WEIGHT VECTORS AND HIGHEST WEIGHT VECTORS IN UNITARY HIGHEST WEIGHT MODULES

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ABSTRACT. The theta correspondence for the dual pair $(\widetilde{\mathbf{U}}_n, \widetilde{\mathbf{U}}_{p,q})$ (where $\widetilde{\mathbf{U}}_n$ and $\widetilde{\mathbf{U}}_{p,q}$ are 2-fold covers of $\mathbf{U}_n$ and $\mathbf{U}_{p,q}$ respectively) is used to find nice bases for a collection of unitary highest weight modules of $\widetilde{\mathbf{U}}_{p,q}$, and also for the highest weight vectors with respect to the obvious maximal compact subgroup. The techniques used involve signed Hibi rings, an extension of the concept of Hibi ring, which has been identified as an underlying structure in constructing standard monomial theories, and in finding straightening laws in certain rings. An extension of the Littlewood-Richardson Rule to tensor products of arbitrary rational representations of $\mathrm{GL}_n(\mathbb{C})$ is required for finding the desired bases, and is also provided.

1. Introduction

Let n, p, q be positive integers, and let

$$W_{n,p,q} = (\mathbb{C}^{n*})^p \oplus (\mathbb{C}^n)^q$$

be the $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ module formed by taking the direct sum of q copies of $\mathbb{C}^n$ and p copies of its dual, $\mathbb{C}^{n*}$. The action of GL_n on $W_{n,p,q}$ induces an action of GL_n on the algebra $\mathcal{P}(W_{n,p,q})$ of polynomial functions on $W_{n,p,q}$ in the usual way. On the other hand, $\mathcal{P}(W_{n,p,q})$ is also a module for the algebra $\mathcal{PD}(W_{n,p,q})$ of polynomial coefficient differential operators on $W_{n,p,q}$. Let $\mathcal{PD}(W_{n,p,q})^{\mathrm{GL}_n}$ be the subalgebra of $\mathcal{PD}(W_{n,p,q})$ consisting of those operators which commute with the action of GL_n on $\mathcal{P}(W_{n,p,q})$. It is well known ([Ho2]) that $\mathcal{PD}(W_{n,p,q})^{\mathrm{GL}_n}$ is generated by a Lie algebra $\mathfrak{g}'$ which is isomorphic to the general linear Lie algebra $\mathfrak{gl}_{p+q} = \mathfrak{gl}_{p+q}(\mathbb{C})$. Thus, we have a joint action by $\mathrm{GL}_n \times \mathfrak{g}'$ on $\mathcal{P}(W_{n,p,q})$. Under this joint action, $\mathcal{P}(W_{n,p,q})$ decomposes as a direct sum ([Ho2, KV])

$$\mathcal{P}(W_{n,p,q}) \cong \bigoplus_{\lambda \in \Lambda_n^+(p,q)} \rho_n^\lambda \otimes \pi_{p,q}^\lambda \qquad (1.0.1)$$

where

(i) $\Lambda_n^+(p,q)$ is a set of non-increasing sequences $\lambda = (\lambda_1, ..., \lambda_n)$ of integers, of which at most p are positive and at most q are negative; and

(ii) for each λ in $\Lambda_n^+(p,q)$, ρ_n^λ is the irreducible rational representation of GL_n labeled in the standard way ([GW]) by λ, and $\pi_{p,q}^\lambda$ is an (infinite dimensional) irreducible module for $\mathfrak{g}'$.

