THE HASSE NORM PRINCIPLE FOR ABELIAN EXTENSIONS – CORRIGENDUM

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Abstract. The proofs of Theorem 1.1 and Theorem 1.5(2) in the authors' paper *The Hasse norm principle for abelian extensions* are incorrect. We point out the mistakes and provide correct proofs, using techniques of the original paper.

Contents

1. Introduction

1.1. **The mistakes.** This is a corrigendum to the paper [3]. The paper contains the following independent mistakes:

1. The proof of Theorem 5.1 has a gap in the dominated convergence argument, since the last part of Lemma 5.3 is false.
2. In Lemma 6.7 the implication '$\implies$' is false. (A corrected statement is Lemma C6 in this document.)
3. The final statement in Lemma 6.9 is false. Therefore Lemma 6.12 is false.
4. Typo: sign error in equation (4.15). The Euler factor should start $1 + (Q^\beta - 1)q_v^{-1}$ in the case $q \equiv 1 \pmod{Q}$.

The typo (4) is inconsequential but mistakes (1)–(3) have the following consequences for the paper:

- The proof of Theorem 1.1 is incomplete (it uses Theorem 5.1 and Lemma 6.12).
- The proof of Theorem 1.5(2) is incomplete (it uses Lemma 6.7).
- The proof of Theorem 5.2(2) is incomplete (it uses Theorem 5.1).

Example C1. Here is an explicit counter-example to Lemma 6.7 and Lemma 6.9. Consider the number field $K/\mathbb{Q}$ given in [6, Number Field 8.0.10070523904.2]. This is a $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ extension of $\mathbb{Q}$ whose only non-cyclic decomposition group occurs at 7 and is $(\mathbb{Z}/2\mathbb{Z})^2$. However, one checks that the map

$$\mathbb{Z}/2\mathbb{Z} = \mathrm{H}^3(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{H}^3((\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

is the zero map, which implies that weak approximation holds. We are grateful to André Macedo for providing this illustrative example.

We remark that none of these issues affect the subsequent paper [4] which involved only the Hasse norm principle (no weak approximation) and counting by conductor, which is substantially easier than counting by discriminant with the harmonic analysis approach.

1.2. **The fix.** We focus on providing complete correct proofs for the main results (Theorems 1.1 and 1.5(2)) from the introduction of [3]. Their statements are as follows. Let k be a number field.

Theorem C2 ([3, Theorem 1.1]). *Let $n, r \in \mathbb{Z}$ with $n > 1$ and $r \geq 0$. Let Q be the smallest prime dividing n and let $G = \mathbb{Z}/n\mathbb{Z} \oplus (\mathbb{Z}/Q\mathbb{Z})^r$. Then 0% of G-extensions of k fail the Hasse norm principle, when ordered by discriminant.*

Theorem C3 ([3, Theorem 1.5(2)]). *Let G be a non-trivial finite abelian group, let Q be the smallest prime dividing $|G|$ and suppose that the Q-Sylow subgroup of G is not cyclic. Then as φ varies over all G-extensions of k, ordered by discriminant, 0% of the tori $\mathrm{R}^1_{K_\varphi/k}\mathbb{G}_\mathrm{m}$ satisfy weak approximation.*

Proving these results turns out to be actually quite delicate; the dominated convergence argument used in the proof of [3, Theorem 5.1] is fatally flawed and cannot be rescued. Instead, we prove Theorems C2 and C3 by exhibiting explicit cancellation between the poles of different Dirichlet series which arise through Möbius inversion. Koymans and Rome have recently found alternative proofs of [3, Theorem 1.1], see [8] (for k an arbitrary number field), and [3, Theorem 1.5], see [7] (only for $k = \mathbb{Q}$).

We now state the result which will imply both theorems. Let k be a number field and let Ω_k denote the set of all places of k. We fix embeddings $\bar{k} \subset \bar{k}_v$ for all $v \in \Omega_k$. For any non-archimedean place v, let $I_v \subset \mathrm{Gal}(\bar{k}_v/k_v)$ be the inertia subgroup. Then the coset $\mathrm{Frob}_v I_v$ is independent of the choice of Frobenius element $\mathrm{Frob}_v \in \mathrm{Gal}(\bar{k}_v/k_v)$.

We will write our groups multiplicatively. For $n \in \mathbb{Z}$, we denote the group of n-th roots of unity in $\mathbb{C}$ by μ_n. Let G be a finite abelian group and Q the smallest prime dividing the order of G. For cyclic G, Theorem C2 follows from the Hasse norm theorem. Therefore, we assume throughout this corrigendum that the Q-Sylow subgroup of G is not cyclic; this means that we may write $G = M \times \mu_{Q^{a_1}} \times \cdots \times \mu_{Q^{a_t}}$ where $Q \nmid |M|$ and $t \geq 2$. Let $e_1, \dots, e_t$ generate $\mu_{Q^{a_1}}, \dots, \mu_{Q^{a_t}}$, respectively.

As in [3], by a *sub-G-extension* of a field F with separable closure $\bar{F}$, we mean a continuous homomorphism $\varphi : \mathrm{Gal}(\bar{F}/F) \to G$, and we call a surjective sub-G-extension of F a *G-extension* of F. Hence, a sub-G-extension φ of k induces a sub-G-extension φ_v of k_v at every place v.

Theorem C4. *Let S be a finite set of places of k containing all the archimedean places. Let $i, j \in \{1, \dots, t\}$ be distinct. For $v \notin S$, define Λ_v to be the set of all sub-G-extensions φ_v of k_v for which the following implication holds:*

$$e_i^{Q^{a_i-1}} \in \varphi_v(I_v) \implies \varphi_v(\mathrm{Frob}_v I_v) \subseteq \langle M, e_1, e_2, \dots, e_{j-1}, e_j^Q, e_{j+1}, \dots, e_t \rangle.$$

Then only 0% of all G-extensions φ of k satisfy $\varphi_v \in \Lambda_v$ for all $v \in \Omega_k \setminus S$, when ordered by discriminant.

We use this to prove our results as follows.

1.3. **Proof of Theorem C2.** This theorem concerns groups of the form $G = M \times \mu_Q^{t-1} \times \mu_{Q^a}$, where $t \geq 2$, $a \geq 1$, Q is a prime, and M is a cyclic group such that all prime divisors of $|M|$ are larger than Q. We need to show that only 0% of G-extensions of bounded discriminant fail the Hasse norm principle. We use the notation of Theorem C4.

If a G-extension φ of k fails the Hasse norm principle, then by [4, Lemma 4.2] there is a maximal proper subgroup Υ of $\wedge^2 G$ such that, for all places v of k, the image of the natural map $\wedge^2(\operatorname{Im}\varphi_v) \to \wedge^2 G$ is contained in Υ. Hence, it is enough to show that, for every such Υ, this occurs for 0% of all G-extensions of k.

Henceforth, for a subgroup H of G, we abuse notation by writing $\wedge^2 H \subseteq \Upsilon$ to mean that the image of the natural map $\wedge^2 H \to \wedge^2 G$ is contained in Υ.

Lemma C5. *Let $\Upsilon \subset \wedge^2 G$ be a maximal proper subgroup. Then there exist $e_1, \dots, e_t \in G$ such that e_t has order Q^a, $e_1, \dots, e_{t-1}$ have order Q and $e_1, \dots, e_t$ generate G/M, and indices $i, j \in \{1, \dots, t\}$ with $i \leq t-1$ and the following property: for every finite place v of k and G-extension φ of k such that $\wedge^2(\operatorname{Im}\varphi_v) \subseteq \Upsilon$, we have*

$$e_i \in \varphi_v(I_v) \implies \varphi_v(\operatorname{Frob}_v I_v) \subseteq \langle M, e_1, \dots, e_{j-1}, e_j^Q, e_{j+1}, \dots, e_t \rangle. \tag{1.1}$$

Proof. We split the proof into two cases. First assume that $\wedge^2(\mu_Q^{t-1}) \not\subseteq \Upsilon$, so that there are order Q elements $\epsilon_i, \epsilon_j \in \mu_Q^{t-1}$ such that $\epsilon_i \wedge \epsilon_j \notin \Upsilon$. Define $L_i := \{g \in G \mid \epsilon_i \wedge g \in \Upsilon\}$. Then, if $\epsilon_i \in \varphi_v(I_v)$, the assumption $\wedge^2(\operatorname{Im}\varphi_v) \subseteq \Upsilon$ implies $\varphi_v(\operatorname{Frob}_v I_v) \subseteq L_i$. We will now find generators of G for which L_i becomes the set on the right-hand side of (1.1).

Note that $M \subseteq L_i$ since $\epsilon_i \wedge M = 1$. Furthermore, L_i is a proper subgroup of G since $\epsilon_j \notin L_i$. By maximality of Υ, we have $|(\wedge^2 G)/\Upsilon| = Q$ and hence $|(\epsilon_i \wedge G)/(\epsilon_i \wedge L_i)| \mid Q$, as $\epsilon_i \wedge L_i = \Upsilon \cap (\epsilon_i \wedge G)$. Since $\epsilon_i \wedge \epsilon_j \notin \Upsilon$, we have $\epsilon_i \wedge L_i \neq \epsilon_i \wedge G$ and hence $|(\epsilon_i \wedge G)/(\epsilon_i \wedge L_i)| = Q$. In the case $a \geq 2$, suppose for contradiction that $L_i \subseteq M \times G[Q^{a-1}]$. Then $\epsilon_i \wedge L_i \subseteq \epsilon_i \wedge G[Q^{a-1}]$ and since

$$|(\epsilon_i \wedge G)/(\epsilon_i \wedge L_i)| = Q = |(\epsilon_i \wedge G)/(\epsilon_i \wedge G[Q^{a-1}])|,$$

it follows that $\epsilon_i \wedge L_i = \epsilon_i \wedge G[Q^{a-1}]$. This contradicts the fact that $\epsilon_i \wedge \epsilon_j \notin \Upsilon$. Therefore, L_i contains an element of order Q^a. Therefore, both when $a \geq 2$ and when $a = 1$, we deduce that $L_i \cong M \times \mu_Q^s \times \mu_{Q^a}$ for some $s \leq t-2$, because L_i is a proper subgroup of G containing M. Since $\epsilon_i \in L_i$, it follows that $\epsilon_i \wedge L_i \cong \mathbb{F}_Q^s$. Now $\epsilon_i \wedge G \cong \mathbb{F}_Q^t$ and $|(\epsilon_i \wedge G)/(\epsilon_i \wedge L_i)| = Q$ together yield $s = t-1$. Thus,

$$G \cong \mu_Q \times L_i. \tag{1.2}$$

Fix a choice of isomorphism as in (1.2) and use it to identify G with $\mu_Q \times L_i$. Now choosing generators e_1 for the ‘extra’ copy of μ_Q on the right-hand side of (1.2) and $\epsilon_i = e_2, \dots, e_t$ for the Q-primary part of L_i (where $e_1, \dots, e_{t-1}$ each have order Q and e_t has order Q^a) gives a choice of generators $e_1, \dots, e_t$ for the Q-primary part of G such that $L_i = \langle M, e_2, \dots, e_t \rangle$, completing the proof in this case (with $(i, j) = (2, 1)$).

Now assume that $\wedge^2(\mu_Q^{t-1}) \subseteq \Upsilon$. Let $e_1, \dots, e_{t-1}$ generate μ_Q^{t-1} and let e_t generate μ_{Q^a}. Since $\Upsilon \neq \wedge^2 G$ and the $e_m \wedge e_n$ for $1 \leq m < n \leq t$ generate $\wedge^2 G$, there exists $i \leq t-1$ such that $e_i \wedge e_t \notin \Upsilon$. Suppose $e_i \in \varphi_v(I_v)$. The assumption $\wedge^2(\operatorname{Im}\varphi_v) \subseteq \Upsilon$ implies that

$$\varphi_v(\operatorname{Frob}_v I_v) \subseteq L_i := \{g \in G \mid e_i \wedge g \in \Upsilon\}.$$

Note that L_i is a proper subgroup of G because $e_t \notin L_i$. Moreover, L_i contains the maximal proper subgroup $\langle M, e_1, \dots, e_{t-1}, e_t^Q\rangle$ since $\wedge^2\mu_Q^{t-1} \subseteq \Upsilon$ and $e_i \wedge e_t^Q = e_i \wedge M = 1$. Thus, $L_i = \langle M, e_1, \dots, e_{t-1}, e_t^Q\rangle$, completing the proof. □

Theorem C2 now follows immediately from Theorem C4, noting that $a_i = 1$ in the notation of Theorem C4.

1.4. **Proof of Theorem C3.** We require the following corrected version of [3, Lemma 6.7].

Lemma C6. *Let G be a finite abelian group and φ a G-extension of k with associated field K. Then $\mathrm{R}^1_{K/k}\mathbb{G}_\mathrm{m}$ satisfies weak approximation if and only if the induced map*

$$\wedge^2(\operatorname{Im}\varphi_v) \to \wedge^2 G$$

is the trivial map for all places v of k.

