PROPAGATION ESTIMATES AND FREDHOLM ANALYSIS FOR THE TIME-DEPENDENT SCHRÖDINGER EQUATION

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Abstract. We study the time-dependent Schrödinger operator $P = D_t + \Delta_g + V$ acting on functions defined on $\mathbb{R}^{n+1}$, where, using coordinates $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$, D_t denotes $-i\partial_t$, Δ_g is the positive Laplacian with respect to a time dependent family of non-trapping metrics $g_{ij}(z,t)dz^i dz^j$ on $\mathbb{R}^n$ which are equal to the Euclidean metric outside of a compact set in spacetime, and $V = V(z,t)$ is a potential function which is also compactly supported in spacetime. In this paper we introduce a new approach to studying P, by finding pairs of Hilbert spaces between which the operator acts invertibly.

Using this invertibility it is straightforward to solve the 'final state problem' for the time-dependent Schrödinger equation, that is, find a global solution $u(z,t)$ of $Pu = 0$ having prescribed asymptotics as $t \to +\infty$. These asymptotics are of the form

$$u(z,t) \sim t^{-n/2} e^{i|z|^2/4t} f_+\big(\frac{z}{2t}\big), \quad t \to +\infty$$

where f_+, the 'final state' or outgoing data, is an arbitrary element of a suitable function space $\mathcal{W}^k(\mathbb{R}^n)$; here k is a regularity parameter simultaneously measuring smoothness and decay at infinity. We can of course equally well prescribe asymptotics as $t \to -\infty$; this leads to incoming data f_-. We consider the 'Poisson operators' $\mathcal{P}_\pm : f_\pm \to u$ and precisely characterise the range of these operators on $\mathcal{W}^k(\mathbb{R}^n)$ spaces. Finally we show that the scattering map, mapping f_- to f_+, preserves these spaces.

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The authors were supported in part by the Australian Research Council through grant DP180100589. The first author is supported in part by ARC grant DP210103242.

1. Introduction and statement of results

1.1. **Introduction.** In this article we develop Fredholm theory for the time-dependent Schrödinger equation on $\mathbb{R}^{n+1} = \mathbb{R}^n_z \times \mathbb{R}_t$, with time-dependent coefficients. We begin by studying the inhomogeneous problem

$$Pu(z,t) := \left(\frac{1}{i} \frac{\partial}{\partial t} + \Delta_{g(t)} + V(z,t) \right) u(z,t) = v(z,t) \tag{1.1}$$

where $\Delta_{g(t)}$ denotes the positive Laplacian with respect to the metric $g(t)$, and we assume that

$$g_{ij}(t) - \delta_{ij} \text{ and } V \text{ are compactly supported in spacetime, and} \\ g(t) \text{ is a nontrapping metric on } \mathbb{R}^n \text{ for every time } t. \tag{1.2}$$

The condition of compact support is chosen for convenience; it could be weakened to symbolic-type decay estimates (in spacetime) of $g_{ij} - \delta_{ij}$ and the potential function V. The stronger assumption is made in order to introduce the Fredholm approach to the time-dependent Schrödinger operator in a relatively simple, but still variable-coefficient, setting. We do not make the assumption that V is real valued.

It was proven by Lascar in [25] that solutions u to (1.1) satisfy a propagation of singularities result which is analogous to the classical theorem of Hörmander [16] but adapted to the parabolic nature of the equation, i.e. where time derivatives are first order but spatial derivatives are second order, so both D_t and Δ_g contribute to the principal symbol of P. Lascar's result implies in particular that if Pu is smooth, then singularities of u propagate along $g(t)$-geodesics in space, at a fixed time t, i.e. with 'infinite speed'.

In this paper we prove microlocal propagation estimates for P that are valid uniformly out to spacetime infinity. This is done by adapting propagation estimates, including so-called 'radial point estimates', of Melrose [27] and Vasy [36] (which are themselves a microlocal version of the classical Mourre estimate [28]) to this setting. In particular, we prove estimates for u in terms of $v = Pu$ in weighted parabolic Sobolev spaces $H^{s,l}_{par}(\mathbb{R}^{n+1})$, defined in Section 2, thus taking into account both the (parabolic) regularity, as measured by s, and spacetime growth or decay, as measured by l. One essential feature is that we need to work with variable spacetime orders, which will vary 'microlocally', that is, vary in phase space not just physical spacetime, as will be explained shortly. We will always use a sans-serif font such as r to denote variable orders. We define, for arbitrary fixed differential order s and suitable variable orders $\mathsf{r}_\pm$, Hilbert spaces $\mathcal{Y}^{s,\mathsf{r}_\pm} = H^{s,\mathsf{r}_\pm}_{\mathrm{par}}(\mathbb{R}^{n+1})$ and $\mathcal{X}^{s,\mathsf{r}_\pm}$, given by

$$\mathcal{X}^{s,\mathsf{r}_\pm} = \{u \in H^{s,\mathsf{r}_\pm}_{\mathrm{par}}(\mathbb{R}^{n+1}) \mid Pu \in H^{s-1,\mathsf{r}_\pm+1}_{\mathrm{par}}(\mathbb{R}^{n+1})\}, \tag{1.3}$$

with the corresponding inner products and norms. Our first main result is then the following mapping property for P:

Theorem 1.1. *Assume that $g(t)$ and V satisfy the conditions above. For all $s \in \mathbb{R}$, for each choice of sign $\pm$ and all weight functions $\mathsf{r}_\pm$ satisfying the conditions in Section 5.2, the map*

$$P : \mathcal{X}^{s,\mathsf{r}_\pm} \to \mathcal{Y}^{s-1,\mathsf{r}_\pm+1} \tag{1.4}$$

is invertible.

Theorem 1.1, proved in Section 6.1 below, implies the existence of two inverses (which we will call *propagators*) of P, namely

$$P_+^{-1} : \mathcal{Y}^{s-1,\mathsf{r}_++1} \to \mathcal{X}^{s,\mathsf{r}_+}$$

and

$$P_-^{-1} : \mathcal{Y}^{s-1,\mathsf{r}_-+1} \to \mathcal{X}^{s,\mathsf{r}_-}.$$

These are in fact 'forward' and 'backward' propagators. To explain this, consider $v \in C_c^\infty(\mathbb{R}^{n+1})$. If one denotes by $T_\pm$ the initial and final times of the support of v,

$$T_+ = \sup\{t : \exists (z,t) \in \operatorname{supp} v\},\ T_- = \inf\{t : \exists (z,t) \in \operatorname{supp} v\},$$

there are two special solutions to (1.1), the forward solution u_+ and the backward solution u_-, which are the unique solutions satisfying (respectively) $\operatorname{supp} u_+ \subset \{t \geq T_-\}$ and $\operatorname{supp} u_- \subset \{t \leq T_+\}$. The inverse mappings $P_\pm^{-1}$ of (1.4), which we refer to as the forward (+)/ backward (+) propagators, take v to $P_+^{-1}v = u_+, P_-^{-1}v = u_-$, in this instance lying in $\mathcal{X}^{s,\mathsf{r}_\pm}$ for arbitrary s (since v is assumed smooth). The asymptotic behavior of $u_\pm$ as $t \to \pm\infty$ in the regions $|z|/|t| \leq C$ is (see Section 6.3)

$$(4\pi it)^{-n/2} e^{i|z|^2/4t} f_\pm\Big(\frac{z}{2t}\Big), \quad t \to \pm\infty, \quad \frac{|z|}{|t|} \leq C < \infty \tag{1.5}$$

where $f_\pm$ are Schwartz. The role of the spacetime decay/growth weight function $\mathsf{r}_\pm$ is precisely to allow only one of these behaviours; namely, for example choosing +, the weight function r_+ is subject to a threshold condition which allow expansions such as (1.5) for $t \to +\infty$, but not such expansions as $t \to -\infty$. For r_-, the reverse is true. See Section 5.2 for the precise conditions on $\mathsf{r}_\pm$.

Let us elaborate on how the variable spacetime orders $\mathsf{r}_\pm$ allow or disallow expansions such as (1.5). Given $f_\pm \in \mathcal{S}(\mathbb{R}^n)$, we note that (1.5) is in the weighted space $\langle z,t\rangle^{-l} L^2(\mathbb{R}^{n+1})$ for $l < -1/2$, but not for $l = -1/2$. The value $-1/2$ is thus a threshold value; whether the spacetime weight is greater or less than this threshold value determines whether (1.5) for $f \neq 0$ is possible for a function in the corresponding weighted space. Microlocally the expression (1.5) is concentrated at the 'outgoing radial set' $\mathcal{R}_+$ for $t \to +\infty$ and the 'incoming radial set' $\mathcal{R}_-$ for $t \to -\infty$;

these are the limiting points of bicharacteristics of P at spacetime infinity, that is, the initial/final points of bicharacteristics of P on the compactified phase space. (By a *bicharacteristic* of P we mean an integral curve of the Hamilton vector field of $p = \sigma(P)$ contained within the characteristic set $\mathrm{char}(P) = \{p = 0\}$.) The key property of the weight r_+ is therefore that r_+ is less than $-1/2$ on $\mathcal{R}_+$ but greater than $-1/2$ on $\mathcal{R}_-$, thereby allowing elements of $\mathcal{X}^{s,\mathsf{r}_+}$ to have asymptotics (1.5) as $t \to +\infty$ but not for $t \to -\infty$. For r_-, the reverse is true.

The above discussion may give the impression that the $\mathsf{r}_\pm$ could be chosen to depend only on t, and therefore, these weights do not need to vary 'microlocally'. In fact, this is not the case. Along bicharacteristics at fibre-infinity (that is, where the frequency variables are infinite), the time t is fixed and can be any finite value, so the value of t cannot be used to distinguish between these two sets. What determines whether we are at $\mathcal{R}_\pm$ is the relative orientation of the spatial variable z and its dual variable ζ. Let $\hat{z} = z/|z|$ and $\hat{\zeta} = \zeta/|\zeta|$. At the incoming radial set, we have $\hat{z} \cdot \hat{\zeta} = -1$ while at the outgoing radial set, we have $\hat{z} \cdot \hat{\zeta} = +1$. Thus $\mathsf{r}_\pm$ need to be functions of both z and ζ (at least), and in particular, the weights must vary nontrivially in phase space, not just in spacetime.

Theorem 1.1 immediately implies that we can solve the 'final state problem' with prescribed outgoing data $f_\pm$ in the sense of (1.5). For now we only consider Schwartz $f_\pm$; in Theorem 1.3 we will treat distributional $f_\pm$.

Theorem 1.2. *Given $f_+ \in \mathcal{S}(\mathbb{R}^n)$ there is a unique solution to the equation $Pu = 0$ with asymptotics* (1.5) *as $t \to +\infty$. Moreover, for every real s, and variable orders $\mathsf{r}_\pm$ satisfying the conditions in Section 5.2, u lies in the space*

$$\mathcal{X}^{s,\mathsf{r}_+} + \mathcal{X}^{s,\mathsf{r}_-}. \tag{1.6}$$

We write $u = \mathcal{P}_+ f_+$ and refer to $\mathcal{P}_+$ as the outgoing Poisson operator.

Similarly, given $f_- \in \mathcal{S}(\mathbb{R}^n)$ there is a unique solution $\tilde{u}$ to $P\tilde{u} = 0$, also lying in (1.6)*, with asymptotics* (1.5) *as $t \to -\infty$. We write $\tilde{u} = \mathcal{P}_- f_-$ and call $\mathcal{P}_-$ the incoming Poisson operator.*

See Section 6.3 for the simple proof in the case of Schwartz data. Thus, global solutions to $Pu = 0$ do not lie in one or other of our function spaces, but in the sum of the two. Given a (suitable) global solution, this decomposition is easily effected using a microlocal partition of unity, $\mathrm{Id} = Q_- + Q_+$, where Q_- is microlocally equal to the identity near $\mathcal{R}_-$ and microlocally trivial near $\mathcal{R}_+$. Then $u = Q_+ u + Q_- u$ is a decomposition such that $Q_\pm u \in \mathcal{X}^{s,\mathsf{r}_\pm}$.

One of the main goals of this work is to establish foundational theory for a microlocal/Fredholm approach to the *nonlinear* Schrödinger equation along the lines of [10, 11] for the nonlinear Helmholtz equation, which in turn was inspired by works [15], [8] on nonlinear wave equations. The first step in this direction has been taken in [9]. To this end, we include *module regularity estimates* which typically arise in microlocal approaches to nonlinear analysis. The notion of module regularity was formalized in [12] although it goes back much further; for example, the definition of Lagrangian distribution given by Hörmander in [17] (which he credits to Melrose) is in terms of module regularity. In the present context, it is closely related to Klainerman's vector field method [24]. Thus we also prove refined mapping properties for P in which the spaces $\mathcal{X}^{s,\mathsf{r}_\pm}, \mathcal{Y}^{s-1,\mathsf{r}_\pm+1}$ are replaced by spaces in which regularity is measured with respect to iterated application of elements in the module of operators which are characteristic on $\mathcal{R}_\pm$. We distinguish between the module $\mathcal{N}$ of operators which are characteristic at both radial sets $\mathcal{R}_\pm$ simultaneously and the larger modules of operators $\mathcal{M}_+$ and $\mathcal{M}_-$ vanishing at $\mathcal{R}_+$ and $\mathcal{R}_-$, respectively. The module $\mathcal{N} = \mathcal{M}_+ \cap \mathcal{M}_-$ is generated (up to precomposition by globally elliptic operators) by a finite collection of operators which correspond directly to the natural invariance properties of the free Schrödinger equation; these are the generators of translation, of rotations, and of Galilean transformations. We let $H_\pm^{s,l;\kappa,k}$ denote the elements of the space $H_{\mathrm{par}}^{s,l}(\mathbb{R}^{n+1})$ with k orders of small module regularity and κ additional orders of module regularity with respect to $\mathcal{M}_\pm$ (see Definition 4.4). Defining, analogously to (1.3), $\mathcal{Y}_\pm^{s,l;\kappa,k} = H_\pm^{s,l;\kappa,k}$ and

$$\mathcal{X}_\pm^{s,l;\kappa,k} = \{u \in H_\pm^{s,l;\kappa,k} \mid Pu \in H_\pm^{s-1,l+1;\kappa,k}\}, \tag{1.7}$$

then for suitable l and κ we also obtain a Hilbert space isomorphism

$$P : \mathcal{X}_\pm^{s,l;\kappa,k} \to \mathcal{Y}_\pm^{s-1,l+1;\kappa,k}; \tag{1.8}$$

see Proposition 6.3.

If κ is at least 1, then it turns out that we can take the spacetime order l in (1.8) to be *constant* in the range $-3/2 < l < -1/2$. The reason for this is that (choosing the + sign arbitrarily) the $\mathcal{M}_+$ module is elliptic at $\mathcal{R}_-$, so $\mathcal{M}_+$-module regularity of order $\kappa \geq 1$ in effect raises the spacetime regularity weight at $\mathcal{R}_-$ by one, thus raising it above the threshold value of $-1/2$ (since $l > -3/2$). On the other hand, the regularity at $\mathcal{R}_+$ will still be below threshold (since $l < -1/2$) as $\mathcal{M}_+$ is characteristic there. Being able to take a constant spacetime order is advantageous when proving multiplicative properties, as has been explained in [15], and will be important in our planned future work on the nonlinear Schrödinger equation.

Moreover, the consideration of module regularity spaces enables us to prove a precise scattering result in terms of natural spaces $\mathcal{W}^k(\mathbb{R}^n)$ of incoming/outgoing data of solutions to $Pu = 0$. Here $k \in \mathbb{Z}$ is a regularity index, measuring both smoothness and decay at infinity, and is such that $\cap_k \mathcal{W}^k(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$, while $\cup_k \mathcal{W}^k(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n)$. The $\mathcal{W}^k(\mathbb{R}^n)$ for $k \geq 1$ are themselves module regularity spaces, relative to a module $\widehat{\mathcal{N}}$ induced by the small module $\mathcal{N}$ mentioned above (see Section 7.1). In fact, these modules are such that the free Poisson operator $\mathcal{P}_0$, i.e. the Poisson operator from Theorem 1.2 for the flat Euclidean metric with $V \equiv 0$, intertwines $\widehat{\mathcal{N}}$ and $\mathcal{N}$. We then show that the Poisson operators $\mathcal{P}_\pm$ extend from Schwartz space to all tempered distributions, and we precisely characterise the range of $\mathcal{P}_\pm$ on $\mathcal{W}^k(\mathbb{R}^n)$. In the theorem below, the weight functions $\mathsf{r}_\pm$ are chosen as in Section 5.2, in particular they are equal to $-1/2$ off a neighborhood of the radial sets on the characteristic set:

Theorem 1.3. *For $k \in \mathbb{N}$, the range of the Poisson operator $\mathcal{P}_+$ on $\mathcal{W}^k(\mathbb{R}^n)$ is precisely*

$$\{u \in \mathcal{X}_+^{1/2,\mathsf{r}_+;k,0}(\mathbb{R}^{n+1}) + \mathcal{X}_-^{1/2,\mathsf{r}_-;k,0}(\mathbb{R}^{n+1}) \mid Pu = 0\},$$

i.e. that is, those elements of $\mathcal{X}^{1/2,\mathsf{r}_+} + \mathcal{X}^{1/2,\mathsf{r}_-}$ in the kernel of P having module regularity of order k.

For $k \leq -1$, the range of $\mathcal{P}_+$ on $\mathcal{W}^k(\mathbb{R}^n)$ is precisely the elements of $\ker P$ *in*

$$\{u \in H_{\mathrm{par}}^{k+1/2,k-1/2}(\mathbb{R}^{n+1}) \mid Pu = 0\}.$$

Provided that $k \geq 2$, the global solution $u = \mathcal{P}_+ f_+$ admits the asymptotic (1.5) *in the precise sense that*

$$\lim_{t\to+\infty} (4\pi i t)^{n/2} e^{-it|\zeta|^2} u_+(2t\zeta, t) = f_+(\zeta) \tag{1.9}$$

as a limit in the space $\langle\zeta\rangle^{1/2+\varepsilon}\mathcal{W}^{k-2}(\mathbb{R}^n_\zeta)$.

Remark 1.1. The difference between the two cases $k \geq 0$ and $k \leq -1$ in Theorem 1.3 is that, in the latter case, the spacetime regularity $k - 1/2$ is everywhere below threshold, so nothing special happens at the radial sets, while in the former case, the spacetime regularity must drop to below threshold at the radial sets. In both cases, given a solution to $Pu = 0$, its outgoing data is in $\mathcal{W}^k(\mathbb{R}^n)$ if and only if it is microlocally in $H_{\mathrm{par}}^{k+1/2,k-1/2}(\mathbb{R}^{n+1})$ away from the radial sets.

Moreover, we show that the scattering map, which maps the incoming data f_- of global solutions u to the outgoing data f_+, preserves the spaces $\mathcal{W}^k(\mathbb{R}^n)$:

Theorem 1.4. *The scattering map S, initially defined for $f_- \in \mathcal{S}(\mathbb{R}^n)$, extends to a bounded map from $\mathcal{W}^k(\mathbb{R}^n)$ to itself for each $k \in \mathbb{Z}$.*

1.2. **Parabolic calculus.** We begin by developing the calculus of parabolic pseudodifferential operators on $\mathbb{R}^{n+1}$. These are quantizations of symbols $S_{\mathrm{par}}^{m,l}(\mathbb{R}^{n+1})$, of fibre (or differential) order m and spacetime order l, defined using the standard spacetime weight function $(1 + |z|^2 + t^2)^{1/2}$ and the *parabolic* weight function $(1 + |\zeta|^4 + \tau^2)^{1/4}$ in the dual variables — see (2.2) for the precise definition. Thus, unlike usual pseudodifferential operators, classical parabolic pseudodifferential operators do not have principal symbols that are homogeneous functions on τ, ζ in the standard sense but instead are homogeneous with respect to the *parabolic scaling*

$$(\zeta, \tau) \mapsto (c\zeta, c^2\tau), \quad c > 0. \tag{1.10}$$

In addition, the behavior in the spacetime variables (z, t) is assumed to be uniformly symbolic in the usual sense. All of this is accomplished through the introduction of radial compactification of spacetime and parabolic compactification of the scattering cotangent bundle, ${}^{sc}\overline{T}^*_{\mathrm{par}}(\mathbb{R}^{n+1})$. As in other Fredholm analysis of non-elliptic operators, it is convenient to have variable order Sobolev spaces at our disposal, and in our case, it is only necessary to have variable spacetime decay order. We thus define spaces of pseudodifferential operators, $\Psi_{\mathrm{par}}^{s,\mathsf{r}}(\mathbb{R}^{n+1})$, for constant $s \in \mathbb{R}$ and r a classical symbol of order $(0, 0)$ (see Definition 2.3). Here s is the order of (parabolic) differential regularity and r is the spacetime decay order. Then choosing an elliptic and invertible element A of this space, we define

$$H_{\mathrm{par}}^{s,\mathsf{r}} = \{u \in H_{\mathrm{par}}^{-M,-N} \mid Au \in L^2\}, \tag{1.11}$$

for any sufficiently large M, N. In this space, the (parabolic) differential order is fixed at s, but the order of decay at spacetime infinity varies microlocally.

The Schrödinger operator is a differential operator of order $(2, 0)$ lying in this parabolic calculus, whose characteristic set $\mathrm{char}(P)$ contains two disjoint 'radial sets' $\mathcal{R}_\pm$ mentioned above. These are submanifolds of sources $(-)$ and sinks $(+)$ for the rescaled Hamilton flow on the characteristic set $\mathrm{char}(P)$. We show that this parabolic calculus enjoys structures and features similar to those of Melrose's scattering calculus, including extensions of the notions of characteristic set and rescaled-Hamilton flow to the boundaries, introduced by compactification, at both spacetime and fiber infinity. As in the scattering calculus, this allows one to formulate and prove propagation estimates, including at the radial sets $\mathcal{R}_\pm$, uniformly up to spacetime and fibre infinity. By adapting positive commutator estimates introduced originally by Hörmander, and developed by Melrose [27] and Vasy [36], to the parabolic calculus, we prove microlocal propagation estimates for P in each region of phase space, which we put together to obtain global Fredholm estimates. These take the form, in the setting of Theorem 1.1,

$$\|u\|_{H_{\mathrm{par}}^{s,\mathsf{r}_+}} \leq C\Big(\|Pu\|_{H_{\mathrm{par}}^{s-1,\mathsf{r}_+ +1}} + \|u\|_{H_{\mathrm{par}}^{M,N}}\Big), \quad u \in \mathcal{X}^{s,\mathsf{r}_+}, \tag{1.12}$$

together with the dual estimate

$$\|u\|_{H_{\mathrm{par}}^{s',\mathsf{r}_-}} \leq C\Big(\|P^*u\|_{H_{\mathrm{par}}^{s'-1,\mathsf{r}_- +1}} + \|u\|_{H_{\mathrm{par}}^{M',N'}}\Big), \quad u \in \mathcal{X}^{s',\mathsf{r}_-}. \tag{1.13}$$

Here s is arbitrary and M, N should be thought of as very negative, so that $H_{\mathrm{par}}^{s,\mathsf{r}_+}$ embeds compactly into $H_{\mathrm{par}}^{M,N}$ (thus, we require $M < s$ and $N < \inf \mathsf{r}_+$, and similarly for the second estimate). For convenience, we assume here that the potential function V is real, so that $P = P^*$. As shown by Vasy, these estimates imply that P is a Fredholm map from

$$\{u \in H_{\mathrm{par}}^{s,\mathsf{r}_\pm} \mid Pu \in H_{\mathrm{par}}^{s-1,\mathsf{r}_\pm +1}\} \to H_{\mathrm{par}}^{s-1,\mathsf{r}_\pm +1},$$

that is, between $\mathcal{X}^{s,\mathsf{r}_\pm}$ and $\mathcal{Y}^{s-1,\mathsf{r}_\pm+1}$ in our notation. Moreover, the index of P, mapping between these spaces, is zero. So P is invertible between these spaces if and only if its null space is trivial. This triviality is easy to show by considering the evolution in time of the spatial L^2 norm of global solutions — see Section 6.1.

1.3. **Relation to previous literature.** There have been many approaches to solving variable-coefficient time-dependent Schrödinger equations, including via ODE methods in Banach spaces [22], approximating Feynman integrals [7], via oscillatory integrals [34, 21, 18, 13] or via the FBI transform (e.g. [32]). Our approach is, to the best of our knowledge, essentially different to any previous method for treating the time-dependent Schrödinger equation, although inspired by previous Fredholm treatments of non-elliptic problems for the wave equation [36, 2, 15, 8] and the Helmholtz equation [11]. The first example of Fredholm theory used to treat a non-elliptic problem appears to be Faure and Sjöstrand's treatment of Anosov flows in [6]. This appears at first sight to be very different in nature to subsequent treatments, but Dyatlov and Zworski [5] showed that this example in fact fits into the general framework set out by Vasy in [36]. Fredholm theory in a Lorentzian (hence non-elliptic) setting was considered by Bär and Strohmaier in [1]. Very recently, Sussman has used microlocal propagation estimates to study the Klein-Gordon equation [33].

Our work builds off the results of Lascar on inhomogeneous pseudodifferential operators and the geometric microlocal scattering theory of Melrose [25, 27]. The former develops a general theory of operators with inhomogeneous symbols, and extends many of the standard structures and theorem in microlocal analysis to these operators, including propagation of singularities. Lascar's work is local in nature, and does not lead directly to quantitative global estimates. The global microlocal perspective imparts exactly that; as in Melrose's work, the notion of (parabolic) wavefront set of distributions on spacetime can be extended up to and including the introduced boundary via radial compactification. We also use the module regularity formalism introduced by the third author with Melrose and Vasy in [12].

There is a vast literature on scattering theory for the Schrödinger equation, that we will not attempt to discuss here. See for example the monographs [29], [40, 41] or [3]. Relatively little of this literature treats the case of time dependent metrics or potentials. Yafaev [19, 39] wrote several studies on wave operators for time-dependent potentials (including periodic potentials), and Chapter 3 of [3] is devoted to time-decaying potentials. Rodnianski and Schlag [30] considered rough and time-dependent potentials, and more recently Soffer and Wu [31] proved local decay for NLS with time-dependent potentials.

Asymptotic decay of solutions to Schrödinger's equation is a widely studied topic, going back at least to Jensen and Kato [20]. There, as in the general results on the scattering operator in [23], the Hamiltonians under consideration are time-independent.

This work is intended to be a foundation for a wide-ranging program of research into nonlinear Schrödinger operators with nonlinearity polynomial in u and $\overline{u}$. We expect that our method will be advantageous for analyzing the large-time asymptotics of solutions. Indeed, combining the linear theory in the present work with a multiplication result for module regularity spaces, along the lines of [11] in the Helmholtz case, has led directly to a small-data result for NLS [9]. We expect large data results in the defocusing case should be achievable by combining our techniques with a priori estimates on solutions provided by Strichartz or Morawetz estimates. In the focusing case, we anticipate that the method — after some further development — will be effective in analyzing the interaction of solitons and radiation. This will require developing a Fredholm approach to 'three-body-type' potentials, which is a topic of independent interest and one currently being pursued.

1.4. **Structure of the paper.** In Section 2, we discuss the compactification of phase space and set up the parabolic scattering calculus. We obtain standard results for composition, L^2-boundedness for zeroth order operators, and elliptic parametrices, and define weighted parabolic Sobolev spaces including variable order weights.

In Section 3, we discuss the geometry of the characteristic variety $\mathrm{char}(P)$ in our compactified phase space, and particularly properties of the Hamilton vector field relative the radial sets $\mathcal{R}_\pm$. This geometry, particularly the fact that $\mathcal{R}_-$ is a source, and $\mathcal{R}_+$ a sink, for the (rescaled) bicharacteristic flow, is crucial for the estimates in Section 5.

In Section 4 we introduce the modules $\mathcal{M}_\pm$ and $\mathcal{N}$ with respect to which we shall prove module regularity estimates, and derive a basic positivity property that makes the iterative module regularity argument possible.

In Section 5 we give the microlocal propagation estimates that we need to assemble the global Fredholm estimate, as in (1.12), (1.13). These estimates are actually proved in Section 8 for general operators in the calculus obeying some structural conditions.

In Section 6 we show invertibility of P both in the case of weighted parabolic Sobolev spaces with variable weights, and in the case of module regularity spaces. This establishes Theorem 1.1. We deduce solvability of the final state problem for Schwartz outgoing data, proving Theorem 1.2.

In Section 7, we define the spaces $\mathcal{W}^k(\mathbb{R}^n)$ of incoming and outgoing boundary data, and analyze the Poisson operator and scattering map on these spaces, proving Theorems 1.3 and 1.4.

In the appendix, Section 8, we prove various propagation estimates which we apply to P, including radial points estimates and module regularity propagation estimates. To maximize the utility of these results, we work in a general setting analogous to that of [38], in which we assume only that the operator under consideration has a non-degenerate characteristic set with smooth submanifolds of radial sets.

1.5. **Acknowledgements.** The authors thank Andras Vasy and Peter Hintz for their encouragement and for several enlightening conversations. They also thank MATRIX for its hospitality during the workshop "Hyperbolic Differential Equations in Geometry and Physics" during April 2022.

2. Scattering calculus and parabolic scattering calculus

2.1. **Definition of the parabolic scattering calculus.** In order to make use of propagation estimates in our present setting, it is necessary for us to work with a calculus of pseudodifferential operators which contains $P = D_t + \Delta$, $D_t = -i\partial_t$ as an operator of principal type.

Such a calculus must be anisotropic, so that D_t and Δ can be viewed as operators of the same order. Anisotropic calculi with this property are considered in [25], [16], as well as propagation estimates at "interior" points. We shall require a "scattering" (in the sense of Melrose) version of this calculus in order to obtain propagation estimates along bicharacteristics lying in the boundary of the radial compactification (in each factor) of $T^*\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$.

We denote the elements of $T^*\mathbb{R}^{n+1}$ as (z, t, ζ, τ) where $z, \zeta \in \mathbb{R}^n$ and $t, \tau \in \mathbb{R}$. Our parabolic pseudodifferential calculus will consist of operators which are quantizations of symbols defined with respect to an anisotropic weight function $R = R(\zeta, \tau)$, defined by

$$R^4 = |\zeta|^4 + \tau^2, \quad R > 0. \tag{2.1}$$

Definition 2.1. For $m, l \in \mathbb{R}$, $a \in C^\infty(T^*\mathbb{R}^{n+1})$, define the seminorms,

$$\|a\|_{S^{m,l}_{\mathrm{par}},N} = \sum_{|\alpha|+k+|\beta|+j \le N} \sup_{T^*\mathbb{R}^{n+1}} \left| \langle (z,t)\rangle^{-(l-|\alpha|-k)} \langle R\rangle^{-(m-|\beta|-2j)} \partial_z^\alpha \partial_t^k \partial_\zeta^\beta \partial_\tau^j a(z,t,\zeta,\tau) \right|.$$

We denote the Fréchet space defined by these seminorms by

$$S^{m,l}_{\mathrm{par}}(T^*\mathbb{R}^{n+1}) = \{a \in \mathcal{C}^\infty(T^*\mathbb{R}^{n+1}) : \|a\|_{S^{m,l}_{\mathrm{par}},N} < \infty, \text{ for all } N \in \mathbb{N}_0\}. \tag{2.2}$$

This is a statement about decay of a and its derivatives in both the spacetime and momentum variables. In the momentum variables, the parabolic nature of the calculus is encoded by the use of the 'parabolic radial variable' R on the RHS. Note also that differentiation in τ produces additional decay $\sim \langle R\rangle^{-2}$ compared to $\langle R\rangle^{-1}$ produced by differentiation in ζ. We remark that R is comparable to $\max(\langle\zeta\rangle, \langle\sqrt{|\tau|}\rangle)$. Thus, for example, an $a \in S^{1,0}_{\mathrm{par}}$ grows at most like $|\tau|^{1/2}$ in the region $|\zeta|^2/|\tau| < 2$ and at most like $|\zeta|$ in the region $|\tau|/|\zeta|^2 < 2$.

Note that the residual symbol space is exactly the Schwartz functions:

$$S^{-\infty,-\infty}_{\mathrm{par}}(T^*\mathbb{R}^{n+1}) := \bigcap_{m,l} S^{m,l}_{\mathrm{par}}(T^*\mathbb{R}^{n+1}) = \mathcal{S}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}).$$

A careful development of the local properties of the pseudodifferential calculus obtained by quantizing these symbols can be found in [25], [16]. We briefly summarize some of its properties.

Definition 2.2. The class $\Psi^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$ of parabolic pseudodifferential operators corresponding to $S^{m,l}_{\mathrm{par}}(T^*\mathbb{R}^{n+1})$ are the operators $\mathrm{Op}(a)$ with Schwartz kernels

$$\mathrm{Op}(a)(z,t,z',t') = (2\pi)^{-n-1} \int_{\mathbb{R}^{n+1}} e^{i(z-z')\cdot\zeta + (t-t')\tau} a(z,t,\zeta,\tau)\, d\zeta\, d\tau \tag{2.3}$$

for some $a \in S^{m,l}_{\mathrm{par}}(T^*\mathbb{R}^{n+1})$, where Op denotes left quantization and the integral (2.3) is interpreted in the distributional sense.

We also denote $\mathrm{Diff}^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$ denote the set of differential operators in $\Psi^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$.

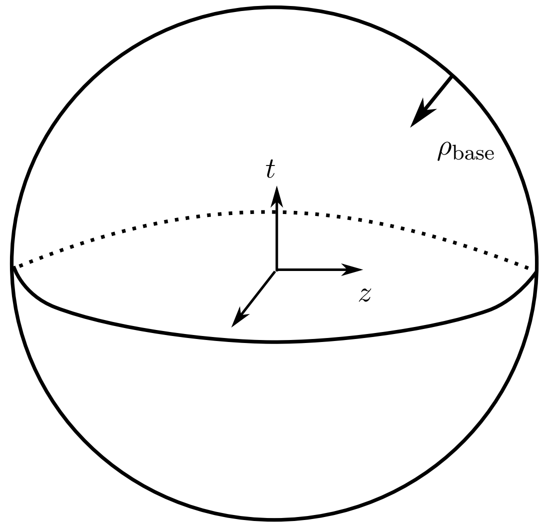

FIGURE 1. The radially compactified spacetime. The function $\rho_{\mathrm{base}} = (1 + t^2 + |z|^2)^{-1/2}$ defines (i.e. vanishes at) spacetime infinity. The radial set $\mathcal{R}_+$ lies over the top hemisphere, while the radial set $\mathcal{R}_-$ lies over the bottom hemisphere.

