirichlet L-functions in the rational function field setting.

Let $\mathbb{F}_q$ be a finite field of odd characteristic. For $m \in \mathbb{F}_q[x]$, $m \neq 0$, set $|m| = q^{\deg m}$, and for $d, m \in \mathbb{F}_q[x]$, $d \neq 0$ and m monic, let $\chi_d(m) = (d/m)$ denote the quadratic symbol. We define $\chi_d(1) = 1$, and if $d \in \mathbb{F}_q^\times$, we have $\chi_d(m) = \mathrm{sgn}(d)^{\deg m}$, for all non-constant $m \in \mathbb{F}_q[x]$, where $\mathrm{sgn}(d) = 1$ or -1 according as $d \in (\mathbb{F}_q^\times)^2$ or not.

For d square-free, the L-function associated to the primitive character χ_d is defined by

$$L(s, \chi_d) = \sum_{\substack{m\in\mathbb{F}_q[x]\\ m-\text{monic}}} \chi_d(m)|m|^{-s} = \prod_{p-\text{monic \& irreducible}} \left(1 - \chi_d(p)|p|^{-s}\right)^{-1}$$

for complex s, with $\Re(s) > 1$. When d is non-constant this L-function turns out to be a polynomial in q^{-s} of degree $\deg d - 1$, and if $d \in \mathbb{F}_q^\times$,

$$L(s, \chi_d) = \frac{1}{1 - \operatorname{sgn}(d) q^{1-s}};$$

when $d \in (\mathbb{F}_q^\times)^2$, the L-function is just $\zeta(s)$–the zeta-function. In addition, if one defines $\gamma_q(s, d)$ by

$$\gamma_q(s, d) = q^{\frac{1}{2}\left(3 + (-1)^{\deg d}\right)\left(s - \frac{1}{2}\right)} \left(1 - \operatorname{sgn}(d) q^{-s}\right)^{\frac{1}{2}\left(1 + (-1)^{\deg d}\right)} \left(1 - \operatorname{sgn}(d) q^{s-1}\right)^{-\frac{1}{2}\left(1 + (-1)^{\deg d}\right)}$$

then $L(s, \chi_d)$ satisfies the functional equation

$$L(s, \chi_d) = \gamma_q(s, d) |d|^{\frac{1}{2} - s} L(1 - s, \chi_d).$$

We now use the coefficients $a(\mathrm{k}; u)$ to build the relevant family of multiple Dirichlet series.

Let $\mathbb{F}_q$ be a finite field of odd characteristic, and fix an element $\theta_0 \in \mathbb{F}_q^\times \smallsetminus (\mathbb{F}_q^\times)^2$. For $\mathbf{s} = (s_1, \ldots, s_5)$ and $a_1, a_2 \in \{1, \theta_0\}$, we define $\mathscr{W}(\mathbf{s}; a_2, a_1)$ by a series (summed over monics in $\mathbb{F}_q[x]$) of the form

$$\mathscr{W}(\mathbf{s}; a_2, a_1) \quad = \sum_{\substack{m_1, \ldots, m_4, d\text{–monic} \\ d = d_0 d_1^2,\ d_0 \text{ square free}}} \frac{\chi_{a_1 d_0}(\widehat{m}_1 \widehat{m}_2 \widehat{m}_3 \widehat{m}_4) \chi_{a_2}(d_0) A(m_1, m_2, m_3, m_4, d)}{|m_1|^{s_1} |m_2|^{s_2} |m_3|^{s_3} |m_4|^{s_4} |d|^{s_5}} \tag{41}$$

where $\widehat{m}_i$ ($i = 1, \ldots, 4$) is the part of m_i coprime to d_0. The coefficients $A(m_1, m_2, m_3, m_4, d)$ are completely determined by the following two conditions:

(i) If p is monic irreducible, then

$$A\left(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_4}, p^l\right) = a\left(k_1, k_2, k_3, k_4, l; q^{-(\deg p)/2}\right).$$

(ii) For monic m_1, m_2, m_3, m_4, d, we have

$$A(m_1, m_2, m_3, m_4, d) \;=\; \prod_{\substack{p^{k_i} \,\|\, m_i \\ p^l \,\|\, d}} A\left(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_4}, p^l\right)$$

the product being taken over monic irreducibles.

Lemma 7.1. — *The series defining $\mathscr{W}(\mathbf{s}; a_2, a_1)$ is absolutely convergent for $\Re(s_i) > 1$, $i = 1, \ldots, 5$.*

Proof. The proof is similar to that of [23, Lemma 6.5]. We first show that the series (41) is absolutely convergent when $\Re(s_i)$ is sufficiently large. To see this, by condition (i) and the estimate of the coefficients $a(\mathrm{k}; u)$, we have

$$\left|A\left(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_4}, p^{k_5}\right)\right| \ll_{\epsilon, q} |p|^{\epsilon|\mathrm{k}|}$$

for every monic irreducible p, the implied constant depending upon q and ϵ, but it is independent of the degree of p. Choosing $n \geqslant 1$ such that this constant is smaller than q^n, it follows from (ii) that

$$|A(m_1, m_2, m_3, m_4, d)| < |m_1 m_2 m_3 m_4 d|^{n+\epsilon}$$

for all monic polynomials m_1, m_2, m_3, m_4, d. Since

$$|\mathscr{W}(\mathbf{s}; a_2, a_1)| \leqslant \sum_{m_1,\dots,m_4,\,d} \frac{|A(m_1, m_2, m_3, m_4, d)|}{|m_1|^{\Re(s_1)}|m_2|^{\Re(s_2)}|m_3|^{\Re(s_3)}|m_4|^{\Re(s_4)}|d|^{\Re(s_5)}}$$

we have absolute convergence as long as $\Re(s_i) > n+1+\epsilon$. On the other hand, the right-hand side of the last inequality decomposes as

$$\prod_p \left(1 + \sum_{\mathrm{k}\neq \mathbf{0}} \left|a\left(\mathrm{k}; |p|^{-1/2}\right)\right| |p|^{-\langle \mathrm{k}, \Re(\mathbf{s})\rangle}\right)$$

where $\Re(\mathbf{s}) := (\Re(s_1), \dots, \Re(s_5))$, and $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^5$. Taking the logarithm, our assertion now follows by a simple comparison with the series

$$\sum_p \sum_{\mathrm{k}\neq \mathbf{0}} |p|^{-\lambda|\mathrm{k}|}$$

which is easily seen to be convergent when $\lambda > 1$. □

For simplicity, we shall assume from now on that $q \equiv 1 \pmod 4$. Following [23, Section 3], we can write

$$\mathscr{W}(\mathbf{s}; a_2, a_1) = \sum_{d=d_0d_1^2} \frac{\prod_{i=1}^4 L\left(s_i + \frac{1}{2}, \chi_{a_1d_0}\right) \cdot \chi_{a_2}(d_0) P_d(\mathbf{s}'; \chi_{a_1d_0})}{|d|^{s_5}} \tag{42}$$

where $P_d(\mathbf{s}'; \chi_{a_1d_0})$ ($\mathbf{s}' := (s_1, \dots, s_4)$) is the Dirichlet polynomial defined by

$$\begin{aligned} P_d(s_1, \dots, s_4; \chi_{a_1d_0}) &= \prod_{\substack{p^l \,\|\, d \\ l \equiv 1 \pmod 2}} P_l\left(|p|^{-s_1}, \dots, |p|^{-s_4}; q^{-(\deg p)/2}\right) \\ &\cdot \prod_{\substack{p \,|\, d_1 \\ p^l \,\|\, d \\ l \equiv 0 \pmod 2}} P_l\left(\chi_{a_1d_0}(p)|p|^{-s_1}, \dots, \chi_{a_1d_0}(p)|p|^{-s_4}; q^{-(\deg p)/2}\right). \end{aligned} \tag{43}$$

Here $P_l(\underline{x}; u)$ are the polynomials in Lemma 5.2. The functions $P_d(\mathbf{s}'; \chi_{a_1d_0})$ are symmetric in all variables, and by (31), they satisfy a functional equation as $s_i \to -s_i$ for each $1 \leqslant i \leqslant 4$. It is clear that (42) converges absolutely for arbitrary $s_1, \dots, s_4 \in \mathbb{C} \setminus \{1/2\}$ as long as s_5 has sufficiently large real part.

We can also write

$$\mathscr{W}(\mathbf{s}; a_2, a_1) = \sum_{m_1m_2m_3m_4 = n_0n_1^2} \frac{L\left(s_5 + \frac{1}{2}, \chi_{a_2n_0}\right)\chi_{a_1}(n_0) Q_{\underline{m}}(s_5; \chi_{a_2n_0})}{|m_1|^{s_1}|m_2|^{s_2}|m_3|^{s_3}|m_4|^{s_4}} \tag{44}$$

where, for $\underline{m} = (m_1, \dots, m_4)$, the Dirichlet polynomial $Q_{\underline{m}}(s_5; \chi_{a_2n_0})$ is given by

$$Q_{\underline{m}}(s_5; \chi_{a_2n_0}) = \prod_{\substack{p^{k_i} \,\|\, m_i \\ |\underline{k}| \equiv 1 \pmod 2}} Q_{\underline{k}}\left(|p|^{-s_5}; q^{-(\deg p)/2}\right) \cdot \prod_{\substack{p \,|\, n_1 \\ p^{k_i} \,\|\, m_i \\ |\underline{k}| \equiv 0 \pmod 2}} Q_{\underline{k}}\left(\chi_{a_2n_0}(p)|p|^{-s_5}; q^{-(\deg p)/2}\right).$$

Again, by (31), the polynomials $Q_{\underline{m}}(s_5; \chi_{a_2n_0})$ satisfy a functional equation as $s_5 \to -s_5$, and for every $s_5 \in \mathbb{C} \setminus \{1/2\}$, the series (44) converges absolutely as long as all $s_1, \dots, s_4$ have sufficiently large real parts.