Although this situation can be understood on a purely algebraic level, and is of interest in its own right as an extension of results on invariant theory stemming from the work

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of Hermann Weyl [Wy], it is also worthwhile to look at it in the context of the theta correspondence for dual pairs in the symplectic group $\mathrm{Sp}_{2m}(\mathbb{R})$ (or more properly, its two-fold cover, the metaplectic group Mp_{2m}) [Ho1, Ho3, A]. In this context, $m = 2n(p+q)$, and GL_n is the (slightly modified) complexification of a two-fold cover $\widetilde{\mathbf{U}}_n$ of the unitary group $\mathbf{U}_n$. Its centralizer in Mp_{2m} is $\widetilde{\mathbf{U}}_{p,q}$, and $\mathfrak{gl}_{p+q}$ is the (slightly modified) Lie algebra of the complexification of the Lie algebra of $\widetilde{\mathbf{U}}_{p,q}$. Also, $\mathrm{GL}_p \times \mathrm{GL}_q$ is the (slightly modified) complexification of $\widetilde{\mathbf{U}}_p \times \widetilde{\mathbf{U}}_q$, the maximal compact subgroup of $\widetilde{\mathbf{U}}_{p,q}$. (All the modifications mentioned affect only the centers of these groups or Lie algebras.) The polynomial ring $\mathcal{P}(W_{n,p,q})$ serves as the space of $\widetilde{\mathbf{U}}_n \times (\widetilde{\mathbf{U}}_p \times \widetilde{\mathbf{U}}_q)$ - finite vectors in the Fock model [Kn, Fo] for the Weil (aka, oscillator) representation of Mp_{2m}.

In this context, the associated representations of $\widetilde{\mathbf{U}}_n$ and $\widetilde{\mathbf{U}}_{p,q}$ are unitary. The representations of $\widetilde{\mathbf{U}}_n$ are of course finite dimensional, while those of $\widetilde{\mathbf{U}}_{p,q}$ are infinite dimensional. They are of the special type known as *holomorphic*, or *highest weight* representations. It was shown in [EHW] that, as n varies, the representations of $\widetilde{\mathbf{U}}_{p,q}$ that appear exhaust the full family of holomorphic unitary representations. For n small, the representations are also in some sense small, and in particular, do not contribute to the Plancherel formula for $\widetilde{\mathbf{U}}_{p,q}$. However, for n large, the representations that appear are all discrete series, and these repeat over and over as n increases. For more details about the theta correspondence for unitary groups, including cases for which neither member of the pair is compact, we refer to [P1, P2]. There are similar decompositions that describe the action of the other classical groups, O_n or Sp_{2n} on the polynomials on several copies of their standard modules, and there should be results analogous to the ones we establish for GL_n in this paper. However, these present extra technical difficulties, and their study must await another paper.

The purpose of this paper is two-fold:

(1) to describe a basis for each $\pi_{p,q}^{\lambda}$; and
(2) to describe a basis for the $\widetilde{\mathbf{U}}_p \times \widetilde{\mathbf{U}}_q$ highest weight vectors in $\pi_{p,q}^{\lambda}$.

Item (1) refines the description given by Kashiwara and Vergne ([KV]) of the representations of $\widetilde{\mathbf{U}}_{p,q}$ realized via the dual pair $(\mathbf{U}_n, \mathbf{U}_{p,q})$. (The paper [KV] was written before the concept of dual pair had been recognized.) Particular interest attaches to the cases when $p+q > n$, since these involve singular holomorphic representations. Item (2) gives an extension of the results of [HTW1] to the more difficult situation when non-polynomial representations are permitted as factors in a tensor product.

We now briefly describe our approach to (1) and (2). If G is a group and V is a G module, then V^G shall denote the subspace of V consisting of all G-invariants in V, i.e.

$$V^G = \{v \in V : g.v = v \ \forall g \in G\}.$$

Let U_n be the standard maximal unipotent subgroup of GL_n, consisting of upper triangular matrices with 1 on the diagonal. (Note: U_n should not be confused with the unitary group $\mathbf{U}_n$.) Let A_n be the subgroup of GL_n consisting of diagonal matrices in GL_n. We study the algebra

$$\mathcal{R} = \mathcal{R}_{n,p,q} := \mathcal{P}(W_{n,p,q})^{U_n}$$

of all U_n invariant polynomials in $\mathcal{P}(W_{n,p,q})$. It is a module for $A_n \times \mathfrak{g}'$, and decomposes as a direct sum

$$\mathcal{R} \cong \bigoplus_{\lambda \in \Lambda_n^+(p,q)} (\rho_n^\lambda)^{U_n} \otimes \pi_{p,q}^\lambda,$$