Some results about operators $A \in \Psi^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$ can be obtained easily from the containment

$$S^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1}) \subset \begin{cases} S^{m,l}_{1/2,0}(\mathbb{R}^{n+1}), & m \geq 0 \\ S^{m/2,l}_{1/2,0}(\mathbb{R}^{n+1}), & m \leq 0 \end{cases} \tag{2.4}$$

where $S^{m,l}_{\delta,\delta'}(\mathbb{R}^{n+1})$ are the standard scattering symbol spaces, as follows immediately from the definition. Thus e.g. we conclude that A maps $\mathcal{S}(\mathbb{R}^{n+1})$ to itself.

2.2. **Compactification of phase space.** As in the standard scattering calculus (see [27]), we compactify $T^*\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ in each factor, but, as we describe now, we do so inhomogeneously in the dual (momentum) variables.

We compactify the spacetime factor using the standard radial compactification, namely using the map $Z = (z, t) \in \mathbb{R}^{n+1}$

$$\psi_1(Z) = \left(\frac{1}{(1 + |Z|^2)^{1/2}}, \frac{Z}{(1 + |Z|^2)^{1/2}} \right) \in S^{n+1}_+, \tag{2.5}$$

where S^{n+1} is the unit sphere in $\mathbb{R}^{n+2}$ and S^{n+1}_+ the half sphere with first coordinate nonnegative. Since ψ_1 is a diffeomorphism onto the interior of $(S^{n+1}_+)^\circ$, we may take the closure, that is, S^{n+1}_+ itself, as our compactification. Topologically, the compactification is a closed ball — see Figure 1. We abuse notation and allow Z to denote the corresponding interior point $\psi_1(Z)$. With this notation, the function

$$\rho_{\mathrm{base}} = \frac{1}{(1 + |Z|^2)^{1/2}} \tag{2.6}$$

extends smoothly to all of S^{n+1}_+ and is a boundary defining function (bdf) of ∂S^{n+1}_+ (meaning $\partial S^{n+1}_+ = \{\rho_{\mathrm{base}} = 0\}$ and $d\rho_{\mathrm{base}}$ is non-vanishing over ∂S^{n+1}_+). A function $f \in C^\infty(S^{n+1}_+)$ thus equivalently satisfies that $f(Z)$ is smooth and on $|Z| > C$, $f \in C^\infty([0, 1)_{\rho_{\mathrm{base}}} \times \partial S^{n+1}_+)$. It is straightforward to show that away from $Z = 0$,

$1/|Z|$ is also a boundary defining function, so smoothness of f means that, with $\hat{Z} = Z/|Z|$,

$$f(1/|Z|, \hat{Z}) \sim \sum_{j=0}^{\infty} |Z|^{-j} a_j(\hat{Z})$$

where $\hat{Z} \in \partial S_+^{n+1}$ and $a_j \in C^\infty(\partial S_+^{n+1})$.

For the other factor, $\mathbb{R}^{n+1}_{\zeta,\tau}$, we consider a 'parabolic sphere' given by

$$S^{n+1}_{\mathrm{par},+} = \{(\zeta_0, \zeta, \tau) \in \mathbb{R}^{n+2} \mid \zeta_0^4 + |\zeta|^4 + \tau^2 = 1,\ \zeta_0 \geq 0\}. \tag{2.7}$$

Consider the smooth mapping

$$\psi_2 \colon \mathbb{R}^{n+1}_{\zeta,\tau} \longrightarrow S^{n+1}_{par,+}$$

given by

$$\psi_2(\zeta, \tau) = \left(\frac{1}{(1 + R^4)^{1/4}}, \frac{\zeta}{(1 + R^4)^{1/4}}, \frac{\tau}{(1 + R^4)^{1/2}} \right) \in S^{n+1}_{\mathrm{par},+} \tag{2.8}$$

where R is as in (2.1). This is a diffeomorphism onto the interior $(S^{n+1}_{\mathrm{par},+})^\circ$ (using the fact that R^4 is a smooth function). This defines our parabolic compactification of the fibres $R^{n+1}_{\zeta,\tau}$, namely we take the closed parabolic half-sphere $S^{n+1}_{\mathrm{par},+}$ as the compactification. This shows that

$$\rho_{\mathrm{fib}} = (1 + R^4)^{-1/4} \tag{2.9}$$

is a boundary defining function for the boundary of this compactified space, which we shall call 'fibre-infinity'. Natural 'angular' variables are induced by smooth coordinates on the boundary of the parabolic half-sphere, extended into the interior by requiring them to be invariant under the parabolic scaling (1.10). In the region $|\zeta|^2/|\tau| < 2$, $|\tau| \geq 1$ we can use the angular coordinates $\zeta_i/\sqrt{|\tau|}$ and the 'radial' (i.e. homogeneous of degree 1 with respect to the parabolic scaling) coordinate $\sqrt{|\tau|}$, while in the region $|\tau|/|\zeta|^2 < 2$, $|\zeta| \geq 1$, we can assume without loss of generality that $|\zeta_1| \geq \max_i |\zeta_i|/2$ locally, and then we can use angular coordinates ζ_j/ζ_1, $j = 2, \ldots, n$ together with $\tau\zeta_1^{-2}$, and the radial coordinate ζ_1. We will call the first region τ-dominant, and the second type of region ζ-dominant. (Of course there is an overlap region which is both τ-dominant and ζ-dominant.)

We thereby obtain a compactification $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$

$$\psi = (\psi_1, \psi_2) \colon T^*\mathbb{R}^{n+1} \longrightarrow S_+^{n+1} \times S_+^{n+1} =: \overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}. \tag{2.10}$$

This is a manifold with corners of codimension two, the boundary being a union of two boundary hypersurfaces

$$\partial\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}} = \{\rho_{\mathrm{base}} = 0\} \cup \{\rho_{\mathrm{fib}} = 0\} = \partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}} \cup \partial_{\mathrm{fib}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}, \tag{2.11}$$

where

$$\partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}} \simeq \partial S_+^{n+1} \times S^{n+1}_{\mathrm{par},+} \text{ is “spacetime infinity”}$$

and

$$\partial_{\mathrm{fib}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}} \simeq S_+^{n+1} \times \partial S^{n+1}_{\mathrm{par},+} \text{ is “fiber infinity”}.$$

Similar to how we write the variables (z, t) collectively as Z, we shall write (ζ, τ) collectively as Φ. We also write

$$|\Phi| = R = (|\zeta|^4 + \tau^2)^{1/4}$$

(this would more correctly be denoted $|\Phi|_{\mathrm{par}}$ or similar, but we prefer the simpler notation), and somewhat imprecisely write $\hat{\Phi}$ for a set of n angular variables, which can take various forms as described above. We also write $\langle\Phi\rangle := (1+R^4)^{1/4}$.

As in the standard scattering calculus, classical symbols can be defined using smooth functions on $\overline{T}^*_{\mathrm{par}}\mathbb{R}^{n+1}$. To be absolutely concrete, a function a_0 lies in $C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}})$ if and only if a_0 is smooth in the interior of $\overline{T}^*_{\mathrm{par}}\mathbb{R}^{n+1}$ and satisfies the following,

- **Spatial infinity/momentum interior:** On sets $|Z| > C$, $|\Phi| < C$, the function a_0 is a smooth funtion of $\rho_{\mathrm{base}} = \langle Z\rangle^{-1}, \hat{Z}, \Phi$;
- **Spatial interior/momentum infinity:** On sets $|Z| < C$, $|\Phi| > C$, a_0 is a smooth function of $Z, \rho_{\mathrm{fib}} = 1/\langle\Phi\rangle, \hat{\Phi}$;
- **Spatial infinity/momentum infinity (the corner):** On sets $|Z| > C$, $|\Phi| > C$, a_0 is a smooth function of $\rho_{\mathrm{base}}, \rho_{\mathrm{fib}}, \hat{Z}, \hat{\Phi}$.

It is straightforward to show that if $a_0 \in C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}})$ then $a_0\circ\psi^{-1} \in C^\infty(T^*\mathbb{R}^{n+1})$ in fact lies in $S^{0,0}_{\mathrm{par}}$.

Definition 2.3. Still with $\Phi = (\zeta,\tau)$, a symbol $a \in S^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$ is said to be classical if

$$\langle Z\rangle^{-l}\langle\Phi\rangle^{-m}a \equiv \rho^l_{\mathrm{base}}\,\rho^m_{\mathrm{fib}}\,a \;\in C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}),$$

meaning it extends to the boundary as a smooth function on this space. We denote the set of classical symbols by $S^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$ and the set of pseudodifferential operators obtained by quantizing such operators by $\Psi^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$.

Remark 2.1. The more general symbols in Definition 2.1 can also be characterized by a regularity condition when they are thought of as functions on $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$, namely they are conormal to the boundary with the appropriate weights.

2.3. **Symbols, ellipticity, operator wavefront sets and Hamilton vector fields.** We have the principal symbol mapping

$$\sigma_{m,l}\colon \Psi^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1}) \longrightarrow S^{m,l}_{\mathrm{par}}/S^{m-1,l-1}_{\mathrm{par}} = S^{[m,l]}_{\mathrm{par}}, \tag{2.12}$$

with kernel equal to $\Psi^{m-1,l-1}_{\mathrm{par}}$. Restricting attention to classical operators, given $\mathrm{Op}(a) = A, \mathrm{Op}(b) = B \in \Psi^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$, we have

$$\sigma_{m,l}(A) = \sigma_{m,l}(B) \iff \Big(\langle Z\rangle^{-l}\langle\Phi\rangle^{-m}a\Big)|_{\partial\overline{T}^*_{\mathrm{par}}\mathbb{R}^{n+1}} \equiv \Big(\langle Z\rangle^{-l}\langle\Phi\rangle^{-m}b\Big)|_{\partial\overline{T}^*_{\mathrm{par}}\mathbb{R}^{n+1}}.$$

This allows us to view the principal symbol of a classical operator — renormalized by suitable powers of the boundary defining functions of fibre and spacetime infinity — as a function on the boundary of compactified phase space, $\partial\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$.

The appropriate notion of ellipticity in this calculus is uniform in the spacetime weight function. Thus, we say $A \in \Psi^{m,l}_{\mathrm{par}}$ is *globally* elliptic if

$$\sigma_{m,l}(A)(Z,\Phi) \geq C\langle Z\rangle^l\langle\Phi\rangle^m. \tag{2.13}$$

More generally we consider microlocal ellipticity at a boundary point of the compactified parabolic cotangent bundle, $q \in \partial\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$. We say that $A \in \Psi^{m,l}_{\mathrm{par}}$ is (microlocally) elliptic at q, and write $q \in \mathrm{ell}_{m,l}(A)$, if this estimate holds in a neighborhood of q in $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$. To clarify this we can write down the estimates at the three regions of phase space we considered above.

(1) For q in the spacetime interior (and therefore at fibre infinity), points in phase space are in $\mathrm{ell}_{m,l}(A)$ if and only if A is elliptic there in the standard parabolic sense, i.e. for $q = (Z_0, \hat{\Phi}_0)$, $q \in \mathrm{ell}_{m,l}(A)$ if and only if for some $C, \varepsilon > 0$, $a(Z, \Phi) \geq C|\Phi|^m$ for $|\Phi| > \varepsilon^{-1}, |Z - Z_0|, |\hat{\Phi} - \hat{\Phi}_0| < \varepsilon$.

(2) At spacetime infinity with Φ finite, $q = (\hat{Z}_0, \Phi_0)$ lies in $\mathrm{ell}_{m,l}(A)$ if and only if for some $C, \varepsilon > 0$, $a(Z, \Phi) \geq C|Z|^l$ for $|Z| > \varepsilon^{-1}, |\hat{Z} - \hat{Z}_0|, |\Phi - \Phi_0| < \varepsilon$.

(3) At the corner, $q = (\hat{Z}_0, \hat{\Phi}_0)$ lies in $\mathrm{ell}_{m,l}(A)$ if and only if for some $C, \varepsilon > 0$, $a(Z, \Phi) \geq C|Z|^l|\Phi|^m$ for $|Z|, |\Phi| > \varepsilon^{-1}, |\hat{Z} - \hat{Z}_0|, |\hat{\Phi} - \hat{\Phi}_0| < \varepsilon$.

The elliptic set of A is by definition an open set. The characteristic set is simply the complement of the elliptic set (hence closed):

$$\mathrm{char}_{m,l}(A) = \partial \overline{T}^*_{\mathrm{par}} \mathbb{R}^{n+1} \setminus \mathrm{ell}_{m,l}(A). \tag{2.14}$$

We will almost always drop the subscripts m, l and write $\mathrm{ell}(A)$ and $\mathrm{char}(A)$ for these sets.

For classical operators, it is convenient to think of the properties of these symbols in terms of the boundary restriction of the reweighted function $a_0 = \langle Z \rangle^{-l} \langle \Phi \rangle^{-m} a$. Specifically, for $A = \mathrm{Op}(a) \in \Psi^{m,l}_{\mathrm{par,cl}}$ and a_0 as above we have

$$\begin{aligned} \mathrm{ell}(A) &= \{q \in \partial \overline{T}^*_{\mathrm{par}} \mathbb{R}^{n+1} : a_0(q) \neq 0\}, \\ \mathrm{char}(A) &= \{q \in \partial \overline{T}^*_{\mathrm{par}} \mathbb{R}^{n+1} : a_0(q) = 0\}, \end{aligned} \tag{2.15}$$

thus $\mathrm{char}(A)$ is simply the vanishing locus of the smooth function a_0.

It is sometimes convenient to reweight a classical symbol only in the base variables. Hence, for $A = \mathrm{Op}(a)$, $a \in S^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$, we shall define

$$\sigma_{\mathrm{base},l}(A) := [\rho^l_{\mathrm{base}} a] \,|_{\partial_{\mathrm{base}} \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}}, \tag{2.16}$$

which is a classical symbol of order m on the fibres of $\partial_{\mathrm{base}} \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}$. (This was denoted $\sigma_{\mathrm{base},m,l}(A)$ in [11].)

In the standard scattering calculus, the fiber principal symbol is, in a sense, the usual principal symbol, and, if the symbol is classical, it can be represented by a homogeneous function. In our parabolic setting a related statement is true, namely in bounded spacetime sets, away from zero momentum, the principal symbol of fibre order m of a classical symbol in this calculus is represented by a unique function homogeneous of degree m, in the parabolic sense, i.e.

$$\lambda > 0, \ \tilde{a}(t, z, \lambda^2 \tau, \lambda \zeta) = \lambda^m \tilde{a}(t, z, \tau, \zeta). \tag{2.17}$$

Indeed, this function is the unique function a homogeneous of degree m such that $\langle \Phi \rangle^{-m} a$ has the appropriate boundary value at fibre infinity. However, if the symbol is not classical, the symbol at spacetime infinity is usually *not* represented by a homogeneous function in the fiber variables.

The concept of operator wavefront set, also known as microlocal support, also carries over directly. Namely, if $A = \mathrm{Op}(a)$, then the operator wavefront set $\mathrm{WF}'(A)$ is the essential support of a, i.e. the subset of $\partial \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}$ whose complement consist of points q such that "a is trivial in an open neighborhood of q". The meaning of this statement is that there is a neighbourhood $U \subset \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}$ of q such that $a|_{U \cap T^* \mathbb{R}^{n+1}}$ is Schwartz, i.e. vanishes to all orders together with all its derivatives. In particular, for $A \in \Psi^{m,l}_{\mathrm{par}}$,

$$\mathrm{WF}'(A) = \varnothing \implies A \in \Psi^{-\infty,-\infty}_{\mathrm{par}},$$

meaning $A = \mathrm{Op}(a)$ for $a \in \mathcal{S}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ and therefore $\mathrm{WF}'(A) = \varnothing$ implies that A maps tempered distributions to Schwartz functions, $A\colon \mathcal{S}'(\mathbb{R}^{n+1}) \longrightarrow \mathcal{S}(\mathbb{R}^{n+1})$.

As in the standard scattering setting, given $a \in S^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$, an appropriate rescaling of the standard Hamilton vector field H_a extends smoothly to the whole of $\overline{T}^*_{\mathrm{par}}\mathbb{R}^{n+1}$. This can be seen from the following lemma which characterizes the asymptotic behaviour of the linear vector fields on $\mathbb{R}^{n+1}$.

Lemma 2.4. *Let $\rho_{\mathrm{base}}, \rho_{\mathrm{fib}}$ be boundary defining functions for $C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}})$ (see below (2.10)) and let $\partial_{\hat{Z}_i}$ and $\partial_{\hat{\zeta}_i}$ denote vector fields tangent to the spheres $\mathbb{S}^n_{\hat{z}}$ and $\mathbb{S}^n_{\hat{\zeta}}$, respectively. Then*

$$\partial_{Z_j}, \partial_{\zeta_j} \in \mathrm{span}_{C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}})}\langle \rho^2_{\mathrm{base}}\partial_{\rho_{\mathrm{base}}},\, \rho_{\mathrm{base}}\partial_{\hat{Z}_i},\, \rho^2_{\mathrm{fib}}\partial_{\rho_{\mathrm{fib}}},\, \rho_{\mathrm{fib}}\partial_{\hat{\zeta}_i}\rangle,$$

while

$$\partial_\tau \in \mathrm{span}_{C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}})}\langle \rho^3_{\mathrm{fib}}\partial_{\rho_{\mathrm{fib}}},\, \rho^2_{\mathrm{fib}}\partial_{\hat{\zeta}_i}\rangle$$

Proof. Since the spacetime compactification here is the standard radial compactification, the statement for the spacetime vector fields ∂_Z follows from the standard scattering case, and indeed there are no $\rho^2_{\mathrm{fib}}\partial_{\rho_{\mathrm{fib}}}$ or $\rho_{\mathrm{fib}}\partial_{\hat{\zeta}_i}$ terms. Here one simply writes the vector fields in polar coordinates $|Z|, \hat{Z}$ with $\rho_{\mathrm{base}} = 1/\langle Z\rangle$.

For the fiber variables, differentiating (2.1) shows

$$\partial_\tau \rho_{\mathrm{fib}} \in \rho^3_{\mathrm{fib}} C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}), \quad \partial_\tau(\zeta_i\rho_{\mathrm{fib}}),\ \partial_\tau(\tau\rho^2_{\mathrm{fib}}) \in \rho^2_{\mathrm{fib}} C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}})$$

which implies the statement for ∂_τ, and

$$\partial_{\zeta_j} \rho_{\mathrm{fib}} \in \rho^2_{\mathrm{fib}} C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}), \quad \partial_{\zeta_j}(\zeta_i\rho_{\mathrm{fib}}),\ \partial_{\zeta_j}(\tau\rho^2_{\mathrm{fib}}) \in \rho_{\mathrm{fib}} C^\infty(\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}),$$

which implies the statement for ∂_{ζ_j}. □

Given a classical symbol $a \in S^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$, recalling that

$$H_a = \frac{\partial a}{\partial \tau}\frac{\partial}{\partial t} - \frac{\partial a}{\partial t}\frac{\partial}{\partial \tau} + \sum_{j=1}^n \big(\frac{\partial a}{\partial \zeta_j}\frac{\partial}{\partial z_j} - \frac{\partial a}{\partial z_j}\frac{\partial}{\partial \zeta_j}\big),$$

define the (parabolically) rescaled Hamilton vector field vector field

$$H^{m,l}_a := \rho^{m-1}_{\mathrm{fib}}\rho^{l-1}_{\mathrm{base}} H_a \tag{2.18}$$

From Lemma 2.4, we see that $H^{m,l}_a$ *extends smoothly to $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ and is tangent to the boundary.* Moreover, the fact that the components of ∂_τ vanish an order faster at fiber infinity than the components of the other vector fields implies that at fiber infinity, the terms with ∂_τ do not contribute to leading order there, i.e.

$$H^{m,l}_a|_{\{\rho_{\mathrm{fib}}=0\}} = \rho^{m-1}_{\mathrm{fib}}\rho^{l-1}_{\mathrm{base}}\left(\sum_{j=1}^n \partial_{\zeta_j} a \partial_{z_j} - \partial_{z_j} a \partial_{\zeta_j}\right)$$

We also have, as in the standard scattering setting, that the flow of $H^{m,l}_a$ preserves the characteristic set of $\mathrm{Op}(a)$.

Proposition 2.5. *Let $a \in S^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$ be classical. Then $H^{m,l}_a$ is tangent to the characteristic set $\mathrm{char}_{m,l}(\mathrm{Op}(a))$, provided that this set is a submanifold.*

Definition 2.6. For $A \in \Psi^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$, $A = \mathrm{Op}(a)$, the **radial set** of A is defined by

$$\mathcal{R}(A) = \{q \in \mathrm{char}_{m,l}(A) : H^{m,l}_a \text{ vanishes at } q\}. \tag{2.19}$$

Equivalently, $\mathcal{R}(A)$ is the set of stationary points of the flow of $H^{m,l}_a$ on $\mathrm{char}_{m,l}(A)$.

We will also require parabolic pseudodifferential operators of variable spacetime order.

Definition 2.7. Given a *weight function* $\mathsf{r} \in S^{0,0}_{\mathrm{par,cl}}$, a *classical* symbol of order $(0,0)$, and $\delta \in (0,1/2)$, we define the following weighted symbol class

$$S^{m,\mathsf{r}}_{\delta,\mathrm{par}} = \{a \in \mathcal{C}^\infty(T^*\mathbb{R}^{n+1}) : |\partial_z^\alpha \partial_t^k \partial_\zeta^\beta \partial_\tau^j a| \lesssim \langle Z\rangle^{\mathsf{r}-(1-\delta)(|\alpha|+k)+\delta(|\beta|+2j)} \langle \Phi\rangle^{m-|\beta|-2j}\}. \tag{2.20}$$

We will variously call r a spacetime weight, or a (variable) spacetime order.

Remark 2.2. One can see the necessity of the $\delta > 0$ loss to treat non-constant $\mathsf{r} \in S^{0,0}_{\mathrm{par,cl}}$ by considering $\langle z\rangle^{\mathsf{r}}$. Indeed,

$$\partial_{z_i}\langle z\rangle^{\mathsf{r}} = \mathsf{r}\langle z\rangle^{\mathsf{r}-1}\partial_{z_i}\langle z\rangle + \langle z\rangle^{\mathsf{r}}\log\langle z\rangle \partial_{z_i}\mathsf{r}$$

and similar estimates for further derivatives show that this function lies in $S^{0,\mathsf{r}}_{\delta,\mathrm{par}}$ for any $\delta > 0$, but not $\delta = 0$ when r is nonconstant.

Quantisation of symbols in $S^{m,\mathsf{r}}_{\delta,\mathrm{par}}$ for arbitrary $\delta \in (0,1/2)$ works in exactly the same way as for constant order symbols, giving rise to a class $\Psi^{m,\mathsf{r}}_{\delta,\mathrm{par}}$ of variable order operators in our parabolic pseudodifferential calculus. We note that we have a similar containment as in (2.4), namely

$$S^{m,\mathsf{r}}_{\mathrm{par}}(\mathbb{R}^{n+1}) \subset \begin{cases} S^{m,\mathsf{r}}_{1/2,\delta}(\mathbb{R}^{n+1}), & m \geq 0 \\ S^{m/2,\mathsf{r}}_{1/2,\delta}(\mathbb{R}^{n+1}), & m \leq 0 \end{cases}. \tag{2.21}$$

2.4. **Composition, L^2-boundedness, Sobolev spaces, and elliptic regularity.** We derive standard properties of the parabolic scattering calculus.

Proposition 2.8. *Let* $\mathrm{Op}(a) = A \in \Psi^{m,\mathsf{r}}_{\delta,\mathrm{par}}(\mathbb{R}^{n+1})$ *and* $\mathrm{Op}(b) = B \in \Psi^{m',\mathsf{r}'}_{\delta,\mathrm{par}}(\mathbb{R}^{n+1})$. *Then*

- $AB \in \Psi^{m+m',\mathsf{r}+\mathsf{r}'}_{\delta,\mathrm{par}}(\mathbb{R}^{n+1})$;
- $\sigma_{m+m',\mathsf{r}+\mathsf{r}'}(AB) = \sigma_{m,\mathsf{r}}(A)\sigma_{m',\mathsf{r}'}(B)$;
- *The commutator* $[A,B]$ *is in the space* $\Psi^{m+m'-1,\mathsf{r}+\mathsf{r}'-1+\delta}_{\delta,\mathrm{par}}$, *and*

$$\sigma_{m+m'-1,\mathsf{r}+\mathsf{r}'-1+\delta}([A,B]) = \frac{1}{i}\{a,b\}$$

where $\{a,b\}$ *denotes the Poisson bracket:*

$$\{a,b\} = H_a b' \tag{2.22}$$

Proof. This can be obtained cheaply from (2.21) and the standard expansion of the symbol of a product.

□

The elliptic set $\mathrm{ell}_{m,\mathsf{r}}(A)$ for variable order operators A is defined just as for the constant spacetime order case. The microlocal elliptic parametrix construction goes through in this context and we conclude:

Proposition 2.9. *Let $A \in \Psi^{m,\mathsf{r}}_{\delta,\mathrm{par}}(\mathbb{R}^{n+1})$ and let K be a compact subset of* $\mathrm{ell}_{m,\mathsf{r}}(A)$*. Then there is $B \in \Psi^{-m,-\mathsf{r}}_{\delta,\mathrm{par}}(\mathbb{R}^{n+1})$ such that*

$$K \cap \mathrm{WF}'(AB - Id) = K \cap \mathrm{WF}'(BA - Id) = \varnothing.$$

The global version of this proposition and the Hörmander "square root trick" then imply that

$$A \in \Psi^{0,0}_{\mathrm{par}}(\mathbb{R}^{n+1}) \implies A\colon L^2 \longrightarrow L^2 \text{ is bounded.} \tag{2.23}$$

Note this also follows from the containment (2.4).

We define the parabolic weighted Sobolev spaces, initially with constant orders, analogously to the standard scattering spaces by

Definition 2.10.

$$H^{m,l}_{\mathrm{par}} := \{u \in \mathcal{S}'(\mathbb{R}^{n+1}) : Au \in L^2(\mathbb{R}^{n+1}) \text{ for all } A \in \Psi^{m,l}_{\mathrm{par}}\}.$$

Note that the operator

$$\Lambda_{m,l} = \mathrm{Op}(\langle Z\rangle^{-m}\langle \Phi\rangle^{-l}) = \langle z, t\rangle^{-m}\mathcal{F}^{-1}\langle\Phi\rangle^{-l}\mathcal{F}.$$

lies in $\Psi^{-m,-l}$ and is manifestly invertible. For any $M, N \in \mathbb{R}$ it therefore defines an isomorphism

$$\Lambda_{m,l}\colon H^{M,L}_{\mathrm{par}} \longrightarrow H^{M+m,L+l}_{\mathrm{par}},$$

and we define a topology on $H^{M,L}_{\mathrm{par}}$ by

$$\|u\|_{H^{m,l}_{\mathrm{par}}} = \|\Lambda_{-m,-l}u\|_{L^2}. \tag{2.24}$$

For any $A \in \Psi^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$ and $u \in H^{M,L}_{\mathrm{par}}$, (2.23) and Proposition 2.8 yield the estimate

$$\|Au\|_{H^{M-m,L-l}_{\mathrm{par}}} = \|(\Lambda_{-M+m,-L+l}A\Lambda_{M,L})\Lambda_{-M,-L}u\|_{L^2} \leq C\|u\|_{H^{M,L}_{\mathrm{par}}},$$

and thus:

Proposition 2.11. *Let $A \in \Psi^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$. Then for any $M, L \in \mathbb{R}$, $A\colon H^{M,L}_{\mathrm{par}} \longrightarrow H^{M-m,L-l}_{\mathrm{par}}$ is bounded.*

Propositions 2.9 and 2.11 together yield the following result.

Proposition 2.12. *Suppose $P \in \Psi^{m,l}_{\mathrm{par}}$ and $Q, G \in \Psi^{0,0}_{\mathrm{par}}$ such that P and G are elliptic on* $\mathrm{WF}'(Q)$*. Then if $GPu \in H^{s-m,r-l}_{\mathrm{par}}$, we have $Qu \in H^{s,r}$ with the estimate*

$$\|Qu\|_{H^{s,r}_{\mathrm{par}}} \leq C(\|GPu\|_{H^{s-m,r-l}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}) \tag{2.25}$$

for any $M, N \in \mathbb{R}$.

Variable order Sobolev spaces can be defined similarly.

Definition 2.13. Let r be a classical symbol in $S^{0,0}_{\mathrm{par}}$ satisfying $\mathsf{r} \geq l$ and let A be a fixed, classical, globally elliptic element of $\Psi^{m,\mathsf{r}}_{\delta,\mathrm{par}}$ with $\delta \in (0, 1/2)$. We define

$$H^{m,\mathsf{r}}_{\mathrm{par}} := \{u \in H^{m,l}_{\mathrm{par}} : Au \in L^2(\mathbb{R}^{n+1})\}$$

We equip $H^{m,\mathsf{r}}_{\mathrm{par}}$ with the norm

$$\|u\|_{H^{m,\mathsf{r}}_{\mathrm{par}}} := \|u\|_{H^{m,l}_{\mathrm{par}}} + \|Au\|_{L^2}. \tag{2.26}$$

This imparts a Hilbert space structure on $H^{m,\mathsf{r}}_{\mathrm{par}}$, and moreover this structure is independent of the choice of l and A. To see the independence we note that by the standard elliptic parametrix construction we can choose B such that $I = BA + R$

with $B \in \Psi_{\mathrm{par}}^{-m,-\mathrm{l}}$ and $R \in \Psi_{\mathrm{par}}^{-\infty,-\infty}$, for any other $\tilde{A} \in \Psi^{m,\mathrm{r}}$ we can write $\tilde{A}u = \tilde{A}BAu + \tilde{A}Ru$ and use that $\tilde{A}B \in \Psi_{\delta,\mathrm{par}}^{0,0}$ to bound

$$\|\tilde{A}u\| \lesssim \|u\|_{H_{\mathrm{par}}^{m,l}} + \|Au\|_{L^2}.$$

Remark 2.1. Proposition 2.11 and Proposition 2.12 hold for variable order operators $A, P \in \Psi_{\mathrm{par}}^{m,\mathrm{r}}$ with only minor modifications to the proof. See for example [38, Proposition 5.15].

3. Geometry of the time-dependent Schrödinger equation

3.1. **Characteristic variety.** Let P denote the operator $D_t + \Delta_g + V$, where $D_t = -i\partial_t$, Δ_g is the (positive) Laplacian on $\mathbb{R}^n$ with respect to a metric $g = g(t)$ and the metric g and potential V are as in Section 1.1. The operator P lies in $\mathrm{Diff}_{\mathrm{par}}^{2,0}$ and has principal symbol (written using the Einstein summation convention)

$$\sigma_{2,0}(P) = p(z,t,\zeta,\tau) = \tau + g^{ij}(z,t)\zeta_i\zeta_j.$$

In this section we will study this operator using the structures developed in the previous section; in particular we will identify its (parabolic) characteristic set, its radial set, and explain the Hamiltonian dynamics thereon.

Recall that the characteristic set, defined in (2.14), is a subset of the boundary of the compactified parabolic cotangent bundle, and therefore has a component at spacetime infinity and a component at fibre infinity, which intersect at the corner (both spacetime and fibre infinity).

By assumption, in a neighbourhood of spacetime infinity the operator P coincides with $P_0 = D_t + \Delta_0$, where Δ_0 is the (positive) flat Laplacian. The symbol of this operator is $\tau + |\zeta|^2$, and thus at spacetime infinity and in regions of bounded $\Phi = (\zeta, \tau)$, the characteristic set $\mathrm{char}(P_0)$ is given simply by $\tau = -|\zeta|^2$. Obviously, where where (τ, ζ) is finite, this set is a smooth submanifold of $\overline{T_{\mathrm{par}}^*\mathbb{R}^{n+1}}$. Near fibre infinity (and still near spacetime infinity), we use coordinates ρ_{fib} and 'angular' variables τ/R^2 and ζ/R, where $R = (|\zeta|^4 + \tau^2)^{1/4}$ as usual. In these coordinates, the characteristic set is

$$\{\tau/R^2 = -|\zeta/R|^2\} \cap \partial\overline{T_{\mathrm{par}}^*\mathbb{R}^{n+1}}.$$

Noting that $(\tau/R^2)^2 + (|\zeta|^2/R^2)^2 = 1$, we see that $\tau/R^2 = 2^{-1/2}$ and $|\zeta|/R = 2^{-1/4}$ on the characteristic set, so the differential of the function $\tau/R^2 + |\zeta/R|^2$ is nonvanishing and hence the zero locus is a smooth submanifold in a neighbourhood of fibre infinity (near spacetime infinity), meeting fibre infinity transversally.

In a bounded spacetime region, near fibre infinity, the characteristic variety can be similarly written

$$\{\tau/R^2 = -g^{ij}(z,t)(\zeta/R)_i(\zeta/R)_j\} \cap \partial\overline{T_{\mathrm{par}}^*\mathbb{R}^{n+1}}, \tag{3.1}$$

which again shows that the characteristic variety is a smooth codimension one submanifold of $\partial\overline{T_{\mathrm{par}}^*\mathbb{R}^{n+1}}$ in this region.