Proposition 7.2. — *Let $\mathscr{Z}(\mathbf{x})$ denote the function $\mathscr{W}(\mathbf{s};1,1)$ after substituting $x_i = q^{-s_i}$, $i = 1,\ldots,5$. Then the functions $\mathscr{Z}(\mathbf{x})$ and $\tilde{Z}_W(\mathbf{x};\sqrt{q})$ satisfy the same functional equation* (28).

Proof. This follows at once from (31) and the functional equation of $L(s,\chi_d)$. $\square$

Remark 7.1. This functional equation was stated first in [13], in terms of the vector function

$$\boldsymbol{\mathscr{W}}(\mathbf{s}) = \frac{1}{2}\begin{pmatrix} \mathscr{W}(\mathbf{s};1,1) + \mathscr{W}(\mathbf{s};\theta_0,1) \\ \mathscr{W}(\mathbf{s};1,1) - \mathscr{W}(\mathbf{s};\theta_0,1) \\ \mathscr{W}(\mathbf{s};1,\theta_0) + \mathscr{W}(\mathbf{s};\theta_0,\theta_0) \end{pmatrix}$$

and predated the Chinta-Gunnells action. To make the comparison with early work easier, and because it generalizes more readily to number fields (for which the Chinta-Gunnells action is not available), we also state the vector form of (28). The vector function obtained by substituting $x_i = q^{-s_i}$ ($i = 1,\ldots,5$) in $\boldsymbol{\mathscr{W}}(\mathbf{s})$ satisfies the same functional equation as the vector

$$\tilde{\mathbf{Z}}_1(\mathbf{x};u) := \begin{pmatrix} (\tilde{Z}_W)_1^+(\mathbf{x};u) \\ (\tilde{Z}_W)_1^-(\mathbf{x};u) \\ (\tilde{Z}_W)_1^+(\varepsilon_5\mathbf{x};u) \end{pmatrix} = V\overline{\tilde{Z}_W}(\mathbf{x};u), \quad \text{where } V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

namely $\tilde{\mathbf{Z}}_1(\mathbf{x};u) = \mathrm{M}_w(\mathbf{x};u)\tilde{\mathbf{Z}}_1(w\mathbf{x};u)$ for all $w \in W$, for the 1-cocycle $\mathrm{M}_w(\mathbf{x};u)$ defined on generators by $\mathrm{M}_{\sigma_i}(\mathbf{x};u) = -\frac{1}{x_i^2}V\Lambda_{\sigma_i}(\mathbf{x};u)V^{-1}$.

Remark 7.2. Expressions similar to (42), (44), and the conclusions of Lemma 7.1, and Proposition 7.2 all still hold if instead of $\mathscr{W}(\mathbf{s};a_2,a_1)$, one takes the multiple Dirichlet series constructed in the same way using the coefficients of $Z_W^{\mathrm{CG}}(\mathbf{x};u)$. More generally, one can start with any power series $F(z,u)$, and take the multiple Dirichlet series whose p-parts are $F\big(|p|^{-\delta(\mathbf{s})},|p|^{-1/2}\big)\tilde{Z}_W\big(|p|^{-s_1},\ldots,|p|^{-s_5};|p|^{-1/2}\big)$, where $\delta(\mathbf{s}) = s_1 + s_2 + s_3 + s_4 + 2s_5$. It is easy to see that this multiple Dirichlet series is, in fact,

$$\prod_p F\big(|p|^{-\delta(\mathbf{s})},|p|^{-1/2}\big)\cdot\mathscr{W}(\mathbf{s};a_2,a_1). \tag{45}$$

Choosing $F(z,u)$ so that the product converges absolutely when $\Re(\delta(\mathbf{s})) > 6$, one sees that the multiple Dirichlet series can be expressed as in (42) and (44) for $\Re(s_i) > 1$, $i = 1,\ldots,5$, and that it satisfies the required functional equations. Thus to determine a canonical normalization of $\tilde{Z}_W(\mathbf{x};u)$ (or equivalently $Z_W^{\mathrm{CG}}(\mathbf{x};u)$), which eventually allows us to establish the analytic properties of the corresponding multiple Dirichlet series, some additional conditions must be imposed.

There is yet another multiple Dirichlet series

$$Z(\mathbf{x};q) = \sum_{\mathrm{k}\in\mathbb{N}^{r+1}} b(\mathrm{k};q)\mathbf{x}^{\mathrm{k}}$$

associated with moments of L-functions that is worth considering; as before, x_i stands for q^{-s_i}, $i = 1,\ldots,$ $r+1$, and the coefficients $b(\mathrm{k};q)$ are finite sums $\sum_\lambda c_\lambda\lambda$ over q-Weil algebraic integers of weights $\nu_\lambda \in \mathbb{N}$, subject to the following three conditions:

(a) Each q-Weil integer λ occurs in the sum together with all its complex conjugates.

(b) The coefficients c_λ are rational numbers, and if λ,λ' are conjugates over $\mathbb{Q}$, then $c_\lambda = c_{\lambda'}$.

(c) For $|\mathrm{k}| = k_1 + \cdots + k_{r+1} > 1$, we have that $|\mathrm{k}| + 2 \leqslant \nu_\lambda \leqslant 2|\mathrm{k}|$ for all λ occurring in the sum.

The lower bound of ν_λ in the last condition will be referred to as ***dominance***. Let $\overline{\mathbb{F}}_q$ be a fixed algebraic closure of $\mathbb{F}_q$. We require that the multiple Dirichlet series to be of the form (41), with multiplicative coefficients $B(m_1, \ldots, m_r, d)$, such that the following conditions are also satisfied:

(A1) The sub-series

$$\sum_{\underline{k}\in\mathbb{N}^r} b(\underline{k}, 0; q)\underline{x}^{\underline{k}} = \prod_{i=1}^{r} \frac{1}{1 - qx_i}$$

where, as before, $\underline{x} := (x_1, \ldots, x_r)$ and $\underline{k} := (k_1, \ldots, k_r)$. In addition,

$$\sum_{\underline{k}\in\mathbb{N}^r} b(\underline{k}, 1; q)\underline{x}^{\underline{k}} = q \quad \text{and} \quad \sum_{l\geqslant 0} b(\underline{0}, l; q)x_{r+1}^{l} = \frac{1}{1 - qx_{r+1}}.$$

In particular, $B(1, \ldots, 1) = b(0, \ldots, 0; q) = 1$.

(A2) The coefficients $b(\mathrm{k}; q^n)$ corresponding to $Z(\mathbf{x}; q^n)$ over any finite field extension $\mathbb{F}_{q^n} \subset \overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ are given by

$$b(\mathrm{k}; q^n) = \sum_{\lambda} c_\lambda \lambda^n.$$

(A3) For every monic irreducible $p \in \mathbb{F}_q[x]$ of degree $e \geqslant 1$, the coefficients $B\big(p^{k_1}, \ldots, p^{k_{r+1}}\big)$ are given by

$$B\big(p^{k_1}, \ldots, p^{k_{r+1}}\big) = q^{e|\mathrm{k}|} \sum_{\lambda} c_\lambda \lambda^{-e}.$$

In [25], the first two authors established the existence of a unique multiple Dirichlet series satisfying all these conditions. This axiomatic construction was generalized by Whitehead [49] to simply laced affine root systems, and by Sawin [47], using geometric methods, in a more general setting.

One should notice that it is not a priori clear that this multiple Dirichlet series satisfies a group of functional equations. To see that this is indeed the case, one can adapt the proof of [49, Proposition 2.2.1]. However, in the next section, this will be clarified when $r = 4$.

8. Comparison

Let notations be as above. Our goal in this section is to compare the functions $\mathscr{Z}(\mathbf{x})$ and $\tilde{Z}_W(\mathbf{x}; \sqrt{q})$, and deduce from this comparison (combined with Theorem 5.1) the meromorphic continuation of $\mathscr{Z}(\mathbf{x})$ to the maximal possible region $|\mathbf{x}^\delta| < 1$.

One checks that the functions $\mathscr{Z}(\mathbf{x})$ and $\tilde{Z}_W(\mathbf{x}; \sqrt{q})$ satisfy the following conditions:

(i) Both $\mathscr{Z}(\mathbf{x})$ and $\tilde{Z}_W(\mathbf{x}; \sqrt{q})$ are holomorphic for $|x_i| < 1/q$ ($i = 1, \ldots, 5$).

(ii) Both $D(\mathbf{x}; \sqrt{q})\mathscr{Z}(\mathbf{x})$ and $D(\mathbf{x}; \sqrt{q})\tilde{Z}_W(\mathbf{x}; \sqrt{q})$ are power series that converge absolutely if either $\underline{x} \in \mathbb{C}^4$ and $|x_5|$ is sufficiently small, or $x_5 \in \mathbb{C}$ and $|x_1|, \ldots, |x_4|$ are sufficiently small.

