INTERSECTIONS IN LUBIN-TATE SPACE AND BIQUADRATIC FUNDAMENTAL LEMMAS

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Abstract. We compute the intersection multiplicities of special cycles in Lubin-Tate spaces, and formulate a new arithmetic fundamental lemma relating these intersections to derivatives of orbital integrals.

Contents

1. Introduction

We compute the intersection multiplicity of special cycles in Lubin-Tate deformation spaces. Interpolating ideas from [HS19, Li21, Zh12], we formulate a new arithmetic fundamental lemma expressing the intersection as the derivative of an orbital integral. The results and conjectures of [Li21] are special cases of those presented here.

1.1. **An intersection problem.** Fix a nonarchimedean local field F (of any characteristic), let $\breve{F}$ be the completion of its maximal unramified extension, and let $\boldsymbol{k}$ be the residue field of $\mathcal{O}_{\breve{F}}$.

Suppose $\mathcal{G}$ is a one-dimensional formal $\mathcal{O}_F$-module of height $2h$ over $\boldsymbol{k}$, and denote by $X \to \mathrm{Spf}(\mathcal{O}_{\breve{F}})$ the Lubin-Tate deformation space of $\mathcal{G}$. Being (noncanonically) isomorphic to the formal spectrum of a power series ring $\mathcal{O}_{\breve{F}}[[T_1, \ldots, T_{2h-1}]]$, it is a formal scheme of dimension $2h$.

Let K_1 and K_2 be separable quadratic field extensions of F, and fix $\mathcal{O}_F$-algebra maps

$$\Phi_1 : \mathcal{O}_{K_1} \to \mathrm{End}_{\mathcal{O}_F}(\mathcal{G}), \qquad \Phi_2 : \mathcal{O}_{K_2} \to \mathrm{End}_{\mathcal{O}_F}(\mathcal{G}). \tag{1.1.1}$$

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The deformation space of $\mathcal{G}$ with its extra $\mathcal{O}_{K_i}$-action is a closed formal subscheme $f_i : Y_i \hookrightarrow X$.

The formal subschemes Y_1 and Y_2 each have dimension h, and so it is natural to consider the intersection multiplicity

$$I(\Phi_1, \Phi_2) = \mathrm{len}_{\mathcal{O}_{\breve{F}}} H^0(X, f_{1*}\mathcal{O}_{Y_1} \otimes_{\mathcal{O}_X} f_{2*}\mathcal{O}_{Y_2}). \tag{1.1.2}$$

In this generality (1.1.2) could be infinite; this would be the case, for example, if $K_1 = K_2$ and $\Phi_1 = \Phi_2$.

One of the main results of this paper is a formula for the above intersection multiplicity. The formula involves a polynomial invariant associated to the pair (Φ_1, Φ_2), which we now describe.

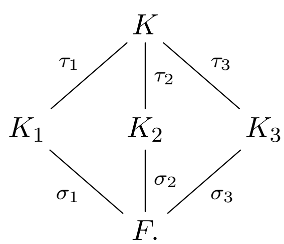
1.2. **Invariant polynomials.** Suppose for the moment that K_1 and K_2 are any quadratic étale F-algebras. Equivalently, K_i is either a quadratic Galois extension, or $K_i \cong F \times F$. Denote by $\sigma_i \in \mathrm{Aut}(K_i/F)$ the nontrivial automorphism.

The automorphism group of the F-algebra $K = K_1 \otimes_F K_2$ contains the Klein four subgroup $\{\mathrm{id}, \tau_1, \tau_2, \tau_3\} \subset \mathrm{Aut}(K/F)$ with nontrivial elements

$$(x \otimes y)^{\tau_1} = x \otimes y^{\sigma_2}, \quad (x \otimes y)^{\tau_2} = x^{\sigma_1} \otimes y, \quad (x \otimes y)^{\tau_3} = x^{\sigma_1} \otimes y^{\sigma_2}.$$

Denote by $K_3 \subset K$ the subalgebra of elements fixed by τ_3. The picture, along with generators of the automorphism groups, is

(1.2.1)



Elementary algebra shows that $K_1 \cong K_2$ if and only if $K_3 \cong F \times F$. If K_1 and K_2 are nonisomorphic field extensions, then K is a biquadratic field extension of F, and K_3 is its unique third quadratic subfield.

Now suppose B is any central simple F-algebra of dimension $4h^2$, and that we are given F-algebra embeddings

$$\Phi_1 : K_1 \to B, \qquad \Phi_2 : K_2 \to B. \tag{1.2.2}$$

From this data we construct in §2.2 a distinguished element $\mathbf{s}$ in the central simple K_3-algebra $C = B \otimes_F K_3$, and show that its reduced characteristic polynomial is a square. The *invariant polynomial*

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)}(T) \in K_3[T]$$

is its unique monic square root.

The pair (Φ_1, Φ_2) is *regular semisimple* (Definition 2.5.1) if $\mathbf{s} \in C^\times$ and the subalgebra $K_3[\mathbf{s}] \subset C$ is an étale K_3-algebra of dimension h. This is our way of making precise the notion that the pair (Φ_1, Φ_2) is in general position; see also Proposition A.2.2. As evidence that the invariant polynomial is a natural quantity to consider, we prove in Corollary 2.5.8 that the $B^\times$-conjugacy class of a regular semisimple pair is completely determined by its invariant polynomial.

1.3. **The intersection formula.** Now return to the setting of §1.1, and apply the constructions of §1.2 to the central division algebra

$$B = \mathrm{End}_{\mathcal{O}_F}(\mathcal{G}) \otimes F$$

to obtain a degree h polynomial $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ with coefficients in K_3.

To any $\mathcal{O}_F$-algebra embeddings

$$\Psi_1 : \mathcal{O}_{K_1} \to M_{2h}(\mathcal{O}_F), \qquad \Psi_2 : \mathcal{O}_{K_2} \to M_{2h}(\mathcal{O}_F),$$

we may associate in the same way a degree h polynomial $\mathrm{Inv}_{(\Psi_1,\Psi_2)}$ with coefficients in K_3. More generally, for any $g \in \mathrm{GL}_{2h}(\mathcal{O}_F)$ we may form the conjugate embedding $g\Psi_2 g^{-1}$ and the polynomial $\mathrm{Inv}_{(\Psi_1,g\Psi_2 g^{-1})}$. Let

$$R(g) = \mathrm{Res}\big(\mathrm{Inv}_{(\Phi_1,\Phi_2)}, \mathrm{Inv}_{(\Psi_1,g\Psi_2 g^{-1})}\big) \in K_3$$

be the resultant. It follows from Proposition 2.3.3 that $R(g)^2 \in F$, and we define

$$|R(g)| = \sqrt{|R(g)^2|}.$$

Here the absolute value on F is normalized by $|\pi| = q^{-1}$, where $\pi \in \mathcal{O}_F$ is a uniformizing parameter and q is the cardinality of the residue field.

Theorem A. *If (Φ_1, Φ_2) is regular semisimple then $|R(g)| \neq 0$ for all $g \in \mathrm{GL}_{2h}(\mathcal{O}_F)$, and the intersection multiplicity (1.1.2) satisfies*

$$I(\Phi_1, \Phi_2) = c(0) \cdot |d_1 d_2|^{-\frac{h^2}{2}} \cdot \int_{\mathrm{GL}_{2h}(\mathcal{O}_F)} \frac{dg}{|R(g)|}.$$

In particular, the left hand side is finite. Here $d_i \in \mathcal{O}_F$ is any generator of the discriminant of K_i/F, and

$$c(0) = \frac{\#\mathrm{GL}_{2h}(\mathcal{O}_F/\pi\mathcal{O}_F)}{\#\mathrm{GL}_h(\mathcal{O}_{K_1}/\pi\mathcal{O}_{K_1}) \cdot \#\mathrm{GL}_h(\mathcal{O}_{K_2}/\pi\mathcal{O}_{K_2})}.$$

Theorem A is a special case of Theorem 4.4.2, which gives a more general intersection formula for cycles on covers of X obtained by adding Drinfeld level structures. When $K_1 = K_2$, it specializes to the main result of [Li21].

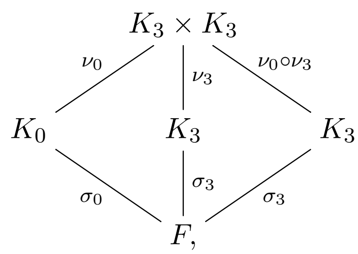
1.4. **Matching and fundamental lemmas.** Let us return to the general situation of §1.2, where K_1 an K_2 are quadratic étale F-algebras. From the pair (K_1, K_2) we constructed a diagram (1.2.1) of étale F-algebras. There is another diagram of étale F-algebras, in some sense dual to the first.

Having already constructed K_3 from the pair (K_1, K_2), we may set

$$K_0 = F \times F$$

and repeat the construction of (1.2.1) with the pair (K_1, K_2) replaced by (K_0, K_3). Using the canonical isomorphism $K_0 \otimes_F K_3 \cong K_3 \times K_3$, the resulting picture is

(1.4.1)



where $(x,y)^{\nu_0} = (x^{\sigma_3}, y^{\sigma_3})$ and $(x,y)^{\nu_3} = (y,x)$. To be completely explicit: the middle inclusion of K_3 into $K_3 \times K_3$ is given by $z \mapsto (z,z)$, while the inclusion on the right is $z \mapsto (z, z^{\sigma_3})$.

The key point is that the pairs (K_1, K_2) and (K_0, K_3) give rise to the *same* third quadratic algebra K_3. The constructions described in §1.2 work equally well with the pair (K_1, K_2) replaced by (K_0, K_3), and associate to any pair of F-algebra embeddings

$$\Phi_0 : K_0 \to M_{2h}(F), \qquad \Phi_3 : K_3 \to M_{2h}(F) \tag{1.4.2}$$

a degree h monic polynomial $\mathrm{Inv}_{(\Phi_0,\Phi_3)}(T) \in K_3[T]$.

Suppose the pairs (Φ_1, Φ_2) and (Φ_0, Φ_3) of (1.2.2) and (1.4.2) are regular semisimple and *matching*, in the sense that they have the same invariant polynomial. Suppose also that the extension K/K_3 is unramified.

In Definition 3.2.4 we associate an *orbital integral* $O_{(\Phi_0,\Phi_3)}(f; s, \eta)$ to any compactly supported function

$$f : \mathrm{GL}_{2h}(\mathcal{O}_F) \backslash \mathrm{GL}_{2h}(F) / \mathrm{GL}_{2h}(\mathcal{O}_F) \to \mathbb{C}. \tag{1.4.3}$$

Here $\eta : K_3^\times \to \{\pm 1\}$ is the unramified quadratic character determined by K/K_3, and s is a complex variable. We propose two conjectures on the behavior of this orbital integral at $s = 0$.

Suppose first that the central simple algebra in (1.2.2) is $B = M_{2h}(F)$. In Definition 3.1.1 we associate to (1.4.3) another orbital integral $O_{(\Phi_1,\Phi_2)}(f)$, and conjecture that

$$O_{(\Phi_0,\Phi_3)}(f; 0, \eta) = \pm O_{(\Phi_1,\Phi_2)}(f). \tag{1.4.4}$$

This is the *biquadratic fundamental lemma* of Conjecture 3.4.1. The ambiguity in sign arises because $O_{(\Phi_0,\Phi_3)}(f; s, \eta)$ depends on some additional choices, which make its value at $s = 0$ well-defined only up to ± 1. When $K_1 = K_2$ the biquadratic fundamental lemma is equivalent to the Guo-Jacquet fundamental lemma proposed in [Guo96].

Now suppose that the central simple algebra in (1.2.2) is the division algebra $B = \mathrm{End}_{\mathcal{O}_F}(\mathcal{G}) \otimes F$ of (1.1.1). Because $\mathrm{End}_{\mathcal{O}_F}(\mathcal{G})$ is a (noncommutative) discrete valuation ring, the pair (Φ_1, Φ_2) of (1.2.2) automatically satisfies the integrality conditions of (1.1.1). In this case, we show in Proposition 3.3.3 that the matching of (Φ_0, Φ_3) with (Φ_1, Φ_2) implies the vanishing of $O_{(\Phi_0,\Phi_3)}(f;s,\eta)$ at $s=0$ for all f. When $f=\mathbf{1}$ is the characteristic function of $\mathrm{GL}_{2h}(\mathcal{O}_F)$, the work of Zhang [Zh12], its descendants [LZ17, Mih17, RTZ13, Zh19], and the relative trace formula approach to the Gross-Kohnen-Zagier theorem [HS19], suggests one should have an *arithmetic biquadratic fundamental lemma* of (roughly) the form

$$\frac{d}{ds}O_{(\Phi_0,\Phi_3)}(\mathbf{1};s,\eta)\big|_{s=0} \stackrel{?}{=} I(\Phi_1,\Phi_2)\log(q). \tag{1.4.5}$$

The equality (1.4.5) was conjectured by the authors in an earlier version of this paper, but it was explained to us by Andreas Mihatsch that this is a bit too naive. The correct conjecture should involve identifying the Lubin-Tate space $X=X^{(0)}$ as one connected component of a larger Rapoport-Zink space

$$X^\bullet = \bigsqcup_{\ell\in\mathbb{Z}} X^{(\ell)},$$

extending the definition of the cycles $Y_i\to X$ to cycles $Y_i^\bullet\to X^\bullet$, and replacing the right hand side of (1.4.5) with an intersection number taking all connected components into account (carefully, as simply adding the intersection multiplicities on all components will always result in ∞). The reader can find the precise statement of this corrected version of (1.4.5) stated as Conjecture 4.5.2.

Remark 1.4.1. When $h\in\{1,2\}$, the authors have verified both the biquadratic fundamental lemma (at least when $f=\mathbf{1}$ is the characteristic function of $\mathrm{GL}_{2h}(\mathcal{O}_F)$) and the arithmetic biquadratic fundamental lemma (1.4.4). The calculations for $h=1$ appear in §5. Calculations for $h=2$ will appear in future work.

1.5. **Global analogues.** The results and conjectures of this work are purely local in nature, but we wish to give the reader at least some indication of their global analogues. As this is purely for motivational purposes, the following discussion will be somewhat impressionistic.

The global problem to which the biquadratic fundamental lemma (1.4.4) should be applied is purely representation-theoretic. Suppose we start with a global field F and a central simple F-algebra B of dimension $4h^2$. Let K_1 and K_2 be quadratic étale F-algebras, let K_0 and K_3 be as in (1.4.1), and suppose we are given pairs of F-algebra embeddings (Φ_1,Φ_2) and (Φ_0,Φ_3) as in (1.2.2) and (1.4.2).

Let $\mathbb{A}$ be the adele ring of F. Suppose we are given a cuspidal automorphic representation π^B of $B_{\mathbb{A}}^\times$, and let π be its Jacquet-Langlands lift to $\mathrm{GL}_{2h}(\mathbb{A})$. The idea is that the biquadratic fundamental lemma should imply period relations of the form

$$\int_{[H_1]} f_1(h_1)\,dh_1 \int_{[H_2]} \overline{f_2(h_2)}\,dh_2 = \int_{[H_0]} f_0(h_0)\,dh_0 \int_{[H_3]} \overline{f_3(h_3)}\cdot\eta(h_3)\,dh_3$$

for suitable $f_1, f_2 \in \pi^B$ and $f_0, f_3 \in \pi$. Here

$$H_1, H_2 \subset B^\times \quad \text{and} \quad H_0, H_3 \subset \mathrm{GL}_{2h}(F)$$

are the centralizers of the various Φ_i's, so that $H_i \cong \mathrm{GL}_{2h}(K_i)$, and

$$[H_i] = H_i(F)\backslash H_i(\mathbb{A})/\mathbb{A}^\times.$$

The character $\eta : H_3(\mathbb{A}) \to \{\pm 1\}$ is determined by the extension K/K_3, as in (3.2.3). In the special case where $B = M_2(F)$, F has positive characteristic, and π is unramified, such a period relation appears as [HS19, Theorem D].

It is expected (and sometimes known) that the four periods above are related to special values of L-functions [FJ93, FMW18, Guo96]. For example, the period integral over $[H_0]$ is related to the central value of the standard L-function of π.

Now we turn to the global analogue of the arithmetic biquadratic fundamental lemma (1.4.5). Here one expects a formula relating central derivatives of L-functions to intersections of special cycles on unitary Shimura varieties, in the spirit of the Gross-Zagier [GZ86] and (especially) Gross-Kohnen-Zagier [GKZ87] theorems on the Néron-Tate pairings of Heegner points in a modular Jacobian.

Take F to be a quadratic imaginary field, and let K_1 and K_2 be quartic CM fields, each containing F. Suppose we are given a hermitian space W_i over K_i of dimension h, whose signature at one archimedean place of its maximal totally real subfield is $(h-1, 1)$, and whose signature at the other place is $(h, 0)$. One can associate to the unitary group $U(W_i)$ a Shimura variety Y_i of dimension $h - 1$.

If we view each W_i as an F-vector space, and apply $\mathrm{Tr}_{K_i/F}$ to the hermitian form, we obtain a hermitian space V_i over F of signature $(2h-1, 1)$. To the unitary group $U(V_i)$ we can associate a Shimura variety X_i of dimension $2h - 1$, and the inclusion $U(W_i) \subset U(V_i)$ defines a morphism $Y_i \to X_i$.

Suppose further that $V_1 \cong V_2$, so that $X_1 \cong X_2$. Call this common hermitian space V, and this common Shimura variety X. We now have two cycles Y_1 and Y_2 of codimension h on X.

The idea, roughly speaking, is that the Beilinson-Bloch height pairing of Y_1 and Y_2 in the codimension h Chow group of X should be related to derivatives of L-functions. For certain cuspidal automorphic representations π of $U(V)$ one should able to project Y_1 and Y_2 onto the π-component of the Chow group, and the height pairing of these two projections should essentially be the central derivative of the standard L-function of π, multiplied by a period of π over a smaller unitary group $U(W_3) \subset U(V)$, where W_3 is a hermitian space of dimension h over K_3. The main result of [HS19] provides some evidence that such a relation should hold.

If one chooses a prime p that is split in F, but with each prime above it nonsplit in both K_1 and K_2, the above global cycles $Y_1, Y_2 \to X$ become (after applying the Rapoport-Zink uniformization theorem) the cycles on Lubin-Tate space whose intersection is the subject of (1.4.5).

1.6. **Acknowledgements.** We thank Andreas Mihatsch for helpful correspondence, and the anonymous referee for a careful reading and constructive comments.

2. Invariants of algebra embeddings

In this section only, we allow F to be any field whatsoever. We will attach a polynomial to any pair of F-algebra embeddings (Φ_1, Φ_2) as in (1.2.2). When the pair is regular semisimple, in a sense we will make precise, the pair is determined up to conjugacy by this polynomial.

2.1. **Noether-Skolem plus epsilon.** Suppose A is a semisimple F-algebra of finite dimension; in other words,

$$A \cong A_1 \times \cdots \times A_r$$

where each A_i is a finite dimensional simple F-algebra (whose center may be strictly larger than F). Let B be a central simple F-algebra.

Theorem 2.1.1 (Noether-Skolem)**.** *Suppose $\varphi, \varphi' : A \to B$ are F-algebra homomorphisms, let $B(\varphi) = B$ with its left A-module structure induced by φ, and let $B(\varphi') = B$ with its left A-module structure induced by φ'. If*

$$B(\varphi) \cong B(\varphi')$$

as A-modules, then φ and φ' are conjugate by an element of $B^\times$.

Proof. If A is a simple algebra then the hypothesis $B(\varphi) \cong B(\varphi')$ is automatically satisfied, and this is the usual Noether-Skolem theorem [SP, Tag 074Q]. The proof of this mild generalization follows the same argument, and we leave the details as an exercise to the reader. □

Suppose L is a finite étale F-algebra; in other words, a finite product of finite separable field extensions.

Definition 2.1.2. By an *F-algebra embedding* $L \to B$ we mean an injective map of F-algebras making B into a free L-module.

Corollary 2.1.3. *Any two F-algebra embeddings $L \to B$ are $B^\times$-conjugate.*

Proof. Take $A = L$ in Theorem 2.1.1. □

Remark 2.1.4. The corollary is false if we drop the freeness condition in Definition 2.1.2. For example, the F-algebra maps $F \times F \to M_3(F)$ defined by

$$(x, y) \mapsto \begin{pmatrix} x & & \\ & x & \\ & & y \end{pmatrix} \quad \text{and} \quad (x, y) \mapsto \begin{pmatrix} x & & \\ & y & \\ & & y \end{pmatrix}$$

are not $\mathrm{GL}_3(F)$-conjugate.

2.2. **Invariant polynomials.** Fix a positive integer h, and let B be a central simple F-algebra of dimension $4h^2$. Let K_1 and K_2 be quadratic étale extensions of F as in (1.2.1). Our goal is to attach to any pair of F-algebra embeddings

$$\Phi_1 : K_1 \to B, \qquad \Phi_2 : K_2 \to B. \tag{2.2.1}$$

a degree h monic polynomial with coefficients in K_3. This generalizes constructions of [HS19] in the special case $h = 1$, and constructions of [Guo96, Li21] in the special case $K_1 \cong K_2$. See §A.2.

The construction of this polynomial uses the K_3-algebra

$$C = B \otimes_F K_3.$$

The canonical maps

$$K_1 \otimes_F K_3 \xrightarrow{x_1\otimes x_3 \mapsto x_1x_3} K, \qquad K_2 \otimes_F K_3 \xrightarrow{x_2\otimes x_3 \mapsto x_2x_3} K \tag{2.2.2}$$

are both isomorphisms, and so the F-algebra embeddings (2.2.1) extend uniquely to K_3-algebra embeddings

$$\Phi_1, \Phi_2 : K \to C. \tag{2.2.3}$$

Lemma 2.2.1. *If $y \in K$ is a K_3-algebra generator then*

$$\Phi_1(y) - \Phi_2(y^{\tau_3}) = \Phi_2(y) - \Phi_1(y^{\tau_3}), \tag{2.2.4}$$

and for all $x \in K$ we have

$$\begin{aligned} (\Phi_1(y) - \Phi_2(y^{\tau_3})) \cdot \Phi_2(x) &= \Phi_1(x) \cdot (\Phi_1(y) - \Phi_2(y^{\tau_3})) \\ (\Phi_1(y) - \Phi_2(y^{\tau_3})) \cdot \Phi_1(x) &= \Phi_2(x) \cdot (\Phi_1(y) - \Phi_2(y^{\tau_3})) \\ (\Phi_1(y) - \Phi_2(y)) \cdot \Phi_2(x) &= \Phi_1(x^{\tau_3}) \cdot (\Phi_1(y) - \Phi_2(y)) \\ (\Phi_1(y) - \Phi_2(y)) \cdot \Phi_1(x) &= \Phi_2(x^{\tau_3}) \cdot (\Phi_1(y) - \Phi_2(y)). \end{aligned} \tag{2.2.5}$$

Proof. The two maps (2.2.3) have the same restriction to K_3, and so

$$\begin{aligned} \Phi_1(y + y^{\tau_3}) &= \Phi_2(y + y^{\tau_3}) \\ \Phi_1(yy^{\tau_3}) &= \Phi_2(yy^{\tau_3}). \end{aligned}$$

The relation (2.2.4) is clear from the first of these. If we write $x \in K$ as $x = ay + b$ with $a, b \in K_3$, then the first equality of (2.2.5) follows from

$$\begin{aligned} (\Phi_1(y) - \Phi_2(y^{\tau_3})) \cdot \Phi_2(y) &= \Phi_1(y)\Phi_2(y) - \Phi_2(yy^{\tau_3}) \\ &= \Phi_1(y)\Phi_2(y) - \Phi_1(yy^{\tau_3}) \\ &= \Phi_1(y) \cdot (\Phi_1(y) - \Phi_2(y^{\tau_3})). \end{aligned}$$

The other equalities in (2.2.5) are proved in the same way. □

Fix a $y \in K$ such that $K = K_3[y]$. Noting that $(y - y^{\tau_3})^2 \in K_3^\times$, define

$$\mathbf{s} = \frac{(\Phi_1(y) - \Phi_2(y^{\tau_3}))^2}{(y - y^{\tau_3})^2} \in C \tag{2.2.6}$$

and

$$\mathbf{t} = \frac{\Phi_1(y)\Phi_2(y) - \Phi_2(y)\Phi_1(y)}{(y - y^{\tau_3})^2} \in C. \tag{2.2.7}$$

Proposition 2.2.2. *The elements* (2.2.6) *and* (2.2.7) *do not depend on the choice of K_3-algebra generator $y \in K$, and satisfy*

$$\mathbf{s} \cdot \Phi_i(x) = \Phi_i(x) \cdot \mathbf{s}, \qquad \mathbf{t} \cdot \Phi_i(x) = \Phi_i(x^{\tau_3}) \cdot \mathbf{t} \tag{2.2.8}$$

for all $x \in K$. In particular, the first equality implies that $\mathbf{st} = \mathbf{ts}$.