3.2. **Hamilton vector field.** The Hamilton vector field of P is

$$H_p = \frac{\partial}{\partial t} + 2g^{ij}(z,t)\zeta_i\frac{\partial}{\partial z_j} - \frac{\partial g^{ij}}{\partial t}\zeta_i\zeta_j\frac{\partial}{\partial \tau} - \frac{\partial g^{ij}}{\partial z_k}\zeta_i\zeta_j\frac{\partial}{\partial \zeta_k}. \tag{3.2}$$

In bounded regions of spacetime, the fiber rescaled Hamilton vector field is simply $\frac{1}{R}H_p$. Restricting this to fibre infinity, that is, taking the limit as $R \to \infty$, we see that the coefficients of ∂_t and ∂_τ vanish. Thus the flow at fibre-infinity takes place at one moment of time, say $t = t_0$ (reflecting 'infinite propagation speed' for the time-dependent Schrödinger equation). If we let $\zeta' = \zeta/R$ then we obtain the rescaled flow equations

$$\dot{z}^i = 2g^{ij}(z,t_0)\zeta'_j, \quad \dot{\zeta}'_k = -\frac{\partial g^{ij}(z,t_0)}{\partial z_k}\zeta'_i\zeta'_j$$

which we recognize as the geodesic equations for the metric $g(t_0)$ at the fixed time t_0. Moreover, from (3.1) and the uniform comparability of $g(t_0)$ with the flat metric on bounded regions of spacetime, we see that $|\zeta'|$ is bounded below, so this rescaled Hamilton vector field is nonvanishing over the spacetime interior. Recalling also the nontrapping assumption (1.2), we see that the operator P is an operator of real principal type in the sense of Duistermaat-Hörmander [4, Definition 6.3.2] over the spacetime interior: the Hamilton vector field is nonvanishing everywhere, and the bicharacteristics exit every compact set in a finite time.

3.3. **Radial sets.** We now determine the radial set (see Definition 2.6) for P, and the nature of the Hamilton flow in a neighbourhood of the radial set.

As has just been shown, the rescaled Hamilton vector field in the spacetime interior is nonvanishing. Radial points, if they exist, therefore lie over spacetime infinity, that is, in $\partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$. We will show

Proposition 3.1. *The radial set $\mathcal{R}$ of P is a disjoint union $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ of smooth submanifolds of $\partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ of dimension n. The component $\mathcal{R}_+$ is a family of global sinks for the rescaled Hamilton vector field $H_p^{2,0}$, and $\mathcal{R}_-$ is a family of global sources for this rescaled Hamilton vector field. Every bicharacteristic $\gamma(s)$ of P (meaning a flowline of $H_p^{2,0}$ within the characteristic variety* $\mathrm{char}(P)$*) converges to $\mathcal{R}_+$ as $s \to \infty$ and to $\mathcal{R}_-$ as $s \to -\infty$.*

Proof. To prove this, we start by noting that in a neighbourhood of spacetime infinity, P coincides with $P_0 = D_t + \Delta_0$, where Δ_0 is the flat (positive) Laplacian on $\mathbb{R}^n$. Thus we only need to consider the Hamilton flow for this flat model, which is

$$H_{p_0} = \frac{\partial}{\partial t} + 2\zeta \cdot \frac{\partial}{\partial z}. \tag{3.3}$$

We first consider the region spacetime region $\{|t| \geq \varepsilon|z|, t \geq C\}$ for arbitrary $C, \varepsilon > 0$. In terms of Figure 1, this is strictly in the 'northern hemisphere'. In this region, $w := z/t$ is a coordinate on the spacetime boundary, we take $\rho_{\mathrm{base}} = 1/t$ as a boundary defining function, and we rescale the Hamilton vector field by dividing by ρ_{base}, or equivalently multiplying by t; that is, we consider

$$tH_{p_0} = t\frac{\partial}{\partial t} + 2t\zeta \cdot \frac{\partial}{\partial z}.$$

Using coordinates $(\rho_{\mathrm{base}}, w, \zeta, \tau)$ which are valid near spacetime infinity and for bounded $\Phi = (\zeta, \tau)$, this is

$$-\rho_{\mathrm{base}}\frac{\partial}{\partial \rho_{\mathrm{base}}} + (2\zeta - w) \cdot \frac{\partial}{\partial w}.$$

Then changing coordinates to $(\rho_{\mathrm{base}}, \tilde{w}, \zeta, \tau)$ where $\tilde{w} = w - 2\zeta$, we obtain

$$-\rho_{\mathrm{base}}\frac{\partial}{\partial \rho_{\mathrm{base}}} - \tilde{w} \cdot \frac{\partial}{\partial \tilde{w}}.$$

Thus the radial set in this region, which we denote $\mathcal{R}_+$, is given by $\rho_{\rm base} = 0, \tilde{w} = 0$ and $\tau + |\zeta|^2 = 0$ (which is just the condition of lying in $\mathrm{char}(P_0)$). It is clear that the rescaled Hamilton vector field is a sink near $\mathcal{R}_+$. Thus, in this region, we have

$$\mathcal{R}_+ = \{\rho_{\rm base} = 0,\ \zeta = \frac{w}{2},\ \tau = -\frac{|w|^2}{4}\}, \quad w = \frac{z}{t}. \tag{3.4}$$

Noting that w is a coordinate on spacetime infinity in this region, we see that the radial set $\mathcal{R}_+$ is a *graph over spacetime infinity* in this region. Since $\zeta = w/2$, we can equally well use ζ as a coordinate. This structure reflects the fully *dispersive* nature of the Schrödinger equation: each frequency propagates in a different direction or at a different speed, and therefore ends up at a different point of spacetime infinity.

There is an analogous radial set over the 'southern hemisphere' (in terms of Figure 1), where we restrict to $\{|t| \geq \varepsilon|z|, t \leq -C\}$. As we will shortly show, this is a different component of the radial set, which we denote $\mathcal{R}_-$. If we now redefine our coordinates so that $\rho_{\tilde{\rm base}} = -1/t$ (so that it is a nonnegative function), $w = z/|t|$ and $\tilde{w} = w + 2\zeta$, we have

$$\mathcal{R}_- = \{\rho_{\rm base} = 0,\ \zeta = -\frac{w}{2},\ \tau = -\frac{|w|^2}{4}\} \tag{3.5}$$

which is a graph over (part of) the southern hemisphere. The rescaled Hamilton vector field $|t|H_{p_0}$ takes the form

$$\rho_{\rm base}\frac{\partial}{\partial\rho_{\rm base}} + \tilde{w}\cdot\frac{\partial}{\partial\tilde{w}},$$

i.e. it is a source near $\mathcal{R}_-$.

We now consider the case where $|z| \geq C|t|$ and $|z| \geq C$, that is, near the 'equator' in terms of Figure 1. Working in a small neighbourood of the equator, we may suppose without loss of generality that the first spatial coordinate z_1 is positive, and satisfies $z_1 \geq 1/2 \max_i |z_i|$. In that case, we may take the spacetime boundary defining function $\rho_{\rm base}$ to be $1/z_1$. We also write $s = t/z_1$ and $v_j = z_j/z_1$ for $j \geq 2$. First working in a region where Φ is bounded, we rescale the Hamilton vector field by dividing by $\rho_{\rm base}$, that is, multiplying by z_1. Using coordinates $(\rho_{\rm base}, s, v_j, \zeta, \tau)$, we have

$$\rho_{\rm base}^{-1}H_{p_0} = \big(1 - 2s\zeta_1\big)\frac{\partial}{\partial s} + \sum_{j\geq 2}\big(2\zeta_j - 2\zeta_1 v_j\big)\frac{\partial}{\partial v_j} - 2\zeta_1\rho_{\rm base}\frac{\partial}{\partial\rho_{\rm base}}.$$

We are interested in the region where s is small, otherwise the previous calculation applies. In order for this vector field to vanish, we see from the ∂_s coefficient that when s is small, necessarily $|\zeta_1|$ is large, and either positive or negative depending on the sign of s. First taking the case that ζ_1 is large and positive, we use fibre boundary defining function $\rho_{\rm fib} = 1/\zeta_1$ and coordinates $\omega_j = \zeta_j/\zeta_1$ and $\sigma = \tau/|\zeta|^2$. We further rescale the Hamilton vector field by multiplying by $\rho_{\rm fib}$. An easy computation shows

$$\rho_{\rm base}^{-1}\rho_{\rm fib}H_{p_0} = \big(\rho_{\rm fib} - 2s\big)\frac{\partial}{\partial s} + \sum_{j\geq 2}\big(2\omega_j - 2v_j\big)\frac{\partial}{\partial v_j} - 2\rho_{\rm base}\frac{\partial}{\partial\rho_{\rm base}}.$$

Changing variables to $\tilde{s} = 2s - \rho_{\rm fib}$, $\tilde{v}_j = v_j - \omega_j$, this vector field expressed in coordinates $(\tilde{s}, \tilde{v}_j, \rho_{\rm base}, \rho_{\rm fib}, \omega_j, \sigma)$ takes the form

$$-2\tilde{s}\frac{\partial}{\partial\tilde{s}} - 2\sum_{j\geq 2}\tilde{v}_j\frac{\partial}{\partial\tilde{v}_j} - 2\rho_{\rm base}\frac{\partial}{\partial\rho_{\rm base}}. \tag{3.6}$$

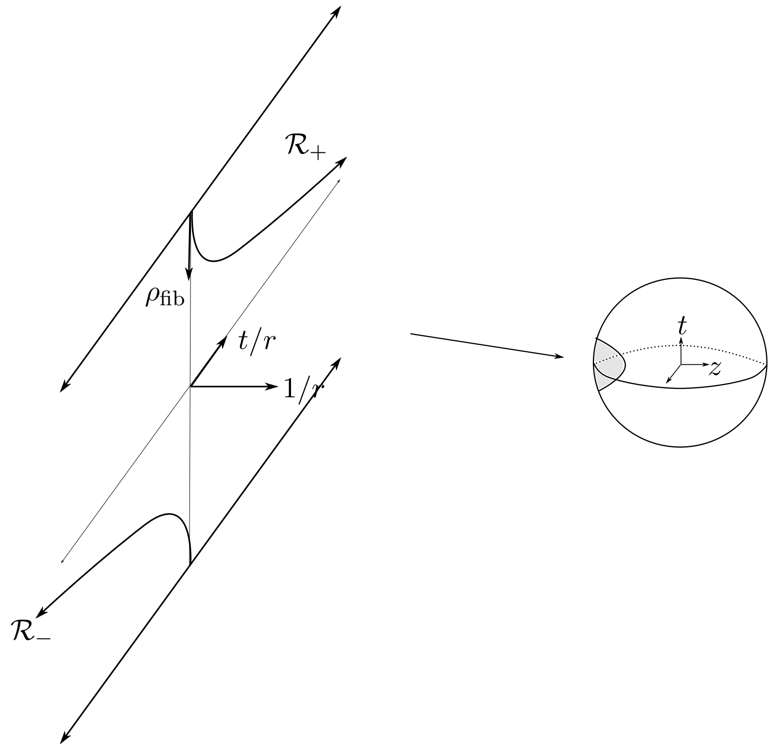

FIGURE 2. Near the "equator" $t = 0$ in the boundary, the radial sets remain disjoint and intersect the corner normally. In this figure the fibre over the equator is represented by the vertical line, with fibre infinity represented by the two straight bold lines. The sign of t/r determines whether we are over the northern (+) or the southern (−) hemisphere.

In this region, where $\zeta_1 >> 0$, the radial set is given by $\{\rho_{\mathrm{base}} = 0, \tilde{s} = 0, \tilde{v}_j = 0, \sigma = -1\}$. These equations define a submanifold of dimension n inside $\partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$, which is transverse to fibre infinity. It is not hard to check that where s is strictly positive, this set coincides with the set $\mathcal{R}_+$ defined in (3.4). We write the equations for $\mathcal{R}_+$ using more natural coordinates in this region as

$$\mathcal{R}_+ = \{\rho_{\mathrm{base}} = 0,\ s = \rho_{\mathrm{fib}}/2,\ \hat{\zeta} = \hat{z},\ \tau/|\zeta|^2 = -1\}. \tag{3.7}$$

The rescaled Hamilton vector field in (3.6) is clearly a sink at $\mathcal{R}_+$ in this region. Notice that the projection to spacetime infinity gives the closed northern hemisphere, since $s = \rho_{\mathrm{fib}}/2 \geq 0$ in (3.7). Wherever s is strictly positive, $\mathcal{R}_+$ is a graph, but it fails to be so at the boundary $s = 0$: the graph 'turns vertical' at the equator, and has a boundary at fibre-infinity. See Figure 2.

We next take the case that ζ_1 is very negative. In this case we use fibre boundary defining function $\rho_{\mathrm{fib}} = -1/\zeta_1$ and coordinates $\omega_j = \zeta_j/|\zeta_1|$ and $\sigma = \tau/|\zeta|^2$. We rescale the Hamilton vector field by multiplying by (the new) ρ_{fib} and repeat the calculation. We obtain the vector field

$$\rho_{\mathrm{base}}^{-1}\rho_{\mathrm{fib}}H_{p_0} = \big(\rho_{\mathrm{fib}} + 2s\big)\frac{\partial}{\partial s} + \sum_{j\geq 2}\big(2\omega_j + 2v_j\big)\frac{\partial}{\partial v_j} + 2\rho_{\mathrm{base}}\frac{\partial}{\partial \rho_{\mathrm{base}}}.$$

This time we change variables to $\overline{s} = 2s + \rho_{\mathrm{fib}}$, $\overline{v}_j = v_j + \omega_j$ then in coordinates $(\overline{s}, \overline{v}_j, \rho_{\mathrm{base}}, \rho_{\mathrm{fib}}, \omega_j, \sigma)$ we have the vector field

$$2\overline{s}\frac{\partial}{\partial \overline{s}} + 2\sum_{j\geq 2} \overline{v}_j \frac{\partial}{\partial \overline{v}_j} + 2\rho_{\mathrm{base}}\frac{\partial}{\partial \rho_{\mathrm{base}}}. \tag{3.8}$$

This part of the radial set coincides with $\mathcal{R}_-$ where $s < 0$, and can be expressed as

$$\mathcal{R}_- = \{\rho_{\mathrm{base}} = 0,\ s = -\rho_{\mathrm{fib}}/2,\ \hat{\zeta} = -\hat{z},\ \tau/|\zeta|^2 = -1\}. \tag{3.9}$$

Comparing (3.7) and (3.9), we see that the two sets are disjoint. In fact, we have $\hat{\zeta} = \hat{z}$ for $\mathcal{R}_+$, while $\hat{\zeta} = -\hat{z}$ for $\mathcal{R}_-$. The rescaled Hamilton vector field is, visibly from (3.6) and (3.8), a source at $\mathcal{R}_-$, and a sink at $\mathcal{R}_+$ in this region (which is another way to see the disjointness of the two components). Moreover, $s \geq 0$ for $\mathcal{R}_+$, while $s \leq 0$ for $\mathcal{R}_-$, so $\mathcal{R}_+$ lies over the closed northern hemisphere and $\mathcal{R}_+$ lies over the closed southern hemisphere, with both reaching fibre-infinity (but in disjoint sets as just described) over the equator. This is illustrated in Figure 2.

It remains to prove the last statement in the Proposition. For this we need the nontrapping assumption, that is, that for each fixed t_0, the metric $g(t_0)$ is nontrapping: every geodesic for $g(t_0)$ in $T^*\mathbb{R}^n$ reaches spacetime infinity both forwards and backwards. That means, in particular, that every bicharacteristic $\gamma(s)$ for P coincides with a bicharacteristic for P_0 when s is sufficiently negative or sufficiently positive. Recalling the discussion at the end of Section 3.1, it follows that it suffices to prove the statement for P_0.

Now because the rescaled Hamilton vector field has a smooth extension to the boundary of the compactified parabolic cotangent bundle, which is tangent to the boundary, it suffices to prove the statement for flow lines in the interior, and take a limit as the flow lines approach the boundary. So we consider an interior flow line for P_0, contained within $\{p_0 = 0\}$. These take the form for some t_0, z_0, ζ_0,

$$t(s) = t_0 + s, \quad z(s) = z_0 + 2\zeta_0 s, \quad \zeta = \zeta_0, \quad \tau = -|\zeta_0|^2.$$

As $s \to +\infty$ we see that z/t converges to $2\zeta_0$, while τ is fixed at $-|\zeta_0|^2$. Moreover, $\hat{z}(s) \to \hat{\zeta}_0$. We see that this converges to a point of $\mathcal{R}_+$. Similarly, as $s \to -\infty$, we have $z/|t| \to -2\zeta_0$, $\tau = -|\zeta_0|^2$ and $\hat{z}(s) \to -\hat{\zeta}_0$ so this converges to a point of $\mathcal{R}_-$. □

The final task in this section is to observe the Lagrangian nature of the radial sets. Let $\tilde{\mathcal{R}}_\pm$ denote the $(n+1)$-dimensional submanifold of $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ uniquely determined by the following two conditions:

- $\tilde{\mathcal{R}}_\pm$ are invariant under spacetime dilation, $Z \mapsto aZ$ for $a \in \mathbb{R}_+$, and
- The intersection of $\tilde{\mathcal{R}}_\pm$ with $\partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ is $\mathcal{R}_\pm$.

Lemma 3.2. *The submanifolds $\tilde{\mathcal{R}}_\pm$ defined above are Lagrangian submanifolds for the standard symplectic form $\sum_j d\zeta_j \wedge dz_j + d\tau \wedge dt$ on $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$.*

Proof. Since $\tilde{\mathcal{R}}_\pm$ are smooth submanifolds with boundary, of the correct dimension, it is only necessary to verify the Lagrangian condition, i.e. that the symplectic form vanishes when restricted to $\tilde{\mathcal{R}}_\pm$, in the interior of $\tilde{\mathcal{R}}_\pm$. In this region we can use the coordinates $(w = z/t, \rho = 1/t, \zeta, \tau)$, as in the beginning of the proof of Proposition 3.1. In these coordinates $\tilde{\mathcal{R}}_\pm$ is given by

$$\tilde{\mathcal{R}}_\pm = \{\zeta = \frac{w}{2},\ \tau = -\frac{|w|^2}{4},\ \pm\rho > 0\},$$

and the symplectic form restricted to $\tilde{\mathcal{R}}_\pm$ is

$$d\big(\frac{z}{2t}\big) \wedge dz + d\big(- \frac{|z|^2}{4t^2}\big) \wedge dt = -\frac{z}{2t^2} dt \wedge dz - \frac{z}{2t^2} dz \wedge dt = 0.$$

□

4. Module regularity

4.1. **Test modules.** In addition to the parabolic scattering Sobolev spaces considered in Definition 2.10 and their variable order analogues in Definition 2.13, we shall require the notion of iterated regularity with respect to a test module of pseudodifferential operators, introduced in [12]. In this section we shall work exclusively with pseudodifferential operators with classical symbols, $\Psi^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$ (see Definition 2.3).

Definition 4.1. A test module of operators contained in $\Psi^{1,1}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$ is a vector subspace of $\Psi^{1,1}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$ that contains and is a module over $\Psi^{0,0}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$, is finitely generated over $\Psi^{0,0}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$, and is closed under commutators.

Consider an arbitrary test module $\mathcal{M} \subset \Psi^{1,1}_{\mathrm{par,cl}}$, with generating set $\mathbf{A} = (\mathbf{A}_j)_{j=0}^N \subset \Psi^{1,1}_{\mathrm{par,cl}}$ where $\mathbf{A}_0 = \mathrm{Id}$. Powers of the module can be defined in the natural way.

Definition 4.2. For arbitrary $\kappa \in \mathbb{N}$ we define $\mathcal{M}^\kappa$ to be the $\Psi^{0,0}_{\mathrm{par}}$-module generated by the set

$$\{A_\alpha = \mathbf{A}^\alpha : |\alpha| \le \kappa\}, \tag{4.1}$$

where $\alpha = (\alpha_0, \dots, \alpha_N)$ is a multi-index and $\mathbf{A}^\alpha$ denotes the composition

$$\mathbf{A}^\alpha := \mathbf{A}_0^{\alpha_0} \mathbf{A}_1^{\alpha_1} \dots \mathbf{A}_N^{\alpha_N}.$$

Notice that the ordering of the factors is immaterial due to the fact that the module is by definition closed under commutators.

Equivalently, $\mathcal{M}^\kappa$ is the module generated by all κ-fold products of elements of $\mathcal{M}$.

Let $\mathcal{M}$ be a test module. We define Sobolev spaces of functions with additional regularity with respect to $\mathcal{M}$ as follows.

Definition 4.3. Let s and l be real numbers, and κ a natural number. We define the space of functions with $\mathcal{M}$-module regularity of order κ in $H^{s,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$ by

$$H^{s,l;\kappa}_{\mathcal{M}} := \{u \in H^{s,l}_{\mathrm{par}}(\mathbb{R}^{n+1}) : Au \in H^{s,l}_{\mathrm{par}} \text{ for all } A \in \mathcal{M}^\kappa\}. \tag{4.2}$$

Concretely, $u \in H^{s,l;\kappa}_{\mathcal{M}}$ if and only if $\mathbf{A}^\alpha u \in H^{s,l}_{\mathrm{par}}$ for all α with $|\alpha| \le \kappa$. It shall be useful to assume additional regularity with respect to a fixed submodule $\mathcal{N} \subset \mathcal{M}$ with generating set $\mathbf{B} = (\mathbf{B}_j)_{j=0}^{N'}$. To this end, we introduce the the following refinement of (4.2).

Definition 4.4. We define the space of functions with $\mathcal{M}$-module regularity of order κ and $\mathcal{N}$-module regularity of order k in $H^{s,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$ by

$$H^{s,l;\kappa,k}_{\mathcal{M}} := \{u \in H^{s,l}_{\mathrm{par}}(\mathbb{R}^{n+1}) : ABu \in H^{s,l}_{\mathrm{par}} \text{ for all } A \in \mathcal{M}^\kappa \text{ and } B \in \mathcal{N}^k.\} \tag{4.3}$$

We suppress the $\mathcal{N}$ in our notation, as in this paper it will only ever be used with the specific module $\mathcal{N}$ from Definition 4.5. Concretely, $u \in H_{\mathcal{M}}^{s,l;\kappa,k}$ if and only if $\mathbf{A}^\alpha \mathbf{B}^\beta u \in H_{\mathrm{par}}^{s,l}$ for all α, β with $|\alpha| \le \kappa$ and $|\beta| \le k$.

We equip the spaces $H_{\mathcal{M}}^{s,l;\kappa}, H_{\mathcal{M}}^{s,l;\kappa,k}$ with a Hilbert space structure by fixing a choice of generators $\mathbf{A}, \mathbf{B}$ and taking

$$\|u\|^2_{H_{\mathcal{M}}^{s,l;\kappa}} := \sum_{|\alpha|\le\kappa} \|\mathbf{A}^\alpha u\|^2_{H_{\mathrm{par}}^{s,l}} \tag{4.4}$$

and

$$\|u\|^2_{H_{\mathcal{M}}^{s,l;\kappa,k}} := \sum_{|\alpha|\le\kappa} \sum_{|\beta|\le k} \|\mathbf{A}^\alpha \mathbf{B}^\beta u\|^2_{H_{\mathrm{par}}^{s,l}}. \tag{4.5}$$

Remark 4.1. Definition 4.3 and Definition 4.4 generalise in the natural way to the case of variable spacetime weight $\mathsf{r} \in \Psi_{\mathrm{par}}^{0,0}$.

4.2. **The modules $\mathcal{M}_\pm$ and $\mathcal{N}$.** We now introduce two specific modules $\mathcal{M}_\pm$, and a common submodule $\mathcal{N}$ that shall be the modules of interest in this paper.

Definition 4.5. If $\mathcal{R} = \mathcal{R}(P) = \mathcal{R}_+ \cup \mathcal{R}_-$ is the radial set for the operator P considered in (1.1) and (1.2), we define $\mathcal{M}_\pm$ by

$$\mathcal{M}_\pm := \{A \in \Psi_{\mathrm{par,cl}}^{1,1}(\mathbb{R}^{n+1}) \mid \mathcal{R}_\pm \subset \mathrm{char}(A)\}. \tag{4.6}$$

We also define

$$\mathcal{N} := \mathcal{M}_+ \cap \mathcal{M}_- \tag{4.7}$$

and we write $H_\pm^{s,l;\kappa,k}$ for the module regularity space (4.3) when $\mathcal{M} = \mathcal{M}_\pm$ is as in (4.5) and $\mathcal{N}$ is as in (4.7).

Remark 4.2. Since $\mathcal{R}$ is contained in $\partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$, the condition (4.6) is simply the condition that $\sigma_{\mathrm{base},1}(a)$ vanishes on $\mathcal{R}_\pm$ (see discussion at the beginning of Section 2.3, up to equation (2.16)), where $A = \mathrm{Op}(a)$.

Proposition 4.6. *The modules $\mathcal{M}_\pm$, and therefore also $\mathcal{N}$, are test modules in the sense of Definition 4.1.*

Proof. It is clear that $\mathcal{M}_\pm$ are vector subspaces of $\Psi_{\mathrm{par,cl}}^{1,1}(\mathbb{R}^{n+1})$ that contain and are modules over $\Psi_{\mathrm{par,cl}}^{0,0}$. It remains to show that these are finitely generated and are closed under commutators.

Closedness under commutators follows from Lemma 3.2. Indeed, if A_1 and A_2 are two module elements, then we can find $\tilde{A}_i$, $i = 1, 2$, that differ from A_i by an element of $\Psi_{\mathrm{par,cl}}^{0,0}(\mathbb{R}^{n+1})$ and have symbol invariant under the scaling $Z \mapsto aZ$ near spacetime infinity. Indeed, recalling the definition of $\sigma_{\mathrm{base},l}$ from (2.16), we just take the symbol of $\tilde{A}_i$ so that $\sigma_{\mathrm{base},1}(\tilde{A}_i)$ is invariant under the scaling and to agree with $\sigma_{\mathrm{base},1}(A_i)$ at spacetime infinity. It is clear that $\tilde{A}_i$ are also in the module. Then the symbols of $\tilde{A}_1$ and $\tilde{A}_2$ vanish on $\tilde{\mathcal{R}}_\pm$. We now use the standard fact in symplectic geometry that if a Hamiltonian is constant on a Lagrangian submanifold, then its Hamilton vector field is tangent to this submanifold. (Proof: it suffices to show for a model Lagrangian, say $L = \{(0, \xi)\} \subset T^*\mathbb{R}^n_{x,\xi}$. Any Hamiltonian constant on L takes the form $\sum x_j b_j(x, \xi)$. Then the Hamiltonian vector field is $b_j \partial_{\xi_j} + x_j H_{b_j}$ which is tangent to L.) Denoting the symbols of $\tilde{A}_i$ by $\tilde{a}_i$, it follows that $H_{\tilde{a}_1}\tilde{a}_2$ vanishes on $\tilde{\mathcal{R}}_\pm$. This is the principal symbol of $i[\tilde{A}_1, \tilde{A}_2]$, so it follows that $\sigma_{\mathrm{base},1}(i[\tilde{A}_1, \tilde{A}_2])$ vanishes at $\mathcal{R}_\pm$, so this operator is also in the module. But $[A_1, A_2]$ differs from

$[\tilde{A}_1, \tilde{A}_2]$ by an operator of order $(0,0)$, so we see that $[A_1, A_2]$ also lies in the module $\mathcal{M}_\pm$.

The proof that $\mathcal{M}_\pm$ are finitely generated is postponed to Proposition 4.7. □

To show that the modules $\mathcal{M}_\pm$ are finitely generated, we will exhibit an explicit set of generators. To begin with, we introduce some useful cutoff functions. Taking $\chi(s) \in \mathcal{C}^\infty(\mathbb{R})$ to be 0 for $s \in (-\infty, 1/3]$ and 1 for $s \in [2/3, \infty)$, we define

$$\chi_{\mathrm{pol},+} := \chi\left(\frac{t}{\langle Z\rangle}\right) \tag{4.8}$$

$$\chi_{\mathrm{pol},-} := \chi\left(-\frac{t}{\langle Z\rangle}\right) \tag{4.9}$$

$$\chi_{\mathrm{eq}} := 1 - \chi_{\mathrm{pol},+} - \chi_{\mathrm{pol},-} \tag{4.10}$$

where $\chi(1/|w|)$ is extended to be 1 at $w = 0$. These functions can be regarded as cutoffs to neighbourhoods of polar and equatorial regions of the boundary of space-time.

Due to the parabolic nature of our calculus, we also need to work with microlocal square roots of D_t. This requires pseudodifferential cutoffs based on the sign of τ. To this end we introduce

$$\chi_\tau = \chi\left(-\frac{2\tau}{\langle R^2\rangle}\right). \tag{4.11}$$

We also define an elliptic element $E_s = \mathrm{Op}(\langle R^2\rangle^{s/2}) \in \Psi^{s,0}_{\mathrm{par,cl}}$ and operators

$$B_\pm := \mathrm{Op}(b_\pm), \quad b_\pm = \big(z\cdot\xi \mp |z|\sqrt{-\tau}\big)\chi_\tau\chi_{\mathrm{eq}} \in S^{1,1}_{\mathrm{par,cl}}. \tag{4.12}$$

Our candidate generating sets are then as follows.

$$\mathcal{G}_\pm := \{z_i D_{z_j} - z_j D_{z_i},\ tD_{z_i} - z_i/2,\ \langle Z\rangle E_{-1}P,\ E_1,\ t\chi_{\mathrm{pol},\mp}E_1,\ B_\pm\}. \tag{4.13}$$

Proposition 4.7. *The sets $\mathcal{G}_\pm$ generate $\mathcal{M}_\pm$. The set $\mathcal{G}_+ \cap \mathcal{G}_-$, consisting of all but the last two generators in* (4.13)*, generates $\mathcal{N}$.*

Proof. We omit the proof that $\mathcal{G}_-$ generates $\mathcal{M}_-$ as it is similar to the proof that $\mathcal{G}_+$ generates $\mathcal{M}_+$.

We recall from Section 3.3 that in the 'north polar' region $w = z/t \le C$, $t \ge 0$, $\mathcal{R}_-$ is empty and $\mathcal{R}_+$ is given by (3.4):

$$\mathcal{R}_+ = \{\rho_{\mathrm{base}} = 0,\ \zeta = \frac{w}{2},\ \tau = -\frac{|w|^2}{4}\} = \{\rho_{\mathrm{base}} = 0,\ w = 2\zeta,\ \tau = -|\zeta|^2\}. \tag{4.14}$$

Similarly in the 'south polar' region $w = z/|t| \le C$, $t \le 0$, $\mathcal{R}_+$ is empty and $\mathcal{R}_-$ is given by (3.4):

$$\mathcal{R}_- = \{\rho_{\mathrm{base}} = 0,\ \zeta = -\frac{w}{2},\ \tau = -\frac{|w|^2}{4}\} = \{\rho_{\mathrm{base}} = 0,\ w = -2\zeta,\ \tau = -|\zeta|^2\}. \tag{4.15}$$

On the other hand, in the region near the equator, assuming without loss of generality that $z_1 >> 0$ is dominant, then $\mathcal{R}$ is contained where ζ_1 is dominant, but it may be either positive (near $\mathcal{R}_+$) or negative (near $\mathcal{R}_-$). In either case we can use coordinates $\rho_{\mathrm{base}} = 1/z_1$, $\rho_{\mathrm{fib}} = \pm 1/\zeta_1$, $v_j = z_j/z_1$ and $\omega_j = \zeta_j/\zeta_1$ for $j \ge 2$, and $s = t/z_1$, $\sigma = \tau/|\zeta|^2$. Then , $\mathcal{R}_\pm$ is given by (3.7),

$$\mathcal{R}_\pm = \{\rho_{\mathrm{base}} = 0,\ s = \pm\rho_{\mathrm{fib}}/2,\ \hat{\zeta} = \pm\hat{z},\ \tau/|\zeta|^2 = -1\}. \tag{4.16}$$

The proof of Proposition 4.7 is in several steps. We first record that

$$\text{each element of } \mathcal{G}_\pm \text{ is characteristic on } \mathcal{R}_\pm. \tag{4.17}$$

We omit this straightforward computation. It follows that each element of $\mathcal{G}_+ \cap \mathcal{G}_-$ is characteristic on $\mathcal{R}$.

We next claim that, to show that $\mathcal{G}_+$ generates $\mathcal{M}_+$ and $\mathcal{G}_+ \cap \mathcal{G}_-$ generates $\mathcal{N}$, it suffices to prove three properties of these generating sets:

$$\text{For all } q \in \partial_{\mathrm{base}} \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}} \setminus \mathcal{R}, \text{ there is an element of } \mathcal{G}_+ \cap \mathcal{G}_- \text{ elliptic at } q, \tag{4.18}$$

$$\text{For all } q \in \mathcal{R}_-, \text{ there is an element of } \mathcal{G}_+ \text{ elliptic at } q, \text{ and} \tag{4.19}$$

$$\begin{gathered}\text{For any } q \in \mathcal{R}, \text{ there are } n+1 \text{ elements of } \mathcal{G}_+ \cap \mathcal{G}_-, \text{ say } A_1, \dots, A_{n+1}, \\ \text{such that the functions } \rho_{\mathrm{base}} \rho_{\mathrm{fib}} \sigma_{1,1}(A_i), \text{ viewed as functions} \\ \text{on } \partial_{\mathrm{base}} \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}, \text{ have linearly independent differentials at } q.\end{gathered} \tag{4.20}$$

We now prove this claim in the case of $\mathcal{G}_+$ and $\mathcal{M}_+$. By (4.17), all elements of the module generated by $\mathcal{G}_+$ are characteristic at $\mathcal{R}_+$ and hence belong to $\mathcal{M}_+$ by (4.6). Conversely, we must show that any element of $\mathcal{M}_+$ is a linear combination of elements of $\mathcal{G}_+$ with coefficients in $\Psi^{0,0}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$. So let A be an arbitrary element of $\mathcal{M}_+$. It suffices to show that OA is in the module generated by $\mathcal{G}_+$ for any $O \in \Psi^{0,0}_{\mathrm{par,cl}}$ with arbitrarily small microsupport. Thus it suffices to prove assuming that the microsupport of A is contained in a small neighbourhood of a point $q \in \partial_{\mathrm{base}} \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}$.

If q is not in $\mathcal{R}_+$, then by (4.18) and (4.19), there is an element E of $\mathcal{G}_+$ elliptic at q, and we can assume the microsupport of A is contained in the elliptic set of E. Then by the standard elliptic construction, we have $A = QE + R$ where $Q \in \Psi^{0,0}_{\mathrm{par,cl}}$ and $R \in \Psi^{-\infty,-\infty}_{\mathrm{par,cl}}$, hence A is in the module generated by $\mathcal{G}_+$.

