

# RIEMANN-HILBERT HIERARCHIES FOR HARD EDGE PLANAR ORTHOGONAL POLYNOMIALS

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**ABSTRACT.** We obtain a full asymptotic expansion for orthogonal polynomials with respect to weighted area measure on a Jordan domain  $\mathcal{D}$  with real-analytic boundary. The weight is fixed and assumed to be real-analytically smooth and strictly positive, and for any given precision  $\varkappa$ , the expansion holds with an  $O(N^{-\varkappa-1})$  error in  $N$ -dependent neighborhoods of the exterior region as the degree  $N$  tends to infinity. The main ingredient is the derivation and analysis of Riemann-Hilbert hierarchies – sequences of scalar Riemann-Hilbert problems – which allows us to express all higher order correction terms in closed form. Indeed, the expansion may be understood as a Neumann series involving an explicit operator. The expansion theorem leads to a semiclassical asymptotic expansion of the corresponding hard edge probability wave function in terms of distributions supported on  $\partial\mathcal{D}$ .

## 1. INTRODUCTION AND MAIN RESULTS

**1.1. Weighted planar orthogonal polynomials.** Denote by  $\mathcal{D}$  a bounded Jordan domain with analytic boundary in the complex plane  $\mathbb{C}$ , and fix a non-negative continuous weight function  $\omega$  on  $\mathcal{D}$  such that  $\log \omega$  extends to a real-analytically smooth function in a neighborhood of the boundary  $\partial\mathcal{D}$ . We denote by  $dA$  the standard area element  $dA(z) := (2\pi i)^{-1} dz \wedge d\bar{z}$ , and by  $d\sigma$  the arc length element  $d\sigma(z) := (2\pi)^{-1} |dz|$ , where we have chosen the normalizations so that the unit disk  $\mathbb{D}$  and the unit circle  $\mathbb{T}$  have unit area and length, respectively. The standard  $L^2$ -space with respect to the measure  $1_{\mathcal{D}}\omega dA$  is denoted by  $L^2(\mathcal{D}, \omega)$ .

We consider analytic polynomials in the complex plane

$$(1.1.1) \quad P(z) = c_N z^N + c_{N-1} z^{N-1} + \dots + c_0,$$

where the coefficients  $c_0, c_1, \dots, c_N$  are complex numbers. If the coefficient  $c_N$  is non-zero,  $P$  is said to have degree  $N$ , and we refer to  $c_N$  as the *leading coefficient* of  $P$ . If  $c_N = 1$ , the polynomial  $P$  is called *monic*. The space of all polynomials of the form (1.1.1) is denoted by  $\text{Pol}_N$ , and we supply it with the Hilbert space structure of  $L^2(\mathcal{D}, \omega dA)$ . The resulting space is the polynomial Bergman space, denoted  $\text{Pol}_N^2(\mathcal{D}, \omega)$ . Note that the dimension of  $\text{Pol}_N$  equals  $N + 1$ .

We now define the system  $(P_N(z))_{N \in \mathbb{N}}$  of *normalized planar orthogonal polynomials (ONPs)* with respect to the measure  $1_{\mathcal{D}}\omega dA$  recursively by applying the standard Gram-Schmidt algorithm to the sequence  $(z^N)_{N \in \mathbb{N}}$  of monomials. The normalization

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condition means that  $\|P_N\|_{L^2(\mathcal{D}, \omega)} = 1$ , and in addition we ask that the leading coefficient  $\kappa_N$  of  $P_N$  is positive. We also consider the *monic orthogonal polynomial of degree  $N$* , denoted  $\pi_N$ , so that  $P_N = \kappa_N \pi_N$ . These orthogonal polynomials are variously referred to as *Bergman polynomials* and *Carleman polynomials* in the literature, the latter referring mainly to the constant weight case.

We obtain a full asymptotic description of the polynomials  $\pi_N(z)$  and  $P_N(z)$  as the degree  $N$  tends to infinity in a shrinking neighborhood of the exterior region  $\mathbb{C} \setminus \mathcal{D}$ . We remark that our results do not give detailed asymptotic information about the orthogonal polynomials deep inside  $\mathcal{D}$ , where the zeros are located. To obtain such a description would require methods that go beyond the present work.

**1.2. Asymptotic expansion of orthogonal polynomials.** Denote by  $V$  the Szegő function of  $\omega$  relative to  $\mathcal{D}$ , defined as the unique outer function  $V$  on  $\mathbb{C} \setminus \mathcal{D}$  which satisfies  $2\operatorname{Re} V = -\log \omega$  on  $\partial\mathcal{D}$  and is real-valued at infinity. We denote by  $\varphi$  the conformal mapping of  $\mathbb{C} \setminus \mathcal{D}$  onto the exterior disk  $\mathbb{D}_e := \{z \in \mathbb{C} : |z| > 1\}$ , normalized so that the point at infinity is preserved with  $\varphi'(\infty) > 0$ ; we call such a conformal map *orthostatic*. In fact, it is known that  $\varphi'(\infty) = \frac{1}{\operatorname{cap}(\mathcal{D})}$ , where  $\operatorname{cap}(\mathcal{D})$  is the logarithmic capacity of  $\mathcal{D}$ . Due to our smoothness assumptions, both the functions  $V$  and  $\varphi$  extend holomorphically across the boundary  $\partial\mathcal{D}$ , the latter in addition as a univalent function.

**Theorem 1.2.1** (Pointwise asymptotic expansion). *There exist bounded holomorphic functions  $B_j$  with  $B_0 \equiv 1$  and  $B_j(\infty) = 0$  for  $j \geq 1$ , defined in an open neighborhood of  $\mathbb{C} \setminus \mathcal{D}$  such for any given  $\varkappa \in \mathbb{N}$  and  $A > 0$ , the monic orthogonal polynomial  $\pi_N$  admits the asymptotic expansion*

$$\pi_N(z) = C_N \varphi'(z) \varphi^N(z) e^{V(z)} \left( \sum_{j=0}^{\varkappa} N^{-j} B_j(z) + O(N^{-\varkappa-1}) \right)$$

as  $N$  tends to infinity, valid for all  $z$  with  $\operatorname{dist}_{\mathbb{C}}(z, \mathcal{D}^c) \leq AN^{-1} \log N$ . Here, the constant  $C_N$  is given by

$$C_N = (\varphi'(\infty))^{-N-1} e^{-V(\infty)} = \operatorname{cap}(\mathcal{D})^{N+1} \exp\left(\frac{1}{2} \int_{\mathbb{T}} \log \omega \circ \varphi^{-1} d\sigma\right)$$

for  $N \geq 1$ .

*Remark 1.2.2.* A corresponding asymptotic formula for the *normalized orthogonal polynomial* is obtained by recalling  $P_N = \kappa_N \pi_N$ . The leading coefficient  $\kappa_N$  is given by  $\kappa_N = C_N^{-1} D_N$ , where  $D_N = 1 + \sum_{j=1}^{\varkappa} N^{-j} d_j + O(N^{-\varkappa-1})$  for some sequence  $(d_j)_{j \geq 1}$  of real constants.

The pointwise asymptotic expansion of Theorem 1.2.1 is derived from an  $L^2$ -version of the asymptotic expansion. Denote by  $\chi_0$  an appropriate smooth cut-off function which vanishes deep inside  $\mathcal{D}$  but equals one in a fixed neighborhood of  $\mathbb{C} \setminus \mathcal{D}$ , say, on  $\mathcal{D}_\rho := \mathbb{C} \setminus \varphi^{-1}(\mathbb{D}_e(0, \rho))$  for some constant  $0 < \rho < 1$ . Here and in the sequel, we use the notation  $\mathbb{D}_e(0, \rho) = \{z : |z| > \rho\}$ . We put

$$F_N(z) = D_N \varphi'(z) \varphi^N(z) e^{V(z)} \sum_{j=0}^{\varkappa} N^{-j} B_j(z),$$

where the constant  $D_N$  is as in Remark 1.2.2.

**Theorem 1.2.3** (Asymptotic expansion in the  $L^2$ -sense). *If  $\chi_0$  and  $F_N$  are as above, then for any given  $\varkappa \in \mathbb{N}$  we have*

$$\int_{\mathcal{D}} |P_N(z) - \chi_0(z)F_N(z)|^2 \omega(z) dA = O(N^{-\varkappa-1}),$$

as  $N \rightarrow +\infty$ .

As an application of Theorems 1.2.1 and 1.2.3, we obtain an asymptotic expansion of the wave function  $1_{\mathcal{D}}|P_N|^2\omega$  as  $N \rightarrow +\infty$  in terms of distributions supported on the boundary  $\partial\mathcal{D}$ . For the details, see Theorem 2.1.1 below.

**1.3. Algorithmic aspects and Riemann-Hilbert hierarchies.** We define the operator  $\mathbf{T}$  by

$$(1.3.1) \quad \mathbf{T} := \mathbf{M}_{\Omega}^{-1}(z\partial_z + \mathbf{I})\mathbf{M}_{\Omega},$$

so that

$$\mathbf{T}f(z) = \frac{1}{\Omega(z)}(z\partial_z + \mathbf{I})(f(z)\Omega(z)),$$

where  $\mathbf{M}_{\Omega}$  stands for the operator of multiplication by the modified weight  $\Omega = e^{2\operatorname{Re} V \circ \varphi^{-1}} \omega \circ \varphi^{-1}$ , which has  $\Omega|_{\mathbb{T}} = 1$ . Note that if  $f$  is  $C^\infty$ -smooth (or  $C^\omega$ -smooth) in a neighborhood of  $\mathbb{T}$ , then the same holds for  $\mathbf{T}f$ . Let  $\mathbf{P}$  be the orthogonal projection of  $L^2(\mathbb{T})$  onto the conjugate Hardy space  $H_{-,0}^2$  with constants removed, and let  $\mathbf{R} = \mathbf{R}_{\mathbb{T}}$  denote the restriction to the unit circle. We put  $\mathbf{Q} = \mathbf{P}\mathbf{R}$ , and agree to think of  $\mathbf{Q}f$  as a holomorphic function on  $\overline{\mathbb{D}}_e$  for  $f$  real-analytically smooth in a neighborhood of the unit circle  $\mathbb{T}$ . As it turns out, the coefficients  $B_j$  are expressed in terms of these operators. This is done by deriving and solving recursively the sequence of *collapsed orthogonality conditions*

$$(1.3.2) \quad X_j \in H_{-,0}^2 \cap (-\Xi_j + H^2), \quad j = 1, 2, 3, \dots,$$

where  $\Xi_j$  is expressed in terms of previous coefficients:  $\Xi_j = \sum_{l \leq j-1} (-1)^{j-l} \mathbf{R}\mathbf{T}^{j-l} X_l$  for  $j \geq 1$ . The conditions (1.3.2) form a sequence of classical scalar Riemann-Hilbert problems with jump across the circle, which we refer to as a *Riemann-Hilbert hierarchy*. We return to this connection in Subsection 1.4 below. The solution of (1.3.2) is as follows.

**Theorem 1.3.1.** *The coefficient  $B_j$  is given by  $B_j = X_j \circ \varphi$ , where the function  $X_0$  is the constant  $X_0(z) \equiv 1$ , and for any  $j \geq 1$ ,  $X_j$  is determined by the recursive condition (1.3.2), with solution*

$$X_j = \mathbf{Q}\mathbf{T}[(\mathbf{Q} - \mathbf{I})\mathbf{T}]^{j-1} X_0, \quad j = 1, 2, 3, \dots$$

If we put  $S_N = S_{N,\varkappa} = \sum_{j \leq \varkappa} N^{-j} X_j$ , we have

$$S_N = X_0 + \frac{1}{N} \mathbf{Q}\mathbf{T} \sum_{j=0}^{\varkappa} N^{-j} [(\mathbf{Q} - \mathbf{I})\mathbf{T}]^j X_0.$$

*Remark 1.3.2.* (a) We realize that the above expression is a partial Neumann series, which suggests the formal identity

$$S_N \cong \sum_{k=0}^{\infty} N^{-k} X_k = \left[ \mathbf{I} + \frac{1}{N} \mathbf{Q}\mathbf{T} \left( I - \frac{1}{N} (\mathbf{Q} - \mathbf{I})\mathbf{T} \right)^{-1} \right] X_0.$$

(b) In Carleman's classical asymptotic formula for  $\Omega(z) \equiv 1$  (see Subsection 1.5 below), no higher order corrections  $B_j$  for  $j \geq 1$  appear. This may be seen from the following properties of the operators  $\mathbf{T}$  and  $\mathbf{Q}$ : the operator  $\mathbf{T}$  acts on constant

functions as the identity operator, while  $\mathbf{Q}$  annihilates the constants, and we start with  $X_0 \equiv 1$ .

(c) After  $X_0 \equiv 1$ , the first nontrivial term is given by  $X_1 = \mathbf{P}(z\partial \log \Omega(z)|_{z \in \mathbb{T}})$ . This allows us to express  $B_1 = X_1 \circ \varphi$  as  $B_1 = -(\partial_\rho V_\rho - \partial_\rho V_\rho(\infty))|_{\rho=1}$ , where  $V_\rho$  denotes the Szegő function of  $\omega$  relative to the domain  $\mathbb{C} \setminus \mathcal{D}_\rho$ . Here, the domain  $\mathcal{D}_\rho$  is the bounded Jordan domain whose boundary curve is implicitly defined by the condition that  $|\varphi(z)| = \rho$ .