(iii) Both $\mathscr{Z}(\mathbf{x})$ and $\tilde{Z}_W(\mathbf{x}; \sqrt{q})$ are symmetric in $x_1, \ldots, x_4$, and satisfy the same functional equation (28).

It follows from [13, Theorem 3.7] that

$$\mathscr{Z}(\mathbf{x}) = C(\mathbf{x}^\delta)\tilde{Z}_W(\mathbf{x};\sqrt{q}) \tag{46}$$

for some function C of one complex variable.

Theorem 8.1. — *We have the equalities*

$$\mathscr{Z}(\mathbf{x}) = \tilde{Z}_W(\mathbf{x};\sqrt{q}) = Z(q^{-1/2}\mathbf{x};q).$$

Proof. We first show that $\mathscr{Z}(\mathbf{x})$ and $\tilde{Z}_W(\mathbf{x};\sqrt{q})$ have the same residue at $x_5 = 1/\sqrt{q}$ (hence $C(\mathbf{x}^\delta)$ in (46) is identically 1).

From the expression (44) of $\mathscr{W}(\mathbf{s};a_2,a_1)$, we see that this function has a (simple) pole at $s_5 = \frac{1}{2}$ only if $a_2 = 1$, and the part contributing to this pole is

$$\zeta\left(s_5+\tfrac{1}{2}\right)\cdot\sum_{m_1m_2m_3m_4=\square}\frac{Q_{\underline{m}}(s_5;1)}{|m_1|^{s_1}|m_2|^{s_2}|m_3|^{s_3}|m_4|^{s_4}} = \zeta\left(s_5+\tfrac{1}{2}\right)\cdot\prod_p\left(\sum_{|\underline{k}|\equiv 0 \pmod 2}\frac{Q_{\underline{k}}(|p|^{-s_5};|p|^{-1/2})}{|p|^{k_1s_1+k_2s_2+k_3s_3+k_4s_4}}\right).$$

Note that the local factor of the product in the right-hand side is

$$\sum_{|\underline{k}|\equiv 0 \pmod 2} Q_{\underline{k}}(x_5;u)\underline{x}^{\underline{k}}$$

where, for a monic irreducible p, we set $x_i = |p|^{-s_i}$ ($i=1,\ldots,5$), and $u = |p|^{-1/2}$. By (35), this local factor evaluated at $x_5 = u$ is just $R(u\underline{x};u)$, and thus

$$\lim_{s_5\to\frac{1}{2}}\left(1-q^{\frac{1}{2}-s_5}\right)\mathscr{W}(\mathbf{s};1,a_1) = \prod_p R\left(|p|^{-\frac{1}{2}-s_1},|p|^{-\frac{1}{2}-s_2},|p|^{-\frac{1}{2}-s_3},|p|^{-\frac{1}{2}-s_4};|p|^{-\frac{1}{2}}\right).$$

By using the formula of $R(\underline{x};u)$ in Theorem 6.1, the product equals $R(q^{-s_1},q^{-s_2},q^{-s_3},q^{-s_4};\sqrt{q})$, that is, $\mathscr{Z}(\mathbf{x})$ and $\tilde{Z}_W(\mathbf{x};\sqrt{q})$ have the same residue at $x_5 = 1/\sqrt{q}$. This shows that

$$\mathscr{Z}(\mathbf{x}) = \tilde{Z}_W(\mathbf{x};\sqrt{q}).$$

To prove the second equality, set $u=\sqrt{q}$. It is clear that $\tilde{Z}_W(u\mathbf{x};u)$ satisfies the conditions (a) and (b) in the previous section. The upper bound in condition (c) follows easily from (12) and the fact that the coefficients of the power series expansion of $J(x/u,u,\varepsilon)$ are polynomials in u^{-1}.

To show that the coefficients of $\tilde{Z}_W(u\mathbf{x};u)$ satisfy the dominance condition, we have to show that the coefficients $a(\mathrm{k};u)$ in (40) are divisible by u^2 if $|\mathrm{k}|>1$. This follows from Corollary 5.4.

The first and third conditions in (A1) can be verified using Lemma 5.2; the generating function of $b(\underline{k},1;q)$ can be computed from the term $1|\sigma_5(\mathbf{x};u)$ in $Z_W(\mathbf{x};u)$. Conditions (A2) and (A3) can be easily verified directly. This completes the proof. $\square$

Remark 8.1. The fact that $\mathscr{Z}(\mathbf{x})$ coincides with $Z(q^{-1/2}\mathbf{x};q)$ shows that our choice of the p-part $\widetilde{Z}_W$ for the Weyl group multiple Dirichlet series associated with the 4-th moment of quadratic L-functions is canonical; this comparison was the sole reason for introducing the axiomatic multiple Dirichlet series at the end of the previous section. The uniqueness of $Z(\mathbf{x};q)$ implies, for example, that the multiple Dirichlet series with all the p-parts equal to $Z_W^{\mathrm{CG}}\left(|p|^{-s_1},\ldots,|p|^{-s_5};|p|^{-1/2}\right)$ cannot satisfy the full set of conditions that $\mathscr{Z}(\mathbf{x})$ satisfies. In fact, the reader can check directly using (45) in Remark 7.2 that condition (A3) (i.e., the local-to-global principle) is ***not*** satisfied for this choice of the p-parts.

9. Applications

In this final section of the paper, we begin with two straightforward consequences of Theorem 8.1. First, by taking coefficients of the function $\tilde{Z}_W(\mathbf{x};\sqrt{q})$ with respect to the variable x_5, we obtain an exact formula for the 4-th moments of quadratic Dirichlet L-functions over rational function fields, weighted by the polynomials P_d defined in Section 7 and Lemma 5.2. Then, from the analytic properties of $\tilde{Z}_W(\mathbf{x};\sqrt{q})$, we also deduce an asymptotic formula for these moments, completely analogous to that conjectured in [28] for arbitrary moments. For the remaining of the section, we study in some detail the secondary terms in the asymptotic formula. In particular, our analysis will show that all these terms are non-zero.

Theorem 9.1. — *For $D \geqslant 1$, we have the exact formula:*

$$\sum_{\deg d = D} L\left(\tfrac{1}{2}, \chi_{d_0}\right)^4 P_d(\chi_{d_0}) = \mathrm{Coeff}_{\xi^D} \tilde{Z}_W(\underline{1}, \xi; \sqrt{q})$$

where $P_d(\chi_{d_0}) = P_d(0, \ldots, 0; \chi_{d_0})$, *and* $(\underline{1}, \xi) := (1, 1, 1, 1, \xi)$.

Proof. If we put $\xi = q^{-s_5}$, then by (42) we can write

$$\mathscr{W}(\mathbf{s}; 1, 1) = \sum_{D \geqslant 0} \left\{ \sum_{\deg d = D} \prod_{i=1}^{4} L\left(s_i + \tfrac{1}{2}, \chi_{d_0}\right) P_d(\mathbf{s}'; \chi_{d_0}) \right\} \xi^D.$$

Setting $s_i = 0$, the asserted formula follows at once from the equality $\mathscr{Z}(\underline{1}, \xi) = \tilde{Z}_W(\underline{1}, \xi; \sqrt{q})$. □

However, one cannot expect to have an analogue of this result for general function fields, or number fields. For this reason, we shall also give the asymptotic formula for the fourth moment sums that is, indeed, expected to generalize to any global field. This asymptotic formula has also the advantage of separating the contributions corresponding to the singularities of the function $\mathscr{Z}(\underline{1}, \xi)$.

For $n \geqslant 1$, let, as in [28, Section 6],

$$\Phi_n = \left\{ \sum_{i=1}^{5} n_i \alpha_i \in \Phi^+_{\mathrm{re}} : n_5 = n \right\}$$

and, for $D \geqslant 1$, define $Q_n(D, q)$ by

$$Q_n(D, q) = \lim_{\underline{x} \to \underline{1}} \left(\sum_{\alpha \in \Phi_n} \sum_{\zeta^{2n} = 1} R_{\alpha, \zeta}(\underline{x}; \sqrt{q}) \zeta^D \mathbf{x}^{D\alpha'/n} \right)$$

where $\alpha' = \alpha - n\alpha_5$, and $R_{\alpha,\zeta}(\underline{x}; \sqrt{q})$ is given by (34). We have the following:

Theorem 9.2. — *For $D, N \geqslant 1$ and $(N+1)^{-1} < \Theta < N^{-1}$, we have the asymptotic formula*

$$\sum_{\deg d = D} L\left(\tfrac{1}{2}, \chi_{d_0}\right)^4 P_d(\chi_{d_0}) = \sum_{n \leqslant N} Q_n(D, q) q^{\frac{D}{2n}} + O_{\Theta, q}\left(q^{\frac{D\Theta}{2}}\right).$$

Proof. The asymptotic formula follows from Theorem 8.1 and a straightforward application of the residue theorem to the integral

$$\frac{1}{2\pi i}\oint_{\partial\mathscr{A}_\Theta}\frac{\mathscr{Z}(\underline{1},\xi)}{\xi^{D+1}}\,d\xi$$

where $\mathscr{A}_\Theta = \{\xi \in \mathbb{C} : q^{-2} \leqslant |\xi| \leqslant q^{-\Theta/2}\}$ — see also [28, Theorem 6.1]. □

Remark 9.1. One might be puzzled by the discrepancy of a factor of $q^{D/2}$ in this asymptotic formula. This is simply explained by our normalization $\mathbf{x} \to q^{-1/2}\mathbf{x}$ of the function $\tilde{Z}_W(\mathbf{x};\sqrt{q})$ which is causing the correction polynomials $P_d(\chi_{d_0})$ to be off by a factor of $|d|^{1/2}$. For instance, when $d = d_0$ is square-free, then

$$P_d(\chi_{d_0}) = |d|^{-1/2}$$

instead of 1.

