LARGE FOURIER COEFFICIENTS OF HALF-INTEGER WEIGHT MODULAR FORMS

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Abstract. This article is concerned with the Fourier coefficients of cusp forms (not necessarily eigenforms) of half-integer weight lying in the plus space. We give a soft proof that there are infinitely many fundamental discriminants D such that the Fourier coefficients evaluated at $|D|$ are non-zero. By adapting the resonance method, we also demonstrate that such Fourier coefficients must take quite large values.

1. Introduction

Let k be a positive integer, and let $S_{k+\frac{1}{2}}$ denote the space of cusp forms of half-integral weight $k+\frac{1}{2}$ for the group $\Gamma_0(4)$ (which consists of the elements of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ with lower left entry divisible by 4). The theory of such forms was developed by Shimura [18], and such a form g has a Fourier expansion

$$g(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z} \tag{1.1}$$

with z in the upper half plane. We shall restrict attention to forms g in the *plus subspace* $S^+_{k+\frac{1}{2}}$ of those forms whose Fourier coefficients $c(n)$ are zero unless $(-1)^k n \equiv 0, 1 \pmod 4$ (see Kohnen [11]). This paper is concerned with the coefficients $c(|D|)$ where D is a fundamental discriminant with $|D| = (-1)^k D > 0$. In particular, we wish to show that these coefficients must be non-zero infinitely often, and indeed occasionally get large in terms of $|D|$. For the sake of simplicity, we have restricted attention to level 4 and to holomorphic forms of half-integer weight, and it should be possible to extend these results to general level, or to non-holomorphic Maass forms. Indeed recently Jääsaari, Lester, and Saha [10] have extended our methods to higher level, and applied those results to the study of fundamental Fourier coefficients of Siegel cusp forms.

When g is a Hecke eigenform, Waldspurger's famous theorem (see [23], and in fully explicit form [12]) states that the squares $|c(|D|)|^2$ are proportional to the values $|D|^{k-\frac{1}{2}} L(f, \chi_D, k)$, where f is a normalized Hecke eigenform in the space S_{2k} of cusp forms of weight $2k$ on Γ corresponding to g under the Shimura correspondence. Here, $L(f, \chi_D, s)$ denotes the Hecke L-function of f twisted with the primitive quadratic character χ_D attached to the fundamental discriminant D, and $s = k$ is its central point. In this case, the problems of non-vanishing and producing large values of $|c(|D|)|$ amount to the well studied problems of non-vanishing and omega results for central values in this family of L-functions (see, for example, [6, 8, 15, 16, 21]). More recently, Hulse et al. [7] have studied sign changes in the coefficients $c(|D|)$'s (when normalized to be all real), and further progress on that problem is due to Lester and Radziwiłł [13].

Our main interest, however, is in the situation where g is a general cusp form, and not necessarily a Hecke eigenform. In the case when g is a linear combination of two eigenforms, the problem of non-vanishing was resolved by Luo and Ramakrishnan [14]. The general case was resolved in the work of Saha [17] who showed that for *any* non-zero g in $S^+_{k+\frac{1}{2}}$ there are infinitely many fundamental discriminants D with $(-1)^k D > 0$ such that $c(|D|)$ is not zero. In this paper we give two proofs of this result, showing further that $|c(|D|)|$ gets large for many fundamental discriminants D. Our first proof introduces a new Dirichlet series built out of the coefficients $c(|D|)$ and Dirichlet L-functions attached to the character χ_D. This proof is qualitative and soft, and makes no use of the Waldspurger formula. The second proof is based on Waldspurger's formula and the connection to L-values. It uses the resonance method, developed in [19], to show that linear combinations of L-values can be made large. The resonance method proceeds by comparing the average of L-values weighted by a carefully chosen resonator Dirichlet polynomial with the average of the resonator polynomial itself. If the ratio of these averages can be made large, then one concludes that the L-values must get large. The new feature in our work is to show that a resonator that makes the twists of one L-function large does not correlate with twists of other L-functions, allowing one to obtain large values of linear combinations of L-functions.

Theorem 1.1. *Let g be a non-zero element of $S^+_{k+\frac{1}{2}}$ with Fourier expansion as in* (1.1).

(a) There are infinitely many fundamental discriminants D with $(-1)^k D > 0$ such that $c(|D|) \neq 0$.

(b) Let $\epsilon > 0$ be given, and X be large. There are at least $X^{1-\epsilon}$ fundamental discriminants D with $X < (-1)^k D \le 2X$ such that

$$|c(|D|)| \ge |D|^{\frac{k}{2}-\frac{1}{4}} \exp\Big(\frac{1}{82}\frac{\sqrt{\log|D|}}{\sqrt{\log\log|D|}}\Big).$$

If g is an eigenform, then as mentioned earlier $|c(|D|)|^2$ is proportional to $|D|^{k-\frac{1}{2}}L(f,\chi_D,k)$. The Lindelöf hypothesis then implies that $|c(|D|)| \ll |D|^{\frac{k}{2}-\frac{1}{4}+\epsilon}$ for any $\epsilon > 0$. Writing a general $g \in S^+_{k+\frac{1}{2}}$ as a linear combination of eigenforms, we arrive at the conjecture that

$$|c(|D|)| \ll_{g,\varepsilon} |D|^{\frac{k}{2}-\frac{1}{4}+\varepsilon} \qquad (\varepsilon > 0).$$

This is an analogue of the Ramanujan-Petersson conjecture in integral weight, and remains an outstanding open problem. Indeed, conjectures on the maximal size of L-functions (see [4]) suggest that for fundamental discriminants D, perhaps even the following stronger bound holds (for some $C > 0$):

$$|c(|D|)| \ll_g |D|^{\frac{k}{2}-\frac{1}{4}} \exp\Big(C\sqrt{\log|D|\log\log|D|}\Big).$$

The resonance method [19] produces large values of L-functions in very general settings. Although this is not one of the examples worked out in [19], the resonance method shows that for Hecke eigenforms f of integer weight $2k$, there are infinitely many fundamental discriminants D such that $L(f,\chi_D,k) \gg \exp(c\sqrt{\log|D|/\log\log|D|})$ for a positive constant c; a somewhat weaker result may be found in [6]. Thus for an eigenform $g \in S^+_{k+\frac{1}{2}}$, one would get corresponding lower bounds for $|c(|D|)|$. Theorem 1.1(b) establishes a similar bound for general $g \in S^+_{k+\frac{1}{2}}$, and the key is to adapt the resonance method to show that one can produce large values of twists of a particular L-function while keeping the twists of all other L-functions of Hecke eigenforms of weight $2k$ small. Work of Bondarenko and Seip [1] gives an improvement of the resonance method of [19], producing still larger values of $|\zeta(\frac{1}{2}+it)|$, and a similar improvement for values of $|L(\frac{1}{2},\chi)|$ has been obtained in [2]. However, this method exploits positivity (of coefficients, and of orthogonality relations) in crucial ways, and does not seem to extend to L-functions in other families, such as the family of quadratic twists of an eigenform. Thus, apart from the constant $\frac{1}{82}$ (which we have made no attempt to optimize), the lower bounds furnished in Theorem 1.1 (b) are the best currently known, even in the situation of eigenforms g.