Proof. Results of Tate and Voskresenskiĭ ([3, Theorems 6.1, 6.2]) imply that weak approximation holds if and only if

$$\mathrm{H}^3(G,\mathbb{Z}) = \ker\left(\mathrm{H}^3(G,\mathbb{Z}) \to \prod_v \mathrm{H}^3(\operatorname{Im}\varphi_v,\mathbb{Z})\right).$$

Dualising and applying [3, Lemma 6.4], this is equivalent to

$$\operatorname{Im}\left(\prod_v \wedge^2(\operatorname{Im}\varphi_v) \to \wedge^2 G\right) = \{1\},$$

which is equivalent to the statement of the lemma. □

We use this to prove Theorem C3 as follows. This theorem concerns finite abelian groups G for which the Q-Sylow subgroup is not cyclic. Such groups can be written in the form $G = M \times \mu_{Q^{a_1}} \times \cdots \times \mu_{Q^{a_t}}$ where $t \geq 2$ and all primes dividing $|M|$ are greater than Q. Let $e_1, \dots, e_t$ generate $\mu_{Q^{a_1}}, \dots, \mu_{Q^{a_t}}$, respectively. We need to show that for only 0% of G-extensions φ of k the torus $\mathrm{R}^1_{K_\varphi/k}\mathbb{G}_\mathrm{m}$ satisfies weak approximation. By Lemma C6 this happens if and only if $\operatorname{Im}(\wedge^2(\operatorname{Im}\varphi_v) \to \wedge^2 G) = \{1\}$ for all places v of k. Thus the result follows immediately from Lemma C6, Theorem C4, and the following.

Lemma C7. *Let $i, j \in \{1, \dots, t\}$ be any two distinct indices with $a_i \leq a_j$. Then, for any finite place v of k that satisfies $\operatorname{Im}(\wedge^2(\operatorname{Im}\varphi_v) \to \wedge^2 G) = \{1\}$, we have*

$$e_i^{Q^{a_i-1}} \in \varphi_v(I_v) \implies \varphi_v(\operatorname{Frob}_v I_v) \subseteq \langle M, e_1, \dots, e_{j-1}, e_j^Q, e_{j+1}, \dots, e_t\rangle.$$

Proof. The element $e_i \wedge e_j$ has order Q^{a_i} in $\wedge^2 G$. Suppose that $e_i^{Q^{a_i-1}} \in \varphi_v(I_v)$ and let $x \in \varphi_v(\operatorname{Frob}_v I_v)$. Since $\operatorname{Im}(\wedge^2(\operatorname{Im}\varphi_v) \to \wedge^2 G) = \{1\}$, we have $e_i^{Q^{a_i-1}} \wedge x = 1 \in \wedge^2 G$. Write $x = m e_1^{b_1} \dots e_t^{b_t}$ for some $m \in M$ and $b_1, \dots, b_t \in \mathbb{Z}$. Now use the fact that the Q-primary part of $\wedge^2 G$ is $\bigoplus_{1 \leq k < \ell \leq t} \mu_{Q^{a_k}} \otimes \mu_{Q^{a_\ell}}$, where $\mu_{Q^{a_k}} \otimes \mu_{Q^{a_\ell}}$ is generated by $e_k \wedge e_\ell$, to see that $Q \mid b_j$. □

The remainder of the corrigendum is devoted to the proof of Theorem C4. We begin with some preliminary results.

2. Preliminaries

2.1. **Tauberian theorem.** We require the following special case of Delange's Tauberian theorem.

Theorem C8. *Let* $a > 0$ *and let* $\mathfrak{f}(s) = \sum_{n=1}^{\infty} a_n/n^s$ *be a Dirichlet series with real non-negative coefficients which converges on* $\operatorname{Re} s > a$. *Suppose that there exists a real number* $\omega > 0$ *such that the function* $\mathfrak{g}(s) = \mathfrak{f}(s)(s-a)^{\omega}$ *admits an extension to a continuous function on the domain* $\operatorname{Re} s \geq a$ *with* $\mathfrak{g}(a) > 0$. *Assume that there exists* $\delta \in (0,1]$ *such that*

$$\mathfrak{g}(s) = \mathfrak{g}(a) + O((s-a)^{\delta}), \quad \text{as } s \to a \quad \text{for } \operatorname{Re} s > a.$$

If $0 < \omega < 1$ *assume also that the derivative* $\mathfrak{g}'$ *admits an extension to a continuous function on the domain* $\operatorname{Re} s \geq a$. *Then*

$$\sum_{n \leq x} a_n \sim \frac{\mathfrak{g}(a)}{a\Gamma(\omega)} x^a (\log x)^{\omega - 1}, \qquad \text{as } x \to \infty.$$

Proof. We prove this as an application of [1, Thm. I], closely following the proof of [1, Thm. III, §5.2.1], which imposes the stronger assumption that $\mathfrak{g}$ admits a holomorphic continuation to $s = a$.

We begin with a standard reduction. The result [1, Thm. I] states a Tauberian theorem for a suitable function $\alpha(t)$ in terms of the analytic properties of its Laplace transform $f(s) = \int_0^{\infty} e^{-st}\alpha(t)\mathrm{d}t$. For our application we take $\alpha(t) = \sum_{n \leq e^t} a_n$, which for $\operatorname{Re} s > a$ gives

$$f(s) = \int_0^{\infty} e^{-st} \sum_{n \leq e^t} a_n \mathrm{d}t = \int_1^{\infty} x^{-s-1} \sum_{n \leq x} a_n \mathrm{d}x = \frac{1}{s} \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \frac{1}{s}\mathfrak{f}(s).$$

Thus to obtain our asymptotic formula we will apply [1, Thm. I] to $\alpha(\log x)$ (the factor $1/s$ gives rise to the factor $1/a$ in our statement).

Set $g(s) = \mathfrak{g}(s)/s$. In the notation of [1, Thm. I] we take $A = 1$, l any positive real number less than 1 and

$$\beta(t) = \begin{cases} 0, & 0 \leq t < 1, \\ g(a)t^{\omega-1}/\Gamma(\omega), & t \geq 1. \end{cases}$$

In the notation of *loc. cit.* we then have $G(s-a) = (s-a)^{-\omega}g(a) + h(s)$ where $h(s)$ is an entire function in s (see [1, Lem. 4]). We need to consider

$$F(s) = f(s) - G(s-a) = (s-a)^{-\omega}(g(s) - g(a)) - h(s).$$

This admits a continuous extension to $\operatorname{Re} s \geq a$ away from $s = a$, by our assumptions on $\mathfrak{g}$.

We first consider the case where $\omega \geq 1$. In the notation of [1, Thm. I] we take $\gamma(u) = u^{\omega-1}/g(a)$. As $s \to 0$ in the half-plane $\operatorname{Re} s > 0$ we have

$$F(s+a) = O\left(\frac{|s|^{\delta}}{|s|^{\omega}}\right) = O\left(\frac{|s|^{\delta-1}}{|s|^{\omega-1}}\right)$$

thus the hypotheses of *loc. cit.* hold with $\psi(r) = r^{\delta-1}$.

For the case where $0 < \omega < 1$ we take $\gamma(u) = u^\omega/\omega g(a)$. The derivative $F'(s)$ admits a continuous extension to $\operatorname{Re} s \geq a$ away from $s = a$, by our assumptions. As $s \to 0$ in the half-plane $\operatorname{Re} s > 0$ we have

$$F'(s+a) = -\omega s^{-\omega-1}(g(s+a) - g(a)) + s^{-\omega} g'(s+a) - h'(s+a) = O\left(\frac{|s|^{\delta-1}}{|s|^\omega}\right)$$

thus the hypotheses of *loc. cit.* hold with $\psi(r) = r^{\delta-1}$. □

The following lemma assists with applications of the Tauberian theorem.

Lemma C9. *Suppose that $g(s)$ is holomorphic on $\operatorname{Re} s > a$ and both g and g' extend to continuous functions on $\operatorname{Re} s \geq a$. Define*

$$\begin{aligned} g_0 : \mathbb{R} &\to \mathbb{C} \\ t &\mapsto g(a+ti). \end{aligned}$$

Then g_0 is differentiable with derivative $g_0'(t) = ig'(a+ti)$.

Proof. Write $g(x+iy) = u(x,y) + iv(x,y)$. For $t \in \mathbb{R}$ and $h > 0$ the mean value theorem shows that

$$\begin{aligned} \operatorname{Re}(g_0(t+h) - g_0(t)) &= u(a, t+h) - u(a+h, t+h) \\ &\quad + u(a+h, t+h) - u(a+h, t) \\ &\quad + u(a+h, t) - u(a, t) \\ &= -hu_x(a+\xi_1, t+h) + hu_y(a+h, t+\xi_2) \\ &\quad + hu_x(a+\xi_3, t), \end{aligned}$$

where $\xi_1, \xi_2, \xi_3 \in (0, h)$. By the continuity of g' (and thus of u_x, u_y, v_x, v_y), we get that

$$\lim_{h \to 0^+} \frac{\operatorname{Re}(g_0(t+h) - g_0(t))}{h} = -u_x(a,t) + u_y(a,t) + u_x(a,t) = u_y(a,t),$$

as desired. Together with the analogous argument for the imaginary part, this shows that

$$\begin{aligned} \lim_{h \to 0^+} \frac{g_0(t+h) - g_0(t)}{h} &= u_y(a,t) + iv_y(a,t) = \lim_{h \to 0} (u_y(a+h,t) + iv_y(a+h,t)) \\ &= \lim_{h \to 0} ig'(a+h+ti) = ig'(a+ti), \end{aligned}$$

as desired. An analogous argument works for the left-sided limit. □

2.2. **Bounds for L-functions.** Recall that a Dirichlet character of k is a finite order Hecke character, i.e. a continuous homomorphism $\chi : \mathbf{A}^*/k^* \to S^1$ with finite image. Let $\mathfrak{f}$ be the conductor of χ, then χ defines the L-function $L(s;\chi) = \prod_{v \nmid \mathfrak{f}\infty}(1 - \chi(\pi_v) q_v^{-s})^{-1}$ for $\operatorname{Re} s > 1$. When $\chi \neq 1$, this extends to an entire function satisfying the usual functional equation.

Lemma C10. *Let k be a number field and $\chi : \mathbf{A}^*/k^* \to S^1$ a non-trivial Dirichlet character with conductor $\mathfrak{f}$. Then, for $\operatorname{Re} s \geq 1$, we have*

$$L(s;\chi) \ll_{k,\epsilon} (N_{k/\mathbb{Q}}\mathfrak{f} \cdot (1 + |\operatorname{Im} s|))^\epsilon \quad \textit{and} \quad \frac{L'}{L}(s;\chi) \ll_{k,\epsilon} (N_{k/\mathbb{Q}}\mathfrak{f} \cdot (1 + |\operatorname{Im} s|))^\epsilon.$$

Proof. The first bound follows from Rademacher's classical convexity estimate, see e.g. [9, III, Theorem 14A].

For the second bound, inserting the Hadamard factorisation of the completed L-function $\Lambda(s;\chi)$ in the functional equation and taking the logarithmic derivative yields the crude bound (see [5, (5.28), (5.31)])

$$\frac{L'}{L}(s;\chi) = \sum_{|s-\rho|<1} \frac{1}{s-\rho} + O_{k,\epsilon}((N_{k/\mathbb{Q}}\mathfrak{f}\cdot(1+|\operatorname{Im} s|))^{\epsilon}),$$

where the sum runs through all zeros ρ of $L(s;\chi)$ with $|s-\rho|<1$. By [5, (5.27)], the number of such zeros is $\ll_{k,\epsilon} (N_{k/\mathbb{Q}}\mathfrak{f}\cdot(1+|\operatorname{Im} s|))^{\epsilon}$. The classical zero-free region [5, Theorem 5.35] implies that for each such zero ρ, with at most one exception,

$$\frac{1}{s-\rho} \ll_{k,\epsilon} (N_{k/\mathbb{Q}}\mathfrak{f}\cdot(1+|\operatorname{Im} s|))^{\epsilon}.$$

If there is an exceptional zero ρ, it is real and satisfies

$$1-\rho \gg_{k,\epsilon} (N_{k/\mathbb{Q}}\mathfrak{f})^{-\epsilon}$$

(with an ineffective implied constant) by [2]. These estimates taken together show the required bound for $\frac{L'}{L}(s;\chi)$. □

3. Proof of Theorem C4

We now begin the proof of Theorem C4. We may enlarge the finite set S of places of k, and thus assume that it contains all places dividing $|G|$, enough places to ensure that the ring of S-integers $\mathcal{O}_S$ of k is a PID, and all places v with small q_v, where the meaning of "small" will become clear during the proof. All implied constants in $\ll$- and O-notation are allowed to depend on k, G, S.

3.1. The counting function.

We will prove Theorem C4 with $i=1$ and $j=t$. This clearly suffices.