Notice also that Corollary 5.5 and definition (43) of the Dirichlet polynomials $P_d(s_1,\ldots,s_4;\chi_{a_1d_0})$ imply that $P_d(\chi_{d_0})$ is non-negative for all d. In particular, we have the inequality

$$\sum_{\substack{d-\text{monic \& sq. free}\\ \deg d = D}} L\big(\tfrac{1}{2},\chi_d\big)^4\,|d|^{-1/2} \leqslant \sum_{\substack{d-\text{monic}\\ \deg d = D}} L\big(\tfrac{1}{2},\chi_{d_0}\big)^4 P_d(\chi_{d_0}).$$

Moreover, [28, Conjecture 1.2] predicts an asymptotic formula for the (traditional) moment sum in the left-hand side of the above inequality, similar to that in Theorem 9.2. (This similarity between the two asymptotics should still persist when considering the analogues of these moment sums over any global field.) For these reasons, the presence of the correction factors $P_d(\chi_{d_0})$ in our fourth moment sum is harmless for all practical purposes. That is, if an analogue of the asymptotic formula in Theorem 9.2 is proved in the number field setting, it would have the same applications as the corresponding asymptotic formula for the traditional fourth moment sum.

9.1. An explicit formula for $Q_n(D,q)$

We now give explicit formulas for the terms $Q_n(D,q)$ in Theorem 9.2, following closely [28]. In *loc. cit.*, conjectural formulas were given for the first two terms Q_1 and Q_2 in the asymptotics of the r-th moment of quadratic Dirichlet L-functions, summed over square-free monic polynomials. It is interesting to note that we obtain the same formulas, except for the so-called "arithmetic factor" which is simpler when summing over all monic polynomials. Here, we are able to treat all the terms $Q_n(D,q)$ since the sets Φ_n can be explicitly described for $D_4^{(1)}$.

First, we rewrite the double sum in the formula of $Q_n(D,u^2)$ by grouping together the terms with $\pm\zeta$. Denote by $f^e = f_1^+$ (resp. $f^o = f_1^-$) the even (resp. odd) part of the function $f(\mathbf{x})$ with respect to the sign function ε_1, and let μ_k be the set of k-th roots of unity in $\mathbb{C}$. We have

$$Q_n(D,u^2) = \frac{1}{n}\sum_{\zeta\in\mu_{2n}/\{\pm1\}}\zeta^D I_{n,\zeta}(D,u)$$

with

$$I_{n,\zeta}(D,u) = \lim_{\underline{x}\to\underline{1}}\sum_{\alpha\in\Phi_n}\mathbf{x}^{D\alpha'/n}\big\{R(w_\alpha\mathbf{x};u)f_{w_\alpha}^{a_D}(\mathbf{x};u)\big\}\big|_{x_5=\zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} \tag{47}$$

where $f_w = (1 + ux_5)\|w$ was defined in Lemma 6.2, and $a_D \in \{e, o\}$ denotes the parity of D. Although the function R does not depend on x_5, we write $R(\mathbf{x}; u)$ for $R(\underline{x}; u)$, hence $R(w\underline{x}; u) = R(w\mathbf{x}; u)$. Notice that $R(w\mathbf{x}; u)$ is even with respect to both ε_1 and ε_5 for all $w \in W$, and that $I_{n,-\zeta}(D, u) = (-1)^D I_{n,\zeta}(D, u)$, so the sum over ζ above is indeed well-defined.

Next, we give integral formulas for the sum in (47), from which it will be clear that the limit exists, and it is a polynomial in D of degree 10 if n is odd, and of degree 7 if n is even.

9.1.1. The case n odd

Let $n = 2k+1$ with $k \geqslant 0$. For $I \subset S := \{1, 2, 3, 4\}$, possibly empty, put $w_I = \prod_{i \in I} \sigma_i$, and $\alpha_I = \sum_{i \in I} \alpha_i$, with the understanding that the empty product is the identity, and the empty sum is 0. Also, let $t = \sigma_1 \cdots \sigma_4 \sigma_5$, for which $t\alpha_5 = \alpha_5 - \delta$. Then

$$\Phi_n = \{\alpha_5 + \alpha_I + k\delta : I \subset S\} \quad \text{and} \quad t^k w_I(\alpha_5 + \alpha_I + k\delta) = \alpha_5.$$

Notice that t^2 is a translation, with $t^2\alpha_i = \alpha_i + \delta$ ($i = 1, \ldots, 4$), and $t^2\alpha_5 = \alpha_5 - 2\delta$.

Lemma 9.3. — *Let $n = 2k + 1$ with $k \geqslant 0$, and let $\alpha = \alpha_5 + k\delta$ and $w_\alpha = t^k$. Then we have a decomposition*

$$R(w_\alpha \mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} = R_{n,\zeta}(\underline{x}; u) \cdot \prod_{1 \leqslant i \leqslant j \leqslant 4} \frac{1}{1 - x_i x_j}$$

with $R_{n,\zeta}(\underline{1}; u) = R_n(u^{-2/n}/\zeta^2)$ for an explicit function $R_n(\varrho)$. Moreover, the function $R_n(\varrho)$ is given by an absolutely convergent power series for $|\varrho| < 1$; when $n = 1$, we have

$$R_1(\varrho) = (\varrho; \varrho)_\infty^{-11}.$$

Due to Remark 9.2 below, we omit the formula for $R_n(\varrho)$ when $n > 1$.

Proof. The assertion follows at once from the explicit formula of $R(\underline{x}; u)$ in Theorem 6.1. ☐

Recall that $f_w = (1 + ux_5)\|w$. For $b \in \{e, o\}$, let $f^b_{n,\zeta}(\underline{x}; u) = f^b_{w_\alpha}(\mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}}$ for α and w_α as in the previous lemma. Since α and w_α are fixed, we omit them from the notation $R_{n,\zeta}$ and $f^b_{n,\zeta}$.

Proposition 9.4. — *The limit (47) is given by the integral*

$$I_{n,\zeta}(D, u) = \frac{1}{4!} \frac{1}{(2\pi i)^4} \oint \cdots \oint h_{n,\zeta,D}(\underline{z}; u) \frac{\prod_{1 \leqslant i < j \leqslant 4} (z_j - z_i)^2 (1 - z_i z_j)}{\prod_{i=1}^4 (1 - z_i)^8} \frac{dz_1}{z_1^4} \cdots \frac{dz_4}{z_4^4}$$

where

$$h_{n,\zeta,D}(\underline{z}; u) = \frac{R_{n,\zeta}(\underline{z}; u)}{\prod_{i=1}^4 z_i^{D/(2n)}} \cdot \begin{cases} f^e_{n,\zeta}(\underline{z}; u) \prod_{i=1}^4 \dfrac{1 - u}{1 - u/z_i} & \text{if } D \text{ even} \\ f^o_{n,\zeta}(\underline{z}; u) \prod_{i=1}^4 z_i^{1/2} & \text{if } D \text{ odd.} \end{cases}$$

Here each path of integration encloses the point $z_j = 1$, but not the points $z_j = 0, u$.

Proof. Let $\alpha \in \Phi_n$ and $w_\alpha \in W$ be as in the previous lemma. By taking $\beta = \alpha + \alpha_I \in \Phi_n$ in (47), with $w_\beta = w_\alpha w_I$, we have

$$f_{w_\alpha w_I}(\mathbf{x}) = f_{w_\alpha} \| w_I(\mathbf{x}) = \prod_{i \in I} \frac{1 - u/x_i}{1 - ux_i} f^e_{w_\alpha}(w_I \mathbf{x}) + \frac{1}{\mathbf{x}^{\alpha_I}} f^o_{w_\alpha}(w_I \mathbf{x}).$$

Assuming D is even (the case D odd being similar), we can then write the sum in (47) as

$$\prod_{i=1}^{4}(1 - u/x_i)x_i^{D/2} \sum_{I \subset S} h(w_I \mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha_I + k\alpha_S})^{-1/n}}$$

where $h(\mathbf{x}; u) = \frac{R(w_\alpha \mathbf{x}; u) f^e_{w_\alpha}(\mathbf{x}; u)}{\prod_{i=1}^4 (1 - u/x_i) x_i^{D/(2n)}}$. Now we use the following:

Fact 9.5. — *For any function $g(\mathbf{x})$ and a fixed index $1 \leqslant j \leqslant 4$, the transformation $x_j \mapsto 1/x_j$ takes*

$$g(\mathbf{x})|_{x_5 = c/x_j^a} \xrightarrow{x_j \mapsto 1/x_j} g(\sigma_j \mathbf{x})|_{x_5 = c/x_j^b}$$

where $a, b \in \mathbb{Q}$ with $a + b = 1$, and c is any function not depending on x_j and x_5.