where, for each $\lambda \in \Lambda_n^+(p,q)$, the space $\left(\rho_n^\lambda\right)^{U_n}$ is the space of U_n invariant vectors in ρ_n^λ. By the highest weight theory, for each $\lambda \in \Lambda_n^+(p,q)$, $\left(\rho_n^\lambda\right)^{U_n}$ is one-dimensional. Thus any nonzero vector in $(\rho_n^\lambda)^{U_n}$ will be an eigenvector for A_n. We denote the character in $\widehat{A}_n$ by which A_n acts on $\left(\rho_n^\lambda\right)^{U_n}$ by ψ_n^λ. This is the highest weight of ρ_n^λ. As a representation of $\mathfrak{g}'$, we will have

$$(\rho_n^\lambda)^{U_n} \otimes \pi_{p,q}^\lambda \simeq \pi_{p,q}^\lambda.$$

That is, we can realize the representation $\pi_{p,q}^\lambda$ of $\mathfrak{g}'$ on the subspace $\left(\rho_n^\lambda\right)^{U_n} \otimes \pi_{p,q}^\lambda$ of $\mathcal{R}$.

The algebra $\mathcal{R}$ is also a module for $A_n \times A_p \times A_q$, so it is graded by the product $\widehat{A}_n \times \widehat{A}_p \times \widehat{A}_q \cong \mathbb{Z}^{n+p+q}$ of character groups of A_n, A_p and A_q. We will define a semigroup Ω with the property that Ω is a disjoint union of a certain collection of subsets, each of which has cardinality equal to the dimension of a certain corresponding homogeneous component of $\mathcal{R}$. The semigroup Ω is a ***signed Hibi cone*** ([Wa]). We will refer to these as *sH* cones. An *sH* cone comes endowed with a canonical set $\mathcal{G}$ of generators. For our cone Ω, we will associate bijectively to $\mathcal{G}$ a set $\tilde{\mathcal{G}}$ of explicit polynomials in $\mathcal{R}$, and we will show that an appropriate set of monomials in the elements of $\tilde{\mathcal{G}}$ forms a basis $\mathcal{B}_{\mathcal{R}}$ for $\mathcal{R}$. We summarize this relationship by saying that $\mathcal{R}$ *has a standard monomial theory with respect to* $\tilde{\mathcal{G}}$. Since the elements of $\mathcal{B}_{\mathcal{R}}$ are all highest weight vectors for GL_n, the subset of $\mathcal{B}_{\mathcal{R}}$ that have weight ψ_n^λ with respect to A_n will form a basis for the representation $\pi_{p,q}^\lambda$ of $\mathfrak{g}'$.

Next, we note that the group $\mathrm{GL}_p \times \mathrm{GL}_q$ also acts on the algebra $\mathcal{P}(W_{n,p,q})$, and this action commutes with the action of GL_n. This action of $\mathrm{GL}_p \times \mathrm{GL}_q$ is a very slight modification of the complexified action of the two-fold cover $\widetilde{\mathbf{K}}$ of $\mathbf{U}_p \times \mathbf{U}_q$ mentioned in goal (2) above. Thus, we have a joint action by $\mathrm{GL}_n \times (\mathrm{GL}_p \times \mathrm{GL}_q)$ on $\mathcal{P}(W_{n,p,q})$. We study the subalgebra

$$\mathcal{Q} = \mathcal{Q}_{n,p,q} := \mathcal{P}(W_{n,p,q})^{U_n \times (U_p \times U_q)}$$

of all $\mathrm{GL}_n \times (\mathrm{GL}_p \times \mathrm{GL}_q)$ highest weight vectors in $\mathcal{P}(W_{n,p,q})$. Note that

$$\mathcal{Q} = \left(\mathcal{P}(W_{n,p,q})^{U_n}\right)^{U_p \times U_q} = \mathcal{R}^{U_p \times U_q},$$

that is, $\mathcal{Q}$ is the algebra of $U_p \times U_q$ invariants in $\mathcal{R}$. So this is a further refinement of the understanding of $\mathcal{R}$.