We first translate the statement to a problem on the idele class group using class field theory. For every place v of k we choose a uniformiser $\pi_v \in k_v^{\times}$. Write $L = \langle M, e_1, e_2, \dots, e_{t-1}, e_t^Q\rangle$. Consider the function $f(\chi) = \prod_v f_v(\chi_v)$ on the group $\operatorname{Hom}(\mathbf{A}^*, G)$ of continuous homomorphisms from $\mathbf{A}^*$ to G, defined as follows:

$$f_v(\chi_v) = \begin{cases} 1 & \text{if } v \in S, \\ 1 & \text{if } v \notin S \text{ and } e_1^{Q^{a_1-1}} \in \chi_v(\mathcal{O}_v^*) \implies \chi_v(\pi_v) \in L, \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi_v \in \operatorname{Hom}(k_v^*, G)$ is the composition of χ with the natural inclusion $k_v^* \to \mathbf{A}^*$. Note that a G-extension φ of k satisfying $\varphi_v \in \Lambda_v$ for all $v \in \Omega_k \setminus S$ (as in Theorem C4 with $i=1$ and $j=t$) corresponds via class field theory to a continuous surjective homomorphism $\chi \in \operatorname{Hom}(\mathbf{A}^*, G)$ such that $e_1^{Q^{a_1-1}} \in \chi_v(\mathcal{O}_v^*)$ implies $\chi_v(\pi_v\mathcal{O}_v^*) \subset L$ for all $v \in \Omega_k \setminus S$. The set of such G-extensions is contained in the set of those that correspond to continuous surjective homomorphisms $\chi \in \operatorname{Hom}(\mathbf{A}^*, G)$ with $f(\chi)=1$. These are counted by the function

$$N^*(G,L;B) = \#\{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, G) \ :\ \chi \text{ surjective}, \ \Delta(\chi) \le B,\ f(\chi)=1\}.$$

Therefore, comparing with the total number of G-extensions of k with discriminant bounded by B (see [3, Theorem 1.7]), it suffices to show that

$$N^*(G,L;B) = o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1}), \tag{3.1}$$

where $\alpha(G) = |G|(1-Q^{-1})$ and $\nu(k,G) = (|G[Q]|-1)/[k(\mu_Q):k]$.

For any subgroup $H \subseteq G$ and $\chi : \mathbf{A}^*/k^* \to H$, recall the definition of $\Phi_H(\chi)$ from [3, §2], in particular that $\Phi_H(\chi) = \Delta(\chi)^{|H|/|\operatorname{Im}\chi|}$, and set

$$N(H,L;B) = \#\{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, H) \ : \ \Phi_H(\chi) \leq B, \ f(\chi) = 1\}.$$

Then

$$N(H,L;B) = \sum_{J \subseteq H} N^*(J,L;B^{|J|/|H|}),$$

and thus by Möbius inversion as in [3, §2] and [11, §2],

$$N^*(G,L;B) = \sum_{H \subseteq G} \mu(G/H) N(H,L;B^{|H|/|G|}). \tag{3.2}$$

For a subgroup $H \subseteq G$, we let β_H be its $\mathbb{F}_Q$-rank, i.e. the dimension of the $\mathbb{F}_Q$-vector space $H \otimes_{\mathbb{Z}} \mathbb{F}_Q$. If $\beta_H < \beta_G$, then $\nu(k,H) < \nu(k,G)$ and

$$N(H,L;B^{|H|/|G|}) \leq N(H;B^{|H|/|G|}) = O(B^{1/\alpha(G)}(\log B)^{\nu(k,H)-1})$$

by [3, Lemma 4.7], where $N(H;B) = \#\{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, H) \ : \ \Phi_H(\chi) \leq B\}$. Hence, it suffices to consider subgroups H with $\beta_H = \beta_G$ in (3.2).

3.2. **Case 1:** $a_t = 1$**.** In this case $L = \langle M, e_1, e_2, \ldots, e_{t-1}\rangle$. Let H be a subgroup of G with $\beta_H = \beta_G$. We show that the individual counting function in (3.2) satisfies $N(H,L;B^{|H|/|G|}) = o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1})$.

For any finite set $T \supset S$ of places of k, we consider the truncation of f,

$$f_T(\chi) = \prod_{v \in T} f_v(\chi_v),$$

with corresponding counting function

$$N_T(H,L;B) = \#\{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, H) \ : \ \Phi_H(\chi) \leq B, \ f_T(\chi) = 1\}.$$

Note that the counting function N_T imposes local conditions at only finitely many places, so the main counting results of [3] apply. Hence, [3, Lemma 4.7] shows that

$$\lim_{B\to\infty} \frac{N(H,L;B^{|H|/|G|})}{B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1}} \leq \lim_{B\to\infty} \frac{N_T(H,L;B^{|H|/|G|})}{B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1}} =: c_{H,T},$$

with a constant $c_{H,T} \geq 0$. We will show that $\lim_{T\to\Omega_k} c_{H,T} = 0$, which is enough to ensure that $N(H,L;B^{|H|/|G|}) = o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1})$.

Let us recall some notation from [3]. We write $k_0 := k(\mu_Q)$. For $\operatorname{Re}(s) > 1$, we set $\zeta_{k_0,v}(s) := 1$ for archimedean v, and $\zeta_{k_0,v}(s) := \prod_{w|v}(1-q_w^{-s})^{-1}$, the product of the local factors of the Dedekind zeta function of k_0 at all places w above v, for non-archimedean v. Moreover, $\widehat{1}_v(\cdot;s)$ is the Fourier transform of the function

$1/\Phi_H(\cdot)^s$ on $\mathrm{Hom}(k_v^*, H)$, and $\widehat{f}_{H,v}(\cdot; s)$ is the Fourier transform of $f_v(\cdot)/\Phi_H(\cdot)^s$. As in [3, (4.13)], we have

$$c_{H,T} \ll \lim_{s\to 1/\alpha(G)^+} \sum_{x\in k^*\otimes H^\wedge} \prod_{v\in T} \frac{\widehat{f}_{H,v}(x_v; s|G|/|H|)}{\zeta_{k_0,v}(\alpha(G)s)^{\nu(k,G)}} \prod_{v\notin T} \frac{\widehat{1}_v(x_v; s|G|/|H|)}{\zeta_{k_0,v}(\alpha(G)s)^{\nu(k,G)}}. \tag{3.3}$$

For $x_v \in \mathcal{O}_v^* \otimes H^\wedge$, we write $x_v \in \mathcal{O}_v^{*Q} \otimes H^\wedge$ to say that x_v is in the image of $\mathcal{O}_v^{*Q} \otimes H^\wedge$ in $\mathcal{O}_v^* \otimes H^\wedge$ under the natural map (which is not injective in general).

By [3, Lemma 4.1], for $v \notin S$, $x_v \in \mathcal{O}_v^* \otimes H^\wedge$, and $\mathrm{Re}(s) \geq 0$

$$\widehat{1}_v(x_v; s|G|/|H|) =$$
$$\begin{cases} 1 + (Q^{\beta_G}\mathbb{1}_{\mathcal{O}_v^{*Q}\otimes H^\wedge}(x_v) - 1)q_v^{-\alpha(G)s} + O(q_v^{-(\alpha(G)+1)s}), & q_v \equiv 1 \bmod Q, \\ 1 + O(q_v^{-(\alpha(G)+1)s}), & q_v \not\equiv 1 \bmod Q. \end{cases}$$

Moreover, [3, Lemma 3.4] shows that

$$\widehat{1}_v(x_v; s|G|/|H|) = 0 \quad \text{for} \quad x_v \notin \mathcal{O}_v^* \otimes H^\wedge. \tag{3.4}$$

For a subgroup A of k_v^*, a subgroup R of H, and $x_v \in k_v^* \otimes H^\wedge$, we abuse notation by writing $x_v \in A \otimes R^\wedge$ to mean that the image of x_v under the natural map $k_v^* \otimes H^\wedge \to k_v^* \otimes R^\wedge$ is in the image of $A \otimes R^\wedge$ under the natural map $A \otimes R^\wedge \to k_v^* \otimes R^\wedge$.

For $\widehat{f}_{H,v}$, we have the following result.

Lemma C11. *Let* $\mathrm{Re}(s) \geq 0$. *Write* $L_H = L \cap H$ *and write* V *for the subgroup of order* Q *of* G *generated by* $e_1^{Q^{a_1-1}}$. *Let* $v \notin S$ *and* $x_v \in k_v^* \otimes H^\wedge$. *If* $q_v \not\equiv 1 \bmod Q$, *then*

$$\widehat{f}_{H,v}(x_v; s|G|/|H|) = \mathbb{1}_{\mathcal{O}_v^*\otimes H^\wedge}(x_v) + O(q_v^{-(\alpha(G)+1)s}). \tag{3.5}$$

If $q_v \equiv 1 \bmod Q$, *then* $\widehat{f}_{H,v}(x_v; s|G|/|H|)$ *is equal to*

$$\mathbb{1}_{\mathcal{O}_v^*\otimes H^\wedge}(x_v) + q_v^{-\alpha(G)s}\Big(\mathbb{1}_{\mathcal{O}_v^*\otimes H^\wedge}(x_v)\Big(Q^{\beta_G} \cdot \mathbb{1}_{\mathcal{O}_v^{*Q}\otimes H^\wedge}(x_v) - Q \cdot \mathbb{1}_{\mathcal{O}_v^{*Q}\otimes V^\wedge}(x_v)\Big)$$
$$+ \mathbb{1}_{\mathcal{O}_v^*\otimes L_H{}^\wedge}(x_v)\Big(\mathbb{1}_{\mathcal{O}_v^{*Q}\otimes V^\wedge}(x_v) - \frac{1}{Q}\Big)\Big) + O(q_v^{-(\alpha(G)+1)s}).$$

Moreover, if $x_v \notin \mathcal{O}_v^* \otimes L_H{}^\wedge$ *then* $\widehat{f}_{H,v}(x_v; s) = 0$.

Remark C12. Note that since $\beta_H = \beta_G$, the Q-torsion subgroup $G[Q]$ is contained in H and hence $V \subseteq H$.

Proof of Lemma C11. Writing $k_v^* = \mathcal{O}_v^* \oplus \langle \pi_v \rangle$, we identify $k_v^*/\mathcal{O}_v^*$ with $\langle \pi_v \rangle$ and hence $\mathrm{Hom}(k_v^*, H)$ with $\mathrm{Hom}(\mathcal{O}_v^*, H) \oplus \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, H)$. By [3, Lemma 3.3],

$$\widehat{f}_{H,v}(x_v; s|G|/|H|) = \sum_{l|(\exp(H), q_v-1)} \left(\sum_{\substack{\chi_v \in \mathrm{Hom}(\mathcal{O}_v^*, H) \\ \ker \chi_v = \mathcal{O}_v^{*l}}} \langle \chi_v, x_v \rangle \tau_{f_v}(\chi_v, x_v) \right) q_v^{-|G|(1-1/l)s},$$

where

$$\tau_{f_v}(\chi_v, x_v) := \frac{1}{|H|} \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, H)} f_v(\chi_v\psi_v)\langle \psi_v, x_v \rangle.$$

Taking $l = 1$ gives $\mathbb{1}_{\mathcal{O}_v^* \otimes H^\wedge}(x_v)$. If $q_v \not\equiv 1 \bmod Q$, then all other terms are $O(q_v^{-(\alpha(G)+1)s})$, so we have shown (3.5).

Suppose that $q_v \equiv 1 \bmod Q$. Then the contribution from $l = Q$ is

$$\begin{aligned} & q_v^{-\alpha(G)s} \frac{1}{|H|} \sum_{\substack{\chi_v \in \mathrm{Hom}(\mathcal{O}_v^*, H) \\ \ker \chi_v = \mathcal{O}_v^{*Q}}} \langle \chi_v, x_v \rangle \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, H)} f_v(\chi_v \psi_v) \langle \psi_v, x_v \rangle \\ =& q_v^{-\alpha(G)s} \frac{1}{|H|} \sum_{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \hookrightarrow V} \langle \chi_v, x_v \rangle \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, H)} f_v(\chi_v \psi_v) \langle \psi_v, x_v \rangle \qquad (3.6) \\ & + q_v^{-\alpha(G)s} \frac{1}{|H|} \sum_{\substack{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \hookrightarrow H \\ \mathrm{Im}\, \chi_v \not\subset V}} \langle \chi_v, x_v \rangle \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, H)} f_v(\chi_v \psi_v) \langle \psi_v, x_v \rangle. \qquad (3.7) \end{aligned}$$

By the definitions of f_v and L_H, and our identification $k_v^*/\mathcal{O}_v^* = \langle \pi_v \rangle$, (3.6) is equal to

$$\begin{aligned} & q_v^{-\alpha(G)s} \frac{1}{|H|} \sum_{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \hookrightarrow V} \langle \chi_v, x_v \rangle \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, L_H)} \langle \psi_v, x_v \rangle \\ =& q_v^{-\alpha(G)s} \frac{|L_H|}{|H|} \mathbb{1}_{\mathcal{O}_v^* \otimes L_H^\wedge}(x_v) \sum_{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \hookrightarrow V} \langle \chi_v, x_v \rangle \\ =& q_v^{-\alpha(G)s} \frac{1}{Q} \mathbb{1}_{\mathcal{O}_v^* \otimes L_H^\wedge}(x_v) \Big(-1 + \sum_{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \to V} \langle \chi_v, x_v \rangle \Big) \\ =& q_v^{-\alpha(G)s} \frac{1}{Q} \mathbb{1}_{\mathcal{O}_v^* \otimes L_H^\wedge}(x_v) \Big(-1 + Q \cdot \mathbb{1}_{\mathcal{O}_v^{*Q} \otimes V^\wedge}(x_v) \Big). \qquad (3.8) \end{aligned}$$

Furthermore, by the definition of f_v, (3.7) is equal to

$$\begin{aligned} & q_v^{-\alpha(G)s} \frac{1}{|H|} \sum_{\substack{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \hookrightarrow H \\ \mathrm{Im}\, \chi_v \not\subset V}} \langle \chi_v, x_v \rangle \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, H)} \langle \psi_v, x_v \rangle \\ =& q_v^{-\alpha(G)s} \mathbb{1}_{\mathcal{O}_v^* \otimes H^\wedge}(x_v) \sum_{\substack{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \to H \\ \mathrm{Im}\, \chi_v \not\subset V}} \langle \chi_v, x_v \rangle \\ =& q_v^{-\alpha(G)s} \mathbb{1}_{\mathcal{O}_v^* \otimes H^\wedge}(x_v) \left(\sum_{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \to H} \langle \chi_v, x_v \rangle - \sum_{\chi_v : \mathcal{O}_v^*/\mathcal{O}_v^{*Q} \to V} \langle \chi_v, x_v \rangle \right) \\ =& q_v^{-\alpha(G)s} \mathbb{1}_{\mathcal{O}_v^* \otimes H^\wedge}(x_v) \left(Q^{\beta_G} \cdot \mathbb{1}_{\mathcal{O}_v^{*Q} \otimes H^\wedge}(x_v) - Q \cdot \mathbb{1}_{\mathcal{O}_v^{*Q} \otimes V^\wedge}(x_v) \right). \qquad (3.9) \end{aligned}$$

Taking the sum of (3.8) and (3.9) completes the proof of the formula for $\widehat{f}_{G,v}(x_v; s)$. To see that $x_v \notin \mathcal{O}_v^* \otimes L_H{}^\wedge$ implies $\widehat{f}_{H,v}(x_v; s|G|/|H|) = 0$, we also have to consider the remaining terms, i.e. $l|(\exp(H), q_v - 1)$ with $l > Q$.