Proof. For the independence of $\mathbf{s}$ and $\mathbf{t}$ on y, we will actually prove something slightly stronger: the elements

$$\mathbf{s} = \frac{-(y_1 y_2^{\tau_3} + y_2 y_1^{\tau_3}) + \Phi_1(y_1)\Phi_2(y_2) + \Phi_2(y_2^{\tau_3})\Phi_1(y_1^{\tau_3})}{(y_1 - y_1^{\tau_3})(y_2 - y_2^{\tau_3})} \tag{2.2.9}$$

and

$$\mathbf{t} = \frac{\Phi_1(y_1)\Phi_2(y_2) - \Phi_2(y_2)\Phi_1(y_1)}{(y_1 - y_1^{\tau_3})(y_2 - y_2^{\tau_3})} \tag{2.2.10}$$

are independent of the choices of K_3-algebra generators $y_1, y_2 \in K$, and agree with (2.2.6) and (2.2.7). Indeed, by direct calculation one can see that the right hand sides of (2.2.9) and (2.2.10) are unchanged by substitutions of the form $y_1 \mapsto a_1 y_1 + b_1$ and $y_2 \mapsto a_2 y_2 + b_2$ with $a_1, a_2 \in K_3^\times$ and $b_1, b_2 \in K_3$, and hence both are independent of the choices of y_1 and y_2. If we set $y_1 = y_2 = y$ then (2.2.9) simplifies to

$$\mathbf{s} = \frac{(\Phi_1(y) - \Phi_2(y^{\tau_3})) \cdot (\Phi_2(y) - \Phi_1(y^{\tau_3}))}{(y - y^{\tau_3})^2}$$

which is equal to (2.2.6) by (2.2.4). Similarly, the right hand side of (2.2.10) simplifies to (2.2.7).

It follows from (2.2.5) that $\Phi_i(x)$ commutes with $(\Phi_1(y) - \Phi_2(y^{\tau_3}))^2$, and so also commutes with $\mathbf{s}$. This proves the first equality in (2.2.8). The second is proved in the same way, after noting that (2.2.7) can be rewritten as

$$\mathbf{t} = \frac{(\Phi_1(y) - \Phi_2(y))(\Phi_1(y) - \Phi_2(y^{\tau_3}))}{(y - y^{\tau_3})^2}. \tag{2.2.11}$$

□

Proposition 2.2.3. *The reduced characteristic polynomial $P_{\mathbf{s}} \in K_3[T]$ of $\mathbf{s} \in C$ is a square, and if we write $P_{\mathbf{s}} = Q_{\mathbf{s}}^2$ then $Q_{\mathbf{s}}(\mathbf{s}) = 0$ in C.*

Proof. The claim can be checked after extending scalars to a separable extension, so we may assume that F itself is separably closed. Fix isomorphisms $K_1 \cong F \times F$ and $B \cong M_{2h}(F)$ in such a way that $\Phi_1 : K_1 \to M_{2h}(F)$ is

$$\Phi_1(a, b) = \begin{pmatrix} aI & \\ & bI \end{pmatrix}$$

where $I \in M_h(F)$ is the identity matrix. Using (2.2.8), one can see that $\mathbf{s}, \mathbf{t} \in C \cong M_{2h}(K_3)$ have the form

$$\mathbf{s} = \begin{pmatrix} \mathbf{s}_+ & \\ & \mathbf{s}_- \end{pmatrix} \qquad \mathbf{t} = \begin{pmatrix} & \mathbf{u} \\ \mathbf{v} & \end{pmatrix}$$

for some $\mathbf{u}, \mathbf{v}, \mathbf{s}_\pm \in M_h(K_3)$. The condition that $\mathbf{s}$ and $\mathbf{t}$ commute translates to $\mathbf{s}_+\mathbf{u} = \mathbf{u}\mathbf{s}_-$ and $\mathbf{v}\mathbf{s}_+ = \mathbf{s}_-\mathbf{v}$.

Let $P_{\mathbf{s}}^{\pm}$ be the characteristic polynomial of $\mathbf{s}_\pm$. As $P_{\mathbf{s}} = P_{\mathbf{s}}^+ P_{\mathbf{s}}^-$, to prove the proposition it suffices to prove $P_{\mathbf{s}}^+ = P_{\mathbf{s}}^-$. If $\mathbf{t} \in C$ is invertible then so are $\mathbf{u}$ and $\mathbf{v}$, hence $\mathbf{s}_+$ and $\mathbf{s}_-$ are similar. The general case is by a Zariski density argument. For any $g \in \mathrm{GL}_{2h}(F)$ we may form the elements $\mathbf{s}, \mathbf{t} \in M_{2h}(K_3)$ associated to the pair $(\Phi_1, g\Phi_2 g^{-1})$, and view them as functions of g. We have seen that $P_{\mathbf{s}}^+ = P_{\mathbf{s}}^-$ for all g in the open dense subset defined by $\det(\mathbf{t}) \neq 0$, so the same equality also holds at $g = I$. □

We define the invariant polynomial of (Φ_1, Φ_2) to be the degree h polynomial $Q_{\mathbf{s}}$ of Proposition 2.2.3. In other words:

Definition 2.2.4. The *invariant polynomial*

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)} \in K_3[T] \tag{2.2.12}$$

of the pair (2.2.1) is the unique monic square root of the reduced characteristic polynomial of $\mathbf{s} \in C$.

Remark 2.2.5. If Φ_1 and Φ_2 are simultaneously conjugated by an element of $B^\times$, then $\mathbf{s}, \mathbf{t} \in C$ are simultaneously conjugated by that same element. Thus the invariant polynomial only depends on the $B^\times$-conjugacy class of (Φ_1, Φ_2).

Remark 2.2.6. The centralizer $C(\Phi_i)$ of the image of $\Phi_i : K \to C$ is a central simple K-algebra of rank h^2, and $\mathbf{s} \in C(\Phi_i)$ by (2.2.8). The same argument used in the proof of Proposition 2.2.3 shows that the invariant polynomial is equal to reduced characteristic polynomial of $\mathbf{s}$ when viewed as an element of either one of the K-algebras $C(\Phi_1)$ or $C(\Phi_2)$.

Remark 2.2.7. We always allow the possibility that $K_1 = K_2$, but be warned: when $K_1 = K_2$ the two isomorphisms in (2.2.2) are not equal, as they arise from embeddings $K_1 \to K$ and $K_2 \to K$ with different images. Because of this, even if the two F-algebra maps in (2.2.1) are equal, the K_3-algebra maps in (2.2.3) will not be.

2.3. **The functional equation.** We will show that the invariant polynomial $\mathrm{Inv}_{(\Phi_1,\Phi_2)}(T) \in K_3[T]$ of Definition 2.2.4 satisfies a functional equation in $T \mapsto 1 - T$.

By mild abuse of notation, we denote again by σ_3 the F-linear automorphism $\mathrm{id} \otimes \sigma_3$ of $C = B \otimes_F K_3$.

Lemma 2.3.1. *The element* $\mathbf{s} \in C$ *of* (2.2.6) *satisfies*

$$\mathbf{s}^{\sigma_3} = \frac{(\Phi_1(y) - \Phi_2(y))^2}{(y - y^{\tau_3})^2}.$$

Proof. Recall from Proposition 2.2.2 that $\mathbf{s}$ is independent of the choice of K_3-algebra generator $y \in K$ in (2.2.6). As y^{τ_1} is also such a generator, we therefore have

$$\mathbf{s} = \frac{(\Phi_1(y^{\tau_1}) - \Phi_2(y^{\tau_2}))^2}{(y^{\tau_1} - y^{\tau_2})^2}.$$

Now note that every $a \in K$ satisfies $\Phi_i(a)^{\sigma_3} = \Phi_i(a^{\tau_i})$, so that

$$\big[\Phi_1(y^{\tau_1}) - \Phi_2(y^{\tau_2})\big]^{\sigma_3} = \Phi_1(y) - \Phi_2(y),$$

and that $\sigma_3 = \tau_1|_{K_3}$, so that

$$\big[(y^{\tau_1} - y^{\tau_2})^2\big]^{\sigma_3} = \big[(y^{\tau_1} - y^{\tau_2})^2\big]^{\tau_1} = (y - y^{\tau_3})^2.$$

Applying σ_3 to the above expression for $\mathbf{s}$ therefore proves the claim. □

Proposition 2.3.2. *The elements* $\mathbf{s}$ *and* $\mathbf{t}$ *satisfy*

$$\mathbf{s} + \mathbf{s}^{\sigma_3} = 1, \qquad \mathbf{s}\mathbf{s}^{\sigma_3} + \mathbf{t}^2 = 0.$$

Proof. Abbreviate $\alpha = \Phi_1(y) - \Phi_2(y)$ and, recalling (2.2.4),

$$\beta = \Phi_1(y) - \Phi_2(y^{\tau_3}) = \Phi_2(y) - \Phi_1(y^{\tau_3}).$$

The relation (2.2.5) shows that

$$\alpha\beta = \Phi_1(y)\beta - \Phi_2(y)\beta = \beta\Phi_2(y) - \beta\Phi_1(y) = -\beta\alpha,$$

from which it follows that $\alpha^2 + \beta^2 = (\alpha+\beta)^2 = (y - y^{\tau_3})^2$. Combining this with Lemma 2.3.1 yields

$$\mathbf{s} + \mathbf{s}^{\sigma_3} = \frac{\beta^2}{(y - y^{\tau_3})^2} + \frac{\alpha^2}{(y - y^{\tau_3})^2} = 1.$$

In particular $\mathbf{s}$ and $\mathbf{s}^{\sigma_3}$ commute. Using the formula for $\mathbf{t}$ found in (2.2.11), we obtain

$$\mathbf{s}^{\sigma_3}\mathbf{s} + \mathbf{t}^2 = \frac{\alpha^2\beta^2}{(y - y^{\tau_3})^4} + \frac{\alpha\beta\alpha\beta}{(y - y^{\tau_3})^4} = \frac{\alpha(\alpha\beta + \beta\alpha)\beta}{(y - y^{\tau_3})^4} = 0,$$

completing the proof. □

Proposition 2.3.3. *The invariant polynomial* $\mathrm{Inv} = \mathrm{Inv}_{(\Phi_1,\Phi_2)}$ *satisfies the functional equation*

$$(-1)^h \cdot \mathrm{Inv}(1 - T) = \mathrm{Inv}^{\sigma_3}(T),$$

where Inv^{σ_3} *is obtained from* Inv *by applying the nontrivial automorphism* $\sigma_3 \in \mathrm{Aut}(K_3/F)$ *coefficient-by-coefficient.*

Proof. In general, if $a, b \in C$ satisfy $a + b = 1$, then their reduced characteristic polynomials $P_a, P_b \in K_3[T]$ satisfy $P_a(1-T) = P_b(T)$. Indeed, after extending scalars one may assume that $C \cong M_{2h}(K_3)$, and that a and b are upper triangular matrices. In this case the claim is obvious.

Applying this to the relation $\mathbf{s} + \mathbf{s}^{\sigma_3} = 1$ of Proposition 2.3.2 shows that

$$P_{\mathbf{s}}(1-T) = P_{\mathbf{s}}^{\sigma_3}(T),$$

and taking the unique monic square root of each side proves the claim. □

2.4. **The elements w and z.** The elements $\mathbf{s}$ and $\mathbf{t}$ defined by (2.2.6) and (2.2.7), being independent of the choice of $y \in K$ used in their construction, are canonically attached to the pair of embeddings (2.2.1). However, they have the disadvantage that they live in the base change $C = B \otimes_F K_3$ rather than in B itself.

We now construct substitutes for $\mathbf{s}$ and $\mathbf{t}$ that live in B, but which depend on noncanonical choices. Namely, fix F-algebra generators $x_1 \in K_1$ and $x_2 \in K_2$, and define

$$\mathbf{w} = \Phi_1(x_1)\Phi_2(x_2) + \Phi_2(x_2^{\sigma_2})\Phi_1(x_1^{\sigma_1}) \in B \tag{2.4.1}$$

$$\mathbf{z} = \Phi_1(x_1)\Phi_2(x_2) - \Phi_2(x_2)\Phi_1(x_1) \in B. \tag{2.4.2}$$

If we view both $x_1 = x_1 \otimes 1$ and $x_2 = 1 \otimes x_2$ as elements of $K = K_1 \otimes K_2$, then

$$x_1 x_2^{\sigma_2} + x_2 x_1^{\sigma_1} \in K_3 \qquad \text{and} \qquad (x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2}) \in K_3^\times,$$

and so both may be viewed as central elements in C. Similarly, we view $\mathbf{w}, \mathbf{z} \in C$.

Proposition 2.4.1. *The elements* (2.2.6) *and* (2.2.7) *satisfy*

$$\mathbf{s} = \frac{-(x_1 x_2^{\sigma_2} + x_2 x_1^{\sigma_1}) + \mathbf{w}}{(x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2})} \quad \text{and} \quad \mathbf{t} = \frac{\mathbf{z}}{(x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2})}.$$

Proof. Take $y_1 = x_1$ and $y_2 = x_2$ in (2.2.9) and (2.2.10). □

Proposition 2.4.2. *The elements* $\mathbf{w}$ *and* $\mathbf{z}$ *commute, and satisfy*

$$\mathbf{w} \cdot \Phi_i(x) = \Phi_i(x) \cdot \mathbf{w}, \qquad \mathbf{z} \cdot \Phi_i(x) = \Phi_i(x^{\sigma_i}) \cdot \mathbf{z}$$

for all $x \in K_i$. *Moreover, they are related by*

$$\mathbf{z}^2 = \mathbf{w}^2 - \mathrm{Tr}(x_1)\mathrm{Tr}(x_2)\mathbf{w} + \mathrm{Tr}(x_1^2)\mathrm{Nm}(x_2) + \mathrm{Tr}(x_2^2)\mathrm{Nm}(x_1),$$

where Tr *and* Nm *denote the trace and norm from* K_1 *or* K_2 *to* F, *as appropriate.*

Proof. Everything except the final claim follows from Proposition 2.4.1 and the analogous properties of $\mathbf{s}$ and $\mathbf{t}$ proved in Proposition 2.2.2. For the final claim define elements $u, v \in K_3$ by

$$u = x_1 x_2^{\sigma_2} + x_2 x_1^{\sigma_1} \quad \text{and} \quad v = (x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2}),$$

so that $v\mathbf{s} = \mathbf{w} - u$ and $\mathbf{z}^2 = v^2\mathbf{t}^2$ by Proposition 2.4.1. Using the relation $\mathbf{s}\mathbf{s}^{\sigma_3} = -\mathbf{t}^2$ from Proposition 2.3.2 and $v^{\sigma_3} = v^{\tau_1} = -v$, we compute

$$\mathbf{z}^2 = v^2\mathbf{t}^2 = (v\mathbf{s})(v\mathbf{s})^{\sigma_3} = (\mathbf{w} - u)(\mathbf{w} - u^{\sigma_3}).$$

To obtain the desired formula for $\mathbf{z}^2$, expand the right hand side and use the relation $u^{\sigma_3} = u^{\tau_1} = x_1^{\sigma_1}x_2^{\sigma_2} + x_2x_1$. □

2.5. **Regular semisimple pairs.** We have already noted in Remark 2.2.5 that the invariant polynomial of the pair (Φ_1, Φ_2) fixed in §2.2 depends only on its $B^\times$-conjugacy class. In this subsection we will show that for a pair that is regular semisimple, in the following sense, the invariant polynomial of Definition 2.2.4 determines the conjugacy class.

Definition 2.5.1. The pair (Φ_1, Φ_2) is *regular semisimple* if $\mathbf{s} \in C^\times$ and $K_3[\mathbf{s}] \subset C$ is an étale K_3-subalgebra of dimension h.

Remark 2.5.2. In Proposition A.2.2 we will show that Definition 2.5.1 is compatible with the usual notion of regular semisimple from geometric invariant theory.

Remark 2.5.3. Using the relation $\mathbf{t}^2 = -\mathbf{s}\mathbf{s}^{\sigma_3}$ of Proposition 2.3.2, we see that $\mathbf{s} \in C^\times$ if and only if $\mathbf{t} \in C^\times$.

The following proposition shows that one can characterize regular semisimple pairs using the invariant polynomial.

Proposition 2.5.4. *The pair (Φ_1, Φ_2) is regular semisimple if and only if for every F-algebra map $\rho : K_3 \to F^{\mathrm{alg}}$ the image*

$$\mathrm{Inv}^{\rho}_{(\Phi_1,\Phi_2)}(T) \in F^{\mathrm{alg}}[T]$$

of the invariant polynomial has h distinct nonzero roots.

Proof. Recall from Proposition 2.2.3 that $\mathbf{s} \in C$ is a zero of the invariant polynomial $Q_{\mathbf{s}} = \mathrm{Inv}_{(\Phi_1,\Phi_2)}$, and so there is a surjection

$$K_3[T]/(Q_{\mathbf{s}}) \xrightarrow{T\mapsto \mathbf{s}} K_3[\mathbf{s}]. \tag{2.5.1}$$

If (Φ_1, Φ_2) is regular semisimple then the surjection (2.5.1) is an isomorphism for dimension reasons, and so the domain is an étale F-algebra in which T is a unit. It follows that $Q^{\rho}_{\mathbf{s}}$ has h distinct nonzero roots for any $\rho : K_3 \to F^{\mathrm{alg}}$.

For the converse, suppose that $Q^{\rho}_{\mathbf{s}}$ has h distinct nonzero roots for any ρ. Let $M_{\mathbf{s}} \in K_3[T]$ be the minimal polynomial of $\mathbf{s} \in C$, so that $M_{\mathbf{s}} \mid Q_{\mathbf{s}}$, and (2.5.1) factors as

$$K_3[T]/(Q_{\mathbf{s}}) \to K_3[T]/(M_{\mathbf{s}}) \xrightarrow{T\mapsto \mathbf{s}} K_3[\mathbf{s}]$$

with the second arrow an isomorphism. By elementary linear algebra, the roots of $M^{\rho}_{\mathbf{s}}$ in F^{alg} are the same as the roots of the characteristic polynomial $P^{\rho}_{\mathbf{s}} = Q^{\rho}_{\mathbf{s}} \cdot Q^{\rho}_{\mathbf{s}}$ of $\mathbf{s} \in C^{\rho} \cong M_{2h}(F^{\mathrm{alg}})$, which are same as the roots of $Q^{\rho}_{\mathbf{s}}$. It now follows from our assumptions on $Q_{\mathbf{s}}$ that $M_{\mathbf{s}} = Q_{\mathbf{s}}$, and so (2.5.1) is an isomorphism whose domain is an étale F-algebra of dimension h in which T is a unit. Thus (Φ_1, Φ_2) is regular semisimple. $\square$

The F-subalgebra generated by $\Phi_1(K_1) \cup \Phi_2(K_2) \subset B$ is denoted

$$F(\Phi_1, \Phi_2) \subset B.$$

Recall the elements $\mathbf{w}, \mathbf{z} \in F(\Phi_1, \Phi_2)$ of (2.4.1) and (2.4.2), and set

$$L = F[\mathbf{w}] \subset B.$$

Although the element $\mathbf{w}$ depends on the choices of $x_1 \in K_1$ and $x_2 \in K_2$, the subalgebra L does not. For example, using Proposition 2.4.1 we see that

$$L \otimes_F K_3 = K_3[\mathbf{w}] = K_3[\mathbf{s}] \subset C \tag{2.5.2}$$

does not depend on the choices of x_1 and x_2 (because $\mathbf{s}$ does not), and L is characterized as the elements in (2.5.2) fixed by the F-linear automorphism $\sigma_3 = \mathrm{id} \otimes \sigma_3$ of $C = B \otimes_F K_3$.

Proposition 2.5.5. *If the pair (Φ_1, Φ_2) is regular semisimple, the following properties hold.*

(1) *The element $\mathbf{z} \in B$ is a unit.*
(2) *The F-algebra L is étale of dimension h.*
(3) *The centralizer of $F(\Phi_1, \Phi_2) \subset B$ is L.*
(4) *The centralizer of $L \subset B$ is $F(\Phi_1, \Phi_2)$.*
(5) *The ring $F(\Phi_1, \Phi_2)$ is a quaternion algebra over its center L. Writing $L \cong \prod L_i$ as a product of fields, this means that each $F(\Phi_1, \Phi_2) \otimes_L L_i$ is a central simple L_i-algebra of dimension 4.*
(6) *The ring B is free as an $F(\Phi_1, \Phi_2)$-module.*

Proof. For property (1), note that $\mathbf{t} \in C^\times$ by Remark 2.5.3, and so $\mathbf{z} \in B^\times$ by Proposition 2.4.1. For property (2), the K_3-algebra (2.5.2) is étale of dimension h by hypothesis, and so L is étale over F of the same dimension. The remaining properties rely on the following lemma.

Lemma 2.5.6. *There is a separable field extension F'/F such that, abbreviating*

$$B' = B \otimes_F F', \qquad L' = L \otimes_F F',$$

there exists an isomorphism $B' \cong M_{2h}(F')$ identifying

$$L' = \left\{ \begin{pmatrix} x_1 I_2 & & \\ & \ddots & \\ & & x_h I_2 \end{pmatrix} : x_1, \ldots, x_h \in F' \right\}.$$

Here $I_2 \in M_2(F')$ is the 2×2 identity matrix.

Proof. Let $P_{\mathbf{w}} \in F[T]$ be reduced characteristic polynomial of $\mathbf{w} \in B$, and let $Q_{\mathbf{w}} \in F[T]$ be its minimal polynomial. We know from Proposition 2.4.1 that

$$\mathbf{w} = c + d\mathbf{s} \in C$$

for scalars $c \in K_3$ and $d \in K_3^\times$. This implies that $P_{\mathbf{w}}(c + dT) \in K_3[T]$ is the reduced characteristic polynomial of $\mathbf{s}$, while $Q_{\mathbf{w}}(c + dT) \in K_3[T]$ is its minimal polynomial. These latter polynomials are precisely the $P_{\mathbf{s}}$ and $Q_{\mathbf{s}}$ of Proposition 2.2.3 (the first by definition of $P_{\mathbf{s}}$, and the second by the proof of Proposition 2.5.4). As $P_{\mathbf{s}} = Q_{\mathbf{s}}^2$, we deduce that $P_{\mathbf{w}} = Q_{\mathbf{w}}^2$.

Choose any F'/F large enough that $B' \cong M_{2h}(F')$. Let V be the unique simple left B'-module; in other words, the standard representation of $M_{2h}(F')$. It follows from $P_{\mathbf{w}} = Q_{\mathbf{w}}^2$ that the list of invariant factors of the matrix $\mathbf{w} \in B'$ can only be $Q_{\mathbf{w}} \mid Q_{\mathbf{w}}$. Thus

$$V \cong L' \oplus L' \tag{2.5.3}$$

as a left modules over the subring $L' = F'[\mathbf{w}] \subset B'$.