The algebra $\mathcal{Q}$ is also graded by $\widehat{A}_n \times \widehat{A}_p \times \widehat{A}_q$, and can be decomposed into homogeneous components for this grading as

$$\mathcal{Q} = \bigoplus_{(\lambda,G,H)} \mathcal{Q}_{\lambda,G,H}$$

where in the ordered triple (λ, G, H), the component λ belongs to $\Lambda_n^+(p,q)$, and G and H are Young diagrams with at most p rows and at most q rows respectively. We show that, for each ordered triple (λ, G, H) which appears in the direct sum, the homogeneous component $\mathcal{Q}_{(\lambda,G,H)}$ of $\mathcal{Q}$ can be identified with the set of $\widetilde{\mathbf{K}}$ highest weight vectors in $\pi_{p,q}^\lambda$ of a certain specific weight, determined by G and H. On the other hand, by a duality phenomenon, $\mathcal{Q}_{(\lambda,G,H)}$ can also be identified with the set of GL_n highest weight vectors of weight ψ_n^λ in the tensor product $\rho_n^{H^*} \otimes \rho_n^G$, where $\rho_n^{H^*}$ is the representation dual to ρ_n^H. It follows that the dimension of $\mathcal{Q}_{(\lambda,G,H)}$ is equal to the multiplicity $c^{\lambda}_{H^*,G}$ of ρ_n^{λ} in $\rho_n^{H^*} \otimes \rho_n^{G}$. In order to describe $c^{\lambda}_{H^*,G}$, we extend the classical Littlewood-Richardson Rule to give a formula that describes the multiplicities in the tensor product of any two irreducible rational representations of GL_n.

Finally, we construct a basis $\mathbf{B}_{(\lambda,G,H)}$ for $\mathcal{Q}_{(\lambda,G,H)}$ by following a scheme developed in [HTW1] and [HL1]. We remark that a basis for $\mathcal{Q}_{(\lambda,G,H)}$ was given in [HL1] under some restrictions on p and q (the "stable range" condition).

Here is the plan of the rest of the paper. In Section 2, we establish most of the further notation we will need in the paper, and also review basic facts about the representation theory of GL_n. Sections 3 and 4 are devoted to constructing the desired bases for $\mathcal{R}$ and $\mathcal{Q}$, respectively. For the readers' convenience, we have also included a review on signed Hibi cones in the Appendix.

2. Preliminaries

In this section, we set up some notation and review some basic facts on the representations of the general linear group GL_n.

2.1. **Some notation.** As in Section 1, we let A_n be the diagonal torus in GL_n and let U_n be the maximal unipotent subgroup of GL_n consisting of all $n \times n$ upper triangular matrices with 1's on the diagonal. Then $B_n = A_nU_n$ is the standard Borel subgroup of GL_n. We know by the standard highest weight theory ([GW]), the irreducible rational representations of GL_n are in bijective correspondence with the semigroup $\widehat{A}_n^+$ of dominant weights for GL_n with respect to B_n. For each $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$, define $\psi_n^{\lambda} : A_n \longrightarrow \mathbb{C}^{\times}$ by

$$\psi_n^{\lambda}(\mathrm{diag}(a_1, ..., a_n)) = a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n} \tag{2.1.1}$$

where $\mathrm{diag}(a_1, ..., a_n)$ is the diagonal matrix with diagonal entries $a_1, ..., a_n$. Then

$$\widehat{A}_n^+ = \{\psi_n^{\lambda} : \lambda \in \Lambda_n^+\} \quad \text{where} \quad \Lambda_n^+ = \{\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}. \tag{2.1.2}$$

For $\lambda = (\lambda_1, ..., \lambda_n) \in \Lambda_n^+$, we denote by ρ_n^{λ} the irreducible representation of GL_n with highest weight ψ_n^{λ}, and denote by $(\rho_n^{\lambda})^*$ the contragredient representation of ρ_n^{λ}. Then we have ([GW])

$$(\rho_n^{\lambda})^* \cong \rho_n^{\lambda^*} \quad \text{where} \quad \lambda^* = (-\lambda_n, -\lambda_{n-1}, ..., -\lambda_1). \tag{2.1.3}$$