Taking for $g(\mathbf{x})$ the function $h(\mathbf{x}; u)$ and applying [28, Lemma 7.1] (the case $m = 0$ of Lemma 9.9 below) yields an integral representation for the sum above. Now one can take $\underline{x} \to \underline{1}$ inside the integral, giving the above expression for $I_{n,\zeta}(D, u)$. □

From the integral representation, a standard argument (e.g., [28, Prop. 7.7]) leads to the following:

Corollary 9.6. — *When n is odd, $I_{n,\zeta}(D, u)$ is a polynomial in D of degree 10, with leading term given by*

$$D^{10} R_{n,\zeta}(\underline{1}; u) f^{a_D}_{n,\zeta}(\underline{1}; u) \frac{1}{4!(2n)^{10}} \frac{1}{(2\pi i)^4} \oint_{|t_4|=1} \cdots \oint_{|t_1|=1} \prod_{i=1}^{4} e^{-t_i} \cdot \frac{\prod_{1 \leqslant i < j \leqslant 4}(t_i - t_j)^2 (t_i + t_j)}{\prod_{i=1}^4 t_i^8} dt_1 \cdots dt_4.$$

The integral (with the factor $(2\pi i)^{-4}$ included) equals $8/1575$. When $n = 1$, following [36, Prop. 2.1], we can express the leading term of $Q_1(D, q)$ as

$$D^{10} R_1(q^{-1}) \frac{1}{2^r} \prod_{j=0}^{r-1} \frac{(2j)!}{(r+j)!} \qquad (r = 4).$$

It agrees with the analogous main term in the conjectural asymptotic formula of the fourth moment over the rationals (see [24], [21], or [36, Theorem 1.1]), apart from the "arithmetic factor", which in our case is $R_1(\varrho) = (\varrho; \varrho)_\infty^{-11}$.

9.1.2. The case n even

Let $n = 2k$ with $k \geqslant 1$, and let $t = \sigma_1 \cdots \sigma_5$ be as before, with $t\alpha_5 = \alpha_5 - \delta$. Then

$$\Phi_n = \{\pm\alpha_i + k\delta : i \in S\} \quad \text{and} \quad t^{k-1}\sigma_i t(-\alpha_i + k\delta) = t^{k-1}\sigma_i t \sigma_i(\alpha_i + k\delta) = \alpha_5.$$

Lemma 9.7. — *Let $n = 2k$ with $k \geqslant 1$, and let $\alpha = -\alpha_4 + k\delta \in \Phi_n$ and $w_\alpha = t^{k-1}\sigma_4 t$. Then we have a decomposition*

$$R(w_\alpha \mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} = R_{n,\zeta}(\underline{x}; u) \cdot \frac{1}{1 - x_4^2} \prod_{j=1}^{3} \frac{1}{(1 - x_4 x_j)(1 - x_4/x_j)}$$

with $R_{n,\zeta}(\underline{1}; u) = R_n(u^{-2/n}/\zeta^2)$ for an explicit function $R_n(\varrho)$. In addition, the function $R_n(\varrho)$ is given by an absolutely convergent power series for $|\varrho| < 1$; when $n = 2$, we have

$$R_2(\varrho) = (\varrho; \varrho)_\infty^{-8} (\varrho; \varrho^2)_\infty^{-6} (1 - 1/\varrho)^{-7}.$$

Proof. The assertion follows at once from the explicit formula of $R(\underline{x}; u)$ in Theorem 6.1. □

For $b \in \{\mathscr{e}, \mathscr{o}\}$, let $f_{n,\zeta}^b(\underline{x}; u) = f_{w_\alpha}^b(\mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}}$ for α and w_α as in the previous lemma.

Proposition 9.8. — *The limit $I_{n,\zeta}(D, u)$ is given by the integral*

$$\frac{1}{2^3 3!} \frac{1}{(2\pi i)^4} \oint \cdots \oint h_{n,\zeta,D}(\underline{z}; u) \frac{\prod_{1 \leqslant i < j \leqslant 4} (z_i - z_j)^{\mathrm{e}_{ij}} (1 - z_i z_j) \prod_{1 \leqslant k \leqslant l \leqslant 3} (1 - z_k z_l) \prod_{i=1}^3 z_i}{\prod_{i=1}^4 (1 - z_i)^8} \frac{dz_1}{z_1^4} \cdots \frac{dz_4}{z_4^4}$$

where $\mathrm{e}_{ij} = 1$ or 2 according as $j = 4$ or not, and

$$h_{n,\zeta,D}(\underline{z}; u) = \frac{R_{n,\zeta}(\underline{z}; u)}{z_4^{D/n}} \cdot \begin{cases} f_{n,\zeta}^{\mathscr{e}}(\underline{z}; u) \prod_{i=1}^4 \dfrac{1 - u}{1 - u/z_i} & \text{if } D \text{ even} \\ f_{n,\zeta}^{\mathscr{o}}(\underline{z}; u) \prod_{i=1}^4 z_i^{1/2} & \text{if } D \text{ odd.} \end{cases}$$

The paths of integration are as in Proposition 9.4.

Proof. Let $\alpha \in \Phi_n$ and $w_\alpha \in W$ be as in the previous lemma, and set $\alpha^+ = \sigma_4 \alpha \in \Phi_n$, and $w_{\alpha^+} = w_\alpha \sigma_4$. As before, assume that D is even, as the case D odd is entirely similar. We have $f_{w_{\alpha^+}} = f_{w_\alpha} \| \sigma_4$, and the two terms of the sum in (47) corresponding to α and α^+ can be written as

$$\prod_{i=1}^{4} (1 - u/x_i) x_i^{D/2} \cdot \left(h(\mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} + h(\sigma_4 \mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha^{+'}})^{-1/n}} \right)$$

where $h(\mathbf{x}; u) = \frac{R(w_\alpha \mathbf{x}; u) f_{w_\alpha}^{\mathscr{e}}(\mathbf{x}; u)}{x_4^{D/n} \prod_{i=1}^4 (1 - u/x_i)}$ and $\alpha^{+'}$ denotes the sum of the first four components of α^+. By Fact 9.5, the two terms inside the parenthesis are interchanged by the transformation $x_4 \mapsto 1/x_4$. Moreover, the function $h(\mathbf{x}; u)$ is clearly symmetric in the first three variables, and we claim that the first term in the parenthesis is also invariant under the transformations $x_j \mapsto 1/x_j$, for $j = 1, 2, 3$. Indeed, fixing one such j, by applying again Fact 9.5, we have that

$$h(\mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} \xrightarrow{x_j \mapsto 1/x_j} h(\sigma_j \mathbf{x}; u)|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}}.$$

One checks that $h(\sigma_j \mathbf{x}; u) = \frac{R(w_\alpha \sigma_j \mathbf{x}; u) f_{w_\alpha \sigma_j}^{\mathscr{e}}(\mathbf{x}; u)}{x_4^{D/n} \prod_{i=1}^4 (1 - u/x_i)}$. Since $w_\alpha \sigma_j \alpha = w_\alpha \alpha = \alpha_5$, our claim follows now by Lemma 6.2.

Using these symmetries and the lemma below with $m = 3$, we can express the sum in (47) as an integral. Our assertion follows now by taking $\underline{x} \to 1$ inside the integral. □

Lemma 9.9. — *Let $a_1, \ldots, a_r$ be distinct non-zero complex numbers such that $a_i a_j \neq 1$ for all $1 \leqslant i, j \leqslant r$. Suppose h is a function of r complex variables, holomorphic on a domain containing $\left(a_1^{\delta_1}, \ldots, a_r^{\delta_r}\right)$ for all $(\delta_1, \ldots, \delta_r) \in \{\pm 1\}^r$. For $0 \leqslant m < r$, define $K_m(\mathrm{z})$ by*

$$K_m(\mathrm{z}) = \frac{h(\mathrm{z})}{\prod_{k=1}^{m} \prod_{l=m+1}^{r} (1 - z_k z_l)(1 - z_k^{-1} z_l) \prod_{m+1 \leqslant k \leqslant l \leqslant r} (1 - z_k z_l)}, \quad \mathrm{z} := (z_1, \ldots, z_r).$$

Then we have

$$\sum_{\sigma \in \mathbb{S}_r} \sum_{\delta_{\sigma(i)} = \pm 1} K_m\left(a_{\sigma(1)}^{\delta_{\sigma(1)}}, \ldots, a_{\sigma(r)}^{\delta_{\sigma(r)}}\right)$$
$$= \frac{(-1)^{r(r+1)/2}}{\left(2\pi\sqrt{-1}\right)^r} \oint \cdots \oint h(\mathrm{z}) \cdot \frac{\prod_{1 \leqslant i < j \leqslant r} (z_i - z_j)^{\mathrm{e}_{ij}} (1 - z_i z_j) \cdot \prod_{1 \leqslant k \leqslant l \leqslant m} (1 - z_k z_l) \prod_{i=1}^{m} z_i^{r-m}}{\prod_{i,j=1}^{r} \left(1 - z_i a_j\right)\left(1 - z_i a_j^{-1}\right)} \frac{d\mathrm{z}}{\mathrm{z}^r}$$

where $\mathrm{e}_{ij} = 1$ or 2 according as $i \leqslant m$ and $j \geqslant m+1$ or not, $\mathrm{z}^r := z_1^r \cdots z_r^r$, and where each path of integration encloses the $a_j^{\pm 1}$, but not $z_j = 0$.