The conditions (1.3.2) appear through asymptotic analysis of integrals of the type appearing in Theorem 2.1.1 below, whose mass concentrate to a small one-sided neighborhood of  $\partial\mathcal{D}$ . Although different in that it is local near a point, a related asymptotic analysis using Laplace's method appear in the asymptotic analysis of Bergman kernels in  $\mathbb{C}^d$  with exponentially varying weights, and more generally in Kähler geometry. In this connection we should mention the works of Engliš [20], Charles [12], Loi [35] and Xu [53].

**1.4. Connections with classical Riemann-Hilbert problems.** We return to why the collapsed orthogonality conditions (1.3.2) consist of scalar Riemann-Hilbert problems on the circle. For  $j \geq 1$  we consider the function

$$Z_j := \begin{cases} X_j & \text{on } \mathbb{D}_e, \\ \Xi_j + X_j & \text{on } \mathbb{D}, \end{cases}$$

where functions in  $H^2$  are thought of as holomorphic in  $\mathbb{D}$ , while functions in  $H^2_-$  are holomorphic in the exterior  $\mathbb{D}_e$ . As  $X_j \in H^2_{-,0}$  and  $\Xi_j + X_j \in H^2$  by (1.3.2), the function  $Z_j$  solves the scalar Riemann-Hilbert problem on the Riemann sphere with jump  $\Xi_j$  across the circle  $\mathbb{T}$  and normalization  $Z_j(\infty) = 0$ .

In addition, there is the approach of Its and Takhtajan [30] (see also [31]), which expresses planar orthogonal polynomials as solutions of a  $(2 \times 2)$ -matrix  $\bar{\partial}$ -problem, or a *soft Riemann-Hilbert problem*. This approach follows developments in the theory of orthogonal polynomials on the real line, which saw a major breakthrough with the introduction of matrix Riemann-Hilbert techniques and the Deift-Zhou steepest descent method, see [21], [22], [18], and [17]. In [27, Section 7] we discuss the connection between the Its-Takhtajan approach and the orthogonal foliation flow method (the latter is described in Subsection 1.7 below). A corresponding soft Riemann-Hilbert problem is readily formulated for fixed weights as well. In our smooth setting, the fact that we only need to solve scalar Riemann-Hilbert problems may be thought of as a kind of *diagonalization of the matrix  $\bar{\partial}$ -problem* (the  $\bar{\partial}$ -problems for  $\pi_N$  and  $\pi_{N-1}$  “disconnect”), which intuitively corresponds to having the zeros buried inside the domain  $\mathcal{D}$ . In settings with corners, cusps, or weights with singular boundary points, we would expect that the zeros protrude to the corresponding points on  $\partial\mathcal{D}$ . This suggests that a deeper understanding of the matrix  $\bar{\partial}$ -problem is necessary.

After the appearance of the first version of this paper, the soft Riemann-Hilbert approach to planar orthogonal polynomials with exponentially varying weights was developed further in [24] and [28].

We should also mention that the Deift-Zhou method has found applications in the context of planar orthogonal polynomials for particular exponentially varying weights, see e.g. [5, 6, 9, 33].

**1.5. Historical remarks.** The study of planar orthogonal polynomials begins with the pioneering work of Carleman [10] (see also the collected works edition [11]) where he studies the case with constant weight  $\omega \equiv 1$ . For this reason, when the weight is

constant,  $P_N$  is sometimes called the  $N$ -th weighted Carleman polynomial. Carleman was motivated by the contemporary result of Szegő on the orthogonal polynomials for arc-length measure on the boundary curve [49], described in the monograph [50]. For further developments in the weighted setting on the unit circle, we refer to Simon's monographs [44, 45].

In the unweighted case, Carleman finds the asymptotic formula

$$P_N(z) = (N+1)^{\frac{1}{2}} \varphi'(z) \varphi^N(z) (1 + O(\rho^N)), \quad z \in \mathbb{C} \setminus \mathcal{D}_{\rho_0}.$$

Here,  $\rho_0 < \rho < 1$  and  $\mathcal{D}_{\rho_0}$  is the image of  $\mathbb{D}_e(0, \rho_0)$  under the inverse mapping  $\varphi^{-1}$ , so that in particular the formula holds in a neighborhood of the closed exterior domain  $\mathbb{C} \setminus \mathcal{D}$ . Carleman's technique is based on a small miracle of Green's formula, which allows for switching the integration over  $\mathcal{D}$  to integration over the exterior domain. This miracle does not carry over to the weighted setting.

Later on, building on a modification of Carleman's theorem due to Korovkin [32], Suetin considers the case of *weighted* planar orthogonal polynomials, and shows that Carleman's formula generalizes appropriately. Suetin finds only the leading term, but compensates by allowing for lower degrees of smoothness (Hölder continuity). More recently, further improvement of Carleman's analysis in the unweighted case has been made possible by the efforts of several contributors, including Beckermann, Dragnev, Gustafsson, Levin, Lubinsky, Miña-Díaz, Putinar, Saff, Stahl, Stylianopoulos and Totik [7, 19, 23, 34, 36, 37, 41, 48]. These results help to give a more accurate description of the behavior of  $P_N$  deeper inside  $\mathcal{D}$  closer to the zeros, and alternatively allow for a lower degree of smoothness of  $\partial\mathcal{D}$  as well as a more complicated topology of the exterior domain. In addition, the works [23] and [41] highlight real-world applications of the study of planar orthogonal polynomials in the field of image analysis (domain recovery from complex moments).

To the best of our knowledge, no higher order correction term past  $B_0 \equiv 1$  in the context of Theorem 1.2.1 has been identified previously.

For particular domains and weights, more is known about the orthogonal polynomials. For the unit disk and weights of the form  $\omega = |p|^2$  where  $p$  is an analytic polynomial, Miña-Díaz has analyzed the orthogonal polynomials [38]. The analysis gives better asymptotics inside the disk compared with what we obtain in the present work. We could also mention the recent work [1], where Akemann-Nagao-Parra-Vernizzi study a one-parameter family of weights on ellipses, and identify the corresponding orthogonal polynomials as the classical Gegenbauer polynomials.

In light of the simple iterative nature of the formulæ of Theorem 1.3.1 and Remark 1.3.2 (b), the growth of  $B_j$  as  $j$  increases may be controlled. Likely, this control is strong enough to allow for our approach to be pushed to yield an exponentially decaying error term in a fixed neighborhood of  $\mathbb{C} \setminus \mathcal{D}$  as in Carleman's theorem. That would be analogous to the recent strengthening of the asymptotic expansions of Bergman kernels (e.g. by Tian, Catlin, Zelditch, and Berman-Berndtsson-Sjöstrand) by Rouby, Sjöstrand and Vu Ngoc [40] for real-analytic exponentially varying weights, improving on the methods of [8] (see also the more recent works of Hezari and Xu [29], Charles [13] and Deleporte, Hitrik and Sjöstrand [14]). If the expansion holds with such an error term, the zeros of  $P_N$  would consequently stay away from  $\partial\mathcal{D}$  for large  $N$ .

**1.6. Fixed vs varying weights.** In related work [27], [26], we obtain asymptotic expansions for orthogonal polynomials and partial Bergman kernels with respect to

exponentially varying planar measures  $e^{-2mQ}dA$ . That work was motivated by problems in random matrix theory (see, e.g., [52, 25, 2, 3, 4]) with relations to weighted potential theory (see the monographs by Saff-Totik [42] and Stahl-Totik [46]). While the general approach in the present work is somewhat analogous, there are important differences. Indeed, for the above variable weights the measure is supported on the entire plane, and a compact set  $\mathcal{S}_{n/m}$  where most of the mass of the *probability wave functions*  $|P_n|^2 e^{-2mQ}$  is concentrated, appears as the solution of a free boundary problem. The wave function associated to an orthonormal polynomial  $P_n = P_{n,mQ}$  takes the shape of a Gaussian ridge which peaks along the boundary  $\partial\mathcal{S}_\tau$  with  $\tau = n/m$ . In the present case, the domain  $\mathcal{D}$  is given and the probability wave functions  $1_{\mathcal{D}}|P_N|^2\omega$  are truncated at  $\partial\mathcal{D}$  and decay exponentially as we protrude into  $\mathcal{D}$ . In terms of the corresponding polynomial Bergman kernels and the associated determinantal Coulomb gas model, the truncation corresponds to confining the particles in the model to the domain  $\mathcal{D}$  with a hard edge (see Subsection 2.2 below).

From one point of view the present problem is more straightforward, as there is no free boundary problem. From the other point of view, the main method (the orthogonal foliation flow, see Subsection 1.7 below) needs to be adapted to more restrictive initial conditions, which requires new insight.

A related difference between exponentially varying and fixed weights is seen in the one-dimensional situation, as illustrated by the two seminal contributions [16] and [15] by Deift, Kriecherbauer, McLaughlin, Venakides and Zhou treating the cases of fixed and varying weights on the real line  $\mathbb{R}$ , respectively. In the fixed-weight case, they have no truncation to an interval, which would correspond to our domain  $\mathcal{D}$ , so the spectrum grows with  $N$ . The planar analogue of this global fixed-weight problem remains to be investigated.

**1.7. Outline of the main ideas.** To derive the coefficients in the asymptotic expansion, (Theorem 1.3.1) we use the fact that the mass of  $1_{\mathcal{D}}|P_N|^2\omega$  concentrates to a small (one-sided) neighborhood of  $\partial\mathcal{D}$ . In particular, the orthogonality conditions

$$\int_{\mathcal{D}} Q(z) \overline{P_N(z)} \omega(z) dA(z) = 0, \quad Q \in \text{Pol}_{N-1}$$

may be understood, asymptotically, as orthogonality conditions on  $\partial\mathcal{D}$ . In a nutshell, the coefficient functions obey a *Riemann-Hilbert hierarchy*, that is, a recursive sequence of Riemann-Hilbert problems. An important aspect of the present work is the solution of this hierarchy in closed form.

The underlying idea for the proof of Theorem 1.2.1, developed in detail in Section 4 below, begins with the disintegration formula

$$(1.7.1) \quad \int_{\mathcal{D}} F(z) \omega(z) dA(z) = 2 \int_T \int_{\gamma_t} F(z) \omega(z) v(z) d\sigma(z) dt$$

valid for appropriately integrable functions  $F$ , where  $\mathcal{D}$  is smoothly foliated by a curve family  $(\gamma_t)_{t \in T}$ , the symbol  $v(z)$  denotes the normal velocity of the flow as a curve passes through  $z$ .

The goal would be to find a foliation  $(\gamma_{N,t})_t$  of  $\mathcal{D}$ , such that the orthonormal polynomial  $P_{N,t}$  with respect to the measure  $\omega v d\sigma$  on  $\gamma_t$  is stationary in the flow parameter  $t$  up to a constant multiple:

$$P_{N,t} = c(t) P_{N,0}, \quad t \in T.$$

In view of the disintegration formula (1.7.1) we would then find that  $P_N$  is a constant multiple of, say,  $P_{N,0}$ . As the orthogonal polynomials for a fixed weight on a given

analytic curve is well understood following Szegő, this would provide a way to find  $P_N$ .

This procedure cannot be carried out to the letter, but if we allow for an error in the stationarity condition, as well as for a truncation of the domain  $\mathcal{D}$ , it is possible to find an algorithm which solves this problem approximately in a self-improving fashion. One would begin with a crude initial guess for  $P_N$  and the foliation, and then obtain appropriate correction terms by the requirements that the flow should cover a sufficiently large region with a given error while leaving  $P_{N,t}$  stationary.

**1.8. Notational conventions.** We denote by  $C^k$ ,  $C^\infty$  and  $C^\omega$  the spaces of  $k$  times differentiable, infinitely differentiable and real-analytic functions, respectively. By  $H^\infty(D)$  we denote the class of bounded holomorphic functions on  $D$ .

We use the notation  $E^c$ ,  $E^\circ$  and  $\bar{E}$  for the complement, interior and closure of a set  $E$ . We use the standard  $O$  and  $o$ -notation (alternatively, the  $f = O(g)$  is replaced by  $f \lesssim g$ ). The symbol  $f \asymp g$  means that  $f = O(g)$  and  $g = O(f)$  hold simultaneously.

By  $u_N \cong v_N$  we mean that  $u_N$  and  $v_N$  agree at the level of formal asymptotic expansions, meaning that if such an expansion is truncated at an arbitrary level then  $u_N$  and  $v_N$  agree up to the indicated error.

We use the standard complex derivatives  $\partial$  and  $\bar{\partial}$  defined by

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial}_z = \frac{1}{2}(\partial_x + i\partial_y)$$

where  $z = x + iy$ . The (quarter) Laplacian  $\Delta = \frac{1}{4}(\partial_x^2 + \partial_y^2)$  then factorizes as  $\Delta = \partial\bar{\partial}$ .

We use the notation  $\partial_x^\times := x\partial_x$  where  $\partial_x$  is the usual (partial) differential operator with respect to the variable  $x$ . For the complex Wirtinger derivatives, we use the notation  $\partial_z^\times := z\partial_z$  and the conjugate operator  $\bar{\partial}_z^\times := \bar{z}\bar{\partial}_z$ .