If q is in $\mathcal{R}_+$, then there exist $A_1, \dots, A_{n+1} \in \mathcal{G}_+$ as in (4.20). Since $\mathcal{R}_+ \subset \partial_{\mathrm{base}} \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}$ is a smooth submanifold of codimension $n+1$, it follows that $\tilde{a}_1 = \rho_{\mathrm{base}} \rho_{\mathrm{fib}} \sigma_{1,1}(A_1), \dots, \tilde{a}_{n+1} = \rho_{\mathrm{base}} \rho_{\mathrm{fib}} \sigma_{1,1}(A_{n+1})$ are defining functions for $\mathcal{R}_+$ locally. That means that any function $\tilde{a}$ vanishing on $\mathcal{R}_+$ and supported sufficiently close to q can be expressed

$$\tilde{a} = \sum_{j=1}^{n+1} b_j \tilde{a}_j, \tag{4.21}$$

locally near q, for some smooth functions b_j. In particular, given arbitrary $A \in \mathcal{M}_+$, this is true for $\tilde{a} = \rho_{\mathrm{base}} \rho_{\mathrm{fib}} \sigma_{1,1}(A)$. We can extend the b_j, which are smooth functions on $\partial_{\mathrm{base}} \overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}$, to classical symbols of order $(0,0)$ on $\overline{T^*_{\mathrm{par}} \mathbb{R}^{n+1}}$; let B_j be the left quantization of these symbols. Then (4.21) implies

$$A = \sum_{j=1}^{n+1} B_j A_j + A', \quad A' \in \Psi^{1,0}_{\mathrm{par,cl}}.$$

Since $E_1 \in \mathcal{G}_+$ is elliptic as an element of $\Psi^{1,0}_{\mathrm{par,cl}}$, similarly to the first case, we can write $A' = Q' E_1 + R'$ where $Q' \in \Psi^{0,0}_{\mathrm{par,cl}}$ and $R' \in \Psi^{-\infty,-\infty}_{\mathrm{par,cl}}$. Putting these together we have

$$A = \sum_{j=1}^{n+1} B_j A_j + Q' E_1 + R'$$

which shows that A is in the module generated by $\mathcal{G}_+$, proving the claim in this case.

The proof of the claim for $\mathcal{G}_+ \cap \mathcal{G}_-$ and $\mathcal{N}$ is similar, but we only need to use (4.18) and (4.20) in this case.

It remains to show (4.18) — (4.20). We consider different regions of the boundary of phase space in turn.

First, consider the interior of fibre infinity. This region is disjoint from $\mathcal{R}$. We observe that generator $E_1 \in \mathcal{G}_+ \cap \mathcal{G}_-$ is elliptic, establishing (4.18) in this region (and therefore, vacuously, also (4.20)). This reduces us to the study of $\partial_{\text{base}} \overline{T^*_{\text{par}} \mathbb{R}^{n+1}}$, that is, to spacetime infinity.

Second, consider the north polar region where $|w| \leq C$, $w = z/t$, and $t \geq 0$. In this region, $\mathcal{R} = \mathcal{R}_+$ locally and is given by (4.14). Consider the $n+1$ generators $A_i = 2tD_{z_i} - z_i$, $i = 1 \ldots n$, and $A_{n+1} = \langle Z \rangle E_{-1} P$, from $\mathcal{G}_+ \cap \mathcal{G}_-$. As we are away from fibre-infinity we can consider $\sigma_{\text{base},1}(A_i) = t^{-1} \sigma_{1,1}(A_i)$. These functions are $2\zeta_i - w_i$ and $\tau + |\zeta|^2$, up to a smooth nonvanishing factor. The set of common zeroes is therefore precisely $\mathcal{R}$. Moreover, these functions have linearly independent differentials at each point of $\mathcal{R}$. This proves (4.18) and (4.20) in this region, while (4.19) is vacuously true in this region.

Third, consider the south polar region where $|w| \leq C$ and $t \leq 0$. In this region, $\mathcal{R} = \mathcal{R}_-$ locally. The $n+1$ generators in the previous paragraph as in the previous paragraph are such that the common zero set of their symbols is precisely $\mathcal{R}_-$, and their differentials are linearly independent. This proves (4.18) and (4.20) in this region. To prove (4.19) it is straightforward to compute that for every $q \in \mathcal{R}_-$ in this region, either $t\chi_{\text{pol},\mp} E_1$ or B_+ is elliptic at q. This is clear for the first operator, wherever $\chi_{\text{pol},\mp} > 0$. If this factor vanishes, necessarily we are close to the equator. Then consider B_+. We express the symbol b_+ of this operator, using (4.12), as

$$b_+ = |z||\zeta| \Big((\hat{z} \cdot \hat{\zeta} - 1) + \frac{(|\zeta|^2 + \tau)}{|\zeta|(|\zeta| + \sqrt{-\tau})} \Big) \chi_\tau \chi_{\text{eq}}. \tag{4.22}$$

Dividing by $|z||\zeta|$ we obtain a smooth function on $\overline{T^*_{\text{par}} \mathbb{R}^{n+1}}$ in this region. The second term vanishes on $\mathcal{R}_+ + \mathcal{R}_-$ due to the factor $|\zeta|^2 + \tau$. The second factor vanishes on $\mathcal{R}_+$ but is nonzero on $\mathcal{R}_-$, as $\hat{z} \cdot \hat{\zeta} - 1 = -2$ on $\mathcal{R}_-$. This establishes (4.19).

Fourth, consider the region near the equator where without loss of generality, we assume $z_1 > 0$ is a dominant z-variable. Then we use coordinates as described above (4.16). We notice that if $q \in \partial_{\text{base}} \overline{T^*_{\text{par}} \mathbb{R}^{n+1}}$ in this region is such that ζ_1 is not dominant as a momentum variable at q, then $q \notin \mathcal{R}$ as $\hat{z} = \pm \hat{\zeta}$ according to (4.16). We now observe that if ζ_j, $j \geq 2$, is dominant at q then the operator $z_1 D_{z_j} - z_j D_{z_1}$ is elliptic at q, while if no ζ variable is dominant at q then $|\zeta(q)|^2/\tau(q) = 0$ and hence $\langle Z \rangle E^{-1} P$ is elliptic at q, proving (4.18). So consider the case where ζ_1 is dominant and $\rho_{\text{fib}} = \pm 1/\zeta_1$ where the sign is taken so that $\rho_{\text{fib}} \geq 0$ in either case. Then we consider the $n-1$ generators $z_1 D_{z_j} - z_j D_{z_1}$, together with $\langle Z \rangle E^{-1} P$ and $tD_{z_1} - z_1/2$. After multiplying by $\rho_{\text{base}} \rho_{\text{fib}}$, the symbols of these operators become $\pm \omega_j - v_j$, $\sigma + 1$ and $s \mp \rho_{\text{fib}}/2$. We see that the reweighted symbols of these $n+1$ generators have common zero locus equal to $\mathcal{R}$, and their differentials are linearly independent. This establishes (4.18) and (4.20) in this region. To complete the argument we observe that B_+ is elliptic for $q \in \mathcal{R}_-$ in this region, as follows from the discussion of (4.22), which establishes (4.19).

□

4.3. **Positivity properties.** When proving positive commutator estimates for module regularity spaces, the following notions of positivity are extremely useful.

Definition 4.8. Let $\mathcal{M}$ be a finitely generated $\Psi^{0,0}_{\mathrm{par}}$-module, with generators $A_0 = \mathrm{Id}, A_1, \ldots, A_N \subset \Psi^{1,1}_{\mathrm{par,cl}}$. We say $\mathcal{M}$ is *P-positive* on the subset $S \subseteq \mathrm{char}(P)$ of spacetime infinity if for each j, there exist $C_{jk} \in \Psi^{1,0}_{\mathrm{par}}$ and $C'_j \in \Psi^{0,1}_{\mathrm{par}}$ such that we have

$$i\langle Z\rangle[A_j, P] = \sum_{k=0}^{N} C_{jk}A_k + C'_j P \tag{4.23}$$

with

$$\sigma_{\mathrm{base},1,0}(C_{jk})|_S = 0 \text{ for } j \neq k \tag{4.24}$$

and

$$\mathrm{Re}(\sigma_{\mathrm{base},1,0}(C_{jj}))|_S \geq 0. \tag{4.25}$$

Similarly, we say that $\mathcal{M}$ is *P-negative* on S if the same conditions are satisfied with the inequality (4.25) reversed. We say $\mathcal{M}$ is *P-critical* on S if $\mathcal{M}$ is both *P-positive* and *P-negative* on S.

We conclude the section by showing that the modules $\mathcal{M}_\pm$ and $\mathcal{N}$ satisfy positivity properties phrased using Definition 4.8. We shall exploit these in Section 5.4.

Proposition 4.9. *The modules defined in Definition 4.5 enjoy the following positivity properties.*

(i) $\mathcal{M}_+$ is P-positive at $\mathcal{R}_+$;
(ii) $\mathcal{M}_-$ is P-negative at $\mathcal{R}_-$;
(iii) $\mathcal{N}$ is P-critical at $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$.

Proof. The commutators of the first two differential generators with $P_0 = \Delta_0 + D_t$, where Δ_0 is the flat Euclidean (positive) Laplacian, are as follows:

$$[z_i D_{z_j} - z_j D_{z_i}, P_0] = 0, \tag{4.26}$$

$$[tD_{z_j} - z_j/2, P_0] = 0, \tag{4.27}$$

As g is Euclidean outside of a compact set in space-time, we have that $P_0 - P$ is a compactly supported differential operator in $\Psi^{2,-\infty}_{\mathrm{par}}$. This implies that the commutators

$$i\langle Z\rangle[z_i D_{z_j} - z_j D_{z_i}, P] \tag{4.28}$$

and

$$i\langle Z\rangle[tD_{z_j} - z_j/2, P] \tag{4.29}$$

are compactly supported differential operators in $\Psi^{2,-\infty}_{\mathrm{par}}$. They can both be written in the form $CE + R$ for some $C \in \Psi^{1,-\infty}_{\mathrm{par}}$ and $R \in \Psi^{-\infty,-\infty}_{\mathrm{par}}$ by using the ellipticity of the generator E, moreover $\sigma_{\mathrm{base},1,1}(C)$ vanishes on $\mathcal{R}$ in both instances as $C \in \Psi^{1,-\infty}_{\mathrm{par}}$. The generator involving P has commutator

$$i\langle Z\rangle[\langle Z\rangle E_{-1}P, P] = i\langle Z\rangle[\langle Z\rangle E_{-1}, P]P \tag{4.30}$$

which is also of the required form as $[\langle Z\rangle E_{-1}, P] \in \Psi^{0,0}_{\mathrm{par}}$.

The elliptic generator E_1 itself is of lower order $(1,0)$, and so the corresponding commutator can also be written in the form

$$i\langle Z\rangle[E_1, P] = CE_1 + R \tag{4.31}$$

where $C \in \Psi^{1,0}_{\mathrm{par}}$ and $R \in \Psi^{-\infty,-\infty}_{\mathrm{par}}$. As $P = P_0$ outside of a compact set in space-time, the Poisson bracket $\{\sigma(E), \sigma(P)\}$ vanishes identically near the boundary of the space-time compactification. Consequently C has base symbol vanishing on $\mathcal{R}$. These computations establish that $\mathcal{N}$ is P-critical at $\mathcal{R}$.

We now consider the two additional generators of $\mathcal{M}_+$. For $A \in \Psi^{1,1}_{\mathrm{par}} = \mathrm{Op}(a)$, we have $\sigma_{2,0}([A,P]) = iH_p a = i(\partial_t + 2\zeta \cdot \partial_z)a$ in a neighbourhood of spacetime infinity, and so applying this to the generator with symbol

$$a = t\langle R^2\rangle^{1/2}\chi_{\mathrm{pol},-}$$

we obtain

$$\begin{aligned} &i\langle Z\rangle\sigma_{2,0}([A,P]) && (4.32)\\ &= -\langle R^2\rangle^{1/2}\left[\chi_{\mathrm{pol},-}\langle Z\rangle + \chi'\left(\frac{-t}{\langle Z\rangle}\right)\left(\frac{-t-t|z|^2+2t^2z\cdot\zeta}{\langle Z\rangle^2}\right)\right] && (4.33)\\ &= \frac{\langle Z\rangle}{-t}a - \langle R^2\rangle^{1/2}\chi'\left(\frac{-t}{\langle Z\rangle}\right)\left(\frac{-t-t|z|^2+2t^2z\cdot\zeta}{\langle Z\rangle^2}\right) && (4.34)\end{aligned}$$

The prefactor of a is non-negative on the support of a. The term $\langle R^2\rangle^{1/2}\chi'\cdot\left(-\frac{t}{\langle Z\rangle^2}\right)$ is of lower order $(-\infty,-1)$. The remaining terms can be written as

$$\langle R^2\rangle^{1/2}\chi'\left(\frac{-t}{\langle Z\rangle^2}\right)\cdot\left(\frac{2t^2z\cdot\zeta - t|z|^2}{\langle Z\rangle}\right) \tag{4.35}$$

and since

$$\frac{2t^2z\cdot\zeta - t|z|^2}{\langle Z\rangle^2} = \frac{2tz}{\langle Z\rangle^2}\cdot(t\zeta - z/2) \tag{4.36}$$

the term in (4.35) is a sum of $S^{1,0}_{\mathrm{par}}$-multiples of $\sigma_{2,0}(tD_{z_i} - z_i/2)$, with each of the $S^{1,0}_{\mathrm{par}}$ coefficients vanishing on $\mathcal{R}_+$ due to the cutoff factor.

Thus we have shown

$$i\langle Z\rangle\sigma_{2,0}([A,P]) = \sum_{k=0}^{N} c_{jk}\sigma_{1,0}(A_k) \tag{4.37}$$

for $c_{jk} \in S^{1,0}_{\mathrm{par}}$ with $\sigma_{\mathrm{base},1,0}(c_{jk}) = 0$ on $\mathcal{R}_+$. Quantising each symbol in this identity, we see that $tE_1\chi_{\mathrm{pol},-}$ satisfies the required positivity condition.

Finally we compute $i\langle Z\rangle\sigma([B_+,P])$ using the Poisson bracket. We have

$$\begin{aligned} &i\langle Z\rangle\sigma([B_+,P])\\ &= -\langle Z\rangle(\partial_t + 2\zeta\cdot\partial_z)\left((z\cdot\zeta - |z|\sqrt{-\tau})\chi_\tau\chi_{\mathrm{eq}}\right)\\ &= -\chi_\tau\langle Z\rangle(\partial_t + 2\zeta\cdot\partial_z)\left((z\cdot\zeta - |z|\sqrt{-\tau})\chi_{\mathrm{eq}}\right)\\ &= -\chi_\tau\langle Z\rangle\left(\left(2|\zeta|^2 - \frac{2\sqrt{-\tau}z\cdot\zeta}{|z|}\right)\chi_{\mathrm{eq}}\right.\\ &\qquad \left. - (z\cdot\zeta - |z|\sqrt{-\tau})\chi'\left(\frac{|t|}{\langle Z\rangle}\right)\left(\frac{1+|z|^2-2tz\cdot\zeta}{\langle Z\rangle^3}\right)\cdot\mathrm{sgn}(t)\right).\end{aligned} \tag{4.38}$$

The second term in the final equation is in the form $\sum_{j=1}^n c_j(t\zeta_j - z_j/2) + c'$ with $c_j, c' \in S^{1,0}_{\mathrm{par}}$ and with every c_j and c' vanishing on $\mathcal{R}_+$. The remaining term in (4.38) can be written in the form

$$-2\chi_\tau\chi_{\mathrm{eq}}\langle Z\rangle\left(|\zeta|^2 - \frac{\sqrt{-\tau}z\cdot\zeta}{|z|}\right) = 2\chi_\tau\chi_{\mathrm{eq}}\langle Z\rangle\frac{\sqrt{-\tau}}{|z|}(z\cdot\zeta - |z|\sqrt{-\tau}) \tag{4.39}$$

$$- 2\chi_\tau\chi_{\mathrm{eq}}\langle Z\rangle(\tau + |\zeta|^2). \tag{4.40}$$

Since the first term is a positive $S^{1,0}_{\mathrm{par}}$ multiple of $\sigma(B_+)$ and the second is a $S^{0,1}_{\mathrm{par}}$ multiple of $\sigma(P)$, quantising this identity leads to the required positivity condition for B_+.

Taking the opposite sign choices in the final two generators leads to an almost identical computation, with the different sign leading to a conclusion of P-negativity rather than P-positivity. ☐

Remark 4.3. The notion of module regularity can be generalised to the setting where the $\mathbf{A}_j$ are only assumed to lie in $\Psi^{s_j,l_j}_{\mathrm{par}}$ for collections of positive integers $(s_j)_{j=0}^N$ and $(l_j)_{j=0}^N$. In this setting, it can be useful to work with a *reduced* version $\mathcal{M}^{(\kappa)}$ of the module powers in Definition 4.2 where the indices α in the generating set are restricted to those with $\mathbf{A}^\alpha \in \Psi^{\kappa,\kappa}_{\mathrm{par}}$. This approach has been pursued in [9].

4.4. **Density.** For use in the proof of Theorem 6.3, we prove the following density result for module regularity spaces $H_{\pm}^{s,\mathsf{r};\kappa,k}$.

Proposition 4.10. *Suppose that the variable order* r *is constant in a neighbourhood of the radial sets. Then the space* $\mathcal{S}(\mathbb{R}^{n+1})$ *of Schwartz functions is dense in* $H_{\pm}^{s,\mathsf{r};\kappa,k}$ *for all* $s \in \mathbb{R}$ *and* $k, \kappa \in \mathbb{N}$*.*

Proof. By microlocalizing, we can reduce to the following special cases:

(i) Proving the same statement for constant spacetime weight r;
(ii) Proving the statement for variable order but for $k = \kappa = 0$, that is, with module regularity absent.

Indeed, near the radial sets our spacetime weight r is constant by assumption, while away from the radial sets, both the large and small modules are elliptic (see (4.18) and (4.19)), in which case the module regularity space is microlocally identical to $H_{\mathrm{par}}^{s+k+\kappa,\mathsf{r}+k+\kappa}$.

In case (i), we first consider the case $s = \mathsf{r} = 0$. We let T_ε, for $\varepsilon > 0$, be a family of parabolic scattering pseudodifferential operators of order $(-\infty, -\infty)$ such that the $S^{0,0}$-seminorms of T_ε are uniformly bounded, and $T_\varepsilon \to \mathrm{Id}$ strongly. For example, one can take $T_\varepsilon = \mathrm{Op}(e^{-\varepsilon(|z|^2+t^2)}e^{-\varepsilon(|\zeta|^4+\tau^2)})$. Then it is not difficult to show that for any $A_1, \dots, A_j \in \Psi^{1,1}_{\mathrm{par,cl}}$, the multi-commutator $[A_1, [A_2, \dots [A_j, T_\varepsilon] \dots]]$ tends to zero strongly; we omit the proof. Given $u \in H_{\pm}^{0,0;\kappa,k}$, we define $u_j = T_{j^{-1}} u \in \mathcal{S}$. Then $u_j \to u$ in L^2, since $T_\varepsilon \to \mathrm{Id}$ strongly. Moreover, for any product $A_1 \dots A_q B_1 \dots B_{q'}$ of at most k elements of $\mathcal{M}_\pm$ and at most κ elements of $\mathcal{N}$, we find that

$$A_1 \dots A_q B_1 \dots B_{q'} u_j = A_1 \dots A_q B_1 \dots B_{q'} T_{j^{-1}} u = T_{j^{-1}} A_1 \dots A_q B_1 \dots B_{q'} u \\ + \text{ commutator terms.} \quad (4.41)$$

The first term on the RHS tends to $A_1 \dots A_q B_1 \dots B_{q'} u$ as $j \to \infty$. The commutator factors all tend to zero strongly. We move these factors to the left, at the cost of double commutators, which we move to the left at the cost of triple commutators, and so on. Eventually, we arrive at a sum of terms, the left factor of which is a multicommutator of module elements with $T_{j^{-1}}$ and the remaining factors are module elements. All of these multicommutator factors tend to zero strongly, and they act on a fixed function in L^2, using the fact that u has module regularity of order (k, κ). All terms other than the first one above therefore tend to zero in L^2 as $j \to \infty$. We deduce that

$$A_1 \dots A_q B_1 \dots B_{q'} u_j \to A_1 \dots A_q B_1 \dots B_{q'} u \text{ as } j \to \infty.$$

We deduce that $u_j \to u$ in the topology of $H_{\pm}^{0,0;\kappa,k}$, proving the density in the case $s = \mathsf{r} = 0$.

For general constant s and r, we choose an elliptic, invertible operator $F \in \Psi^{s,r}_{\mathrm{par,cl}}$. (To do this, we start with an elliptic operator of the form $\mathrm{Op}^w(f)$ where f is a real elliptic symbol of order (s,r); then $\mathrm{Op}^w(f)$ is formally self-adjoint and Fredholm, hence has a finite dimensional kernel, which consists of Schwartz functions due to elliptic regularity. Then $F = \mathrm{Op}^w(f) + \Pi$, where Π is orthogonal projection onto the null space, is invertible, and Π is an operator of order $(-\infty,-\infty)$, so $F \in \Psi^{s,r}_{\mathrm{par,cl}}$ as required.) Given $u \in H^{s,r;\kappa,k}_{\pm}$, we have $Fu \in H^{0,0;\kappa,k}_{\pm}$. We choose Schwartz u_j converging to Fu in $H^{0,0;\kappa,k}_{\pm}$. Then we claim that $F^{-1}u_j$ converges to u in $H^{s,r;\kappa,k}_{\pm}$. The proof is a standard commutation argument, which we omit. This completes the proof in case (i).

In case (ii), so now r can be a variable order, we choose an elliptic invertible operator of order (s,r) as above. Then given $u \in H^{s,\mathsf{r}}_{\mathrm{par}}$, Fu is in L^2. We approximate Fu in L^2 by the Schwartz sequence $u_j = T_{j^{-1}}Fu$ as above, and then $F^{-1}u_j$ converges to u in the topology of $H^{s,\mathsf{r}}_{\mathrm{par}}$. This proves case (ii). □

5. Fredholm estimates

In this section, we show that the operator $P = D_t + \Delta_g + V$ is a Fredholm map between suitable function spaces, following closely the methodology introduced in [36], and followed in [11], in which microlocal estimates, including radial points propagation estimates, are combined to prove global Fredholm estimates.

5.1. **Microlocal propagation estimates.** Here we collect together various microlocal estimates for P. These are proved in Section 8 for a general class of operators. Here we restate these estimates in the special case of the operator P under consideration.

We can distinguish four different estimates, each valid in a particular microlocal region. The first region is the elliptic region $\mathrm{ell}(P)$. In this case we obtain an estimate without loss of spacetime or differential order. This was already stated as Proposition 2.12 but for ease of reference we restate it here. The second region is near the characteristic variety $\mathrm{char}(P)$ and away from the radial sets. This is the region of principal-type propagation, and is essentially Hörmander's original 'propagation of singularities' (really propagation of regularity) estimate from [16]. The third region is near the radial sets. In this case, there are two estimates required, depending on whether the spacetime regularity order is greater than or less than the threshold value of $-1/2$ (see the discussion in the Introduction). From the technical point of view, the significance of the threshold value is precisely the different form that the radial point estimates necessarily take in the two cases.

The elliptic estimate, Proposition 2.12 in the particular case of our Schrödinger operator $P = D_t + \Delta_g + V$ takes the following form.

Proposition 5.1 (Elliptic estimate)**.** *Suppose that $Q, G \in \Psi^{0,0}_{\mathrm{par}}(\mathbb{R}^{n+1})$ are such that P and G are elliptic on $\mathrm{WF}'(Q)$, let r be an arbitrary spacetime order, and let $s, M, N \in \mathbb{R}$. Then there exists $C > 0$ such that, if $GPu \in H^{s-2,\mathsf{r}}_{\mathrm{par}}$, we have $Qu \in H^{s,\mathsf{r}}_{\mathrm{par}}$ with an estimate*

$$\|Qu\|_{H^{s,\mathsf{r}}_{\mathrm{par}}} \leq C(\|GPu\|_{H^{s-2,\mathsf{r}}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{5.1}$$

Remark 5.1. For non-experts in microlocal analysis, we mention that this estimate is the microlocal analogue of the standard elliptic estimate in classical PDE theory: if q, g are two C^∞_c functions with $\mathrm{supp}\, q \subset \{g > 0\}$ and if P is a differential operator of order 2 with smooth coefficients that is elliptic on the support of g, then we have for any $M \in \mathbb{R}$ the estimate (in standard Sobolev spaces)

$$\|qu\|_{H^s} \leq C(\|gPu\|_{H^{s-2}} + \|u\|_{H^M}). \tag{5.2}$$

The estimate is of course only interesting when M is smaller than s. We think of the symbols of Q and G in (5.1) as cutoff functions, analogous to q and g in (5.2), but on phase space rather than just on spacetime.

The propagation of singularities (regularity) estimate, Proposition 8.1, reads as follows. Note the loss of one order of regularity in both the spacetime order r and the differential order s, reflecting the fact that the characteristic variety $\mathrm{char}(P)$ meets both spacetime-infinity and fibre-infinity.

Proposition 5.2 (Propagation of regularity)**.** *Let $Q, Q', G \in \Psi^{0,0}_{\mathrm{par}}$ be operators of order $(0,0)$ with G elliptic on $\mathrm{WF}'(Q)$. Let s, M and N be real numbers and let r be a variable spacetime order that is non-increasing in the direction of the bicharacteristic flow of P.*

Furthermore, suppose that for every $\alpha \in \mathrm{WF}'(Q) \cap \mathrm{char}(P)$ there exists α' such that Q' is elliptic at α' and there is a forward bicharacteristic curve γ of P from α' to α such that G is elliptic on γ.

Then there exists $C > 0$ such that, if $GPu \in H^{s-1,\mathsf{r}+1}_{\mathrm{par}}$ and $Q'u \in H^{s,\mathsf{r}}_{\mathrm{par}}$, we have $Qu \in H^{s,\mathsf{r}}_{\mathrm{par}}$ with an estimate

$$\|Qu\|_{H^{s,\mathsf{r}}_{\mathrm{par}}} \leq C(\|Q'u\|_{H^{s,\mathsf{r}}_{\mathrm{par}}} + \|GPu\|_{H^{s-1,\mathsf{r}+1}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{5.3}$$

Remark 5.2. Figure 3 illustrates the setup of Proposition 5.2. In words, the Proposition states that regularity of the function u (in both the spacetime and differential order) propagates from the microsupport of Q', that is $\mathrm{WF}'(Q')$, to the microsupport of Q, provided that the regularity is not greater at $\mathrm{WF}'(Q)$ than at the corresponding points of $\mathrm{WF}'(Q')$ (we cannot inexplicably gain regularity!) and provided that Pu is sufficiently regular in a microlocal neighbourhood of all the bicharacteristics that traverse between $\mathrm{WF}'(Q')$ and $\mathrm{WF}'(Q)$.

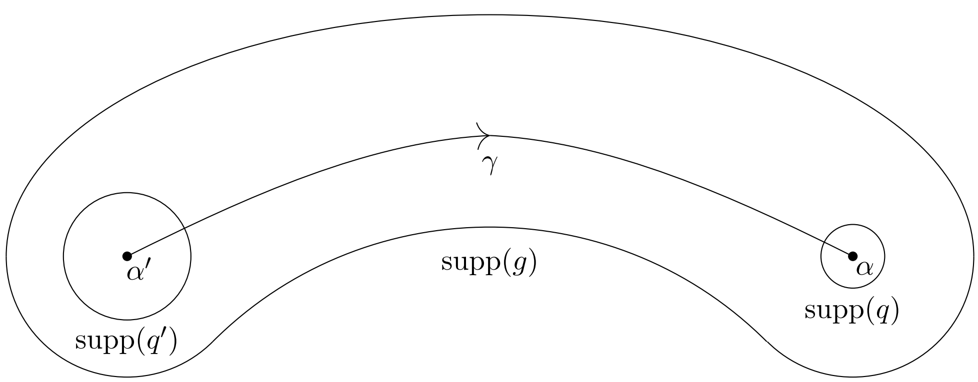

FIGURE 3. The hypotheses of Proposition 5.2 is that for each $\alpha \in \mathrm{supp}(q) \cap \mathrm{char}(P)$ there is a bicharacteristic segment γ, contained within $\mathrm{supp}(g)$, connecting $\alpha' \in \mathrm{supp}(q')$ to α. Notice that we only need this condition for $\alpha \in \mathrm{char}(P)$, otherwise the stronger elliptic estimate is available.

Notice that the estimate above only gives a trivial estimate if $\mathrm{WF}'(Q)$ meets the radial set. This is because the bicharacteristic flow is stationary on the radial sets, so we would need Q' also elliptic at the radial set, which means the conclusion would be no stronger than the assumption.

For estimates valid near the radial points, we refer to Propositions 8.3 and 8.2 in the Appendix, based on estimates due to Melrose [27] and Vasy [36], adapted here to the parabolic calculus. Specializing to the case of the time dependent Schrödinger operator $P = \Delta_g + D_t + V \in \Psi^{2,0}_{\mathrm{par}}$, and iterating the results to give an arbitrary gain of regularity compared to the background regularity assumption, gives the following results. We state them for constant orders for simplicity, as that is all that our arguments require.

Proposition 5.3 (Below threshold radial point estimate)**.** *Let s, r, M, N be real numbers with $r < -\frac{1}{2}$. Assume there exists a neighbourhood U of $\mathcal{R}_\pm$ and $Q', G \in \Psi^{0,0}_{\mathrm{par}}$ such that for every $\alpha \in \mathrm{char}(P) \cap U \setminus \mathcal{R}_\pm$ the bicharacteristic γ through α enters $\mathrm{ell}(Q')$ whilst remaining in $\mathrm{ell}(G)$. Then there exists $Q \in \Psi^{0,0}_{\mathrm{par}}$ elliptic on $\mathcal{R}_\pm$ and a constant $C > 0$ such that if $u \in H^{M,N}_{\mathrm{par}}$, $Q'u \in H^{s,r}_{\mathrm{par}}$ and $GPu \in H^{s-1,r+1}_{\mathrm{par}}$, then $Qu \in H^{s,r}_{\mathrm{par}}$ with an estimate*

$$\|Qu\|_{H^{s,r}_{\mathrm{par}}} \le C(\|Q'u\|_{H^{s,r}_{\mathrm{par}}} + \|GPu\|_{H^{s-1,r+1}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{5.4}$$

Proposition 5.4 (Above threshold radial point estimate)**.** *Suppose s, s', r, r', M and N be real numbers satisfying $r > r' > -\frac{1}{2}$ and $s > s'$. Assume that $G \in \Psi^{0,0}_{\mathrm{par}}$ is elliptic at $\mathcal{R}_\pm$. Then there exists $Q \in \Psi^{0,0}_{\mathrm{par}}$ elliptic at $\mathcal{R}_\pm$ and a constant $C > 0$ such that, if $u \in H^{M,N}_{\mathrm{par}}$, $Gu \in H^{s',r'}_{\mathrm{par}}$ and $GPu \in H^{s-1,r+1}_{\mathrm{par}}$, then $Qu \in H^{s,r}_{\mathrm{par}}$ with an estimate*

$$\|Qu\|_{H^{s,r}_{\mathrm{par}}} \le C(\|GPu\|_{H^{s-1,r+1}_{\mathrm{par}}} + \|Gu\|_{H^{s',r'}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{5.5}$$

Remark 5.3. We see that the below threshold estimate (5.4) looks the same as (5.3) but with the additional assumption that r is below the threshold value of $-1/2$. On the other hand, the above threshold estimate is a bit different: we do *not* need to assume that we have microlocal regularity at the same order (s, r) on some other set $\mathrm{WF}'(Q)$, but instead, we *do* need to assume a priori that we have regularity at some order (s', r') where r' is already above threshold. The proposition then tells us we can bootstrap this to (s, r)-regularity, provided that Pu is suitably regular. This difference is crucial as it means that we have a starting place for proving (s, r)-regularity: that is, we can deduce (s, r) regularity, say for a solution to $Pu = 0$, without having to already know it somewhere else. This explains why our function spaces introduced below in (5.7), (5.8) impose above threshold regularity at one of the radial sets. On the other hand, to propagate regularity all the way to the other radial set the regularity needs to be below threshold at the other radial set so that Proposition 5.3 can be applied.

5.2. Global Fredholm estimate — variable order case.

In this subsection, we combine the estimates in the preceding subsection into a single global estimate that will suffice to establish the Fredholm property for $P = \Delta_g + D_t + V$ as a map between two suitable variable order Sobolev spaces. We choose real two constants l, m with $l < -1/2 < m$ — that is, l is below, and m above, threshold — and fix a weight function $\mathsf{r}_+ \in S^{0,0}_{\mathrm{par}}$ with the properties

(i) $\mathsf{r}_+(z, \zeta) \in [l, m]$;
(ii) $\mathsf{r}_+ = l$ in a neighbourhood $\mathcal{U}_+$ of $\mathcal{R}_+$ and $\mathsf{r}_+ = m$ in a neighbourhood $\mathcal{U}_-$ of $\mathcal{R}_-$;
(iii) r_+ is nonincreasing along the bicharacteristics of P

and take

$$\mathsf{r}_- = -1 - \mathsf{r}_+. \tag{5.6}$$

In some cases, it is convenient to assume additionally that

(iv) $l > -3/2$ and $m \leq l + 1$.

Remark 5.4. Proposition 3.1 shows that assumption (iii) is compatible with (i) and (ii).

We then define the variable order Sobolev spaces

$$\mathcal{Y}^{s,\mathsf{r}_\pm} = H_{\mathrm{par}}^{s,\mathsf{r}_\pm} \tag{5.7}$$

and

$$\mathcal{X}^{s,\mathsf{r}_\pm} = \{u \in \mathcal{Y}^{s,\mathsf{r}_\pm} : Pu \in H_{\mathrm{par}}^{s-1,\mathsf{r}_\pm+1}\} \tag{5.8}$$

We then have the following global Fredholm estimate for P.

Proposition 5.5. *For the above choice of weight functions $\mathsf{r}_\pm$ satisfying (i) – (iii) above, arbitrary $s \in \mathbb{R}$, $M < s$, $N < l$, and all $u \in \mathcal{X}^{s,\mathsf{r}_\pm}$ we have*

$$\|u\|_{H_{\mathrm{par}}^{s,\mathsf{r}_\pm}} \leq C(\|Pu\|_{H_{\mathrm{par}}^{s-1,\mathsf{r}_\pm+1}} + \|u\|_{H_{\mathrm{par}}^{M,N}}) \tag{5.9}$$

Proof. We shall prove (5.9) in the case of the weight function r_+. The proof for r_- is essentially identical, with the roles of $\mathcal{R}_\pm$ swapped.