Similarly to the calculations above, the contribution from l is

$$\begin{aligned}
&q_v^{-\alpha(G)s}\frac{1}{|H|}\sum_{\substack{\chi_v\in\mathrm{Hom}(\mathcal{O}_v^*,H)\\ \ker\chi_v=\mathcal{O}_v^{*l}}}\langle\chi_v,x_v\rangle\sum_{\psi_v\in\mathrm{Hom}(k_v^*/\mathcal{O}_v^*,H)}f_v(\chi_v\psi_v)\langle\psi_v,x_v\rangle\\
=&q_v^{-\alpha(G)s}\frac{1}{|H|}\sum_{\substack{\chi_v\in\mathrm{Hom}(\mathcal{O}_v^*,H)\\ \ker\chi_v=\mathcal{O}_v^{*l}\\ V\subset\mathrm{Im}\,\chi_v}}\langle\chi_v,x_v\rangle\sum_{\psi_v\in\mathrm{Hom}(k_v^*/\mathcal{O}_v^*,H)}f_v(\chi_v\psi_v)\langle\psi_v,x_v\rangle \qquad (3.10)\\
&+q_v^{-\alpha(G)s}\frac{1}{|H|}\sum_{\substack{\chi_v\in\mathrm{Hom}(\mathcal{O}_v^*,H)\\ \ker\chi_v=\mathcal{O}_v^{*l}\\ V\not\subset\mathrm{Im}\,\chi_v}}\langle\chi_v,x_v\rangle\sum_{\psi_v\in\mathrm{Hom}(k_v^*/\mathcal{O}_v^*,H)}f_v(\chi_v\psi_v)\langle\psi_v,x_v\rangle. \qquad (3.11)
\end{aligned}$$

By the definition of f_v, the inner sum in (3.10) becomes

$$\sum_{\psi_v\in\mathrm{Hom}(k_v^*/\mathcal{O}_v^*,L_H)}\langle\psi_v,x_v\rangle=|L_H|\cdot\mathbb{1}_{\mathcal{O}_v^*\otimes L_H{}^\wedge}(x_v),$$

which is zero for $x_v\notin\mathcal{O}_v^*\otimes L_H{}^\wedge$. The inner sum in (3.11) becomes

$$\sum_{\psi_v\in\mathrm{Hom}(k_v^*/\mathcal{O}_v^*,H)}\langle\psi_v,x_v\rangle=|H|\cdot\mathbb{1}_{\mathcal{O}_v^*\otimes H^\wedge}(x_v),$$

which is zero unless $x_v\in\mathcal{O}_v^*\otimes H^\wedge$, which is even stronger than $x_v\in\mathcal{O}_v^*\otimes L_H{}^\wedge$. □

By (3.4), the summand in (3.3) is zero unless $x\in\mathcal{O}_T^*\otimes H^\wedge$. Hence, the sum over x is finite and

$$c_{H,T}\ll\sum_{x\in\mathcal{O}_T^*\otimes H^\wedge}\prod_{v\in T}\frac{\widehat{f}_{H,v}(x_v;1/\alpha(H))}{\zeta_{k_0,v}(1)^{\nu(k,G)}}\lim_{s\to1/\alpha(G)^+}\prod_{v\in\Omega_k\smallsetminus T}\frac{\widehat{1}_v(x_v;s|G|/|H|)}{\zeta_{k_0,v}(\alpha(G)s)^{\nu(k,G)}}. \qquad (3.12)$$

We now define k_x/k_0 to be the elementary abelian Q-extension associated to $x\in k^*\otimes H^\wedge$ as in [3, Section 4.4]. Explicitly, taking a presentation of the Q-primary part of H as a direct sum of cyclic groups, we can write

$$k^*\otimes H^\wedge=k^*\otimes D^\wedge\times k^*/k^{*Q^{c_1}}\times\cdots\times k^*/k^{*Q^{c_t}}$$

for some subgroup D of M and $c_1,\ldots,c_t\in\mathbb{Z}_{>0}$. We write $x\in\mathcal{O}_T^*\otimes H^\wedge$ as $x=(y,x_1,\ldots,x_t)$ with $y\in\mathcal{O}_T^*\otimes D^\wedge$ and $x_i\in\mathcal{O}_T^*/\mathcal{O}_T^{*Q^{c_i}}$ for $1\le i\le t$. Then

$$k_x=k_0(\sqrt[Q]{x_1},\ldots,\sqrt[Q]{x_t}).$$

With this setup, it follows from [3, Lemma 4.5] that the function

$$\widehat{1}_T(x;s|G|/|H|)=\prod_{v\in\Omega_k\smallsetminus T}\widehat{1}_v(x_v;s|G|/|H|)$$

has a pole of order strictly smaller than $\nu(k,G)=(Q^{\beta_G}-1)/[k_0:k]$ at $s=1/\alpha(G)$, unless $k_x=k_0$. Hence, the summand in (3.12) vanishes unless $k_x=k_0$.

Lemma C13. *For $x\in k^*\otimes H^\wedge$, we have $k_x=k_0$ if and only if $x\in k^{*Q}\otimes H^\wedge$.*

Proof. The "if" follows directly from the definition of k_x. For the "only if", suppose $k_x = k_0$, let $i \in \{1, \dots, t\}$ and fix a representative of x_i. The polynomial $X^Q - x_i$ cannot be irreducible over k, as it has roots in k_0 and $[k_0 : k] \leq Q - 1$. As Q is prime, this already implies $x_i \in k^{*Q}$. Moreover, for the first factor $k^* \otimes D^\wedge$ of $k^* \otimes H^\wedge$, we see that every $y \in k^* \otimes D^\wedge$ is a Q-th power, as $(Q, |D|) = 1$. □

Given $x \in k^{*Q} \otimes H^\wedge$, let $v \in \Omega_k \smallsetminus S$. By Lemma C11, we have $\widehat{f}_{H,v}(x_v; s) = 0$ unless $x_v \in \mathcal{O}_v^* \otimes L_H{}^\wedge$. Since $\beta_H = \beta_G$, the Q-torsion subgroup $G[Q]$ is contained in H. Since $a_t = 1$, this implies that $e_t \in H$ and $H = L_H \oplus \langle e_t \rangle$. Therefore, as e_t has order Q and $x \in k^{*Q} \otimes H^\wedge$, the condition that $x_v \in \mathcal{O}_v^* \otimes L_H{}^\wedge$ is equivalent to $x_v \in \mathcal{O}_v^{*Q} \otimes H^\wedge$. Therefore, we may further restrict the sum in (3.12) to the finite set $\mathcal{O}_S^{*Q} \otimes H^\wedge$, which is independent of T. This yields

$$c_{H,T} \ll \sum_{x \in \mathcal{O}_S^{*Q} \otimes H^\wedge} \prod_{v \in T \smallsetminus S} \frac{\widehat{f}_{H,v}(x_v; 1/\alpha(H))}{\widehat{1}_v(x_v; 1/\alpha(H))} \lim_{s \to 1/\alpha(G)^+} \prod_{v \in \Omega_k \smallsetminus S} \frac{\widehat{1}_v(x_v; s|G|/|H|)}{\zeta_{k_0,v}(\alpha(G)s)^{\nu(k,G)}}.$$

Now observe from Lemma C11 and the description of $\widehat{1}_v(x_v; s|G|/|H|)$ preceding Lemma C11, that

$$\prod_{v \in T \smallsetminus S} \frac{\widehat{f}_{H,v}(x_v; 1/\alpha(H))}{\widehat{1}_v(x_v; 1/\alpha(H))} =$$
$$\prod_{\substack{v \in T \smallsetminus S \\ q_v \not\equiv 1 \bmod Q}} \left(1 + O(q_v^{-1-1/\alpha(G)})\right) \cdot \prod_{\substack{v \in T \smallsetminus S \\ q_v \equiv 1 \bmod Q}} \left(1 - (Q - 2 + 1/Q)q_v^{-1} + O(q_v^{-1-1/\alpha(G)})\right)$$

for $x \in \mathcal{O}_S^{*Q} \otimes H^\wedge$. With, e.g., the Chebotarev density theorem, this shows that $\lim_{T \to \Omega_k} c_{H,T} = 0$, as desired. We have thus proved (3.1), and hence Theorem C4, in the case $a_t = 1$.

3.3. Case 2: $a_t \geq 2$.

This is the hard case, where the individual counting functions in (3.2) are not $o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1})$. Instead, we match up the subgroups H giving the highest-order contribution in (3.2) and show cancellation between them. We already observed at the end of Section 3.1 that these subgroups H have maximal $\mathbb{F}_Q$-rank, i.e. $\beta_H = \beta_G$. Hence, it is enough to consider subgroups contained in the set

$$W := \{H \subseteq G \ : \ \beta_H = \beta_G, \ \mu(G/H) \neq 0\}.$$

Recall that $L = \langle M, e_1, e_2, \dots, e_{t-1}, e_t^Q \rangle$ and partition W into the sets

$$W_1 := \{H \in W \ : \ H \not\subseteq L\} \quad \text{and} \quad W_2 := \{J \in W \ : \ J \subseteq L\}.$$

Lemma C14.

1. *For $H \in W_1$, we have $H/(H \cap L) \cong \mathbb{Z}/Q\mathbb{Z}$ and $L/(H \cap L) \cong G/H$.*
2. *The map $H \mapsto H \cap L$ induces a surjection $\phi : W_1 \to W_2$.*
3. *For $J \in W_2$, the map $\ell \mapsto \langle J, e_t \cdot \ell \rangle$ induces a bijection between $(L/J)[Q]$ and $\phi^{-1}(J)$.*

Proof. (1): Since $H \in W_1$, we have $H \neq H \cap L$ and therefore the natural injection $H/(H \cap L) \hookrightarrow G/L \cong \mathbb{Z}/Q\mathbb{Z}$ is an isomorphism. Similarly, the cokernel of the natural map $L/(H \cap L) \hookrightarrow G/H$ is trivial because G/L has order Q and $H \not\subseteq L$.

(2) *and* (3): First, we claim that if $A, B \in W$ then $A \cap B \in W$. Note that having maximal $\mathbb{F}_Q$-rank is equivalent to containing $G[Q]$. Moreover, suppose that $\mu(G/A)$ and $\mu(G/B)$ are nonzero. Then for every prime p, G/A and G/B contain no element of order p^2. Suppose for contradiction that $G/(A\cap B)$ contains an element of order p^2, represented by $g \in G$. Then, since $\mu(G/A)$ and $\mu(G/B)$ are nonzero, $g^p \in A$ and $g^p \in B$, so g has order at most p in $G/(A \cap B)$, which is the desired contradiction. Now, since $a_t \geq 2$, we have $L \in W$ and therefore the map ϕ is well defined.

Let $H \in \phi^{-1}(J)$, so $H \in W_1$ and $H \cap L = J$. By (1), the natural injection $H/J \hookrightarrow G/L \cong \mathbb{Z}/Q\mathbb{Z}$ is an isomorphism. Since G/L is generated by the image of e_t, this implies that $H = \langle J, e_t \cdot \ell\rangle$ for some $\ell \in L$. Since $J \in W_2$, we have $\mu(G/J) \neq 0$ and hence $e_t^Q \in J$. Since H/J has order Q, we also have $(e_t \cdot \ell)^Q \in J$, and hence $\ell^Q \in J$. So far, we have shown that any element of $\phi^{-1}(J)$ is of the form $\langle J, e_t \cdot \ell\rangle$ for some $\ell \in L$ such that $\ell^Q \in J$. Now we show that any group of this form is in $\phi^{-1}(J)$. Let $\ell \in L$ be such that $\ell^Q \in J$ and let $H = \langle J, e_t \cdot \ell\rangle$. It is clear that $H \in W_1$, since $J \in W$ and $e_t \cdot \ell \notin L$. We must show that $H \cap L = J$. Clearly, $J \subseteq H \cap L$. As noted above, since $J \in W_2$, we have $e_t^Q \in J$. So H/J has order at most Q. Moreover, $H/(H \cap L)$ has order Q by (1), and hence $J = H \cap L$, as required. In particular, we have shown that $\phi^{-1}(J)$ is non-empty for any $J \in W_2$, whereby ϕ is surjective and we have proved (2).