Using (2), we may enlarge F' to assume that $L' \cong F' \times \cdots \times F'$ (h factors). If $e_1, \ldots, e_h \in L'$ are the orthogonal idempotents inducing this decomposition, it follows from (2.5.3) that

$$V = e_1 V \oplus \cdots \oplus e_h V$$

with each summand a 2-dimensional F'-subspace on which L' acts through scalars. Choose a basis for each summand, and use this to identify $B' \cong \mathrm{End}_{F'}(V) \cong M_{2d}(F')$. This isomorphism has the desired properties. □

Now we complete the proof of Proposition 2.5.5. Keep the notation of the lemma, and abbreviate

$$F'(\Phi_1, \Phi_2) = F(\Phi_1, \Phi_2) \otimes_F F'.$$

By Proposition 2.4.2, $\mathbf{w}$ commutes with the image of $\Phi_i : K_i \to B$, and hence $F(\Phi_1, \Phi_2)$ is contained in the centralizer of L. Hence $F'(\Phi_1, \Phi_2)$ is contained in the centralizer of L', or, in other words,

$$F'(\Phi_1, \Phi_2) \subset \underbrace{M_2(F') \times \cdots \times M_2(F')}_{h \text{ times}}, \tag{2.5.4}$$

embedded block diagonally into $M_{2h}(F')$.

The F'-subalgebra on the left hand side of (2.5.4) contains

$$L' \cong \underbrace{F' \times \cdots \times F'}_{h \text{ times}},$$

as well as $\Phi_1(K_1)$, and a unit $\mathbf{z} \in F'(\Phi_1, \Phi_2)$ such that $\mathbf{z}\Phi_1(x) = \Phi_1(x^{\sigma_1})\mathbf{z}$ for all $x \in K_1$ (Proposition 2.4.2). From this, it is not hard to see first that equality holds in (2.5.4), and then that the F'-algebras

$$L' \subset F'(\Phi_1, \Phi_2) \subset B'$$

satisfy the obvious analogues of properties (3)-(6). The F-algebras

$$L \subset F(\Phi_1, \Phi_2) \subset B$$

therefore satisfy those properties, completing the proof. □

We next show that when (Φ_1, Φ_2) is regular semisimple, the isomorphism class of $F(\Phi_1, \Phi_2)$ as an F-algebra is completely determined by the invariant polynomial.

Proposition 2.5.7. *Assume that (Φ_1, Φ_2) is regular semisimple. If B' is a central simple F-algebra of the same dimension as B, and if we are given F-algebra embeddings*

$$\Phi_1' : K_1 \to B', \qquad \Phi_2' : K_2 \to B'$$

such that $\mathrm{Inv}_{(\Phi_1,\Phi_2)} = \mathrm{Inv}_{(\Phi_1',\Phi_2')}$, then there is an isomorphism of F-algebras

$$F(\Phi_1, \Phi_2) \cong F(\Phi_1', \Phi_2')$$

identifying $\Phi_1 = \Phi_1'$ and $\Phi_2 = \Phi_2'$.

Proof. The validity of the proposition does not depend on the choices of $x_1 \in K_1$ and $x_2 \in K_2$ made in §2.4. If $\mathrm{char}(F) \neq 2$ we choose them so that

$$x_1^{\sigma_1} = -x_1 \quad \text{and} \quad x_2^{\sigma_2} = -x_2.$$

If $\mathrm{char}(F) = 2$, we use the surjectivity of $\mathrm{Tr} : K_i \to F$ to choose them so that

$$x_1^{\sigma_1} = x_1 + 1 \quad \text{and} \quad x_2^{\sigma_2} = x_2 + 1.$$

In either case, for $i \in \{1, 2\}$ we define $a_i, b_i \in F$ by

$$x_i^{\sigma_i} = a_i x_i + b_i \in K_i.$$

Using only the data $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ and the choices of x_1 and x_2, we will construct an abstract F-algebra $\mathcal{F}(\varphi_1, \varphi_2)$ and embeddings $\varphi_i : K_i \to \mathcal{F}(\varphi_1, \varphi_2)$ in such a way that

$$\mathcal{F}(\varphi_1, \varphi_2) \cong F(\Phi_1, \Phi_2)$$

and φ_i is identified with Φ_i. Once this is done, the claim follows immediately: by symmetry (note that (Φ_1', Φ_2') is also regular semisimple, by Proposition 2.5.4 and the assumption on the matching of invariant polynomials), there is an analogous isomorphism $\mathcal{F}(\varphi_1, \varphi_2) \cong F(\Phi_1', \Phi_2')$.

Let $\mathbf{w}, \mathbf{z} \in B$ be the elements (2.4.1) and (2.4.2). We saw in the proof of Lemma 2.5.6 that the minimal polynomial $Q_{\mathbf{w}} \in F[T]$ of $\mathbf{w}$ is related to the invariant polynomial $\mathrm{Inv}_{(\Phi_1,\Phi_2)} \in K_3[T]$ by a change of variables

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)}(T) = Q_{\mathbf{w}}(c + dT),$$

where the scalars $c \in K_3$ and $d \in K_3^\times$ depend on the choices of $x_1 \in K_1$ and $x_2 \in K_2$, but not on the pair (Φ_1, Φ_2). Thus the F-algebra

$$\mathcal{L} = F[W]/(Q_{\mathbf{w}}(W))$$

of dimension h, where $F[W]$ is a polynomial ring in one variable, depends only on $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ and the choices of x_1 and x_2. There is a surjective morphism $\mathcal{L} \to L$ characterized by $W \mapsto \mathbf{w}$.

Now let

$$\mathcal{F}(\varphi_1, \varphi_2) = \mathcal{L}[Z, X_1, X_2]$$

be the free $\mathcal{L}$-algebra generated by three noncommuting variables, modulo the two-sided ideal generated by the relations

- $X_i^2 - \mathrm{Tr}(x_i)X_i + \mathrm{Nm}(x_i) = 0$,
- $Z^2 = W^2 - \mathrm{Tr}(x_1)\mathrm{Tr}(x_2)W + \mathrm{Tr}(x_1^2)\mathrm{Nm}(x_2) + \mathrm{Tr}(x_2^2)\mathrm{Nm}(x_1)$,
- $ZX_i = (a_iX_i + b_i)Z$,
- $Z = W - [a_2X_2 + b_2][a_1X_1 + b_1] - X_2X_1$.

The first of these relations allows us to define

$$\varphi_i : K_i \to \mathcal{F}(\varphi_1, \varphi_2)$$

by $\varphi_i(x_i) = X_i$. Moreover, there is a unique surjection

$$\mathcal{F}(\varphi_1, \varphi_2) \to F(\Phi_1, \Phi_2), \tag{2.5.5}$$

satisfying $W \mapsto \mathbf{w}$, $Z \mapsto \mathbf{z}$, and $X_i \mapsto \Phi_i(x_i)$. Indeed, one only needs to check that $\mathbf{w}$, $\mathbf{z}$, and $\Phi_i(x_i)$ satisfy the same four relations as W, Z, and X_i. The first relation is clear, the second and third are found in Proposition 2.4.2, and the fourth follows directly from (2.4.1) and (2.4.2), which imply

$$\begin{aligned}\mathbf{w} - \mathbf{z} &= \Phi_2(x_2^{\sigma_2})\Phi_1(x_1^{\sigma_1}) + \Phi_2(x_2)\Phi_1(x_1)\\ &= [a_2\Phi_2(x_2) + b_2][a_1\Phi_1(x_1) + b_1] + \Phi_2(x_2)\Phi_1(x_1).\end{aligned}$$

We next claim that $\mathcal{F}(\varphi_1, \varphi_2)$ is generated as an $\mathcal{L}$-algebra by Z and X_1 alone. To see this, rewrite

$$Z = W - [a_2X_2 + b_2][a_1X_1 + b_1] - X_2X_1$$

as

$$W - Z = X_2 \cdot (a_1a_2X_1 + X_1 + b_1a_2) + (a_1b_2X_1 + b_1b_2), \tag{2.5.6}$$

and recall our particular choices of x_1 and x_2. If $\mathrm{char}(F) \neq 2$ then

$$a_1a_2x_1 + x_1 + b_1a_2 = 2x_1 \in K_1^\times,$$

while if $\mathrm{char}(F) = 2$ then

$$a_1a_2x_1 + x_1 + b_1a_2 = 1 \in K_1^\times.$$

In either case, applying φ_1 shows that $a_1a_2X_1 + X_1 + b_1a_2$ is a unit in $F[X_1]$, allowing us to solve (2.5.6) for X_2 in terms of W, Z, and X_1.

Finally, from the relations satisfied by X_1 and Z it is clear that

$$\mathcal{L}[Z, X_1] = \mathcal{L} + \mathcal{L}X_1 + \mathcal{L}Z + \mathcal{L}X_1Z$$

as $\mathcal{L}$-modules, and so the F-dimension of $\mathcal{L}[Z, X_1] = \mathcal{F}(\varphi_1, \varphi_2)$ is at most $4h$. As the F-algebra $F(\Phi_1, \Phi_2)$ has dimension $4h$ by Proposition 2.5.5, the surjection (2.5.5) is an isomorphism. □

Corollary 2.5.8. *Suppose we are given F-algebra embeddings*

$$\Phi_1, \Phi_1' : K_1 \to B, \qquad \Phi_2, \Phi_2' : K_2 \to B$$

such that (Φ_1, Φ_2) and (Φ_1', Φ_2') are both regular semisimple. Then (Φ_1, Φ_2) and (Φ_1', Φ_2') have the same invariant polynomial if and only if they lie in the *same $B^\times$-conjugacy class. In other words, if and only if there is a $b \in B^\times$ such that*

$$b\Phi_1 b^{-1} = \Phi_1' \quad \textit{and} \quad b\Phi_2 b^{-1} = \Phi_2'.$$

Proof. One direction is Remark 2.2.5, so assume that (Φ_1, Φ_2) and (Φ_1', Φ_2') have the same invariant polynomial. By Proposition 2.5.7, there is an isomorphism

$$F(\Phi_1, \Phi_2) \cong F(\Phi_1', \Phi_2')$$

identifying $\Phi_1 = \Phi_1'$ and $\Phi_2 = \Phi_2'$. By Theorem 2.1.1 (whose hypotheses are satisfied by Proposition 2.5.5), the inclusions $F(\Phi_1, \Phi_2) \to B$ and $F(\Phi_1', \Phi_2') \to B$ are $B^\times$-conjugate, and the proposition follows. □

Remark 2.5.9. If $h = 1$, so that B is a quaternion algebra over F, the invariant polynomial has the form

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)}(T) = T - \xi$$

for some $\xi \in K_3$ satisfying (by Proposition 2.3.3) $\mathrm{Tr}_{K_3/F}(\xi) = 1$. The proof that ξ determines the $B^\times$-conjugacy class of (Φ_1, Φ_2) already appears in [HS19], although the construction of ξ described there is quite different.

3. The biquadratic fundamental lemma

From this point on we assume that F is a nonarchimedean local field, and fix quadratic étale F-algebras K_1 and K_2. In this section we define two kinds of orbital integrals, and formulate a conjectural fundamental lemma relating them.

3.1. Orbital integrals for (Φ_1, Φ_2).

Given F-algebra embeddings

$$\Phi_1 : K_1 \to M_{2h}(F), \qquad \Phi_2 : K_2 \to M_{2h}(F),$$

let $H_i \subset \mathrm{GL}_{2h}(F)$ be the centralizer of $\Phi_i(K_i)^\times$. If we use Φ_i to view F^{2h} as a K_i-module, the natural action of H_i on F^{2h} determines isomorphisms

$$H_1 \cong \mathrm{GL}_h(K_1), \qquad H_2 \cong \mathrm{GL}_h(K_2),$$

well-defined up to conjugacy.

Assume that (Φ_1, Φ_2) is regular semisimple, in the sense of Definition 2.5.1. This implies, by Proposition 2.5.5, that $F(\Phi_1, \Phi_2) \subset M_{2h}(F)$ has centralizer an étale F-algebra $L \subset M_{2h}(F)$ of dimension h. In particular

$$L^\times = H_1 \cap H_2 \tag{3.1.1}$$

is a torus.

Definition 3.1.1. Given a compactly supported f as in (1.4.3), define the *orbital integral*

$$O_{(\Phi_1,\Phi_2)}(f) = \int_{H_1\cap H_2\backslash H_1\times H_2} f(g_1^{-1}h_1^{-1}h_2g_2)\, dh_1\, dh_2,$$

where $g_1, g_2 \in \mathrm{GL}_{2h}(F)$ are chosen to satisfy the integrality condition

$$\Phi_i(\mathcal{O}_{K_i}) \subset g_i M_{2h}(\mathcal{O}_F) g_i^{-1},$$

and $H_1 \cap H_2 \subset H_1 \times H_2$ is embedded diagonally.

Remark 3.1.2. The $\Phi_i(\mathcal{O}_{K_i})$-stable lattices $\Lambda \subset F^{2h}$ form a single H_i-orbit, and the Haar measure on H_i is normalized by

$$\mathrm{vol}(\mathrm{Stab}_{H_i}(\Lambda)) = 1.$$

The Haar measure on $H_1 \cap H_2$ is normalized, using (3.1.1), by $\mathrm{vol}(\mathcal{O}_L^\times) = 1$.

Remark 3.1.3. The orbital integral is independent of the choice of g_1 and g_2. Moreover, if we set $g = g_1^{-1} g_2$ and $H_i' = g_i^{-1} H_i g_i$, the change of variables $h_i \mapsto g_i h_i g_i^{-1}$ puts the orbital integral into the more familiar form

$$O_{(\Phi_1,\Phi_2)}(f) = \int_{\{(h_1,h_2)\,:\,gh_2 = h_1 g\}\backslash H_1' \times H_2'} f(h_1^{-1} g h_2)\, dh_1\, dh_2.$$

3.2. **Orbital integrals for** (Φ_0, Φ_3)**.** As in §1.4, we may set

$$K_0 = F \times F$$

and repeat the construction of (1.2.1) with the pair (K_1, K_2) replaced by (K_0, K_3). This gives another diagram of F-algebras (1.4.1). Repeating the constructions of §2.2, we associate to any pair of F-algebra embeddings

$$\Phi_0 : K_0 \to M_{2h}(F), \qquad \Phi_3 : K_3 \to M_{2h}(F)$$

a degree h monic polynomial $\mathrm{Inv}_{(\Phi_0,\Phi_3)} \in K_3[T]$ satisfying the functional equation

$$(-1)^h \cdot \mathrm{Inv}_{(\Phi_0,\Phi_3)}(1-T) = \mathrm{Inv}^{\sigma_3}_{(\Phi_0,\Phi_3)}(T).$$

Denote by $H_i \subset \mathrm{GL}_{2h}(F)$ the centralizer of $\Phi_i(K_i)^\times$. If we use Φ_i to view F^{2h} as a K_i-module, then choices of K_i-bases determine isomorphisms

$$H_0 \cong \mathrm{GL}_h(K_0), \qquad H_3 \cong \mathrm{GL}_h(K_3). \tag{3.2.1}$$

Composing the absolute value $|\cdot| : F^\times \to \mathbb{R}^\times$ with the character

$$H_0 \cong \mathrm{GL}_h(K_0) = \mathrm{GL}_h(F) \times \mathrm{GL}_h(F) \xrightarrow{(a,b)\mapsto \frac{\det(a)}{\det(b)}} F^\times$$

determines a character, again denoted

$$|\cdot| : H_0 \to \mathbb{R}^\times. \tag{3.2.2}$$

We denote by $\eta : K_3^\times \to \{\pm 1\}$ the character associated to the étale quadratic extension K/K_3 by class field theory (if $K_3 \cong F \times F$, then class field theory associates to K/K_3 a quadratic character on each copy of $F^\times$, and η is defined as their product), and note that $\eta|_{F^\times}$ is the trivial character. Composing η with the determinant $H_3 \cong \mathrm{GL}_h(K_3) \to K_3^\times$ yields a character, again denoted

$$\eta : H_3 \to \{\pm 1\}. \tag{3.2.3}$$

Remark 3.2.1. The character (3.2.3) admits two natural extensions to $\mathrm{GL}_{2h}(F)$, given by the compositions

$$\mathrm{GL}_{2h}(F) \xrightarrow{\det} F^\times \xrightarrow{\eta_{K_i/F}} \{\pm 1\}$$

for $i \in \{1,2\}$, where $\eta_{K_i/F}$ is the quadratic character associated to K_i/F.

Remark 3.2.2. In contrast, the character (3.2.2) does not extend to $\mathrm{GL}_{2h}(F)$. Indeed, if $z \in \mathrm{GL}_{2h}(F)$ is any element such that $z \cdot \Phi_0(x) = \Phi_0(x^{\sigma_0}) \cdot z$ for all $x \in K_0$, then conjugation by z preserves H_0, but

$$|zh_0z^{-1}| = |h_0|^{-1}.$$

For the remainder of this subsection we assume that (Φ_0, Φ_3) is regular semisimple. By Proposition 2.5.5, the subalgebra $F(\Phi_0, \Phi_3) \subset M_{2h}(F)$ has centralizer an étale F-algebra L of dimension h, and hence

$$L^\times = H_0 \cap H_3 \tag{3.2.4}$$

is a torus.

Lemma 3.2.3. *The characters* (3.2.2) *and* (3.2.3) *are trivial on* (3.2.4)*.*

Proof. We used choices of isomorphisms (3.2.1) to define the characters in question, but it is easy to see that the resulting characters do not depend on these choices. We are therefore free to make them in a particular way.

Using the embedding $\Phi_i : K_i \to M_{2h}(F)$, we may view the standard representation F^{2h} as a free K_i-module of rank h. We claim that there is an F-subspace $V \subset F^{2h}$ such that the natural maps

$$K_0 \otimes_F V \to F^{2h} \quad \text{and} \quad K_3 \otimes_F V \to F^{2h}$$

are isomorphisms. Indeed, using Proposition 2.5.5 it is not hard to see that F^{2h} is free of rank one over both $K_0 \otimes_F L$ and $K_3 \otimes_F L$. There is therefore a Zariski dense set of elements $e \in F^{2h}$ such that

$$(K_0 \otimes_F L)e = F^{2h} = (K_3 \otimes_F L)e,$$

and for any such e the subspace $V = Le$ has the required properties.

Let $f_1, \ldots, f_h$ be an F-basis for the subspace $V \subset F^{2h}$. These same vectors form a basis for both the K_0-module and K_3-module structures on F^{2h}. If we use this basis to define the isomorphisms (3.2.1), then both isomorphisms identify $L^\times = H_0 \cap H_3$ with a subgroup of $\mathrm{GL}_h(F)$. It is clear from their constructions that (3.2.2) and (3.2.3) have trivial restriction to this subgroup. □

The preceding lemma allows us to make the following definition.

Definition 3.2.4. For every compactly supported function (1.4.3), define the *orbital integral*

$$O_{(\Phi_0,\Phi_3)}(f; s, \eta) = \int_{H_0 \cap H_3 \backslash H_0 \times H_3} f(g_0^{-1} h_0^{-1} h_3 g_3) \cdot |h_0|^s \cdot \eta(h_3)\, dh_0\, dh_3,$$

where $g_0, g_3 \in \mathrm{GL}_{2h}(F)$ are chosen to satisfy the integrality conditions

$$\Phi_i(\mathcal{O}_{K_i}) \subset g_i M_{2h}(\mathcal{O}_F) g_i^{-1}, \tag{3.2.5}$$

and $H_0 \cap H_3 \subset H_0 \times H_3$ is embedded diagonally.

Remark 3.2.5. The Haar measures on H_0, H_3, and $H_0 \cap H_3$ are normalized as in Remark 3.1.2.

Remark 3.2.6. The orbital integral depends on the choices of g_0 and g_3 satisfying (3.2.5), but not in a significant way. Different choices have the form $g_i' = z_i g_i k_i$ with $z_i \in H_i$ and $k_i \in \mathrm{GL}_{2h}(\mathcal{O}_F)$, and such a change multiplies the orbital integral by $\eta(z_3)|z_0|^{-s}$. In particular, the value of the orbital integral at $s = 0$ is well-defined up to ± 1.

Proposition 3.2.7. *If K/K_3 is ramified, then $O_{(\Phi_0,\Phi_3)}(f; s, \eta) = 0$ for every f as in* (1.4.3).

Proof. For any $u \in H_3 \cap g_3 \mathrm{GL}_{2h}(\mathcal{O}_F) g_3^{-1}$, making the change of variables $h_3 \mapsto h_3 u$ in Definition 3.2.4 shows that

$$O_{(\Phi_0,\Phi_3)}(f; s, \eta) = \eta(u) \cdot O_{(\Phi_0,\Phi_3)}(f; s, \eta).$$

Recalling that g_3 was chosen so that

$$\Phi_3(\mathcal{O}_{K_3}) \subset g_3 M_{2h}(\mathcal{O}_F) g_3^{-1},$$

we may use Φ_3 to view $g_3 \mathcal{O}_F^{2h}$ as a free $\mathcal{O}_{K_3}$-module of rank h. Our assumption on η guarantees the surjectivity of

$$H_3 \cap g_3 \mathrm{GL}_{2h}(\mathcal{O}_F) g_3^{-1} = \mathrm{Aut}_{\mathcal{O}_{K_3}}(g_3 \mathcal{O}_F^{2h}) \xrightarrow{\det} \mathcal{O}_{K_3}^\times \xrightarrow{\eta} \{\pm 1\},$$

and so we may choose u as above with $\eta(u) = -1$. □

Still assuming that (Φ_0, Φ_3) is regular semisimple, we show that the orbital integral of Definition 3.2.4 satisfies a functional equation in $s \mapsto -s$. This will be needed in the proof of Proposition 3.3.3 below.

Let $g_0, g_3 \in \mathrm{GL}_{2h}(F)$ be as in (3.2.5). We can use both the embedding $\Phi_i : K_i \to M_{2h}(F)$ and its Galois conjugate $\Phi_i \circ \sigma_i$ to make the $\mathcal{O}_F$-lattice $g_i \mathcal{O}_F^{2h}$ into an $\mathcal{O}_{K_i}$-module. These two $\mathcal{O}_{K_i}$-modules are isomorphic, and it follows that there is a $u_i \in g_i \mathrm{GL}_{2h}(\mathcal{O}_F) g_i^{-1}$ such that

$$u_i \Phi_i(x) u_i^{-1} = \Phi_i(x^{\sigma_i})$$

for all $x \in K_i$.

As in (2.4.2), fix an F-algebra generators $x_0 \in K_0$ and $x_3 \in K_3$, and define

$$\mathbf{z} = \Phi_0(x_0)\Phi_3(x_3) - \Phi_3(x_3)\Phi_0(x_0) \in M_{2h}(F). \tag{3.2.6}$$

Recall from Proposition 2.3.2 that $\mathbf{z}\Phi_i(x) = \Phi_i(x^{\sigma_i})\mathbf{z}$ for all $x \in K_i$, so that $\mathbf{z}u_i$ commutes with the image of Φ_i. From Proposition 2.5.5 we know that $\det(\mathbf{z}) \neq 0$, and so $\mathbf{z}u_i \in H_i$.