In our setting, it is more convenient also to use diagram notation. Recall that a ***Young diagram*** is an array of square boxes arranged in left-justified horizontal rows, with each row no longer than the one above it ([Fu]), and we will denote it by an upper case letter such as D, E or F. The ***depth*** of a Young diagram D, denoted by $\mathbf{r}(D)$, is the number of rows in D. Let Λ_n^{++} be the subset of Λ_n^+ defined by

$$\Lambda_n^{++} = \{\lambda = (\lambda_1, ..., \lambda_n) \in \Lambda_n^+ : \lambda_n \geq 0\}. \tag{2.1.4}$$

Then the set of Young diagrams with depth at most n can be identified with Λ_n^{++} as follows: A Young diagram D with $\mathbf{r}(D) = m \leq n$ is identified with the element $(d_1, ..., d_n)$ of Λ_n^{++} where for each $1 \leq i \leq m$, d_i is the number of boxes in the ith row of D and $d_i = 0$ for $i > m$. We call any ρ_n^{λ} with λ in Λ_n^{++} a ***polynomial representation*** of GL_n, and we will denote it also by ρ_n^{D}, where D is the diagram corresponding to λ.

For a general element $\lambda = (\lambda_1, ..., \lambda_n)$ of Λ_n^+, we shall associate an ordered pair (D, E) of Young diagrams with λ defined by

$$D = (d_1, ..., d_n) \quad \text{and} \quad E = (e_1, ..., e_n)$$

where for each $1 \le i \le n$,

$$d_i = \max(\lambda_i, 0) \quad \text{and} \quad e_i = -\min(\lambda_{n+1-i}, 0). \tag{2.1.5}$$

In this situation, we shall write $\lambda = (D, E)$, $\psi_n^\lambda = \psi_n^{D,E}$ and $\rho_n^\lambda = \rho_n^{D,E}$. Note that $\lambda^* = (E, D)$, so we have

$$(\rho_n^{D,E})^* \cong \rho_n^{E,D}.$$

2.2. $(\mathrm{GL}_n, \mathrm{GL}_m)$**-duality.** Let n and m be positive integers, and let $\mathrm{M}_{nm} = \mathrm{M}_{nm}(\mathbb{C})$ denote the space of all $n \times m$ complex matrices. For a matrix u, u^t shall denote its transpose. Let $\mathrm{GL}_n \times \mathrm{GL}_m$ act on M_{nm} by

$$\tau_{n,m}(g, h)(x) = (g^{-1})^t x h^{-1} \tag{2.2.1}$$

where $(g, h) \in \mathrm{GL}_n \times \mathrm{GL}_m$ and $x \in \mathrm{M}_{nm}$. Then under this action, we have

$$\mathrm{M}_{nm} \cong \mathbb{C}^{n*} \otimes \mathbb{C}^{m*},$$

where $\mathbb{C}^{n*}$ is the contragredient representation of the standard representation $\mathbb{C}^n$ of GL_n, and similarly for $\mathbb{C}^{m*}$. The action $\tau_{n,m}$ induces the following action (which will again be denoted by $\tau_{n,m}$) of $\mathrm{GL}_n \times \mathrm{GL}_m$ on the algebra $\mathcal{P}(\mathrm{M}_{nm})$ of polynomial functions on M_{nm}: for $(g, h) \in \mathrm{GL}_n \times \mathrm{GL}_m$, $f \in \mathcal{P}(\mathrm{M}_{nm})$ and $x \in \mathrm{M}_{nm}$,

$$[\tau_{n,m}(g, h)(f)](x) = f(\tau_{n,m}(g^{-1}, h^{-1})(x)) = f(g^t x h). \tag{2.2.2}$$