Proof. The same idea as in the proof of [28, Lemma 7.5]. □

The argument in [28, Prop. 7.7] gives the following:

Corollary 9.10. — *When n is even, $I_{n,\zeta}(D, u)$ is a polynomial of degree 7 in D with leading term*

$$\frac{D^7}{7! n^7} R_n(\zeta^{-2} u^{-2/n}) f_{n,\zeta}^{a_D}(\underline{1}; u).$$

Remark 9.2. We omitted providing formulas for the functions $R_n(\varrho)$ in Lemmas 9.3 and 9.7, when $n \geqslant 3$. Instead, we will show in the next subsection that the leading term of $Q_n(D, u^2)$ can be solely expressed in terms of $R_1(\varrho)$ or $R_2(\varrho)$ (according as n is odd or even), and of the matrix $B(\mathbf{x}; u)$ in Theorem 4.3.

9.1.3. The leading term of $Q_n(D, q)$

The leading term of $Q_n(D, q)$, as a polynomial in D, is

$$\frac{D^{10}}{2^4 n^{10}} \prod_{j=0}^{3} \frac{(2j)!}{(4+j)!} S_n(D, \sqrt{q}) \quad \text{or} \quad \frac{D^7}{7! n^7} S_n(D, \sqrt{q})$$

according as n is odd or even. Here we set

$$S_n(D, u) := \frac{1}{n} \sum_{\zeta \in \mu_{2n}/\{\pm 1\}} \zeta^D R_n(\zeta^{-2} u^{-2/n}) f_{w_\alpha}^{a_D}(\underline{1}, \zeta^{-1} u^{-1/n}; u)$$

where $\alpha \in \Phi_n$, $w_\alpha \in W$, and the functions $R_n(\varrho)$ are those occurring in Lemmas 9.3 and 9.7. Recall that $a_D \in \{e, o\}$ stands for the parity of D, and that $f^e = f^{e,e} + f^{o,e}$, $f^o = f^{e,o}$ for $f \in \mathbb{C}(\mathbf{x}, u)_0$.

In what follows, we show that the functions $S_n(D, \sqrt{q})$ are non-zero for all $n \geqslant 1$ and all D, so that all $Q_n(D, q)$ are present in the asymptotic formula in Theorem 9.2. In fact, we prove the stronger result that

$S_n(D, \sqrt{q})$ is given by a power series in $q^{-1/2n}$, whose coefficients are either all positive or all negative, depending on the residue of n modulo 4.

We start by giving an expression for $S_n(D,u)$ involving only the term for $\zeta = 1$ in the summation. To do so, we need a property of the function $f_w = (1+ux_5)\|w$.

Lemma 9.11. — *For all $w \in W$, the functions $f_w^{e,e}$, $f_w^{e,o}$ and $f_w^{o,e}$ are even, odd and odd, respectively, as functions of u.*

Proof. We have

$$\overline{f_w}(\mathbf{x};u) = \tilde{\Lambda}_w(\mathbf{x};u) \cdot {}^t(1, u\mathbf{x}^{w^{-1}\alpha_5}, 0). \tag{48}$$

As functions of u, the entries of $\tilde{\Lambda}_w(\mathbf{x};u)$ have the same parities as the corresponding entries of the matrix $\left(\begin{smallmatrix} 1 & u & u \\ u & 1 & 1 \\ u & 1 & 1 \end{smallmatrix}\right)$. This follows easily by induction on the length $\ell(w)$ of w; the case $\ell(w)=1$ is clear from the formulas of Λ_1 and Λ_2 in Section 4.1, and the product of such matrices is again of the same type, so the cocycle relation gives the induction step. Our assertions now follow from (48). $\square$

For $c \in \mathbb{Z}$, let $\mathscr{U}_{n,c}$ be the operator acting on Laurent series $f(\varrho) = \sum_{k\in\mathbb{Z}} n_k \varrho^k$ with $n_k \in \mathbb{C}$, defined by

$$\mathscr{U}_{n,c}(f) := \sum_{k \equiv c \ (\mathrm{mod}\ n)} n_k \varrho^k = \frac{1}{n}\sum_{\zeta\in\mu_n} \zeta^c f(\zeta^{-1}\varrho).$$

We will apply this operator to Laurent series f which converge absolutely for $0 < |\varrho| < 1$, when it is well-defined and the equality above clearly holds. Note that $\mathscr{U}_{n,c}$ depends only on c modulo n.

Proposition 9.12. — *With α, w_α and $R_n(\varrho)$ from Lemma 9.3 (resp. Lemma 9.7) for n odd (resp. n even), we have:*

$$S_n(D,u) = \mathscr{U}_{n,\,D/2}\Big[R_n(u^{-2/n}) \cdot f_{w_\alpha}^{e,e}(\underline{1}, u^{-1/n}; u)\Big] + u\mathscr{U}_{n,\,D/2}\Big[R_n(u^{-2/n}) \cdot u^{-1} f_{w_\alpha}^{o,e}(\underline{1}, u^{-1/n}; u)\Big]$$

if D is even, and

$$S_n(D,u) = u^{1-1/n}\,\mathscr{U}_{n,\lfloor D/2\rfloor}\Big[R_n(u^{-2/n}) \cdot u^{-1+1/n} f_{w_\alpha}^{e,o}(\underline{1}, u^{-1/n}; u)\Big]$$

if D is odd. Here we view the functions between brackets as Laurent series in $\varrho = u^{-2/n}$, by the previous lemma.

Proof. By Lemma 9.11, we have that

$$f_{w_\alpha}^{e,e}(\underline{1}, x_5; u) = g_1(x_5^2, u^2), \quad f_{w_\alpha}^{e,o}(\underline{1}, x_5; u) = ux_5 g_2(x_5^2, u^2) \quad \text{and} \quad f_{w_\alpha}^{o,e}(\underline{1}, x_5; u) = ug_3(x_5^2, u^2)$$

with $g_i \in \mathbb{Q}(x,u)$. Thus, assuming D to be even (the case D odd being similar), we have the following expression for $S_n(D,u)$:

$$\begin{aligned} S_n(D,u) = S_n(D, \varrho^{-n/2}) &= \frac{1}{n}\sum_{\zeta\in\mu_{2n}/\{\pm1\}} \zeta^D R_n(\varrho/\zeta^2)\big(g_1(\varrho/\zeta^2, \varrho^{-n}) + ug_3(\varrho/\zeta^2, \varrho^{-n})\big) \\ &= \mathscr{U}_{n,\,D/2}[R_n(\varrho) g_1(\varrho, \varrho^{-n})] + u\mathscr{U}_{n,\,D/2}[R_n(\varrho) g_3(\varrho, \varrho^{-n})]. \end{aligned}$$

Note that the functions between brackets converge absolutely as Laurent series in ϱ for $0 < |\varrho| < 1$, by Lemmas 9.3 and 9.7, so the last equality holds. Switching back to the variable u completes the proof. $\square$

This proposition yields explicit formulas for the leading term of $Q_n(D,q)$, which, for small n, can be used to check that this term is given by a non-zero power series with coefficients of the same sign. To prove this property for all n, in Corollary 9.15 below, we shall give different expressions for the functions between brackets in the proposition, using the extra functional equation introduced in Section 4.2.

Recall the extension of the cocycle $\tilde{\Lambda}$ to $W \oplus \mathbb{Z}$ from the beginning of Section 6.3. In particular, we have

$$\tilde{\Lambda}_{\tau^{-1}}(\mathbf{x};u) = \tilde{B}(\mathbf{x};u) = B(\mathbf{x};u)/(1-u^2x^\delta)^2$$

and the vector function $\tilde{\mathbf{Z}} = \overline{\tilde{Z}_W}$ satisfies the functional equation (25), with the cocycle $\tilde{\Lambda}$ in place of Λ. Consider the vector version of the residue in Lemma 6.2:

$$\mathbf{C}_{\alpha,\zeta}(\underline{x};u) := \lim_{x_5 \to \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} \left(1 - \zeta u^{1/n}\mathbf{x}^{\alpha/n}\right)\tilde{\mathbf{Z}}(\mathbf{x};u)$$

for any $\alpha = n\alpha_5 + \alpha' \in \Phi_n$ and $\zeta \in \mu_{2n}$. The residue vector $\mathbf{C}_{\alpha,\zeta}(\underline{x};u)$ appears in the proof of Lemma 6.2, where it is shown that

$$\mathbf{C}_{\alpha,\zeta}(\underline{x};u) = \frac{1}{2n}\left\{R(w_\alpha \underline{x};u)\overline{f_{w_\alpha}}(\mathbf{x};u)\right\}\Big|_{x_5=\zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} \tag{49}$$

with $w_\alpha \in W$ such that $w_\alpha \alpha = \alpha_5$.

Assuming now that α is chosen as in Lemmas 9.3 and 9.7, we give a formula for $\mathbf{C}_{\alpha,\zeta}(\underline{x};u)$ in terms of the above extended cocycle.