## 2. EXTENSIONS AND APPLICATIONS

**2.1. Distributional asymptotic expansion.** In addition to the pointwise expansion and the expansion in the  $L^2$ -sense supplied by Theorems 1.2.1 and Theorem 1.2.3, respectively, we find another expansion of the orthogonal polynomials as distributions. At the intuitive level, this is led by the considerations of Section 3 below, where we see that the orthogonality relations which define  $P_N$  naturally collapse to conditions on the boundary  $\partial\mathcal{D}$ . In accordance with this observation, the distributions involved in the expansion are supported on  $\partial\mathcal{D}$ .

To describe the result, we introduce the operator  $\Lambda_N$ , which incorporates the structure of the orthogonal polynomials:

$$\Lambda_N f(z) = \varphi'(z)\varphi^N(z)e^V f \circ \varphi.$$

This operator is discussed in detail in Subsection 3.1 below. Here, we may mention that  $\Lambda_N$  acts isometrically between  $L^2$ -spaces with weights  $|z|^{2N}\Omega(z)$  and  $\omega(z)$  on  $\mathbb{D} \setminus \mathbb{D}(0, \rho)$  and  $\mathcal{D} \setminus \mathcal{D}_\rho$ , respectively.

Denote by  $G$  a bounded  $C^\infty$ -smooth function on the plane  $\mathbb{C}$ . By [4, Lemma 5.1], which relies on work of Whitney and Seeley on extensions of smooth functions (see [43], [51]), we may split  $G$  as a sum of three bounded and  $C^\infty$ -smooth functions

$$(2.1.1) \quad G = G_+ + G_- + G_0$$

where  $G_+$  is holomorphic and  $G_-$  is conjugate holomorphic on  $\mathbb{C} \setminus \mathcal{D}$ , respectively, and where  $G_0$  vanishes along  $\partial\mathcal{D}$ . For  $\varkappa \geq 1$ , we consider the index set

$$I_\varkappa = \{(\nu, j, k) \in \mathbb{N}^3 : \nu \geq 1 \text{ and } \nu + j + k \leq \varkappa\}.$$

**Theorem 2.1.1** (Distributional asymptotic expansion). *With  $\mathbf{T}$  given by (1.3.1), and  $G$  a bounded smooth function on  $\mathbb{C}$  decomposed according to (2.1.1), we have for any given positive integer  $\varkappa$  the asymptotics*

$$\begin{aligned} \int_{\mathcal{D}} G(z) |P_N(z)|^2 \omega(z) dA(z) &= G_+(\infty) + G_-(\infty) \\ &+ D_N^2 \sum_{(\nu, j, k) \in I_\varkappa} \frac{1}{N^{\nu+j+k}} \int_{\mathbb{T}} \left(-\frac{r}{2} \partial_r\right)^\nu g_0(e^{it}) \mathbf{W}_{N, \nu, \varkappa} [X_j \bar{X}_k](e^{it}) \frac{dt}{2\pi} + O(N^{-\varkappa-1}), \end{aligned}$$

as  $N \rightarrow \infty$ , where  $g_0 = G_0 \circ \varphi^{-1}$  and the operator  $\mathbf{W}_{N, \nu, \varkappa}$  is given by

$$\mathbf{W}_{N, \nu, \varkappa} = \mathbf{R} \sum_{\mu=0}^{\varkappa-\nu} N^{-\mu} \binom{\nu+\mu}{\nu} \left(-\frac{r}{2} \partial_r - \mathbf{I}\right)^\mu \mathbf{M}_\Omega.$$

*Remark 2.1.2.* In particular, we have that

$$\int_{\mathcal{D}} G(z) |P_N(z)|^2 \omega(z) dA(z) = G_+(\infty) + G_-(\infty) + O(N^{-1})$$

as  $N \rightarrow \infty$ , which says that the wave function  $1_{\mathcal{D}} |P_N|^2 \omega$  approximates harmonic measure for  $\mathbb{C} \setminus \overline{\mathcal{D}}$  relative to the point at infinity.

We will obtain Theorem 2.1.1 below in Section 6.

**2.2. Constrained Coulomb gases and off-spectral asymptotics of polynomial Bergman kernels.** Given a positive integer  $N$ , we denote by  $K_N(z, w)$  the polynomial Bergman kernel for the space  $\text{Pol}_N^2(\mathcal{D}, \omega)$ . Such kernels appear as correlation kernels for determinantal Coulomb gas models. The weights considered here appear in connection with constrained (or conditioned) Coulomb gases, where the particles are confined to the domain  $\mathcal{D}$  by a *hard edge*. Specifically, our situation corresponds to potentials of the form  $Q_N = Q^0 + N^{-1}Q^1$  where  $Q^0$  is constant on  $\mathcal{D}$ . From a Coulomb gas perspective, it would be natural to consider confined weights of the form  $e^{-NQ} 1_{\mathcal{D}}$  where the potential  $Q$  is a more general smooth subharmonic function. The analysis of that problem will require a better understanding of an associated Laplacian growth problem with a fixed wall and a moving free boundary. We expect the methods developed here to be helpful in obtaining asymptotics of the orthogonal polynomial  $P_{N,n}$  of degree  $n$  with respect to the measure  $e^{-2NQ} \chi_{\mathcal{D}} dA$  when the degree  $n$  is large compared to  $N$ .

The polynomial Bergman kernel may be expressed in terms of the orthogonal polynomials

$$K_N(z, w) = \sum_{j=0}^N P_j(z) \overline{P_j(w)}, \quad (z, w) \in \mathbb{C}^2,$$

so the asymptotic expansion of  $P_N$  obtained above gives (at least in principle) information about the kernel  $K_N$ . As observed in [26], there is also a direct way to obtain asymptotics of  $K_N(z, w)$  when  $w$  is fixed in the *off-spectral region*, which in this case equals  $\mathbb{C} \setminus \overline{\mathcal{D}}$ . We define the *normalized reproducing kernel*

$$k_{N,w}(z) = \frac{K_N(z, w)}{\sqrt{K_N(w, w)}}, \quad z \in \mathbb{C}.$$

For fixed  $w \in \mathbb{C}$ ,  $k_{N,w}$  is the unique element in the unit sphere of  $\text{Pol}_N^2(\mathcal{D}, \omega)$  which maximizes the point evaluation functional  $\text{Re } f(w)$ . For a given off-spectral point



$w \in \mathbb{C} \setminus \overline{\mathcal{D}}$ , we denote by  $\varrho_w$  the unique outer function on  $\mathbb{C} \setminus \overline{\mathcal{D}}$  which is positive at the point  $w$  with boundary values

$$(2.2.1) \quad |\varrho_w(z)|^2 = \frac{|\varphi(w)|^2 - 1}{|\varphi(z) - \varphi(w)|^2}, \quad z \in \partial\mathcal{D}.$$

The following result gives the behavior of  $k_{N,w}$ .

**Proposition 2.2.1.** *Under the assumptions of Theorem 1.2.1, there exist constants*

$$D_{N,w} = e^{-i(N \arg \varphi(w) + \arg \varphi'(w) + \operatorname{Im} V(w))} (1 + N^{-1} d_{1,w} + \dots), \quad d_{j,w} \in \mathbb{R},$$

and bounded holomorphic functions  $B_{j,w}$  with  $B_{0,w} \equiv 1$  and  $B_{j,w}(\infty) = 0$  for  $j \geq 1$  defined in an open neighborhood of  $\mathbb{C} \setminus \mathcal{D}$ , such that for any fixed,  $\varkappa \in \mathbb{N}$  and positive real numbers  $A, \delta > 0$  we have the asymptotics

$$k_{N,w}(z) = D_{N,w} N^{\frac{1}{2}} \varrho_w(z) \varphi'(z) \varphi^N(z) e^{V(z)} \left( \sum_{j=0}^{\varkappa} N^{-j} B_{j,w}(z) + O(N^{-\varkappa-1}) \right),$$

as  $N \rightarrow \infty$ , valid for  $z, w$  with  $\operatorname{dist}_{\mathbb{C}}(z, \mathcal{D}^c) \leq AN^{-1} \log N$  and  $\operatorname{dist}_{\mathbb{C}}(w, \mathcal{D}) \geq \delta$ , respectively.

As in Theorem 1.3.1, it is possible to obtain closed form expressions for the coefficients  $B_{j,w}$  in terms of iterates of a corresponding operator  $\mathbf{T}_w$ .

The proof of Proposition 2.2.1 is along the lines of the proof of Theorem 1.2.1. The main difference is that the Berezin kernel  $|k_{N,w}|^2 \omega$  should approximate the harmonic measure in  $\mathbb{C} \setminus \mathcal{D}$  for the point  $w$  instead of the harmonic measure for the point at infinity, which explains the presence of the factor  $\varrho_w(z)$ . In Subsection 5.4 below, we discuss the necessary changes in the proof.

### 3. HIGHER ORDER CORRECTIONS VIA RIEMANN-HILBERT HIERARCHIES

**3.1. Canonical positioning.** We begin with the algorithmic aspects of the asymptotic expansion, and in particular we compute the coefficient functions  $(B_j)_{j \in \mathbb{N}}$ . This is done under the assumption that Theorem 1.2.3 holds, and amounts to collapsing the planar orthogonality relations into orthogonality relations on the unit circle.

According to Theorem 1.2.3 there exists a holomorphic function  $F_N$  (a truncated asymptotic expansion, also called a  $\varkappa$ -abschnitt) of polynomial growth

$$F_N(z) = D_N N^{\frac{1}{2}} \varphi'(z) \varphi^N(z) e^{V(z)} \sum_{j=0}^{\varkappa} N^{-j} B_j(z),$$

where  $B_j$  are bounded and holomorphic functions on  $\mathbb{D}_e(0, \rho) = \{z \in \mathbb{C} : |z| > \rho\}$  for some  $0 < \rho < 1$ , and  $D_N = 1 + d_1 N^{-1} + \dots + d_{\varkappa} N^{-\varkappa}$  is a real positive constant, such that as  $N \rightarrow +\infty$ ,

$$\|P_N - \chi_0 F_N\|_{L^2(\mathcal{D}, \omega)} = O(N^{-\varkappa-1}).$$

Here, we recall that  $\chi_0$  is a cut-off function which vanishes deep inside  $\mathcal{D}$  but is identically one in a neighborhood of the exterior domain  $\mathbb{C} \setminus \mathcal{D}$ . In particular, for  $q \in \operatorname{Pol}_{N-1}$  we have the approximate orthogonality

$$(3.1.1) \quad \int_{\mathcal{D}} \chi_0^2(z) q(z) \overline{F_N(z)} \omega(z) dA(z) = O(N^{-\varkappa-1} \|q\|_{L^2(\mathcal{D}, \omega)}),$$

while  $\|\chi_0^2 F_N\|_{L^2(\mathcal{D}, \omega)} = 1 + O(N^{-\varkappa-1})$  holds by the triangle inequality.

*Remark 3.1.1.* The approximate orthogonality relation (3.1.1) holds more generally for all holomorphic functions  $q$  on  $\mathbb{C} \setminus \mathcal{D}_\rho$  in  $L^2(\mathcal{D} \setminus \mathcal{D}_\rho, \omega)$  of polynomial growth  $|q(z)| = O(|z|^{N-1})$  by Proposition 5.1.1 below. The norm on the right-hand side should then be replaced by  $\|\chi_0 q\|_{L^2(\mathcal{D}, \omega)}$ .