Proposition 3.2.8. *For any Hecke function f as in (1.4.3), we have the functional equation*

$$O_{(\Phi_0,\Phi_3)}(f;s,\eta) = |\mathbf{z}u_0|^s \cdot \eta(\mathbf{z}u_3) \cdot O_{(\Phi_0,\Phi_3)}(f;-s,\eta).$$

Proof. If we define another Hecke function f^* by

$$f^*(g) = f(g_0^{-1}u_0^{-1}g_0 \cdot g \cdot g_3^{-1}u_3g_3),$$

then

$$\begin{aligned} &O_{(\Phi_0,\Phi_3)}(f;s,\eta) \\ &= \int_{H_0\cap H_3\backslash H_0\times H_3} f(g_0^{-1}h_0^{-1}h_3g_3)\cdot|h_0|^s\cdot\eta(h_3)\,dh_0\,dh_3 \\ &= \int_{H_0\cap H_3\backslash H_0\times H_3} f^*(g_0^{-1}u_0h_0^{-1}h_3u_3^{-1}g_3)\cdot|h_0|^s\cdot\eta(h_3)\,dh_0\,dh_3 \\ &= |\mathbf{z}u_0|^s\eta(\mathbf{z}u_3)\int_{H_0\cap H_3\backslash H_0\times H_3} f^*(g_0^{-1}\mathbf{z}^{-1}h_0^{-1}h_3\mathbf{z}g_3)\cdot|h_0|^s\cdot\eta(h_3)\,dh_0\,dh_3, \end{aligned}$$

where the final equality is obtained by the substitution $h_i \mapsto h_i\mathbf{z}u_i$. Making the further substitution $h_i \mapsto \mathbf{z}h_i\mathbf{z}^{-1}$, and using Remarks 3.2.1 and 3.2.2, yields

$$O_{(\Phi_0,\Phi_3)}(f;s,\eta) = |\mathbf{z}u_0|^s \cdot \eta(\mathbf{z}u_3) \cdot O_{(\Phi_0,\Phi_3)}(f^*;-s,\eta).$$

Now note that $g_0^{-1}u_0^{-1}g_0$ and $g_3^{-1}u_3^{-1}g_3$ lie in $\mathrm{GL}_{2h}(\mathcal{O}_F)$, and so $f^* = f$. □

3.3. Matching pairs.

Suppose we are given F-algebra embeddings

$$\begin{aligned} \Phi_0 &: K_0 \to M_{2h}(F), & \Phi_1 &: K_1 \to B \\ \Phi_3 &: K_3 \to M_{2h}(F), & \Phi_2 &: K_2 \to B, \end{aligned}$$

where B is a central simple F-algebra of dimension $4h^2$. Assume that both (Φ_0,Φ_3) and (Φ_1,Φ_2) are regular semisimple.

Definition 3.3.1. The pairs (Φ_0,Φ_3) and (Φ_1,Φ_2) *match* if

$$\mathrm{Inv}_{(\Phi_0,\Phi_3)} = \mathrm{Inv}_{(\Phi_1,\Phi_2)}.$$

Denote by $\mathbf{w}_{12},\mathbf{z}_{12} \in B$ the elements (2.4.1) and (2.4.2) corresponding to the pair (Φ_1,Φ_2), to distinguish them from the similarly defined elements $\mathbf{w}_{03},\mathbf{z}_{03} \in M_{2h}(F)$ corresponding to (Φ_0,Φ_3). These depend on choices of F-algebra generators $x_i \in K_i$ for $i \in \{0,1,2,3\}$, which we now choose in a compatible way. If $\mathrm{char}(F) \neq 2$, first choose x_1 and x_2 in such a way that $x_i^{\sigma_i} = -x_i$ and then set

$$x_0 = (1,-1) \in K_0, \qquad x_3 = x_1 \otimes x_2 \in K_3 \subset K_1 \otimes_F K_2.$$

If $\mathrm{char}(F) = 2$, first choose x_1 and x_2 so that $x_i^{\sigma_i} = x_i + 1$ and then set

$$x_0 = (0,1) \in K_0, \qquad x_3 = x_1 \otimes 1 + 1 \otimes x_2 \in K_3 \subset K_1 \otimes_F K_2.$$

We now have F-subalgebras

$$F[\mathbf{w}_{12},\mathbf{z}_{12}] \subset B, \qquad F[\mathbf{w}_{03},\mathbf{z}_{03}] \subset M_{2h}(F).$$

Our particular choices of generators are justified by the following result.

Lemma 3.3.2. *If* (Φ_0, Φ_3) *and* (Φ_1, Φ_2) *match, there is an isomorphism of* F*-algebras*

$$F[\mathbf{w}_{12}, \mathbf{z}_{12}] \cong F[\mathbf{w}_{03}, \mathbf{z}_{03}]$$

sending $\mathbf{w}_{12} \mapsto \mathbf{w}_{03}$ *and* $\mathbf{z}_{12} \mapsto \mathbf{z}_{03}$*.*

Proof. The proof of Lemma 2.5.6 explains how to reconstruct the minimal polynomials of $\mathbf{w}_{12} \in B$ and $\mathbf{w}_{03} \in M_{2h}(F)$ from the invariant polynomials $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ and $\mathrm{Inv}_{(\Phi_0,\Phi_3)}$, using the scalars $c_{12}, d_{12}, c_{03}, d_{03} \in K_3$ characterized by the equalities

$$\mathbf{w}_{12} = c_{12} + d_{12}\mathbf{s}_{12}, \qquad \mathbf{w}_{03} = c_{03} + d_{03}\mathbf{s}_{03}$$

of Proposition 2.4.1. We have chosen (x_1, x_2) and (x_0, x_3) in such a way that $c_{12} = c_{03}$ and $d_{12} = d_{03}$, and so the matching of invariant polynomials implies the matching of minimal polynomials. Therefore there exists an isomorphism of (étale, by Proposition 2.5.5) F-algebras

$$F[\mathbf{w}_{12}] \cong F[\mathbf{w}_{03}] \tag{3.3.1}$$

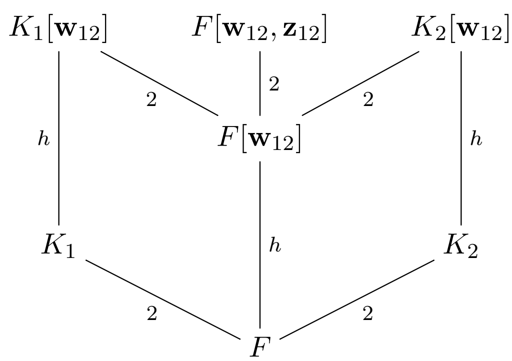
sending $\mathbf{w}_{12} \mapsto \mathbf{w}_{03}$. Using Proposition 2.4.2 one sees that $\mathbf{z}_{12}$ and $\mathbf{z}_{03}$ have the same (quadratic) minimal polynomial over (3.3.1), and the lemma follows. □

Proposition 3.3.3. *Assume that* B *is a division algebra. If* (Φ_0, Φ_3) *and* (Φ_1, Φ_2) *match, then*

$$O_{(\Phi_0,\Phi_3)}(f; 0, \eta) = 0$$

for all f *as in* (1.4.3)*.*

Proof. Identifying K_1 and K_2 with their images under $\Phi_i : K_i \to B$, and using Propositions 2.3.2 and 2.5.5, we find inside of B the diagram of field extensions



of the indicated degrees. Let σ_i be the nontrivial automorphism of $K_i[\mathbf{w}_{12}]$ fixing $F[\mathbf{w}_{12}]$, and recall from Proposition 2.4.2 that

$$\mathbf{z}_{12}^2 \in F[\mathbf{w}_{12}]$$

and $\mathbf{z}_{12}a = a^{\sigma_i}\mathbf{z}_{12}$ for all $a \in K_i[\mathbf{w}_{12}]$.

If $\mathbf{z}_{12}^2$ were a norm from $K_i[\mathbf{w}_{12}]$, say $\mathbf{z}_{12}^2 = aa^{\sigma_i}$, then

$$(\mathbf{z}_{12} - a)(\mathbf{z}_{12} + a^{\sigma_i}) = 0$$

would contradict B being a division algebra. This gives the final equality in

$$\begin{aligned} \eta_{K_i/F}(-\det(\mathbf{z}_{03})) &= \eta_{K_i/F}(-\mathrm{Nm}_{F[\mathbf{w}_{03},\mathbf{z}_{03}]/F}(\mathbf{z}_{03})) \\ &= \eta_{K_i/F}(-\mathrm{Nm}_{F[\mathbf{w}_{12},\mathbf{z}_{12}]/F}(\mathbf{z}_{12})) \\ &= \eta_{K_i/F}(\mathrm{Nm}_{F[\mathbf{w}_{12}]/F}(\mathbf{z}_{12}^2)) \\ &= \eta_{K_i[\mathbf{w}_{12}]/F[\mathbf{w}_{12}]}(\mathbf{z}_{12}^2) \\ &= -1, \end{aligned}$$

where $\eta_\bullet$ denotes the character associated to a quadratic extension by local class field theory, and we have used Lemma 3.3.2.

To complete the proof, we may assume (after Proposition 3.2.7) that K/K_3 is unramified. This implies that at least one of K_1 or K_2 is unramified over F. Without loss of generality, let us suppose K_1 is unramified.

If $u_3 \in \mathrm{GL}_{2h}(F)$ is as in Proposition 3.2.8, so that $\det(u_3) \in \mathcal{O}_F^\times$, then Remark 3.2.1 provides the first equality in

$$\eta(\mathbf{z}_{03}u_3) = \eta_{K_1/F}(\det(\mathbf{z}_{03}u_3)) = \eta_{K_1/F}(\det(\mathbf{z}_{03})) = -1.$$

The claim now follows from the functional equation of Proposition 3.2.8. □

3.4. **The central value conjecture.** Assume that K/K_3 is unramified, and suppose we are given F-algebra embeddings

$$\Phi_0 : K_0 \to M_{2h}(F), \qquad \Phi_1 : K_1 \to M_{2h}(F),$$
$$\Phi_3 : K_3 \to M_{2h}(F), \qquad \Phi_2 : K_2 \to M_{2h}(F).$$

Conjecture 3.4.1 (Biquadratic fundamental lemma)**.** *If the pairs (Φ_0, Φ_3) and (Φ_1, Φ_2) are regular semisimple and matching (Definitions 2.5.1 and 3.3.1), then for any f as in* (1.4.3) *we have*

$$\pm O_{(\Phi_0,\Phi_3)}(f; 0, \eta) = O_{(\Phi_1,\Phi_2)}(f).$$

Remark 3.4.2. When $K_1 \cong K_2$, Proposition A.3.3 implies that this conjecture is equivalent to the Guo-Jacquet fundamental lemma, which was proved by Guo [Guo96] when $f = \mathbf{1}$ is the characteristic function of $\mathrm{GL}_2(\mathcal{O}_F)$.

There is one case in which the biquadratic fundamental lemma is trivial.

Proposition 3.4.3. *If either one of K_1 or K_2 is isomorphic to $F \times F$, then Conjecture 3.4.1 is true.*

Proof. Assume for simplicity that $K_1 \cong F \times F$, the other case being entirely similar. In this case there are isomorphisms

$$K_1 \cong K_0, \quad K_2 \cong K_3, \quad K \cong K_2 \times K_2$$

(which we now fix), and the character $\eta : K_3^\times \to \{\pm 1\}$ is trivial. Thus we need only prove the equality

$$\int_{H_0\cap H_3\backslash H_0\times H_3} f(g_0^{-1}h_0^{-1}h_3g_3)\, dh_0\, dh_3 = \int_{H_1\cap H_2\backslash H_1\times H_2} f(g_1^{-1}h_1^{-1}h_2g_2)\, dh_1\, dh_2$$

for any Hecke function (1.4.3).

Recalling Remark 3.2.6, both integrals are independent of the choices of $g_0, g_1, g_2, g_3 \in \mathrm{GL}_{2h}(F)$ satisfying

$$\Phi_i(\mathcal{O}_{K_i}) \subset g_i M_{2h}(\mathcal{O}_F) g_i^{-1}.$$

More to the point, the integral on the left is unchanged if we replace the pair (Φ_0, Φ_3) by a pair that is $\mathrm{GL}_{2h}(F)$-conjugate to it. Corollary 2.5.8 implies that (Φ_0, Φ_3) and (Φ_1, Φ_2) are conjugate, and the equality of integrals follows. □

4. Intersections in Lubin-Tate space

We continue to let F be a nonarchimedean local field, and now assume that K_1 and K_2 are separable quadratic field extensions of F.

In the Lubin-Tate deformation space of a formal $\mathcal{O}_F$-module, one can construct cycles of formal $\mathcal{O}_{K_1}$-modules and $\mathcal{O}_{K_2}$-modules. We prove a formula expressing the intersection multiplicities of such cycles in terms of the invariant polynomials of §2.2; when $K_1 = K_2$, this recovers the main result of [Li21]. We then state a conjectural arithmetic fundamental lemma, in the spirit of [Zh12], relating the intersection multiplicity to the central derivative of an orbital integral.

4.1. **Initial data.** Let $\breve{F}$ be the completion of the maximal unramified extension of F, and let $\boldsymbol{k}$ be the residue field of $\mathcal{O}_{\breve{F}}$. Choose an extension of the reduction $\mathcal{O}_F \to \boldsymbol{k}$ to a ring homomorphism

$$\mathcal{O}_K \to \boldsymbol{k}. \tag{4.1.1}$$

In particular, $\boldsymbol{k}$ is both an $\mathcal{O}_{K_1}$-algebra and an $\mathcal{O}_{K_2}$-algebra.

Fix a pair $(\mathcal{G}, \mathcal{M})$ in which

- $\mathcal{G}$ is a formal $\mathcal{O}_F$-module over $\boldsymbol{k}$ of dimension 1 and height $2h$,
- $\mathcal{M}$ is a free $\mathcal{O}_F$-module of rank $2h$.

Fix also pairs $(\mathcal{H}_1, \mathcal{N}_1)$ and $(\mathcal{H}_2, \mathcal{N}_2)$ in which

- $\mathcal{H}_i$ is a formal $\mathcal{O}_{K_i}$-module over $\boldsymbol{k}$ of dimension 1 and height h,
- $\mathcal{N}_i$ is a free $\mathcal{O}_{K_i}$-module of rank h.

Remark 4.1.1. A one dimensional formal $\mathcal{O}_F$-module G over an $\mathcal{O}_F$-algebra A is always assumed to be *strict*, in the sense that the induced action $\mathcal{O}_F \to \mathrm{End}_A(\mathrm{Lie}(G)) \cong A$ agrees with the structure map. A similar assumption is made for formal $\mathcal{O}_{K_i}$-modules, which is why we have fixed an $\mathcal{O}_K$-algebra structure (4.1.1) on $\boldsymbol{k}$.

There are $\mathcal{O}_F$-linear isomorphisms $\mathcal{G} \cong \mathcal{H}_i$ and $\mathcal{M} \cong \mathcal{N}_i$. Rather than fixing such isomorphisms, we work in slightly more generality and fix pairs (ϕ_1, ψ_1) and (ϕ_2, ψ_2) of invertible elements

$$\begin{aligned} \phi_1 &\in \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{H}_1, \mathcal{G})[1/\pi], & \psi_1 &\in \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{N}_1, \mathcal{M})[1/\pi], \\ \phi_2 &\in \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{H}_2, \mathcal{G})[1/\pi], & \psi_2 &\in \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{N}_2, \mathcal{M})[1/\pi]. \end{aligned} \tag{4.1.2}$$

Denote by $\mathrm{ht}(\phi_i)$ the height of ϕ_i as a quasi-isogeny of formal $\mathcal{O}_F$-modules, and define

$$\mathrm{ht}(\psi_i) = \mathrm{ord}_F(\det(\eta_i \circ \psi_i))$$

for any $\mathcal{O}_F$-linear isomorphism $\eta_i : \mathcal{M} \cong \mathcal{N}_i$.

Remark 4.1.2. Although for now we work in the generality described above, eventually we will assume that

$$\phi_i : \mathcal{H}_i \cong \mathcal{G} \quad \text{and} \quad \psi_i : \mathcal{N}_i \cong \mathcal{M}$$

are $\mathcal{O}_F$-linear isomorphisms, so that $\mathrm{ht}(\phi_i) = 0 = \mathrm{ht}(\psi_i)$. The reader will lose little by restricting to this special case throughout.

Define central simple F-algebras

$$B(\mathcal{G}) = \mathrm{End}_{\mathcal{O}_F}(\mathcal{G})[1/\pi], \qquad B(\mathcal{M}) = \mathrm{End}_{\mathcal{O}_F}(\mathcal{M})[1/\pi]$$

of dimension $4h^2$. The first is a division algebra. The second is isomorphic to $M_{2h}(F)$, but we do not fix such an isomorphism.

The data (4.1.2), together with the natural actions of $\mathcal{O}_{K_i}$ on $\mathcal{H}_i$ and $\mathcal{N}_i$, determine F-algebra embeddings

$$\begin{aligned} \Phi_1 &: K_1 \to B(\mathcal{G}), & \Psi_1 &: K_1 \to B(\mathcal{M}) \\ \Phi_2 &: K_2 \to B(\mathcal{G}), & \Psi_2 &: K_2 \to B(\mathcal{M}). \end{aligned} \tag{4.1.3}$$

The constructions of (2.2.12) then provide us with monic degree h polynomials

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)} \in K_3[t], \qquad \mathrm{Inv}_{(\Psi_1,\Psi_2)} \in K_3[t].$$

We denote by

$$R(\Phi_1, \Phi_2, \Psi_1, \Psi_2) = \mathrm{Res}\left(\mathrm{Inv}_{(\Phi_1,\Phi_2)}, \mathrm{Inv}_{(\Psi_1,\Psi_2)}\right) \in K_3 \tag{4.1.4}$$

their resultant, and, when no confusion can arise, abbreviate this to

$$R = R(\Phi_1, \Phi_2, \Psi_1, \Psi_2).$$

It follows from the functional equations (Proposition 2.3.3) satisfied by both invariant polynomials that $R^{\sigma_3} = (-1)^h \cdot R$. In particular $R^2 \in F$, and we abbreviate $|R| = \sqrt{|R^2|}$.

Proposition 4.1.3. *If* (Φ_1, Φ_2) *is regular semisimple, then* $|R| \neq 0$.

Proof. First suppose that $K_1 \not\cong K_2$, so that K_3 is a field. If $|R| = 0$ then $R = 0$, and so the polynomials $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ and $\mathrm{Inv}_{(\Psi_1,\Psi_2)}$ share a common root in an algebraic closure of K_3.

Consider the F-subalgebra

$$F(\Phi_1, \Phi_2) \subset B(\mathcal{G})$$

generated by $\Phi_1(K_1) \cup \Phi_2(K_2)$. As $B(\mathcal{G})$ is a division algebra, Proposition 2.5.5 implies that $F(\Phi_1, \Phi_2)$ is a quaternion division algebra over its center $L = F[\mathbf{w}]$, which is a degree h field extension of F. In particular, the minimal polynomial of $\mathbf{w}$ over F is irreducible. As in the proof of Lemma 2.5.6, this minimal polynomial is related to the invariant polynomial $Q_{\mathbf{s}} = \mathrm{Inv}_{(\Phi_1,\Phi_2)}$ by a change of variables, and so $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ is itself irreducible.

As $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ and $\mathrm{Inv}_{(\Psi_1,\Psi_2)}$ have the same degree, share a common root, and the first is irreducible, they must be equal. By Proposition 2.5.7, the F-subalgebra

$$F(\Psi_1, \Psi_2) \subset B(\mathcal{M})$$

is also quaternion division algebra over L, and so any representation of it has F-dimension a multiple of $4h$. As $B(\mathcal{M}) \cong M_{2h}(F)$, we have arrived at a contradiction.

Now suppose that $K_1 \cong K_2$, so that $K_3 \cong F \times F$. This case is not really different. If we write $R = (R_1, R_2) \in F \times F$, the relation $R^{\sigma_3} = (-1)^h R$ noted above implies $R_1 = (-1)^h R_2$. Thus if $|R| = |R_1| \cdot |R_2|$ vanishes both $R_1 = 0$ and $R_2 = 0$.

As $K_3[T] \cong F[T] \times F[T]$, we may view $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ and $\mathrm{Inv}_{(\Psi_1,\Psi_2)}$ as pairs of polynomials with coefficients in F. The argument above shows that each component of $\mathrm{Inv}_{(\Phi_1,\Phi_2)}$ is irreducible, and the vanishing of R_1 and R_2 imply that those components agree with the components of $\mathrm{Inv}_{(\Psi_1,\Psi_2)}$. This again implies that $F(\Psi_1, \Psi_2)$ is a quaternion division algebra over a degree h field extension of F, contradicting $F(\Psi_1, \Psi_2) \subset M_{2h}(F)$. □

4.2. **Height calculations.** Define a formal $\mathcal{O}_F$-module

$$\mathbf{X} = \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{M}, \mathcal{G})$$

over $\boldsymbol{k}$, noncanonically isomorphic to $\mathcal{G}^{2h}$. That is to say, $\mathbf{X}$ is the formal scheme over $\boldsymbol{k}$ whose functor of points assigns to an Artinian $\boldsymbol{k}$-algebra the $\mathcal{O}_F$-module

$$\mathbf{X}(A) = \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{M}, \mathcal{G}(A)).$$

Similarly, for $i \in \{1, 2\}$ define formal $\mathcal{O}_F$-modules

$$\mathbf{Y}_i = \mathrm{Hom}_{\mathcal{O}_{K_i}}(\mathcal{N}_i, \mathcal{H}_i), \qquad \overline{\mathbf{Y}}_i = \mathrm{Hom}_{\mathcal{O}_{K_i}}(\mathcal{N}_i, \overline{\mathcal{H}}_i),$$

where $\overline{\mathcal{H}}_i = \mathcal{H}_i$ endowed with its conjugate $\mathcal{O}_{K_i}$-action. Each is noncanonically isomorphic to $\mathcal{G}^h$. There are natural morphisms

$$\begin{array}{ccc} \mathbf{Y}_i & & \overline{\mathbf{Y}}_i \\ & {\scriptstyle s_i}\searrow \quad \swarrow{\scriptstyle \overline{s}_i} & \\ & \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{N}_i, \mathcal{H}_i), & \end{array}$$

and the composition

$$\mathbf{Y}_i \times \overline{\mathbf{Y}}_i \xrightarrow{s_i \times \overline{s}_i} \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{N}_i, \mathcal{H}_i) \xrightarrow{x \mapsto \phi_i \circ x \circ \psi_i^{-1}} \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{M}, \mathcal{G}) = \mathbf{X} \tag{4.2.1}$$

defines a quasi-isogeny

$$\Delta_i \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_i \times \overline{\mathbf{Y}}_i, \mathbf{X})[1/\pi]. \tag{4.2.2}$$

Proposition 4.2.1. *The quasi-isogeny* (4.2.2) *has height*

$$\mathrm{ht}(\Delta_i) = h^2 \cdot \mathrm{ord}_F(d_i) + 2h \cdot \mathrm{ht}(\phi_i) - 2h \cdot \mathrm{ht}(\psi_i),$$

where $d_i \in \mathcal{O}_F$ is a generator of the discriminant of K_i/F.