To complete the proof of (3), we assert that two groups $H_1 = \langle J, e_t \cdot \ell_1\rangle$ and $H_2 = \langle J, e_t \cdot \ell_2\rangle$ in $\phi^{-1}(J)$ are equal if and only if $\ell_1 \cdot \ell_2^{-1} \in J$. Simply observe that if $H_1 = H_2$ then $\ell_1 \cdot \ell_2^{-1} \in H_1 \cap L = J$. The other direction is clear. □

Note that for $J \subseteq L$, we have $f(\chi) = 1$ for all $\chi : \mathbf{A}^*/k^* \to J$, and hence $N(J, L; B^{|J|/|G|}) = N(J; B^{|J|/|G|})$. Thus, Lemma C14 shows that (3.2) equals

$$\begin{aligned}
&\sum_{H\in W_1} \mu(G/H)N(H, L; B^{|H|/|G|}) + \sum_{J\in W_2} \mu(G/J)N(J; B^{|J|/|G|}) \\
&\qquad + o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1}) \\
&= \sum_{J\in W_2} \left(\mu(G/J)N(J; B^{|J|/|G|}) + \mu(L/J) \sum_{H\in\phi^{-1}(J)} N(H, L; B^{|H|/|G|}) \right) \\
&\qquad + o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1}). \qquad (3.13)
\end{aligned}$$

Lemma C15. *For $J \in W_2$, we have $\mu(G/J) = -|(L/J)[Q]| \cdot \mu(L/J)$.*

Proof. Recall that the Möbius function on isomorphism classes of finite abelian groups satisfies $\mu(G) = 0$ if G has a cyclic subgroup of order p^n, with p a prime and $n \geq 2$, $\mu(G_1 \times G_2) = \mu(G_1)\mu(G_2)$ if G_1 and G_2 have coprime order, and $\mu((\mathbb{Z}/p\mathbb{Z})^n) = (-1)^n p^{n(n-1)/2}$ for a prime p and $n \in \mathbb{Z}_{>0}$.

Since $J \in W_2$, we have $\mu(G/J) \neq 0$ and hence $G/J \cong T \times (\mathbb{Z}/Q\mathbb{Z})^s$ where $Q \nmid |T|$ and $s \in \mathbb{Z}_{>0}$. Since $[G : L] = Q$, it follows that L/J is an index Q subgroup of G/J, and hence $L/J \cong T \times (\mathbb{Z}/Q\mathbb{Z})^{s-1}$. Now $\mu(G/J) = (-1)^s Q^{s(s-1)/2}\mu(T)$ and $\mu(L/J) = (-1)^{s-1}Q^{(s-1)(s-2)/2}\mu(T)$, whence the result. □

Plugging the result of Lemma C15 into (3.13) gives

$$\sum_{J\in W_2}\mu(L/J)\left(-|(L/J)[Q]|\cdot N(J;B^{|J|/|G|})+\sum_{H\in\phi^{-1}(J)}N(H,L;B^{|H|/|G|})\right)$$
$$+\,o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1}). \tag{3.14}$$

Lemma C14(3) shows that the number of terms in the sum over $H\in\phi^{-1}(J)$ is $|(L/J)[Q]|$. Our aim is thus to show that for each of these terms we have

$$N(H,L;B^{|H|/|G|})=N(J;B^{|J|/|G|})+o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1}), \tag{3.15}$$

which will prove (3.1). We begin this proof of this, which culminates in Lemma C24, by studying the counting functions $N(H,L;B)$ through their zeta functions

$$F_{H,f}(s)=\sum_{\chi\in\mathrm{Hom}(\mathbf{A}^*/k^*,H)}\frac{f(\chi)}{\Phi_H(\chi)^s},$$

converging absolutely for $\mathrm{Re}\,s>1/\alpha(H)$. Henceforth, we will fix an element J of W_2 and an element H of $\phi^{-1}(J)$, so $J=H\cap L$. For the corresponding zeta functions, we prove the following proposition. Note that $Q\cdot|J|=|H|$ by Lemma C14(1). We write again $k_0=k(\mu_Q)$.

Proposition C16. *There are holomorphic functions* $g_1(s)$ *and* $g_2(s)$ *on* $\mathrm{Re}\,s>1/\alpha(H)$, *such that*

$$F_{H,f}(s)=\zeta_{k_0}(\alpha(H)s)^{\nu(k,H)}\cdot g_1(s), \tag{3.16}$$
$$F_{J,f}(Qs)=\zeta_{k_0}(\alpha(H)s)^{\nu(k,H)}\cdot g_2(s), \tag{3.17}$$

for $\mathrm{Re}\,s>1/\alpha(H)$. *The functions* g_1,g_2 *extend continuously to* $\mathrm{Re}\,s\geq 1/\alpha(H)$ *and satisfy, for some* $\delta\in(0,1]$ *and all* s *in some compact convex neighbourhood* C *of* $1/\alpha(H)$ *in this half-plane, the condition*

$$|g_i(s)-g_i(1/\alpha(H))|\ll_C|s-1/\alpha(H)|^\delta.$$

Moreover, $g_1(1/\alpha(H))=g_2(1/\alpha(H))\neq 0$.

Almost all that remains of this corrigendum is devoted to the proof of Proposition C16, which we start now. The natural inclusion of J in H gives a short exact sequence

$$1\to J\xrightarrow{\mathrm{incl}}H\to H/J\to 1, \tag{3.18}$$

where $H/J\cong\mu_Q$ by Lemma C14(1). This induces an exact sequence

$$1\to\mathrm{Hom}(\mathbf{A}^*/k^*,J)\xrightarrow{\mathrm{incl}}\mathrm{Hom}(\mathbf{A}^*/k^*,H)\to\mathrm{Hom}(\mathbf{A}^*/k^*,H/J). \tag{3.19}$$

For any $\eta_v\in\mathrm{Hom}(k_v^*,H)$ and $\chi_v\in\mathrm{Hom}(k_v^*,J)$, we define $f_{\eta_v,v}(\chi_v):=f_v(\chi_v\eta_v)$. Similarly, for $\eta\in\mathrm{Hom}(\mathbf{A}^*,H)$ and $\chi\in\mathrm{Hom}(\mathbf{A}^*,J)$, we let

$$f_\eta(\chi):=f(\chi\eta)=\prod_v f_{\eta_v,v}(\chi_v).$$

Note that, for all places $v\notin S$ with $\eta_v|_{\mathcal{O}_v^*}=1$, we have

$$f_{\eta_v,v}|_{\mathrm{Hom}(k_v^*/\mathcal{O}_v^*,J)}=f_v|_{\mathrm{Hom}(k_v^*/\mathcal{O}_v^*,J)}=1.$$

We fix a system $R \subseteq \operatorname{Hom}(\mathbf{A}^*/k^*, H)$ of representatives for the quotient $\operatorname{Hom}(\mathbf{A}^*/k^*, H)/\operatorname{Hom}(\mathbf{A}^*/k^*, J)$, such that $1 \in R$. Then for $\operatorname{Re} s > 1/\alpha(H)$,

$$F_{H,f}(s) = \sum_{\eta \in R} \sum_{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, J)} \frac{f_\eta(\chi)}{\Phi_H(\chi\eta)^s}. \tag{3.20}$$

Our next goal is to analyse the sum over $\operatorname{Hom}(\mathbf{A}^*/k^*, J)$ via Poisson summation.

Let $s \in \mathbb{C}$. We start by computing the local Fourier transforms $h_{\eta_v,v}(x_v; s)$ of the functions $\chi_v \mapsto f_{\eta_v,v}(\chi_v)\Phi_H(\chi_v\eta_v)^{-s}$ on $\operatorname{Hom}(k_v^*, J)$. With the Haar measure on $\operatorname{Hom}(k_v^*, J)$ normalised as in [3, §3.2], they are given for $x_v \in k_v^* \otimes J^\wedge$ by the formula

$$h_{\eta_v,v}(x_v; s) = \sum_{\chi_v \in \operatorname{Hom}(k_v^*, J)} \frac{f_{\eta_v,v}(\chi_v)\langle \chi_v, x_v\rangle}{\Phi_H(\chi_v\eta_v)^s} \cdot \begin{cases} 1 & \text{if } v \mid \infty, \\ |J|^{-1} & \text{if } v \nmid \infty. \end{cases} \tag{3.21}$$

As in [3, §4.3], we denote the local Fourier transform of $f_v\Phi_J^{-s}$ by $\widehat{f}_{J,v}(x_v; s)$.

As in the proof of Lemma C11, we use our uniformiser π_v to split the sequence $1 \to \mathcal{O}_v^* \to k_v^* \to k_v^*/\mathcal{O}_v^* \to 1$, thus writing $\operatorname{Hom}(k_v^*, J) = \operatorname{Hom}(\mathcal{O}_v^*, J) \times \operatorname{Hom}(k_v^*/\mathcal{O}_v^*, J)$. Hence, we write $\chi_v \in \operatorname{Hom}(k_v^*, J)$ as $\chi_v = \chi_{v,\mathrm{r}}\chi_{v,\mathrm{ur}}$ with $\chi_{v,\mathrm{r}} \in \operatorname{Hom}(\mathcal{O}_v^*, J)$ and $\chi_{v,\mathrm{ur}} \in \operatorname{Hom}(k_v^*/\mathcal{O}_v^*, J)$. Recall, moreover, the abuse of notation introduced before the statement of Lemma C11.

Lemma C17. *Let* $\operatorname{Re}(s) \geq 0$, *let* v *be a place of* k, *let* $\eta_v \in \operatorname{Hom}(k_v^*, H)$ *and let* $x_v \in k_v^* \otimes J^\wedge$. *Write* V *for the subgroup of order* Q *of* G *generated by* $e_1^{Q^{a_1-1}}$.

(1) *If* $v \notin S$ *and* $x_v \notin \mathcal{O}_v^* \otimes J^\wedge$, *then* $h_{\eta_v,v}(x_v; s) = 0$.

(2) *If* $\eta_v \in \operatorname{Hom}(k_v^*, J)$, *then*
$$h_{\eta_v,v}(x_v; s) = \langle \eta_v^{-1}, x_v\rangle \widehat{f}_{J,v}(x_v; Qs).$$

(3) *If* $v \notin S$ *with* $q_v \not\equiv 1 \bmod Q$, $x_v \in \mathcal{O}_v^* \otimes J^\wedge$ *and* $\eta_v|_{\mathcal{O}_v^*} \in \operatorname{Hom}(\mathcal{O}_v^*, J)$, *then*
$$h_{\eta_v,v}(x_v; s) = \langle (\eta_v|_{\mathcal{O}_v^*})^{-1}, x_v\rangle + O(q_v^{-(\alpha(H)+1)s}).$$

(4) *If* $v \notin S$ *with* $q_v \equiv 1 \bmod Q$, $x_v \in \mathcal{O}_v^* \otimes J^\wedge$ *and* $\eta_v|_{\mathcal{O}_v^*} \in \operatorname{Hom}(\mathcal{O}_v^*, J)$, *but* $\eta_v \notin \operatorname{Hom}(k_v^*, J)$, *then*
$$\begin{aligned} h_{\eta_v,v}(x_v; s) = \langle (\eta_v|_{\mathcal{O}_v^*})^{-1}, x_v\rangle \left(1 + \left(Q^{\beta_H}\mathbb{1}_{\mathcal{O}_v^{*Q}\otimes J^\wedge}(x_v) - Q\mathbb{1}_{\mathcal{O}_v^{*Q}\otimes V^\wedge}(x_v)\right) q_v^{-\alpha(H)s}\right) \\ + O(q_v^{-(\alpha(H)+1)s}). \end{aligned}$$

(5) *If* $v \notin S$ *and* $\eta_v|_{\mathcal{O}_v^*} \notin \operatorname{Hom}(\mathcal{O}_v^*, J)$, *then*
$$h_{\eta_v,v}(x_v; s) = O(q_v^{-(\alpha(H)+1)s}).$$

Proof. Note that $\beta_G = \beta_J$ implies that $V \subseteq J$.

(1): Recall that $J = H \cap L$. For $\chi_v \in \operatorname{Hom}(k_v^*, H)$ and $\psi_v \in \operatorname{Hom}(k_v^*/\mathcal{O}_v^*, J)$, we have $(\chi_v\psi_v)(\mathcal{O}_v^*) = \chi_v(\mathcal{O}_v^*)$ and $(\chi_v\psi_v)(\pi_v) \in J$ if and only if $\chi_v(\pi_v) \in J$. Hence, $f_v(\chi_v) = f_v(\chi_v\psi_v)$. Therefore, $h_{\eta_v,v}(x_v; s)$ equals

$$\frac{1}{|J|} \sum_{\chi_{v,\mathrm{r}} \in \operatorname{Hom}(\mathcal{O}_v^*, J)} \frac{f_v(\chi_{v,\mathrm{r}}\eta_v)\langle \chi_{v,\mathrm{r}}, x_v\rangle}{\Phi_H(\chi_{v,\mathrm{r}}\eta_v)^s} \sum_{\chi_{v,\mathrm{ur}} \in \operatorname{Hom}(k_v^*/\mathcal{O}_v^*, J)} \langle \chi_{v,\mathrm{ur}}, x_v\rangle.$$

The inner sum in the last expression is 0 for $x_v \notin \mathcal{O}_v^* \otimes J^\wedge$ by character orthogonality.