Lemma 9.13. — *a). If $n = 2k+1$ and $\alpha = \alpha_5 + k\delta \in \Phi_n$, then*

$$\mathbf{C}_{\alpha,\zeta}(\underline{x};u) = \frac{1}{2n}\left\{R(\underline{x};u\mathbf{x}^{k\delta})\tilde{\Lambda}_{\tau^{-k}}(\mathbf{x};u)\cdot v_0\right\}\Big|_{x_5=\zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}}$$

where $v_0 = {}^t(1,1,0)$.

b). If $n = 2k+2$ and $\alpha = -\alpha_4 + (k+1)\delta \in \Phi_n$, then

$$\mathbf{C}_{\alpha,\zeta}(\underline{x};u) = \frac{1}{2n}\left\{R(\sigma_4 t\underline{x};u\mathbf{x}^{k\delta})\tilde{\Lambda}_{\tau^{-k}}(\mathbf{x};u)\cdot\overline{f_{\sigma_4 t}}(\mathbf{x};u\mathbf{x}^{k\delta})\right\}\Big|_{x_5=\zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}}$$

where $t = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$.

Proof. We use the functional equation (25), written for $\tilde{\mathbf{Z}}$ and the cocycle $\tilde{\Lambda}$, to compute the residue vector $\mathbf{C}_{\alpha,\zeta}(\underline{x};u)$. In part a), we take the element τ^{-k} in (25), while in part b) we take the element $\sigma_4 t\tau^{-k}$. Note that by Theorem 4.3 and the cocycle relation, the matrix $\tilde{\Lambda}_{\tau^{-k}}(\mathbf{x};u)$ has poles only if $u\mathbf{x}^\beta = \pm 1$ for some $\beta \in \Phi^+_{\mathrm{re}}$ with $\beta < k\delta$. Since $\alpha > k\delta$, it can be evaluated at $x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}$, that is, when $u\mathbf{x}^\alpha = \pm 1$. □

The decompositions in Lemmas 9.3 and 9.7 have the following analogues:

Lemma 9.14. — *a). For $n = 2k+1 \geqslant 1$ and $\alpha = \alpha_5 + k\delta \in \Phi_n$, we have*

$$R(\underline{x};u\mathbf{x}^{k\delta})\big|_{x_5=\zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} = R_{1,1}\left(\underline{x};u\mathbf{x}^{k\delta}\big|_{x_5=\zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}}\right)\cdot\prod_{1\leqslant i\leqslant j\leqslant 4}\frac{1}{1-x_ix_j}$$

where the first factor in the right-hand side becomes $R_1(u^{-2/n}/\zeta^2)$ *when evaluated at* $\underline{x} = \underline{1}$. *Here* $R_{1,1}(\underline{x};u)$ *and* $R_1(\varrho)$ *are the same functions as in Lemma 9.3.*

b). For $n = 2k+2 \geqslant 2$ *and* $\alpha = -\alpha_4 + (k+1)\delta \in \Phi_n$, *we have*

$$R(\sigma_4 t\underline{x}; u\mathbf{x}^{k\delta})|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} = R_{2,1}\Big(\underline{x}; u\mathbf{x}^{k\delta}|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}}\Big) \cdot \frac{1}{1-x_4^2}\prod_{j=1}^{3}\frac{1}{(1-x_4x_j)(1-x_4/x_j)}$$

where the first factor in the right-hand side becomes $R_2(u^{-2/n}/\zeta^2)$ *when evaluated at* $\underline{x} = \underline{1}$. *Here* $R_{2,1}(\underline{x};u)$ *and* $R_2(\varrho)$ *are the same functions as in Lemma 9.7.*

Proof. The assertions follow at once from the cases $n=1$, $\zeta = 1$ and $n=2$, $\zeta=1$ of the two aforementioned lemmas, after making the substitution

$$u \mapsto u\mathbf{x}^{k\delta}|_{x_5 = \zeta^{-1}(u\mathbf{x}^{\alpha'})^{-1/n}} = \begin{cases} \zeta^{1-n}(u\mathbf{x}^{\alpha'})^{1/n} & \text{if } n \text{ odd} \\ \zeta^{2-n}x_4^{1-2/n}u^{2/n} & \text{if } n \text{ even.}\end{cases}$$

Note that $R(\underline{x};u)$ is even in u, and so the factors $\zeta^n = \pm 1$ can be ignored in the above substitution. □

We now give simpler formulas for the functions inside brackets in Proposition 9.12, in which the dependence upon n appears only in the cocycle matrix $\tilde{\Lambda}_{\tau^{-\lfloor (n-1)/2\rfloor}}$. We express the arguments in terms of $\varrho = u^{-2/n}$, choosing the square root $\varrho^{1/2} := u^{-1/n}$.

Corollary 9.15. — *Let* $\alpha \in \Phi_n$, $w_\alpha \in W$ *and* $R_n(\varrho)$ *be as in Lemma 9.3, for* n *odd, or as in Lemma 9.7, for* n *even. Then:*

a). If $n = 2k+1$, *we have*

$$R_n(\varrho)\overline{f_{w_\alpha}}(\underline{1}, \varrho^{1/2}; \varrho^{-n/2}) = R_1(\varrho)\tilde{\Lambda}_{\tau^{-k}}(\underline{1}, \varrho^{1/2}; \varrho^{-n/2}) \cdot v_0$$

where we recall that $R_1(\varrho) = (\varrho;\varrho)_\infty^{-11}$.

b). If $n = 2k+2$, *we have*

$$R_n(\varrho)\overline{f_{w_\alpha}}(\underline{1}, \varrho^{1/2}; \varrho^{-n/2}) = R_2(\varrho)\tilde{\Lambda}_{\tau^{-k}}(\underline{1}, \varrho^{1/2}; \varrho^{-n/2}) \cdot \overline{f_{\sigma_4 t}}(\underline{1}, \varrho^{1/2}; \varrho^{-1})$$

where we recall that $R_2(\varrho) = (\varrho;\varrho)_\infty^{-8}(\varrho;\varrho^2)_\infty^{-6}(1-1/\varrho)^{-7}$. *We also have*

$$\overline{f_{\sigma_4 t}}(\underline{1}, \varrho^{1/2}; \varrho^{-1}) = {}^t\Big(\tfrac{\varrho^4+7\varrho^3+13\varrho^2+7\varrho+1}{\varrho^4}, \tfrac{\varrho^3+7\varrho^2+7\varrho+1}{\varrho^{7/2}}, \tfrac{3\varrho^2+7\varrho+3}{\varrho^3}\Big).$$

Proof. The identities follow by expressing $\mathbf{C}_{\alpha,1}(\underline{x};u)$ in two ways, using (49) and Lemma 9.13. After cancelling the singular parts, one can then evaluate at $\underline{x} = \underline{1}$ by using Lemma 9.3 (resp. Lemma 9.7) for n odd (resp. n even), and Lemma 9.14. □

We are now ready to show the following:

Proposition 9.16. — *For* $n \geqslant 1$, *put* $\varrho = q^{-1/n} < 1$. *Then the leading coefficient of* $Q_n(D,q)$ *is*

$$(-1)^{\lfloor n/2\rfloor}\varrho^{3\lfloor n/2\rfloor\cdot\lfloor (n+1)/2\rfloor} \cdot g_{n,D}(\sqrt{\varrho})$$

where $g_{n,D} \neq 0$ *is a power series with non-negative coefficients depending only upon* $\lfloor D/2\rfloor$ *modulo* n.

Proof. As before, put $k = \lfloor (n-1)/2 \rfloor$. When $k = 0$ (i.e., $n = 1, 2$), our assertion can be verified directly from Proposition 9.12 and Corollary 9.15; the cocycle $\tilde{\Lambda}_{\tau^{-k}}$ is the identity matrix in this case. If $k > 0$, we first observe that the entries of the matrix $\tilde{\Lambda}_{\tau^{-k}}(\underline{1}, \varrho^{1/2}; \varrho^{-n/2})$ in Corollary 9.15 are of the form $(-1)^k \varrho^{3k(n-k)}$ times non-zero power series in $\varrho^{1/2}$ with non-negative coefficients. Indeed, by the cocycle relation, we can write

$$\tilde{\Lambda}_{\tau^{-k}}(\underline{1}, x_5; 1/x_5^n) = \tilde{B}(\underline{1}, x_5; 1/x_5^n)\tilde{B}(\underline{1}, x_5; 1/x_5^{n-2}) \cdot \ldots \cdot \tilde{B}(\underline{1}, x_5; 1/x_5^{n-2k+2})$$

and note that $n - 2k + 2 \geqslant 3$. Now, for $i \geqslant 3$, Lemma A.1 gives the functional equation

$$\tilde{B}(\underline{1}, x_5; 1/x_5^i) = -x_5^{6(i-1)} \cdot {}^t\tilde{B}(\underline{1}, x_5; x_5^{i-2}). \tag{50}$$

By Theorem 4.3, the function $\tilde{B}(\underline{1}, x_5; u)$ has poles only if $ux_5^j = \pm 1$ ($j = 0, 1, 2$), and so both sides of (50) are defined. Furthermore, by Proposition A.2, the entries of the matrix in the right-hand side of (50) are non-zero power series in x_5 with positive coefficients, and thus the assertion about $\tilde{\Lambda}_{\tau^{-k}}(\underline{1}, \varrho^{1/2}; \varrho^{-n/2})$ follows at once by taking $x_5 = \varrho^{1/2}$ and summing the exponents of ϱ coming from (50).