Proof. By choosing an $\mathcal{O}_F$-basis $1, \eta \in \mathcal{O}_{K_i}$, we identify the natural map

$$\mathrm{Hom}_{\mathcal{O}_{K_i}}(\mathcal{O}_{K_i}, \mathcal{H}_i) \times \mathrm{Hom}_{\mathcal{O}_{K_i}}(\mathcal{O}_{K_i}, \overline{\mathcal{H}}_i) \to \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{O}_{K_i}, \mathcal{H}_i) \tag{4.2.3}$$

with the morphism of formal $\mathcal{O}_F$-modules

$$\mathcal{H}_i \times \mathcal{H}_i \xrightarrow{(x,y) \mapsto (x+y, \eta x + \eta^{\sigma_i} y)} \mathcal{H}_i \times \mathcal{H}_i. \tag{4.2.4}$$

The different $\mathfrak{D}_i$ of K_i/F is generated by $\eta - \eta^{\sigma_i}$, and so the kernel of (4.2.4) is the image of

$$\mathcal{H}_i[\mathfrak{D}_i] \xrightarrow{x \mapsto (x, -x)} \mathcal{H}_i \times \mathcal{H}_i.$$

It follows that the isogenies (4.2.3) and (4.2.4) have $\mathcal{O}_F$-height $h \cdot \mathrm{ord}_F(d_i)$. As $\mathcal{N}_i$ is free of rank h over $\mathcal{O}_{K_i}$, we deduce that that first arrow in (4.2.1) is an isogeny of $\mathcal{O}_F$-height $h^2 \cdot \mathrm{ord}_F(d_i)$. The claim follows easily from this. □

The quasi-isogenies (4.2.2) for $i \in \{1, 2\}$ determine an element

$$\Delta_2^{-1} \circ \Delta_1 \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_1 \times \overline{\mathbf{Y}}_1, \mathbf{Y}_2 \times \overline{\mathbf{Y}}_2)[1/\pi],$$

which is encoded by four components. The two that interest us are

$$\alpha_{12} \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_1, \overline{\mathbf{Y}}_2)[1/\pi], \qquad \beta_{12} \in \mathrm{Hom}_{\mathcal{O}_F}(\overline{\mathbf{Y}}_1, \mathbf{Y}_2)[1/\pi]. \tag{4.2.5}$$

Similarly,

$$\Delta_1^{-1} \circ \Delta_2 \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_2 \times \overline{\mathbf{Y}}_2, \mathbf{Y}_1 \times \overline{\mathbf{Y}}_1)[1/\pi]$$

is encoded by four components, and the two that interest us are

$$\alpha_{21} \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_2, \overline{\mathbf{Y}}_1)[1/\pi], \qquad \beta_{21} \in \mathrm{Hom}_{\mathcal{O}_F}(\overline{\mathbf{Y}}_2, \mathbf{Y}_1)[1/\pi]. \tag{4.2.6}$$

Lemma 4.2.2. *We have the equalities*

$$\mathrm{ht}(\alpha_{12}) = \mathrm{ht}(\beta_{12}) \quad \textit{and} \quad \mathrm{ht}(\alpha_{21}) = \mathrm{ht}(\beta_{21}).$$

Proof. We first claim that there is a $\gamma \in B(\mathcal{G})^\times$ such that

$$\Phi_1(x^{\sigma_1}) = \gamma \cdot \Phi_1(x) \cdot \gamma^{-1}, \qquad \Phi_2(y^{\sigma_2}) = \gamma \cdot \Phi_2(y) \cdot \gamma^{-1}.$$

for all $x \in K_1$ and $y \in K_2$. (We remark that if the pair (Φ_1, Φ_2) is regular semisimple, the element $\mathbf{z} \in B(\mathcal{G})$ defined by (2.4.2) is a unit by Proposition 2.5.5, and Proposition 2.4.2 allows us to take $\gamma = \mathbf{z}$). Abbreviate $B = B(\mathcal{G})$. The embeddings Φ_1 and Φ_2 of (4.1.3) determine two $\mathbb{Z}/2\mathbb{Z}$-gradings, exactly as in (A.1.2),

$$B = B_+^{\Phi_1} \oplus B_-^{\Phi_1}, \qquad B = B_+^{\Phi_2} \oplus B_-^{\Phi_2}.$$

These satisfy

$$B_-^{\Phi_1} = (B_+^{\Phi_1})^\perp, \qquad B_-^{\Phi_2} = (B_+^{\Phi_2})^\perp,$$

where $\perp$ is orthogonal complement with respect to the nondegenerate bilinear form $(b_1, b_2) \mapsto \mathrm{Trd}(b_1 b_2)$ determined by the reduced trace $\mathrm{Trd} : B \to F$. If $B_-^{\Phi_1} \cap B_-^{\Phi_2} = 0$ then $B_-^{\Phi_1} + B_-^{\Phi_2} = B$. Applying $\perp$ to both sides of this last equality yields $B_+^{\Phi_1} \cap B_+^{\Phi_2} = 0$, which contradicts $1 \in B_+^{\Phi_1} \cap B_+^{\Phi_2}$. Any nonzero $\gamma \in B_-^{\Phi_1} \cap B_-^{\Phi_2}$ is contained in $B^\times$ (recall that B is a division algebra) and satisfies the desired properties.

Using the quasi-isogenies of (4.1.2), we obtain quasi-isogenies

$$\gamma_1 = \phi_1^{-1} \circ \gamma \circ \phi_1 \in \mathrm{End}_{\mathcal{O}_F}(\mathcal{H}_1)[1/\pi]$$
$$\gamma_2 = \phi_2^{-1} \circ \gamma \circ \phi_2 \in \mathrm{End}_{\mathcal{O}_F}(\mathcal{H}_2)[1/\pi],$$

which do not commute with the actions of $\mathcal{O}_{K_1}$ and $\mathcal{O}_{K_2}$. Instead, they define quasi-isogenies

$$\gamma_1 \in \mathrm{Hom}_{\mathcal{O}_{K_1}}(\mathcal{H}_1, \overline{\mathcal{H}}_1)[1/\pi], \qquad \overline{\gamma}_1 \in \mathrm{Hom}_{\mathcal{O}_{K_1}}(\overline{\mathcal{H}}_1, \mathcal{H}_1)[1/\pi],$$
$$\gamma_2 \in \mathrm{Hom}_{\mathcal{O}_{K_2}}(\mathcal{H}_2, \overline{\mathcal{H}}_2)[1/\pi], \qquad \overline{\gamma}_2 \in \mathrm{Hom}_{\mathcal{O}_{K_2}}(\overline{\mathcal{H}}_2, \mathcal{H}_2)[1/\pi],$$

which in turn determine quasi-isogenies

$$\Gamma_1 \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_1, \overline{\mathbf{Y}}_1)[1/\pi], \qquad \overline{\Gamma}_1 \in \mathrm{Hom}_{\mathcal{O}_F}(\overline{\mathbf{Y}}_1, \mathbf{Y}_1)[1/\pi],$$
$$\Gamma_2 \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_2, \overline{\mathbf{Y}}_2)[1/\pi], \qquad \overline{\Gamma}_2 \in \mathrm{Hom}_{\mathcal{O}_F}(\overline{\mathbf{Y}}_2, \mathbf{Y}_2)[1/\pi],$$

all of the same height, and making the diagram

$$\begin{array}{ccc}
\mathbf{Y}_1 \times \overline{\mathbf{Y}}_1 & \xrightarrow{(a,b)\mapsto(\overline{\Gamma}_1(b),\Gamma_1(a))} & \mathbf{Y}_1 \times \overline{\mathbf{Y}}_1 \\
\downarrow{\scriptstyle \Delta_1} & & {\scriptstyle \Delta_1}\downarrow \\
\mathbf{X} & \xrightarrow{\gamma} & \mathbf{X} \\
\uparrow{\scriptstyle \Delta_2} & & {\scriptstyle \Delta_2}\uparrow \\
\mathbf{Y}_2 \times \overline{\mathbf{Y}}_2 & \xrightarrow[(a,b)\mapsto(\overline{\Gamma}_2(b),\Gamma_2(a))]{} & \mathbf{Y}_2 \times \overline{\mathbf{Y}}_2
\end{array}$$

commute. The commutativity implies that $\beta_{12} \circ \Gamma_1 = \overline{\Gamma}_2 \circ \alpha_{12}$, so $\mathrm{ht}(\alpha_{12}) = \mathrm{ht}(\beta_{12})$. The equality $\mathrm{ht}(\alpha_{21}) = \mathrm{ht}(\beta_{21})$ is proved similarly. □

Proposition 4.2.3. *The heights of* (4.2.5) *and* (4.2.6) *are related to the resultant* (4.1.4) *by*

$$\mathrm{ht}(\alpha_{12}) + \mathrm{ht}(\alpha_{21}) = \mathrm{ord}_F(R^2) = \mathrm{ht}(\beta_{12}) + \mathrm{ht}(\beta_{21}).$$

Proof. If we define idempotent elements

$$e_1, f_1, e_2, f_2 \in \mathrm{End}_{\mathcal{O}_F}(\mathbf{X})[1/\pi]$$

by the commutativity of the diagrams

$$\begin{array}{ccc} \mathbf{Y}_i \times \overline{\mathbf{Y}}_i & \xrightarrow{(a,b)\mapsto(a,0)} & \mathbf{Y}_i \times \overline{\mathbf{Y}}_i \\ \Delta_i \downarrow & & \downarrow \Delta_i \\ \mathbf{X} & \xrightarrow{e_i} & \mathbf{X} \end{array} \qquad \begin{array}{ccc} \mathbf{Y}_i \times \overline{\mathbf{Y}}_i & \xrightarrow{(a,b)\mapsto(0,b)} & \mathbf{Y}_i \times \overline{\mathbf{Y}}_i \\ \Delta_i \downarrow & & \downarrow \Delta_i \\ \mathbf{X} & \xrightarrow{f_i} & \mathbf{X}, \end{array}$$

then the composition

$$\mathbf{Y}_1 \times \overline{\mathbf{Y}}_1 \xrightarrow{\Delta_1} \mathbf{X} \xrightarrow{f_2e_1+e_2f_1} \mathbf{X} \xrightarrow{\Delta_2^{-1}} \mathbf{Y}_2 \times \overline{\mathbf{Y}}_2$$

is given by $(a,b) \mapsto (\beta_{12}(b), \alpha_{12}(a))$. It follows that

$$\mathrm{ht}(\alpha_{12}) + \mathrm{ht}(\beta_{12}) = \mathrm{ht}(f_2e_1 + e_2f_1) + \mathrm{ht}(\Delta_1) - \mathrm{ht}(\Delta_2).$$

The same equality holds with the indices 1 and 2 switched everywhere. Adding these together and using Lemma 4.2.2 shows that

$$2\mathrm{ht}(\alpha_{12}) + 2\mathrm{ht}(\alpha_{21}) = \mathrm{ht}(f_2e_1 + e_2f_1) + \mathrm{ht}(f_1e_2 + e_1f_2). \tag{4.2.7}$$

Some elementary algebra shows that e_i and f_i are the images of $(1,0)$ and $(0,1)$, respectively, under

$$K_i \times K_i \cong K_i \otimes_F K_i \xrightarrow{\Phi_i \otimes \Psi_i} B(\mathcal{G}) \otimes_F B(\mathcal{M})^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{O}_F}(\mathbf{X})[1/\pi]$$

where the first isomorphism is the inverse of

$$K_i \otimes_F K_i \xrightarrow{a \otimes b \mapsto (ab, ab^{\sigma_i})} K_i \times K_i,$$

and op indicates the opposite algebra.

Extend (4.1.3) to K_3-algebra embeddings

$$\Phi_i : K \to C(\mathcal{G}) = B(\mathcal{G}) \otimes_F K_3$$
$$\Psi_i : K \to C(\mathcal{G}) = B(\mathcal{G}) \otimes_F K_3$$

as in (2.2.3), and denote by $\mathrm{Nrd} : C(\mathcal{G}) \otimes_{K_3} C(\mathcal{M})^{\mathrm{op}} \to K_3$ the reduced norm as a central simple K_3-algebra. If we fix any K_3-algebra generator $y \in K$, the images of these idempotents in $C(\mathcal{G}) \otimes_{K_3} C(\mathcal{M})^{\mathrm{op}}$ are given by the explicit formulas

$$e_i = [\Phi_i(y - y^{\tau_3}) \otimes 1]^{-1} \cdot [\Phi_i(y) \otimes 1 - 1 \otimes \Psi_i(y^{\tau_3})]$$
$$f_i = [\Phi_i(y - y^{\tau_3}) \otimes 1]^{-1} \cdot [\Phi_i(y) \otimes 1 - 1 \otimes \Psi_i(y)].$$

In general, if $a \in C(\mathcal{G})$ and $b \in C(\mathcal{M})^{\mathrm{op}}$ have reduced characteristic polynomials $P_a, P_b \in K_3[t]$, then

$$\mathrm{Res}(P_a, P_b) = \mathrm{Nrd}(a \otimes 1 - 1 \otimes b).$$

Indeed, after extending scalars we may assume that both $C(\mathcal{G})$ and $C(\mathcal{M})^{\mathrm{op}}$ are matrix algebras. If a and b are diagonalizable then the equality is obvious, and the general case follows by a Zariski density argument.

The construction (2.2.6) attaches to (Φ_1, Φ_2) and (Ψ_1, Ψ_2) elements

$$\mathbf{s}_\Phi \in C(\mathcal{G}) \quad \text{and} \quad \mathbf{s}_\Psi \in C(\mathcal{M})^{\mathrm{op}},$$

whose reduced characteristic polynomials $P_{\mathbf{s}_\Phi}, P_{\mathbf{s}_\Psi} \in K_3[T]$ are the squares of the invariant polynomials of (Φ_1, Φ_2) and (Ψ_1, Ψ_2), respectively. Thus

$$\begin{aligned} R^4 &= \mathrm{Res}(\mathrm{Inv}^2_{(\Phi_1,\Phi_2)}, \mathrm{Inv}^2_{(\Psi_1,\Psi_2)}) \\ &= \mathrm{Res}(P_{\mathbf{s}_\Phi}, P_{\mathbf{s}_\Psi}) \\ &= \mathrm{Nrd}(\mathbf{s}_\Phi \otimes 1 - 1 \otimes \mathbf{s}_\Psi). \end{aligned}$$

It follows from Proposition 2.3.2 that

$$-(\mathbf{s}_\Phi \otimes 1 - 1 \otimes \mathbf{s}_\Psi) = (1 - \mathbf{s}_\Phi) \otimes 1 - 1 \otimes (1 - \mathbf{s}_\Psi) = \mathbf{s}_\Phi^{\sigma_3} \otimes 1 - 1 \otimes \mathbf{s}_\Psi^{\sigma_3},$$

and using this and Lemma 2.3.1 we find

$$R^4 = \mathrm{Nrd}\left[\frac{(\Phi_1(y) - \Phi_2(y))^2}{(y - y^{\tau_3})^2} \otimes 1 - 1 \otimes \frac{(\Psi_1(y) - \Psi_2(y))^2}{(y - y^{\tau_3})^2} \right]. \tag{4.2.8}$$

Define $S, T \in C(\mathcal{G}) \otimes_{K_3} C(\mathcal{M})^{\mathrm{op}}$ by

$$\begin{aligned} S &= [\Phi_1(y) - \Phi_2(y)] \otimes 1 + 1 \otimes [\Psi_1(y) - \Psi_2(y)] \\ &= [\Phi_1(y - y^{\tau_3}) \otimes 1] \cdot e_1 - [\Phi_2(y - y^{\tau_3}) \otimes 1] \cdot e_2 \\ T &= [\Phi_1(y) - \Phi_2(y)] \otimes 1 - 1 \otimes [\Psi_1(y) - \Psi_2(y)] \\ &= [\Phi_1(y - y^{\tau_3}) \otimes 1] \cdot f_1 - [\Phi_2(y - y^{\tau_3}) \otimes 1] \cdot f_2. \end{aligned}$$

On the one hand, we have

$$ST = [\Phi_1(y) - \Phi_2(y)]^2 \otimes 1 - 1 \otimes [\Psi_1(y) - \Psi_2(y)]^2,$$

and comparison with (4.2.8) shows that

$$R^4 = \frac{\mathrm{Nrd}(ST)}{\mathrm{Nrd}(\Phi_1(y - y^{\tau_3}) \otimes 1) \cdot \mathrm{Nrd}(\Phi_2(y - y^{\tau_3}) \otimes 1)}. \tag{4.2.9}$$

On the other hand,

$$\begin{aligned} ST &= STe_2 + STf_2 \\ &= -[\Phi_2(y - y^{\tau_3}) \otimes 1] \cdot e_2 \cdot f_1 \cdot [\Phi_1(y - y^{\tau_3}) \otimes 1] \cdot e_2 \\ &\quad - [\Phi_2(y - y^{\tau_3}) \otimes 1] \cdot f_2 \cdot e_1 \cdot [\Phi_1(y - y^{\tau_3}) \otimes 1] \cdot f_2 \\ &= -[\Phi_2(y - y^{\tau_3}) \otimes 1] \cdot (e_2 f_1 + f_2 e_1) \cdot [\Phi_1(y - y^{\tau_3}) \otimes 1] \cdot (e_1 f_2 + f_1 e_2). \end{aligned}$$

Combining this with (4.2.9) shows that

$$R^4 = \mathrm{Nrd}(f_2 e_1 + e_2 f_1) \cdot \mathrm{Nrd}(f_1 e_2 + e_1 f_2),$$

and combining this with (4.2.7) proves the proposition. □

4.3. **Cycles on a formal module.** We now use the calculations of §4.2 to compute the intersection multiplicity of two cycles on the formal scheme

$$\mathbf{X} \cong \mathrm{Spf}(\boldsymbol{k}[[x_1, \ldots, x_{2h}]]).$$

For $i \in \{1, 2\}$, we use the quasi-isogeny Δ_i of (4.2.2) to define

$$f_i \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_i, \mathbf{X})[1/\pi] \tag{4.3.1}$$

as the composition

$$\mathbf{Y}_i \xrightarrow{y \mapsto (y,0)} \mathbf{Y}_i \times \overline{\mathbf{Y}}_i \xrightarrow{\Delta_i} \mathbf{X}.$$

Choosing $k_i \in \mathbb{Z}$ large enough that π^{k_i} clears the denominator in (4.3.1), we obtain finite morphisms

$$\begin{array}{ccc} \mathbf{Y}_1 & & \mathbf{Y}_2 \\ & {\scriptstyle \pi^{k_1} f_1} \searrow \quad \swarrow {\scriptstyle \pi^{k_2} f_2} & \\ & \mathbf{X}. & \end{array} \tag{4.3.2}$$

Remark 4.3.1. If ϕ_i and ψ_i are chosen as in Remark 4.1.2 then one can take $k_i = 0$, and the resulting maps $f_i : \mathbf{Y}_i \to \mathbf{X}$ are closed immersions.

We wish to compute the intersection multiplicity of $\mathbf{Y}_1$ and $\mathbf{Y}_2$, viewed as codimension h cycles on $\mathbf{X}$. This is, by definition, the dimension of the $\boldsymbol{k}$-vector space of global sections of the $\mathcal{O}_{\mathbf{X}}$-module tensor product of

$$\mathcal{F}_1 = (\pi^{k_1} f_1)_* \mathcal{O}_{\mathbf{Y}_1} \quad \text{and} \quad \mathcal{F}_2 = (\pi^{k_2} f_2)_* \mathcal{O}_{\mathbf{Y}_2}.$$

Of course the sheaf $\mathcal{F}_i$ depends on the choices of k_i, ϕ_i, and ψ_i, but we suppress this from the notation. The following theorem gives an explicit formula for this dimension in terms of the resultant (4.1.4).

Theorem 4.3.2. *Recall that $q = |\pi|^{-1}$ is the cardinality of the residue field of $\mathcal{O}_F$, and that $d_1, d_2 \in \mathcal{O}_F$ generate the discriminants of K_1/F and K_2/F. The intersection multiplicity of the cycles* (4.3.2) *is*

$$\dim_{\boldsymbol{k}} H^0\big(\mathbf{X}, \mathcal{F}_1 \otimes \mathcal{F}_2\big) = \frac{q^{2h^2(k_1+k_2)} \cdot q^{h\cdot[\mathrm{ht}(\phi_1)+\mathrm{ht}(\phi_2)-\mathrm{ht}(\psi_1)-\mathrm{ht}(\psi_2)]}}{|R| \cdot |d_1 d_2|^{h^2/2}}.$$

In particular, the left hand side is finite if and only if $|R| \neq 0$.

Proof. The proof will follow easily from the height calculations of §4.2 and the following lemma, which shows that f_i admits a particularly nice factorization

$$\mathbf{Y}_i \xrightarrow{A_i} \mathbf{Z}_i \hookrightarrow \mathbf{X}$$

as a quasi-isogeny of formal $\mathcal{O}_F$-modules followed by a closed immersion.

Lemma 4.3.3. *There is a (non-unique) decomposition $\mathbf{X} = \mathbf{Z}_i \times \overline{\mathbf{Z}}_i$ of formal $\mathcal{O}_F$-modules such that the quasi-isogeny Δ_i has the form*

$$\begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix} : \mathbf{Y}_i \times \overline{\mathbf{Y}}_i \to \mathbf{Z}_i \times \overline{\mathbf{Z}}_i$$

for some

$$A_i \in \mathrm{Hom}_{\mathcal{O}_F}(\mathbf{Y}_i, \mathbf{Z}_i)[1/\pi]$$
$$B_i \in \mathrm{Hom}_{\mathcal{O}_F}(\overline{\mathbf{Y}}_i, \mathbf{Z}_i)[1/\pi]$$
$$D_i \in \mathrm{Hom}_{\mathcal{O}_F}(\overline{\mathbf{Y}}_i, \overline{\mathbf{Z}}_i)[1/\pi].$$

Proof. Let $\overline{\mathcal{N}}_i = \mathcal{N}_i$, but endowed with its conjugate $\mathcal{O}_{K_i}$ action. As $\mathrm{End}_{\mathcal{O}_F}(\mathcal{G})$ is the unique maximal order in $B(\mathcal{G})$, the map Φ_i of (4.1.3) restricts to $\Phi_i : \mathcal{O}_{K_i} \to \mathrm{End}_{\mathcal{O}_F}(\mathcal{G})$. Let $\mathcal{G}_i = \mathcal{G}$ endowed with this action of $\mathcal{O}_{K_i}$. Tracing through the definitions, we have canonical identifications of quasi-isogenies

$$\begin{array}{ccc} \mathbf{Y}_i \times \overline{\mathbf{Y}}_i & = & \mathrm{Hom}_{\mathcal{O}_{K_i}}(\mathcal{N}_i \times \overline{\mathcal{N}}_i, \mathcal{H}_i) \\ \downarrow{\scriptstyle \Delta_i} & & \downarrow{\scriptstyle x \mapsto \phi_i \circ x \circ \gamma_i} \\ \mathbf{X} & = & \mathrm{Hom}_{\mathcal{O}_{K_i}}(\mathcal{O}_{K_i} \otimes_{\mathcal{O}_F} \mathcal{M}, \mathcal{G}_i), \end{array}$$

where γ_i is the K_i-linear composition

$$K_i \otimes_F \mathcal{M}[1/\pi] \xrightarrow{\mathrm{id} \otimes \psi_i^{-1}} K_i \otimes_F \mathcal{N}_i[1/\pi] \xrightarrow{x \otimes n \mapsto (xn, x^{\sigma_i} n)} \mathcal{N}_i[1/\pi] \times \overline{\mathcal{N}}_i[1/\pi].$$

Using the Iwasawa decomposition in $\mathrm{GL}_{2h}(K_i)$, we may find a decomposition of $\mathcal{O}_{K_i}$-modules

$$\mathcal{O}_{K_i} \otimes_{\mathcal{O}_F} \mathcal{M} \cong \mathcal{P} \times \mathcal{Q}$$

such that $\gamma_i(\mathcal{Q}[1/\pi]) \subset \overline{\mathcal{N}}_i[1/\pi]$. This induces a decomposition of the lower right corner of the diagram, and the induced decomposition of the lower left corner has the desired properties. □

The isogeny $\pi^{k_i} A_i : \mathbf{Y}_i \to \mathbf{Z}_i$ is a finite flat morphism of degree

$$e_i \stackrel{\mathrm{def}}{=} q^{2h^2 k_i + \mathrm{ht}(A_i)}.$$

Thus $(\pi^{k_i} f_i)_* \mathcal{O}_{\mathbf{Y}_i} \cong \mathcal{O}_{\mathbf{Z}_i}^{e_i}$ as $\mathcal{O}_{\mathbf{X}}$-modules, and

$$\mathcal{F}_1 \otimes \mathcal{F}_2 \cong (\mathcal{O}_{\mathbf{Z}_1} \otimes \mathcal{O}_{\mathbf{Z}_2})^{e_1 e_2}.$$