(2): Upon replacing χ_v by $\chi_v\eta_v$ and observing that $\Phi_H(\chi_v) = \Phi_J(\chi_v)^Q$, we see from (3.21) that

$$\begin{aligned} h_{\eta_v,v}(x_v;s) &= \langle \eta_v^{-1}, x_v\rangle \sum_{\chi_v \in \mathrm{Hom}(k_v^*,J)} \frac{f_v(\chi_v)\langle \chi_v, x_v\rangle}{\Phi_J(\chi_v)^{Qs}} \cdot \begin{cases} 1 & \text{if } v \mid \infty, \\ |J|^{-1} & \text{if } v \nmid \infty. \end{cases} \\ &= \langle \eta_v^{-1}, x_v\rangle \widehat{f}_{J,v}(x_v;Qs). \end{aligned}$$

(3) and *(4)*: Again identifying $k_v^*/\mathcal{O}_v^* = \langle \pi_v\rangle$, we write $\eta_v = \eta_{v,\mathrm{r}}\eta_{v,\mathrm{ur}}$ with $\eta_{v,r} \in \mathrm{Hom}(\mathcal{O}_v^*, J)$ and $\eta_{v,\mathrm{ur}} \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, H)$. Upon noting that $\Phi_H(\chi_v\eta_v) = \Phi_J(\chi_v\eta_{v,\mathrm{r}})^Q$ and replacing the variable χ_v by $\chi_v\eta_{v,\mathrm{r}}$, we see from (3.21) that

$$h_{\eta_v,v}(x_v;s) = \frac{\langle \eta_{v,\mathrm{r}}^{-1}, x_v\rangle}{|J|} \sum_{\chi_v \in \mathrm{Hom}(k_v^*,J)} \frac{f_v(\chi_v\eta_{v,\mathrm{ur}})\langle \chi_v, x_v\rangle}{\Phi_J(\chi_v)^{Qs}}.$$

Hence, $h_{\eta_v,v}(x_v;s)$ is just $\langle (\eta_v|_{\mathcal{O}_v^*})^{-1}, x_v\rangle$ times the Fourier transform of the function $f_{\eta_{v,\mathrm{ur}},v}\Phi_J^{-Qs}$, which we may evaluate using [3, Lemma 3.3]. This shows that $h_{\eta_v,v}(x_v;s)$ is equal to $\langle (\eta_v|_{\mathcal{O}_v^*})^{-1}, x_v\rangle$ times

$$\sum_{d \mid (\exp(J), q_v - 1)} \left(\sum_{\substack{\chi_v \in \mathrm{Hom}(\mathcal{O}_v^*,J) \\ \ker \chi_v = \mathcal{O}_v^{*d}}} \langle \chi_v, x_v\rangle \tau_{f_{\eta_{v,\mathrm{ur}},v}}(\chi_v, x_v) \right) q_v^{-|J|(1-1/d)Qs}, \tag{3.22}$$

where

$$\tau_{f_{\eta_{v,\mathrm{ur}},v}}(\chi_v, x_v) := \frac{1}{|J|} \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*,J)} f_v(\chi_v\psi_v\eta_{v,\mathrm{ur}})\langle \psi_v, x_v\rangle.$$

Clearly, for $\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*, J)$ we have $f_v(\psi_v\eta_{v,\mathrm{ur}}) = 1$, as $(\psi_v\eta_{v,\mathrm{ur}})(\mathcal{O}_v^*) = \{1\}$. Hence, the summand for $d = 1$ in (3.22) is equal to

$$\tau_{f_{\eta_{v,\mathrm{ur}},v}(1,x_v)} = \frac{1}{|J|} \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*,J)} \langle \psi_v, x_v\rangle = 1.$$

If $q_v \not\equiv 1 \bmod Q$, there is no summand with $d = Q$, and the further summands are $\ll q_v^{-(\alpha(H)+1)s}$, as desired in *(3)*.

If $q_v \equiv 1 \bmod Q$, then the next summand appears for $d = Q$. Our assumptions on η_v in *(4)* imply that $\eta_{v,\mathrm{ur}}(\pi_v) \notin J$. Hence, $f_v(\chi_v\psi_v\eta_{v,\mathrm{ur}}) = 1$ if and only if $e_1^{Q^{a_1-1}} \notin \chi_v(\mathcal{O}_v^*)$. Hence, the summand for $d = Q$ in (3.22) is

$$\begin{aligned} &\frac{q_v^{-\alpha(H)s}}{|J|} \sum_{\substack{\chi_v \in \mathrm{Hom}(\mathcal{O}_v^*,J) \\ \ker \chi_v = \mathcal{O}_v^{*Q} \\ \chi_v(\mathcal{O}_v^*) \neq V}} \langle \chi_v, x_v\rangle \sum_{\psi_v \in \mathrm{Hom}(k_v^*/\mathcal{O}_v^*,J)} \langle \psi_v, x_v\rangle \\ =&q_v^{-\alpha(H)s} \left(\sum_{\chi_v \in \mathrm{Hom}(\mathcal{O}_v^*/\mathcal{O}_v^{*Q},J)} \langle \chi_v, x_v\rangle - \sum_{\chi_v \in \mathrm{Hom}(\mathcal{O}_v^*/\mathcal{O}_v^{*Q},V)} \langle \chi_v, x_v\rangle \right) \\ =& q_v^{-\alpha(H)s}(Q^{\beta_H}\mathbb{1}_{\mathcal{O}_v^{*Q}\otimes J^\wedge}(x_v) - Q\mathbb{1}_{\mathcal{O}_v^{*Q}\otimes V^\wedge}(x_v)). \end{aligned}$$

Here we have repeatedly used that $\mathcal{O}_v^*/\mathcal{O}_v^{*Q}$ is a cyclic group of order Q for $v \notin S$.

(5): For all $\chi_v \in \operatorname{Hom}(k_v^*, J)$, we have $(\chi_v\eta_v)|_{\mathcal{O}_v^*} \in \operatorname{Hom}(\mathcal{O}_v^*, H) \smallsetminus \operatorname{Hom}(\mathcal{O}_v^*, J)$. Hence, $(\chi_v\eta_v)(\mathcal{O}_v^*)$ contains an element of order Q^{a_t}, which shows that $\ker((\chi_v\eta_v)|_{\mathcal{O}_v^*}) = \mathcal{O}_v^{*d}$ for some d divisible by Q^{a_t}, so in particular $Q^2 \mid d$. This shows that

$$\Phi_H(\chi_v\eta_v) = \prod_{\psi \in (\mathcal{O}_v^*/\mathcal{O}_v^{*d})^\wedge} \Phi(\psi)^{|H|/d} = q_v^{|H|(1-1/d)}.$$

Hence, every summand in (3.21) is $\ll q_v^{-|H|(1-1/d)s} \ll q_v^{-(\alpha(H)+1)s}$. □

Note that $f_v(\chi_v) = 1$ for all $v \notin S$ and $\chi_v \in \operatorname{Hom}(k_v^*, J)$. Hence, by [3, Lemma 4.1], the local Fourier transforms $\widehat{f}_{J,v}(x_v; Qs)$ appearing in case *(2)* of Lemma C17 take, for $v \notin S$ and $x_v \in \mathcal{O}_v^* \otimes J^\wedge$, the shape

$$\begin{cases} 1 + (Q^{\beta_H}\mathbb{1}_{\mathcal{O}_v^{*Q}\otimes J^\wedge}(x_v) - 1)q_v^{-\alpha(H)s} + O(q_v^{-(\alpha(H)+1)s}), & q_v \equiv 1 \bmod Q, \\ 1 + O(q_v^{-(\alpha(H)+1)s}), & q_v \not\equiv 1 \bmod Q. \end{cases} \tag{3.23}$$

Lemma C18. *Let* $\eta \in \operatorname{Hom}(\mathbf{A}^*, H)$.

(1) Let $x = (x_v)_v \in \mathbf{A}^* \otimes J^\wedge$. *Then the product* $h_\eta(x; s) := \prod_v h_{\eta_v,v}(x_v; s)$ *converges absolutely for* $\operatorname{Re} s > 1/\alpha(H)$ *and defines a holomorphic function in this half-plane.*

(2) Let $\operatorname{Re} s > 1/\alpha(H)$. *Then the function* $x \mapsto h_\eta(x; s)$ *on* $\mathbf{A}^* \otimes J^\wedge$ *is the Fourier transform of the function* $\chi \mapsto f_\eta(\chi)\Phi_H(\chi\eta)^{-s}$ *on* $\operatorname{Hom}(\mathbf{A}^*, J)$.

Proof. (1): Almost all places v of k satisfy $v \notin S$, $x_v \in \mathcal{O}_v^* \otimes J$ and $\eta_v|_{\mathcal{O}_v^*} = 1$. For such v, Lemma C17 and (3.23) show that $h_{\eta_v,v}(x_v; s) = 1 + O(q_v^{-\alpha(H)s})$, which is sufficient to prove *(1)*.

(2): For all $v \notin S$ with $\eta_v|_{\mathcal{O}_v^*} = 1$, we see that $f_{\eta_v,v}(\chi_v)\Phi_H(\chi_v\eta_v)^{-s} = 1$ for all $\chi_v \in \operatorname{Hom}(k_v^*/\mathcal{O}_v^*, J)$. Hence, *(2)* follows from *(1)* by standard tools in Fourier analysis on restricted direct products, see e.g. [9, I, Lemma 4]. □

We require the following version of Poisson summation.

Lemma C19. *Let* $\operatorname{Re} s > 1/\alpha(H)$ *and* $\eta \in \operatorname{Hom}(\mathbf{A}^*, H)$. *Then*

$$\sum_{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, J)} \frac{f_\eta(\chi)}{\Phi_H(\chi\eta)^s} = \frac{1}{|\mathcal{O}_k^* \otimes J^\wedge|} \sum_{x \in k^* \otimes J^\wedge} h_\eta(x; s). \tag{3.24}$$

The sum on the right-hand side has only finitely many non-zero terms and defines a holomorphic function on $\operatorname{Re} s > 1/\alpha(H)$.

Proof. We know from Lemma C18 that $h_\eta(x; s)$ is the Fourier transform of the summand on the left-hand side.

We consider the finite set of places $T = S \cup \{v \ : \ \eta_v|_{\mathcal{O}_v^*} \neq 1\}$. For $v \notin T$ and $\chi_v \in \operatorname{Hom}(k_v^*, J)$, we have $\Phi_H(\chi_v\eta_v)^s = \Phi_H(\chi_v)^s = \Phi_J(\chi_v)^{Qs}$. As, moreover, the function $f_{\eta_v,v}(\cdot)$ on $\operatorname{Hom}(k_v^*, J)$ is invariant under multiplication of its argument by elements of $\operatorname{Hom}(k_v^*/\mathcal{O}_v^*, J)$, we see that

$$h_{\eta_v,v}(x_v; s) = \\ \frac{1}{|J|} \sum_{\chi_{v,\mathrm{r}} \in \operatorname{Hom}(\mathcal{O}_v^*, J)} \frac{f_{\eta_v}(\chi_{v,r})\langle \chi_{v,r}, x_v\rangle}{\Phi_J(\chi_{v,r})^{Qs}} \sum_{\chi_{v,\mathrm{ur}} \in \operatorname{Hom}(k_v^*/\mathcal{O}_v^*, J)} \langle \chi_{v,\mathrm{ur}}, x_v\rangle.$$

By character orthogonality, we obtain for $x_v \in \mathcal{O}_v^* \otimes J^\wedge$ that

$$h_{\eta_v,v}(x_v;s) = \sum_{\chi_{v,r}\in \mathrm{Hom}(\mathcal{O}_v^*,J)} \frac{f_{\eta_v}(\chi_{v,r})\langle \chi_{v,r}, x_v\rangle}{\Phi_J(\chi_{v,r})^{Qs}},$$

and otherwise $h_{\eta_v,v}(x_v;s) = 0$. In particular, the sum over x on the right-hand side in (3.24) is finite. Now the desired result follows by exchanging the order of summation, as in the proof of [4, Proposition 3.9]. □

Point *(1)* of Lemma C17 shows that $h_\eta(x;s) = 0$ for all η, unless $x \in \mathcal{O}_S^* \otimes J^\wedge$. Continuing from (3.20), Lemma C19 thus shows that

$$F_{H,f}(s) = \frac{1}{|\mathcal{O}_k^* \otimes J^\wedge|} \sum_{\eta \in R} \sum_{x \in \mathcal{O}_S^* \otimes J^\wedge} h_\eta(x;s). \tag{3.25}$$

Next, we analyse the individual Fourier transforms $h_\eta(x;s)$ in terms of various zeta functions.

Recall that $k_0 = k(\mu_Q)$ and let k_x/k_0 be the elementary abelian Q-extension associated to $x \in k^* \otimes J^\wedge$ as in [3, Section 4.4]. We also define $k_{x,1}/k_0$ to be the extension associated to $x \in k^* \otimes J^\wedge$ that is induced via Kummer theory by any homomorphism $\psi : \mu_Q \to J$ whose image is $\langle e_1^{Q^{a_1-1}} \rangle = V$. Namely, ψ induces a map $\Psi \in \mathrm{Hom}(k^* \otimes J^\wedge, k_0^*/k_0^{*Q})$ as follows:

$$\Psi : k^* \otimes J^\wedge \xrightarrow{\iota \otimes \psi^\wedge} k_0^* \otimes {\mu_Q}^\wedge = k_0^*/k_0^{*Q},$$

where ι denotes the natural inclusion $k^* \to k_0^*$. The image of $x \in k^* \otimes J^\wedge$ under Ψ defines the extension $k_{x,1}/k_0$ by Kummer theory.