While Theorem 1.1 (b) produces occasional large values of $|c(|D|)|$, "typical" values of $|c(|D|)|$ tend to be much smaller. Central values of L-functions are conjectured to be log-normal with a suitable mean and variance, which is a conjectured analogue of the classical work of Selberg on the log-normality of $|\zeta(\frac{1}{2}+it)|$. Radziwiłł and Soundararajan [16] have established one sided central limit theorems for central values of quadratic twists of elliptic curves with positive sign of the functional equation. These arguments carry over to quadratic twists of eigenforms of larger integer weight, and establish that for all but $o(X)$ fundamental discriminants D with $X \le (-1)^k D \le 2X$ one has

$$|c(|D|)| \ll_{g,\epsilon} |D|^{\frac{k}{2}-\frac{1}{4}}(\log D)^{-\frac{1}{4}+\epsilon},$$

where $\epsilon > 0$. The connection with L-functions first establishes such a result for eigenforms g, and then the same conclusion holds for any $g \in S^+_{k+\frac{1}{2}}$ by decomposing g in terms of eigenforms.

In our discussion above, we have confined ourselves to $c(|D|)$ where D is a fundamental discriminant. These are the fundamental objects of interest, and the problem of obtaining large values of $c(n)$ for n not arising from fundamental discriminants is of a different flavor (and comparatively easier). For example, fixing a fundamental discriminant D_0, and varying m, the Shimura lift implies that for eigenforms g finding large values of $c(|D_0|m^2)$ amounts to finding large values of the Hecke eigenvalues $a(m)$ of the Shimura lift. For work in this direction see [5], [3].

Acknowledgments. S.G. would like to acknowledge MTR/2018/000201 and DAE number theory plan project for partial financial support. K.S. is partially supported through a grant from the National Science Foundation, and a Simons Investigator Grant from the Simons Foundation. This work was carried out while K.S. was a senior Fellow at the ETH Institute for Theoretical Studies, whom he thanks for their warm and generous hospitality. The authors also thank the referees for their careful reading.

2. Notation and review

2.1. **Half integer weight forms.** Throughout let $g_1, \ldots, g_r$ denote a basis of Hecke eigenforms for $S^+_{k+\frac{1}{2}}$. Denote the Fourier expansions of g_ν by

$$g_\nu(z) = \sum_{n=1}^{\infty} c_\nu(n) e^{2\pi i n z}.$$

Let g be a general cusp form in the space $S^+_{k+\frac{1}{2}}$, and write

$$g = \sum_{\nu=1}^{r} \lambda_\nu g_\nu$$

for some constants $\lambda_\nu \in \mathbb{C}$. Thus the Fourier coefficients $c(n)$ of g are also linear combinations of the Fourier coefficients $c_\nu(n)$:

$$c(n) = \sum_{\nu=1}^{r} \lambda_\nu c_\nu(n).$$

The Fourier coefficients $c(n)$ satisfy the usual Hecke bound

$$c(n) \ll_g n^{\frac{k}{2}+\frac{1}{4}},$$

while they are expected to satisfy the analogue of the Ramanujan bound namely $c(n) \ll_g n^{\frac{k}{2}-\frac{1}{4}+\epsilon}$ (which we discussed earlier in the case when $n = (-1)^k D$ for a fundamental discriminant D).

We associate to g the Hecke L-series

$$L(g, s) = \sum_{n \geq 1} c(n) n^{-s},$$

which by the Hecke bound for $c(n)$ converges absolutely when $\sigma > \frac{k}{2} + \frac{5}{4}$. Further from [18] we know that $L(g, s)$ has holomorphic continuation to $\mathbb{C}$ and satisfies the functional equation

$$\Lambda(g|W_4,\ k + \tfrac{1}{2} - s) = \Lambda(g, s). \tag{2.1}$$

Here

$$g \mapsto g|W_4,\ (g|W_4)(z) := (-2iz)^{-k-\frac{1}{2}} g\Big(-\frac{1}{4z}\Big)$$

is the Fricke involution on $S_{k+\frac{1}{2}}$, and

$$\Lambda(g, s) := \pi^{-s}\Gamma(s)L(g, s).$$

2.2. **Dirichlet L-functions.** Associated to a fundamental discriminant D is a primitive Dirichlet character $\pmod{|D|}$ which we denote by χ_D. To χ_D we may associate the Dirichlet L-function

$$L(\chi_D, s) = \sum_{n=1}^{\infty} \chi_D(n) n^{-s},$$

which converges absolutely in the half-plane $\sigma > 1$ and extends analytically to $\mathbb{C}$ (except for a pole at $s = 1$ in the case $D = 1$ corresponding to $\zeta(s)$). Put $\delta = 0$ if $D > 0$ (so that $\chi_D(-1) = 1$) and $\delta = 1$ if $D < 0$ (so that $\chi_D(-1) = -1$). Then the completed L-function

$$\Lambda(\chi_D, s) := \Big(\frac{|D|}{\pi}\Big)^{\frac{s+\delta}{2}} \Gamma\Big(\frac{s+\delta}{2}\Big) L(\chi_D, s)$$

satisfies the functional equation

$$\Lambda(\chi_D, 1 - s) = \Lambda(\chi_D, s). \tag{2.2}$$

For these classical facts, see for example Iwaniec and Kowalski [9].

2.3. **The Shimura lift and integer weight Hecke eigenforms.** The Shimura correspondence associates to every eigenform $g_\nu \in S^+_{k+\frac{1}{2}}$ a Hecke eigenform f_ν of weight $2k$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ (see [11]). We normalize f_ν to have first coefficient 1, so that it has a Fourier expansion

$$f_\nu(z) = \sum_{n=1}^{\infty} a_\nu(n) e^{2\pi i n z},$$

with $a_\nu(1) = 1$. The Fourier coefficients $a_\nu(n)$, which are also the eigenvalues of the Hecke operators, satisfy multiplicative Hecke relations, and satisfy the Deligne bound $|a_\nu(n)| \leq d(n) n^{k-\frac{1}{2}}$ with $d(n)$ denoting the number of divisors of n. Associated to the Hecke eigenform f_ν is the L-function

$$L(f_\nu, s) = \sum_{n=1}^{\infty} \frac{a_\nu(n)}{n^s} = \prod_p \Big(1 - \frac{a_\nu(p)}{p^s} + \frac{p^{2k-1}}{p^{2s}}\Big)^{-1},$$

which converges absolutely for $\sigma > k + \frac{1}{2}$, extends analytically to $\mathbb{C}$, and satisfies the functional equation

$$\Lambda(f_\nu, s) = (2\pi)^{-s} \Gamma(s) L(f_\nu, s) = (-1)^k \Lambda(f_\nu, 2k - s). \tag{2.3}$$

The coefficients of g_ν and its Shimura lift f_ν are related by means of the identity

$$c_\nu(n^2|D|) = c_\nu(|D|) \sum_{d|n} \mu(d) \chi_D(d) d^{k-1} a_\nu(n/d),$$

where D is a fundamental discriminant with $(-1)^k D > 0$ and $n \geq 1$ or equivalently by means of the Dirichlet series identity

$$L(\chi_D, s - k + 1) \sum_{n=1}^{\infty} c_\nu(|D| n^2) n^{-s} = c_\nu(|D|) L(f_\nu, s). \tag{2.4}$$