We recall that  $\Omega$  is the modified weight function given by

$$(3.1.2) \quad \Omega = e^{2\operatorname{Re} V \circ \varphi^{-1}} \omega \circ \varphi^{-1},$$

where  $\varphi : \mathcal{D}^c \rightarrow \mathbb{D}_e$  is the Riemann mapping with the standard normalization at infinity, which we recall extends across  $\partial\mathcal{D}$ . The function  $\Omega$  is defined and real-analytic on the annulus  $\mathbb{D} \setminus \mathbb{D}(0, \rho)$  for some parameter  $\rho$  with  $0 < \rho < 1$ . By possibly increasing  $\rho$  slightly, we may assume that  $\Omega \geq \epsilon_0$  on the annulus  $\mathbb{D} \setminus \mathbb{D}(0, \rho)$  for some constant  $\epsilon_0 > 0$ . In view of the definition of the Szegő function  $V$ , we have  $\Omega|_{\mathbb{T}} \equiv 1$ . We recall that  $X_j = B_j \circ \varphi^{-1}$  so that the functions  $X_j$  are holomorphic on  $\mathbb{D}_e(0, \rho)$  with  $X_j(\infty) = 0$  for all  $j \geq 1$ , and put

$$f_N = D_N N^{\frac{1}{2}} \sum_{j=0}^{\infty} N^{-j} X_j.$$

If  $\mathbf{\Lambda}_N$  is the *canonical positioning operator*

$$(3.1.3) \quad \mathbf{\Lambda}_N f(z) = \varphi'(z) \varphi(z)^N e^{V(z)} (f \circ \varphi)(z),$$

we have  $F_N = \mathbf{\Lambda}_N[f_N]$ . By the change-of-variables formula,  $\mathbf{\Lambda}_N$  acts isometrically and isomorphically

$$(3.1.4) \quad \mathbf{\Lambda}_N : L^2(\mathbb{D} \setminus \mathbb{D}(0, \rho), r_N \Omega \, dA) \rightarrow L^2(\mathcal{D} \setminus \mathcal{D}_\rho, \omega)$$

where we use  $r_N$  to denote  $r_N(z) = |z|^{2N}$ . Moreover,  $\mathbf{\Lambda}_N$  preserves holomorphicity, and we have asymptotically

$$|\mathbf{\Lambda}_N f(z)| \asymp |z^N f(z)| \quad \text{as } |z| \rightarrow +\infty.$$

**3.2. Collapsing the orthogonality relations.** We apply the relation (3.1.1) to the family of functions  $q = \mathbf{\Lambda}_N[e_k]$ , where  $e_k(w) = w^{-k}$  for  $k \geq 1$ . Since these functions are not necessarily polynomials, we interpret (3.1.1) in the generalized sense of Remark 3.1.1. We agree to interpret the product  $\chi_0 q$  as zero wherever the cut-off function  $\chi_0$  vanishes, also where  $q$  is undefined. As a consequence, the product  $\chi_0 q$  gets to be defined globally on  $\mathbb{C}$ . Since the modified weight  $\Omega$  is bounded it is evident that for any fixed  $k \in \mathbb{N}$  we have

$$(3.2.1) \quad \|\chi_0 q\|_{L^2(\mathcal{D}, \omega)}^2 = \int_{\mathcal{D}} |\chi_0^2 \mathbf{\Lambda}_N[e_k]|^2 \omega \, dA \lesssim \int_{\mathcal{D} \setminus \mathcal{D}_\rho} |\varphi|^{2(N-k)} \, dA \lesssim N^{-1}.$$

By the isometric property (3.1.4) of  $\mathbf{\Lambda}_N$  we apply (3.1.1) while taking Remark 3.1.1 and the norm bound (3.2.1) into account, to find

$$(3.2.2) \quad \begin{aligned} \int_{\mathcal{D}} \chi_0^2(z) F_N(z) \overline{\mathbf{\Lambda}_N[e_k](z)} \omega(z) \, dA(z) &= \int_{\mathbb{D}} \chi_1^2(w) f_N(w) \bar{w}^{-k} |w|^{2N} \Omega(w) \, dA(w) \\ &= \int_0^{2\pi} \int_0^\infty \chi_1^2(e^{-s+it}) f_N(e^{-s+it}) e^{k(s+it)} \Omega(e^{-s+it}) e^{-2(N+1)s} \frac{ds dt}{\pi} = O(N^{-\kappa - \frac{3}{2}}), \end{aligned}$$

where we used an anti-holomorphic exponential change of variables  $w = e^{-s+it}$  and where  $\chi_1 := \chi_0 \circ \varphi^{-1}$  is another cut-off function. To make things as simple as possible, let us agree that  $\chi_1(e^{-s+it})$  is radial, so that  $\chi_1(e^{-s+it}) = \chi_1(e^{-s})$ , and that for some

constant  $\alpha > 1$  with  $0 \leq \rho < \alpha^2 \rho < 1$  we have  $\chi_1 \equiv 0$  on  $\mathbb{D}(0, \alpha\rho)$  while  $\chi_1 \equiv 1$  on  $\mathbb{D} \setminus \mathbb{D}(0, \alpha^2 \rho)$ . We integrate first in the  $s$ -variable, and notice that (3.2.2) reads

$$(3.2.3) \quad \int_0^{2\pi} e^{ikt} \left( \int_0^\infty G_{t,N,k}(s) e^{-2Ns} ds \right) dt = O(N^{-\varkappa - \frac{3}{2}}), \quad k = 1, 2, 3, \dots,$$

where

$$G_{t,N,k}(s) = \chi_1(e^{-s}) f_N(e^{-s+it}) e^{(k-2)s} \Omega(e^{-s+it}).$$

The expression (3.2.3) is suitable for asymptotic analysis.

**Proposition 3.2.1.** *Fix  $\varkappa \in \mathbb{N}$  and let  $G \in C^\infty([0, \infty))$  with  $G^{(\varkappa+1)} \in L^\infty(\mathbb{R}_+)$ . Then we have*

$$\int_0^\infty G(s) e^{-\lambda s} ds = \frac{G(0)}{\lambda} + \frac{G'(0)}{\lambda^2} + \frac{G''(0)}{\lambda^3} + \dots + \frac{G^{(\varkappa)}(0)}{\lambda^{\varkappa+1}} + O\left(\frac{1}{\lambda^{\varkappa+2}} \|G^{(\varkappa+1)}\|_{L^\infty(\mathbb{R}_+)}\right)$$

as  $\lambda \rightarrow +\infty$ .

*Proof.* By iterated integration by parts, we find the formula

$$(3.2.4) \quad \int_0^\infty G(s) e^{-\lambda s} ds = \sum_{j=0}^{\varkappa} \frac{G^{(j)}(0)}{\lambda^{j+1}} + \frac{1}{\lambda^{\varkappa+1}} \int_0^\infty G^{(\varkappa+1)}(s) e^{-\lambda s} ds$$

which holds since  $G^{(\varkappa+1)} \in L^\infty(\mathbb{R}_+)$ . Moreover, we may estimate the integral in the right-hand side of (3.2.4):

$$\left| \int_0^\infty G^{(\varkappa+1)}(s) e^{-\lambda s} ds \right| \leq \|G^{(\varkappa+1)}\|_{L^\infty(\mathbb{R}_+)} \int_0^\infty e^{-\lambda s} ds \leq \lambda^{-1} \|G^{(\varkappa+1)}\|_{L^\infty(\mathbb{R}_+)},$$

which yields the assertion.  $\square$

**3.3. Derivation of the Riemann-Hilbert hierarchy.** We let  $h_N$  be the function  $h_N(z) = f_N(e^{-z}) \Omega(e^{-z})$ . Since  $f_N$  has an asymptotic expansion in terms of the functions  $X_j$  and  $\Omega$  is a fixed function, the function  $h_N$  admits an asymptotic expansion  $h_N = D_N N^{\frac{1}{2}} \sum N^{-j} Y_j$  with  $Y_j(z) = (X_j \Omega)(e^{-z})$ , and where we recall that  $D_N = 1 + O(N^{-1})$ . We define the cut-off function  $\chi_2(s) := \chi_1(e^{-s})$ , so that the inner integral in (3.2.3) takes the form

$$\int_0^\infty G_{t,N,k}(s) e^{-2Ns} ds = \int_0^\infty \chi_2^2(s) h_N(s+it) e^{(k-2)s} e^{-2Ns} ds.$$

We use the asymptotic expansion of  $h_N$  and Proposition 3.2.1 with  $\lambda = 2N$  to find

$$\begin{aligned} & \int_0^\infty \chi_2^2(s) h_N(s+it) e^{(k-2)s} e^{-2Ns} ds \\ &= D_N N^{\frac{1}{2}} \sum_{l=0}^{\varkappa} N^{-l} \int_0^\infty \chi_2^2(s) Y_l(s+it) e^{(k-2)s} e^{-2Ns} ds \\ &= D_N N^{\frac{1}{2}} \sum_{j+l \leq \varkappa} \frac{1}{2^{j+1} N^{j+l+1}} \partial_s^j \left( Y_l(s+it) e^{(k-2)s} \right) \Big|_{s=0} + O(N^{-\varkappa - \frac{3}{2}}), \end{aligned}$$

so that by Leibniz formula we have

$$\begin{aligned}
(3.3.1) \quad D_N^{-1} \int_0^\infty \chi_2^2(s) h_N(s+it) e^{(k-2)s} e^{-2Ns} ds \\
= N^{\frac{1}{2}} \sum_{j+l \leq \varkappa} \frac{1}{2^{j+1} N^{j+l+1}} \sum_{r=0}^j \binom{j}{r} (k-2)^r \partial_s^{j-r} Y_l(s+it) \Big|_{s=0} + O(N^{-\varkappa-\frac{3}{2}}) \\
= \frac{N^{\frac{1}{2}}}{2} \sum_{p=0}^\varkappa \frac{1}{N^{p+1}} \sum_{l=0}^p \sum_{r=0}^{p-l} \frac{(k-2)^r}{2^{p-l}} \binom{p-l}{r} \partial_s^{p-l-r} Y_l(s+it) \Big|_{s=0} + O(N^{-\varkappa-\frac{3}{2}}).
\end{aligned}$$

We notice that for polynomials  $P$  we have

$$(3.3.2) \quad \int_0^{2\pi} e^{ikt} P(k) f(t) dt = \int_0^{2\pi} e^{ikt} P(i\partial_t) f(t) dt.$$

In view of (3.3.2), the condition (3.2.3) combined with (3.3.1) asserts that for all  $p$  with  $0 \leq p \leq \varkappa$  we have

$$\forall k \geq 1 : \quad \sum_{l=0}^p \sum_{r=0}^{p-l} \frac{1}{2^{p-l}} \binom{p-l}{r} \int_0^{2\pi} e^{ikt} (i\partial_t - 2\mathbf{I})^r \partial_s^{p-l-r} Y_l(s+it) \Big|_{s=0} dt = 0,$$

and since  $\varkappa$  is arbitrary, the above condition in fact holds for each  $p \geq 0$ . Contracting the inner sum using the binomial theorem, we hence find that

$$(3.3.3) \quad \forall p \geq 0, \forall k \geq 1 : \quad \sum_{l=0}^p \int_{\mathbb{T}} e^{ikt} \left( \frac{1}{2} (\partial_s + i\partial_t) - \mathbf{I} \right)^{p-l} Y_l(s+it) \Big|_{s=0} dt.$$

Now, expressing this in terms of the original function  $f$  we claim that

$$(3.3.4) \quad \left( \frac{1}{2} (\partial_s + i\partial_t) - \mathbf{I} \right)^j Y_l(s+it) \Big|_{s=0} = (-1)^j \mathbf{T}^j X_l|_{\mathbb{T}}, \quad j = 0, 1, 2, \dots,$$

where the operator  $\mathbf{T}$  is given by

$$\mathbf{T} = \mathbf{M}_\Omega^{-1} (\partial_z^\times + \mathbf{I}) \mathbf{M}_\Omega,$$

$\mathbf{M}_\Omega$  is the operator of multiplication by  $\Omega$ , and where we recall that  $\partial_z^\times = z\partial_z$ . Indeed, we write out the relation  $Y(s+it) = (X\Omega)(e^{-s+it})$  and notice that

$$\begin{aligned}
\left( \frac{1}{2} (\partial_s + i\partial_t) - \mathbf{I} \right) Y(s+it) &= \left( \frac{1}{2} (\partial_s + i\partial_t) - \mathbf{I} \right) (X\Omega)(e^{-s+it}) \\
&= \left( -\frac{1}{2} (\partial_r^\times - i\partial_\theta) - \mathbf{I} \right) (X\Omega)(re^{i\theta}) = -(\partial_z^\times + \mathbf{I})(X\Omega)(z).
\end{aligned}$$

By iteration this gives the relation (3.3.4). In conclusion, the condition (3.3.3) becomes

$$(3.3.5) \quad \forall p \geq 0, \forall k \geq 1 : \quad \sum_{l=0}^p (-1)^{p-l} \int_{\mathbb{T}} e^{ikt} \mathbf{T}^{p-l} X_l(e^{it}) dt = 0.$$

When  $p = 0$  this says that  $X_0 \in H^2$ . But we know a priori that  $B_0$  is a bounded analytic function in  $\mathbb{C} \setminus \mathcal{D}$  so that  $X_0 \in H_-^2$ . Hence  $X_0$  must be a constant, and we choose  $X_0 \equiv 1$ . For  $p \geq 1$ ,  $B_p$  is a bounded analytic function in  $\mathbb{C} \setminus \mathcal{D}$  with  $B_p(\infty) = 0$ , so that  $X_p \in H_{-,0}^2$  for  $p \geq 1$ . As a consequence, we see that the condition (3.3.5) for  $p \geq 1$  is equivalent to the Riemann-Hilbert hierarchy

$$X_p \in H_{-,0}^2 \cap (-\Xi_p + H^2), \quad \Xi_p = \sum_{l=0}^{p-1} (-1)^{p-l} \mathbf{R} \mathbf{T}^{p-l} X_l, \quad p \geq 1,$$

where  $\mathbf{R}$  is the restriction operator to  $\mathbb{T}$ . Hence, in view of [27, Proposition 2.5.1] the recursive solution to these conditions is

$$(3.3.6) \quad \forall p \geq 1 : \quad X_p = -\mathbf{P}\Xi_p = \sum_{l=0}^{p-1} (-1)^{p-l+1} \mathbf{Q}\mathbf{T}^{p-l} X_l,$$

where  $\mathbf{Q} = \mathbf{P}\mathbf{R}$  and  $\mathbf{P} = \mathbf{P}_{H_{-,0}^2}$  is the orthogonal projection onto the Hardy space of functions on  $\mathbb{D}_e$  vanishing at infinity.