Theorem 2.2.3. ($(\mathrm{GL}_n, \mathrm{GL}_m)$-duality [Ho2, Ho4, GW]) *Under the action $\tau_{n,m}$ by $\mathrm{GL}_n \times \mathrm{GL}_m$, $\mathcal{P}(\mathrm{M}_{nm})$ admits the following decomposition as a $\mathrm{GL}_n \times \mathrm{GL}_m$ module:*

$$\mathcal{P}(\mathrm{M}_{nm}) \cong \mathcal{P}(\mathbb{C}^{n*} \otimes \mathbb{C}^{m*}) \cong \bigoplus_{\mathbf{r}(D) \le \min(n,m)} \rho_n^D \otimes \rho_m^D.$$

In particular, if $m = 1$, then we have

$$\mathcal{P}(\mathrm{M}_{n1}) \cong \mathcal{P}(\mathbb{C}^{n*} \otimes \mathbb{C}^*) \cong \mathcal{P}(\mathbb{C}^{n*}) \cong \bigoplus_{d \ge 0} \rho_n^{(d)} \otimes \rho_1^{(d)} \tag{2.2.4}$$

as a $\mathrm{GL}_n \times \mathrm{GL}_1$ module. Here, (d) denotes the Young diagram which has only 1 row with d boxes.

Next, we compose the action $\tau_{n,m}$ with the automorphism $g \to (g^{-1})^t$ of GL_n to obtain a new action $\tau'_{n,m}$. Specifically, for $(g, h) \in \mathrm{GL}_n \times \mathrm{GL}_m$ and $x \in \mathrm{M}_{nm}$, let

$$\tau'_{n,q}(g, h)(x) = g x h^{-1}. \tag{2.2.5}$$

Then under the action $\tau'_{n,m}$, we have

$$\mathrm{M}_{nm} \cong \mathbb{C}^n \otimes \mathbb{C}^{m*}$$

as a $\mathrm{GL}_n \times \mathrm{GL}_m$ module, and $\tau'_{n,m}$ induces an action (also denoted by $\tau'_{n,m}$) by $\mathrm{GL}_n \times \mathrm{GL}_m$ on $\mathcal{P}(\mathrm{M}_{nm})$ given by

$$[\tau'_{n,q}(g, h)(f)](x) = f(g^{-1} x h) \tag{2.2.6}$$

where $f \in \mathcal{P}(\mathrm{M}_{nm})$, $(g, h) \in \mathrm{GL}_n \times \mathrm{GL}_m$ and $x \in \mathrm{M}_{nm}$. By this change of action, Theorem 2.2.3 is converted to the following equivalent result.

Theorem 2.2.7. *Under the action $\tau'_{n,m}$ by $\mathrm{GL}_n \times \mathrm{GL}_m$, $\mathcal{P}(\mathrm{M}_{nm})$ admits the following decomposition as a $\mathrm{GL}_n \times \mathrm{GL}_m$ module:*

$$\mathcal{P}(\mathrm{M}_{nm}) \cong \mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^{m*}) \simeq \bigoplus_{\mathbf{r}(D)\leq \min(n,m)} \rho_n^{D^*} \otimes \rho_m^{D}.$$

In particular, if $m = 1$, then we have

$$\mathcal{P}(\mathrm{M}_{n1}) \cong \mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^*) \cong \mathcal{P}(\mathbb{C}^n) \cong \bigoplus_{d\geq 0} \rho_n^{(d)*} \otimes \rho_1^{(d)} \tag{2.2.8}$$

as a $\mathrm{GL}_n \times \mathrm{GL}_1$ module.