A deeper relation between the coefficients of g_ν and the Shimura lift f_ν is given by the Waldspurger formula. If D is a fundamental discriminant, the L-series of the D-th quadratic twist of f_ν is given by

$$L(f_\nu, \chi_D, s) = \sum_{n=1}^{\infty} a_\nu(n) \chi_D(n) n^{-s}.$$

It converges absolutely for $\sigma > k + \frac{1}{2}$, extends analytically to $\mathbb{C}$, and satisfies the functional equation

$$\Lambda(f_\nu, \chi_D, s) = \Big(\frac{|D|}{2\pi}\Big)^s \Gamma(s) L(f_\nu, \chi_D, s) = (-1)^k \chi_D(-1) \Lambda(f_\nu, \chi_D, 2k - s). \tag{2.5}$$

Note that if D is a fundamental discriminant with $(-1)^k D < 0$, then the sign of the functional equation above is -1, and so the central value $L(f_\nu, \chi_D, k)$ equals zero. In the complementary case $(-1)^k D > 0$ (which dovetails with the definition of the plus space $S^+_{k+\frac{1}{2}}$), Waldspurger's formula gives

$$|c_\nu(|D|)|^2 = C_\nu |D|^{k-\frac{1}{2}} L(f, \chi_D, k). \tag{2.6}$$

Here C_ν is a constant, which Kohnen and Zagier [12] obtained in the elegant form

$$C_\nu = \frac{(k-1)!}{\pi^k} \frac{\langle g_\nu, g_\nu \rangle}{\langle f_\nu, f_\nu \rangle}, \tag{2.7}$$

where $\langle g_\nu, g_\nu\rangle$ and $\langle f_\nu, f_\nu\rangle$ denote the normalized Petersson norms of g_ν and f_ν.

Finally, we record a consequence of Rankin–Selberg theory for the coefficients $a_\nu(p)$. Namely, as $x\to\infty$

$$\sum_{p\le x}\frac{|a_\nu(p)|^2}{p^{2k-1}}\log p\sim x, \tag{2.8}$$

whereas if $\nu_1\neq\nu_2$ then

$$\sum_{p\le x}\frac{a_{\nu_1}(p)\overline{a_{\nu_2}(p)}}{p^{2k-1}}\log p=o(x). \tag{2.9}$$

3. Non-vanishing of Fourier coefficients

In this section we establish part (a) of Theorem 1.1, and show that if $g\in S^+_{k+\frac12}$ is not identically zero, then there are infinitely many fundamental discriminants D with $|D|=(-1)^kD>0$ such that $c(|D|)\neq 0$. Our proof will be based on the following Dirichlet series:

$$D_g(s):=\sum_{n=1}^\infty\frac{\alpha(n)}{n^s}, \tag{3.1}$$

where, $\alpha(n)=0$ if $(-1)^kn\equiv 2,3 \bmod 4$, and when $(-1)^kn\equiv 0,1\bmod 4$ we set (writing n uniquely as $n=|D|m^2$ with D a fundamental discriminant as above),

$$\alpha(n):=c(|D|)\mu(m)\chi_D(m)m^{k-1}.$$

The Hecke bound $|c(|D|)|\ll_g |D|^{\frac k2+\frac14}$ gives $|\alpha(n)|\ll_g n^{\frac k2+\frac14}$ so that the Dirichlet series $D_g(s)$ converges absolutely in $\sigma>\frac k2+\frac54$, and defines a holomorphic function in that half-plane.

In this half-plane of absolute convergence $\sigma>\frac k2+\frac54$, we may rewrite $D_g(s)$ as

$$D_g(s)=\sum_{(-1)^kD>0}\sum_{m=1}^\infty\frac{c(|D|)\chi_D(m)\mu(m)}{|D|^sm^{2s-k+1}}=\sum_{(-1)^kD>0}\frac{c(|D|)}{|D|^sL(\chi_D,2s-k+1)}, \tag{3.2}$$

upon recalling that in the half-plane $\mathrm{Re}(z)>1$ one has

$$\frac{1}{L(\chi_D,z)}=\sum_{m=1}^\infty\frac{\mu(m)\chi_D(m)}{m^z}.$$

Since $g=\sum_\nu\lambda_\nu g_\nu$ we have

$$D_g(s)=\sum_{\nu=1}^r\lambda_\nu D_{g_\nu}(s).$$

Now from (2.4) (taking there $2s$ in place of s) we have

$$\frac{c_\nu(|D|)}{|D|^sL(\chi_D,2s-k+1)}=\frac{1}{L(f_\nu,2s)}\sum_{m=1}^\infty\frac{c_\nu(|D|m^2)}{|D|^sm^{2s}},$$

and summing this over all D with $(-1)^kD>0$ we conclude that

$$D_{g_\nu}(s)=\frac{L(g_\nu,s)}{L(f_\nu,2s)}.$$

Thus

$$D_g(s) = \sum_{\nu=1}^{r} \lambda_\nu \frac{L(g_\nu, s)}{L(f_\nu, 2s)}. \tag{3.3}$$

In particular, $D_g(s)$ has meromorphic continuation to $\mathbb{C}$ and is holomorphic for $\sigma > \frac{1}{2} + \frac{1}{4}$ (since in that half plane $L(f_\nu, 2s)$ has an absolutely convergent Euler product, and is therefore non-zero).

Suppose now that $g \in S^+_{k+\frac{1}{2}}$ has only finitely many fundamental discriminants D with $c(|D|) \neq 0$. We seek to show that g must be identically zero; that is, all the λ_ν equal zero. The proof is in two stages: First, we show that $D_g(s)$ must be identically zero (that is, all the coefficients $c(|D|)$ are zero). The key input here is that if only finitely many $c(|D|)$ are non-zero, then from (3.2) $D_g(s)$ inherits a functional equation arising from the one for Dirichlet L-functions. But this turns out to be inconsistent with the functional equation for $D_g(s)$ arising from (3.3) and the functional equations for $L(g_\nu, s)$ and $L(f_\nu, 2s)$. In the second stage, using these functional equations again, we show that $\sum_{\nu=1}^r \lambda_\nu c_\nu(|D|) a_\nu(p)$ must vanish for all fundamental discriminants D with $4|D$ and $(-1)^k D > 0$ and all odd primes p. By invoking Rankin-Selberg relations for $a_\nu(p)$ together with the fact that for each eigenform g_ν there exists a fundamental discriminant D with $4|D$ and $c_\nu(|D|) \neq 0$ (see [11]), we finally find that $g = 0$; a contradiction.

3.1. **Showing that $D_g(s) = 0$.** From the functional equations (2.1) and (2.3) we see that

$$\frac{L(g_\nu, s)}{L(f_\nu, 2s)} = \gamma(s) \frac{L(g_\nu | W_4, k + \frac{1}{2} - s)}{L(f_\nu, 2k - 2s)}$$

where, upon using the duplication formula for the Γ-function,

$$\gamma(s) = (-1)^k \cdot 2^{2k-4s} \cdot \pi^{k-\frac{1}{2}-2s} \cdot \frac{\Gamma(2s)\Gamma(k+\frac{1}{2}-s)}{\Gamma(s)\Gamma(2k-2s)} = (-1)^k \pi^{k-\frac{1}{2}-2s} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(k-s)}.$$

Thus we have the functional equation

$$D_g(s) = \sum_{\nu=1}^{r} \lambda_\nu \frac{L(g_\nu, s)}{L(f_\nu, 2s)} = \gamma(s) \sum_{\nu=1}^{r} \lambda_\nu \frac{L(g_\nu | W_4, k+\frac{1}{2}-s)}{L(f_\nu, 2k-2s)}. \tag{3.4}$$