The various conditions on q_v and x appearing in Lemma C17 can be interpreted in terms of splitting conditions in these extensions of k as follows. Let $v \notin S$ and $x \in \mathcal{O}_S^* \otimes J^\wedge$. Then

$$\begin{aligned} v \text{ splits completely in } k_0/k &\iff q_v \equiv 1 \bmod Q, \\ v \text{ splits completely in } k_{x,1}/k &\iff q_v \equiv 1 \bmod Q \text{ and } x_v \in \mathcal{O}_v^{*Q} \otimes V^\wedge, \\ v \text{ splits completely in } k_x/k &\iff q_v \equiv 1 \bmod Q \text{ and } x_v \in \mathcal{O}_v^{*Q} \otimes J^\wedge. \end{aligned}$$

Moreover, for $\eta \in \mathrm{Hom}(\mathbf{A}^*/k^*, H)$, let k_η be the fixed field of $\mathrm{Im}\,\eta \cap J$ inside the Galois extension of k defined by η. In other words, k_η is the extension corresponding to the image of η under the map $\mathrm{Hom}(\mathbf{A}^*/k^*, H) \to \mathrm{Hom}(\mathbf{A}^*/k^*, H/J)$ induced by the natural map $H \to H/J$. Thus, k_η/k is a Galois extension with $\mathrm{Gal}(k_\eta/k) = \mathrm{Im}\,\eta/(\mathrm{Im}\,\eta \cap J) \hookrightarrow H/J \cong \mathbb{Z}/Q\mathbb{Z}$, and any $v \notin S$ satisfies:

$$\begin{aligned} v \text{ is unramified in } k_\eta/k &\iff \eta_v(\mathcal{O}_v^*) \subseteq J, \\ v \text{ splits completely in } k_\eta/k &\iff \eta_v(k_v^*) \subseteq J. \end{aligned}$$

For a place $v \nmid \infty$ of k and an extension $K \supset k$, we write

$$\mathrm{spl}(K,v) := \begin{cases} 1 & \text{if } v \text{ splits completely in } K/k, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma C20. *Let* $\operatorname{Re} s \geq 0$*,* $v \notin S$*,* $x \in \mathcal{O}_S^* \otimes J^\wedge$ *and* $\eta \in \operatorname{Hom}(\mathbf{A}^*/k^*, H)$ *such that* $\eta_v|_{\mathcal{O}_v^*} \in \operatorname{Hom}(\mathcal{O}_v^*, J)$*. Then*

$$\begin{aligned} h_{\eta_v,v}(x_v; s) =& \langle (\eta_v|_{\mathcal{O}_v^*})^{-1}, x_v\rangle \Big(1 \\ &+ \left(Q^{\beta_H} \operatorname{spl}(k_x, v) - Q \operatorname{spl}(k_{x,1}, v) + Q \operatorname{spl}(k_\eta k_{x,1}, v) - \operatorname{spl}(k_\eta k_0, v)\right) q_v^{-\alpha(H)s} \\ &+ O(q_v^{-(\alpha(H)+1)s})\Big). \end{aligned}$$

Proof. This follows from Lemma C17 together with the description of $\widehat{f}_{J,v}(x_v; Qs)$ in (3.23) and the characterisation of the various splitting conditions immediately preceding the statement of Lemma C20. □

For any Galois extension K/k, let $\zeta_K(s)$ be its Dedekind zeta function. For an archimedean place $v \in \Omega_k$, let $\zeta_{K,v}(s) := 1$. For v non-archimedean, we define the local factor at v of ζ_K as

$$\zeta_{K,v}(s) := \prod_{\substack{w \in \Omega_K \\ w \mid v}} \frac{1}{1 - q_v^{-f(v)s}} = 1 + \#\{w \mid v\} q_v^{-f(v)s} + O_{[K:k]}\left(q_v^{-2f(v)s}\right),$$

where $f(v)$ is the inertia degree of v in K. Lemma C20 shows that, for $\operatorname{Re} s > 1$, $v \notin S$, $x \in \mathcal{O}_S^* \otimes J^\wedge$ and $\eta \in \operatorname{Hom}(\mathbf{A}^*/k^*, H)$ with $\eta_v|_{\mathcal{O}_v^*} \in \operatorname{Hom}(\mathcal{O}_v^*, J)$, we have

$$h_{\eta_v,v}(x_v; s) = \langle (\eta_v|_{\mathcal{O}_v^*})^{-1}, x_v\rangle \left(L_v(\eta, x; \alpha(H)s) + O(q_v^{-(\alpha(H)+1)s})\right), \tag{3.26}$$

where $L_v(\eta, x; s)$ is the local factor at v of the function

$$L(\eta, x; s) := \zeta_{k_x}(s)^{\frac{Q^{\beta_H}}{[k_x:k]}} \zeta_{k_{x,1}}(s)^{\frac{-Q}{[k_{x,1}:k]}} \zeta_{k_\eta k_{x,1}}(s)^{\frac{Q}{[k_\eta k_{x,1}:k]}} \zeta_{k_\eta k_0}(s)^{\frac{-1}{[k_\eta k_0:k]}}.$$

Write

$$\nu(\eta, x) := \frac{1}{[k_{x,1} : k]}\left(\frac{Q^{\beta_H}}{[k_x : k_{x,1}]} - Q\right) + \frac{1}{[k_\eta k_0 : k]}\left(\frac{Q}{[k_\eta k_{x,1} : k_\eta k_0]} - 1\right).$$

As $[k_x : k_{x,1}] \mid Q^{\beta_H - 1}$ and $[k_\eta k_{x,1} : k_\eta k_0] \mid Q$, it is clear that $\nu(\eta, x) \geq 0$.

Lemma C21. *Let* $\eta \in \operatorname{Hom}(\mathbf{A}^*/k^*, H)$ *and* $x \in \mathcal{O}_S^* \otimes J^\wedge$*. Then we have*

$$\nu(\eta, x) \leq \frac{Q^{\beta_H} - 1}{[k_0 : k]} = \nu(k, H),$$

with equality precisely when $\eta \in \operatorname{Hom}(\mathbf{A}^*/k^*, J)$ *and* $k_x = k_0$*.*

Proof. Directly from the definition we see that $\nu(\eta, x)$ is maximal if and only if all the occurring degrees are minimal. Hence the upper bound, which is attained precisely when $k_x = k_0$ and $k_\eta \subseteq k_0$. As $\operatorname{Gal}(k_\eta/k) = \operatorname{Im}\eta/(\operatorname{Im}\eta \cap J) \hookrightarrow H/J \cong \mathbb{Z}/Q\mathbb{Z}$ and $[k_0 : k] \mid Q - 1$, the latter condition holds if and only if $k_\eta = k$, which means that $\operatorname{Im}\eta \subseteq J$. □

For $\eta \in \operatorname{Hom}(\mathbf{A}^*/k^*, H)$, we write

$$T(\eta) := \{v \in \Omega_k \ : \ v \notin S,\ \eta_v|_{\mathcal{O}_v^*} \notin \operatorname{Hom}(\mathcal{O}_v^*, J)\}.$$

Note that $T(\eta)$ is the set of places $v \notin S$ where k_η/k is ramified, in particular it is finite.

Lemma C22. *Let* $\eta \in \operatorname{Hom}(\mathbf{A}^*/k^*, H)$ *and* $x \in \mathcal{O}_S^* \otimes J^\wedge$. *Then there is a function* $g_0(\eta, x; s)$, *holomorphic in an open neighbourhood of the half-plane* $\operatorname{Re} s \geq 1$, *such that, for* $\operatorname{Re} s > 1$,

$$L(\eta, x; s) = \zeta_{k_0}(s)^{\nu(\eta,x)} \cdot g_0(\eta, x; s).$$

Moreover, for any $\epsilon > 0$ *the function* $g_0(\eta, x; s)$ *satisfies on* $\operatorname{Re} s \geq 1$ *the bounds*

$$g_0(\eta, x; s) \ll_\epsilon (1 + |\operatorname{Im} s|)^\epsilon \prod_{v \in T(\eta)} q_v^\epsilon,$$

$$g_0'(\eta, x; s) \ll_\epsilon (1 + |\operatorname{Im} s|)^\epsilon \prod_{v \in T(\eta)} q_v^\epsilon.$$

Proof. We start by observing that, for $\operatorname{Re} s > 1$,

$$\begin{aligned} \frac{L(\eta, x; s)}{\zeta_{k_0}(s)^{\nu(\eta,x)}} = \left(\frac{\zeta_{k_x}}{\zeta_{k_0}}(s)\right)^{\frac{1}{[k_{x,1}:k]}\left(\frac{Q^\beta H}{[k_x:k_{x,1}]} - Q\right)} \left(\frac{\zeta_{k_x}}{\zeta_{k_{x,1}}}(s)\right)^{\frac{Q}{[k_{x,1}:k]}} \\ \left(\frac{\zeta_{k_\eta k_{x,1}}}{\zeta_{k_0}}(s)\right)^{\frac{1}{[k_\eta k_0:k]}\left(\frac{Q}{[k_\eta k_{x,1}:k_\eta k_0]} - 1\right)} \left(\frac{\zeta_{k_\eta k_{x,1}}}{\zeta_{k_\eta k_0}}(s)\right)^{\frac{1}{[k_\eta k_0:k]}}. \end{aligned} \tag{3.27}$$

Recall that for a tower $F/K/k_0$ of number fields with F/k_0 abelian, we have

$$\frac{\zeta_F}{\zeta_K}(s) = \prod_\chi L(s; \chi),$$

where χ runs through the characters of $\mathbf{A}_{k_0}^*/k_0^* N_{F/k_0} \mathbf{A}_F^*$ that are non-trivial on $N_{K/k_0}\mathbf{A}_K^*$. In particular, each of the occurring L-functions is entire. Next, we observe that the Kummer extensions k_x/k_0 and $k_\eta k_{x,1}/k_0$ are ramified only at places of k_0 lying above places in $S \cup T(\eta)$. Outside S, the ramification is tame, hence the conductor of each of these extensions has absolute norm $\ll \prod_{v \in T(\eta)} q_v^{[k_0:k]} \ll \prod_{v \in T(\eta)} q_v^{Q-1}$.

Each appearing L-function $L(s; \chi)$ is zero-free on some open neighbourhood of the half-plane $\operatorname{Re} s \geq 1$, and thus has holomorphic $[k_{x,1}k_\eta : k]$-th roots. Hence, the function

$$g_0(\eta, x; s) = \frac{L(\eta, x; s)}{\zeta_{k_0}(s)^{\nu(\eta,x)}},$$

as a product of non-negative integer powers of such roots for all occurring $L(s; \chi)$, is holomorphic on an open neighbourhood of $\operatorname{Re} s \geq 1$.

Now let $\operatorname{Re} s \geq 1$. Lemma C10, applied to all the $L(s; \chi)$, shows that

$$g_0(\eta, x; s) \ll_\epsilon (1 + |\operatorname{Im} s|)^\epsilon \prod_{v \in T(\eta)} q_v^\epsilon.$$

Using Lemma C10 again, the simple observation that, for arbitrary exponents $\beta \in \mathbb{R}$,

$$(L(s; \chi)^\beta)' = \beta \frac{L'}{L} L^\beta(s; \chi), \tag{3.28}$$

and the product rule, we see that also

$$g_0'(\eta, x; s) \ll_\epsilon (1 + |\operatorname{Im} s|)^\epsilon \prod_{v \in T(\eta)} q_v^\epsilon,$$

as desired. □

Lemma C23. *Let* $\eta \in \mathrm{Hom}(\mathbf{A}^*/k^*, H)$ *and* $x \in \mathcal{O}_S^* \otimes J^\wedge$. *Then there is a function* $g(\eta, x; s)$, *holomorphic in an open neighbourhood of* $\mathrm{Re}\, s \geq 1/\alpha(H)$, *such that, for* $\mathrm{Re}\, s > 1/\alpha(H)$,

$$h_\eta(x; s) = \zeta_{k_0}(\alpha(H)s)^{\nu(\eta,x)} \cdot g(\eta, x; s).$$

Moreover, for any $\epsilon > 0$, *the function* $g(\eta, x; s)$ *satisfies on* $\mathrm{Re}\, s \geq 1/\alpha(H)$ *the bounds*

$$g(\eta, x; s) \ll_\epsilon (1 + |\operatorname{Im} s|)^\epsilon \prod_{v \in T(\eta)} q_v^{-(\alpha(H)+1)s+\epsilon} \tag{3.29}$$

$$g'(\eta, x; s) \ll_\epsilon (1 + |\operatorname{Im} s|)^\epsilon \prod_{v \in T(\eta)} q_v^{-(\alpha(H)+1)s+\epsilon}. \tag{3.30}$$

Proof. From Lemma C17 and (3.26), we see that $h_\eta(x; s)$ takes the form

$$\prod_{v \in S} h_{\eta_v, v}(x_v; s) \prod_{v \in T(\eta)} O(q_v^{-(\alpha(H)+1)s}) \prod_{\substack{v \notin S \cup T(\eta) \\ \eta_v|_{\mathcal{O}_v^*} \neq 1}} \langle (\eta_v|_{\mathcal{O}_v^*})^{-1}, x_v \rangle$$
$$\prod_{v \notin S \cup T(\eta)} (L_v(\eta, x; \alpha(H)s) + O(q_v^{-(\alpha(H)+1)s})).$$