**3.4. Recursive solution of the Riemann-Hilbert hierarchy.** The recursion can be solved as follows.

**Proposition 3.4.1.** *For each bounded holomorphic function  $X_0 \in H^\infty(\mathbb{D}_e(0, \rho))$ , the recursion (3.3.6) has a unique solution which is given by*

$$X_p = \mathbf{Q}\mathbf{T}[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p-1} X_0, \quad p \geq 1.$$

*Proof.* Due to the triangular nature of the recursion, it is clear that it admits a unique solution for each choice of  $X_0 \in H^\infty(\mathbb{D}_e(0, \rho))$ .

It remains to verify that the claimed solution meets the recursion. We prove this by induction. For the base case  $p = 1$  this holds trivially, since  $X_1 = \mathbf{Q}\mathbf{T}X_0$ .

We assume the formula for  $X_p$  is valid for  $p \leq p_0 - 1$ , and proceed to show that if we define  $X_{p_0}$  by the given formula, then the recursion holds for  $p = p_0$  as well. Hence, we compute

$$\begin{aligned} X_{p_0} &= \mathbf{Q}\mathbf{T}[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-1} X_0 = \mathbf{Q}\mathbf{T}[\mathbf{Q}\mathbf{T} - \mathbf{T}][\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-2} X_0 \\ &= \mathbf{Q}\mathbf{T}\mathbf{Q}\mathbf{T}[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-2} X_0 - \mathbf{Q}\mathbf{T}^2[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-2} X_0 \\ &= \mathbf{Q}\mathbf{T}X_{p_0-1} - \mathbf{Q}\mathbf{T}^2[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-2} X_0, \end{aligned}$$

where we use the induction hypothesis to replace the first term on the second line by  $-\mathbf{Q}\mathbf{T}X_{p_0-1}$ . If  $p_0 = 2$ , we are done. For  $p_0 \geq 2$ , this procedure may be repeated with the last term, to give

$$\begin{aligned} X_{p_0} &= \mathbf{Q}\mathbf{T}X_{p_0-1} - \mathbf{Q}\mathbf{T}^2\mathbf{Q}\mathbf{T}[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-3} X_0 + \mathbf{Q}\mathbf{T}^3[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-3} X_0 \\ &= \mathbf{Q}\mathbf{T}X_{p_0-1} - \mathbf{Q}\mathbf{T}^2X_{p_0-2} + \mathbf{Q}\mathbf{T}^3[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-3} X_0, \end{aligned}$$

and so on. In each step, we use the equality  $(0 \leq k < p_0 - 1)$

$$\mathbf{Q}\mathbf{T}^k[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-k} X_0 = \mathbf{Q}\mathbf{T}^k X_{p_0-k} - \mathbf{Q}\mathbf{T}^{k+1}[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-k-1} X_0,$$

which holds in view of the induction hypothesis. The procedure ends when  $k = p_0 - 1$ , for which the relevant identity reads

$$\mathbf{Q}\mathbf{T}^{p_0-1}[\mathbf{Q}\mathbf{T} - \mathbf{T}]X_0 = \mathbf{Q}\mathbf{T}^{p_0}X_0 - \mathbf{Q}\mathbf{T}^{p_0}\mathbf{Q}X_0 = \mathbf{Q}\mathbf{T}^{p_0}X_0.$$

We then get for the full expression

$$\mathbf{Q}\mathbf{T}[\mathbf{Q}\mathbf{T} - \mathbf{T}]^{p_0-1} X_0 = - \sum_{k=1}^{p_0} (-1)^k \mathbf{Q}\mathbf{T}^k X_{p_0-k} = \sum_{k=0}^{p_0-1} (-1)^{p_0-k+1} \mathbf{Q}\mathbf{T}^{p_0-k} X_k,$$

which completes the verification that  $X_{p_0}$  given by the desired formula satisfies the recursion.  $\square$

*Proof of Theorem 1.3.1.* The conclusion of the theorem is now immediate in view of the Riemann-Hilbert conditions (3.3.6) and Proposition 3.4.1.  $\square$

## 4. THE ORTHOGONAL FOLIATION FLOW

**4.1. The asymptotic expansion in  $L^2$ .** We obtain Theorem 1.2.1 from its  $L^2$ -analogue Theorem 1.2.3. We proceed in two steps. First, we construct a family of *approximately orthogonal quasipolynomials* (cf. [27, Subsection 3.2]), and then we show that these approximate well the true orthogonal polynomials in norm. The latter part of the proof is based on Hörmander  $L^2$ -estimates for the  $\bar{\partial}$ -operator with polynomial growth control for the solution. The quasipolynomials  $F_N$  are of the form  $F_N = \mathbf{\Lambda}_N[f_N]$ , where

$$f_N = D_N N^{\frac{1}{2}} \sum_{j=0}^{\infty} N^{-j} X_j$$

for some bounded holomorphic functions  $X_j$  defined on  $\mathbb{C} \setminus \mathcal{D}_\rho$ , where  $\mathbf{\Lambda}_N$  is the canonical positioning operator (3.1.3), and the radius  $\rho$  has  $0 < \rho < 1$ .

**4.2. The main lemma.** We put  $s = N^{-1}$  and consider it as a positive continuous parameter. After passage to the unit disk via the operator  $\mathbf{\Lambda}_N$ , the fact that  $N$  is an integer will be inessential. By a slight abuse of notation, we put

$$(4.2.1) \quad f_s = D_s s^{-\frac{1}{2}} \sum_{j=0}^{\infty} s^j X_j.$$

We consider a smooth family of orthostatic conformal mappings  $\psi_{s,t}$  of the closed exterior disk  $\overline{\mathbb{D}}_e$ , indexed by non-negative parameters  $s$  and  $t$ . For a fixed  $s$ , we assume that the smooth boundary loops  $\psi_{s,t}(\mathbb{T})$  foliate a domain  $\mathcal{E}_s := \bigcup_{0 \leq t \leq \delta} \psi_{s,t}(\mathbb{T})$  (below we will use  $\delta = \delta_s = s |\log s|^2$ ). We think of a foliation as a simple cover of a set. We consider sets  $\mathcal{E}_s \subset \mathbb{D}$  located near the boundary  $\mathbb{T}$ , and define the *flow density*  $\Upsilon_{s,t}$  by

$$(4.2.2) \quad \Upsilon_{s,t}(\zeta) := |f_s \circ \psi_{s,t}|^2 |\psi_{s,t}|^{2/s} \Omega \circ \psi_{s,t} \operatorname{Re}(-\bar{\zeta} \partial_t \psi_{s,t} \overline{\psi'_{s,t}}), \quad \zeta \in \mathbb{T}.$$

While somewhat daunting, this expression comes about naturally through the following disintegration identity:

$$(4.2.3) \quad \int_{\mathcal{E}_s} q(z) \overline{F_N(z)} \omega(z) dA = \int_0^\delta \int_{\mathbb{T}} h_s(\psi_{s,t}(\zeta)) \Upsilon_{s,t}(\zeta) d\sigma(\zeta) dt,$$

where the function  $q = \mathbf{\Lambda}_N[h_s]$  (recall  $s = N^{-1}$ ). In particular, we have that  $h_s$  is a bounded holomorphic function in a neighborhood of  $\mathbb{C} \setminus \mathbb{D}$ , and whenever  $|q(z)| = o(|z|^N)$  we have  $h_s(\infty) = 0$ . If the flow density  $\Upsilon_{s,t}(\zeta)$  would be constant as a function of  $\zeta \in \mathbb{T}$ , then the integral in (4.2.3) vanishes by the mean value property of harmonic functions. The formula (4.2.3) is a consequence of the quasiconformal change-of-variables

$$\Psi_s(z) = \psi_{s,1-|z|}(z/|z|).$$

The factor  $\operatorname{Re}(-\bar{\zeta} \partial_t \psi_{s,t} \overline{\psi'_{s,t}})$  is a constant multiple of the Jacobian of this transform. For the necessary details we refer to [27, Subsection 3.4].

We may not be able to solve the equation  $\Upsilon_{s,t}(\zeta) \equiv \operatorname{const}(s, t)$  exactly, but we look for a solution in an approximate sense. We consider a family of conformal mappings  $\psi_{s,t}$ , with an asymptotic expansion jointly in  $s$  and  $t$  given by the ansatz

$$(4.2.4) \quad \psi_{s,t}(\zeta) = e^{-t} \zeta \exp\left(t \sum_{j=1}^{\infty} s^j \eta_{j,t}(\zeta)\right),$$

with  $\eta_{j,t}$  bounded and holomorphic in a neighborhood of  $\overline{\mathbb{D}}_e$  and real at infinity, so that the following initial condition at  $t = 0$  is met:

$$\psi_{s,0}(\zeta) = \zeta, \quad \zeta \in \mathbb{D}_e.$$

It is clear from (4.2.4) that the mappings  $\psi_{s,t}$  are small perturbations of the identity mapping. These perturbations  $\psi_{s,t}$  should be orthostatic conformal mappings of the closed exterior disk (cf. [27, Lemma 6.2.5]) with  $\psi_{s,t}(\mathbb{T}) \subset \mathbb{D}$  for small positive  $s$  and  $t$ . In view of (4.2.4) this holds provided that the coefficient functions  $\eta_{j,t}$  are bounded in a neighborhood of  $\overline{\mathbb{D}}_e$  and depend smoothly on  $t$ . We also look for bounded holomorphic functions  $f_s$  of the form (4.2.1) which extend holomorphically to a neighborhood of  $\overline{\mathbb{D}}_e$ . These two families of functions  $\psi_{s,t}$  and  $f_s$  should be chosen such that the *approximate flow equation*

$$(4.2.5) \quad \Pi_{s,t}(\zeta) := \log \Upsilon_{s,t}(\zeta) + \log s + s^{-1}t = O(s^{\varkappa+1}), \quad \zeta \in \mathbb{T}, \quad 0 \leq t \leq \delta_s$$

is met as  $s \rightarrow 0$ . We will see that there is a choice of the coefficient functions such that  $\Pi_{s,t}$  is smooth in  $s$  and  $t$ , while

$$(4.2.6) \quad \partial_s^j \Pi_{s,t}(\zeta)|_{s=0} = 0 \quad \text{for } j = 0, \dots, \varkappa,$$

and then apply Taylor's formula to obtain (4.2.5).

**Lemma 4.2.1.** *Given  $\varkappa \geq 0$ , there exists a number  $0 < \rho < 1$ , bounded holomorphic functions  $f_s$  of the form (4.2.1) on  $\mathbb{D}_e(0, \rho)$  and orthostatic conformal mappings  $\psi_{s,t}$  of the forms (4.2.4) which extend holomorphically to  $\mathbb{D}_e(0, \rho)$  and univalently to some  $\mathbb{D}_e(0, \rho_\varkappa)$  with  $\rho \leq \rho_\varkappa < 1$ , such that the flow equation (4.2.5) holds.*

The proof of this lemma is carried out in Subsection 4.3 and Subsection 4.4 below.

**4.3. Preliminary simplification of the flow equation.** It is advantageous to work with  $g_s = \log(s|f_s|^2) = 2\operatorname{Re} \log f_s + \log s$ , which should have an asymptotic expansion

$$(4.3.1) \quad g_s = \sum_{j=0}^{\varkappa} s^{-j} \nu_j$$

for bounded harmonic  $\nu_j$ . We recall the function  $\Omega$  which was defined previously in (3.1.2) and put

$$(4.3.2) \quad U = \log \Omega = 2\operatorname{Re} V \circ \varphi^{-1} + \log \omega \circ \varphi^{-1}.$$

By the defining property of the Szegő function  $V$ , the function  $U$  vanishes identically on the unit circle. In terms of the functions  $g_s$  and  $U$  defined in (4.3.1) and (4.3.2), respectively, the *logarithmic flow density*  $\Pi_{s,t}$  may be rewritten as

$$(4.3.3) \quad \begin{aligned} \Pi_{s,t}(\zeta) = & 2t \sum_{j=0}^{\varkappa} s^j \operatorname{Re} \eta_{j+1}(\zeta) + \sum_{j=1}^{\varkappa} s^j \nu_j \circ \psi_{s,t}(\zeta) \\ & + U \circ \psi_{s,t}(\zeta) + \log \operatorname{Re} (-\bar{\zeta} \partial_t \psi_{s,t}(\zeta) \overline{\psi'_{s,t}(\zeta)}). \end{aligned}$$

We recall that the flow equation is given by (4.2.5).

**4.4. Algorithmic solution of the flow equation.** The coefficients  $\nu_j$  and  $\eta_{j,t}$  are found in an algorithmic fashion as follows.