The tensor product on the right is the structure sheaf of

$$\mathbf{Z}_1 \times_{\mathbf{X}} \mathbf{Z}_2 = \ker(\mathbf{Z}_1 \to \mathbf{X} \to \overline{\mathbf{Z}}_2),$$

which, by the definition of height, is the formal spectrum of a $\boldsymbol{k}$-algebra of dimension $q^{\mathrm{ht}(\mathbf{Z}_1 \to \mathbf{X} \to \overline{\mathbf{Z}}_2)}$. Thus

$$\begin{aligned} \dim_{\boldsymbol{k}} H^0\big(\mathbf{X}, \mathcal{F}_1 \otimes \mathcal{F}_2\big) &= e_1 e_2 \cdot \dim_{\boldsymbol{k}} H^0(\mathbf{X}, \mathcal{O}_{\mathbf{Z}_1} \otimes \mathcal{O}_{\mathbf{Z}_2}) \\ &= q^{2h^2(k_1+k_2)} \cdot q^{\mathrm{ht}(A_1)+\mathrm{ht}(A_2)} \cdot q^{\mathrm{ht}(\mathbf{Z}_1 \to \mathbf{X} \to \overline{\mathbf{Z}}_2)}. \end{aligned}$$

The composition

$$\mathbf{Y}_1 \xrightarrow{A_1} \mathbf{Z}_1 \to \mathbf{X} \to \overline{\mathbf{Z}}_2 \xrightarrow{D_2^{-1}} \overline{\mathbf{Y}}_2$$

is precisely the map α_{12} of (4.2.5), and so

$$\begin{aligned} \mathrm{ht}(\mathbf{Z}_1 \to \mathbf{X} \to \overline{\mathbf{Z}}_2) &= \mathrm{ht}(\alpha_{12}) - \mathrm{ht}(A_1) + \mathrm{ht}(D_2) \\ &= \mathrm{ht}(\alpha_{12}) - \mathrm{ht}(A_1) - \mathrm{ht}(A_2) + \mathrm{ht}(\Delta_2). \end{aligned}$$

This leaves us with

$$\dim_{\boldsymbol{k}} H^0\big(\mathbf{X}, \mathcal{F}_1 \otimes \mathcal{F}_2\big) = q^{2h^2(k_1+k_2)} q^{\mathrm{ht}(\alpha_{12})+\mathrm{ht}(\Delta_2)}. \tag{4.3.3}$$

As the same reasoning holds with the indices 1 and 2 reversed throughout,

$$\dim_{\boldsymbol{k}} H^0\big(\mathbf{X}, \mathcal{F}_1 \otimes \mathcal{F}_2\big) = q^{2h^2(k_1+k_2)} q^{\mathrm{ht}(\alpha_{21})+\mathrm{ht}(\Delta_1)}. \tag{4.3.4}$$

Multiplying (4.3.3) and (4.3.4) together and using Proposition 4.2.3 yields

$$\begin{aligned} \dim_{\boldsymbol{k}} H^0\big(\mathbf{X}, \mathcal{F}_1 \otimes \mathcal{F}_2\big) &= q^{2h^2(k_1+k_2)} q^{\frac{\mathrm{ht}(\alpha_{12})+\mathrm{ht}(\alpha_{21})+\mathrm{ht}(\Delta_1)+\mathrm{ht}(\Delta_2)}{2}} \\ &= q^{2h^2(k_1+k_2)} q^{\frac{\mathrm{ord}_F(R^2)+\mathrm{ht}(\Delta_1)+\mathrm{ht}(\Delta_2)}{2}} \\ &= |R|^{-1} \cdot q^{2h^2(k_1+k_2)} q^{\frac{\mathrm{ht}(\Delta_1)+\mathrm{ht}(\Delta_2)}{2}}. \end{aligned}$$

Theorem 4.3.2 now follows from the formulas for $\mathrm{ht}(\Delta_1)$ and $\mathrm{ht}(\Delta_2)$ found in Proposition 4.2.1. □

4.4. **Cycles on the Lubin-Tate tower.** Assume now that the elements (4.1.2) are chosen as in Remark 4.1.2. In other words, for $i \in \{1,2\}$ we fix $\mathcal{O}_F$-linear isomorphisms

$$\phi_i : \mathcal{H}_i \cong \mathcal{G} \quad \text{and} \quad \psi_i : \mathcal{N}_i \cong \mathcal{M}.$$

Let $\mathrm{Nilp}(\mathcal{O}_{\breve{F}})$ be the category of $\mathcal{O}_{\breve{F}}$-schemes on which the uniformizer $\pi \in \mathcal{O}_F$ is locally nilpotent. For any $S \in \mathrm{Nilp}(\mathcal{O}_{\breve{F}})$ we abbreviate

$$\bar{S} = S \times_{\mathrm{Spec}(\mathcal{O}_{\breve{F}})} \mathrm{Spec}(\boldsymbol{k}).$$

Associated to the pair $(\mathcal{G}, \mathcal{M})$ and an integer $m \ge 0$ one has the Lubin-Tate deformation space

$$X(\pi^m) \to \mathrm{Spf}(\mathcal{O}_{\breve{F}})$$

classifying triples (G, ϱ, t_m) over $S \in \mathrm{Nilp}(\mathcal{O}_{\breve{F}})$ consisting of

- a formal $\mathcal{O}_F$-module G over S,
- an $\mathcal{O}_F$-linear quasi-isogeny $\varrho : \mathcal{G}_{\bar{S}} \to G_{\bar{S}}$ of height 0,
- an $\mathcal{O}_F$-linear Drinfeld level structure $t_m : \pi^{-m}\mathcal{M}/\mathcal{M} \to G[\pi^m]$.

For each $i \in \{1,2\}$ the above isomorphisms ϕ_i and ψ_i determine a closed formal subscheme

$$f_i(m) : Y_i(\pi^m) \hookrightarrow X(\pi^m), \tag{4.4.1}$$

defined as the locus of points (G, ϱ, t_m) for which there exists a (necessarily unique) action $\mathcal{O}_{K_i} \to \mathrm{End}_{\mathcal{O}_F}(G)$ making both

$$\mathcal{H}_{i,\overline{S}} \xrightarrow{\phi_i} \mathcal{G}_{\overline{S}} \xrightarrow{\varrho} G_{\overline{S}}$$

and

$$\pi^{-m}\mathcal{N}_i/\mathcal{N}_i \xrightarrow{\psi_i} \pi^{-m}\mathcal{M}/\mathcal{M} \xrightarrow{t_m} G[\pi^m]$$

$\mathcal{O}_{K_i}$-linear.

We think of $Y_1(\pi^m)$ and $Y_2(\pi^m)$ as cycles on $X(\pi^m)$. Their intersection multiplicity is, by definition, the length of the $\mathcal{O}_{\breve{F}}$-module of global sections of the tensor product of coherent $\mathcal{O}_{X(\pi^m)}$-modules

$$\mathcal{F}_1(\pi^m) = f_1(m)_* \mathcal{O}_{Y_1(\pi^m)} \quad \text{and} \quad \mathcal{F}_2(\pi^m) = f_2(m)_* \mathcal{O}_{Y_2(\pi^m)}.$$

Proposition 4.4.1. *If $|R| \neq 0$, then*

$$\mathrm{len}_{\mathcal{O}_{\breve{F}}} H^0\big(X(\pi^m), \mathcal{F}_1(\pi^m) \otimes \mathcal{F}_2(\pi^m)\big) = |R|^{-1} \cdot |d_1 d_2|^{-h^2/2}$$

for all $m \gg 0$.

Proof. Because we have now chosen the data (4.1.2) as in Remark 4.1.2, we may take $k_1 = k_2 = 0$ throughout §4.3. Theorem 4.3.2 implies that the $\boldsymbol{k}$-vector space $H^0(\mathbf{X}, \mathcal{F}_1 \otimes \mathcal{F}_2)$ has finite dimension, and so [Li21, Theorem 4.1] implies that there is an isomorphism of $\mathcal{O}_{\breve{F}}$-modules

$$H^0\big(X(\pi^m), \mathcal{F}_1(\pi^m) \otimes \mathcal{F}_2(\pi^m)\big) \cong H^0(\mathbf{X}, \mathcal{F}_1 \otimes \mathcal{F}_2)$$

for all $m \gg 0$. Now use the equality of Theorem 4.3.2. □

For any $g \in B(\mathcal{M})^\times$, we can replace the embedding $\Psi_2 : K_2 \to B(\mathcal{M})$ with its conjugate $g\Psi_2 g^{-1}$ in (4.1.4) to define

$$R(g) = R(\Phi_1, \Phi_2, \Psi_1, g\Psi_2 g^{-1}) \in K_3. \tag{4.4.2}$$

As in §4.1, $R(g)^2 \in F$ and we abbreviate

$$|R(g)| = \sqrt{|R(g)^2|}.$$

If (Φ_1, Φ_2) is regular semisimple, then $|R(g)| \neq 0$ by Proposition 4.1.3.

Theorem 4.4.2. *Suppose $m \geq 0$, and set*

$$U(\pi^m) = \ker\big(\mathrm{Aut}_{\mathcal{O}_F}(\mathcal{M}) \to \mathrm{Aut}_{\mathcal{O}_F}(\mathcal{M}/\pi^m\mathcal{M})\big) \subset B(\mathcal{M})^\times.$$

If the pair (Φ_1, Φ_2) is regular semisimple, then

$$\mathrm{len}_{\mathcal{O}_{\breve{F}}} H^0\big(X(\pi^m), \mathcal{F}_1(\pi^m) \otimes \mathcal{F}_2(\pi^m)\big) = \frac{c(m) \cdot |d_1 d_2|^{-h^2/2}}{\mathrm{vol}(U(\pi^m))} \int_{U(\pi^m)} \frac{dg}{|R(g)|}.$$

In particular, the left hand side is finite. Here we have defined $c(m) = 1$ if $m > 0$, and

$$c(0) = \frac{\#\mathrm{Aut}_{\mathcal{O}_F}(\mathcal{M}/\pi\mathcal{M})}{\#\mathrm{Aut}_{\mathcal{O}_{K_1}}(\mathcal{N}_1/\pi\mathcal{N}_1) \cdot \#\mathrm{Aut}_{\mathcal{O}_{K_2}}(\mathcal{N}_2/\pi\mathcal{N}_2)}.$$

Proof. This follows from Proposition 4.4.1, exactly as in the proof of [Li21, Proposition 6.6]. □

4.5. **The central derivative conjecture.** We change the setup slightly from §4.1. Fix $\mathcal{O}_F$-linear isomorphisms

$$\phi_1 : \mathcal{H}_1 \cong \mathcal{G}, \qquad \phi_2 : \mathcal{H}_2 \cong \mathcal{G}$$

such that the induced F-algebra embeddings

$$\Phi_1 : K_1 \to B(\mathcal{G}), \qquad \Phi_2 : K_2 \to B(\mathcal{G})$$

form a regular semisimple pair (Φ_1, Φ_2).

Suppose we are also given F-algebra embeddings

$$\Phi_0 : K_0 \to M_{2h}(F), \qquad \Phi_3 : K_3 \to M_{2h}(F)$$

such that the pair (Φ_0, Φ_3) is regular semisimple and matches (Φ_1, Φ_2) in the sense of Definition 3.3.1. As $B(\mathcal{G})$ is a division algebra, Proposition 3.3.3 implies that

$$O_{(\Phi_0,\Phi_3)}(f; 0, \eta) = 0$$

for all f as in (1.4.3). In particular, this holds when f is the characteristic function of $\mathrm{GL}_{2h}(\mathcal{O}_F)$, which we denote by $\mathbf{1} : \mathrm{GL}_{2h}(F) \to \mathbb{C}$.

We now consider a modified version of the constructions of §4.4, in which no Drinfeld level structure is added, but the quasi-isogeny ϱ in the moduli problem is allowed to be of arbitrary height.

Let $X^\bullet$ be the formal scheme over $\mathcal{O}_{\breve{F}}$ classifying pairs (G, ϱ) consisting of a formal $\mathcal{O}_F$-module G over $S \in \mathrm{Nilp}(\mathcal{O}_{\breve{F}})$ and an $\mathcal{O}_F$-linear quasi-isogeny $\varrho : \mathcal{G}_{\bar{S}} \to G_{\bar{S}}$. For every $\ell \in \mathbb{Z}$ the open and closed formal subscheme

$$X^{(\ell)} \subset X^\bullet$$

defined by $\mathrm{ht}(\varrho) = \ell$ is isomorphic to the formal spectrum of a power series ring in $2h - 1$ variables over $\mathcal{O}_{\breve{F}}$.

For $i \in \{1, 2\}$, let $\breve{K}_i$ be the completion of the maximal unramified extension of K_i. Consider the formal $\mathcal{O}_{\breve{K}_i}$-scheme $Y_i^\bullet$ classifying pairs (H, ϱ) consisting of a formal $\mathcal{O}_{K_i}$-module H over $S \in \mathrm{Nilp}(\mathcal{O}_{\breve{K}_i})$ and an $\mathcal{O}_{K_i}$-linear quasi-isogeny $\varrho : \mathcal{H}_{i,\bar{S}} \to H_{\bar{S}}$. Denote by

$$f_i : Y_i^\bullet \to X^\bullet \tag{4.5.1}$$

the morphism of $\mathcal{O}_{\breve{F}}$-schemes sending (H, ϱ) to the formal $\mathcal{O}_F$-module $G = H$ endowed with the quasi-isogeny

$$\mathcal{G}_{\bar{S}} \xrightarrow{\phi_1^{-1}} \mathcal{H}_{i,\bar{S}} \xrightarrow{\varrho} H_{\bar{S}} = G_{\bar{S}}.$$

The restriction of $Y_i^\bullet$ to the connected component $X^{(\ell)}$ is denoted

$$Y_i^{(\ell)} = Y_i^\bullet \times_{X^\bullet} X^{(\ell)}.$$

Note that some $Y_i^{(\ell)}$ may be empty.

Proposition 4.5.1. *The morphism (4.5.1) is a closed immersion. Moreover, the pair $Y_i^{(0)} \subset X^{(0)}$ is canonically identified with the pair $Y_i(\pi^m) \subset X(\pi^m)$ from the previous subsection determined by $m = 0$.*

Proof. Similar to (4.4.1), we may define a closed formal subscheme

$$Y_{i,\mathrm{naive}}^\bullet \subset X^\bullet$$

as the locus of points (G, ϱ) in $X^\bullet$ for which there exists a (necessarily unique) action $\mathcal{O}_{K_i} \to \mathrm{End}_{\mathcal{O}_F}(G)$ making

$$\mathcal{H}_{i,\overline{S}} \xrightarrow{\phi_i} \mathcal{G}_{\overline{S}} \xrightarrow{\varrho} G_{\overline{S}}$$

$\mathcal{O}_{K_i}$-linear. Essentially by definition, the morphism (4.5.1) factors through this closed formal subscheme.

The ring $\mathcal{O}_{K_i}$ acts on the Lie algebra of the universal object over $Y^\bullet_{i,\mathrm{naive}}$. As this Lie algebra is a line bundle on $Y^\bullet_{i,\mathrm{naive}}$, this action must be through some $\mathcal{O}_F$-algebra morphism

$$\mathcal{O}_{K_i} \to \mathcal{O}_{Y^\bullet_{i,\mathrm{naive}}}.$$

This morphism endows $Y^\bullet_{i,\mathrm{naive}}$ with the structure of a formal scheme not just over $\mathcal{O}_{\breve{F}}$, but over $\mathcal{O}_{K_i} \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$. One can check (this is essentially the strictness condition implicit in the definition of $Y^\bullet_i$, as in Remark 4.1.1) that the diagram

$$\begin{array}{ccc} Y^\bullet_i & \xrightarrow{f_i} & Y^\bullet_{i,\mathrm{naive}} \\ \downarrow & & \downarrow \\ \mathrm{Spf}(\mathcal{O}_{\breve{K}_i}) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_{K_i} \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}) \end{array}$$

is cartesian, where the bottom horizontal arrow is induced by the map $\mathcal{O}_{K_i} \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}} \to \mathcal{O}_{\breve{K}_i}$ sending $\alpha \otimes x \mapsto \alpha x$. As this bottom arrow is a closed immersion, so is the top one, and hence so is (4.5.1).

If K_i/F is ramified, the bottom horizontal arrow in the diagram is an isomorphism, and hence so is the top horizontal arrow. On the other hand, if K_i/F is unramified then

$$\mathrm{Spf}(\mathcal{O}_{\breve{K}_i}) \to \mathrm{Spf}(\mathcal{O}_{K_i} \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}) \cong \mathrm{Spf}(\mathcal{O}_{\breve{F}}) \sqcup \mathrm{Spf}(\mathcal{O}_{\breve{F}})$$

is an isomorphism onto one of the two components, and the top horizontal arrow is an open and closed immersion.

When $m = 0$ the equality $X(\pi^m) = X^{(0)}$ holds simply by definition, and similarly for

$$Y_i(\pi^m) = Y^\bullet_{i,\mathrm{naive}} \otimes_{X^\bullet} X^{(0)}.$$

The previous paragraph shows that

$$Y_i^{(0)} \subset Y^\bullet_{i,\mathrm{naive}} \otimes_{X^\bullet} X^{(0)}$$

as a union of connected components. As the underlying reduced scheme of $X^{(0)}$ is a point, *any* closed formal subscheme of it is connected. In particular the right hand side of the above inclusion is connected. The formal scheme $Y_i^{(0)}$ is nonempty, as it contains the $\boldsymbol{k}$-valued point $(\mathcal{H}_i, \mathrm{id})$, and so the inclusion is an equality. ☐

We would like to relate the derivative of $O_{(\Phi_0,\Phi_3)}(f; s, \eta)$ at $s = 0$ to the intersection multiplicity of the cycles $Y^\bullet_1$ and $Y^\bullet_2$ on $X^\bullet$, but this is not defined because $Y^\bullet_1 \times_{X^\bullet} Y^\bullet_2$ is an infinite disjoint union of Artinian schemes. We must carefully take connected components into account.

Let $L \subset B(\mathcal{G})$ be the the centralizer of F-subalgebra generated by $\Phi_1(K_1) \cup \Phi_2(K_2)$. Recall from Proposition 2.5.5 that it is an étale F-algebra of dimension h. The group $B(\mathcal{G})^\times$ acts on $X^\bullet$ by changing the quasi-isogeny ϱ in the moduli problem, and the action of the subgroup $L^\times$ preserves each of the closed subschemes $Y^\bullet_i \subset X^\bullet$.

Conjecture 4.5.2 (Arithmetic biquadratic fundamental lemma)**.** *If K/K_3 is unramified then*

$$\frac{\pm 1}{\log(q)} \frac{d}{ds} O_{(\Phi_0,\Phi_3)}(\mathbf{1}; s, \eta)\big|_{s=0} = \sum_{X^{(\ell)} \in L^\times \backslash \pi_0(X^\bullet)} \mathrm{len}_{\mathcal{O}_{\breve{F}}} H^0\big(X^{(\ell)}, f_{1*}\mathcal{O}_{Y_1^{(\ell)}} \otimes_{\mathcal{O}_{X^{(\ell)}}} f_{2*}\mathcal{O}_{Y_1^{(\ell)}}\big),$$

where the sum is over a set of representatives $X^{(\ell)}$ for the $L^\times$ orbits of connected components of $X^\bullet$. We remark that there are only finitely many such orbits, as there are $2h$ orbits under the action of the subgroup $F^\times$.

Remark 4.5.3. In the special case where $K_1 \cong K_2$, Conjecture 4.5.2 is equivalent to the linear arithmetic fundamental lemma of [Li21, Conjecture 1] (which should be corrected to incorporate all connected components of $X^\bullet$, as we have done above). This equivalence uses Proposition A.3.3 below, as the orbital integrals $O_{(\Phi_0,\Phi_3)}(f; s, \eta)$ defined here do not quite agree with the orbital integrals of [Li21].

5. Calculations when $h = 1$

Assume that $h = 1$. We will prove Conjecture 3.4.1 when $f = \mathbf{1}$ is the characteristic function of $\mathrm{GL}_2(\mathcal{O}_F)$, and also prove Conjecture 4.5.2. Throughout, we assume that F is a local field, and that K_1 and K_2 are quadratic étale extensions such that K/K_3 is unramified.

5.1. **Preliminaries.** In §5.1 we assume that K_1 and K_2 are fields, with K_1/F unramified and K_2/F ramified. In particular, $K = K_1 \otimes_F K_2$ is a biquadratic field extension of F. Fix $\mathcal{O}_F$-algebra generators $x_1 \in \mathcal{O}_{K_1}$ and $x_2 \in \mathcal{O}_{K_2}$ with

$$\mathrm{ord}_{K_1}(x_1) = 0, \qquad \mathrm{ord}_{K_2}(x_2) = 1.$$

Let B be a central simple F-algebra of dimension 4. Thus B is either the algebra $M_2(F)$, or the unique quaternion division algebra over F. Fix F-algebra embeddings

$$\Phi_1 : K_1 \to B, \qquad \Phi_2 : K_2 \to B.$$

As in (2.4.1) and (2.4.2), define elements of B by

$$\mathbf{w} = \Phi_1(x_1)\Phi_2(x_2) + \Phi_2(x_2^{\sigma_2})\Phi_1(x_1^{\sigma_1})$$
$$\mathbf{z} = \Phi_1(x_1)\Phi_2(x_2) - \Phi_2(x_2)\Phi_1(x_1).$$

Similarly, let $\mathbf{s} \in C = B \otimes_F K_3$ be as in (2.2.6),

Proposition 5.1.1. *We have $\mathbf{s} \in K_3^\times$, and* $\mathrm{Inv}_{(\Phi_1,\Phi_2)}(T) = T - \mathbf{s}$.

Proof. Recall from §2.2 that $\mathrm{Inv}_{(\Phi_1,\Phi_2)}(T) \in K_3[T]$ is monic of degree $h = 1$, and satisfies $\mathrm{Inv}_{(\Phi_1,\Phi_2)}(\mathbf{s}) = 0$. Hence $\mathbf{s} \in K_3$ and

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)}(T) = T - \mathbf{s}.$$

The relation $\mathbf{s} + \mathbf{s}^{\sigma_3} = 1$ of Proposition 2.3.2 shows that $\mathbf{s} \neq 0$. As our assumptions on K_1 and K_2 imply that K_3 is a field, we have $\mathbf{s} \in K_3^\times$. □

Corollary 5.1.2. *The pair* (Φ_1, Φ_2) *is regular semisimple.*

Proof. Use Proposition 5.1.1 and the criterion of Proposition 2.5.4. □

Lemma 5.1.3. *We have* $\mathbf{w}, \mathbf{z}^2 \in F$, *and*

$$\mathbf{z}^2 = \mathbf{w}^2 + \pi \cdot (a\mathbf{w} + b) \tag{5.1.1}$$

for some $a, b \in \mathcal{O}_F$ *satisfying* $\mathrm{ord}(a) \geq \mathrm{ord}(b) = 0$. *Moreover,* $\mathbf{z} \in B^\times$.