If $\mathrm{Re}\, s > 1/(\alpha(H)+1)$, then $h_{\eta_v, v}(x_v; s) \ll 1$ and $1 \ll \zeta_{K/k_0, v}(\alpha(H)s) \ll 1$ for all $v \in \Omega_k$ and all Galois extensions K/k_0 of degree $[K : k_0] \leq Q^{\beta_H+1}$, which covers all extensions occurring in the definition of $L(\eta, x; s)$. Therefore, there is a function $g_1(\eta, x; s)$, holomorphic on $\mathrm{Re}(s) > 1/(\alpha(H)+1)$ and satisfying

$$g_1(\eta, x; s) \ll_\epsilon \prod_{v \in T(\eta)} q_v^{-(\alpha(H)+1)s+\epsilon} \tag{3.31}$$

on $\mathrm{Re}(s) \geq 1/(\alpha(H)+1) + \epsilon$ for any small $\epsilon > 0$, such that, for $\mathrm{Re}(s) > 1/\alpha(H)$,

$$h_\eta(x; s) = L(\eta, x; \alpha(H)s) \cdot g_1(\eta, x; s) = \zeta_{k_0}(\alpha(H)s)^{\nu(\eta,x)} g_0(\eta, x; \alpha(H)s) g_1(\eta, x; s),$$

where $g_0(\eta, x; s)$ is as in Lemma C22. Hence, the function

$$g(\eta, x; s) := g_0(\eta, x; \alpha(H)s) g_1(\eta, x; s)$$

is holomorphic in an open neighbourhood of $\mathrm{Re}\, s \geq 1/\alpha(H)$ and satisfies (3.29). Next, we use Cauchy's integral formula to bound $g_1'(\eta, x; s)$ in terms of our bound (3.31) for $g_1(\eta, x; s)$. Together with the bounds for g_0 and g_0' from Lemma C22 and the product rule, this shows (3.30). □

Completion of the proof of Proposition C16. We start with the expression (3.25) for $F_{H,f}(s)$. Recall that our representative in R for the trivial class $\mathrm{Hom}(\mathbf{A}^*/k^*, J)$ is $\eta = 1$.

Consider the finite set

$$M := \{\nu(\eta, x) \;:\; \eta \in \mathrm{Hom}(\mathbf{A}^*/k^*, H),\; x \in \mathcal{O}_S^* \otimes J^\wedge\}.$$

We sort the values of η and x in (3.25) according to the value of $\nu(\eta, x)$ and obtain from Lemma C23, for $\operatorname{Re} s > 1/\alpha(H)$, that

$$F_{H,f}(s) = \sum_{\nu \in M} \zeta_{k_0}(\alpha(H)s)^{\nu} g_{\nu}(s), \tag{3.32}$$

where

$$g_{\nu}(s) := \frac{1}{|\mathcal{O}_k^* \otimes J^\wedge|} \sum_{x \in \mathcal{O}_S^* \otimes J^\wedge} \sum_{\substack{\eta \in R \\ \nu(\eta,x)=\nu}} g(\eta, x; s).$$

Let us fix ν and study the function $g_\nu(s)$. The sum over x is finite, so let us focus on the sum over η.

Note that the map induced by $H \to H/J$ embeds R into $\operatorname{Hom}(\mathbf{A}^*/k^*, H/J)$. Denote the image of η under this map by $\bar{\eta}$ and recall that k_η/k is the extension corresponding to $\bar{\eta}$. The places in $T(\eta)$ are exactly those places not in S where k_η/k is ramified. Therefore, $\prod_{v \in T(\eta)} q_v \asymp \Phi(\bar{\eta})$, the conductor of $\bar{\eta}$. We obtain, for $\operatorname{Re}(s) \geq 1/\alpha(H)$, that

$$\prod_{v \in T(\eta)} q_v^{-(\alpha(H)+1)\operatorname{Re} s + \epsilon} \ll \Phi(\bar{\eta})^{-(1+1/\alpha(H))+\epsilon}.$$

Recall that $H/J \simeq \mu_Q$ by Lemma C14(1). For sufficiently small ϵ, the sum

$$\sum_{\psi \in \operatorname{Hom}(\mathbf{A}^*/k^*, \mu_Q)} \Phi(\psi)^{-(1+1/\alpha(H))+\epsilon}$$

converges, as it is equal to the value at $s = 1 + 1/\alpha(H) - \epsilon$ of the Dirichlet series

$$\sum_{\psi \in \operatorname{Hom}(\mathbf{A}^*/k^*, \mu_Q)} \frac{1}{\Phi(\psi)^s},$$

which converges absolutely for $\operatorname{Re} s > 1$ by [10, Lemma 2.10]. By the dominated convergence theorem, the function $g_\nu(s)$ and its derivative extend continuously to $\operatorname{Re} s \geq 1/\alpha(H)$. Now we take

$$g_1(s) := \sum_{\nu \in M} \zeta_{k_0}(\alpha(H)s)^{\nu - \nu(k,H)} g_\nu(s), \tag{3.33}$$

so (3.16) follows from (3.32). As $\zeta_{k_0}^{-1}(\alpha(H)s)$ has no poles and only the zero at $s = 1/\alpha(H)$ in a neighbourhood of $\operatorname{Re} s \geq 1/\alpha(H)$, the negative powers of $\zeta_{k_0}(\alpha(H)s)$ in (3.33) extend continuously to $\operatorname{Re} s \geq 1/\alpha(H)$. Hence, the same holds for $g_1(s)$.

Let $\delta_0 := \nu(k,H) - \max(M \smallsetminus \{\nu(k,H)\}) > 0$ be the difference between $\nu(k,H)$ and the second biggest element of M. We set $a := 1/\alpha(H)$ and show that in a small compact convex neighbourhood C of a in $\operatorname{Re} s \geq a$, each summand of $g_1(s)$ satisfies

$$|\zeta_{k_0}(\alpha(H)s)^{\nu-\nu(k,H)} g_\nu(s) - \zeta_{k_0}(1)^{\nu-\nu(k,H)} g_\nu(a)| \ll_C |s-a|^{\min\{1,\delta_0\}}. \tag{3.34}$$

Then clearly the same will follow for $g_1(s)$, as desired. Let $s \in C$. In case $\nu = \nu(k,H)$, we consider the function $g_0(t) := g_{\nu(k,H)}(a + t(s-a))$ on $[0,1]$. This function is differentiable on $(0,1)$ with derivative

$$g_0'(t) = (s-a) g_{\nu(k,H)}'(a + t(s-a)) \ll_C |s-a|.$$

Note that to obtain this when $\operatorname{Re} s = a$, we have used Lemma C9. By the mean value theorem,

$$|g_{\nu(k,H)}(s) - g_{\nu(k,H)}(a)| = |g_0(1) - g_0(0)| \leq g_0'(\xi) \ll_C |s - a|,$$

where $\xi \in (0,1)$. This shows (3.34) in the case $\nu = \nu(k,H)$. When $\nu < \nu(k,H)$, then $\zeta_{k_0}(1)^{\nu-\nu(k,H)} = 0$ and

$$|\zeta_{k_0}(\alpha(H)s)^{\nu-\nu(k,H)} g_\nu(s)| \ll_C |\zeta_{k_0}(\alpha(H)s)^{-1}|^{\min\{1,\delta_0\}},$$

if C is sufficiently small so that $|\zeta_{k_0}(\alpha(H)s)^{-1}| \leq 1$ in C. Hence, (3.34) follows from the bound

$$|\zeta_{k_0}(\alpha(H)s)^{-1}| \ll_C |s - a|,$$

coming again from the mean value theorem and the fact that $\zeta_{k_0}(\alpha(H)s)^{-1}$ is holomorphic in a neighbourhood of $\operatorname{Re} s \geq a$.

Next, we isolate the summands for the representative $\eta = 1$ of the trivial class,

$$g_2(s) := \frac{1}{|\mathcal{O}_k^* \otimes J^\wedge|} \sum_{x \in \mathcal{O}_S^* \otimes J^\wedge} \zeta_{k_0}(\alpha(H)s)^{\nu(1,x)-\nu(k,H)} g(1,x;s).$$

This sum is finite, and Lemma C19 shows that for $\operatorname{Re} s > 1/\alpha(H)$,

$$\begin{aligned}\zeta_{k_0}(\alpha(H)s)^{\nu(k,H)} \cdot g_2(s) &= \frac{1}{|\mathcal{O}_k^* \otimes J^\wedge|} \sum_{x \in \mathcal{O}_S^* \otimes J^\wedge} h_1(x;s) \\ &= \sum_{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, J)} \frac{f(\chi)}{\Phi_H(\chi)^s} = F_{J,f}(Qs),\end{aligned}$$

as desired. Finally, we observe that

$$\begin{aligned}g_1(1/\alpha(H)) &= g_2(1/\alpha(H)) \\ &\quad + \frac{1}{|\mathcal{O}_k^* \otimes J^\wedge|} \sum_{\substack{\eta \in R \\ \eta \neq 1}} \sum_{x \in \mathcal{O}_S^* \otimes J^\wedge} \zeta_{k_0}(1)^{\nu(\eta,x)-\nu(k,H)} g(\eta, x; 1/\alpha(H)) \\ &= g_2(1/\alpha(H)),\end{aligned}$$

by Lemma C21. We still need to argue that $g_2(1/\alpha(H)) \neq 0$. For this, note that $f = 1$ on all of $\operatorname{Hom}(\mathbf{A}^*/k^*, J)$, as $J \subset L$, and moreover that $\beta_J = \beta_G = \beta_H$. From [11, Proposition 5.5] or [3, Lemma 4.7], we know that

$$F_{J,f}(Qs) = \sum_{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, J)} \frac{1}{\Phi_J(\chi)^{Qs}}$$

has a pole of order at most $\nu(k,J) = \nu(k,H)$ at $s = 1/(Q\alpha(J)) = 1/\alpha(H)$. On the other hand,

$$\{\chi \in \operatorname{Hom}(\mathbf{A}^*/k^*, J) \ : \ \Phi_J(\chi) \leq B^{1/Q}\} \supseteq \{\chi \in J\text{-ext}(k) \ : \ \Delta(\chi) \leq B^{1/Q}\},$$

and the counting function of the latter set grows with order $B^{1/\alpha(H)}(\log B)^{\nu(k,J)-1}$ by [11, Theorem I.2]. Hence, the pole can not be of order smaller than $\nu(k,J)$. □

What remains is to deduce the desired cancellation (3.15) from Proposition C16.

Lemma C24. *We have*

$$N(H, L; B^{|H|/|G|}) = N(J; B^{|J|/|G|)}) + o(B^{1/\alpha(G)}(\log B)^{\nu(k,G)-1}).$$

Proof. We apply Delange's Tauberian theorem (Theorem C8) to the results of Proposition C16.

We take $a := 1/\alpha(H)$ and $\omega := \nu(k, H) \in \mathbb{Z}_{\geq 1}$. Moreover, we write $u(s) := ((s-a)\zeta_{k_0}(\alpha(H)s))^{\omega}$. As $\zeta_{k_0}(\alpha(H)s)$ has a pole of order 1 at $s = a$ and no other poles or zeros in an open neighbourhood of $\mathrm{Re}(s) \geq a$, $u(s)$ extends to a holomorphic function on this neighbourhood with $u(a) \neq 0$. For $\mathfrak{f}(s) = F_{H,f}(s)$, we obtain

$$\mathfrak{g}(s) = (s-a)^{\omega}\mathfrak{f}(s) = u(s)g_1(s),$$

which is holomorphic on $\mathrm{Re}\, s > a$ and extends continuously to $\mathrm{Re}\, s \geq a$. We have

$$\mathfrak{g}(a) = u(a)g_1(a) \neq 0.$$

Moreover, fix a sufficiently small compact convex neighbourhood C of a in $\mathrm{Re}\, s \geq a$. For any $s \in C$, we get

$$\begin{aligned} |\mathfrak{g}(s) - \mathfrak{g}(a)| &\leq |u(s)| \cdot |g_1(s) - g_1(a)| + |g_1(a)| \cdot |u(s) - u(a)| \\ &\ll_C |s-a|^{\delta}. \end{aligned}$$

We have verified the requirements of Theorem C8, and hence

$$N(H, L; B^{|H|/|G|}) = \sum_{\Phi_H(\chi) \leq B^{|H|/|G|}} f(\chi) \sim \frac{\mathfrak{g}(a)}{a\Gamma(\omega)} \left(\frac{|H|}{|G|}\right)^{\omega-1} B^{1/\alpha(G)}(\log B)^{\omega-1}.$$

Noting that $f(\chi) = 1$ for all $\chi : \mathbf{A}^*/k^* \to J$, an analogous argument with $\mathfrak{f}(s) = F_{J,f}(Qs)$ shows that

$$N(J; B^{|J|/|G|}) = \sum_{\Phi_J(\chi)^Q \leq B^{|H|/|G|}} f(\chi) \sim \frac{\mathfrak{g}(a)}{a\Gamma(\omega)} \left(\frac{|H|}{|G|}\right)^{\omega-1} B^{1/\alpha(G)}(\log B)^{\omega-1},$$

indeed with the same value of $\mathfrak{g}(a)$, as $g_1(1/\alpha(H)) = g_2(1/\alpha(H))$ by Proposition C16. ☐

Plugging the results of Lemma C24 into (3.14) shows (3.1) in the case $a_t \geq 2$, thereby completing the proof of Theorem C4.

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