The remaining factors in the right-hand sides of the formulas in Corollary 9.15 are also power series with non-negative coefficients, except for the factor

$$(1 - 1/\varrho)^{-7} = -\varrho^7 \left(1 + \varrho + \varrho^2 + \varrho^3 + \cdots\right)^7$$

in the formula of $R_2(\varrho)$. Thus, when n is even, the product $-\varrho^7 \overline{f_{\sigma_4 t}}(\underline{1}, \varrho^{1/2}; \varrho^{-1})$ gives an additional factor of $(-\varrho)^3$. Finally, to see that the series obtained after applying Proposition 9.12 is not identically zero, one just needs to notice that the power series of $R_1(\varrho)$ and $R_2(\varrho)$ contain powers ϱ^j with non-zero coefficients for all residue classes j modulo n (coming for example from the expansion of $(1-\varrho)^{-1}$). This completes the proof. □

Remark 9.3. The estimate of the power of ϱ dividing the leading term in the previous proposition is essentially optimal. To see this, let us assume that D is even and that $D/2 \equiv 3\lfloor n/2 \rfloor \cdot \lfloor (n+1)/2 \rfloor \pmod{n}$. Then the leading coefficient of the polynomial $Q_n(D, q)$, expanded as a power series in $\varrho^{1/2} = q^{-1/2n}$, is asymptotically

$$(-1)^{\lfloor n/2 \rfloor} \varrho^{3\lfloor n/2 \rfloor \cdot \lfloor (n+1)/2 \rfloor}(1 + O(\varrho)) \cdot \begin{cases} \frac{1}{2^4 n^{10}} \prod_{j=0}^{3} \frac{(2j)!}{(4+j)!} & \text{if } n \text{ odd} \\ \frac{1}{7! n^7} & \text{if } n \text{ even.} \end{cases}$$

Indeed, by Lemma 5.3, one has $\tilde{B}(\mathbf{x}; 0) = \tilde{B}_0$, and the formula follows at once from the argument of the proof of the previous proposition, combined with Proposition 9.12 and the formulas at the beginning of this subsection.

A. Properties of the matrix $B(\mathbf{x}; u)$

We collect here two properties of the matrix $B(\mathbf{x}; u)$ in Theorem 4.3 that were needed in the proofs of Corollary 5.5 and Proposition 9.16.

From the explicit formula of the matrix $B(\mathbf{x}; u) = A^{-1}(\mathbf{x}; u\mathbf{x}^\delta)$, with A given in [1], one verifies the following functional equation.

Lemma A.1. — *The matrix* $\tilde{B}(\mathbf{x};u) = B(\mathbf{x};u)/(1-u^2x^\delta)^2$ *satisfies*

$$ {}^t\tilde{B}\big(\mathbf{x};1/u\mathbf{x}^\delta\big) = -u^6\mathbf{x}^{3\delta}\tilde{B}(\mathbf{x};u). $$

The main result of this appendix is the following:

Proposition A.2. — *Each entry of* $\tilde{B}(\underline{1},x_5;u)$ *is of the form*

$$ \sum_{\substack{a,b,c\geqslant 0\\ a+b+c=11}} \frac{p_{a,b,c}(x_5,u)}{(1-u^2)^a(1-u^2x_5^2)^b(1-u^2x_5^4)^c} \neq 0 $$

where $p_{a,b,c}(x,y)$ *are polynomials with non-negative coefficients.*

Let $\varrho = x_5^2$, $q = u^2$, and set $r = \varrho q$. We have:

$$ \tilde{B}(\underline{1},x_5;u) = \begin{pmatrix} \frac{b_{11}}{(q-1)^4(r-1)^{11}(\varrho r-1)^4} & \frac{\sqrt{r}\,b_{12}}{(q-1)^4(r-1)^{11}} & \frac{\sqrt{q}\,b_{13}}{(q-1)^4(r-1)^{10}(\varrho r-1)^4} \\ \frac{\sqrt{r}\,b_{21}}{(\varrho r-1)^4(r-1)^{11}} & \frac{b_{22}}{(r-1)^{11}} & \frac{\sqrt{\varrho}\,b_{23}}{(\varrho r-1)^4(r-1)^{10}} \\ \frac{\sqrt{q}\,b_{31}}{(q-1)^4(r-1)^{10}(\varrho r-1)^4} & \frac{\sqrt{\varrho}\,b_{32}}{(q-1)^4(r-1)^{10}} & \frac{b_{33}}{(q-1)^4(r-1)^9(\varrho r-1)^4} \end{pmatrix} $$

where the numerators $b_{ij} = b_{ij}(\varrho,q)$ are explicit non-zero polynomials, given in [1]. The proposition follows immediately from the following decompositions of b_{ij}.

Lemma A.3. — *We have* $b_{22}(\varrho,q) = p_{22}(r)$*, where the polynomial* p_{22} *has negative coefficients, and the other numerators decompose as follows*

$$ b_{11} = \vec{a}_8\cdot\vec{p}_{11}, \quad b_{12} = \vec{e}_4\cdot\vec{p}_{12}, \quad b_{21} = \vec{f}_4\cdot\vec{p}_{21}, $$

$$ b_{13} = (q-1)\varrho\vec{a}_6\cdot\vec{p}_{13} + (r-1)^3\vec{a}_4\cdot\vec{q}_{13}, \quad b_{31} = (\varrho r-1)\vec{a}_6\cdot\vec{p}_{13} + (r-1)^3\vec{a}_4\cdot\vec{q}_{13}, $$

$$ b_{23} = \vec{f}_3\cdot\vec{p}_{23}, \quad b_{32} = \vec{e}_3\cdot\vec{p}_{32}, \quad b_{33} = \vec{a}_6\cdot\vec{p}_{33} $$

in terms of the vectors $\vec{a}_{2k} = [(r-1)^{2k-2i}(q-1)^i(\varrho r-1)^i]_{i=0,\ldots,k}$,

$$ \vec{e}_k = \big[(r-1)^{k-i}(q-1)^i\big]_{i=0,\ldots,k}, \quad \vec{f}_k = \big[(r-1)^{k-i}(\varrho r-1)^i\big]_{i=0,\ldots,k} $$

where $\vec{p}_{ij}$ *and* $\vec{q}_{13}$ *denote vectors of non-zero polynomials with negative coefficients of the same size as the first vector in the scalar products.*

Proof. The decompositions can be verified using the explicit formulas of the vectors $\vec{p}_{ij}$ and $\vec{q}_{13}$ in [1]. By Lemma A.1, the polynomials $b_{ji}(\varrho,q)$ can be expressed in terms of $b_{ij}(\varrho,1/q\varrho^2)$, and so we only have to consider half of the off-diagonal entries of $\tilde{B}$. We illustrate how one arrives at these decompositions by considering an example.

For the entry b_{33}, we notice that $b_{33}(\varrho,1/\varrho^n) = (\varrho-1)^6 g_n(\varrho)$ for $n = 0,1,2,\ldots$, where g_n are Laurent polynomials with negative coefficients. Therefore we look for a decomposition involving the factors in the denominator of the $(3,3)$ entry of the form:

$$ b_{33}(\varrho,q) = \sum_{i=0}^{3}(r-1)^{6-2i}(q-1)^i(\varrho r-1)^i h_i(r) $$

with $h_i(r)$ polynomials in $r = \varrho q$. (This assumption on h_i and the type of decomposition is suggested by the symmetry of the diagonal entries under $q \mapsto 1/q\varrho^2$.) Under this assumption, the polynomials h_i can be determined recursively, by letting $h_0(r) = b_{33}(r,1)/(r-1)^6$, and repeating the same procedure with $b_{33}(\varrho, q)$ replaced by

$$\frac{b_{33}(\varrho,q) - b_{33}(r,1)}{(q-1)(\varrho r - 1)}$$

etc. All the coefficients of the polynomials $h_0(r)$ and $h_3(r)$ thus found are already negative, and only two of the coefficients of $h_1(r)$ and $h_2(r)$ are positive. To eliminate the latter, we replace $h_1(r)$ and $h_2(r)$ in the above decomposition by

$$h_2'(r) = h_2(r) - (r-1)^2(r+r^7), \quad h_1'(\varrho,q) = h_1(r) + (r+r^7)(q-1)(\varrho r - 1)$$

so that $h_2'(r)$ has now all coefficients negative. In the same way, we replace $h_1'(\varrho,q)$ and $h_0(r)$ by $h_1''(\varrho,q)$ and $h_0'(\varrho,q)$, respectively, so that $h_1''(\varrho,q)$ has only negative coefficients, and find that $h_0'(\varrho,q)$ has only negative coefficients as well.

The same method can be used to decompose the entries b_{11}, b_{12} and b_{32} in terms of the entries of the corresponding vector $\vec{a}_{2k}$ or $\vec{e}_k$, as in the statement of the lemma. For b_{13}, we decompose both

$$(b_{13}(\varrho,q) \pm b_{31}(\varrho,q))/(\varrho \pm 1)$$

in terms of the entries of $\vec{a}_6$, with coefficients that are again polynomials of r alone. We recover $b_{13}(\varrho,q)$ as a linear combination of these decompositions. □

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