**Step 1.** The formula for  $\Pi_{s,t}$  evaluated at  $s = 0$  reads

$$\Pi_{s,t}|_{s=0} = 2t \operatorname{Re} \eta_{1,t} + \nu_0 \circ \psi_{0,t} + U \circ \psi_{0,t} - t,$$

where  $\psi_{0,t}(\zeta) = e^{-t}\zeta$ . Choose now  $\nu_0 \equiv 0$ , so that

$$(4.4.1) \quad \Pi_{s,t}(\zeta)|_{s=0} = 2t \operatorname{Re} \eta_{1,t} + U \circ \psi_{0,t} - t.$$

**Step 2.** Since  $U|_{\mathbb{T}} = 0$  and  $\psi_{0,t}(\zeta) = e^{-t}\zeta$ , the function

$$\mathfrak{F}_{0,t}(\zeta) = \frac{1}{2t} (U \circ \psi_{0,t} - t) = \frac{1}{2t} U \circ \psi_{0,t} - \frac{1}{2}$$

extends to a real-analytic function of  $(\zeta, t)$  in  $\mathbb{A}(\rho) \times [0, t_1]$  for some parameters  $0 < \rho < 1$  and  $t_1$ , where  $\mathbb{A}(\rho)$  is the annulus

$$\mathbb{A}(\rho) := \{z \in \mathbb{C} : \rho < |z| < \frac{1}{\rho}\}.$$

We define the function  $\eta_{1,t}$  for  $0 \leq t \leq r_1$  by the modified flow

$$\Pi_{s,t}(\zeta)|_{\zeta \in \mathbb{T}, s=0} = 0,$$

which by (4.4.1) is equivalent to

$$\operatorname{Re} \eta_{1,t}(\zeta) = -\mathfrak{F}_{0,t}(\zeta), \quad \zeta \in \mathbb{T}.$$

In view of the real-analyticity of  $\mathfrak{F}_{0,t}(\zeta)$  found in Step 1, this equation may be solved by (see e.g. [27, Subsection 2.5])

$$\eta_{1,t} = -\mathbf{H}_{\mathbb{D}_e}[\mathfrak{F}_{0,t}],$$

where  $\mathbf{H}_{\mathbb{D}_e}$  is the usual *Herglotz transform*

$$\mathbf{H}_{\mathbb{D}_e} f(z) = \int_{\mathbb{T}} \frac{z + \zeta}{z - \zeta} f(\zeta) d\sigma(\zeta), \quad z \in \mathbb{D}_e.$$

We now enter the iterative Steps 3 and 4 of the algorithm, which will continue to loop until all the remaining coefficient functions have been found and the approximate flow equation (4.2.5) has been solved.

**Step 3.** We enter this step with  $j = j_0$  with  $1 \leq j_0 \leq \varkappa$ , and assume that there exist bounded holomorphic functions  $\eta_{j,t}(\zeta)$  for  $1 \leq j \leq j_0$  and bounded harmonic functions  $\nu_j$  for  $j < j_0$  such that the following equations hold:

$$\partial_s^j \Pi_{s,t}(\zeta)|_{\zeta \in \mathbb{T}, s=0} \equiv 0, \quad 0 \leq j < j_0,$$

where  $\Pi_{s,t}$  is given by (4.3.3). Here, the functions  $\eta_{j,t}(\zeta)$  are required to be bounded and holomorphic on  $\mathbb{D}_e(0, \rho)$  and real-analytically smooth in  $t$  for  $t \in [0, t_j]$  for some  $t_j > 0$ , while the functions  $\nu_j(\zeta)$  are bounded and harmonic on  $\mathbb{D}_e(0, \rho)$  (see the argument in the proof of Lemma 4.2.1 below for a comment on the uniformity of these parameters).

The goal of Steps 3 and 4 taken together is to solve the equation

$$(4.4.2) \quad \partial_s^{j_0} \Pi_{s,t}|_{\mathbb{T}, s=0} = 0,$$



by determining suitable functions  $\nu_{j_0}$  and  $\eta_{j_0+1,t}$ . To see what is required, we differentiate in (4.3.3) and obtain

$$\begin{aligned} \partial_s^{j_0} \Pi_{s,t} \Big|_{\mathbb{T}, s=0} &= 2t j_0! \operatorname{Re} \eta_{j_0+1,t} \\ &+ \partial_s^{j_0} \left( U \circ \psi_{s,t} + \log \operatorname{Re} (-\zeta \partial_t \psi_{s,t} \overline{\psi'_{s,t}}) \right) \Big|_{\mathbb{T}, s=0} + \sum_{j=0}^{j_0} \partial_s^{j_0} (s^j \nu_j \circ \psi_{s,t}) \Big|_{\mathbb{T}, s=0}, \end{aligned}$$

where the sum is truncated at  $j_0$  due to the higher order vanishing of the remaining terms. In particular, the equation (4.4.2) should hold for  $t = 0$ , which in view of the above equation entails that

$$\begin{aligned} \nu_{j_0} \Big|_{\mathbb{T}} &= -\frac{1}{j_0!} \partial_s^{j_0} \left( U \circ \psi_{s,t} + \log \operatorname{Re} (-\zeta \partial_t \psi_{s,t} \overline{\psi'_{s,t}}) \right) \Big|_{\mathbb{T}, s=t=0} \\ &- \frac{1}{j_0!} \sum_{j=0}^{j_0-1} \partial_s^{j_0} (s^j \nu_j \circ \psi_{s,t}) \Big|_{\mathbb{T}, s=t=0} =: \mathfrak{G}_{j_0}. \end{aligned}$$

The function  $\mathfrak{G}_{j_0}$  is real-valued and real-analytically smooth on  $\mathbb{T}$ , and given in terms of the already known data set  $(\eta_{j+1,t}$  and  $\nu_j$  for  $0 \leq j \leq j_0 - 1$ ), so that in terms of the Herglotz kernel for the exterior disk we have

$$(4.4.3) \quad \nu_{j_0} := \operatorname{Re} \mathbf{H}_{\mathbb{D}_e} [\mathfrak{G}_{j_0}].$$

**Step 4.** In view of (4.4.3), the equation (4.4.2) holds for  $t = 0$ , which allows us to define a real-analytic function of  $(\zeta, t) \in \mathbb{A}(\rho) \times [0, t_{j_0}]$  for some  $t_{j_0} > 0$  by

$$\mathfrak{F}_{j_0,t}(\zeta) = \frac{1}{2t} \left\{ \partial_s^{j_0} \left( U \circ \psi_{s,t} + \log \operatorname{Re} (-\zeta \partial_t \psi_{s,t} \overline{\psi'_{s,t}}) \right) + \sum_{j=0}^{j_0} \partial_s^{j_0} (s^j \nu_j \circ \psi_{s,t}) \right\} \Big|_{s=0},$$

where we note that  $\mathfrak{F}_{j_0,t}$  is an expression in terms of the data set

$$\{\eta_{j,t}, \nu_j : 1 \leq j \leq j_0\}$$

if we agree that  $\eta_{0,t} = 0$ .

Returning to the equation (4.4.2), we see that it asserts that

$$2t j_0! \operatorname{Re} \eta_{j_0+1,t}(\zeta) + 2t \mathfrak{F}_{j_0,t}(\zeta) = 0, \quad \zeta \in \mathbb{T}.$$

We solve this equation with the help of the Herglotz transform:

$$(4.4.4) \quad \eta_{j_0+1,t} := -(j_0!)^{-1} \mathbf{H}_{\mathbb{D}_e} [\mathfrak{F}_{j_0,t}].$$

In summary, Steps 3 and 4 provide us with the functions  $\nu_{j_0}$  and  $\eta_{j_0+1,t}$  given by (4.4.3) and (4.4.4), respectively, in such a way that the equation (4.4.2) holds. This puts us in a position to return back to Step 3 with  $j_0$  increased to  $j_0 + 1$ .

*Proof of Lemma 4.2.1.* The above described algorithm supplies us with the coefficient functions  $\nu_j$  and  $\eta_{j+1,t}$  for  $0 \leq j \leq \varkappa$ . The functions  $g_s$  and  $\psi_{s,t}$  are then obtained by the equations (4.3.1) and (4.2.4), respectively. Finally, we obtain the function  $f_s$  by the formula

$$(4.4.5) \quad f_s = s^{-\frac{1}{2}} \exp \left( \sum_{j=1}^{\varkappa} s^j \mathbf{H}_{\mathbb{D}_e} [\nu_j|_{\mathbb{T}}] \right) = D_s s^{-\frac{1}{2}} \exp \left( \sum_{j=1}^{\varkappa} s^j \mathbf{H}_{\mathbb{D}_e} [\nu_j|_{\mathbb{T}} - \nu_j(\infty)] \right),$$

where the positive constant  $D_s$  is given by

$$D_s = \exp \left( \sum_{j=1}^{\varkappa} s^j \nu_j(\infty) \right).$$

It is clear from (4.4.5) that  $f_s$  admits an asymptotic expansion of the form (4.2.1), which implicitly defines the coefficients  $X_j$ . We have now established (4.2.6) which gives (4.2.5) by Taylor's formula.

It remains to explain why the radius  $\rho$  with  $0 < \rho < 1$  may be chosen independently of  $\varkappa$ . Since the modified weight  $\Omega$  is real-analytic in neighborhood of the circle  $\mathbb{T}$ , it admits a polarization  $\Omega(z, w)$  which is holomorphic in  $(z, \bar{w})$  for  $(z, w)$  in the  $2\sigma$ -fattened diagonal annulus

$$\widehat{\mathbb{A}}(\rho, \sigma) = \{(z, w) \in \mathbb{C}^2 : (z, w) \in \mathbb{A}(\rho) \times \mathbb{A}(\rho) \text{ and } |z - w| < 2\sigma\}.$$

Here,  $\sigma$  is a strictly positive parameter and  $0 < \rho < 1$ , and  $\rho$  is chosen so that  $\rho \geq (\sqrt{1 + \sigma^2} + \sigma)^{-1}$ . More generally, if  $\mathfrak{F}$  is a real-analytic function which also polarizes to  $\widehat{\mathbb{A}}(\rho, \sigma)$ , then the restriction  $\mathfrak{F}|_{\mathbb{T}}$  has a Laurent series which is convergent in  $\mathbb{A}(\rho)$ . It follows from this that the Herglotz transform  $\mathbf{H}_{\mathbb{D}_e}[\mathfrak{F}|_{\mathbb{T}}]$  represents a holomorphic function in the exterior disk  $\mathbb{D}_e(0, \rho)$ . In particular, the radius  $\rho$  is preserved in the above presented iteration procedure. For more details see [27, Subsections 6.1, 6.3 and 6.12].  $\square$

## 5. EXISTENCE OF ASYMPTOTIC EXPANSIONS

**5.1. A preliminary estimate.** By applying Lemma 4.2.1 with  $s = N^{-1}$ , we obtain functions  $f_s$  and orthostatic conformal maps  $\psi_{s,t}$ , all holomorphic on  $\mathbb{D}_e(0, \rho)$  for some radius  $\rho$  with  $0 < \rho < 1$ . By slight abuse of notation, we denote these functions by  $f_N$  and  $\psi_{N,t}$ , respectively. We put

$$F_N = \mathbf{\Lambda}_N[f_N] = \varphi' \varphi^N e^V f_N \circ \varphi,$$

and for each holomorphic function  $q$  on  $\mathbb{C} \setminus \mathscr{D}_\rho$  of polynomial growth  $|q(z)| = o(|z|^N)$ , we define the function  $h_N$  by

$$h_N = \mathbf{\Lambda}_N^{-1}[q]$$

and denote by  $\mathcal{E}_N = \cup_{0 \leq t \leq \delta_N} \psi_{N,t}(\mathbb{T})$  the region covered by the flow up to time  $\delta_N := N^{-1}(\log N)^2$ . Then  $h_N$  is bounded and holomorphic in  $\mathbb{D}_e(0, \rho)$  with  $h_N(\infty) = 0$ , and we have that

$$(5.1.1) \quad \Upsilon_{N,t}(\zeta) = N e^{-Nt} (1 + O(N^{-\varkappa-1})), \quad 0 \leq t \leq \delta_N, \quad \zeta \in \mathbb{T},$$

where  $\Upsilon_{N,t} = \Upsilon_{s,t}$  is the flow density defined in (4.2.2). As a consequence, we obtain the following estimate.

We recall that  $\chi_0$  is a smooth cut-off function with  $\chi_0 = 1$  on a neighborhood of  $\mathbb{C} \setminus \mathscr{D}$  and  $\chi_0 = 0$  on a neighborhood of  $\overline{\mathscr{D}}_\rho$ .

**Proposition 5.1.1.** *For  $q$  holomorphic on  $\mathbb{C} \setminus \mathscr{D}_\rho$  with  $|q(z)| = o(|z|^N)$  as  $|z| \rightarrow +\infty$ , we have that  $\|\chi_0 F_N\|_{L^2(\mathscr{D}, \omega)}^2 = 1 + O(N^{-\varkappa-1})$ , and*

$$\int_{\mathscr{D}} \chi_0^2(z) q(z) \overline{F_N(z)} \omega(z) dA(w) = O(N^{-\varkappa-1} \|\chi_0 q\|_{L^2(\mathscr{D}, \omega)}).$$

*Proof.* To compute the norm of  $F_N$ , we use the disintegration formula (4.2.3) to obtain

$$\begin{aligned}\|\chi_0 F_N\|_{L^2(\mathcal{D}, \omega)}^2 &= \int_{\mathcal{D}} \chi_0^2(z) |F_N(z)|^2 \omega(w) dA(z) \\ &= \int_{\mathbb{D}} \chi_1^2(w) |f_N(w)|^2 |w|^{2N} \Omega(w) dA(w) \\ &= \int_0^{\delta_N} \int_{\mathbb{T}} \Upsilon_{N,t}(\zeta) d\sigma(\zeta) dt + \int_{\mathbb{D} \setminus \mathcal{E}_N} \chi_1^2(w) |f_N(w)|^2 |w|^{2N} \Omega(w) dA(w),\end{aligned}$$

where we recall that  $\chi_1 = \chi_0 \circ \varphi^{-1}$ , so that  $\chi_1 = 1$  on  $\mathcal{E}_N$  provided that  $N$  is large enough. Since  $|f_N|^2 = O(N)$  on  $\text{supp}(\chi_1)$  it is clear that the integral over  $\mathbb{D} \setminus \mathcal{E}_N$  is, e.g., of order  $O(N^{-2\kappa-2})$ . In addition, in view of (5.1.1), it follows that the integral of the flow density  $\Upsilon_{N,t}$  equals  $1 + O(N^{-\kappa-1})$ . This shows that the norm of  $\chi_0 F_N$  has the claimed asymptotics.