We begin by choosing $Q_1, Q_2, Q_3, Q_4, G_1, G_2, G_3, G_4 \in \Psi_{\mathrm{par}}^{0,0}$ such that

(i) G_j is elliptic on $\mathrm{WF}'(Q_j)$,
(ii) P is elliptic on $\mathrm{WF}'(Q_1)$,
(iii) $\mathrm{WF}'(Q_3) \subset \mathrm{WF}'(G_3) \subset \mathcal{U}_+$,
(iv) $\mathrm{WF}'(Q_4) \subset \mathrm{WF}'(G_4) \subset \mathcal{U}_-$,
(v) $\mathrm{WF}'(Q_2)$ is disjoint from $\mathcal{R}_\pm$,
(vi) Every α in a punctured neighbourhood of $\mathcal{R}_+$ on $\mathrm{char}(P)$ lies on forward bicharacteristic γ from a point $\alpha' \in \mathrm{ell}(Q_2)$, with the bicharacteristic lying entirely in $\mathrm{ell}(G_3)$,
(vii) Every $\alpha \in \mathrm{WF}'(Q_2) \cap \mathrm{char}(P)$ lies on a forward bicharacteristic γ from a point $\alpha' \in \mathrm{ell}(Q_4)$, with the bicharacteristic lying entirely in $\mathrm{ell}(G_2)$,
(viii) $Q_1 + Q_2 + Q_3 + Q_4 = \mathrm{Id}$.

We can now apply Proposition 5.4 using the operators Q_4 and G_4. We can replace the spacetime order r with the variable weight r_+ in this estimate, as r_+ is constant (equal to m) in $\mathrm{WF}'(G_4)$. This yields the estimate

$$\|Q_4 u\|_{s,\mathsf{r}_+} \leq C(\|G_4 Pu\|_{s-2,\mathsf{r}_++1} + \|G_4 u\|_{s',r'} + \|u\|_{M,N}) \tag{5.10}$$

for any $M, N, s, s' \in \mathbb{R}$ and $-1/2 < r' < m$. Similarly, we apply Proposition 5.3 to the operators Q_3, Q_2 to give

$$\|Q_3 u\|_{s,\mathsf{r}_+} \leq C(\|Q_2 u\|_{s,\mathsf{r}_+} + \|G_3 Pu\|_{s-1,\mathsf{r}_++1} + \|u\|_{M,N}). \tag{5.11}$$

Away from the radial sets, we can control $\|Q_1 u\|$ and $\|Q_2 u\|$ using the microlocal elliptic estimate of Proposition 5.1 and the real principal type propagation result of Proposition 5.2 respectively. For the latter, we use $\mathrm{WF}'(Q_4)$ as a source of regularity, given the dynamical condition (vii). Consequently we have an estimate

$$\|Q_2 u\|_{s,\mathsf{r}_+} \leq C(\|Q_4 u\|_{s,\mathsf{r}_+} + \|G_2 Pu\|_{s-1,\mathsf{r}_++1} + \|u\|_{M,N}). \tag{5.12}$$

In the elliptic region, we weaken (5.1) to

$$\|Q_1 u\|_{s,\mathsf{r}_+} \leq C(\|G_1 Pu\|_{s-1,\mathsf{r}_++1} + \|u\|_{M,N}) \tag{5.13}$$

so that the norm of $G_1 Pu$ agrees with the norms for $G_i Pu$ with $i = 2 \ldots 4$.

Without loss of generality, we can assume that the constants C in estimates (5.10) — (5.13) are equal, and exceed 1. Then, we estimate

$$\|u\|_{s,\mathsf{r}_+} \leq \|Q_1 u\|_{s,\mathsf{r}_+} + 2C\|Q_2 u\|_{s,\mathsf{r}_+} + \|Q_3 u\|_{s,\mathsf{r}_+} + 4C^2\|Q_4 u\|_{s,\mathsf{r}_+}$$

and combine the estimates in (5.13), (5.12), (5.11) and (5.10). This combination allows us to absorb the $Q_2 u$ and $Q_4 u$ terms on the RHS by those on the LHS. This gives (with a new constant C)

$$\|u\|_{s,\mathsf{r}_+} \leq C(\|Pu\|_{s-1,\mathsf{r}_++1} + \|G_4 u\|_{s',r'} + \|u\|_{M,N}). \tag{5.14}$$

For $r' = (-1/2, m)$ and appropriate choices of $s' \in (M, s)$ and $\eta \in (0,1)$, Sobolev interpolation and Young's inequality then give

$$\begin{aligned} \|G_4 u\|_{s',r'} &\leq \|G_4 u\|_{s,m}^{1-\eta} \|G_4 u\|_{M,N}^{\eta} && (5.15) \\ &\leq \frac{1}{2}\|u\|_{s,m} + C\|u\|_{M,N} && (5.16) \end{aligned}$$

for a suitable constant C. As $\mathsf{r}_+ = m$ on $\mathrm{WF}'(G_4)$, we can replace the constant order m with the weight r_+ and absorb this term into the left-hand side of (5.14), allowing us to conclude (5.9). □

We now show, following [17, Theorem 21.7] and [37, Section 4.3], that the estimate of Proposition 5.5 implies that P is a Fredholm map.

Proposition 5.6. *For $s \in \mathbb{R}$, the map $P : \mathcal{X}^{s,\mathsf{r}_\pm} \to \mathcal{Y}^{s-1,\mathsf{r}_\pm+1}$ for either sign choice is a Fredholm map of index zero.*

Proof. The argument is essentially identical for the two sign choices, and so we can take the positive sign without loss of generality. On $\ker(P) \subset \mathcal{X}^{s,\mathsf{r}_+}$, the estimate (5.9) simplifies to

$$\|u\|_{H^{s,\mathsf{r}_+}_{\mathrm{par}}} \leq C\|u\|_{H^{M,N}_{\mathrm{par}}}. \tag{5.17}$$

From compactness of the embedding $H^{s,\mathsf{r}_+}_{\mathrm{par}} \subset H^{M,N}_{\mathrm{par}}$, it follows that the identity map restricted to $\ker(P) \subset H^{M,N}_{\mathrm{par}}$ is compact, and so $\ker(P)$ is finite-dimensional.

Next we show that the range of P is closed. To this end, we take a sequence of $u_j \in \mathcal{X}^{s,\mathsf{r}_+}$ with $u_j \in \ker(P)^\perp$ and Pu_j converging to some f in $H^{s-1,\mathsf{r}_++1}_{\mathrm{par}}$. Then first we observe that $\|u_j\|_{M,N}$ is uniformly bounded. If this were not the case, then we could pass to a subsequence with $\|u_j\|_{M,N} \to \infty$ and then making the rescaling $\hat{u}_j = u_j/\|u_j\|_{M,N}$, an application of (5.9) to $\hat{u}_j$ together with the compactness of the embedding $H^{s,\mathsf{r}_+}_{\mathrm{par}} \subset H^{M,N}_{\mathrm{par}}$ allows us to deduce convergence in $H^{s,\mathsf{r}_+}_{\mathrm{par}}$ of a subsequence $\hat{u}_j$ to a limit $v \in \ker(P)$. As $u_j \in \ker(P)^\perp$, it follows that $v = 0$, which is a contradiction as we have $\|\hat{u}_j\|_{M,N} = 1$ by construction. The boundedness of $\|u_j\|_{M,N}$ just demonstrated immediately implies boundedness of $\|u_j\|_{s,\mathsf{r}_++1}$ from (5.9). Once more exploiting the compactness of the embedding $H^{s,\mathsf{r}_+}_{\mathrm{par}} \subset H^{M,N}_{\mathrm{par}}$, it follows that a subsequence u_j is convergent in $H^{M,N}_{\mathrm{par}}$. Since Pu_j is convergent in $H^{s-1,\mathsf{r}_++1}_{\mathrm{par}}$, (5.9) implies that this subsequence is convergent to some u in $\mathcal{X}^{s,\mathsf{r}_+}$ with $Pu = f$ hence proving that the range of P is closed.

Next we show that the cokernel of P is finite-dimensional. We may identify $\mathrm{coker}(P)$ with the set of $v \in (H^{s-1,\mathsf{r}_++1}_{\mathrm{par}})^* = H^{1-s,\mathsf{r}_-}_{\mathrm{par}}$ such that $P^*v = 0$. Since P^* is equal to P with the potential replaced by its complex conjugate, the same argument used to establish finite dimensionality of $\ker(P)$ can be used (with s replaced by $1-s$ and with the opposite sign choice for our spacetime weight), provided we take $M < \min(s, 1-s)$.

It remains to show that the index is zero. We note that P^* is simply P with the potential replaced by its complex conjugate. We can form a linear homotopy between P and P^*; this is a continuous family of operators mapping $\mathcal{X}^{s,\mathsf{r}_\pm} \to \mathcal{Y}^{s-1,\mathsf{r}_\pm+1}$ since the potential is smooth and compactly supported. Hence the index is constant along this path. It follows that $\operatorname{ind}(P) = \operatorname{ind}(P^*)$. On the other hand, $\operatorname{ind}(P) = -\operatorname{ind}(P^*)$ for any Fredholm operator, so we conclude that $\operatorname{ind}(P) = 0$. □

5.3. **Module regularity estimates away from radial sets.** The microlocal estimates of Section 5.1 have analogues in the setting of Sobolev spaces with module regularity, as introduced in Definition 4.5. We state these results for the particular operator $P = \Delta_g + D_t + V$. First we prove an analogue of Proposition 5.1.

Proposition 5.7. *Suppose $Q, G \in \Psi^{0,0}_{\mathrm{par}}$ are such that P and G are elliptic on $\mathrm{WF}'(Q)$, let r be an arbitrary variable order, and let $M, N \in \mathbb{R}$. Then if $GPu \in H_\pm^{s-2,\mathsf{r};\kappa,k}$, we have $Qu \in H_\pm^{s,\mathsf{r};\kappa,k}$ with the estimate*

$$\|Qu\|_{H_\pm^{s,\mathsf{r};\kappa,k}} \leq C(\|GPu\|_{H_\pm^{s-2,\mathsf{r};\kappa,k}} + \|u\|_{H_{\mathrm{par}}^{M,N}}). \tag{5.18}$$

Proof. An elliptic parametrix construction allows us to write

$$Q = CGP + R \tag{5.19}$$

where $C \in \Psi^{-2,0}_{\mathrm{par}}$ satisfies $\mathrm{WF}'(C) \subseteq \mathrm{WF}'(Q)$. Then for a collection $A_1, A_2, \dots, A_{n_1}$ of elements of $\mathcal{M}_\pm$ and a collection $B_1, B_2, \dots, B_{n_2}$ of elements of $\mathcal{N}$, we can compute

$$\begin{aligned}
A_1 \dots A_{n_1} B_1 \dots B_{n_2} Qu &= A_1 \dots A_{n_1} B_1 \dots B_{n_2}(CGP + R)u && (5.20)\\
&= CA_1 \dots A_{n_1} B_1 \dots B_{n_2} GPu + \tilde{R}u && (5.21)\\
&+ \sum_{j=1}^{n_1} A_1 \dots A_{j-1}[C, A_j]A_{j+1} \dots A_{n_1} B_1 \dots B_{n_2} GPu && (5.22)\\
&+ \sum_{j=1}^{n_2} A_1 \dots A_{n_1} B_1 \dots B_{j-1}[C, B_j]B_{j+1} \dots B_{n_2} GPu. && (5.23)
\end{aligned}$$

We can move the commutators to the left of the final two terms by incurring terms involving a double commutator and one fewer module generator in the product that does not lie in a commutator. Iterating this process, we obtain

$$\begin{aligned}
A_1 \dots A_{n_1} B_1 \dots B_{n_2} Qu = CA_1 \dots A_{n_1} B_1 \dots B_{n_2} GLu + \tilde{R}u && (5.24)\\
+ \sum_S C_S(\prod_{A \notin S} A)GPu && (5.25)
\end{aligned}$$

where S ranges over all nonempty subsets of $\{A_1, \dots, A_{n_1}, B_1, \dots, B_{n_2}\}$, and C_S is a multi-commutator involving only C and operators from S. In particular, this means that the operator C_S lies in $\Psi^{-2,0}_{\mathrm{par}}$ and so the $H^{s,\mathsf{r}}_{\mathrm{par}}$ norm of the multi-commutator terms as well as that of the first RHS term in (5.24) is controlled by $H_{\mathrm{par}}^{s-2,\mathsf{r};\kappa,k}$.

After fixing $M, N \in \mathbb{R}$, we conclude

$$\|A_1 \dots A_{n_1} B_1 \dots B_{n_2} Qu\|_{H^{s,\mathsf{r}}_{\mathrm{par}}} \leq C(\|GPu\|_{H_\pm^{s-2,\mathsf{r};\kappa,k}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{5.26}$$

Summing over all choices of A_j and B_j from our generating set completes the proof. □

We also have a module regularity version of Proposition 5.2. As we will only apply this result away from the radial set of P, we include this as an additional convenient assumption.

Proposition 5.8. *Let $Q, Q', G \in \Psi^{0,0}_{\mathrm{par}}$ be operators of order $(0,0)$ with G elliptic on $\mathrm{WF}'(Q)$, and such that $\mathrm{WF}'(Q'), \mathrm{WF}'(G)$ are disjoint from the radial set $\mathcal{R}$. Let r be a variable spacetime order that is non-increasing in the direction of the bicharacteristic flow of P.*

Furthermore, suppose that for every $\alpha \in \mathrm{WF}'(Q) \cap \mathrm{char}(P)$ there exists α' such that Q' is elliptic at α' and there is a forward bicharacteristic curve γ of P from α' to α such that G is elliptic on γ.

Then if $GPu \in H^{s-1,\mathsf{r}+1;\kappa,k}$ and $Q'u \in H^{s,\mathsf{r};\kappa,k}$, we have $Qu \in H^{s,\mathsf{r};\kappa,k}$ with the estimate

$$\|Qu\|_{H^{s,\mathsf{r};\kappa,k}_{\pm}} \leq C(\|Q'u\|_{H^{s,\mathsf{r};\kappa,k}_{\pm}} + \|GPu\|_{H^{s-1,\mathsf{r}+1;\kappa,k}_{\pm}} + \|u\|_{H^{M,N}_{\mathrm{par}}})$$

for any $M, N \in \mathbb{R}$.

Proof. From (4.18), for any $B \in \Psi^{0,0}_{\mathrm{par}}$ with $\mathrm{WF}'(B) \cap \mathcal{R} = \emptyset$, we can use a microlocal partition of unity to write $B = B_1 + \ldots + B_m$ where each $\mathrm{WF}'(B_j)$ is contained in $\mathrm{ell}(A_j)$ for some $A_j \in \mathcal{N}$. As such we have that the norms $\|B_j v\|_{H^{s,r;\kappa,k}_{\pm}}$ and $\|B_j v\|_{H^{s+\kappa+k,r+\kappa+k}_{\mathrm{par}}}$ are equivalent. Summing in j we obtain equivalence between $\|Bv\|_{H^{s,r;\kappa,k}_{\pm}}$ and $\|Bv\|_{H^{s+\kappa+k,r+\kappa+k}_{\mathrm{par}}}$. We can then directly apply Proposition 5.2 to complete the proof, noting that the operators Q, Q', G in these two propositions enjoy this same microsupport condition. □

5.4. **Module regularity estimates near the radial sets.** We now adapt Proposition 5.3 and Proposition 5.4 to the module regularity spaces $H^{s,r;\kappa,k}_{\mathcal{M}_{\pm}}$ and the particular operator $P = \Delta_g + D_t + V$.

Proposition 5.9. *Suppose $r < -1/2$. Assume that there exists a neighbourhood U of $\mathcal{R}_+$ and $Q', G \in \Psi^{0,0}_{\mathrm{par}}$ such that for every $\alpha \in \mathrm{char}(P) \cap U \setminus \mathcal{R}_+$ the bicharacteristic γ through α enters $\mathrm{ell}(Q')$ whilst remaining in $\mathrm{ell}(G)$. Then there exists $Q \in \Psi^{0,0}_{\mathrm{par}}$ elliptic on $\mathcal{R}_+$ such that if $u \in H^{M,N}_{\mathrm{par}}$, $Q'u \in H^{s,r;\kappa,k}_{+}$, and $GPu \in H^{s-1,r+1;\kappa,k}_{+}$, then $Qu \in H^{s,r;\kappa,k}_{+}$ with an estimate*

$$\|Qu\|_{H^{s,r;\kappa,k}_{+}} \leq C(\|Q'u\|_{H^{s,r;\kappa,k}_{+}} + \|GPu\|_{H^{s-1,r+1;\kappa,k}_{+}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{5.27}$$

Proposition 5.10. *Suppose $r > r' > -\frac{1}{2}$ and $s > s'$. Assume that $G \in \Psi^{0,0}_{\mathrm{par}}$ is elliptic at $\mathcal{R}_{\pm}$. Then there exists $Q \in \Psi^{0,0}_{\mathrm{par}}$ elliptic at $\mathcal{R}_{\pm}$ such that, if $u \in H^{M,N}_{\mathrm{par}}$, $Gu \in H^{s',r';\kappa,k}_{+}$ and $GPu \in H^{s-1,r+1;\kappa,k}_{+}$, then $Qu \in H^{s,r;\kappa,k}_{+}$ with an estimate*

$$\|Qu\|_{H^{s,r;\kappa,k}_{+}} \leq C(\|GPu\|_{H^{s-1,r+1;\kappa,k}_{+}} + \|Gu\|_{H^{s',r';\kappa,k}_{+}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{5.28}$$

The statements of Proposition 5.9 and Proposition 5.10 also hold with $H_+, \mathcal{R}_{\pm}$ replaced with $H_-, \mathcal{R}_{\mp}$ with the obvious modifications to their proof.

Proof. The proof of Proposition 5.9 and Proposition 5.10 proceeds along similar lines to the proofs of Proposition 5.3 and Proposition 5.4, by iterative use of a positive commutator estimate. The commutator $i([A, P] + (P - P^*)A)$, as in (8.8), where $A \in \Psi^{2s-1,2r+1}_{\mathrm{par}}$ has principal symbol a defined in (8.7), is replaced by

$$i[A_\alpha^* A A_\alpha, P] + 2 \operatorname{Im} V A_\alpha^* A A_\alpha \tag{5.29}$$

where $\alpha = (\alpha', \alpha'') \in \mathbb{N}^{N+1} \times \mathbb{N}^{N'+1}$ and

$$A_\alpha := \mathbf{A}^{\alpha'} \mathbf{B}^{\alpha''} = \prod_{j=0}^{N} \mathbf{A}_j^{\alpha_j} \prod_{k=0}^{N'} \mathbf{B}_k^{\alpha'_k} \tag{5.30}$$

is a product of the generators of $\mathcal{M}_+$ that lies in $\mathcal{M}_+^{\kappa}\mathcal{N}^k$.

We treat the addition of the A_α factors in an inductive manner, and suppose that the conclusions of Proposition 5.9 and Proposition 5.10 hold for all $(\kappa', k') < (\kappa, k)$, that is for all pairs $(\kappa', k') \neq (\kappa, k)$ with $\kappa' \leq \kappa$ and $k' \leq k$. The case $(\kappa, k) = (0, 0)$ is of course provided by Propositions 5.3 and 5.4.

Using (4.23) we obtain

$$\begin{aligned} i[A_\alpha^* A A_\alpha, P] &= A_\alpha^*(A(\sum_j \alpha_j C_{jj})\rho_{\mathrm{base}} A_\alpha + A_\alpha^* \rho_{\mathrm{base}}(\sum_j \alpha_j C_{jj}^*)A)A_\alpha \\ &+ \sum_{|\beta|=|\alpha|\,\beta\neq\alpha} A_\alpha^* A C_{\alpha\beta}\rho_{\mathrm{base}} A_\beta + \sum_{|\beta|=|\alpha|,\beta\neq\alpha} A_\beta^* \rho_{\mathrm{base}} C_{\alpha\beta}^* A A_\alpha \\ &+ A_\alpha^* i[A, P] A_\alpha \\ &+ A_\alpha^* A E_\alpha P + P {E_\alpha}^* A A_\alpha \end{aligned} \tag{5.31}$$

where

(1) $E_\alpha \in \mathcal{M}_+^{\kappa'}\mathcal{N}^{k'}$ with $(\kappa', k') < (\kappa, k)$,
(2) $\sigma_{\mathrm{base},1,1}(C_{\alpha\beta})|_{\mathcal{R}_+}) = 0$,
(3) $\mathrm{Re}(\sigma_{\mathrm{base},1,1}(C_{jj}))|_{\mathcal{R}_+} \geq 0$.

The first term has nonnegative symbol on $\mathcal{R}_+$ from Proposition 4.9, the second term has sign determined by that of $i[A, P]$, which has symbol (8.10), the terms in the third line are characteristic on $\mathcal{R}_+$ by Proposition 4.9 and finally the remaining terms are regarded as error terms. The identity (5.31) is analogous to [11, Eq. (3.23)].

We now assume that we are in the below threshold case, that is $r < -1/2$.

In order to concisely write down the contribution of the first two lines of (5.31) to the commutator estimates, we introduce a matrix of operators in $\Psi_{\mathrm{par}}^{2s,2r}$, with rows and columns indexed by multi-indices α with $|\alpha'| = \kappa$ and $|\alpha''| = k$. We introduce the notation for the indexing set

$$S_{\kappa,k} := \{\alpha = (\alpha', \alpha'') \in \mathbb{N}^{N+1} \times \mathbb{N}^{N'+1} : |\alpha'| = \kappa,\ |\alpha''| = k\}. \tag{5.32}$$

For $\alpha, \beta \in S_{\kappa,k}$, the aforementioned matrix of operators is given by

$$C'_{\alpha\beta} = \begin{cases} (A(\sum_j \alpha_j C_{jj})\rho_{\mathrm{base}} + \rho_{\mathrm{base}}(\sum_j \alpha_j C_{jj}^*)A & (\alpha = \beta) \\ A C_{\alpha\beta}\rho_{\mathrm{base}} + \rho_{\mathrm{base}} C_{\alpha\beta}^* A & (\alpha \neq \beta) \end{cases}. \tag{5.33}$$

Now let $u' \in H_+^{s,l;\kappa,k}$ and take $v_\alpha = A_\alpha u'$.

We compute formally, referring to Section 8 for the regularization arguments needed to justify various steps in the computation. In matrix notation, we have obtained the identity

$$\begin{aligned} \sum_{\alpha\in S_{\kappa,k}} \langle i[A_\alpha^* A A_\alpha, P]u', u'\rangle &= \langle C'v, v\rangle + \langle (i[A, P] \otimes \mathbb{I})v, v\rangle \\ &\qquad + 2\mathrm{Re}(\sum_{\alpha\in S_{\kappa,k}} \langle v_\alpha, A E_\alpha P u'\rangle) \end{aligned} \tag{5.34}$$

where $\mathbb{I}$ indicates the $|S_{\kappa,k}| \times |S_{\kappa,k}|$ identity matrix. Using (8.21), we obtain

$$\begin{aligned} \sum_{\alpha\in S_{\kappa,k}} \langle i[A_\alpha^* A A_\alpha, P]u', u'\rangle &= \langle (C' + (B_1^* B_1 - B_2^* B_2 + F + R) \otimes \mathbb{I})v, v\rangle \\ &\qquad + 2\mathrm{Re}(\sum_{\alpha\in S_{\kappa,k}} \langle v_\alpha, A E_\alpha P u'\rangle). \end{aligned} \tag{5.35}$$

From the nonnegativity conditions on the $C_{jj}, C_{\alpha\beta}$, and the strict positivity of the symbol of B_1, we see that the matrix $C' + B_1^* B_1 \otimes \mathbb{I}$ is diagonal with strictly positive entries on $\mathcal{R}_+$. As such, we may write

$$C' + B_1^* B_1 \otimes \mathbb{I} = B^* B + \tilde{R} \tag{5.36}$$

where the symbol of B is a positive matrix on $\mathcal{R}_+$ and $\tilde{R}$ is a matrix of operators in $\Psi_{\mathrm{par}}^{2s-1,2r-1}$. This allows us to write (5.35) as (dropping the $\mathbb{I}$ tensor factor for brevity)

$$\begin{aligned} \sum_{\alpha \in S_{\kappa,k}} \langle u', i[A_\alpha^* A A_\alpha, P] u' \rangle &= \|Bv\|^2 - \|B_2 v\|^2 - \langle Fv, v\rangle + \langle (-R + \tilde{R})v, v\rangle \\ &\quad + 2\mathrm{Re}\Big(\sum_{\alpha \in S_{\kappa,k}} \langle v_\alpha, A E_\alpha P u' \rangle \Big) \end{aligned} \tag{5.37}$$

We estimate the $\|Bv\|^2$ term by using identity (5.37) and bounding all the other terms that appear there.

We first estimate the commutator term $\langle u', i[A_\alpha^* A A_\alpha, P], u' \rangle$. To do this, we use the identity

$$\langle i[A_\alpha^* A A_\alpha, P] u', u' \rangle = -2\mathrm{Im} \langle A A_\alpha u', A_\alpha P u' \rangle - \langle u', (P - P^*) A_\alpha A A_\alpha u' \rangle.$$

The second term is trivial to estimate, since the symbol of a has disjoint support from that of $V - \overline{V} = P - P^*$, so the operator $(P - P^*) A_\alpha A A_\alpha$ is order $(-\infty, -\infty)$. This term is therefore bounded by $\|u\|^2_{H_{\mathrm{par}}^{M,N}}$ for any M and N. The first term is estimated using

$$2 |\langle A A_\alpha P u', A_\alpha u' \rangle| = 2 |\langle \rho_{\mathrm{base}}^{1/2} A^{1/2} A_\alpha u', \rho_{\mathrm{base}}^{-1/2} A^{1/2} A_\alpha P u' \rangle|. \tag{5.38}$$

Summing over α and applying a weighted Young inequality gives the upper bound for the commutator term of

$$\varepsilon \|\rho_{\mathrm{base}}^{1/2} A^{1/2} v\|^2_{H_{\mathrm{par}}^{1/2,0}} + \varepsilon^{-1} \sum_{\alpha \in S_{\kappa,k}} \|\rho_{\mathrm{base}}^{-1/2} A^{1/2} A_\alpha P u'\|^2_{H_{\mathrm{par}}^{-1/2,0}} + \|v\|^2_{H_{\mathrm{par}}^{M,N}}. \tag{5.39}$$

We choose $Q'' \in \Psi_{\mathrm{par}}^{0,0}$ to be microlocally the identity on $\mathrm{WF}'(A)$. The terms F, R and $\tilde{R}$ in (5.37) are estimated as in (8.18) and (8.19) giving

$$|\langle Fv, v \rangle| + |\langle (-R + \tilde{R})v, v \rangle| \le C \Big(\|GPv\|_{H_{\mathrm{par}}^{s-1,r+1}} + \|Q''v\|_{H_{\mathrm{par}}^{s-1/2,r'}} + \|v\|_{H_{\mathrm{par}}^{M,N}} \Big). \tag{5.40}$$

The term $\|B_2 v\|^2$ is estimated as in (8.21), using the standard propagation estimate of Proposition 8.1. This comes at the cost of a $\|Q'v\|^2_{H_{\mathrm{par}}^{s,r}}$ term on the RHS.

It remains to consider the term in the last line of (5.37). The weighted Young inequality gives

$$2 \sum_{\alpha \in S_{\kappa,k}} |\langle v_\alpha, A E_\alpha P u' \rangle| = 2 \sum_{\alpha \in S_{\kappa,k}} |\langle \rho_{\mathrm{base}}^{1/2} A^{1/2} v_\alpha, \rho_{\mathrm{base}}^{-1/2} A^{1/2} E_\alpha P u' \rangle| \tag{5.41}$$

$$\le \varepsilon \|\rho_{\mathrm{base}}^{1/2} A^{1/2} v\|^2_{H_{\mathrm{par}}^{1/2,0}} + \varepsilon^{-1} \sum_{\alpha \in S_{\kappa,k}} \|\rho_{\mathrm{base}}^{-1/2} A^{1/2} E_\alpha P u'\|^2_{H_{\mathrm{par}}^{-1/2,0}} \tag{5.42}$$

We combine (5.37),(5.39), (5.41) to bound Bv, in particular we have

$$\|Bv\|^2 \leq C(\|Q'v\|^2_{H^{s,r}_{\mathrm{par}}} + \|GPv\|^2_{H^{s-2,r}_{\mathrm{par}}} + \|Q''v\|^2_{H^{s-1/2,r'}_{\mathrm{par}}} + \|v\|^2_{H^{M,N}_{\mathrm{par}}}) \tag{5.43}$$

$$+ 2\varepsilon\|\rho_{\mathrm{base}}^{1/2}A^{1/2}v\|^2_{H^{1/2,0}_{\mathrm{par}}} \tag{5.44}$$

$$+ \varepsilon^{-1}\left(\sum_{\alpha\in S_{\kappa,k}} \|\rho_{\mathrm{base}}^{-1/2}A^{1/2}A_\alpha Pu'\|^2_{H^{-1/2,0}_{\mathrm{par}}} + \sum_{\alpha\in S_{\kappa,k}} \|\rho_{\mathrm{base}}^{-1/2}A^{1/2}E_\alpha Pu'\|^2_{H^{-1/2,0}_{\mathrm{par}}}\right). \tag{5.45}$$

We now choose $Q \in \Psi^{0,0}_{\mathrm{par}}$ to be microlocally equal to the identity near $\mathcal{R}_+$ and such that $\mathrm{WF}'(Q)$ is contained in the elliptic set of B. Then for any M, N there is C such that

$$\|Qv\|_{H^{s,r}_{\mathrm{par}}} \leq \|Bv\|_{H^{s,r}_{\mathrm{par}}} + \|u'\|_{H^{M,N}_{\mathrm{par}}}.$$

From the definition (4.5) of the norm in Sobolev spaces with module regularity, an estimate for $\sum_{\alpha\in S_{\kappa,k}}\|A_\alpha Qu'\|$ in fact gives an estimate on $\|Qu'\|_{H^{s,r;\kappa,k}_+}$. On the other hand, $A_\alpha Qu' = Qv + [A_\alpha, Q]u'$ and the last term is microsupported away from the radial set, so we can estimate the $H^{s,r}_{\mathrm{par}}$ norm of $[A_\alpha, Q]u'$ by

$$\|Q'u'\|_{H^{s,r;\kappa,k}_{\mathrm{par}}} + \|GPu'\|_{H^{s-1,r+1;\kappa,k}_{\mathrm{par}}} + \|u'\|_{H^{M,N}_{\mathrm{par}}}, \tag{5.46}$$

using Proposition 5.8. We argue similarly with the other terms. Hence we obtain the estimate expressed in terms of module regularity spaces:

$$\begin{aligned}\|Qu'\|^2_{H^{s,r;\kappa,k}_+} \leq{}& C(\|Q'u'\|^2_{H^{s,r;\kappa,k}_+} + \|GPu'\|^2_{H^{s-2,r;\kappa,k}_+} + \|Q''u'\|^2_{H^{s-1/2,r';\kappa,k}_+} + \|u'\|^2_{H^{M,N}_{\mathrm{par}}})\\ &+ 2\varepsilon\|\rho_{\mathrm{base}}^{1/2}A^{1/2}u'\|^2_{H^{1/2,0;\kappa,k}_+}\\ &+\varepsilon^{-1}\Big(\sum_{\alpha\in S_{\kappa,k}} \|\rho_{\mathrm{base}}^{-1/2}A^{1/2}A_\alpha Pu'\|^2_{H^{-1/2,0}_{\mathrm{par}}} + \sum_{\alpha\in S_{\kappa,k}} \|\rho_{\mathrm{base}}^{-1/2}A^{1/2}E_\alpha Pu'\|^2_{H^{-1/2,0}_{\mathrm{par}}}\Big).\end{aligned} \tag{5.47}$$

The term in the second line is bounded by

$$2\varepsilon\|Q''u'\|^2_{H^{s,r;\kappa,k}_+} \leq 4\varepsilon\Big(\|Qu'\|^2_{H^{s,r;\kappa,k}_+} + \|(Q''-Q)u'\|^2_{H^{s,r;\kappa,k}_+}\Big) \tag{5.48}$$

and the Qu' term can be absorbed into the left-hand side for sufficiently small ε, while the $(Q''-Q)u'$ term can be estimated as in (5.46) as $Q''-Q$ is microsupported away from $\mathcal{R}_+$. The terms in the final line are controlled by $\|GPu'\|_{H^{s-1,r+1;\kappa,k}_+}$ from the ellipticity of G on $\mathrm{WF}'(A)$. This yields the estimate

$$\begin{aligned}\|Qu'\|_{H^{s,r;\kappa,k}_+} \leq C\Big(&\|Q'u'\|_{H^{s,r;\kappa,k}_+} + \|GPu'\|_{H^{s-1,r+1;\kappa,k}_+} + \|Q''u'\|_{H^{s-1/2,r-1/2;\kappa,k}_+}\\ &+ \|u'\|_{H^{M,N}_{\mathrm{par}}}\Big).\end{aligned} \tag{5.49}$$

Iterating the estimate as in Remark 8.3, the lower-order term is subsumed into the $\|u'\|_{H^{M,N}_{\mathrm{par}}}$ term and we obtain

$$\|Qu'\|_{H^{s,r;\kappa,k}_+} \leq C\Big(\|Q'u'\|_{H^{s,r;\kappa,k}_+} + \|GPu'\|_{H^{s-1,r+1;\kappa,k}_+} + \|u'\|_{H^{M,N}_{\mathrm{par}}}). \tag{5.50}$$

We now consider u satisfying the conditions of Proposition 5.9. By the inductive assumption, we know that Qu is in $H^{s,r;\kappa',k'}_+$ for all $(\kappa', k') < (\kappa, k)$. We now regularize u by letting $u' = u'(\eta) = S_\eta u$ for each $\eta > 0$, where

$$S_\eta = \mathrm{Op}\Big(\frac{\rho_{\mathrm{base}}}{\rho_{\mathrm{base}} + \eta}\,\frac{\rho_{\mathrm{fib}}}{\rho_{\mathrm{fib}} + \eta}\Big).$$

It is easy to check that S_η is in $\Psi^{-1,-1}_{\mathrm{par}}$ for each $\eta > 0$, and $\Psi^{0,0}_{\mathrm{par}}$ in a uniform sense, that is, with seminorms uniformly bounded as $\eta \to 0$. Moreover, S_η tends to the identity operator in the strong operator topology of $\Psi^{0,0}_{\mathrm{par}}$, and in the operator norm topology in $\Psi^{-\varepsilon,-\varepsilon}_{\mathrm{par}}$ for any $\varepsilon > 0$. Then $Qu'(\eta)$ is in $H^{s,r;\kappa,k}_{+}$ for each $\eta > 0$ and the above estimate (5.50) is valid. Then we examine the behaviour of the terms as $\eta \to 0$. Let $\tilde Q$, $\tilde Q'$, $\tilde G$ satisfy the same conditions as Q, Q' and G but with $\mathrm{WF}'(\tilde Q)$ contained in the elliptic set of Q and similarly for the other operators. Then the assumption that $Q'u \in H^{s,r;\kappa,k}_{+}$ implies that $\tilde Q' u'$ is uniformly in $H^{s,r;\kappa,k}_{+}$. Similarly, because GPu is in $H^{s-1,r+1;\kappa,k}_{+}$, $\tilde G P u'$ is uniformly in $H^{s-1,r+1;\kappa,k}_{+}$. We deduce from (5.50) (with operators $\tilde Q$, $\tilde Q'$ and $\tilde G$) that $\tilde Q u'$ is *uniformly* in $H^{s,r;\kappa,k}_{+}$. It follows that $\tilde Q u'$ has a weak limit in $H^{s,r;\kappa,k}_{+}$, as well as converging strongly to $\tilde Q u$ in a weaker topology, say $H^{s-2,r-2;\kappa,k}_{+}$ using the inductive assumption on u and the norm convergence of S_η in $\Psi^{-1,-1}_{\mathrm{par}}$. Now redefining Q to be $\tilde Q$, it follows that Qu is in $H^{s,r;\kappa,k}_{+}$ and satisfies the estimate (5.27).