Proof. Consider the element

$$\frac{\mathrm{Tr}(x_1^2)}{\mathrm{Nm}(x_1)} + \frac{\mathrm{Tr}(x_2^2)}{\mathrm{Nm}(x_2)} = \frac{(x_1 - x_1^{\sigma_1})^2}{x_1 x_1^{\sigma_1}} + \frac{(x_2 + x_2^{\sigma_2})^2}{x_2 x_2^{\sigma_2}}. \tag{5.1.2}$$

The first term on the right hand side lies in $\mathcal{O}_F^\times$, while

$$\mathrm{ord}_{K_2}(x_2 + x_2^{\sigma_2}) = 2\mathrm{ord}_F(x_2 + x_2^{\sigma_2}) > 1 = \mathrm{ord}_{K_2}(x_2)$$

implies that the second term on the right lies in $\pi\mathcal{O}_F$. Thus (5.1.2) lies in $\mathcal{O}_F^\times$. Combining this with the relation

$$\mathrm{Tr}(x_1^2)\mathrm{Nm}(x_2) + \mathrm{Tr}(x_2^2)\mathrm{Nm}(x_1) = \mathrm{Nm}(x_2)\mathrm{Nm}(x_1)\left(\frac{\mathrm{Tr}(x_1^2)}{\mathrm{Nm}(x_1)} + \frac{\mathrm{Tr}(x_2^2)}{\mathrm{Nm}(x_2)}\right),$$

we find that

$$\mathrm{ord}_F\big(\mathrm{Tr}(x_1^2)\mathrm{Nm}(x_2) + \mathrm{Tr}(x_2^2)\mathrm{Nm}(x_1)\big) = 1.$$

Combining this with Proposition 2.4.2 shows that (5.1.1) holds with

$$a = -\frac{\mathrm{Tr}(x_1)\mathrm{Tr}(x_2)}{\pi}, \qquad b = \frac{\mathrm{Tr}(x_1^2)\mathrm{Nm}(x_2) + \mathrm{Tr}(x_2^2)\mathrm{Nm}(x_1)}{\pi}.$$

Recall from Proposition 2.4.2 that $\mathbf{w}$ commutes with both $\Phi(K_1)$ and $\Phi(K_2)$. Each of these subalgebras is equal to its own centralizer in B, and hence $\mathbf{w} \in \Phi(K_1) \cap \Phi(K_2) = F$. The inclusion $\mathbf{z}^2 \in F$ follows from this and (5.1.1). For the final claim, Proposition 5.1.1 tells us that $\mathbf{s} \in K_3^\times$, hence $\mathbf{t} \in C^\times$ by Proposition 2.3.2, hence $\mathbf{z} \in B^\times$ by Proposition 2.4.1. □

Lemma 5.1.4. *If* B *is a matrix algebra then* $\mathrm{ord}_F(\mathbf{z}^2)$ *is even. If* B *is a division algebra then* $\mathrm{ord}_F(\mathbf{z}^2)$ *is odd.*

Proof. The essential point is the relation $\mathbf{z}\Phi_1(x) = \Phi_1(x^{\sigma_1})\mathbf{z}$ of Proposition 2.4.2. If $\mathrm{ord}_F(\mathbf{z}^2)$ is even then, as K_1/F is unramified, there is an $x \in K_1$ such that $xx^{\sigma_1} = \mathbf{z}^2$. This implies

$$(\Phi_1(x) - \mathbf{z})(\Phi_1(x^{\sigma_1}) + \mathbf{z}) = 0,$$

and so $B \cong M_2(F)$. Conversely, if $B \cong M_2(F)$ then pick any nonzero $v \in F^2$. The embedding $\Phi_1 : K_1 \to M_2(F)$ makes F^2 into a K_1-vector space of dimension 1, and so $\mathbf{z}v = \Phi_1(x)v$ for some $x \in K_1$. This implies

$$\mathbf{z}^2 v = \mathbf{z}\Phi_1(x)v = \Phi_1(x^{\sigma_1})\mathbf{z}v = \Phi_1(xx^{\sigma_1})v,$$

and so $\mathbf{z}^2 = xx^{\sigma_1}$. Thus $\mathrm{ord}_F(\mathbf{z}^2)$ is even. □

5.2. **Calculation of an orbital integral.** We keep the notation and assumptions of the previous subsection. In particular, we continue to assume that K_1/F is unramified, while K_2/F is ramified.

If we view $x_1 \in \mathcal{O}_{K_1}$ and $x_2 \in \mathcal{O}_{K_2}$ as elements of K, then

$$x_3 = x_1 x_2^{\sigma_2} + x_1^{\sigma_1} x_2 \in K_3$$

generates K_3 as an F-algebra. In fact, it is easy to see that

$$\mathrm{ord}_{K_3}(x_3) = 1$$

and hence $\mathcal{O}_{K_3} = \mathcal{O}_F[x_3]$, as K_3/F is ramified. Recalling that $K_0 = F \times F$, define

$$\Phi_0 : K_0 \to M_2(F), \qquad \Phi_3 : K_3 \to M_2(F)$$

by

$$\Phi_0(a,b) = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \qquad \Phi_3(x_3) = \begin{pmatrix} \mathbf{w} & 1 \\ \mathbf{m} & \mathrm{Tr}(x_3) - \mathbf{w} \end{pmatrix},$$

where $\mathbf{m} = \mathbf{w} \cdot (\mathrm{Tr}(x_3) - \mathbf{w}) - \mathrm{Nm}(x_3) \in F$.

Lemma 5.2.1. *The pair* (Φ_0, Φ_3) *matches* (Φ_1, Φ_2).

Proof. Set $x_0 = (0,1) \in K_0$, and let

$$\mathbf{w}' = \Phi_0(x_0)\Phi_3(x_3) + \Phi_3(x_3^{\sigma_3})\Phi_0(x_0^{\sigma_0}) \in M_2(F)$$

be the element associated to the pair (Φ_0, Φ_3) by (2.4.1). Using

$$\Phi_0(x_0)\Phi_3(x_3) = \begin{pmatrix} \mathbf{w} & 1 \\ 0 & 0 \end{pmatrix}, \qquad \Phi_3(x_3^{\sigma_3})\Phi_0(x_0^{\sigma_0}) = \begin{pmatrix} 0 & -1 \\ 0 & \mathbf{w} \end{pmatrix}$$

we see that $\mathbf{w}' = \mathbf{w}$.

Recalling Proposition 2.4.1, consider the elements

$$\mathbf{s} = \frac{-(x_1 x_2^{\sigma_2} + x_2 x_1^{\sigma_1}) + \mathbf{w}}{(x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2})}, \qquad \mathbf{s}' = \frac{-(x_0 x_3^{\sigma_3} + x_3 x_0^{\sigma_0}) + \mathbf{w}'}{(x_0 - x_0^{\sigma_0})(x_3 - x_3^{\sigma_3})}$$

of $K_3 \subset M_2(K_3)$ associated to the pairs (Φ_1, Φ_2) and (Φ_0, Φ_3).

Somewhat confusingly, $\mathbf{s}$ and $\mathbf{s}'$ are viewed as elements of the rightmost copies of K_3 in the diagrams (1.2.1) and (1.4.1), which we identify. In particular, one must be mindful of the conventions explained after (1.4.1). If we identify K_0 and K_3 as subalgebras of $K_3 \times K_3$ via $x_0 \mapsto (0,1)$ and $x_3 \mapsto (x_3, x_3)$, then

$$x_0 x_3^{\sigma_3} + x_3 x_0^{\sigma_0} = (x_3, x_3^{\sigma_3}) \in K_3 \times K_3$$

is identified with the element $x_3 = x_1 x_2^{\sigma_2} + x_1^{\sigma_1} x_2$ in the *other* copy of K_3 in the diagram (1.4.1). In other words,

$$x_0 x_3^{\sigma_3} + x_3 x_0^{\sigma_0} = x_1 x_2^{\sigma_2} + x_1^{\sigma_1} x_2.$$

Similarly,

$$(x_0 - x_0^{\sigma_0})(x_3 - x_3^{\sigma_3}) = (x_3^{\sigma_3} - x_3, x_3 - x_3^{\sigma_3}) \in K_3 \times K_3$$

is identified with $x_3^{\sigma_3} - x_3 = (x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2})$ in the rightmost copy of K_3 in the diagram (1.4.1). In other words,

$$(x_0 - x_0^{\sigma_0})(x_3 - x_3^{\sigma_3}) = (x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2}).$$

Having now shown that $\mathbf{s} = \mathbf{s}'$, the pairs (Φ_1, Φ_2) and (Φ_0, Φ_3) match. □

Proposition 5.2.2. *If* $\mathbf{1}$ *is the characteristic function of* $\mathrm{GL}_2(\mathcal{O}_F)$*, then*

$$O_{(\Phi_0,\Phi_3)}(\mathbf{1}; s, \eta) = \begin{cases} 1 & \text{if } \mathrm{ord}_F(\mathbf{w}) = 0 \\ 1 - q^{-s} & \text{if } \mathrm{ord}_F(\mathbf{w}) > 0 \\ 0 & \text{if } \mathrm{ord}_F(\mathbf{w}) < 0. \end{cases}$$

Note that when $\mathrm{ord}_F(\mathbf{w}) \geq 0$*, both* $\Phi_0(\mathcal{O}_{K_0})$ *and* $\Phi_3(\mathcal{O}_{K_3})$ *are contained in* $M_2(\mathcal{O}_F)$*. This allows us to take* $g_0 = g_3 = 1$ *in* (3.2.5)*, and so remove the ambiguity in the orbital integral noted in Remark 3.2.6.*

Proof. In the notation of (3.2.1), we have $H_0 = \Phi_0(K_0^\times)$ and $H_3 = \Phi_3(K_3^\times)$. As K_3/F is ramified, and $x_3 \in K_3$ is a uniformizing parameter, we have

$$F^\times \backslash K_3^\times = \mathcal{O}_F^\times \backslash (\mathcal{O}_{K_3}^\times \sqcup x_3 \mathcal{O}_{K_3}^\times).$$

Choosing a $g_3 \in \mathrm{GL}_2(F)$ such that

$$\Phi_3(\mathcal{O}_{K_3}) \subset g_3 M_2(\mathcal{O}_F) g_3^{-1},$$

the orbital integral of Definition 3.2.4 simplifies to

$$\begin{aligned} O_{(\Phi_0,\Phi_3)}(\mathbf{1}; s, \eta) &= \int_{F^\times \backslash (F^\times \times F^\times \times K_3^\times)} \mathbf{1}(\Phi_0(a,b)^{-1} \Phi_3(x) g_3) \eta(x) |a/b|^s \, da \, db \, dx \\ &= \int_{F^\times \times F^\times \times \mathcal{O}_{K_3}^\times} \mathbf{1}(\Phi_0(a,b)^{-1} \Phi_3(x) g_3) |a/b|^s \, da \, db \, dx \\ &\quad - \int_{F^\times \times F^\times \times \mathcal{O}_{K_3}^\times} \mathbf{1}(\Phi_0(a,b)^{-1} \Phi_3(x_3 x) g_3) |a/b|^s \, da \, db \, dx \\ &= \int_{F^\times \times F^\times} \mathbf{1}(\Phi_0(a,b)^{-1} g_3) |a/b|^s \, da \, db \\ &\quad - \int_{F^\times \times F^\times} \mathbf{1}(\Phi_0(a,b)^{-1} \Phi_3(x_3) g_3) |a/b|^s \, da \, db. \end{aligned} \tag{5.2.1}$$

If $\mathrm{ord}_F(\mathbf{w}) \geq 0$ then we may take $g_3 = 1$, and compute

$$\int_{F^\times \times F^\times} \mathbf{1}(\Phi_0(a,b)^{-1}) |a/b|^s \, da \, db = 1.$$

If $\mathrm{ord}_F(\mathbf{w}) = 0$ then

$$\Phi_3(x_3) \in \begin{pmatrix} 1 & 0 \\ u & \pi \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_F)$$

for some $u \in \mathcal{O}_F^\times$, and one easily checks that

$$\int_{F^\times \times F^\times} \mathbf{1}(\Phi_0(a,b)^{-1} \Phi_3(x_3)) |a/b|^s \, da \, db = 0.$$

If $\mathrm{ord}_F(\mathbf{w}) > 0$ then we instead have

$$\Phi_3(x_3) \in \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \mathrm{GL}_2(\mathcal{O}_F),$$

and one easily checks that

$$\int_{F^\times \times F^\times} \mathbf{1}(\Phi_0(a,b)^{-1}\Phi_3(x_3))|a/b|^s \, da \, db = q^{-s}.$$

Combining these calculations with (5.2.1) proves the claim for $\mathrm{ord}_F(\mathbf{w}) \geq 0$.

Now suppose $\mathrm{ord}_F(\mathbf{w}) < 0$. In this case we may choose

$$g_3 = \begin{pmatrix} 1 & \mathbf{w} \\ 0 & \mathbf{m} \end{pmatrix} \in M_2(F).$$

Direct calculation shows that

$$g_3 \notin \Phi_0(a,b)\mathrm{GL}_2(\mathcal{O}_F), \qquad \Phi_3(x_3)g_3 \notin \Phi_0(a,b)\mathrm{GL}_2(\mathcal{O}_F)$$

for all $a, b \in F^\times$, and so (5.2.1) vanishes. □

5.3. **Central values.** In this subsection we let K_1 and K_2 be any quadratic étale F-algebras, and fix F-algebra embeddings

$$\Phi_0 : K_0 \to M_2(F), \qquad \Phi_1 : K_1 \to M_2(F)$$
$$\Phi_3 : K_3 \to M_2(F), \qquad \Phi_2 : K_2 \to M_2(F).$$

Theorem 5.3.1. *If (Φ_0, Φ_3) and (Φ_1, Φ_2) are regular semisimple and matching, then*

$$\pm O_{(\Phi_0,\Phi_3)}(\mathbf{1}; 0, \eta) = O_{(\Phi_1,\Phi_2)}(\mathbf{1}).$$

Proof. If either one of K_1 or K_2 is the split algebra $F \times F$, the claim is known by Proposition 3.4.3. If K_1 and K_2 are both unramified field extensions, then $K_1 \cong K_2$ and the result is known by work of Guo (Remark 3.4.2). Our assumption that K/K_3 is unramified excludes the possibility that K_1 and K_2 are both ramified field extensions.

This leaves us with the case in which K_1 and K_2 are both fields, and exactly one of them is ramified over F. Under this assumption we will prove a more precise statement: if there is a maximal order in $M_2(F)$ that contains both $\Phi_1(\mathcal{O}_{K_1})$ and $\Phi(\mathcal{O}_{K_2})$, then both sides of the desired equality are equal to 1. If no such maximal order exists, then both sides are 0.

Without loss of generality, assume that K_1 is unramified and K_2 is ramified. Let $x_1 \in K_1$, $x_2 \in K_2$, $\mathbf{w} \in F$ and $\mathbf{z} \in M_2(F)$ be as in §5.1.

Assume there is a maximal order in $M_2(F)$ that contains both $\Phi_1(\mathcal{O}_{K_1})$ and $\Phi(\mathcal{O}_{K_2})$. In particular $\mathrm{ord}_F(\mathbf{w}) \geq 0$, and the calculations of §5.2 that

$$\pm O_{(\Phi_0,\Phi_3)}(\mathbf{1}; 0, \eta) = 1.$$

After conjugating (Φ_1, Φ_2) by an element of $\mathrm{GL}_2(\mathcal{O}_F)$, we may assume that this maximal order is $M_2(\mathcal{O}_F)$, and the orbital integral of Definition 3.1.1

simplifies to

$$\begin{aligned} O_{(\Phi_1,\Phi_2)}(\mathbf{1}) &= \int_{F^\times\backslash(K_1^\times\times K_2^\times)} \mathbf{1}(\Phi_1(h_1^{-1})\Phi_2(h_2))\, dh_1\, dh_2 \\ &= \int_{(\mathcal{O}_F^\times\backslash\mathcal{O}_{K_1}^\times)\times K_2^\times} \mathbf{1}(\Phi_1(h_1^{-1})\Phi_2(h_2))\, dh_1\, dh_2 \\ &= \int_{K_2^\times} \mathbf{1}(\Phi_2(h_2))\, dh_2 \\ &= 1. \end{aligned}$$

Now suppose there is no maximal order in $M_2(F)$ containing both $\Phi_1(\mathcal{O}_{K_1})$ and $\Phi_2(\mathcal{O}_{K_2})$. Replacing the pair (Φ_1,Φ_2) by a conjugate, we may assume that $\Phi_1(\mathcal{O}_{K_1})\subset M_2(\mathcal{O}_F)$, and then

$$\begin{aligned} O_{(\Phi_1,\Phi_2)}(\mathbf{1}) &= \int_{F^\times\backslash(K_1^\times\times K_2^\times)} \mathbf{1}\big(\Phi_1(h_1^{-1})\Phi_2(h_2)g_2\big)\, dh_1\, dh_2 \\ &= \int_{(\mathcal{O}_F^\times\backslash\mathcal{O}_{K_1}^\times)\times K_2^\times} \mathbf{1}\big(\Phi_1(h_1^{-1})\Phi_2(h_2)g_2\big)\, dh_1\, dh_2 \\ &= \int_{K_2^\times} \mathbf{1}\big(\Phi_2(h_2)g_2\big)\, dh_2 \end{aligned} \tag{5.3.1}$$

for any $g_2\in \mathrm{GL}_2(F)$ such that

$$\Phi_2(\mathcal{O}_{K_2})\subset g_2M_2(\mathcal{O}_F)g_2^{-1}.$$

Using the fact that K_2/F is ramified, we may scale g_2 by an element of $\Phi_2(K_2^\times)$ to assume that $\mathrm{ord}_F(\det(g_2))=0$.

If (5.3.1) is nonzero, there there is some $h_2\in K_2^\times$ such that

$$\Phi_2(h_2)g_2\mathcal{O}_F^2=\mathcal{O}_F^2.$$

On the other hand, $\Phi_2(h_2)\in g_2M_2(\mathcal{O}_F)g_2^{-1}$, satisfies

$$\Phi_2(h_2)g_2\mathcal{O}_F^2\subset g_2\mathcal{O}_F^2,$$

and equality holds as

$$\mathrm{ord}_F(\det(\Phi_2(h_2)))=\mathrm{ord}_F(\det(\Phi_2(h_2)g_2))=0$$

Thus $\mathcal{O}_F^2=g_2\mathcal{O}_F^2$, which implies $g_2\in\mathrm{GL}_2(\mathcal{O}_F)$, which implies $\Phi_2(\mathcal{O}_{K_2})\subset M_2(\mathcal{O}_F)$. This contradicts our hypothesis on the pair (Φ_1,Φ_2), and we conclude that (5.3.1) is equal to 0.

Still assuming there is no maximal order in $M_2(F)$ containing both $\Phi_1(\mathcal{O}_{K_1})$ and $\Phi_2(\mathcal{O}_{K_2})$, we claim that

$$\mathrm{ord}_F(\mathbf{w})<0. \tag{5.3.2}$$

Indeed, if $\mathrm{ord}_F(\mathbf{w})\geq 0$ then Lemma 5.1.3 and the relation

$$\mathbf{z}\cdot\Phi_1(x_1)=\Phi_1(x_1^{\sigma_1})\cdot\mathbf{z}$$

of Proposition 2.4.2 imply that

$$\mathrm{Span}_{\mathcal{O}_F}\{1, \mathbf{z}, \Phi_1(x_1), \mathbf{z}\Phi_1(x_1)\} \subset M_2(F)$$

is an $\mathcal{O}_F$-subalgebra. It follows that there is a maximal order $R \subset M_2(F)$ that contains $\mathbf{w}$, $\mathbf{z}$, and all of $\Phi_1(\mathcal{O}_{K_1})$. Our assumption that K_1/F is unramified implies $x_1 - x_1^{\sigma_1} \in \mathcal{O}_F^\times$, and so the relations

$$\begin{aligned}\mathbf{w} - \mathbf{z} &= \Phi_2(x_2^{\sigma_2})\Phi_1(x_1^{\sigma_1}) + \Phi_2(x_2)\Phi_1(x_1) \\ &= \mathrm{Tr}(x_2)\Phi_1(x_1^{\sigma_1}) + \Phi_2(x_2)\Phi_1(x_1 - x_1^{\sigma_1})\end{aligned}$$

imply $\Phi_2(x_2) \in R$. Thus R contains both $\Phi_1(\mathcal{O}_{K_1})$ and $\Phi_2(\mathcal{O}_{K_2})$, contrary to our hypotheses. Hence (5.3.2) holds, and the vanishing of $O_{(\Phi_0,\Phi_3)}(\mathbf{1}; 0, \eta)$ follows from the calculations of §5.2. □

5.4. **Central derivatives.** Now return to the setting of §4.5, with $h = 1$. Thus $\mathcal{G}$ is formal $\mathcal{O}_F$-module over $\boldsymbol{k}$ of height 2, and we are given F-algebra embeddings

$$\Phi_1 : K_1 \to B(\mathcal{G}), \qquad \Phi_2 : K_2 \to B(\mathcal{G})$$

forming a regular semisimple pair (Φ_1, Φ_2), with corresponding closed immersions $f_1 : Y_1^\bullet \to X^\bullet$ and $f_2 : Y_2^\bullet \to X^\bullet$ of formal schemes over $\mathcal{O}_{\breve{F}}$.

The étale F-algebra L appearing in Conjecture 4.5.2, being of dimension $h = 1$, is just F itself. This allows us to take $X^{(0)}$ and $X^{(1)}$ as representatives for the $L^\times$ orbits of $\pi_0(X^\bullet)$, and Conjecture 4.5.2 is a consequence of the following result.

Theorem 5.4.1. *Given F-algebra embeddings*

$$\Phi_0 : K_0 \to M_2(F), \qquad \Phi_3 : K_3 \to M_2(F)$$

with (Φ_0, Φ_3) matching (Φ_1, Φ_2), we have

$$\frac{\pm 1}{\log(q)} \frac{d}{ds} O_{(\Phi_0,\Phi_3)}(\mathbf{1}; s, \eta)\big|_{s=0} = \mathrm{len}_{\mathcal{O}_{\breve{F}}} H^0\big(X^{(0)}, f_{1*}\mathcal{O}_{Y_1^{(0)}} \otimes_{\mathcal{O}_{X^{(0)}}} f_{2*}\mathcal{O}_{Y_2^{(0)}}\big)$$

and

$$H^0\big(X^{(1)}, f_{1*}\mathcal{O}_{Y_1^{(1)}} \otimes_{\mathcal{O}_{X^{(1)}}} f_{2*}\mathcal{O}_{Y_2^{(1)}}\big) = 0.$$

Proof. Note that $B(\mathcal{G})$ is a quaternion division algebra over F, and so K_1 and K_2 are fields. If K_i/F is unramified, then any quasi-isogeny of formal $\mathcal{O}_{K_i}$-modules has even height when viewed as a quasi-isogeny of underlying formal $\mathcal{O}_F$-modules. It follows that the image of

$$f_i : Y_i^\bullet \to X^\bullet$$

only meets those $X^{(\ell)}$ with ℓ even. We deduce that if either of K_1 or K_2 is unramified then

$$f_{1*}\mathcal{O}_{Y_1^{(1)}} \otimes_{\mathcal{O}_{X^{(1)}}} f_{2*}\mathcal{O}_{Y_2^{(1)}} = 0.$$

If K_1 and K_2 are both unramified over F, then $K_1 \cong K_2$ and the claim follows from the calculations of §7 of [Li21]. Our assumption that K/K_3 is unramified excludes the possibility that K_1 and K_2 are both ramified, and so it only remains to consider the case in which one of K_1 and K_2 is ramified and the other is unramified. We will prove that in this case both sides of the first equality in the theorem are equal to 1.