On the other hand, if only finitely many $c(|D|)$ are non-zero, then we may use the functional equation for $L(\chi_D, s)$ (see (2.2)) in the expression (3.2). Thus, with $\delta = 0$ if k is even and $\delta = 1$ if k is odd,

$$\begin{aligned} D_g(s) &= \sum_{(-1)^k D > 0} \frac{c(|D|)}{|D|^s L(\chi_D, 2s-k+1)} \\ &= \pi^{k-\frac{1}{2}-2s} \frac{\Gamma(\frac{2s-k+1+\delta}{2})}{\Gamma(\frac{k-2s+\delta}{2})} \sum_{(-1)^k D>0} \frac{c(|D|)}{|D|^{k-\frac{1}{2}-s} L(\chi_D, k-2s)} \\ &= \pi^{k-\frac{1}{2}-2s} \frac{\Gamma(\frac{2s-k+1+\delta}{2})}{\Gamma(\frac{k-2s+\delta}{2})} D_g(k-\tfrac{1}{2}-s). \end{aligned} \tag{3.5}$$

We warn the reader that, unlike (3.4) which is a true functional equation, the relation (3.5) is predicated on the assumption that only finitely many $c(|D|)$ are non-zero (which we are attempting to disprove). Combining this with (3.4) (evaluated at $k - \frac{1}{2} - s$) we find that

$$D_g(s) = R(s) \sum_{\nu=1}^{r} \lambda_\nu \frac{L(g_\nu | W_4, s+1)}{L(f_\nu, 2s+1)}, \tag{3.6}$$

where

$$R(s) = \pi^{k-\frac{1}{2}-2s} \frac{\Gamma(\frac{2s-k+1+\delta}{2})}{\Gamma(\frac{k-2s+\delta}{2})} \gamma(k - \tfrac{1}{2} - s) = (-1)^k \cdot \frac{\Gamma(k-s)}{\Gamma(\frac{k+\delta}{2} - s)} \cdot \frac{\Gamma(s + \frac{1+\delta-k}{2})}{\Gamma(\frac{1}{2}+s)}.$$

Since k and δ have the same parity, $(k \pm \delta)/2$ is always an integer, and so $R(s)$ is a rational function of s, being the ratio of two polynomials of degree $(k-\delta)/2$.

If $\sigma = \mathrm{Re}(s)$ is large, then using the Hecke bound for Fourier coefficients of cusp forms we see that $L(g_\nu | W_4, s+1)$ is given by an absolutely convergent Dirichlet series. Further, in such a half-plane, using the Euler product, $1/L(f_\nu, 2s+1)$ is also given by an absolutely convergent Dirichlet series. Thus, we may view (3.6) as

$$D_g(s) = R(s) E_g(s), \tag{3.7}$$

where $D_g(s)$ and $E_g(s)$ are both Dirichlet series in s, absolutely convergent in some half plane.

We are now ready to establish our claim that $D_g(s)$ must be identically zero. Suppose not, and consider the relation (3.7) for large real numbers s. For large real s, we have

$$D_g(s) = am^{-s} + O((m+1)^{-s}),$$

where $a \neq 0$ and m^{-s} is the first non-zero term in the Dirichlet series for $D_g(s)$. Similarly

$$E_g(s) = bn^{-s} + O((n+1)^{-s}),$$

for large real s (with $b \neq 0$), and so

$$\frac{a}{m^s} \left(1 + O\left(\left(\frac{m}{m+1} \right)^s \right) \right) = R(s) \cdot \frac{b}{n^s} \left(1 + O\left(\left(\frac{n}{n+1} \right)^s \right) \right).$$

Since $R(s) \to (-1)^{k+\frac{k-\delta}{2}}$ as $s \to \infty$, clearly we must have $m = n$ (and $a = (-1)^{k+\frac{k-\delta}{2}} b$). But then we must have

$$R(s) = (-1)^{k+\frac{k-\delta}{2}} + O\left(\left(\frac{n}{n+1} \right)^s \right),$$

which forces the rational function $R(s)$ to be a constant. Visibly this is a contradiction, and we conclude that $D_g(s)$ is identically zero.

3.2. **Deducing that $g = 0$.** The first stage of our proof has established that $c(|D|) = 0$ for all fundamental discriminants D with $(-1)^k D > 0$. It remains now to establish that g is identically zero, or in other words $\lambda_\nu = 0$ for all ν.

Since $D_g(s)$ is identically zero it follows from (3.6) that

$$\sum_{\nu=1}^{r} \lambda_\nu \frac{L(g_\nu | W_4, s+1)}{L(f_\nu, 2s+1)} = 0 \tag{3.8}$$

for all s. Precisely, we have the relation above for all s not equalling a zero of the rational function $R(s)$, but there are only finitely many such s, and by analytic continuation the relation must hold for all s.

Since g_ν is in the plus subspace we have

$$g_\nu | U_4 W_4 = \Big(\frac{2}{2k+1}\Big) 2^k g_\nu,$$

where U_4 is the operator acting on power series by $\sum_{n\ge 0} c(n)q^n | U_4 = \sum_{n\ge 0} c(4n) q^n$ (see [11] p. 250, and here $(\frac{2}{2k+1})$ denotes the Jacobi symbol). Since W_4 is an involution, applying W_4 to both sides of the above relation we find that

$$g_\nu | U_4 = g_\nu | U_4 W_4 W_4 = \Big(\frac{2}{2k+1}\Big) 2^k g_\nu | W_4.$$

Thus, replacing also $s+1$ with s, we may rewrite (3.8) as

$$\sum_{\nu=1}^{r} \lambda_\nu \frac{L(g_\nu|U_4, s)}{L(f_\nu, 2s-1)} = 0. \tag{3.9}$$

Now recalling the definition of the U_4 operator, and the Euler product for $L(f_\nu, 2s-1)$ we see that for σ sufficiently large

$$\frac{L(g_\nu|U_4, s)}{L(f_\nu, 2s-1)} = \Big(\sum_{n=1}^{\infty} \frac{c_\nu(4n)}{n^s}\Big) \prod_p \Big(1 - \frac{a_\nu(p)}{p^{2s-1}} + \frac{p^{2k-1}}{p^{4s-2}}\Big).$$

Now let D be a fundamental discriminant with $4|D$ and with $(-1)^k D > 0$, and let p be an odd prime. The coefficient of $(|D|p^2/4)^{-s}$ in the Dirichlet series above equals

$$\begin{aligned} c_\nu(|D|p^2) - c_\nu(|D|)\cdot p a_\nu(p) &= c_\nu(|D|)(a_\nu(p) - \chi_D(p)p^{k-1}) - c_\nu(|D|)\cdot p a_\nu(p) \\ &= c_\nu(|D|)\big(a_\nu(p)(1-p) - \chi_D(p)p^{k-1}\big), \end{aligned}$$

where we used the Shimura relation in the middle identity above. From (3.9), and since $c(|D|) = 0$ as we have already established, we find

$$0 = \sum_{\nu=1}^{r} \lambda_\nu c_\nu(|D|)\big(a_\nu(p)(1-p) - \chi_D(p)p^{k-1}\big) = (1-p)\sum_{\nu=1}^{r} \lambda_\nu c_\nu(|D|) a_\nu(p).$$

In other words, we conclude that for all fundamental discriminants D satisfying $4|D$ and $(-1)^k D > 0$, and all odd primes p we have

$$\sum_{\nu=1}^{r} \lambda_\nu c_\nu(|D|) a_\nu(p) = 0. \tag{3.10}$$

Recall that our goal is to show that all the λ_ν must be zero. Suppose not, and (without loss of generality) that $\lambda_1 \neq 0$. We know from ([11], p. 260) that for each ν we can find a fundamental discriminant D_ν with $4|D_\nu$ and $(-1)^k D_\nu > 0$, and such that $c_\nu(|D_\nu|) \neq 0$. Apply (3.10) taking $D = D_1$ there, multiply the relation by $\overline{a_1(p)} p^{1-2k} \log p$, and sum over all $3 \le p \le x$. Then

$$0 = \sum_{\nu=1}^{r} \lambda_\nu c_\nu(|D_1|) \sum_{3\le p\le x} a_\nu(p) \frac{\overline{a_1(p)}}{p^{2k-1}} \log p \sim \lambda_1 c_1(|D_1|) x,$$

by the Rankin–Selberg estimates (2.8) and (2.9). This contradiction completes our proof.