We now turn our attention to the above threshold case, that is Proposition 5.10. From (4.19), the module $\mathcal{M}_+$ is elliptic at $\mathcal{R}_-$ and hence on U_- for sufficiently small U_-. Consequently all functions in (5.28) are microlocalised to regions where the norms $H^{s,r;\kappa,k}_{+}$ and $H^{s+k,r+k;\kappa,0}_{+}$ are equivalent for any $s, r \in \mathbb{R}$ and so it suffices to treat the case $k = 0$.

We can now run the same argument as in the below threshold case, however using (8.13) rather than (8.21) to handle the commutator $[A, P]$ in (5.34) as we are now working in a neighbourhood of the source $\mathcal{R}_-$.

The difference this makes to (5.37) is that both the B and B_2 terms in the first line will now be positive, and so the $B_2 v$ term can be dropped without the need for an application of the propagation theorem Proposition 8.1.

Note that since $k = 0$, we need only consider A_α that are products of elements of $\mathcal{N}$, and so Proposition 4.9 still applies to show that $\sigma_{\mathrm{base},1,1}(C_{jj})|_{\mathcal{R}_-} = 0$ in this case. The rest of the proof proceeds in parallel with the below threshold case.

The analogues of Proposition 5.9 and Proposition 5.10 in the module regularity spaces $H^{s,r;\kappa,k}_{-}$ and switched roles of the radial set components $\mathcal{R}_\pm$ have an almost identical proof. The primary difference is that the module $\mathcal{M}_-$ is P-negative by Proposition 4.9, and so the matrices of operators C' is now negative-definite. In the below-threshold argument, this leads to a change in the sign of the $\|Bv\|^2$ in (5.37). However, we have also switched the roles of the source $\mathcal{R}_-$ and sink $\mathcal{R}_+$, giving corresponding changes to the signs of the second line of (5.37), and so the proof goes through without further changes. The above-threshold argument is adapted similarly. □

5.5. **Global module regularity estimates.** We can combine our microlocal propagation estimates on module regularity spaces in the same way as in Proposition 5.6 to obtain global (semi-)Fredholm estimates.

Proposition 5.11. *(i) Constant spacetime order. Fix constants $s \in \mathbb{R}$ and $l \in (-3/2, -1/2)$. Then for any $k \geq 0$ and $\kappa \geq 1$, any real numbers M and N, and any*

$u \in \mathcal{X}_{\pm}^{s,l;\kappa,k}$, *we have an estimate*

$$\|u\|_{\mathcal{X}_{\pm}^{s,l;\kappa,k}} \le C(\|Pu\|_{\mathcal{Y}_{\pm}^{s-1,l+1;\kappa,k}} + \|u\|_{H_{\mathrm{par}}^{M,N}}). \tag{5.51}$$

(ii) Variable spacetime order. Let $\mathsf{r}_\pm$ *satisfy assumptions (i) – (iii) and* (5.6) *of Section 5.2. Then for any* $k \ge 0$ *and* $\kappa \ge 0$*, any real numbers* M *and* N*, and any* $u \in \mathcal{X}_{\pm}^{s,\mathsf{r}_\pm}$*, we have an estimate*

$$\|u\|_{\mathcal{X}_{\pm}^{s,\mathsf{r}_\pm;\kappa,k}} \le C(\|Pu\|_{\mathcal{Y}_{\pm}^{s-1,\mathsf{r}_\pm+1;\kappa,k}} + \|u\|_{H_{\mathrm{par}}^{M,N}}). \tag{5.52}$$

Proof. We combine our estimates in the same way as in the proof of Proposition 5.5, using Proposition 5.7, Proposition 5.8, Proposition 5.9 and Proposition 5.10 which replace Proposition 5.1, Proposition 5.2, Proposition 5.3 and Proposition 5.4 respectively. We note that in case (i), since l is below threshold, we cannot apply Proposition 5.10 directly at the above-threshold radial set. However, at the above threshold radial set, the module $\mathcal{M}_\pm$ is elliptic. So the estimate is equivalent to the estimate obtained by increasing l by 1 and reducing κ by 1. This is the reason for the assumption that $\kappa \ge 1$: we must have at least one order of module regularity at the radial set $\mathcal{R}_\mp$ to ensure that u is above threshold there. □

6. Solvability of the time-dependent equation

6.1. **Invertibility on variable order spaces.** In this section, we prove Theorem 1.1, which we restate and slightly extend as follows.

Theorem 6.1. *Let* P *be as in* (1.1) *and* (1.2)*, and assume* $\mathsf{r}_\pm$ *are variable orders satisfying (i) – (iii) in Section 5.2. Then for any* $s \in \mathbb{R}$*, the mappings* (1.4) *are invertible.*

The inverse P_+^{-1} *to* (1.4) *with the* $+$ *sign, and the inverse* P_-^{-1} *to* (1.4) *with the* $-$ *sign, are defined independently of the choices of* s *and* $\mathsf{r}_\pm$*, in the following sense. Suppose that* $v \in \mathcal{Y}^{s-1,\mathsf{r}_+ +1} \cap \mathcal{Y}^{s'-1,\mathsf{r}'_+ +1}$ *for* s, s' *and two different choices* $\mathsf{r}_+, \mathsf{r}'_+$*, both satisfying assumptions (i) – (iii) of Section 5.2, and suppose that* $u = P_{\mathsf{r}_+}^{-1} v \in \mathcal{X}^{s,\mathsf{r}_+}$ *and* $u' = P_{\mathsf{r}'_+}^{-1} v \in \mathcal{X}^{s,\mathsf{r}'_+}$*. Then* $u \equiv u'$*, with a similar statement holding for the* $-$ *case.*

Proof. Since Proposition 5.6 established that P is a Fredholm operator of index zero acting from $\mathcal{X}^{s,\mathsf{r}_\pm} \to \mathcal{Y}^{s-1,\mathsf{r}_\pm+1}$, it suffices to show that $\ker(P) = 0$. The argument is essentially identical for the two sign choices so without loss of generality we take a solution $u \in \mathcal{X}^{s,\mathsf{r}_+}$ to $Pu = 0$ and show that $u = 0$.

First, since $Pu = 0$, Proposition 5.1 shows that u is microlocally trivial in the elliptic region. So consider the characteristic set $\mathrm{char}(P)$. As $\mathsf{r}_+ = m > -1/2$ in a neighbourhood of $\mathcal{R}_-$, u is microlocally above threshold in a neighbourhood of $\mathcal{R}_-$. An application of Proposition 5.4 allows us to deduce that in fact u is microlocally in $H_{\mathrm{par}}^{S,L}$ for all $S, L \in \mathbb{R}$ in a neighbourhood of $\mathcal{R}_-$. From Proposition 5.2, it follows in fact this regularity propagates everywhere except, possibly, $\mathcal{R}_+$; that is, u is microlocally in $H_{\mathrm{par}}^{S,L}$ for all $S, L \in \mathbb{R}$ everywhere except possibly at $\mathcal{R}_+$. In particular, u is Schwartz in cones $\{(z,t) \in \mathbb{R}^{n+1} : t < 0, |z|/|t| < C\}$ for arbitrary $C > 0$. Moreover, provided $L < -1/2$, by Proposition 5.3, this regularity propagates into $\mathcal{R}_+$, so that u is in $H_{\mathrm{par}}^{S,-1/2-\varepsilon}$ globally, for any $\varepsilon > 0$. In particular, this tells us that u is (locally in t) a smooth function of t with values in $\langle z \rangle^{1/2+\varepsilon} L^2(\mathbb{R}^n_z)$.

We take the spatial Fourier transform of $u(t,\cdot)$ for each t and write it in the form $e^{-it|\zeta|^2}a(t,\zeta)$. That is,

$$u(z,t) = (2\pi)^{-n} \int e^{i(z\cdot\zeta - t|\zeta|^2)} a(\zeta,t)\, d\zeta. \tag{6.1}$$

Moreover, if $T_- < 0$ is such that the metric $g(t)$ is flat, and the potential $V(t,\cdot)$ vanishes, for $t \le T_-$, then we have $P_0 u = 0$ for $t \le T_-$, which implies that $a(\zeta,t) = a_-(\zeta)$ is independent of t for $t < T_-$.

We now make use of module regularity spaces. In particular, the microlocal triviality of u near $\mathcal{R}_-$ can be interpreted as u having module regularity of all orders microlocally near $\mathcal{R}_-$, that is, $u \in H_+^{S,-1/2-\varepsilon;\mathcal{K},K}$ for all $S, \mathcal{K}, K \ge 0$ microlocally near $\mathcal{R}_-$. Similar to the discussion above, this module regularity propagates, thanks to Propositions 5.7 and Proposition 5.9, everywhere, up to and including $\mathcal{R}_+$, so $u \in H_+^{S,-1/2-\varepsilon;\mathcal{K},K}$ globally for all $S, \mathcal{K}, K \ge 0$.

In particular, we can apply module elements D_{z_j} and $2tD_{z_j} - z_j$ arbitrarily many times to u, while remaining in the space $H_{\mathrm{par}}^{S,-1/2-\varepsilon}$. On the RHS of (6.1) and for $t \le T_-$ this amounts to applying ζ_j and D_{ζ_j}, respectively, to a. We conclude that

$$e^{it|\zeta|^2}\Big(\zeta^\alpha D_\zeta^\beta a_-(\zeta)\Big) \in H^{-1/2-\varepsilon}(\mathbb{R}^n) \text{ for all multi-indices } \alpha, \beta.$$

We conclude that a_- is a Schwartz function.

We have shown that for $t \le T_-$, u can be represented

$$u(z,t) = (2\pi)^{-n} \int e^{i(z\cdot\zeta - t|\zeta|^2)} a_-(\zeta)\, d\zeta,$$

where $a_-(\zeta)$ is Schwartz. Such an integral has a classical stationary phase expansion, with leading term

$$(4\pi i t)^{-n/2} e^{i|z|^2/4t} a_-\big(\frac{z}{2t}\big), \quad t \to -\infty.$$

Since u was previously shown to be Schwartz in the cone $\{(z,t) \in \mathbb{R}^{n+1} : t < 0, |z|/|t| < C\}$, we deduce that $a_-(\zeta) = 0$ for $|\zeta| \le C/2$. As C is arbitrary, we conclude that a_- is identically zero, and hence so is $u(z,t)$ for $t \le T_-$.

Since we can apply module elements D_{z_j} and $2tD_{z_j} - z_j$ arbitrarily many times to u, while remaining in the space $H_{\mathrm{par}}^{S,-1/2-\varepsilon}$, this implies that both $u(t,\cdot)$ and $D_t u(t,\cdot)$ are L^2 in space for each t. Hence the squared norm $E(t) := \|u(\cdot,t)\|^2_{L^2(dg_t)}$ is a non-negative differentiable function of t that vanishes for $t \le -T$. We can write

$$E(t) := \int_{\mathbb{R}^n} |u(z,t)|^2 \rho(z,t)\, dz$$

where $\rho\, dz$ is the Riemannian measure for $g(t)$, with $\rho(z,t)$ a smooth positive function equal to 1 outside a compact set. We can compute $dE(t)/dt$ by differentiating under the integral sign. Cancellation occurs as in the standard proof of conservation of L^2-mass for P_0, and we are left with

$$\frac{dE(t)}{dt} = \int |u(z,t)|^2 \frac{\partial \rho(z,t)}{\partial t}\, dz \le CE(t)$$

for a global constant C. Hence $E = 0$ identically by Grönwall, from which we conclude that $u = 0$ identically. This establishes triviality of $\ker(P)$, and hence invertibility of (1.4).

Finally, we show that the values of P_-^{-1} and P_+^{-1} are defined independently of s and $\mathsf{r}_\pm$ satisfying the assumptions of Section 5.2. Focusing on P_+^{-1}, choose any pair of pairs s, r_+ and s', r'_+ satisfying assumptions (i) – (iii) with constants (l, m) and (l', m') respectively, and let $\mathsf{r}''_+ \in S^{0,0}_{\mathrm{par}}(\mathbb{R}^{n+1})$ be a function satisfying assumptions (i) – (iii) for some weights (l'', m'') such that $\mathsf{r}''_+ \le \min(\mathsf{r}_+, \mathsf{r}'_+)$. Then $P : \mathcal{X}^{s,\mathsf{r}''_+} \to \mathcal{Y}^{s-1,\mathsf{r}''_+ +1}$ is also invertible by the proof above, and since $\mathcal{X}^{s,\mathsf{r}'_+}, \mathcal{X}^{s,\mathsf{r}_+} \subset \mathcal{X}^{s,\mathsf{r}''_+}$, the uniqueness holds.

□

6.2. **Invertibility on module regularity spaces.** We can use Theorem 1.1 to obtain an invertibility theorem regarding P as a map between Sobolev spaces with module regularity, as in (1.7) and (1.8).

First, we record the following inclusion between module regularity spaces and variable order spaces.

Proposition 6.2. *Assume that r_+ and l satisfies assumptions (i) — (iv) of Section 5.2. For $\kappa \ge 1$, we have the inclusion*

$$H^{s,l;\kappa,k}_+ \subset H^{s,\mathsf{r}_+}_{\mathrm{par}}. \tag{6.2}$$

Proof. It suffices to establish (6.2) for $\kappa = 1, k = 0$. Let $u \in H^{s,l;1,0}_+$ and fix a neighbourhood $U \subset \overline{{}^{\mathrm{sc}}T^*\mathbb{R}^{n+1}}$ of $\mathcal{R}_+$ on which $\mathsf{r}_+ = l$, and form a finite cover of $\overline{{}^{\mathrm{sc}}T^*\mathbb{R}^{n+1}}$ consisting of $U, U_1, \dots U_m$, where each U_j is disjoint from $\mathcal{R}_+$ and such that each U_j lies in $\mathrm{ell}(A_j)$ for some $A_j \in \mathcal{M}_+$.

We then quantise a partition of unity subordinate to this cover, and denote the microlocal cutoffs by $Q, Q_1, \dots, Q_m \in \Psi^{0,0}_{\mathrm{par}}$. Now since $u \in H^{s,l}_{\mathrm{par}}$, we have $Qu \in H^{s,l}_{\mathrm{par}}$. Since $\mathsf{r}_+ = l$ on $\mathrm{WF}'(Q)$, it follows that $Qu \in H^{s,\mathsf{r}_+}_{\mathrm{par}}$.

On the other hand, since $A_j u \in H^{s,l}_{\mathrm{par}}$, and $\mathrm{WF}'(Q_j) \subset \mathrm{ell}(A_j)$, microlocal elliptic regularity implies $Q_j u \in H^{s+1,l+1}_{\mathrm{par}} \subset H^{s,\mathsf{r}_+}_{\mathrm{par}}$ for each j, where the final containment is a consequence of $l + 1 \ge m = \max(\mathsf{r}_+)$ using assumption (iv). □

Theorem 6.3. *Fix $s \in \mathbb{R}$ and $l \in (-3/2, -1/2)$. Let $\mathcal{X}^{s,l;\kappa,k}_\pm$ and $\mathcal{Y}^{s-1,l+1;\kappa,k}_\pm$ be as in* (1.7) *and* (1.8)*. Then for any $k \ge 0$ and $\kappa \ge 1$, the map*

$$P : \mathcal{X}^{s,l;\kappa,k}_\pm \to \mathcal{Y}^{s-1,l+1;\kappa,k}_\pm \tag{6.3}$$

is a Hilbert space isomorphism.

For any $k \ge 0$ and $\kappa \ge 0$, provided that $\mathsf{r}_\pm$ satisfy assumptions (i) – (iii) and (5.6)*,*

$$P : \mathcal{X}^{s,\mathsf{r}_\pm;\kappa,k}_\pm \to \mathcal{Y}^{s-1,\mathsf{r}_\pm+1;\kappa,k}_\pm \tag{6.4}$$

are Hilbert space isomorphisms.

Proof. We choose $\mathsf{r}_\pm$ to satisfy assumptions (i) – (iv) of Section 5.2 with respect to l, which is possible since $l > -3/2$. Then Proposition 6.2 gives inclusions $H^{s,l;\kappa,k}_+ \subset H^{s,\mathsf{r}_+}_{\mathrm{par}}$ and $H^{s-1,l+1;\kappa,k}_+ \subset H^{s-1,\mathsf{r}_++1}_{\mathrm{par}}$. Hence by Theorem 1.1 we have the following diagram

$$\begin{array}{ccc} \mathcal{X}^{s,l;\kappa,k}_+ & & \mathcal{Y}^{s-1,l+1;\kappa,k}_+ \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^{s,\mathsf{r}_+}_{\mathrm{par}} & \longrightarrow & \mathcal{Y}^{s-1,\mathsf{r}_++1}_{\mathrm{par}} \end{array} \tag{6.5}$$

We now show the restriction of P to $\mathcal{X}^{s,l;\kappa,k}_+$ yields an isomorphism (6.3) by showing it is a bounded bijection. Boundedness is immediate from the definition of these spaces, and injectivity is immediate from the injectivity of the second row of (6.5).

It remains to prove surjectivity of $P : \mathcal{X}_+^{s,l;\kappa,k} \to \mathcal{Y}_+^{s-1,l+1;\kappa,k}$. Let f be an element of $\mathcal{Y}_+^{s-1,l+1;\kappa,k}$. We exploit the density of Schwartz functions $\mathcal{S}(\mathbb{R}^{n+1})$ in $\mathcal{Y}_+^{s-1,l+1;\kappa,k}$, as shown in Proposition 4.10. So let f_j be Schwartz functions converging to f in $\mathcal{Y}_+^{s-1,l+1;\kappa,k}$. We define $u_j := P_+^{-1} f_j$. Then, according to the propagation estimates of Section 5.1, u_j is microlocally trivial away from the below-threshold radial set $\mathcal{R}_+$. (To see this, note that we can take the order r or r in Propositions 5.1, 5.2 and 5.4 to be arbitrarily large outside any neighbourhood of the below-threshold radial set, here $\mathcal{R}_+$.) Moreover, we can interpret this as arbitrary module regularity away from $\mathcal{R}_+$, and then by Proposition 5.9, this module regularity propagates into $\mathcal{R}_+$. Thus u_j is in $\mathcal{X}_+^{s,l;\kappa',k'}$ for arbitrary (κ', k'). In particular, from (5.51) (taking $M \le s$ and $N < -1/2$) and Theorem 6.1, we have

$$\|u_i - u_j\|_{\mathcal{X}_+^{s,l;\kappa,k}} \to 0 \text{ as } i, j \to \infty. \tag{6.6}$$

Thus, u_j is a Cauchy sequence in $\mathcal{X}_+^{s,l;\kappa,k}$, and hence has a limit $u \in \mathcal{X}_+^{s,l;\kappa,k}$. Finally, since P is continuous $\mathcal{X}_+^{s,l;\kappa,k} \to \mathcal{Y}_+^{s-1,l+1;\kappa,k}$,

$$Pu = P(\lim_{j\to\infty} u_j) = \lim_{j\to\infty} Pu_j = \lim_{j\to\infty} f_j = f,$$

showing that P is surjective on module regularity spaces.

The second statement follows via similar reasoning.

Remark 6.1. The corresponding proof of invertibility on module regularity spaces in [11] has a gap. In [11, Proof of Theorem 2.4], the analogue of estimate (5.51), that is, [11, Equation (3.31)], is asserted without first establishing a priori that u is in the appropriate space $\mathcal{X}_+^{s,\ell;\kappa,k}$. The gap may be filled by arguing as above, that is, using the density of Schwartz functions in this space and then considering a Cauchy sequency of Schwartz functions converging to Pu. The authors thank Yilin Ma for bringing this gap to our attention.

□

6.3. **The final state problem for Schwartz data.** Let f lie in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Define the "free Poisson operator"

$$\mathcal{P}_0(f) = (2\pi)^{-n} \int e^{-it|\zeta|^2} e^{iz\cdot\zeta} f(\zeta) d\zeta = \Big(\mathcal{F}^{-1}_{\zeta\to z}(e^{-it|\zeta|^2} f(\zeta))\Big)(z,t). \tag{6.7}$$

This gives the unique solution to $P_0 u = (D_t + \Delta_0)u = 0$ whose incoming and outgoing data are f, meaning

$$\lim_{t\to\pm\infty} (4\pi i t)^{n/2} e^{-it|\zeta|^2} \mathcal{P}_0 f(2t\zeta, t) \equiv \lim_{\substack{t\to\pm\infty \\ z/2t\to\zeta}} (4\pi i t)^{n/2} e^{-i|z|^2/4t} \mathcal{P}_0 f(z,t) = f(\zeta), \tag{6.8}$$

as follows easily from the stationary phase lemma applied to the ζ integral in (6.7). It is also the operator which solves the free Schrödinger initial value problem for initial data $\hat{f}$.

We now define the Poisson operators for the perturbed operator $P = D_t + \Delta_g + V$.

Definition 6.4. Let P be as in the Introduction. Define the Poisson operators $\mathcal{P}_-, \mathcal{P}_+$ by

$$\begin{aligned} \mathcal{P}_- f &= \mathcal{P}_0 f - P_+^{-1} P \mathcal{P}_0 f = \Big(\mathcal{P}_0 - P_+^{-1}(P - P_0)\mathcal{P}_0\Big) f, \\ \mathcal{P}_+ f &= \mathcal{P}_0 f - P_-^{-1} P \mathcal{P}_0 f = \Big(\mathcal{P}_0 - P_-^{-1}(P - P_0)\mathcal{P}_0\Big) f, \end{aligned} \tag{6.9}$$

where P_+^{-1}, resp. P_-^{-1} are the outgoing, resp. incoming propagators for P (see Theorem 6.1).

Proposition 6.5. *The operator $\mathcal{P}_\pm$ solves the final state problem for $f \in \mathcal{S}(\mathbb{R}^n)$, meaning $P\mathcal{P}_\pm f = 0$ and*

$$\lim_{t\to\pm\infty,\ z/2t\to\zeta} (4\pi i t)^{n/2} e^{-i|z|^2/4t} \mathcal{P}_\pm f(z,t) = f(\zeta) \tag{6.10}$$

Proof. This is a consequence of Theorem 7.11 below, but can be seen directly for Schwartz data quite easily. Indeed, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$P\mathcal{P}_- f = P\mathcal{P}_0 f - (P-P_0)\mathcal{P}_0 f = (P-P_0)\mathcal{P}_0 f - (P-P_0)\mathcal{P}_0 f = 0,$$

where we used $P_0\mathcal{P}_0 = 0$. In the region $|z|/|t| < C, t < T_-$, similar to the proof of Proposition 6.1, the correction term $u_+ := P_+^{-1}(P-P_0)\mathcal{P}_0 f$ is Schwartz. Indeed, we know that u_+ is above threshold at $\mathcal{R}_-$ as it is in the image of P_+^{-1}, so we can take s and r as large as we like in (5.5), since $(P-P_0)\mathcal{P}_0 f$ is compactly supported, giving microlocal regularity of any order in a neighbourhood of $\mathcal{R}_-$. Using Proposition 5.2 this then propagates to $\mathrm{char}(P) \setminus \mathcal{R}_+$, while microlocal regularity in the elliptic region is immediate from Proposition 5.1. Therefore, $\mathcal{P}_- f$ is a solution to the equation which agrees with $\mathcal{P}_0 f$ to infinite order in $|z|/t < C, t < T_-$, for arbitrary C, and thus (6.10) (for $t \to -\infty$) follows from (6.8). A similar argument applies to $\mathcal{P}_+$. □

7. Poisson operator and scattering map

7.1. **Mapping properties of the free Poisson operator.** We will now discuss finer mapping properties of the Poisson operator and scattering operator.

Throughout this section, we assume that $\mathsf{r}_\pm$ satisfy (i) – (iii) at the beginning of Section 5.2, with $l = -1/2 - \varepsilon$ and $m = -1/2 + \varepsilon$ for some small $\varepsilon > 0$, as well as (5.6). In addition, we assume that both $\mathsf{r}_\pm$ are equal to $-1/2$ on $\mathrm{char}(P)$ outside small neighbourhoods of the radial sets. We then define

$$\begin{aligned} \mathsf{r}_{min} &= \min(\mathsf{r}_+, \mathsf{r}_-), \\ \mathsf{r}_{max} &= \max(\mathsf{r}_+, \mathsf{r}_-) \end{aligned} \tag{7.1}$$

and note that $\mathsf{r}_{min} + \mathsf{r}_{max} = -1$ due to (5.6).

We begin with an identity that we will find useful on several occasions. To state it, we choose microlocal cutoffs Q_- and Q_+ such that $Q_- + Q_+ = \mathrm{Id}$, and so that Q_- is microlocally equal to the identity in a neighbourhood of $\mathcal{R}_-$ and microlocally trivial in a neighbourhood of $\mathcal{R}_+$ (and, consequently, vice versa for Q_+).

Lemma 7.1. *Let Q_-, Q_+ be a microlocal partition as described above. Then for any $u \in \mathcal{S}'(\mathbb{R}^{n+1})$ satisfying $Pu = 0$, we have*

$$u = (P_+^{-1} - P_-^{-1})[P, Q_+]u. \tag{7.2}$$

Proof. We observe that $Q_+ u$ is microlocally trivial near $\mathbb{R}_-$. We can therefore find a variable order t_+ satisfying (i) – (iii) of Section 5.2 and a real s such that $Q_+ u \in H_{\mathrm{par}}^{s,\mathsf{t}_+}$. By Theorem 6.1, P_+^{-1} is a left inverse to P on this space, so we have

$$Q_+ u = P_+^{-1} P Q_+ u.$$

Similarly, we have

$$Q_- u = P_-^{-1} P Q_- u.$$

Since $Pu = 0$ we have $PQ_+u = [P, Q_+]u$ and $PQ_-u = [P, Q_-]u$. As we have $Q_+ + Q_- = \mathrm{Id}$, we find $[P, Q_+] = -[P, Q_-]$ and so we obtain

$$u = Q_+u + Q_-u = P_+^{-1}[P, Q_+]u + P_-^{-1}[P, Q_-]u = (P_+^{-1} - P_-^{-1})[P, Q_+]u.$$

□

Due to our assumptions on $\mathsf{r}_\pm$, we can impose the additional assumption that

$$\mathsf{r}_\pm = -1/2 \text{ on } \mathrm{WF}'([P, Q_+]). \tag{7.3}$$

We will then say that Q_-, Q_+ is a *microlocal partition adapted to the variable orders* $\mathsf{r}_\pm$.

The following lemma establishes a fundamental mapping property of the free Poisson operator $\mathcal{P}_0$ on variable order spaces.

Lemma 7.2. *The mapping*

$$\mathcal{P}_0 \colon L^2(\mathbb{R}^n_\xi) \longrightarrow H^{1/2, \mathsf{r}_{min}}_{\mathrm{par}}(\mathbb{R}^{n+1}) \cap \ker(D_t + \Delta_0) \tag{7.4}$$

is a bounded isomorphism.

Proof. We start with the identity

$$\mathcal{P}_0 \mathcal{P}_0^* = i(2\pi)^{-n}\big((P_0)_+^{-1} - (P_0)_-^{-1}\big), \tag{7.5}$$

as can be verified by explicit computation. In fact, both represent the Fourier multiplier $(2\pi)^{1-n}\delta(\tau + |\xi|^2)$. For any $s \in \mathbb{R}$,

$$(P_0)_+^{-1} - (P_0)_-^{-1} \colon H^{s-1, \mathsf{r}_{max}+1}_{\mathrm{par}}(\mathbb{R}^{n+1}) \longrightarrow H^{s, \mathsf{r}_+}_{\mathrm{par}}(\mathbb{R}^{n+1}) + H^{s, \mathsf{r}_-}_{\mathrm{par}}(\mathbb{R}^{n+1}) \subset H^{s, \mathsf{r}_{min}}_{\mathrm{par}}(\mathbb{R}^{n+1}).$$

We apply a TT^* argument to this bounded mapping, for which we require the range to be contained in the dual of the domain, i.e.

$$H^{s, \mathsf{r}_{min}}_{\mathrm{par}}(\mathbb{R}^{n+1}) \subset (H^{s-1, \mathsf{r}_{max}+1}_{\mathrm{par}}(\mathbb{R}^{n+1}))^*.$$

Choosing $s = 1/2$, then since $-\mathsf{r}_{max} - 1 = \mathsf{r}_{min}$, we see that the desired containment holds, and thus $\mathcal{P}_0^*$ maps $H^{-1/2, \mathsf{r}_{max}+1}_{\mathrm{par}}(\mathbb{R}^{n+1})$ into $L^2(\mathbb{R}^n)$. Dually, we conclude that $\mathcal{P}_0$ maps $L^2(\mathbb{R}^n)$ into $H^{1/2, \mathsf{r}_{min}}_{\mathrm{par}}(\mathbb{R}^{n+1})$.

The operator $\mathcal{P}_0$ is obviously injective, as the restriction to $t = 0$ is the inverse Fourier transform. So it remains only to show that it is surjective. Thus, let $u \in H^{1/2, \mathsf{r}_{min}}_{\mathrm{par}}(\mathbb{R}^{n+1}) \cap \ker(D_t + \Delta_0)$. We employ a microlocal partition adapted to the $\mathsf{r}_\pm$ and combine (7.2) (for the free operator P_0) and (7.5) to obtain

$$u = -i(2\pi)^n \mathcal{P}_0 \mathcal{P}_0^*[P_0, Q_+]u.$$

We notice that $[P_0, Q_+]u$ is in $H^{-1/2, 1/2}_{\mathrm{par}}(\mathbb{R}^{n+1})$ using (7.3), and thus is contained in $H^{-1/2, \mathsf{r}_{max}+1}_{\mathrm{par}}(\mathbb{R}^{n+1})$ also by (7.3). Thus, $f := -i(2\pi)^n \mathcal{P}_0^*[P_0, Q_+]u$ is in L^2 using the mapping property of $\mathcal{P}_0^*$ just proved. It follows that $u = \mathcal{P}_0 f$ where $f \in L^2$, proving the surjectivity. □

Recall the small module $\mathcal{N}$ defined in Definition 4.5. Consider the generators

$$\mathrm{Id}, \quad z_k D_{z_j} - z_j D_{z_k}, \quad 2tD_{z_j} - z_j, \quad D_{z_j}, \quad E_{-1}\langle z, t\rangle P_0. \tag{7.6}$$

with $E_{-1} \in \Psi^{-1,0}$ globally elliptic.

We have intertwining relations of these generators with the Poisson operator $\mathcal{P}_0$ (the third and fourth of which were already used in the proof of Theorem 6.1):

$$\begin{aligned} \mathrm{Id}\,\mathcal{P}_0 f &= \mathcal{P}_0(\mathrm{Id}\, f) \\ (z_j D_{z_l} - z_l D_{z_j})\mathcal{P}_0 f &= \mathcal{P}_0((\zeta_j D_{\zeta_l} - \zeta_l D_{\zeta_j}) f) \\ (2t D_{z_j} - z_j)\mathcal{P}_0 f &= \mathcal{P}_0(D_{\zeta_j} f) \\ D_{z_j}\mathcal{P}_0 f &= \mathcal{P}_0(\zeta_j f) \\ E_{-1}\langle z, t\rangle P_0 \mathcal{P}_0 &\equiv 0. \end{aligned} \tag{7.7}$$

The Poisson operator thus intertwines the action of these generators with the following operators on $\mathbb{R}^n_\zeta$:

$$\mathrm{Id}, \quad \zeta_j D_{\zeta_l} - \zeta_l D_{\zeta_j}, \quad D_{\zeta_j}, \quad \zeta_j. \tag{7.8}$$

It is trivial to check that these operators are in $\Psi^{1,1}_{\mathrm{sc}}(\mathbb{R}^n)$ and are closed under commutators. They therefore generate a module which we denote $\widehat{\mathcal{N}}$. We let $\widehat{\mathcal{N}}_{gen}$ denote the finite set of generators in (7.8). (We remark here that we replaced the generator E_1 of $\mathcal{N}$ with the D_{z_j} in (7.6) for convenience. The reason for doing so is that, if we take $E_1 = (1 + D_t^2 + \Delta_z^2)^{1/4}$, then this is intertwined with $(1 + 2|\zeta|^4)^{1/4}$. We find it more convenient to replace this with factors ζ_j which lead to the same module $\widehat{\mathcal{N}}$.)

This leads to the definition of spaces of incoming/outgoing data $\mathcal{W}^k(\mathbb{R}^n_\zeta)$ that will be suitable domain spaces for the free Poisson operator viewed as mapping into module regularity spaces.