Without loss of generality we may assume that K_1 is unramified and K_2 is ramified. Let $x_1 \in \mathcal{O}_{K_1}$ and $x_2 \in \mathcal{O}_{K_2}$ be as in §5.1, and let

$$\mathbf{w} = \Phi_1(x_1)\Phi_2(x_2) + \Phi_2(x_2^{\sigma_2})\Phi_1(x_1^{\sigma_1})$$
$$\mathbf{z} = \Phi_1(x_1)\Phi_2(x_2) - \Phi_2(x_2)\Phi_1(x_1)$$

be the corresponding elements of $B(\mathcal{G})$. The unique maximal order of $B(\mathcal{G})$ must contain both $\Phi_1(\mathcal{O}_{K_1})$ and $\Phi_2(\mathcal{O}_{K_2})$, and so also contains $\mathbf{w}$ and $\mathbf{z}$. By Lemma 5.1.3, we therefore have $\mathbf{w} \in \mathcal{O}_F$ and $\mathbf{z}^2 \in \mathcal{O}_F$, and

$$\mathbf{z}^2 = \mathbf{w}^2 + \pi \cdot (a\mathbf{w} + b)$$

for some $a, b \in \mathcal{O}_F$ satisfying $\mathrm{ord}(a) \geq \mathrm{ord}(b) = 0$. If $\mathrm{ord}_F(\mathbf{w}) = 0$, this would imply $\mathrm{ord}_F(\mathbf{z}^2) = 0$, contradicting Lemma 5.1.4. Therefore

$$\mathrm{ord}_F(\mathbf{w}) > 0, \tag{5.4.1}$$

and the results of §5.2 imply

$$\frac{\pm 1}{\log(q)} \frac{d}{ds} O_{(\Phi_0,\Phi_3)}(\mathbf{1}; s, \eta)\Big|_{s=0} = 1. \tag{5.4.2}$$

Now fix F-algebra embeddings

$$\Psi_1 : K_1 \to M_2(F), \qquad \Psi_2 : K_2 \to M_2(F)$$

satisfying $\Psi_i(\mathcal{O}_{K_i}) \subset M_2(\mathcal{O}_F)$. As in (4.4.2), for any $g \in \mathrm{GL}_2(\mathcal{O}_F)$ let

$$R(g) = \mathrm{Res}\big(\mathrm{Inv}_{(\Phi_1,\Phi_2)}, \mathrm{Inv}_{(\Psi_1, g\Psi_2 g^{-1})}\big) \in K_3.$$

We claim that

$$\mathrm{ord}_{K_3}(R(g)) = -\mathrm{ord}_F(d_1 d_2), \tag{5.4.3}$$

where $d_i \in \mathcal{O}_F$ generates the discriminant of K_i/F. (Of course d_1 is a unit.) Define elements of $M_2(\mathcal{O}_F)$ by

$$\mathbf{w}' = \Psi_1(x_1)\Psi_2(x_2) + \Psi_2(x_2^{\sigma_2})\Psi_1(x_1^{\sigma_1})$$
$$\mathbf{z}' = \Psi_1(x_1)\Psi_2(x_2) - \Psi_2(x_2)\Psi_1(x_1).$$

If $\mathbf{s} \in C$ and $\mathbf{s}' \in M_2(K_3)$ denote the elements constructed from (Φ_1, Φ_2) and (Ψ_1, Ψ_2) as in (2.2.6), then Propositions 5.1.1 and 2.4.1 imply

$$R(1) = \mathrm{Res}(T - \mathbf{s}, T - \mathbf{s}') = \mathbf{s} - \mathbf{s}' = \frac{\mathbf{w} - \mathbf{w}'}{(x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2})}. \tag{5.4.4}$$

As above, Lemma 5.1.3 implies that $\mathbf{w}' \in \mathcal{O}_F$ and $(\mathbf{z}')^2 \in \mathcal{O}_F$ satisfy

$$(\mathbf{z}')^2 = (\mathbf{w}')^2 + \pi \cdot (a\mathbf{w}' + b)$$

for some $a, b \in \mathcal{O}_F$ with $\mathrm{ord}(a) \geq \mathrm{ord}(b) = 0$. If $\mathrm{ord}_F(\mathbf{w}') > 0$ then $\mathrm{ord}_F((\mathbf{z}')^2) = 1$, contradicting Lemma 5.1.4. Hence $\mathrm{ord}_F(\mathbf{w}') = 0$. Combining this with (5.4.1) and (5.4.4) shows that

$$\mathrm{ord}_{K_3}(R(1)) = -\mathrm{ord}_{K_3}((x_1 - x_1^{\sigma_1})(x_2 - x_2^{\sigma_2})).$$

Elementary calculation shows that the right hand side is $-\mathrm{ord}_F(d_1d_2)$, completing the proof of (5.4.3) when $g=1$. The proof for general $g\in \mathrm{GL}_2(\mathcal{O}_F)$ proceeds by replacing Ψ_2 with $g\Psi_2g^{-1}$ throughout the argument.

Using (5.4.3), the $m=0$ case of Theorem 4.4.2 reduces to

$$\mathrm{len}_{\mathcal{O}_{\breve{F}}}\, H^0\big(X^{(0)}, f_{1*}\mathcal{O}_{Y_1^{(0)}} \otimes_{\mathcal{O}_X} f_{2*}\mathcal{O}_{Y_1^{(0)}}\big) = |d_1d_2|^{-\frac{1}{2}} \int_{\mathrm{GL}_2(\mathcal{O}_F)} \frac{dg}{|R(g)|} = 1,$$

and comparison with (5.4.2) completes the proof. □

Appendix A. Comparisons with earlier work

When $K_1=K_2$ our results and conjectures reduce to those of [Li21], but some aspects of this are not completely obvious. In this appendix we provide some results to help guide the reader in the comparison between this paper and [Li21].

One consequence of the comparison is Proposition A.2.2, which shows that our notion of regular semisimplicity from Definition 2.5.1 is equivalent to the more familiar notion from geometric invariant theory.

A.1. **An alternate construction of s.** Return to the setting of §2.2. Thus F is an arbitrary field, B is a central simple F-algebra of dimension $4h^2$, and we are given F-algebra embeddings

$$\Phi_1 : K_1 \to B, \qquad \Phi_2 : K_2 \to B$$

in which each K_i is a quadratic étale F-algebra.

We provide a different construction of the element

$$\mathbf{s} \in C = B\otimes_F K_3$$

defined by (2.2.6). This will allow us to compare our invariant polynomials with the invariant polynomials defined (in the special case $K_1=K_2$) in [Li21], and to compare our definition of regular semisimple pair with the more common one from geometric invariant theory.

The K_3-algebra embeddings $\Phi_1,\Phi_2 : K\to C$ of (2.2.3) are conjugate by Corollary 2.1.3, and hence there is a $c\in C^\times$ such that

$$\Phi_2(x) = c^{-1}\cdot \Phi_1(x)\cdot c. \tag{A.1.1}$$

There is a $\mathbb{Z}/2\mathbb{Z}$-grading $C=C_+\oplus C_-$ in which

$$\begin{aligned} C_+ &= \{a\in C : \forall y\in K,\ \Phi_1(y)\cdot a = a\cdot \Phi_1(y)\} \\ C_- &= \{a\in C : \forall y\in K,\ \Phi_1(y)\cdot a = a\cdot \Phi_1(y^{\tau_3})\}. \end{aligned} \tag{A.1.2}$$

Denote by $c_\pm$ the projection of c to $C_\pm$.

Proposition A.1.1. *The element $c_+-c_-\in C$ is invertible, and*

$$\mathbf{s} = (c_++c_-)^{-1}\cdot c_+\cdot (c_+-c_-)^{-1}\cdot c_+.$$

Proof. If $y \in K^\times$ is any element with $y + y^{\tau_3} = 0$, then

$$c_+ - c_- = \Phi_1(y)^{-1} \cdot (c_+ + c_-) \cdot \Phi_1(y) \in C^\times,$$

proving the first claim.

For the second claim, fix a K_3-algebra generator $y \in K$. The element c was chosen so that $\Phi_1(y) \cdot (c_+ + c_-) = (c_+ + c_-) \cdot \Phi_2(y)$, and therefore

$$c_+(\Phi_1(y) - \Phi_2(y)) = c_-(\Phi_2(y) - \Phi_1(y^{\tau_3})). \tag{A.1.3}$$

Adding $c_+(\Phi_2(y) - \Phi_1(y^{\tau_3}))$ to both sides of (A.1.3), we find

$$c_+\Phi_1(y - y^{\sigma_3}) = (c_+ + c_-)(\Phi_2(y) - \Phi_1(y^{\tau_3})).$$

Subtracting $c_+(\Phi_1(y) - \Phi_2(y^{\tau_3}))$ from both sides of (A.1.3), we find

$$c_+\Phi_2(y - y^{\tau_3}) = (c_+ - c_-)(\Phi_1(y) - \Phi_2(y^{\tau_3})).$$

Rewrite these two equalities as

$$(c_+ + c_-)^{-1} \cdot c_+ = (\Phi_2(y) - \Phi_1(y^{\tau_3})) \cdot \Phi_1(y - y^{\tau_3})^{-1}$$
$$(c_+ - c_-)^{-1} \cdot c_+ = (\Phi_1(y) - \Phi_2(y^{\tau_3})) \cdot \Phi_2(y - y^{\tau_3})^{-1}$$

to see that $(c_+ + c_-)^{-1} \cdot c_+ \cdot (c_+ - c_-)^{-1} \cdot c_+$ is equal to

$$(\Phi_2(y) - \Phi_1(y^{\tau_3})) \cdot \Phi_1(y - y^{\tau_3})^{-1} \cdot (\Phi_1(y) - \Phi_2(y^{\tau_3})) \cdot \Phi_2(y - y^{\tau_3})^{-1},$$

and use (2.2.5) to see that this last expression is equal to (2.2.6). □

A.2. **Invariant polynomials revisited.** We now use Proposition A.1.1 to compare our invariant polynomial with the notion of invariant polynomial from [Li21, Definition 1.1]. Let E be a quadratic étale F-algebra, fix an F-algebra embedding

$$\Phi : E \to B,$$

and let $B = B_+ \oplus B_-$ be the corresponding $\mathbb{Z}/2\mathbb{Z}$-grading, as in (A.1.2).

Definition A.2.1. The *invariant polynomial* M_g of $g \in B^\times$, with respect to $\Phi : E \to B$, is the unique monic square root of the reduced characteristic polynomial of

$$\mathbf{s}_g = (g_+ + g_-)^{-1} g_+ (g_+ - g_-)^{-1} g_+ \in B,$$

where $g_\pm \in B_\pm$ is the projection of g.

To explain the connection with Definition 2.2.4, set $K_1 = E$ and $K_2 = E$, and define

$$\Phi_1 = \Phi : K_1 \to B \quad \text{and} \quad \Phi_2 = g^{-1}\Phi g : K_2 \to B.$$

Let $\sigma \in \mathrm{Aut}(E/F)$ be the nontrivial automorphism, and identify

$$K = K_1 \otimes_F K_2 \cong E \times E$$

via $a \otimes b \mapsto (ab, ab^\sigma)$. The subalgebra $K_3 \subset K$ is identified with $F \times F \subset E \times E$, and so $C = B \times B$. The embeddings

$$\Phi_1, \Phi_2 : E \times E \to B \times B$$

of (2.2.3) take the explicit form

$$\Phi_1(a,b) = (\Phi_1(a), \Phi_1(b)), \qquad \Phi_2(a,b) = (\Phi_2(a), \Phi_2(b^\sigma)).$$

Compare with Remark 2.2.7.

We may choose the element c of (A.1.1) in the form $c = (g, g') \in B^\times \times B^\times$ for some $g' \in B^\times$, and (by Proposition A.1.1) the element $\mathbf{s} \in C$ of (2.2.6) is $\mathbf{s} = (\mathbf{s}_g, \mathbf{s}_{g'})$. What this shows is that the invariant polynomial

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)} \in K_3[T] = F[T] \times F[T]$$

of Definition 2.2.4 is related to that of Definition A.2.1 by

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)} = (M_g, M_{g'}). \tag{A.2.1}$$

Note that $M_{g'}(T) = (-1)^h M_g(1-T)$ by the functional equation of Proposition 2.3.3.

We now use the above discussion and a result of Guo [Guo96] to compare Definition 2.2.4 with the usual notion of regular semisimple from geometric invariant theory. For the rest of this subsection we assume that F is algebraically closed. For $i \in \{1,2\}$ denote by

$$X_i = \{\Phi_i : K_i \to B\}$$

the set of all F-algebra embeddings of K_i into B. There is a natural action of the group $G = B^\times$ on X_i by conjugation, and hence a diagonal action of G on $X_1 \times X_2$.

Proposition A.2.2. *A point $(\Phi_1, \Phi_2) \in X_1 \times X_2$ is regular semisimple in the sense of Definition 2.2.4 if and only if its G-orbit is Zariski closed of maximal dimension.*

Proof. As we are assuming that F is algebraically closed, we may fix isomorphisms $B \cong M_{2h}(F)$ and $K_i \cong F \times F$. There is a standard embedding $\Phi : F \times F \to M_{2h}(F)$ defined by

$$\Phi(a,b) = \begin{pmatrix} aI_h & \\ & bI_h \end{pmatrix},$$

which determines base point $\Phi \in X_i$ with stabilizer

$$H = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} : A, B \in \mathrm{GL}_h(F) \right\} \subset \mathrm{GL}_{2h}(F) \cong G.$$

Using Corollary 2.1.3 we identify $G/H \cong X_i$ as algebraic varieties (one may take this as the definition of the algebraic structure on X_i).

Consider the diagram

$$\begin{array}{ccccc} G\backslash(G \times G) & \longleftarrow & G \times G & \longrightarrow & G/H \times G/H \\ \downarrow{\cong} & & & & \downarrow{\cong} \\ G & & & & X_1 \times X_2 \end{array} \tag{A.2.2}$$

in which the vertical isomorphism on the left sends $(g_1, g_2) \mapsto g_1^{-1} g_2$. The group $H \times H$ acts on G on the right by $g \cdot (h_1, h_2) = h_1^{-1} g h_2$. The group G acts diagonally on $G \times G$ by left multiplication, while $H \times H$ acts by right multiplication. It is easy to see that the above diagram induces bijections

$$\left\{ \begin{array}{c} H \times H \\ \text{orbits in } G \end{array} \right\} \cong \left\{ \begin{array}{c} G \times H \times H \\ \text{orbits in } G \times G \end{array} \right\} \cong \left\{ \begin{array}{c} G \text{ orbits in} \\ X_1 \times X_2 \end{array} \right\}$$

sending $g \mapsto (1, g) \mapsto (\Phi, g\Phi g^{-1})$. Under both bijections, closed orbits correspond to closed orbits (use the fact that both horizontal arrows in (A.2.2) are smooth, so images of open sets are open). The stabilizers (in $H \times H$, $G \times H \times H$, and G, respectively) of $g \in G$, $(g, 1) \in G \times G$ and $(\Phi, g\Phi g^{-1}) \in X_1 \times X_2$ have the same dimension, as all are isomorphic to $H \cap gHg^{-1}$. Thus, under these bijections, closed orbits of maximal dimension correspond to closed orbits of maximal dimension.

The closed $H \times H$ orbits in G of maximal dimension were classified by Guo [Guo96], but we follow the discussion of this classification found in [LM23, §2]. Guo's result, in the form of [LM23, Lemma 2.2], implies that the $H \times H$ orbit of $g \in G$ is closed of maximal dimension if and only if its invariant polynomial (in the sense of Definition A.2.1) with respect to $\Phi : F \times F \to M_{2h}(F)$ has h distinct roots, all different from 0 and 1. Let us denote this invariant polynomial by $M_g(T)$, and note that it agrees with the polynomial $\mathrm{Inv}'(g, T)$ in [LM23, Remark 2.4].

Now start with a pair $(\Phi_1, \Phi_2) \in X_1 \times X_2$ whose G orbit corresponds under the above bijections to the $H \times H$ orbit of $g \in G$. By Proposition 2.5.4, the pair (Φ_1, Φ_2) is regular semisimple if and only if each of the polynomials $M_g(T)$ and $M_g(1-T)$ in

$$\mathrm{Inv}_{(\Phi_1,\Phi_2)}(T) \overset{(\mathrm{A.2.1})}{=} \big(M_g(T), (-1)^h M_g(1-T)\big) \in F[T] \times F[T]$$

has h distinct nonzero roots. This is equivalent to $M_g(T)$ having h distinct roots, all different from 0 and 1. By Guo's result, this last condition is equivalent to the $H \times H$ orbit of $g \in G$ being closed of maximal dimension, which is equivalent to the G orbit of $(\Phi_1, \Phi_2) \in X_1 \times X_2$ having the same property. $\square$

A.3. **The Guo-Jacquet orbital integral.** Here we explain the connection between the orbital integrals of Definition 3.2.4 and those defined in [Guo96, Li21]. This is necessary to justify Remarks 3.4.2 and 4.5.3. Throughout, F is a local field, E is either $F \times F$ or an unramified separable quadratic field extension, and

$$\eta_{E/F} : F^\times \to \{\pm 1\}$$

is the associated quadratic character. As in the proof of Proposition A.2.2, the centralizer in $\mathrm{GL}_{2h}(F)$ of the standard embedding $\Phi : F \times F \to M_{2h}(F)$ is the subgroup

$$H = \left\{ \begin{pmatrix} A & \\ & B \end{pmatrix} : A, B \in \mathrm{GL}_h(F) \right\}.$$

Set $K_1 = E$ and $K_2 = E$. With this choice we have

$$K_3 = F \times F = K_0,$$

and the character $\eta : K_3^\times \to \{\pm 1\}$ determined by the quadratic extension K/K_3 is

$$\eta(a,b) = \eta_{E/F}(ab).$$

Fix a $g \in \mathrm{GL}_{2h}(F)$, and define

$$\Phi_0 : K_0 \to M_{2h}(F) \quad \text{and} \quad \Phi_3 = g\Phi g^{-1} : K_3 \to M_{2h}(F).$$

The centralizers in $\mathrm{GL}_{2h}(F)$ of their images are

$$H_0 = H \quad \text{and} \quad H_3 = gHg^{-1}.$$

We assume throughout that the pair (Φ_0, Φ_3) is regular semisimple in the sense of Definition 2.5.1. By (A.2.1) and Proposition 2.5.4, this is equivalent to the polynomial $M_g(T) \in F[T]$ of Definition A.2.1 having h distinct roots, all different from 0 and 1.

For every compactly supported f as in (1.4.3), define the *Guo-Jacquet orbital integral*

$$O_g(f; s, \eta_{E/F}) = \int_{I_g \backslash (H \times H)} f(h^{-1}gh') \cdot |hh'|^s \cdot \eta_{E/F}(h')\, dh\, dh'. \tag{A.3.1}$$

Here $\eta_{E/F}$ is viewed as a character of H by $\eta_{E/F}(h) = \eta_{E/F}(\det(h))$, the character $|\cdot| : H \to \mathbb{C}^\times$ is defined by

$$\left| \begin{pmatrix} A & \\ & B \end{pmatrix} \right| = \left| \frac{\det(A)}{\det(B)} \right|,$$

and

$$I_g = \{(h,h') \in H \times H : hg = gh'\}.$$

(Our convention for $|h|$ differs from the one used in [Li21, (1.6)] by an inverse, so our orbital integral differs from that one by the substitution $s \mapsto -s$.)

Remark A.3.1. The group I_g is abelian. In fact, $(h,h') \mapsto h$ defines an isomorphism $I_g \cong H_0 \cap H_3$, and the right hand side is the unit group of an étale F-algebra $L \subset M_{2h}(F)$ of dimension h. See (3.1.1).

Remark A.3.2. The integral (A.3.1) is well-defined because the characters

$$(h,h') \mapsto \eta_{E/F}(h'), \qquad (h,h') \mapsto |h|, \qquad (h,h') \mapsto |h'|$$

are all trivial on the subgroup $I_g \subset H \times H$. See Lemma 3.2.3 and the previous remark.

The Guo-Jacquet orbital integral (A.3.1) and the orbital integral of Definition 3.2.4 do *not* agree. Nevertheless, the following holds.

Proposition A.3.3. *We have the equality*

$$O_{(\Phi_0,\Phi_3)}(f;0,\eta) = O_g(f;0,\eta_{E/F}).$$

If (Φ_0,Φ_3) *matches a pair* (Φ_1,Φ_2) *of embeddings of* E *into a division algebra (so that the two sides of the above equality vanish by Proposition 3.3.3), then*

$$\frac{d}{ds} O_{(\Phi_0,\Phi_3)}(f;s,\eta)\big|_{s=0} = \frac{d}{ds} O_g(f;2s,\eta_{E/F})\big|_{s=0}.$$

Proof. We may choose $g_0 = 1$ and $g_3 = g$ in Definition 3.2.4, so that

$$O_{(\Phi_0,\Phi_3)}(f;s,\eta) = \int_{(H_0\cap H_3)\backslash(H_0\times H_3)} f(h_0^{-1}h_3 g)\cdot |h_0|^s \cdot \eta(h_3)\, dh_0\, dh_3.$$

Making the substitution $h = h_0$ and $h' = g^{-1}h_3 g$, we find

$$O_{(\Phi_0,\Phi_3)}(f;s,\eta) = \int_{I_g\backslash(H\times H)} f(h^{-1}gh')\cdot |h|^s \cdot \eta_{E/F}(h')\, dh\, dh',$$

exactly as in Remark 3.1.3. This last expression is not equal to the Guo-Jacquet orbital integral

$$O_g(f;s,\eta_{E/F}) = \int_{I_g\backslash(H\times H)} f(h^{-1}gh')\cdot |hh'|^s \cdot \eta_{E/F}(h')\, dh\, dh',$$

but they visibly agree at $s=0$, proving the first claim.

From now on we assume that (Φ_0,Φ_3) matches a pair (Φ_1,Φ_2) of embeddings of E into a division algebra. Consider the partition $H\times H = \bigsqcup_{m\in\mathbb{Z}} \Omega(m)$ defined by

$$\Omega(m) = \{(h,h')\in H\times H : |h'| = |h|\cdot|\varpi^m|\},$$

where $\varpi \in F^\times$ is a uniformizing parameter. It follows from Remark A.3.2 that each $\Omega(m)\subset H\times H$ is stable under both left and right multiplication by the subgroup I_g, and clearly

$$O_g(f;s,\eta_{E/F}) = \sum_m |\varpi|^{ms} \int_{I_g\backslash\Omega(m)} f(h^{-1}gh')\cdot |h|^{2s}\cdot \eta_{E/F}(h')\, dh\, dh'.$$

If we can prove that

$$\int_{I_g\backslash\Omega(m)} f(h^{-1}gh')\cdot \eta_{E/F}(h')\, dh\, dh' = 0 \tag{A.3.2}$$

for all $m\in\mathbb{Z}$, then we are done by

$$\begin{aligned} &\frac{d}{ds} O_g(f;s,\eta_{E/F})\big|_{s=0} \\ &= \sum_m \frac{d}{ds}\left[\int_{I_g\backslash\Omega(m)} f(h^{-1}gh')\cdot |h|^{2s}\cdot \eta_{E/F}(h')\, dh\, dh'\right]_{s=0} \\ &= \frac{d}{ds} O_{(\Phi_0,\Phi_3)}(f;2s,\eta_{E/F})\big|_{s=0}. \end{aligned}$$

We will prove (A.3.2) by imitating the proof of $O_{(\Phi_0,\Phi_3)}(f;0,\eta)=0$ from Proposition 3.3.3. Let $\mathbf{z}=\mathbf{z}_{03}\in M_{2h}(F)$ be as in (3.2.6), and set

$$u=\begin{pmatrix} & I_h \\ I_h & \end{pmatrix}\in \mathrm{GL}_{2h}(\mathcal{O}_F).$$

As in the discussion leading to Proposition 3.2.8, if we set $u_0=u$ and $u_3=gug^{-1}$ then $\mathbf{z}u_i\in H_i$. This implies both $\mathbf{z}u\in H$ and $g^{-1}\mathbf{z}gu\in H$, and allows us to define

$$\gamma=(\mathbf{z}u, g^{-1}\mathbf{z}ug)\in I_g.$$

Because I_g is abelian (Remark A.3.1), left multiplication by γ commutes with the left multiplication action of I_g on $\Omega(m)$. Making the substitution $(h,h')\mapsto \gamma\cdot(h,h')$ shows that

$$\begin{aligned}&\int_{I_g\backslash\Omega(m)} f(h^{-1}gh')\cdot \eta_{E/F}(h')\,dh\,dh' \\ &= \eta_{E/F}(\mathbf{z}u)\int_{I_g\backslash\Omega(m)} f(h^{-1}gh')\cdot\eta_{E/F}(h')\,dh\,dh'.\end{aligned}$$

The equality (A.3.2) follows from this and the relation $\eta_{E/F}(\mathbf{z}u)=-1$ from the proof of Proposition 3.3.3. □

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