4. Large values of Fourier coefficients

In this section, we begin our proof of part (b) of Theorem 1.1. Using Waldspurger's formula, we recast the problem in terms of producing large values of a particular L-function while keeping other L-values small; see Theorem 4.1 below. We then show how to deduce Theorem 4.1 from two technical propositions, which will be established in the following sections.

Let $g = \sum_\nu \lambda_\nu g_\nu$ be a non-zero element in $S^+_{k+\frac{1}{2}}$. Assume without loss of generality that $\lambda_1 = 1$, and that $|\lambda_\nu| \le 1$ for all $\nu = 2, \dots, r$. By the triangle inequality and Cauchy-Schwarz, we obtain

$$|c(|D|)| \ge |c_1(|D|)| - \sum_{\nu=2}^{r} |c_\nu(|D|)| \ge |c_1(|D|)| - \sqrt{r-1}\Big(\sum_{\nu=2}^{r} |c_\nu(|D|)|^2\Big)^{\frac{1}{2}}.$$

Applying Waldspurger's formula (2.6), it follows that

$$|c(|D|)| \ge \big(C_1|D|^{k-\frac{1}{2}} L(f_1,\chi_D,k)\big)^{\frac{1}{2}} - C\Big(|D|^{k-\frac{1}{2}} \sum_{\nu=2}^{r} L(f_\nu,\chi_D,k)\Big)^{\frac{1}{2}}, \tag{4.1}$$

where C_1 is as in (2.7) (with $\nu = 1$ there), and $C > 0$ is a constant (depending on f_ν, g_ν, but independent of D). Theorem 1.1 may now be deduced from the following result, which exhibits large values of $L(f_1, \chi_D, k)$ while controlling $L(f_\nu, \chi_D, k)$ for $\nu = 2, \dots, r$.

Theorem 4.1. *Let $A > 0$ be a constant, and let X be large. For any $\epsilon > 0$, there are $\gg X^{1-\epsilon}$ fundamental discriminants D with $X < (-1)^k D \le 2X$ such that*

$$L(f_1,\chi_D,k) > A\sum_{\nu=2}^{r} L(f_\nu,\chi_D,k) + \exp\Big(\frac{1}{40}\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big).$$

To establish Theorem 4.1 we shall use the resonance method. Let D be fundamental discriminant with $X < (-1)^k D \le 2X$. We consider the following special value of a "resonator" Dirichlet polynomial at $k - \frac{1}{2}$:

$$R(D) = \sum_{n\le N} r(n)\frac{a_1(n)}{n^{k-\frac{1}{2}}}\chi_D(n), \tag{4.2}$$

where $N = X^{\frac{1}{24}}$ and $r(n)$ is a multiplicative function defined as follows. Set $r(n) = 0$ unless n is square-free, and for primes p define, with $L = \frac{1}{8}\sqrt{\log N \log\log N}$

$$r(p) = \begin{cases} \frac{L}{\sqrt{p}\log p} & \text{if } L^2 \le p \le L^4 \\ 0 & \text{otherwise.}\end{cases} \tag{4.3}$$

The proof of Theorem 4.1 will be based on the following two propositions.

Proposition 4.2. *With notations as above, we have*

$$\sum_{\substack{X<(-1)^k D\le 2X \\ D\equiv 1 \mod 4}} |R(D)|^2 \le \frac{2X}{\pi^2}\mathcal{R} + O(X), \tag{4.4}$$

where

$$\mathcal{R} = \prod_{L^2 \le p \le L^4} \Big(1 + r(p)^2 \frac{a_1(p)^2}{p^{2k-1}}\Big). \tag{4.5}$$

Further

$$\sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} |R(D)|^6 \ll X \exp\Big(O\Big(\frac{\log X}{\log\log X}\Big)\Big). \tag{4.6}$$

Proposition 4.3. *With notations as above, we have*

$$\sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} L(f_1, \chi_D, k)|R(D)|^2 \gg X\mathcal{R} \exp\Big(\Big(\frac{1}{2} + o(1)\Big)\frac{L}{\log L}\Big), \tag{4.7}$$

while for all $2 \le \nu \le r$

$$\sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} L(f_\nu, \chi_D, k)|R(D)|^2 \ll X\mathcal{R} \exp\Big(o\Big(\frac{L}{\log L}\Big)\Big). \tag{4.8}$$

We postpone the proof of these propositions to the next two sections, showing now how to deduce Theorem 4.1 from them.

Proof of Theorem 4.1. Let $\mathcal{S}$ denote the set of fundamental discriminants D with $X < (-1)^k D \le 2X$ and $D \equiv 1 \mod 4$ with

$$L(f_1, \chi_D, k) > A \sum_{\nu=2}^{r} L(f_\nu, \chi_D, k) + \exp\Big(\frac{1}{40}\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big).$$

Note that

$$\begin{aligned} \sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} L(f_1, \chi_D, k)|R(D)|^2 \le & \sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} \Big(A \sum_{\nu=2}^{r} L(f_\nu, \chi_D, k) + \exp\Big(\frac{1}{40}\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big)\Big)|R(D)|^2 \\ & + \sum_{D \in \mathcal{S}} L(f_1, \chi_D, k)|R(D)|^2. \end{aligned} \tag{4.9}$$

Now (4.7) gives a lower bound for the left side above,

$$\begin{aligned} \sum_{\substack{X<(-1)^k D \le 2X \\ D \equiv 1 \mod 4}} L(f_1, \chi_D, k)|R(D)|^2 &\gg X\mathcal{R} \exp\Big(\Big(\frac{1}{2} + o(1)\Big)\frac{L}{\log L}\Big) \\ &= X\mathcal{R} \exp\Big(\Big(\frac{1}{8\sqrt{24}} + o(1)\Big)\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big), \end{aligned}$$

while (4.8) and (4.4) show that the first sum on the right side of (4.9) is negligible in comparison. Thus we may conclude that

$$\sum_{D \in \mathcal{S}} L(f_1, \chi_D, k)|R(D)|^2 \gg X\mathcal{R} \exp\Big(\frac{1}{40}\frac{\sqrt{\log X}}{\sqrt{\log\log X}}\Big). \tag{4.10}$$