Definition 7.3. For $k \in \mathbb{N}$, we define the Hilbert space $\mathcal{W}^k(\mathbb{R}^n_\zeta)$ by

$$\mathcal{W}^k(\mathbb{R}^n_\zeta) = \{ f \in L^2(\mathbb{R}^n_\zeta, d\zeta) \mid A_1 \dots A_j f \in L^2(\mathbb{R}^n_\zeta, d\zeta)\, \forall A_i \in \widehat{\mathcal{N}}_{gen}, 1 \le i \le j \le k \}. \tag{7.9}$$

The norm in this Hilbert space is defined by

$$\|f\|^2 = \sum \|A_1 \dots A_j f\|_2^2,$$

where the sum is over all j-tuples $(A_1, \dots, A_j)$ of generators for $0 \le j \le k$. (when $j = 0$ this is of course just the L^2 norm of f.)

For $k \in \mathbb{N}$, we define the spaces of negative order by

$$\mathcal{W}^{-k}(\mathbb{R}^n_\zeta) = \{ f \in \mathcal{S}'(\mathbb{R}^n_\zeta) \mid f = \sum_{\substack{A_1,\dots,A_j \in \widehat{\mathcal{N}}_{gen} \\ j \le k}} A_1 \dots A_j f_{A_1,\dots,A_j} \mid f_{A_1,\dots,A_j} \in L^2(\mathbb{R}^n_\zeta, d\zeta) \}.$$

The squared norm of f in this Hilbert space is the infimum of

$$\sum_{A_1 \dots A_j} \|f_{A_1,\dots,A_j}\|_2^2$$

over all representations of f in this form, where $(A_1, \dots, A_j)$ are distinct j-tuples of elements of $\widehat{\mathcal{N}}_{gen}$ with $1 \le j \le k$.

Standard considerations show that $\mathcal{W}^{-k}(\mathbb{R}^n)$ is the dual space of $\mathcal{W}^k(\mathbb{R}^n)$. Recalling that $H^{s,\mathsf{r};k}_{\mathcal{N}}(\mathbb{R}^{n+1})$ denotes the module regularity space of order k with respect to $\mathcal{N}$, we show

Proposition 7.4. *For $k \in \mathbb{N}$, the mapping*

$$\mathcal{P}_0 \colon \mathcal{W}^k(\mathbb{R}^n_\zeta) \longrightarrow H^{1/2,\mathsf{r}_{min};k}_{\mathcal{N}}(\mathbb{R}^{n+1}) \cap \ker(D_t + \Delta_0) \tag{7.10}$$

is a bounded isomorphism.

Moreover, if $Q \in \Psi^{0,0}_{\mathrm{par}}(\mathbb{R}^{n+1})$ *is such that the intersection of its microsupport with the characteristic variety is contained in the set where* $\mathsf{r}_{min} = -1/2$ *(which implies that it is microsupported away from the radial sets), then for all integers* $k \in \mathbb{Z}$ *(positive or negative) we have*

$$Q\mathcal{P}_0 \colon \mathcal{W}^k(\mathbb{R}^n_\zeta) \longrightarrow H^{k+1/2,k-1/2}_{\mathrm{par}}(\mathbb{R}^{n+1}) \tag{7.11}$$

is bounded.

Proof. The first statement follows immediately from Lemma 7.2 and commutation identities (7.7). The second statement for $k \geq 0$ follows from the first and the observation that the module $\mathcal{N}$ is elliptic on the microsupport of Q, so k orders of module regularity gains us k in both the differential and spacetime orders of regularity. For $k > 0$, by definition of the space $\mathcal{W}^{-k}(\mathbb{R}^n)$ it suffices to consider f of the form $f = \sum A_1 \dots A_j f'$ where $A_i \in \widehat{\mathcal{N}}_{gen}$, $j \leq k$ and $f' \in L^2$. Using the commutation properties, $\mathcal{P}_0 f$ is equal to a sum of up to k module elements applied to $\mathcal{P}_0 f'$, which we know lies in the space $H^{1/2,\mathsf{r}_{min}}_{\mathrm{par}}(\mathbb{R}^{n+1})$. Since these module elements are order $(1,1)$, we find that $Q\mathcal{P}_0 f$ is in the space $H^{1/2-k,-1/2-k}_{\mathrm{par}}(\mathbb{R}^{n+1})$. □

7.2. **Perturbed Poisson operators.** We now turn to the perturbed Poisson operators $\mathcal{P}_-, \mathcal{P}_+$ from Definition 6.4. We have an analogue (in fact, a slight strengthening) of Proposition 7.4 for the perturbed Poisson operators. The following proposition is the same as the first part of Theorem 1.3.

Proposition 7.5. *For* $k \in \mathbb{N}$*, the range of each Poisson operator* $\mathcal{P}_\pm$ *on* $\mathcal{W}^k(\mathbb{R}^n)$ *is precisely*

$$\{u \in \mathcal{X}^{1/2,\mathsf{r}_+;k,0}_+(\mathbb{R}^{n+1}) + \mathcal{X}^{1/2,\mathsf{r}_-;k,0}_-(\mathbb{R}^{n+1}) \mid Pu = 0\}, \tag{7.12}$$

i.e. that is, those elements of $\mathcal{X}^{1/2,\mathsf{r}_+} + \mathcal{X}^{1/2,\mathsf{r}_-}$ *in the kernel of* P *having module regularity of order* k*.*

For $k \leq -1$*, the range of* $\mathcal{P}_\pm$ *on* $\mathcal{W}^k(\mathbb{R}^n)$ *is precisely*

$$\{u \in H^{k+1/2,k-1/2}_{\mathrm{par}}(\mathbb{R}^{n+1}) \mid Pu = 0\}. \tag{7.13}$$

For either sign of k*, we characterise the range of* $\mathcal{P}_\pm$ *on* $\mathcal{W}^k(\mathbb{R}^n)$ *as those elements of the null space of* P *that are microlocally in* $H^{k+1/2,k-1/2}_{\mathrm{par}}(\mathbb{R}^{n+1})$ *on* $\mathrm{char}(P) \setminus \mathcal{R}$*. That is, provided* $Q \in \Psi^{0,0}_{\mathrm{par}}$ *is microsupported away from the radial sets, the map*

$$Q\mathcal{P}_\pm \colon \mathcal{W}^k(\mathbb{R}^n_\zeta) \longrightarrow H^{k+1/2,k-1/2}_{\mathrm{par}}(\mathbb{R}^{n+1}) \tag{7.14}$$

is bounded.

Proof. We first prove the statement (7.14). This will be deduced from (7.10), the identity (6.9) relating the free and perturbed Poisson operator, and mapping properties of the resolvent. We consider only $\mathcal{P}_+$ as the argument for $\mathcal{P}_-$ is analogous.

Let $f \in \mathcal{W}^k(\mathbb{R}^n)$. By (6.9) we have

$$\mathcal{P}_+ f = \mathcal{P}_0 f - P_-^{-1} P \mathcal{P}_0 f, \tag{7.15}$$

and we have already shown the required regularity for the $\mathcal{P}_0 f$ term on the RHS in Proposition 7.4. We now consider the other term on the RHS.

Since $P - P_0$ is compactly supported in spacetime, one can choose a $G \in \Psi^{0,0}_{\mathrm{par}}$ which is supported near spacetime infinity and microsupported near $\mathcal{R}_+$ such that $G(P - P_0) \equiv 0$. Thus $GP\mathcal{P}_0 f = G(P - P_0)\mathcal{P}_0 f \equiv 0$. Since $u' = P_-^{-1} P \mathcal{P}_0 f$ is above threshold near $\mathcal{R}_+$, the estimate in Proposition 5.4 applies to u', so for any $\tilde{Q}$ with $\mathrm{WF}'(\tilde{Q}) \subset \mathrm{ell}(G)$ we have that for any $K, S, M, N \in \mathbb{R}$ and r' above threshold,

$$\begin{aligned} &\|\tilde{Q} P_-^{-1} P \mathcal{P}_0 f\|_{H^{S,K}_{\mathrm{par}}} \\ &\quad \le C\Big(\|G P P_-^{-1} P \mathcal{P}_0 f\|_{H^{S-1,K+1}_{\mathrm{par}}} + \|G P_-^{-1} P \mathcal{P}_0 f\|_{H^{s',r'}_{\mathrm{par}}} + \|P_+^{-1} P \mathcal{P}_0 f\|_{H^{M,N}_{\mathrm{par}}} \Big) \\ &\qquad\qquad = C\Big(\|G P_-^{-1} (P - P_0) \mathcal{P}_0 f\|_{H^{s',r'}_{\mathrm{par}}} + \|P_-^{-1} (P - P_0) \mathcal{P}_0 f\|_{H^{M,N}_{\mathrm{par}}} \Big) \end{aligned}$$

where the last line follows since $G P P_-^{-1} P \mathcal{P}_0 f = G P \mathcal{P}_0 f \equiv 0$. This estimate shows that after applying $\tilde{Q}$, the second term on the RHS in (7.15) is in $H^{S,K}_{\mathrm{par}}$ for arbitrary S and K and is therefore microlocally trivial close to $\mathcal{R}_+$. Hence the sum u of the two terms on the RHS of (7.15) satisfies the required regularity, namely $H^{k+1/2,k-1/2}_{\mathrm{par}}(\mathbb{R}^{n+1})$ regularity, in a small deleted neighbourhood of $\mathcal{R}_+$. Because $u = \mathcal{P}_+ f$ satisfies $Pu = 0$, we can apply propagation of regularity, that is Theorem 5.2, to deduce the same regularity everywhere on $\mathrm{char}(P) \setminus \mathcal{R}$. This establishes (7.14).

We next show that the range of $\mathcal{P}_+$ is included in (7.12), when $k \ge 0$. Let $u = \mathcal{P}_+ f$. We choose a microlocal partition adapted to r_+ and apply (7.2). Notice that $[P, Q_+]$ has order $(1, -1)$ so the term $[P, Q_+]u$ is in $H^{k-1/2,k+1/2}_{\mathrm{par}}(\mathbb{R}^{n+1})$ using (7.14). This clearly belongs to both $\mathcal{Y}_+^{-1/2,\mathsf{r}_++1;k,0}$ and $\mathcal{Y}_+^{-1/2,\mathsf{r}_-+1;k,0}$ since both $\mathsf{r}_\pm$ are equal to $-1/2$ on the microsupport of $[P, Q_+]$. Applying Theorem 6.3, we find that u lies in (7.12).

To show that the range of $\mathcal{P}_+$ for $k < 0$ lies in (7.13), we start the same way: we use (7.2) again, and the fact that the term $[P, Q_+]u$ is in $H^{k-1/2,k+1/2}_{\mathrm{par}}(\mathbb{R}^{n+1})$ using (7.14). In this case we apply the resolvent mapping property on variable order spaces, Theorem 6.1, with a judicious choice of $\mathsf{r}_\pm$. Namely, for the operator P_+^{-1}, we choose r_+ to be equal to $k - 1/2$ on all bicharacteristic segments between $\mathrm{WF}'([P, Q_+])$ and $\mathcal{R}_+$, and for R_-, we make a similar choice for r_- (with $\mathcal{R}_-$ replacing $\mathcal{R}_+$). The sum of the two spaces $H^{k+1/2,\mathsf{r}_+}_{\mathrm{par}}(\mathbb{R}^{n+1}) + H^{k+1/2,\mathsf{r}_-}_{\mathrm{par}}(\mathbb{R}^{n+1})$ is then equal to (7.13).

It remains to show that the Poisson operator $\mathcal{P}_+$ maps surjectively to the spaces in (7.12) and (7.13). This is postponed until after Proposition 7.10. □

Remark 7.1. There is an apparent problem with (7.15): it looks at first sight as though $\mathcal{P}_+ f$ is less regular (in the differential sense) than $\mathcal{P}_0 f$ since applying P loses two orders of regularity and P_-^{-1} gains back only one order. We circumvent this difficulty by using the compact support of $P - P_0$ and propagation of regularity. If $P - P_0$ were not compactly supported — even if it decayed quite rapidly at spacetime infinity — this argument could not be used. Instead, we would need to use an approximate Poisson operator adapted to P, similarly to what is done for the Helmholtz equation in [26]. The issue is that the free Poisson operator has the 'wrong phase function', adapted to P_0 not P, and only if these two operators agree near spacetime infinity can we effectively treat the Poisson operator $\mathcal{P}_+$ as a perturbation of the free Poisson operator.

Remark 7.2. It is interesting that one can bootstrap from small module regularity (as in (7.10)) to full module regularity (as in (7.14)). This arises because all the modules are elliptic on $\mathrm{char}(P) \setminus \mathcal{R}$, so small module regularity and large module regularity are equivalent there.

Corollary 7.6. *$\mathcal{P}^*$ maps $\mathcal{S}(\mathbb{R}^{n+1})$ to $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Dualizing (7.13), we find that $\mathcal{P}^*$ maps $H^{k-1/2,k+1/2}_{\mathrm{par}}(\mathbb{R}^{n+1})$ to $\mathcal{W}^k(\mathbb{R}^n)$ for $k \geq 1$. Taking the intersection over all such k yields the corollary. □

We now prove an analogue of Proposition 3.4 of [10].

Proposition 7.7. *Let $v \in H^{-1,\mathsf{r}_{max}+1;k}_{par,\mathcal{N}}(\mathbb{R}^{n+1})$ with $k \geq 2$ and $\varepsilon > 0$. Then $u_+ := P_+^{-1} v$ is such that the limits*

$$\mathcal{L}_+ u_+(\zeta) := \lim_{t\to+\infty} (4\pi i t)^{n/2} e^{-it|\zeta|^2} u_+(2t\zeta, t) \tag{7.16}$$

and

$$\mathcal{L}_- u_+(\zeta) := \lim_{t\to-\infty} (4\pi i t)^{n/2} e^{-it|\zeta|^2} u_+(2t\zeta, t) \tag{7.17}$$

exist in $\langle\zeta\rangle^{1/2+\varepsilon}\mathcal{W}^{k-2}(\mathbb{R}^n_\zeta)$, with the limit (7.17) identically zero.

Moreover, we have estimates

$$\begin{aligned} &\|\mathcal{L}_+ u_+\|_{\langle\cdot\rangle^{1/2+\varepsilon}\mathcal{W}^{k-2}} \leq C\|v\|_{H^{1/2,\mathsf{r}_{max}+1;1,k}_{par,+}}, \\ &\|t^{n/2} e^{-it|\zeta|^2/4} u_+(t\zeta, t) - \mathcal{L}_+ u_+\|_{\langle\cdot\rangle^{1/2+\varepsilon}\mathcal{W}^{k-2}} = O(t^{-\varepsilon'}), \quad t \to \infty \end{aligned} \tag{7.18}$$

for ε' sufficiently small. A similar statement is true for $u_- := P_-^{-1} v$, with a zero limit $\mathcal{L}_+ u_-$ as $t \to +\infty$ and a (potentially) nonzero limit $\mathcal{L}_- u_-$ as $t \to -\infty$.

Proof. We prove the statement only for u_+ as the proof for u_- is essentially the same with the incoming and outgoing radial sets switched. Define

$$\tilde{u}(\zeta, t) = (4\pi i t)^{n/2} e^{-it|\zeta|^2} u_+(2t\zeta, t).$$

We will compute the partial derivative of $\tilde{u}(\zeta, t)$, i.e. with ζ fixed, which we denote by $D_t|_\zeta$ to avoid confusion with the partial derivative with respect to t with z fixed.

Then, using $\zeta = z/(2t)$ and $D_t = P - D_z \cdot D_z$, we can write

$$\begin{aligned} D_t|_\zeta \tilde{u}(\zeta, t) &= (4\pi i t)^{n/2} e^{-it|\zeta|^2} \Big(-\frac{in}{2t} - |\zeta|^2 + (\frac{z}{t} \cdot D_z + D_t) \Big) u_+(2t\zeta, t) \\ &= (4\pi i t)^{n/2} e^{-it|\zeta|^2} \Big(P - t^{-2}(tD_z - \frac{z}{2}) \cdot (tD_z - \frac{z}{2}) \Big) u_+(2t\zeta, t) \\ &= (4\pi i t)^{n/2} e^{-it|\zeta|^2} \Big(v(2t\zeta, t) - \Big(t^{-2}(tD_z - \frac{z}{2}) \cdot (tD_z - \frac{z}{2}) \Big) u_+(2t\zeta, t) \Big) \end{aligned} \tag{7.19}$$

We recognize the factor $tD_{z_i} - z_i/2$ as an element of the module $\mathcal{N}$. So we are now in a similar position to the proof in [11]. By Theorem 6.3, $u_+ \in H^{0,\mathsf{r}_+;0,2}$, which allows us to conclude that the second term in the parenthesis has

$$\Big((tD_z - \frac{z}{2}) \cdot (tD_z - \frac{z}{2}) \Big) u_+(2t\zeta, t) \in H^{0,\mathsf{r}_+;0,0}$$

Moreover, from the assumption on v, using: (1) $\Psi^{1,0}_{\mathrm{par}} \subset \mathcal{N}$, (2) that $\mathsf{r}_{max} = -1/2 + \varepsilon$ near the radial sets, and (3) that $\mathcal{N}$ is elliptic away from the radial sets, we have

$$v \in H^{-1,\mathsf{r}_{max}+1;2}_{\mathrm{par},\mathcal{N}}(\mathbb{R}^{n+1}) \subset H^{0,\mathsf{r}_{max}+1;1}_{\mathrm{par},\mathcal{N}}(\mathbb{R}^{n+1}) \subset H^{0,1/2+\varepsilon}_{\mathrm{par}} = \langle(t,z)\rangle^{-1/2-\varepsilon} L^2(dtdz)$$

Thus, on $0 < T < t$

$$\begin{aligned} D_t\tilde{u}(\zeta, t)|_\zeta \in \langle(t,z)\rangle^{-1/2-\varepsilon} t^{n/2} L^2(dtdz) + \langle(t,z)\rangle^{1/2+\varepsilon} t^{n/2} \langle t\rangle^{-2} L^2(dtdz) \\ \subset t^{-1/2-\varepsilon} \langle\zeta\rangle^{1/2+\varepsilon} L^2(dtd\zeta). \end{aligned} \tag{7.20}$$

This is in

$$t^{-\varepsilon'} L^1([T,\infty)_t; \langle\zeta\rangle^{1/2+\varepsilon} L^2(\mathbb{R}^n_\zeta))$$

for $0 < \varepsilon' < \varepsilon$. We can thus integrate the t-derivative, for fixed ζ, of $\tilde{u}$, viewed as a function of t with values in $\langle\zeta\rangle^{1/2+\varepsilon}L^2(\mathbb{R}^n_\zeta)$, out to infinity, showing that the limit exists. Moreover, the convergence is at a rate of $O(t^{-\varepsilon'})$ as we see by integrating $D_t|_\zeta\tilde{u}$ back from $t = \infty$.

Now to prove the result for $k > 2$, we observe that applying module element $2tD_{z_i} - z_i$ to u_+ is equivalent to applying D_{ζ_i} to $\tilde{u}$. In the same way, applying module element $z_iD_{z_j} - z_jD_{z_i}$ to u_+ is equivalent to applying $\zeta_iD_{\zeta_j} - \zeta_jD_{\zeta_i}$ to $\tilde{u}$. Moreover, since $2tD_{z_i} - z_i$ and D_{z_i} are both module elements, it follows that multiplication by $z_i = (2tD_{z_i} - z_i) - 2t\cdot D_{z_i}$ maps u to $\langle t\rangle H^{0,-1/2-\varepsilon}_{\mathrm{par}}$. Since multiplication by $\langle t\rangle$ commutes with both $2tD_{z_i} - z_i$ and D_{z_i}, we can iterate this argument, showing that for $|\alpha| \le k$, and t large, multiplication by z^α maps to $\langle t\rangle^{|\alpha|}H^{0,-1/2-\varepsilon}_{\mathrm{par}}$, and hence, multiplication by ζ^α maps to $H^{0,-1/2-\varepsilon}_{\mathrm{par}}$ for $t \ge 1$. This means that we can apply compositions of up to k generators of $\widehat{\mathcal{N}}$ to $\tilde{u}$, improving (7.20) to

$$D_t|_\zeta\tilde{u} \in t^{-1/2-\varepsilon}\langle\zeta\rangle^{1/2+\varepsilon}\mathcal{W}^{k-2}(dtd\zeta). \tag{7.21}$$

Repeating the argument above shows that the limit (7.16) exists in the $\langle\zeta\rangle^{1/2+\varepsilon}\mathcal{W}^{k-2}$ topology, and thus the limit lies in $\langle\zeta\rangle^{1/2+\varepsilon}\mathcal{W}^{k-2}(\mathbb{R}^n_\zeta)$.

Exactly the same argument shows that the limit of (7.17) exists as $t \to -\infty$. However, because u is obtained by applying the *outgoing* propagator P_+^{-1} to v, u is above threshold (that is, in $H^{0,-1/2+\varepsilon}_{\mathrm{par}}$) microlocally away from $\mathcal{R}_+$. So in any region of the form $t \le -1$, $|\zeta| \le R$, we have $u' \in t^{1/2-\varepsilon}L^2(dtdz)$, which amounts to $\tilde{u}$ being in $t^{1/2-\varepsilon}L^2(dtd\zeta)$ for $(t,\zeta) \in (-\infty,-1] \times B(0,R)$. This is incompatible with the existence of the limit $\mathcal{L}_-u$ in (7.17) unless the limit function vanishes in $B(0,R)$. Since R is arbitrary this proves that $\mathcal{L}_-u = 0$.

□

Corollary 7.8. *The incoming and outgoing Poisson operators $\mathcal{P}_-, \mathcal{P}_+$ satisfy*

$$\mathcal{L}_-\mathcal{P}_- = \mathcal{L}_+\mathcal{P}_+ = \mathrm{Id}. \tag{7.22}$$

Proof. This is an immediate consequence of (6.8) and Proposition 7.7. □

In preparation for Proposition 7.10 we prove the following pairing formula. In order to state the result, recall from Proposition 7.7 that if $Pu = v \in \mathcal{S}(\mathbb{R}^{n+1})$, then the limits $\mathcal{L}_+u$ and $\mathcal{L}_-u$ exist in the space $\langle\zeta\rangle^{1/2+\varepsilon}\mathcal{W}^k$ for arbitrary k.

Lemma 7.9. *Suppose that u_1 and u_2 are two functions on $\mathbb{R}^{n+1}$ such that $Pu_i \in \mathcal{S}(\mathbb{R}^{n+1})$. We denote by $f_i^\pm$ the limits $\mathcal{L}_\pm u_i$. Then the following identity holds:*

$$\int_{\mathbb{R}^{n+1}} \big(u_1(\overline{P^*u_2}) - (Pu_1)\overline{u_2}\big)\,dg(t)\,dt = \frac{-i}{(2\pi)^n}\int_{\mathbb{R}^n_\zeta}\big(f_1^+(\zeta)\overline{f_2^+(\zeta)} - f_1^-(\zeta)\overline{f_2^-(\zeta)}\big)\,d\zeta. \tag{7.23}$$

Proof. According to Theorem 6.3, the u_i are in $H^{1/2,\mathsf{r}_{min};N}_{\mathrm{par},\pm}$ for every N, i.e. they have infinite (small-)module regularity. Following the proof of Proposition 7.7, therefore, u has an expansion as $t \to \pm\infty$, which we write in the form

$$u_i = (4\pi it)^{-n/2}e^{i|z|^2/4t}\Big(f_i^\pm(\zeta) + O_{\mathcal{W}^N_\zeta}(t^{-\varepsilon'})\Big), \quad \zeta = \frac{z}{2t}, \quad t \to \pm\infty. \tag{7.24}$$

In addition we have an L^2-estimate on the size of u_i of the form

$$u_i \in \langle(z,t)\rangle^{1/2+\varepsilon}\langle\zeta\rangle^{-N}L^2(dz\,dt), \quad \zeta = \frac{z}{2t}, \quad R \ge 1. \tag{7.25}$$

The additional factors of $\langle\zeta\rangle^{-N}$ arise from module regularity, as we have already seen in the proof of Proposition 7.7. In addition, we observe that the same estimate holds for any module derivative of u. In particular, we also have

$$D_{z_j} u_i \in \langle (z,t)\rangle^{1/2+\varepsilon} \langle\zeta\rangle^{-N} L^2(dz\, dt), \quad \zeta = \frac{z}{2t}, \quad R \geq 1. \tag{7.26}$$

To prove (7.23) we choose a cutoff function $\chi \in C_c^\infty(\mathbb{R})$ identically equal to 1 near zero, and write

$$\chi_t = \chi(\frac{t}{R}), \quad \chi_z = \chi(\frac{|z|}{2R^2})$$

where R is a large parameter. We can write the LHS of (7.23) as

$$\lim_{R\to\infty} \Big(\langle \chi_t \chi_z u_1, P^* u_2\rangle - \langle P u_1, \chi_t \chi_z u_2\rangle \Big). \tag{7.27}$$

where $\chi(s) \in C_c^\infty(\mathbb{R})$ is identically equal to 1 for $s \in [-1,1]$ and vanishing outside $[-2,2]$. Since P^* is the formal adjoint of P, we can shift derivatives from one side of the inner product to the other and the only non-cancelling terms will be those where a derivative hits one of the χ factors. For large R, the support of the derivative of $\chi(t/R)\chi(|z|/R^2)$ is in the region where $P = P_0$, so it suffices to assume that $P = P_0 = D_t + \sum_j D_{z_j} D_{z_j}$.

First consider shifting one z-derivative. We have

$$\langle \chi_t \chi_z u_1, D_{z_j} D_{z_j} u_2\rangle = \langle \chi_t \chi_z D_{z_j} u_1, D_{z_j} u_2\rangle + \langle \chi_t (D_{z_j}\chi_z) u_1, D_{z_j} u_2\rangle.$$

Adding this to the contribution of the second inner product in (7.27), two terms cancel and we are left with

$$\langle \chi_t (D_{z_j}\chi_z) u_1, D_{z_j} u_2\rangle - \langle D_{z_j} u_1, \chi_t (D_{z_j}\chi_z) u_2\rangle$$

as the contribution arising from integrating by parts in z. We estimate the magnitude of these inner products using Cauchy-Schwartz and (7.25), (7.26). Noticing that $|\zeta| \geq R$ and $\langle (z,t)\rangle \leq CR^2$ on the support of $\chi_t\chi_z$, the $\langle\zeta\rangle^{-N}$ factors can be replaced by R^{-N}, and $\langle (z,t)\rangle^{1/2+\varepsilon}$ is bounded by $(CR^2)^{1/2+\varepsilon}$ on the support of $\chi_t\chi_z$. Taking N sufficiently large, we see that these inner products tend to zero as $R \to \infty$.

What remains is the result when a D_t-derivative hits the χ_t factor, namely

$$\lim_{R\to\infty} -i \int \chi'(\frac{t}{R}) \chi(\frac{|z|}{2R^2}) u_1 \overline{u_2}\, dz\, \frac{dt}{R}.$$

We substitute (7.24) for u_i and notice that only the leading order asymptotic of each contributes to the limit. Moreover, the χ_z factor is 1 on a ball $B(0, 2R)$ in the ζ variable, so this factor tends to 1 pointwise. We further write $dz = (2t)^n d\zeta$, change integration variable to ζ and we obtain (7.23), since

$$\int_0^\infty \chi'(s) ds = 1, \quad \int_{-\infty}^0 \chi'(s) ds = -1.$$

□

We will also need the following operator identity.

Proposition 7.10. *We have the identity*

$$\mathcal{P}_+ \mathcal{P}_+^* = \mathcal{P}_- \mathcal{P}_-^* = i(2\pi)^{-n} \big(P_+^{-1} - P_-^{-1} \big). \tag{7.28}$$

Proof. We first note that we have already shown that each of these three operators is bounded from $H_{\mathrm{par}}^{-1/2,\mathsf{r}_{max}+1}(\mathbb{R}^{n+1})$ to $H_{\mathrm{par}}^{1/2,\mathsf{r}_{min}}(\mathbb{R}^{n+1})$. So to prove the equality, we need only consider the action on a dense subspace, such as $\mathcal{S}(\mathbb{R}^{n+1})$.

We thus consider $v \in \mathcal{S}(\mathbb{R}^{n+1})$ and let $u_+ = P_+^{-1}v$, $u_- = P_-^{-1}v$ and $u = u_+ - u_-$, which therefore solves $Pu = 0$.

We now apply the pairing formula (7.23) with $u_1 = \mathcal{P}_- a$, for some $a \in \mathcal{S}(\mathbb{R}^n)$, and with $u_2 = u_-$ as defined above, i.e. $u_2 = P_-^{-1}v$. Then we find that $Pu_1 = 0$, $f_1^- = a$ and $f_2^+ = 0$, so we obtain

$$\langle \mathcal{P}_- a, v\rangle_{L^2(\mathbb{R}^{n+1})} = i(2\pi)^{-n}\langle a, f_2^-\rangle_{L^2(\mathbb{R}^n)},$$

from which follows $\mathcal{P}_-^* v = i(2\pi)^{-n} f_2^-$. We may also express $u = \mathcal{P}_- f_2^-$ as u is the unique solution to $Pu = 0$ with incoming data f_2^-. We conclude that

$$\mathcal{P}_-\mathcal{P}_-^* v = i(2\pi)^{-n}\mathcal{P}_- f_2^- = i(2\pi)^{-n} u = i(2\pi)^{-n}\big(P_+^{-1} - P_-^{-1}\big)v,$$

proving the proposition. □

Completion of the proof of Proposition 7.5. We need to show that $\mathcal{P}_+$ with domain $\mathcal{W}^k(\mathbb{R}^n)$ surjects onto (7.12) when $k \geq 0$ and (7.13) when $k \leq -1$. To do this, we let u be an element of (7.12) and use (7.2) together with (7.28) to write

$$u = -i(2\pi)^n \mathcal{P}_+\mathcal{P}_+^*[P, Q_+]u. \tag{7.29}$$

Thus, it clearly suffices to show that $\mathcal{P}_+^*[P, Q_+]u$ is in $\mathcal{W}^k$. Observe that, since $[P, Q_+]$ has order $(1, -1)$, and is microsupported away from the radial sets, $[P, Q_+]u$ is in $H_{\mathrm{par}}^{k-1/2,k+1/2}(\mathbb{R}^{n+1})$. We choose a microlocal cutoff Q that is microsupported away from the radial sets, and microlocally the identity on $\mathrm{WF}'([P, Q_+])$, and write

$$u = -i(2\pi)^n\Big(\mathcal{P}_+\mathcal{P}_+^* Q^*[P, Q_+]u + \mathcal{P}_+\mathcal{P}_+^*(\mathrm{Id} - Q^*)[P, Q_+]u\Big). \tag{7.30}$$

Using the dual of (7.14) (with k replaced by $-k$), we find that $\mathcal{P}_+^* Q^*[P, Q_+]u$ is in $\mathcal{W}^k$ as required. On the other hand, by the microlocal support assumptions, $(\mathrm{Id} - Q^*)[P, Q_+]u$ is in $\mathcal{S}(\mathbb{R}^{n+1})$, and by Corollary 7.6, we have $\mathcal{P}_+^*(\mathrm{Id} - Q^*)[P, Q_+]u$ is in $\mathcal{S}(\mathbb{R}^n)$, which is even better.

Surjectivity for (7.13) is proved in exactly the same way. □

Proof of Theorem 1.3. The combination of Propositions 7.5 and 7.7 establishes Theorem 1.3. □

7.3. Scattering map.

The following theorem is a slight elaboration of Theorem 1.4.

Theorem 7.11. *The scattering map, defined initially for $f \in \mathcal{W}^k(\mathbb{R}^n)$ with $k \geq 2$ by*

$$S(f) = \lim_{t\to\infty,\ z/2t\to\zeta} (4\pi i t)^{n/2} e^{-i|z|^2/4t}\mathcal{P}_+ f(z, t) \in \langle\zeta\rangle^{\varepsilon}\mathcal{W}^{k-2}(\mathbb{R}^n)$$

in fact satisfies that

$$S\colon \mathcal{W}^k(\mathbb{R}^n) \longrightarrow \mathcal{W}^k(\mathbb{R}^n) \tag{7.31}$$

is bounded, and extends naturally to a continuous mapping for all $k \in \mathbb{Z}$.

Proof. Let $f \in \mathcal{W}^k(\mathbb{R}^n)$, and let $u = \mathcal{P}_- f$. As in the previous proof, we use (7.2) together with (7.28) to express

$$u = -i(2\pi)^n\mathcal{P}_-\mathcal{P}_-^*[P, Q_+]u = -i(2\pi)^n\mathcal{P}_+\mathcal{P}_+^*[P, Q_+]u. \tag{7.32}$$

It follows that the outgoing data for u is $\mathcal{L}_+ u = -i(2\pi)^n \mathcal{P}_+^* [P, Q_+]u$. That is, the scattering map S has the form (similarly to [35, Proposition 5.1])

$$S = -i(2\pi)^n \mathcal{P}_+^* [P, Q_+] \mathcal{P}_-. \tag{7.33}$$

Next, Corollary 7.6 shows that, up to an operator mapping $\mathcal{W}^k(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$, this is equal to

$$-i(2\pi)^n \mathcal{P}_+^* Q \,[P, Q_+]\, Q' \mathcal{P}_- \tag{7.34}$$

where Q, Q' are operators of order $(0, 0)$ that are microlocally the identity on $\mathrm{WF}'([P, A])$ and microlocally trivial in a neighbourhood of the radial sets.

The mapping property then follows from Proposition 7.5. In fact, $Q'\mathcal{P}_-$ maps $\mathcal{W}^k$ to $H_{\mathrm{par}}^{k+1/2,k-1/2}$; the operator $[P, Q_+]$ is order $(1, -1)$ so maps to $H_{\mathrm{par}}^{k-1/2,k+1/2}$; and then the adjoint operator $\mathcal{P}_+^* Q$ maps to $\mathcal{W}^k$. Thus S (defined this way) extends to a map on all $\mathcal{W}^k$. This completes the proof.

□

8. Appendix: Global propagation of regularity and Fredholm estimates

In this appendix we prove the general propagation of regularity results on the full phase space $\overline{T}^*_{\mathrm{par}}\mathbb{R}^{n+1}$.