Turning to the orthogonality condition, we split the integral according to

$$\int_{\mathcal{D}} \chi_0^2 q \overline{F_N} \omega dA = \int_{\varphi^{-1}(\mathcal{E}_N)} \chi_0^2 q \overline{F_N} \omega dA + \int_{\mathcal{D} \setminus \varphi^{-1}(\mathcal{E}_N)} \chi_0^2 q \overline{F_N} \omega dA =: I_1 + I_2.$$

We start with  $I_1$ , observe that  $\chi_0 = 1$  on  $\varphi^{-1}(\mathcal{E}_N)$ , and use the disintegration formula (4.2.3) to obtain

$$\begin{aligned}(5.1.2) \quad \int_{\varphi^{-1}(\mathcal{E}_N)} \chi_0^2 q \overline{F_N} \omega dA &= \int_{\mathcal{E}_N} h_N(w) \overline{f_N(w)} |w|^{2N} \Omega(w) dA(w) \\ &= \int_0^{\delta_N} \int_{\mathbb{T}} \frac{h_N}{f_N} \circ \psi_{N,t}(\zeta) \Upsilon_{N,t}(\zeta) d\sigma(\zeta) dt \\ &= N \int_0^{\delta_N} \int_{\mathbb{T}} \frac{h_N}{f_N} \circ \psi_{N,t}(\zeta) \left(1 + O(N^{-\kappa-1})\right) d\sigma(\zeta) e^{-Nt} dt \\ &= O\left(N^{-\kappa} \int_0^{\delta_N} \int_{\mathbb{T}} \frac{h_N}{f_N} \circ \psi_{N,t}(\zeta) d\sigma(\zeta) e^{-Nt} dt\right).\end{aligned}$$

In the last step, we use the mean value property for analytic functions and the fact that  $h_N(\infty) = 0$ . Next, we observe that  $e^{-Nt} \leq 2N^{-1} \Upsilon_{s,t}(\zeta)$  for large  $N$ , so that

$$\begin{aligned}N^{-\kappa} \int_0^{\delta_N} \int_{\mathbb{T}} \frac{h_N}{f_N} \circ \psi_{N,t} d\sigma e^{-Nt} dt &\leq 2N^{-\kappa-1} \int_{\mathcal{E}_N} |h_N f_N| r_N \Omega dA \\ &= O(N^{-\kappa-1} \|\chi_0 q\|_{L^2(\mathcal{D}, \omega)}),\end{aligned}$$

using the isometric property of  $\mathbf{A}_N$  and the Cauchy-Schwarz inequality, where we recall the notation  $r_N(w) = |w|^{2N}$ . In addition, if we write  $\mathcal{F}_N = \varphi^{-1}(\mathcal{E}_N)$ , we see that

$$\begin{aligned}\int_{\mathcal{D} \setminus \mathcal{F}_N} \chi_0^2 |q F_N| \omega dA &\leq \|\chi_0 F_N\|_{L^2(\mathcal{D} \setminus \mathcal{F}_N, \omega)} \|\chi_0 q\|_{L^2(\mathcal{D} \setminus \mathcal{F}_N, \omega)} \\ &= O\left(N^{-\kappa-1} \|\chi_0 q\|_{L^2(\mathcal{D}, \omega)}\right),\end{aligned}$$

which holds by Cauchy-Schwarz inequality and the fact that

$$\|\chi_0 F_N\|_{L^2(\mathcal{D} \setminus \mathcal{F}_N, \omega)} = O(N^{-\kappa-1})$$

as shown previously in connection with the analysis of  $\|\chi_0 F_N\|_{L^2(\mathcal{D}, \omega)}^2$ .  $\square$

**5.2. Polynomialization and the  $\bar{\partial}$ -estimate.** In order to obtain Theorem 1.2.3, we need to show that  $\chi_0 F_N$  is well approximated by the orthogonal polynomial  $P_N$ . A first step is to show that  $\chi_0 F_N$  is well approximated by elements of the space  $\text{Pol}_N$ . Since  $\bar{\partial}(\chi_0 F_N) = F_N \bar{\partial} \chi_0$  is supported on a set of the form  $\mathcal{D}_{\rho_1} \setminus \mathcal{D}_\rho$  with  $0 < \rho < \rho_1$  and since  $F_N = \mathbf{A}_N[f_N]$  for  $|f_N|^2 = O(N)$ , we find that

$$\int_{\mathcal{D}} |F_N \bar{\partial} \chi_0|^2 dA \lesssim N \int_{\mathcal{D}_{\rho_1} \setminus \mathcal{D}_\rho} |\varphi|^{2N} |\varphi'|^2 dA = N \int_{\mathbb{D}_{\rho_1} \setminus \mathbb{D}(0, \rho)} r_N dA \leq \rho_1^{2N+2},$$

where we recall  $r_N(z) = |z|^{2N}$  and use the boundedness of the gradient of  $\chi_0$  and  $V$  in the first step. We now apply Hörmander's classical  $\bar{\partial}$ -estimate with weight  $\phi(z) = 2 \log(1 + |z|^2)$ , which tells us that the equation  $\bar{\partial} u = F_N \bar{\partial} \chi_0$  admits a solution  $u$  with

$$\int_{\mathbb{C}} |u|^2 e^{-\phi} dA \leq \int_{\mathcal{D}} |F_N \bar{\partial} \chi_0|^2 \frac{e^{-\phi}}{\Delta \phi} dA \leq \int_{\mathcal{D}} |F_N \bar{\partial} \chi_0|^2 dA = O(\rho_1^{2N})$$

since  $\Delta \phi = 2e^{-\phi}$  on  $\mathcal{D}$ . The function  $u$  is holomorphic in the exterior domain  $\mathbb{C} \setminus \mathcal{D}$ , so the finiteness of  $\|u\|_{L^2(\mathbb{C}, e^{-\phi})}$  implies that  $|u(z)| = O(1)$  as  $|z| \rightarrow +\infty$ . From this it follows that the function  $Q_N$  given by  $Q_N := \chi_0 F_N - u$  is a polynomial of degree  $N$ , and that  $\|Q_N - \chi_0 F_N\|_{L^2(\mathcal{D}, \omega)} = O(\rho_1^{2N})$ .

*Proof of Theorem 1.2.3.* We define a polynomial  $P_N^*$  by

$$P_N^* = (\mathbf{I} - \mathbf{S}_N)Q_N = Q_N - \mathbf{S}_N Q_N,$$

where  $\mathbf{S}_N$  is the orthogonal projection of  $L^2(\mathcal{D}, \omega)$  onto the space  $\text{Pol}_{N-1}^2(\mathcal{D}, \omega)$ . Since  $Q_N$  is a polynomial of degree  $N$ , the polynomial  $P_N^*$  is a constant multiple of the orthogonal polynomial  $P_N$ , say  $P_N^* = c_N P_N$ .

Since  $\|Q_N - \chi_0 F_N\|_{L^2(\mathcal{D}, \omega)} = O(\rho_1^{2N})$ , it follows from Proposition 5.1.1 that  $Q_N$  has norm  $1 + O(N^{-\varkappa-1})$  in  $L^2(\mathcal{D}, \omega)$  and that

$$\int_{\mathcal{D}} q(z) \bar{Q}_N(z) \omega(z) dA(z) = O(N^{-\varkappa-1} \|q\|_{L^2(\mathcal{D}, \omega)}).$$

But by duality, we then see that

$$\|\mathbf{S}_N Q_N\|_{L^2(\mathcal{D}, \omega)} = O(N^{-\varkappa-1}).$$

From these considerations we arrive at

$$\|c_N P_N - Q_N\|_{L^2(\mathcal{D}, \omega)} = \|P_N^* - Q_N\|_{L^2(\mathcal{D}, \omega)} = \|\mathbf{S}_N Q_N\|_{L^2(\mathcal{D}, \omega)} = O(N^{-\varkappa-1}).$$

Since  $P_N$  is normalized and since  $\|Q_N\|_{L^2(\mathcal{D}, \omega)} = 1 + O(N^{-\varkappa-1})$ , we find that

$$|c_N|^2 = \|c_N P_N\|_{L^2(\mathcal{D}, \omega)}^2 = \|Q_N - \mathbf{S}_N Q_N\|_{L^2(\mathcal{D}, \omega)}^2 = 1 + O(N^{-\varkappa-1})$$

The functions  $F_N$ ,  $Q_N$  and  $P_N^*$  all have the same leading coefficient, where we interpret the leading coefficient of  $F_N$  as the limit  $\lim_{z \rightarrow \infty} (F_N(z)/z^N)$ . However,  $F_N$  is chosen to have real and positive leading coefficient, and hence  $|c_N| = c_N$ , and the result follows.  $\square$

**5.3. A Bernstein-Walsh type inequality.** Denote by  $A_{N, \rho}^2(\omega)$  the space of all holomorphic function on the exterior disk  $\mathbb{C} \setminus \bar{\mathcal{D}}_\rho$  subject to the growth condition

$$|f(z)| = O(|z|^N) \quad \text{as } |z| \rightarrow +\infty,$$

endowed with the Hilbert space structure of  $L^2(\mathcal{D} \setminus \mathcal{D}_\rho, \omega)$ .

**Lemma 5.3.1.** *Fix a radius  $\rho_1$  with  $\rho < \rho_1 < 1$ . Then there exists a positive constant  $C = C_{\rho, \rho_1, \omega, \mathcal{D}}$  such that for  $f \in A_{N, \rho}^2(\omega)$  we have*

$$|f(z)| \leq CN \max\{|\varphi(z)|^N, 1\}, \quad z \in \mathbb{C} \setminus \mathcal{D}_{\rho_1},$$

as  $N \rightarrow +\infty$ .

*Proof.* Denote by  $K_N(z, w) = K_{N, w}(z)$  the reproducing kernel for the unweighted space  $A_{N, \rho}^2$  corresponding to  $\omega = 1$ . Using the reproducing property of  $K_N$  and the Cauchy-Schwarz inequality we find that

$$\begin{aligned} |f(z)|^2 &= |\langle f, K_{N, z} \rangle_{L^2(\mathcal{D} \setminus \mathcal{D}_\rho)}|^2 \leq \|f\|_{L^2(\mathcal{D} \setminus \mathcal{D}_\rho)}^2 K_N(z, z) \\ &\lesssim \|f\|_{L^2(\mathcal{D} \setminus \mathcal{D}_{\rho, \omega})}^2 K_N(z, z), \quad z \in \mathbb{C} \setminus \overline{\mathcal{D}_\rho}, \end{aligned}$$

where the implied constant depends on the bound from below of  $\omega$  on the set  $\mathcal{D} \setminus \mathcal{D}_\rho$ . We proceed to estimate the diagonal restriction of the kernel  $K_N$ . An orthonormal basis for  $A_{N, \rho}^2$  is supplied by

$$c_{n, \rho} \varphi^n(z) \varphi'(z),$$

where  $n$  ranges over the integers in the interval  $-\infty < n \leq N$ , and where

$$c_{n, \rho} = \begin{cases} \frac{\sqrt{2n+2}}{\sqrt{1-\rho^{2n+2}}}, & n \neq -1, \\ \frac{1}{\sqrt{|\log \rho^2|}}, & n = -1. \end{cases}$$

As a consequence, the diagonal restriction of the kernel  $K_N$  is given by the formula

$$K_N(z, z) = \left\{ \log \frac{1}{\rho^2} \right\}^{-1} |\varphi(z)|^{-2} |\varphi'(z)|^2 + \sum_{n=-\infty, n \neq -1}^N \frac{2(n+1)}{1-\rho^{2n+2}} |\varphi(z)|^{2n} |\varphi'(z)|^2,$$

for  $z \in \mathbb{C} \setminus \overline{\mathcal{D}_\rho}$ . It is easy to see that for any number  $\rho_1$  with  $\rho < \rho_1 < 1$  we have that

$$\sup_{z \in \mathcal{D} \setminus \mathcal{D}_{\rho_1}} K_N(z, z) \lesssim N^2.$$

Indeed, an explicit calculation in the annular variable  $w = \varphi(z)$  using a trivial bound of the above sum gives this. The result now follows by applying the maximum principle to the function  $f/\varphi^N$  in the domain  $\mathbb{C} \setminus \overline{\mathcal{D}}$ .  $\square$

We proceed to the proof of the main theorem.