Applications of the Cauchy-Schwarz and Hölder inequalities show that the left side above is

$$\le \Big(\sum_{\substack{X<(-1)^k D\le 2X \\ D\equiv 1 \bmod 4}} L(f_1,\chi_D,k)^2\Big)^{\frac12} \Big(\sum_{D\in\mathcal{S}} |R(D)|^4\Big)^{\frac12} \ll (X^{1+\epsilon})^{\frac12} \Big(|\mathcal{S}|\Big)^{\frac16} \Big(\sum_{\substack{X<(-1)^k D\le 2X \\ D\equiv 1 \bmod 4}} |R(D)|^6\Big)^{\frac13}$$
$$\ll X^{\frac56+\epsilon} |\mathcal{S}|^{\frac16}.$$

Here we made use of (4.6) to bound the sum involving $|R(D)|^6$, and used the Perelli–Pomykała bound [15] (obtained from Heath-Brown's large sieve for quadratic characters) of $X^{1+\epsilon}$ for the second moment of L-values. Theorem 4.1 follows. □

We remark that the second moment of the central L-values should be of size $X \log X$, which would lead to a better quantification for the number of large values produced in Theorems 1.1 and 4.1. When we initially wrote the paper, this second moment remained barely out of reach of current technology, with an asymptotic being known assuming GRH (see [21]). The recent work of Xiannan Li [22] establishes the desired asymptotic for the second moment, and thus would allow the $\gg X^{1-\epsilon}$ in Theorem 4.1 to be replaced by $X^{1-C/\log\log X}$ for a suitable constant $C > 0$.

5. Proof of Proposition 4.2

Lemma 5.1. *Let $u \le X$ be an odd natural number. If u is a square then*

$$\sum_{\substack{X<(-1)^k D\le 2X \\ D\equiv 1 \bmod 4}} \chi_D(u) = \frac{X}{2\zeta(2)} \prod_{p|2u} \Big(\frac{p}{p+1}\Big) + O(X^{\frac12+\epsilon} u^{\frac14}),$$

while if u is not a square then

$$\sum_{\substack{X<(-1)^k D\le 2X \\ D\equiv 1 \bmod 4}} \chi_D(u) \ll X^{\frac12+\epsilon} u^{\frac14}.$$

Proof. Let χ_0 and χ_{-4} denote the principal and non-principal characters $\bmod 4$. Note that, for any non-zero integer D,

$$\frac12 (\chi_0(D)+\chi_{-4}(D)) \sum_{\substack{a^2|D \\ a \text{ odd}}} \mu(a) = \begin{cases} 1 & \text{if } D\equiv 1 \bmod 4 \text{ is square-free},\\ 0 & \text{otherwise.}\end{cases}$$

Thus writing $D = a^2 b$ with b square-free, we see that

$$\sum_{\substack{X<(-1)^k D\le 2X \\ D\equiv 1 \bmod 4}} \chi_D(u) = \frac12 \sum_{\substack{a\le\sqrt{2X} \\ a \text{ odd}}} \mu(a) \sum_{X/a^2<(-1)^k b\le 2X/a^2} (\chi_0(b)+\chi_{-4}(b)) \Big(\frac{a^2 b}{u}\Big).$$

If u is not a square, then $\chi_0(\cdot)(\frac{\cdot}{u})$ and $\chi_{-4}(\cdot)(\frac{\cdot}{u})$ are both non-principal Dirichlet characters to the modulus $4u$. Therefore, using the Pólya–Vinogradov bound we obtain

$$\sum_{X/a^2<(-1)^k b\le 2X/a^2} (\chi_0(b)+\chi_{-4}(b)) \Big(\frac{a^2 b}{u}\Big) \ll \min\Big(\sqrt{u}\log(4u), \frac{X}{a^2}\Big).$$

Here the bound $\sqrt{u}\log(4u)$ comes from Pólya–Vinogradov, and the bound X/a^2 by estimating the sum over b trivially. Therefore in this case we obtain

$$\sum_{\substack{X<(-1)^k D\le 2X\\ D\equiv 1 \mod 4}} \chi_D(u) \ll \sum_{\substack{a\le\sqrt{2X}\\ a \text{ odd}}} \min\Big(\sqrt{u}\log(4u), \frac{X}{a^2}\Big) \ll \sum_{\substack{a\le\sqrt{2X}\\ a \text{ odd}}} \Big(\sqrt{u}\log(4u)\frac{X}{a^2}\Big)^{\frac12} \ll X^{\frac12+\epsilon}u^{\frac14}.$$

If u is a square, then $\chi_0(\cdot)(\frac{\cdot}{u})$ is a principal character (the contribution of the non-principal character $\chi_{-4}(\cdot)(\frac{\cdot}{u})$ is negligible as above), which contributes

$$\frac12 \sum_{\substack{a\le\sqrt{2X}\\ (a,2u)=1}} \mu(a) \sum_{\substack{X/a^2<(-1)^k b\le 2X/a^2\\ (b,2u)=1}} 1 = \frac12 \sum_{\substack{a\le\sqrt{2X}\\ (a,2u)=1}} \mu(a)\Big(\frac{X}{a^2}\frac{\phi(2u)}{2u} + O(u^\epsilon)\Big)$$
$$= \frac{X}{2\zeta(2)}\prod_{p|2u}\Big(\frac{p}{p+1}\Big) + O(X^{\frac12+\epsilon}).$$

This completes the proof of the lemma. □

We are now ready to prove Proposition 4.2. Expanding out the definition of $R(D)$, we obtain

$$\sum_{\substack{X<(-1)^k D\le 2X\\ D\equiv 1 \mod 4}} |R(D)|^2 = \sum_{n_1,n_2\le N} r(n_1)r(n_2)\frac{a_1(n_1)}{n_1^{k-\frac12}}\frac{a_1(n_2)}{n_2^{k-\frac12}} \sum_{\substack{X<(-1)^k D\le 2X\\ D\equiv 1 \mod 4}} \chi_D(n_1n_2),$$

and we now use Lemma 5.1 to estimate the sum over D. Since $r(n)=0$ unless n is odd and square-free, $|r(n)|\le 1$ always, and $|a_1(n)|/n^{k-\frac12}\le d(n) \ll n^\epsilon$, we see that the error terms arising from Lemma 5.1 contribute

$$\ll X^{\frac12+\epsilon}\sum_{n_1,n_2\le N}(n_1n_2)^{\frac14+\epsilon} \ll X^{\frac12+\epsilon}N^{\frac52+\epsilon}.$$

The main term in Lemma 5.1 arises when n_1n_2 is a square, and since n_1 and n_2 are both square-free, this means that $n_1=n_2$. Thus the main term is

$$\frac{X}{2\zeta(2)}\sum_{n\le N} r(n)^2\frac{a_1(n)^2}{n^{2k-1}}\prod_{p|2n}\Big(\frac{p}{p+1}\Big) \le \frac{2X}{\pi^2}\prod_{L^2\le p\le L^4}\Big(1+r(p)^2\frac{a_1(p)^2}{p^{2k-1}}\frac{p}{p+1}\Big) \le \frac{2X}{\pi^2}\mathcal{R},$$

upon extending the sum over n to all natural numbers, and recalling the definition of the multiplicative function r. This proves (4.4).