We treat more general parabolic differential operators $L \in \Psi_{\mathrm{par,cl}}^{m,l}$ and establish two microlocal estimates controlling u in terms of itself and Lu. The first of these estimates, Proposition 8.1, is microlocalised to the subset of the characteristic variety $\mathrm{char}(L)$ where the renormalised Hamiltonian vector field $H^{m,l}$ of L is nonvanishing, and amounts to the standard propagation of regularity theorem in the parabolic setting. The second estimate is microlocalised to a neighbourhood of the radial set $\mathcal{R}$ where $H^{m,l}$ vanishes, and in this region we employ radial set estimates as introduced by Melrose [27].

8.1. Positive commutator estimates away from radial sets.

In the subset of $\mathrm{char}(L)$ where the renormalised Hamiltonian vector field $H^{m,l}$ is nonvanishing, we have positive commutator estimates analogous to Hörmander's propagation theorem for real principal type operators.

Proposition 8.1. *Let $L \in \Psi_{\mathrm{par,cl}}^{k,l}$ be an operator of real principal type and let $Q, Q', G \in \Psi_{\mathrm{par}}^{0,0}$ with G elliptic on $\mathrm{WF}'(Q)$.*

Assume that m is a variable spacetime order that is nonincreasing in the direction of the bicharacteristic flow of L. Furthermore, suppose that for every $\alpha \in \mathrm{WF}'(Q) \cap \mathrm{char}(L)$ there exists α' such that Q' is elliptic at α' and there is a forward bicharacteristic curve γ of L from α' to α such that G is elliptic on γ.

Then if $GLu \in H_{\mathrm{par}}^{s-k+1,\mathsf{m}-l+1}$ and $Q'u \in H_{\mathrm{par}}^{s,\mathsf{m}}$, we have $Qu \in H_{\mathrm{par}}^{s,\mathsf{m}}$ with the estimate

$$\|Qu\|_{H_{\mathrm{par}}^{s,\mathsf{m}}} \le C(\|Q'u\|_{H_{\mathrm{par}}^{s,\mathsf{m}}} + \|GLu\|_{H_{\mathrm{par}}^{s-k+1,\mathsf{m}-l+1}} + \|u\|_{H_{\mathrm{par}}^{M,N}})$$

for any $M, N \in \mathbb{R}$.

The proof of Proposition 8.1 is essentially identical to that of [38, Theorem 5.4]. The only difference is that the boundary defining function for the fiber compactification ρ_{fib} is in our setting given by the quasi-homogeneous $(1 + R^4)^{-1/4}$, as in (2.1), and so the Sobolev spaces in the theorem become the parabolic Sobolev spaces considered in this paper.

Remark 8.1. When $\mathrm{WF}'(Q), \mathrm{WF}'(Q')$ and $\mathrm{WF}'(G)$ are disjoint from the corner of the compactified phase space $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$, Proposition 8.1 can be obtained as a direct consequence of standard propagation theorems valid on the boundary faces $\partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ and $\partial_{\mathrm{fib}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$. The original propagation theorem due to Hörmander [16], as first used by Melrose [27] in the scattering setting, is valid on the interior of $\partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ where the anisotropic nature of Ψ_{par} plays no role. On the other hand, the propagation theorem is proven for Ψ_{par} and general anisotropic pseudodifferential calculi on the interior of the boundary face $\partial_{\mathrm{fib}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ in [25].

8.2. **Positive commutator estimates near the radial sets.** In this section we write down the microlocal propagation estimates for a general operator $L \in \Psi^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$ with real principal symbol near its radial set $\mathcal{R}_L$ (see Definition 2.6).

The proofs of these results are essentially identical to the positive commutator estimates in the standard scattering calculus, and we follow the presentation of [38],[14],[27].

We are interested in the study of propagation estimates near a radial set $\mathcal{R}_L$ extending into the corner of the compactified phase space $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$. We shall consider the case of a radial set $\mathcal{R}_L \subset \partial_{\mathrm{base}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ that meets the other boundary face $\partial_{\mathrm{fib}}\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$ transversally.

Let $L \in \Psi^{m,l}_{\mathrm{par,cl}}$ have real principal symbol p, and let $\hat{p}$ denote $\rho^m_{\mathrm{base}}\rho^l_{\mathrm{fib}}p$ (where ρ_{base} and ρ_{fib} are defined by (2.6) and (2.9)), which by the assumption of classicality of L is a smooth function on $\overline{T^*_{\mathrm{par}}\mathbb{R}^{n+1}}$.

We recall from Section 2 that the vector field

$$H^{m,l} := \rho_{\mathrm{fib}}^{1-m}\rho_{\mathrm{base}}^{1-l}H_L \tag{8.1}$$

extends to a smooth vector field tangent to the boundary faces of $\overline{T^*\mathbb{R}^n}$. By definition, $\mathcal{R}_L$ is the subset of $\mathrm{char}(L)$ where $H^{m,l}$ vanishes. We assume that $\mathcal{R}_L$ is a smooth submanifold of $\mathrm{char}(L)$ of codimension k that meets $\partial_{\mathrm{fib}}\overline{T}^*_{\mathrm{fib}}\mathbb{R}^{n+1}$ transversally. As we have seen, this assumption holds for our specific operator P with $k = n$. As a submanifold of $\partial_{\mathrm{base}}\overline{T^*\mathbb{R}^{n+1}}$, $\mathcal{R}_L$ can be characterized by

$$\mathcal{R}_L = \{\rho_{\mathcal{R}_L} = 0,\ \hat{p} = 0\}, \tag{8.2}$$

where $\rho_{\mathcal{R}_L}$ is a quadratic defining function for $\mathcal{R}_L$ as a submanifold of $\mathrm{char}(L)$, that is, $\rho_{\mathcal{R}} = \sum_{j=1}^k \rho^2_{\mathcal{R}_L,j}$, with the $\rho_{\mathcal{R}_L,j}$ a collection of k smooth functions vanishing on $\mathcal{R}$ with linearly independent differentials.

Our main assumption on $\mathcal{R}_L$ will be that it is either a *source* or a *sink* of the Hamilton vector field. To explain what this means concisely, we adopt the convention in the remainder of this Section that in all future occurrences of $\pm$ and $\mp$, the top sign choice corresponds to the sink situation and the bottom corresponds to the source situation. By a source/sink we mean that $H^{m,l}\rho_{\mathcal{R}_L}$ is nonnegative/nonpositive in a neighbourhood of the radial set, and that $H^{m,l}\rho_{\mathrm{base}}$ is 'strictly' nonnegative/nonpositive in the sense that

$$H^{m,l}\rho_{\mathrm{base}} = \mp\beta\rho_{\mathrm{base}} \tag{8.3}$$

where $\beta \in \mathcal{C}^\infty(\overline{T^*\mathbb{R}^{n+1}})$ is strictly positive on $\mathcal{R}_L$. As a consequence of this first condition, taking $\phi \in \mathcal{C}^\infty_c([0,\infty))$, equal to 1 near 0 and decreasing, then

$$\phi_1 := \sqrt{\pm H^{m,l}\big(\phi(\rho_{\mathcal{R}} + \rho^2_{\mathrm{base}})\big)} \tag{8.4}$$

is non-negative, smooth, and vanishes near the radial set. Furthermore, we require that

$$H^{m,l}\rho_{\rm fib} = \mp 2\beta\beta_1\rho_{\rm fib} \tag{8.5}$$

with $\beta_1 \in \mathcal{C}^\infty(\overline{T^*\mathbb{R}^{n+1}})$ vanishing on $\mathcal{R}$. We introduce $\tilde{p}, q \in \mathcal{C}^\infty(\partial(\overline{T^*\mathbb{R}^{n+1}}))$ by setting

$$\tilde{p} = \sigma_{{\rm par},m-1,l-1}(\frac{1}{2i}(L - L^*)) = \pm\beta q \rho_{\rm fib}^{1-m}\rho_{\rm base}^{1-l} \tag{8.6}$$

Finally we choose $\phi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ identically 1 near, and supported sufficently close to 0. We shall use $\phi_0 \circ \hat{p}$ to localize near the characteristic set $\mathrm{char}(L) = \hat{p}^{-1}(0)$.

In the specific case of the operator $P = D_t + \Delta_g + V$, we have seen via explicit calculation in Section 3.3 that its rescaled Hamilton vector field is a sink near $\mathcal{R}_+$ and a source near $\mathcal{R}_-$. Moreover, those calculations shows that $\beta = 2$ on the radial set, while it is clear that β_1 and q are both zero near the spacetime boundary. Thus all these conditions are fulfilled for the operator P.

In order to prove microlocal estimates near the radial set, we need to come up with an operator A so that its commutator with L, or more exactly the operator on the LHS of (8.8), has a positive symbol at the radial set. To this end, we now define

$$a = \phi(\rho_{\mathcal{R}} + \rho_{\rm base}^2)^2\phi_0(\hat{p})^2\rho_{\rm base}^{-l'}\rho_{\rm fib}^{-m'}, \tag{8.7}$$

which we may assume is supported in a given small neighbourhood U of $\mathcal{R}_L$, and compute the principal symbol

$$\sigma_{m+m'-1,l+l'-1}([A, L] + (L - L^*)A) = -(H_L a + 2\tilde{p}a), \tag{8.8}$$

where $A \in \Psi_{\rm par}^{m',l'}$ is the symmetric operator with principal symbol a given by

$$A = (A^{1/2})^2, \quad A^{1/2} = \frac{\mathrm{Op}(\sqrt{a}) + \mathrm{Op}(\sqrt{a})^*}{2}. \tag{8.9}$$

We have

$$\begin{aligned} H_L a &= \rho_{\rm base}^{1-l}\rho_{\rm fib}^{1-m}H^{m,l}a \\ &= \rho_{\rm base}^{1-l-l'}\rho_{\rm fib}^{1-m-m'}(2\phi\phi_0^2H^{m,l}\rho_{\mathcal{R}} + 2\phi^2\phi_0\phi_0'H^{m,l}\hat{p}) \\ &+ \rho_{\rm base}^{1-l}\rho_{\rm fib}^{1-m}\phi^2\phi_0^2(-l'\rho_{\rm base}^{-l'-1}\rho_{\rm fib}^{-m'}H^{m,l}\rho_{\rm base} - m'\rho_{\rm base}^{-l'}\rho_{\rm fib}^{-m'-1}H^{m,l}\rho_{\rm fib}) \\ &= \rho_{\rm base}^{-l-l'+1}\rho_{\rm fib}^{-m-m'+1}(\pm 2\phi\phi_0^2\phi_1^2 + 2\phi^2\phi_0\phi_0'H^{m,l}\hat{p} \pm l'\phi^2\phi_0^2\beta \pm 2m'\phi^2\phi_0^2\beta\beta_1). \end{aligned}$$

Hence we can express (8.8) as

$$\begin{aligned} &-(H_L a + 2\tilde{p}a) = \rho_{\rm base}^{-l-l'+1}\rho_{\rm fib}^{-m-m'+1} \\ &\quad \times \Big(\mp 2\phi\phi_0^2\phi_1^2 - 2\phi^2\phi_0\phi_0'H^{m,l}\hat{p} \mp l'\phi^2\phi_0^2\beta \mp 2m'\phi^2\phi_0^2\beta\beta_1 \mp 2\beta q\phi^2\phi_0^2\Big). \end{aligned} \tag{8.10}$$

Recall that ϕ_0 cuts off near $\mathrm{char}(L)$, and ϕ cuts off near $\mathcal{R}$ where $\phi_1' = 0$. Hence the first term in (8.10) is supported in a punctured neighbourhood of the radial set, and the second term involving ϕ_0' is supported away from the characteristic set. The latter is easily treated by using microlocal elliptic estimates.

The sum of the final three terms in (8.10) has sign determined by that of

$$\mp(l' + 2m'\beta_1 + 2q). \tag{8.11}$$

In particular, if $l' + 2m'\beta_1 + 2q > 0$ on $\mathcal{R}$, then this sign matches that of the first term in (8.10). (Notice that in the case of the specific operator P, this quantity is just $\mp l'$.)

We require that the quantity (8.11) has a definite sign in order to run the positive commutator argument, drawing different conclusions in the two sign cases. Suppose that we want to estimate u (or a microlocalized version of u) in the $H^{s,r}_{par}$ norm. This requires that we choose m' and l' (the orders of A) to satisfy

$$2s = m + m' - 1, \quad 2r = l + l' - 1 \tag{8.12}$$

and recalling that β_1 vanishes on $\mathcal{R}$, we require (eliminating l' from (8.11) that $r + q - \frac{l-1}{2}$ has definite sign on $\mathcal{R}$. We obtain estimates for both signs, but the estimates have slightly different characters. If $r + q - \frac{l-1}{2}$ is positive, then we obtain microlocal regularity if we assume a priori that u is microlocally in $H^{s,r'}_{\mathrm{par}}$ for some $r' \in [r-1/2, r)$ for which we still have $r' + q - \frac{l-1}{2} > 0$. If this quantity is negative, we obtain instead propagation of regularity 'towards' the radial set from a punctured neighbourhood of the radial set.

In the case of the particular operator P, we have $q = 0$ and 0 so the condition becomes that $r - (-1/2)$ has definite sign. This shows that $r = -1/2$ is a threshold value of the spacetime order, where different behaviours occur above and below this value.

Returning to the general operator L, in the case $r + q > \frac{l-1}{2}$, we use (8.10) and *formally* compute

$$\langle i([A, L] + (L - L^*)A)u, u\rangle$$

(that is, ignoring regularity conditions for pairing distributions, and integrating by parts) to obtain

$$\mp 2\mathrm{Im}\langle Au, Lu\rangle = \|B_1 u\|^2 + \|B_2 u\|^2 + \langle Fu, u\rangle + \langle Ru, u\rangle. \tag{8.13}$$

Here $B_j = \mathrm{Op}(b_j), F = \mathrm{Op}(f)$, where, using (8.12) to replace m' and l' by s and r,

$$b_1 = \phi\phi_0\sqrt{\beta(2r - l + 1 + 2\beta_1(2s - m + 1) + 2q)}\rho_{\mathrm{base}}^{-r}\rho_{\mathrm{fib}}^{-s} \tag{8.14}$$

$$b_2 = \sqrt{2\phi}\phi_0\phi_1\rho_{\mathrm{base}}^{-r}\rho_{\mathrm{fib}}^{-s} \tag{8.15}$$

$$f = \pm 2\phi^2\phi_0\phi_0'(H^{m,l}\hat{p})\rho_{\mathrm{base}}^{-2r}\rho_{\mathrm{fib}}^{-2s} \tag{8.16}$$

and $R \in \Psi^{2s-1,2r-1}_{\mathrm{par}}$.

In the special case $Lu = 0$ for example, (8.13) then yields the estimate

$$\|B_1 u\|^2 \le |\langle Fu, u\rangle| + |\langle Ru, u\rangle| \tag{8.17}$$

Let $Q = \Lambda B_1 \in \Psi^{0,0}_{\mathrm{par}}$ where $\Lambda = \mathrm{Op}(\rho_{\mathrm{base}}^{r}\rho_{\mathrm{fib}}^{s})$ and take $Q'' \in \Psi^{0,0}_{\mathrm{par}}$ elliptic on $\mathrm{WF}'(A)$. We make the assumption that $Q''u \in H^{s-1/2,r'}$ as foreshadowed above. As $\mathrm{WF}'(F) \subset \mathrm{WF}'(A) \subset U$ is disjoint from the characteristic set of L, we may choose $\tilde{Q} \in \Psi^{0,0}_{\mathrm{par}}$ such that $\mathrm{WF}'(\tilde{Q}) \subset \mathrm{ell}(Q'')$ and $\tilde{Q}$ is microlocally equal to the identity on $\mathrm{WF}'(F)$. We can then estimate, for arbitrary M, N,

$$\begin{aligned} |\langle Fu, u\rangle| &\le C\big(|\langle Fu, \tilde{Q}u\rangle| + \|u\|^2_{H^{M,N}_{\mathrm{par}}}\big) \\ &\le C\Big(\|Fu\|^2_{H^{-s+1,-r+1}_{\mathrm{par}}} + \|\tilde{Q}u\|^2_{H^{s-1,r-1}_{\mathrm{par}}} + \|u\|^2_{H^{M,N}_{\mathrm{par}}}\Big) \\ &\le C\Big(\|GLu\|^2_{H^{s-m+1,r-l+1}_{\mathrm{par}}} + \|Q''u\|^2_{H^{s-1,r-1}_{\mathrm{par}}} + \|u\|^2_{H^{M,N}_{\mathrm{par}}}\Big) \end{aligned} \tag{8.18}$$

using the fact that we can invert elliptic operators microlocally — see Proposition 2.9. So, for example, we invert GL microlocally on $\mathrm{WF}'(F)$ to write $F =$

$AGL + R'$ with $A \in \Psi_{\mathrm{par}}^{2s-m,2r-l}$ and $R' \in \Psi_{\mathrm{par}}^{-\infty,-\infty}$. The last term in (8.17) can be estimated similarly:

$$\begin{aligned} |\langle Ru, u\rangle| &\le C\Big(\|Ru\|^2_{H_{\mathrm{par}}^{1/2-s,-r'}} + \|\tilde{Q}u\|^2_{H_{\mathrm{par}}^{s-1/2,r'}} + \|u\|^2_{H_{\mathrm{par}}^{M,N}}\Big) \\ &\le C\Big(\|Q''u\|^2_{H_{\mathrm{par}}^{s-1/2,r'}} + \|u\|^2_{H_{\mathrm{par}}^{M,N}}\Big) \end{aligned} \tag{8.19}$$

where $M, N \in \mathbb{R}$ are again arbitrary. Inserting these estimates into (8.17), we obtain

$$\|Qu\|_{H_{\mathrm{par}}^{s,r}} \le C(\|GLu\|_{H_{\mathrm{par}}^{s-m+1,r-l+1}} + \|Q''u\|_{H_{\mathrm{par}}^{s-1/2,r'}} + \|u\|_{H_{\mathrm{par}}^{M,N}}). \tag{8.20}$$

In the second case with $r + q < \frac{l-1}{2}$, the above calculation is similar, but (8.13) is replaced with

$$\mp 2\mathrm{Im}\langle Au, Lu\rangle = -\|B_1u\|^2 + \|B_2u\|^2 + \langle Fu, u\rangle + \langle Ru, u\rangle \tag{8.21}$$

where (8.14) is replaced by

$$b_1 = \phi\phi_0\sqrt{\beta(l - 2r - 1 - 2\beta_1(2s - m + 1) - 2q)}\rho_{\mathrm{base}}^{-r}\rho_{\mathrm{fib}}^{-s} \tag{8.22}$$

The changed sign of B_2 relative to B_1 means that we additionally need microlocal control of u on $\mathrm{WF}'(B_2)$, which lies in a punctured neighbourhood of the radial set $\mathcal{R}$. This can be achieved by using the standard propagation estimate of Proposition 8.1 away from the radial set, and leads to an additional term $\|Q'u\|_{H_{\mathrm{par}}^{s,r}}$ in the estimate, provided Q' and G satisfy the bicharacteristic condition in Proposition 8.1.

One can also relax the assumption $Lu = 0$ to $GLu \in H_{\mathrm{par}}^{s-m+1,r-l+1}$, which only leads to the additional consideration of the term $\langle Au, Lu\rangle$ in (8.13) and (8.21). We absorb the contribution of this term into the positivity of b_1, by replacing the symbol b_1 with $\tilde{b}_1^2 = b_1^2 - \delta a\rho_{\mathrm{base}}^{l'-2r}\rho_{\mathrm{fib}}^{m'-2s} > 0$ for sufficiently small δ. Then we have

$$\|\tilde{B}_1u\|^2 \le -\delta\|\Lambda A^{1/2}u\|^2 + |\langle Fu, u\rangle| + |\langle Ru, u\rangle| + 2|\mathrm{Im}\langle A^{1/2}u, A^{1/2}Lu\rangle| \tag{8.23}$$

where $\Lambda = \mathrm{Op}(\rho_{\mathrm{base}}^{l'/2-r}\rho_{\mathrm{fib}}^{m'/2-s})$ and $A^{1/2}$ is given by (8.9). Taking $\tilde{\Lambda}$ an elliptic parametrix to Λ, we have

$$|\langle A^{1/2}u, A^{1/2}Lu\rangle| \le |\langle \Lambda A^{1/2}u, \tilde{\Lambda}A^{1/2}Lu\rangle| \tag{8.24}$$

$$\le \frac{\delta}{2}\|\Lambda A^{1/2}u\|^2 + \frac{1}{2\delta}\|\tilde{\Lambda}A^{1/2}Lu\|^2 \tag{8.25}$$

and the first of these terms is absorbed by the first term on the right hand side of (8.23), whilst the latter is bounded (recalling G is elliptic on $U \supset \mathrm{WF}'(A)$) by

$$\frac{C}{2\delta}\|GLu\|_{H_{\mathrm{par}}^{s-m+1,r-l+1}} + C\|u\|_{H_{\mathrm{par}}^{M,N}}.$$

We now state and prove the propagation result in the two cases. Our assumptions are as above for both results; that is, we assume that $L \in \Psi_{\mathrm{par,cl}}^{m,l}(\mathbb{R}^{n+1})$ is a classical pseudodifferential operator in the parabolic scattering calculus that has real principal symbol, such that its radial set $\mathcal{R}_L$ is a codimension k submanifold of $\mathrm{char}(L)$ contained in $\partial_{\mathrm{base}}\overline{T}^*_{\mathrm{fib}}\mathbb{R}^{n+1}$ that meets $\partial_{\mathrm{base}}\overline{T}^*_{\mathrm{fib}}\mathbb{R}^{n+1}$ transversally. We assume that $\mathcal{R}_L$ is either a source or a sink for the rescaled Hamilton vector field in the sense described above, and for either the top (sink) or bottom (source) sign choices in (8.4), (8.3), (8.5), we assume that ϕ_1,β_1 are smooth and vanish near $\mathcal{R}$ and on $\mathcal{R}$ respectively, and that β is smooth and positive on $\mathcal{R}$.

Proposition 8.2. *Suppose $L \in \Psi^{m,l}_{\mathrm{par}}(\mathbb{R}^{n+1})$ is as above. For the q defined in (8.6), suppose that $r + q < \frac{l-1}{2}$ on $\mathcal{R}_L$. Assume that there exists a neighbourhood U of $\mathcal{R}_L$ and $Q', Q'', G \in \Psi^{0,0}_{\mathrm{par}}(\mathbb{R}^{n+1})$, with $U \subset \mathrm{ell}(Q'')$ and such that for every $\alpha \in p^{-1}(0) \cap U \setminus \mathcal{R}_L$ the bicharacteristic γ through α enters $\mathrm{ell}(Q')$ whilst remaining in $\mathrm{ell}(G)$. Then there exists $Q \in \Psi^{0,0}_{\mathrm{par}}(\mathbb{R}^{n+1})$ elliptic on $\mathcal{R}_L$ such that if $u \in H^{M,N}$, $Q'u \in H^{s,r}$, $Q''u \in H^{s-1/2,r-1/2}$ and $GLu \in H^{s-m+1,r-l+1}$, then $Qu \in H^{s,r}$ and there is $C > 0$ such that*

$$\|Qu\|_{H^{s,r}_{\mathrm{par}}} \leq C(\|Q'u\|_{H^{s,r}_{\mathrm{par}}} + \|GLu\|_{H^{s-m+1,r-l+1}_{\mathrm{par}}} + \|Q''u\|_{H^{s-1/2,r-1/2}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{8.26}$$

Proposition 8.3. *Suppose $L \in \Psi^{m,l}_{\mathrm{par,cl}}(\mathbb{R}^{n+1})$ is as above. For the q defined in (8.6), suppose $r + q > \frac{l-1}{2}$ on $\mathcal{R}_L$ and moreover $r' + q > \frac{l-1}{2}$ on $\mathcal{R}_L$ for some $r' \in [r - 1/2, r)$. Assume that $Q'', G \in \Psi^{0,0}_{\mathrm{par}}(\mathbb{R}^{n+1})$ are elliptic at $\mathcal{R}_L$. Then there exists $Q \in \Psi^{0,0}_{\mathrm{par}}(\mathbb{R}^{n+1})$, elliptic at $\mathcal{R}_L$, such that if $u \in H^{M,N}$, $Gu \in H^{s-1/2,r'}_{\mathrm{par}}$ and $GLu \in H^{s-m+1,r-l+1}_{\mathrm{par}}$, then $Qu \in H^{s,r}_{\mathrm{par}}$ and there is $C > 0$ such that*

$$\|Qu\|_{H^{s,r}_{\mathrm{par}}} \leq C(\|GLu\|_{H^{s-m+1,r-l+1}_{\mathrm{par}}} + \|Q''u\|_{H^{s-1/2,r'}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{8.27}$$

Remark 8.2. In the statement of Proposition 8.3, we could take $Q'' = G$. However, in order to treat the proofs of these two results jointly, it helps to state the result as above.

Proof. The proof of these two results largely amounts to regularising the commutator estimates outlined above in order to legitimately obtain the equality (8.13).

Note that a priori, the conditions of Proposition 8.3 only imply that Lu and Au have orders $(s - m + 1, r - l + 1)$ and $(s - 1/2 - m', r' - l')$ in $\mathrm{WF}'(A)$, summing to $(-1/2, r' - r)$ (using (8.12)), and so $\langle Au, Lu\rangle$ is not a priori well-defined. This requires some regularization procedure added to the formal calculations above.

To deal with this issue, we replace the symbol a in (8.7) with

$$a_\varepsilon = \varphi_\varepsilon(\rho_{\mathrm{base}}^{-1})^2 \tilde{\varphi}_\varepsilon(\rho_{\mathrm{fib}}^{-1})^2 a := (1 + \varepsilon\rho_{\mathrm{base}}^{-1})^{r'-r}(1 + \varepsilon\rho_{\mathrm{fib}}^{-1})^{-1/2} a \tag{8.28}$$

for $\varepsilon \geq 0$. Thus for each fixed ε, the order of A_ε obtained from a_ε analogously to (8.9) has been shifted by $(-1/2, r' - r)$ relative to A.

These regularising functions have the property that

$$\varphi_\varepsilon'/\varphi_\varepsilon \leq \frac{r - r'}{2} \cdot \min\{1, \rho_{\mathrm{base}}\} \tag{8.29}$$

and

$$\tilde{\varphi}_\varepsilon'/\tilde{\varphi}_\varepsilon \leq \frac{1}{4} \cdot \min\{1, \rho_{\mathrm{fib}}\}. \tag{8.30}$$

Due to the regularisation, the pairing $\langle A_\varepsilon u, LU\rangle$ is now well defined for $\varepsilon > 0$, however the formal integration by parts, i.e. the identity

$$\langle Lu, A_\varepsilon u\rangle - \langle A_\varepsilon u, Lu\rangle = \langle (A_\varepsilon L - L^* A_\varepsilon)u, u\rangle \tag{8.31}$$

still remains to be justified for fixed $\varepsilon > 0$. To do this, we use the functions φ_ε, $\tilde{\varphi}_\varepsilon$ just defined and raise them to a sufficiently high fixed power K: let $\Gamma_t =$

$\mathrm{Op}(\varphi_t^K(\rho_{\mathrm{base}}^{-1})\tilde{\varphi}_t^K(\rho_{\mathrm{fib}}^{-1}))$, and compute

$$\begin{aligned}\langle Lu, A_\varepsilon u\rangle - \langle A_\varepsilon u, Lu\rangle &= \lim_{t\to 0}\Big(\langle \Gamma_t Lu, A_\varepsilon u\rangle - \langle \Gamma_t A_\varepsilon u, Lu\rangle\Big) \\ &= \lim_{t\to 0}\langle (A_\varepsilon \Gamma_t L - L^*\Gamma_t A_\varepsilon)u, u\rangle\end{aligned}$$

We have

$$A_\varepsilon \Gamma_t L - L^*\Gamma_t A_\varepsilon = \Gamma_t(A_\varepsilon L - L^* A_\varepsilon) + [A_\varepsilon, \Gamma_t]L - [L^*, \Gamma_t]A_\varepsilon. \tag{8.32}$$

As Γ_t is uniformly bounded in $\Psi^{0,0}_{\mathrm{par}}$ (in the sense of having symbol with uniformly bounded seminorms), and converges to Id as $t \to 0$ in $\Psi^{\mu,\mu}_{\mathrm{par}}$ for any $\mu > 0$, we have strong convergence $A_\varepsilon \Gamma_t L - L^*\Gamma_t A_\varepsilon \to A_\varepsilon L - L^* A_\varepsilon$ which implies (8.31) is valid for each $\varepsilon > 0$.

As before, we compute the symbol $-(H_p a_\varepsilon + 2\tilde{p} a_\varepsilon)$ of the commutator expression $i([A_\varepsilon, L] + (L - L^*)A_\varepsilon)$ where A_ε is symmetric with principal symbol a_ε.

The calculation proceeds as before from (8.10), however in each term there is the regularising factor $\varphi_\varepsilon^2\tilde{\varphi}_\varepsilon^2$, and there are two additional terms from $H^{m,l}$ falling on the regularisers. These two terms are:

$$\mp 2\rho_{\mathrm{base}}^{1-l-l'}\rho_{\mathrm{fib}}^{1-m-m'}\phi^2\phi_0^2\varphi_\varepsilon^2\tilde{\varphi}_\varepsilon^2 \cdot (\varphi_\varepsilon'/\varphi_\varepsilon)\cdot \rho_{\mathrm{base}}^{-1}\beta \tag{8.33}$$

and

$$\mp 4\rho_{\mathrm{base}}^{1-l-l'}\rho_{\mathrm{fib}}^{1-m-m'}\phi^2\phi_0^2\varphi_\varepsilon^2\tilde{\varphi}_\varepsilon^2 \cdot (\tilde{\varphi}_\varepsilon'/\tilde{\varphi}_\varepsilon)\cdot \rho_{\mathrm{fib}}^{-1}\beta\beta_1 \tag{8.34}$$

In the case $r + q < \frac{l-1}{2}$, these new terms in fact have the same sign as the 'main' term $\|B_1 u\|^2$, and can thus be dropped from the commutator estimate.

In the case $r + q > \frac{l-1}{2}$, these new terms have the opposite sign of $\|B_1 u\|^2$, but can be absorbed into the b_1 term as in (8.23). To see this, we use (8.29) and (8.30) to obtain the estimate

$$\frac{2\varphi_\varepsilon'}{\varphi_\varepsilon} + \frac{4\beta_1\tilde{\varphi}_\varepsilon'}{\tilde{\varphi}_\varepsilon} \le r - r' + \beta_1. \tag{8.35}$$

Since the expression underneath the square root in (8.14) remains positive near $\mathcal{R}_L$ if we replace the r with an r', it follows that, provided we have $r' + q > \frac{l-1}{2}$, the two additional terms can be absorbed into the positive expression b_1^2.

The remaining terms in (8.13), (8.21) are now generally ε-dependent, which we denote with a subscript. Replacing $\mathrm{WF}'(A)$ and $\mathrm{WF}'(F)$ with their uniform versions $\mathrm{WF}'(A_\varepsilon), \mathrm{WF}'(F_\varepsilon)$ (in the sense of [38, Sect. 4.4]), then all estimates go through uniformly in ε to give

$$\|Q_\varepsilon u\|_{H^{s,r}_{\mathrm{par}}} \le C(\|GLu\|_{H^{s-m+1,r-l+1}_{\mathrm{par}}} + \|Q''u\|_{H^{s-1/2,r'}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}) \tag{8.36}$$

for fixed C and all $\varepsilon > 0$ in the case $r + q > r' + q > \frac{l-1}{2}$.

From weak compactness of the unit ball in $H^{s,r}$, as $\varepsilon \to 0$ we thus obtain a limit $\lim_{\varepsilon_j\to 0} Q_\varepsilon u$ in $H^{s,r}$. However, we know that $Q_\varepsilon u$ converges strongly (and a fortiori weakly) to Qu in $H^{s-\varepsilon',r-\varepsilon'}$ for any $\varepsilon' > 0$. It follows that limit obtained from weak compactness is Qu, and hence Qu lies in $H^{s,r}$ satisfying the required estimate

$$\|Qu\|_{H^{s,r}_{\mathrm{par}}} \le C(\|GLu\|_{H^{s-m+1,r-l+1}_{\mathrm{par}}} + \|Q''u\|_{H^{s-1/2,r'}_{\mathrm{par}}} + \|u\|_{H^{M,N}_{\mathrm{par}}}). \tag{8.37}$$

In the case $r + q < \frac{l-1}{2}$, the exact same argument goes through, with the additional term $\|B_2 u\|^2$ in (8.21) controlled using Proposition 8.1 and giving rise to the additional term $\|Q'u\|_{H^{s,r}_{\mathrm{par}}}$ in Proposition 8.2.

□

Remark 8.3. The conclusions of Propositions 8.3 and 8.2 lend themselves to iteration. In fact, if we know that $Q'u$ (microsupported away from the radial set, say) and GLu have regularity (s^*, r^*) and $(s^* - m + 1, r^* - l + 1)$ respectively, then we can apply (8.26) over and over, starting at the assumed regularity (M, N), to obtain regularity (s^*, r^*) for Qu, provided that $r^* + q < (l-1)/2$. Similarly, if GLu has very high regularity, say of order $(s^* - m + 1, r^* - l + 1)$ where s^* and r^* are very large, then we simply apply (8.27) over and over, gaining up to $1/2$ in both spacetime and fibre regularity until we reach s^* and r^*, provided we know a priori that we have regularity (s_0, r_0) with $r_0 + q > (l - 1)/2$. The results obtained by such iteration are stated for the specific operator P in Propositions 5.4 and 5.3. Notice that the Q'' term is not needed in the below-threshold case as it can be iteratively lowered in regularity until it is subsumed into the $\|u\|_{H^{M,N}}$ term.

Remark 8.4. The example of the free Schrödinger operator P_0 shows that these propositions cannot be improved much. Consider a solution u to $P_0 u = 0$ given by $\mathcal{P}_0 f$ for f Schwartz, as in (6.7). The function u is microlocally trivial outside the radial set $\mathcal{R}$, and is in $H^{s,r}_{\mathrm{par}}$ for every $r < -1/2$, but not for $r \geq -1/2$. It shows that the a priori condition that $u \in H^{s',r'}_{\mathrm{par}}$ for some $r' > -1/2$ in Propositions 8.3 and 5.4 cannot be removed in order to gain additional regularity at the radial set, and also shows that regularity gain in Propositions 8.2 and 5.3 cannot be pushed above the threshold level of $-1/2$.

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