3. A basis for $\mathcal{R} = \mathcal{P}(W_{n,p,q})^{U_n}$

We recall the polynomial algebra $\mathcal{P}(W_{n,p,q})$ defined in Section 1, and shall define an action of $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ on it. We restrict this action to GL_n and consider the algebra $\mathcal{R} = \mathcal{P}(W_{n,p,q})^{U_n}$ consisting of all the polynomials in $\mathcal{P}(W_{n,p,q})$ which are fixed by the standard unipotent subgroup U_n of GL_n. The algebra $\mathcal{R}$ is a ***reciprocity algebra*** in the sense of [HTW2]. Roughly, this means that $\mathcal{R}$ has the following properties: $\mathcal{R}$ is a module for $A_n \times A_p \times A_q$, so it is graded by a subsemigroup of the character group of $A_n \times A_p \times A_q$. Moreover, each of its homogeneous components with respect to this grading consists of highest weight vectors for certain Lie algebras, and the dimensions of the homogeneous components encode two sets of branching rules. Our main goal in this section is to determine a basis for $\mathcal{R}$. It will be seen in Section 3.7 that each homogeneous component of $\mathcal{R}$ is the Harish-Chandra module of a unitary highest weight module of an appropriate double cover $\widetilde{\mathbf{U}}_{p,q}$ of $\mathbf{U}_{p,q}$. So we also obtain a basis for each of these Harish-Chandra modules.

3.1. The polynomial algebra $\mathcal{P}(W_{n,p,q})$.

As in the Introduction, we let n, p, q be positive integers, and let

$$W_{n,p,q} = \mathrm{M}_{np} \oplus \mathrm{M}_{nq} = \{(X, Y) : \ X \in \mathrm{M}_{np}, Y \in \mathrm{M}_{nq}\}$$

be the direct sum of the $n \times p$ complex matrices and the $n \times q$ complex matrices. Let the group $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ act on $W_{n,p,q}$ by

$$(g, h_1, h_2).(X, Y) = ((g^{-1})^t X h_1^{-1}, gY h_2^{-1}), \tag{3.1.1}$$

where $(X, Y) \in W_{n,p,q}$, $g \in \mathrm{GL}_n$, $h_1 \in \mathrm{GL}_p$ and $h_2 \in \mathrm{GL}_q$. If we restrict this action to GL_n, then

$$W_{n,p,q} \cong (\mathbb{C}^{n*})^p \oplus (\mathbb{C}^n)^q. \tag{3.1.2}$$

That is, as a representation for GL_n, $W_{n,p,q}$ is isomorphic to the direct sum of q copies of $\mathbb{C}^n$ and p copies of its dual, $\mathbb{C}^{n*}$.

The action (3.1.1) induces an action of $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ on $\mathcal{P}(W_{n,p,q})$ by the formula

$$[(g, h_1, h_2).f](X, Y) = f(g^t X h_1, g^{-1} Y h_2), \tag{3.1.3}$$

where $g \in \mathrm{GL}_n$, $h_1 \in \mathrm{GL}_p$, $h_2 \in \mathrm{GL}_q$, $X \in \mathrm{M}_{np}$, $Y \in \mathrm{M}_{nq}$ and $f \in \mathcal{P}(W_{n,p,q})$. Here we are interested in the restriction of this action to the subgroup $\mathrm{GL}_n \times A_p \times A_q$ of $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$.

Lemma 3.1.4. *Let $\mathbb{Z}^+$ denote the set of all nonnegative integers. For $(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q$, let*

$$\mathcal{P}(W_{n,p,q})_{(\alpha,\beta)}=\left\{f\in\mathcal{P}(W_{n,p,q}):\ (I_n,h_1,h_2).f=\psi_p^\alpha(h_1)\psi_q^\beta(h_2)f\ \ \forall(h_1,h_2)\in A_p\times A_q\right\},$$

where I_n is the $n\times n$ identity matrix, and ψ_p^α and ψ_q^β are defined in equation (2.1.1).

(i) *The algebra $\mathcal{P}(W_{n,p,q})$ has a direct sum decomposition*

$$\mathcal{P}(W_{n,p,q})=\bigoplus_{(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q}\mathcal{P}(W_{n,p,q})_{(\alpha,\beta)}. \tag{3.1.5}$$

(ii) *For $(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q$, $\mathcal{P}(W_{n,p,q})_{(\alpha,\beta)}$ is invariant under the action by GL_n, and as a representation of GL_n,*

$$\mathcal{P}(W_{n,p,q})_{(\alpha,\beta)}\cong T_n(\alpha,\beta)$$

where

$$T_n(\alpha,\beta)=\left(\bigotimes_{i=1}^p\rho_n^{(\alpha_i)}\right)\otimes\left(\bigotimes_{j=1}^q\rho_n^{(\beta_j)*}\right). \tag{3.1.6}$$