The proof of (4.6) is similar. We expand out $R(D)^6$ and use Lemma 5.1. The error terms that arise are bounded now by

$$\ll X^{\frac12+\epsilon}\sum_{n_1,\dots,n_6\le N}(n_1\cdots n_6)^{\frac14+\epsilon} \ll X^{\frac12+\epsilon}N^{\frac{15}{2}+\epsilon}.$$

The main term arises from terms with $n_1\cdots n_6$ being a square. For such terms note that $a_1(n_1)\cdots a_1(n_6)$ is always non-negative (since n_i are all square-free, a_1 is a multiplicative function, and each prime dividing $n_1\cdots n_6$ divides an even number of n_i). Thus the main term is

$$\frac{2X}{\pi^2}\sum_{\substack{n_1,\dots,n_6\le N\\ n_1\cdots n_6=\square}} r(n_1)\cdots r(n_6)\frac{a_1(n_1)\cdots a_1(n_6)}{(n_1\cdots n_6)^{k-\frac12}}\prod_{p|n_1\cdots n_6}\Big(\frac{p}{p+1}\Big).$$

Extending the sum over n_i to infinity, and using multiplicativity, the above is

$$\ll X \prod_{L^2 \le p \le L^4} \Big(1 + \binom{6}{2} r(p)^2 \frac{a(p)^2}{p^{2k-1}} + \binom{6}{4} r(p)^4 \frac{a(p)^4}{p^{4k-2}} + \binom{6}{6} r(p)^6 \frac{a(p)^6}{p^{6k-3}}\Big)$$

$$\ll X \exp\Big(O\Big(\sum_{L^2 \le p \le L^4} \frac{L^2}{p(\log p)^2}\Big)\Big) \ll X \exp\Big(O\Big(\frac{\log X}{\log\log X}\Big)\Big).$$

This completes the proof of Proposition 4.2.

6. Proof of Proposition 4.3

Lemma 6.1. *Let u be an odd positive integer, and write $u = u_1 u_2^2$ with u_1 square-free. Let Φ denote a smooth function compactly supported in $[1/2, 5/2]$, and with $0 \le \Phi(t) \le 1$ for all t. Then*

$$\sum_{\substack{(-1)^k D > 0 \\ D \equiv 1 \mod 4}} \chi_D(u) L(f_\nu, \chi_D, k) \Phi\Big(\frac{|D|}{X}\Big) = A_\nu h_\nu(u) \Big(\int_0^\infty \Phi(t)dt\Big) \frac{a_\nu(u_1)}{u_1^k} X + O(X^{\frac{7}{8}+\epsilon} u^{\frac{3}{8}}),$$

where A_ν is a non-zero constant, and h_ν is a multiplicative function with $h_\nu(p^t) = 1 + O(1/p)$ for prime powers p^t.

Proof. This is a variant of Proposition 2 of [16] which treats the case of quadratic twists of an elliptic curve. Indeed Proposition 2 of [16] is a little more general in allowing the discriminants D to lie in a given progression modulo the level, and also to restrict D to be multiples of another parameter v. Only minor modifications to that argument are needed to handle eigenforms of weight k instead of elliptic curves. The techniques involved are based on earlier work in the family of quadratic twists, see [8, 20, 21]. Very briefly, we start with an "approximate functional equation"

$$L(f_\nu, \chi_D, k) = 2 \sum_{n=1}^\infty \frac{a_\nu(n)}{n^k} \chi_D(n) W\Big(\frac{n}{|D|}\Big),$$

for a suitable weight function $W(\xi)$, which is approximately 1 for small ξ and decays rapidly as $\xi \to \infty$. Then the sum we wish to evaluate equals

$$2 \sum_{n=1}^\infty \frac{a_\nu(n)}{n^k} \sum_{\substack{(-1)^k D > 0 \\ D \equiv 1 \mod 4}} \chi_D(un) W\Big(\frac{n}{|D|}\Big) \Phi\Big(\frac{|D|}{X}\Big).$$

The main terms arise from the case when un is a perfect square, and the contribution of all other terms can be bounded as in [16]. Since $u = u_1 u_2^2$, the condition un being a square amounts to writing $n = u_1 m^2$, and so the main term equals

$$2 \sum_{m=1}^\infty \frac{a_\nu(u_1 m^2)}{u_1^k m^{2k}} \sum_{\substack{(-1)^k D > 0 \\ D \equiv 1 \mod 4 \\ (D, u_1 u_2 m) = 1}} W\Big(\frac{u_1 m^2}{|D|}\Big) \Phi\Big(\frac{|D|}{X}\Big).$$

Evaluating the sum over D asymptotically, we arrive at a main term

$$X\Big(\int_0^\infty \Phi(t)dt\Big)\frac{4}{\pi^2}\sum_{m=1}^\infty \frac{a_\nu(u_1m^2)}{u_1^k m^{2k}}\prod_{\substack{p|u_1u_2m\\ p>2}}\Big(\frac{p}{p+1}\Big).$$

Using the Hecke relations, this can be put in the form stated in the lemma, and we note that the constant A_ν is closely related to the value of the symmetric square L-function attached to f_ν evaluated at the edge of the critical strip; see Proposition 2 of [16] for further details. □

With this lemma in place, we are ready to evaluate

$$\sum_{\substack{(-1)^kD>0\\ D\equiv 1 \mod 4}} L(f_\nu,\chi_D,k)|R(D)|^2\Phi\Big(\frac{|D|}{X}\Big),$$

for $\nu=1,\ldots,r$, and Φ being a suitable approximation to the indicator function of $[1,2]$. Expanding out $|R(D)|^2$ and using Lemma 6.1 we see that the above equals

$$\begin{aligned}&A_\nu\Big(\int_0^\infty \Phi(t)dt\Big)X\sum_{n_1,n_2\le N} r(n_1)r(n_2)\frac{a_1(n_1)a_1(n_2)}{(n_1n_2)^{k-\frac12}}h_\nu(n_1n_2)\frac{a_\nu(n_1n_2/(n_1,n_2)^2)}{(n_1n_2/(n_1,n_2)^2)^k}\\ &\qquad + O\Big(X^{\frac78+\epsilon}\sum_{n_1,n_2\le N} r(n_1)r(n_2)\frac{|a_1(n_1)a_1(n_2)|}{(n_1n_2)^{k-\frac12}}(n_1n_2)^{\frac38}\Big).\end{aligned}\tag{6.1}$$

In deriving the above expression, we used that n_1 and n_2 are square-free (else $r(n_1)r(n_2)=0$) so that $n_1n_2=(n_1n_2/(n_1,n_2)^2)(n_1,n_2)^2$ with $n_1n_2/(n_1,n_2)^2$ being square-free.

Since $|a_1(n_1)|\le d(n_1)n_1^{k-\frac12}\ll n_1^{k-\frac12+\epsilon}$ by the Deligne bound, and $r(n_1)\le 1$ always, the error term in (6.1) may be bounded by

$$\ll X^{\frac78+\epsilon}N^{\frac{11}{4}+\epsilon}\ll X^{\frac{99}{100}},$$

which is acceptable.

We now analyze the main term in (6.1). First we extend the sums over n_1 and n_2 to all natural numbers and analyze this contribution, and then we show that the contribution of the terms with $\max(n_1,n_2)>N$ is negligible. When the terms over n_1, n_2 are extended to all natural numbers, the resulting sums are multiplicative in nature, and thus these give

$$A_\nu\Big(\int_0^\infty \Phi(t)dt\Big)X\prod_{L^2\le p\le L^4}\Big(1+2r(p)h_\nu(p)\frac{a_1(p)a_\nu(p)}{p^{2k-\frac12}}+r(p)^2h_\nu(p^2)\frac{a_1(p)^2}{p^{2k-1}}\Big).\tag{6.2}$$

In thinking of the Euler product above, the first term corresponds to n_1 and n_2 both not divisible by p, the middle term corresponds to exactly one of n_1 or n_2 being divisible by p, and the last term to both n_1 and n_2 being divisible by p.