*Proof of Theorem 1.2.1.* In view of Theorem 1.2.3 we have that

$$P_N(z) = \chi_0 F_N + N^{-\varkappa-1} v_N,$$

where  $v_N$  is confined to a ball of fixed positive radius in the space  $A_{N, \rho}^2(\mathcal{D}, \omega)$ . In view of Lemma 5.3.1 we have

$$|v_N(z)| \lesssim N \max\{|\varphi(z)|^N, 1\}, \quad z \in \mathbb{C} \setminus \overline{\mathcal{D}_{\rho_1}}.$$

Then for  $z \in \mathbb{C} \setminus \mathcal{D}$  we obtain

$$\begin{aligned} (5.3.1) \quad P_N(z) &= F_N(z) + O(N^{-\varkappa} |\varphi(z)|^N) \\ &= D_N N^{\frac{1}{2}} \varphi'(z) \varphi^N(z) e^{V(z)} \left( \sum_{j=0}^{\varkappa} N^{-j} B_j(z) + O(N^{-\varkappa-\frac{1}{2}}) \right), \end{aligned}$$

while on the annular set  $z \in \mathcal{D} \setminus \overline{\mathcal{D}}_{\rho_1}$  we instead have

$$\begin{aligned} P_N(z) &= F_N(z) + O(N^{-\varkappa}) \\ &= D_N N^{\frac{1}{2}} \varphi'(z) \varphi^N(z) e^{V(z)} \left( \sum_{j=0}^{\varkappa_1} N^{-j} B_j(z) + O(N^{-\varkappa - \frac{1}{2}} e^{-N \log |\varphi(z)|}) \right). \end{aligned}$$

Since  $|\varphi| = 1$  on  $\partial\mathcal{D}$ , for any  $z \in \mathcal{D}$  with  $\text{dist}_{\mathbb{C}}(z, \partial\mathcal{D}) \leq AN^{-1} \log N$  we have by Taylor's formula that

$$N \log \frac{1}{|\varphi(z)|} \leq CA \log N,$$

for some positive constant  $C$  depending on the maximum of  $|\varphi'|$  on  $\partial\mathcal{D}_{\rho_1}$ . As a consequence we obtain

$$(5.3.2) \quad P_N(z) = D_N N^{\frac{1}{2}} \varphi'(z) \varphi^N(z) e^{V(z)} \left( \sum_{j=0}^{\varkappa} N^{-j} B_j(z) + O(N^{-\varkappa - \frac{1}{2} + CA}) \right),$$

for all  $z \in \mathbb{C}$  with  $\text{dist}_{\mathbb{C}}(z, \mathcal{D}^c) \leq AN^{-1} \log N$ .

The expansions (5.3.1) and (5.3.2) are of the desired form, and would give the desired expansion for  $P_N$  except that the error terms are too big. However, since all the coefficients  $B_j$  are bounded in the indicated domain, we are free to jack up  $\varkappa$  to get the desired estimate. For instance, if we apply (5.3.1) and (5.3.2) with  $\varkappa$  replaced by  $\varkappa_1 \geq \varkappa + \frac{1}{2} + CA$  we obtain the error term  $O(N^{-\varkappa-1})$ , as claimed.

Finally, the assertion for the monic polynomials  $\pi_N$  follows from the improved versions of (5.3.1) and (5.3.2) by multiplying  $P_N$  with the appropriate positive constant.  $\square$

**5.4. Off-spectral asymptotics.** We describe next what changes are necessary in order for the asymptotic analysis of  $P_N$  to carry over to the setting of normalized polynomial Bergman kernel  $k_{N,w}$  rooted at an off-spectral point  $w$ .

*Proof of Proposition 2.2.1.* The whole proof scheme of the previous result carries over with minimal changes. That is, one obtains first a version of Theorem 1.2.3 using a slight modification of the main Lemma (Lemma 4.2.1), see below. After this has been done, we can use  $\bar{\partial}$ -estimates and standard Hilbert space techniques to finish the proof as in Subsection 5.3.

The only difference when obtaining Theorem 1.2.3 is that the estimate between the last two lines in (5.1.2) should hold whenever  $h_N$  is bounded and holomorphic in  $\mathbb{C} \setminus \mathbb{D}(0, \rho)$  with  $h_N(\varphi(w)) = 0$ , rather than with  $h_N(\infty) = 0$ . This, in turn, boils down to making the following technical change in the flow Lemma 4.2.1: Instead of choosing the functions  $f_s$  and  $\psi_{s,t}$  such that (4.2.5) holds for the flow density  $\Upsilon_{s,t}$ , we need to choose them so that

$$\Pi_{s,t}^w(\zeta) := \log \Upsilon_{s,t}^w(\zeta) + \log s + s^{-1}t = \log \frac{d\varpi_{\mathbb{D}_e, \varphi(w)}}{d\sigma} + O(s^{\varkappa+1})$$

holds for  $\zeta \in \mathbb{T}$  and  $0 \leq t \leq \delta_s$ . Here,  $\Upsilon_{s,t}^w$  is the analogously defined flow density. If we make the ansatz

$$|f_s|^2 = s^{-1} |\varrho_w \circ \varphi^{-1}|^2 e^{g_{s,w}},$$

where  $\varrho_w$  was defined in (2.2.1) and where  $g_{s,w}$  is bounded and harmonic with an asymptotic expansion as in (4.3.1), then the algorithmic procedure used to obtain Lemma 4.2.1 applies to give the suitably modified orthogonal foliation flow.  $\square$

## 6. THE DISTRIBUTIONAL ASYMPTOTIC EXPANSION

**6.1. The action of the holomorphic wave function on quasipolynomials.** We let  $g$  denote a bounded  $C^\infty$ -smooth on  $\mathbb{C} \setminus \mathbb{D}(0, \rho)$ , which is holomorphic on the exterior disk  $\mathbb{D}_e$ , and consider the (Hermitian) action of  $p_N = \Lambda_N^{-1} P_N$  on  $g$  in the Hilbert space structure inherited from  $L^2(\mathcal{D}, \omega)$ .

**Proposition 6.1.1.** *With  $g$  as above, we have for any given integer  $\varkappa \geq 0$  that*

$$\int_{\mathbb{D} \setminus \mathbb{D}(0, \rho)} g(w) \overline{p_N(w)} |w|^{2N} \Omega(w) dA(w) = D_N^{-1} N^{-\frac{1}{2}} g(\infty) + O(N^{-\varkappa - \frac{3}{2}}),$$

as  $N \rightarrow \infty$ . Here, the implied constant is uniformly bounded provided that the norms  $\|(\partial_r^\times)^j g\|_{L^\infty(\mathbb{D}_e(0, \rho))}$  are all uniformly bounded for  $j \leq \varkappa + 1$ .

*Proof.* Assume first that  $g(\infty) = 0$ . In view Theorem 1.2.1 and Proposition 3.2.1 we have

$$\begin{aligned} & \frac{1}{D_N N^{\frac{1}{2}}} \int_{\mathbb{D} \setminus \mathbb{D}(0, \rho)} g(w) \overline{p_N(z)} |w|^{2N} \Omega(w) dA(w) \\ &= \sum_{j+k \leq \varkappa} \frac{1}{N^{j+k+1} 2^j} \int_{\mathbb{T}} \partial_s^j (g(e^{-s+it}) \bar{X}_k(e^{-s+it}) e^{-2s} \Omega(e^{-s+it})) \Big|_{s=0} \frac{dt}{2\pi} + O(N^{-\varkappa-2}) \\ &= \sum_{\substack{j+k \leq \varkappa \\ 0 \leq l \leq j}} \frac{A(j, k, l)}{N^{j+k+1}} \int_0^{2\pi} (\partial_r^\times)^{j-l} g(re^{it}) (\partial_r^\times + 2\mathbf{I})^l (\bar{X}_k \Omega)(re^{it}) \Big|_{r=1} \frac{dt}{2\pi} + O(N^{-\varkappa-2}), \end{aligned}$$

where the indices  $j, k$  are non-negative integers,  $A(j, k, l)$  denotes the constant

$$A(j, k, l) = (-1)^j 2^{-j} \binom{j}{l},$$

and where we apply the Leibniz rule in the last step. We next write

$$-\partial_r^\times = -2\bar{\partial}_z^\times + i\partial_t$$

where  $z = re^{it}$ . Hence, for bounded  $g$  in  $C^\infty(\mathbb{D}_e(0, \rho))$  with  $\bar{\partial}g = 0$  on  $\mathbb{D}_e$  and  $g(\infty) = 0$ , we find that  $-\partial_r^\times g = i\partial_t g$  holds to infinite order on  $\mathbb{T}$ . As a consequence, we find that

$$\begin{aligned} (6.1.1) \quad & \frac{1}{D_N N^{\frac{1}{2}}} \int_{\mathbb{D} \setminus \mathbb{D}(0, \rho)} g(w) \overline{p_N(w)} |w|^{2N} \Omega(w) dA(w) \\ &= \sum_{j, k \leq \varkappa} \frac{(-1)^j}{N^{j+k+1}} \int_0^{2\pi} g(re^{it}) (\partial_z^\times - \mathbf{I})^j (X_k \Omega)(re^{it}) \Big|_{r=1} \frac{dt}{2\pi} + O(N^{-\varkappa-2}) \\ &= \sum_{k=0}^{\varkappa} N^{-k-1} \int_0^{2\pi} g(e^{it}) \sum_{l=0}^k \mathbf{T}^{k-l} X_l(e^{it}) \frac{dt}{2\pi} + O(N^{-\varkappa-2}) = O(N^{-\varkappa-2}), \end{aligned}$$

where where recall that  $\partial_z^\times = z\partial_z$  for  $z = re^{it}$  and that  $\Omega|_{\mathbb{T}} = 1$ . Here, the last equality in (6.1.1) holds since  $\sum_{l=0}^p \mathbf{T}^{p-l} X_l \in H^2$  by the computations in Section 3.

If  $g(\infty) \neq 0$ , we instead obtain

$$\int_{\mathbb{D} \setminus \mathbb{D}(0, \rho)} g(w) \overline{p_N(z)} |w|^{2N} \Omega(w) dA(w) = c_N g(\infty) + O(N^{-\varkappa - \frac{3}{2}})$$

for some constant  $c_N$ , which we proceed to compute. Since  $p_N$  is approximately orthogonal to holomorphic functions vanishing at infinity (by Theorem 1.2.3) and

since by construction  $p_N(\infty) = D_N N^{\frac{1}{2}}$ , we find that

$$\begin{aligned} 1 &= \|p_N\|_{L^2(\mathbb{D} \setminus \mathbb{D}(0, \rho), r_N \Omega)}^2 + O(N^{-\kappa-1}) \\ &= D_N N^{\frac{1}{2}} \int_{\mathbb{D} \setminus \mathbb{D}(0, \rho)} \overline{p_N(w)} |w|^{2N} \Omega(w) dA(w) + O(N^{-\kappa-1}) \\ &= c_N D_N N^{\frac{1}{2}} + O(N^{-\kappa-1}) \end{aligned}$$

which gives  $c_N = D_N^{-1} N^{-\frac{1}{2}} + O(N^{-\kappa-1})$ , as claimed.  $\square$

**6.2. The wave function as a distribution.** Recall that  $G$  is a smooth bounded test function split according to (2.1.1), and let  $g = G \circ \varphi^{-1}$  have a corresponding decomposition  $g = g_0 + g_+ + g_-$ . The function  $g$  is automatically defined on some exterior disk  $\mathbb{C} \setminus \mathbb{D}(0, \rho_1)$ , where  $0 < \rho_1 < 1$ .

*Proof of Theorem 2.1.1.* We apply Proposition 6.1.1 to  $p_N g_+$ , and the conjugated version of the same proposition to  $\overline{p_N} g_-$ . We recall that  $X_0 = 1$  and that for  $k \geq 1$  we have  $\sum_{l=0}^k \mathbf{T}^{k-l} X_l \in H_0^2$  by Theorem 1.3.1. By Theorem 1.2.1, the mass of the orthogonal polynomial  $P_N$  is concentrated near  $\partial \mathcal{D}$  in the sense that for any fixed  $0 < \rho_2 < 1$  we have

$$\int_{\mathcal{D}_{\rho_2}} |P_N|^2 \omega dA = O(N^{-\kappa-1})$$

as  $N \rightarrow +\infty$ . Hence, we have that

$$\begin{aligned} &\int_{\mathcal{D}} G_+(z) |P_N(z)|^2 \omega(z) dA(z) \\ &= \int_{\mathbb{D} \setminus \mathbb{D}(0, \rho_2)} f_N(w) g_+(w) \overline{f_N(w)} |w|^{2N} \Omega(w) dA(w) + O(N^{-\kappa-1}) \\ &= (N^{\frac{1}{2}} D_N)^{-1} f_N(\infty) g_+(\infty) + O(N^{-\kappa-1}) = G_+(\infty) + O(N^{-\kappa-1}), \end{aligned}$$

where  $\rho_2$  with  $\rho_1 < \rho_2 < 1$  is close enough to 1, and similarly for  $G_-$ . The remaining conclusion follows by applying Proposition 3.2.1 and the Leibniz rule to the function  $g_0(e^{-s+it}) |f_N(e^{-s+it})|^2 \Omega(e^{-s+it})$ , and changing the order of summation (taking the order  $\nu$  of the radial differential operator which hits  $g_0$  as the basic parameter).  $\square$

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