Now it remains to show that the terms with $\max(n_1,n_2)>N$ (which are not present in (6.1) but included in (6.2)) contribute a negligible amount. These terms may be bounded by

$$\ll X \sum_{\max(n_1,n_2)>N} r(n_1)r(n_2)\frac{|a_1(n_1)a_1(n_2)|}{(n_1n_2)^{k-\frac{1}{2}}}|h_\nu(n_1n_2)|\frac{|a_\nu(n_1n_2/(n_1,n_2)^2)|}{(n_1n_2/(n_1,n_2)^2)^k}$$
$$\ll X \sum_{n_1,n_2=1}^{\infty} r(n_1)r(n_2)\frac{|a_1(n_1)a_1(n_2)|}{(n_1n_2)^{k-\frac{1}{2}}}|h_\nu(n_1n_2)|\frac{|a_\nu(n_1n_2/(n_1,n_2)^2)|}{(n_1n_2/(n_1,n_2)^2)^k}\Big(\frac{n_1n_2}{N}\Big)^{\alpha},$$

for any $\alpha > 0$. By multiplicativity the above equals

$$XN^{-\alpha}\prod_{L^2\le p\le L^4}\Big(1+2r(p)p^{\alpha}|h_\nu(p)|\frac{|a_1(p)a_\nu(p)|}{p^{2k-\frac{1}{2}}}+r(p)^2p^{2\alpha}|h_\nu(p^2)|\frac{a_1(p)^2}{p^{2k-1}}\Big).$$

Since $h_\nu(p^t) = 1 + O(1/p)$ and $|a_\nu(p)| \le 2p^{k-\frac{1}{2}}$ this is

$$\ll XN^{-\alpha}\exp\Big(\sum_{L^2\le p\le L^4}\Big(\frac{8Lp^{\alpha}}{p\log p}+\frac{4L^2p^{2\alpha}}{p(\log p)^2}\Big)\Big(1+O\Big(\frac{1}{p}\Big)\Big)\Big).$$

Upon choosing $\alpha = 1/(8\log L)$, and using the prime number theorem, the above is

$$\ll X\exp\Big(-\frac{\log N}{8\log L}+\frac{8L}{\log L}+\frac{2L^2}{(\log L)^2}\Big)\ll X,$$

recalling that $L = \frac{1}{8}\sqrt{\log N\log\log N}$.

From our work above we conclude that

$$\sum_{\substack{(-1)^kD>0\\ D\equiv 1 \bmod 4}} L(f_\nu,\chi_D,k)|R(D)|^2\Phi\Big(\frac{|D|}{X}\Big)$$

(6.3)

$$= A_\nu\Big(\int_0^\infty\Phi(t)dt\Big)X\prod_{L^2\le p\le L^4}\Big(1+2r(p)h_\nu(p)\frac{a_1(p)a_\nu(p)}{p^{2k-\frac{1}{2}}}+r(p)^2h_\nu(p^2)\frac{a_1(p)^2}{p^{2k-1}}\Big)+O(X).$$

Let us compare the product above with the product $\mathcal{R}$. For $L^2 \le p \le L^4$, note that (keeping in mind $r(p) = L/(\sqrt{p}\log p) \le 1/\log p$ is always small, that $h_\nu(p^t) = 1 + O(1/p)$, and that $|a_\nu(p)| \le 2p^{k-\frac{1}{2}}$)

$$\Big(1+2r(p)h_\nu(p)\frac{a_1(p)a_\nu(p)}{p^{2k-\frac{1}{2}}}+r(p)^2h_\nu(p^2)\frac{a_1(p)^2}{p^{2k-1}}\Big)\Big(1+r(p)^2\frac{a_1(p)^2}{p^{2k-1}}\Big)^{-1}$$
$$=1+2r(p)\frac{a_1(p)a_\nu(p)}{p^{2k-\frac{1}{2}}}+O\Big(\frac{r(p)^3}{\sqrt{p}}\Big)$$
$$=\exp\Big(2r(p)\frac{a_1(p)a_\nu(p)}{p^{2k-\frac{1}{2}}}+O\Big(\frac{r(p)^2}{\sqrt{p}}\Big)\Big).$$

Using the prime number theorem we conclude that the product in (6.3) equals

$$\mathcal{R}\exp\Big(\sum_{L^2\le p\le L^4}\Big(2r(p)\frac{a_1(p)a_\nu(p)}{p^{2k-\frac12}}+O\Big(\frac{r(p)^2}{\sqrt{p}}\Big)\Big)\Big)$$

$$=\mathcal{R}\exp\Big(\sum_{L^2\le p\le L^4}2r(p)\frac{a_1(p)a_\nu(p)}{p^{2k-\frac12}}+O\Big(\frac{L}{(\log L)^3}\Big)\Big). \tag{6.4}$$

We are now ready to prove Proposition 4.3. In the case $\nu=1$, take $1\ge\Phi(t)\ge 0$ to be a smooth function supported on $[1,2]$ with $\Phi(t)=1$ on $[1.1,1.9]$. Then

$$\sum_{\substack{X<(-1)^kD\le 2X\\ D\equiv 1 \mod 4}}L(f_1,\chi_D,k)|R(D)|^2\ge\sum_{\substack{(-1)^kD>0\\ D\equiv 1 \mod 4}}L(f_1,\chi_D,k)|R(D)|^2\Phi\Big(\frac{|D|}{X}\Big),$$

and from (6.3) and (6.4), we conclude that this is

$$\ge\frac45 A_1X\mathcal{R}\exp\Big(\sum_{L^2\le p\le L^4}2r(p)\frac{a_1(p)^2}{p^{2k-\frac12}}+O\Big(\frac{L}{(\log L)^3}\Big)\Big)+O(X).$$

Applying the Rankin–Selberg estimate (2.8) and partial summation, we obtain

$$\sum_{L^2\le p\le L^4}2r(p)\frac{a_1(p)^2}{p^{2k-\frac12}}=2L\sum_{L^2\le p\le L^4}\frac{a_1(p)^2}{p^{2k}\log p}=\Big(\frac12+o(1)\Big)\frac{L}{\log L},$$

from which (4.7) follows.

Now we turn to the case $\nu>1$, where we take $1\ge\Phi(t)\ge 0$ to be a smooth function compactly supported on $[1/2,5/2]$ and with $\Phi(t)=1$ on $[1,2]$. Now our work in (6.3) and (6.4) shows that

$$\sum_{\substack{X<(-1)^kD\le 2X\\ D\equiv 1 \mod 4}}L(f_\nu,\chi_D,k)|R(D)|^2\le\sum_{\substack{(-1)^kD>0\\ D\equiv 1 \mod 4}}L(f_\nu,\chi_D,k)|R(D)|^2\Phi\Big(\frac{|D|}{X}\Big)$$

$$\le 2A_\nu X\mathcal{R}\exp\Big(\sum_{L^2\le p\le L^4}2r(p)\frac{a_1(p)a_\nu(p)}{p^{2k-\frac12}}+O\Big(\frac{L}{(\log L)^3}\Big)\Big)+O(X).$$

Here the Rankin–Selberg estimate (2.9) and partial summation give

$$\sum_{L^2\le p\le L^4}2r(p)\frac{a_\nu(p)a_1(p)}{p^{2k-\frac12}}=o\Big(\frac{L}{\log L}\Big),$$

from which (4.8) follows.

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