Proof. For each $(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q$, $\mathcal{P}(W_{n,p,q})_{(\alpha,\beta)}$ is the $\psi_p^\alpha\times\psi_q^\beta$- isotypic component of $\mathcal{P}(W_{n,p,q})$ under $A_p\times A_q$, and equation (3.1.5) is the decomposition of $\mathcal{P}(W_{n,p,q})$ into isotypic components. This gives (i).

Next, by equation (3.1.2),

$$W_{n,p,q}\cong\mathbb{C}_1^{n*}\oplus\cdots\oplus\mathbb{C}_p^{n*}\oplus\mathbb{C}_1^n\oplus\cdots\oplus\mathbb{C}_q^n$$

as a GL_n module, where for each $1\le i\le p$, $\mathbb{C}_i^{n*}$ is a copy of $\mathbb{C}^{n*}$, and for each $1\le j\le q$, $\mathbb{C}_j^n$ is a copy of $\mathbb{C}^n$. It follows that

$$\begin{aligned}
\mathcal{P}(W_{n,p,q}) &\cong \left(\bigotimes_{i=1}^p\mathcal{P}(\mathbb{C}_i^{n*})\right)\otimes\left(\bigotimes_{j=1}^q\mathcal{P}(\mathbb{C}_j^n)\right)\\
&\cong \left\{\bigotimes_{i=1}^p\left(\bigoplus_{\alpha_i\in\mathbb{Z}^+}\rho_n^{(\alpha_i)}\otimes\rho_1^{(\alpha_i)}\right)\right\}\otimes\left\{\bigotimes_{j=1}^q\left(\bigoplus_{\beta_j\in\mathbb{Z}^+}\rho_n^{(\beta_j)^*}\otimes\rho_1^{(\beta_j)}\right)\right\}\\
&\qquad\text{(by equations (2.2.4) and (2.2.8))}\\
&\cong \bigoplus_{\substack{(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q\\ \alpha=(\alpha_1,\dots,\alpha_p),\beta=(\beta_1,\dots,\beta_q)}}\left\{\left(\bigotimes_{i=1}^p\rho_n^{(\alpha_i)}\right)\otimes\left(\bigotimes_{j=1}^q\rho_n^{(\beta_j)^*}\right)\right\}\otimes\left(\bigotimes_{i=1}^p\rho_1^{(\alpha_i)}\right)\otimes\left(\bigotimes_{j=1}^q\rho_1^{(\beta_j)}\right)\\
&\cong \bigoplus_{(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q}T_n(\alpha,\beta)\otimes\mathbb{C}_{\psi_p^\alpha}\otimes\mathbb{C}_{\psi_q^\beta},
\end{aligned} \tag{3.1.7}$$

where for each $(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q$, $\mathbb{C}_{\psi_p^\alpha}=\bigotimes_{i=1}^p\rho_1^{(\alpha_i)}$ and $\mathbb{C}_{\psi_q^\beta}=\bigotimes_{j=1}^q\rho_1^{(\beta_j)}$ are (one-dimensional) irreducible representations of $A_p\cong\mathrm{GL}_1^p$ and $A_q\cong\mathrm{GL}_1^q$ respectively. From this decomposition of $\mathcal{P}(W_{n,p,q})$, we deduce that

$$\mathcal{P}(W_{n,p,q})_{(\alpha,\beta)}\cong T_n(\alpha,\beta)\otimes\mathbb{C}_{\psi_p^\alpha}\otimes\mathbb{C}_{\psi_q^\beta}$$

as a representation for $\mathrm{GL}_n \times A_p \times A_q$. Part (ii) then follows by restricting the action to GL_n. □

3.2. The algebra $\mathcal{R}$.

Recall that

$$\mathcal{R} = \mathcal{R}_{n,p,q} = \mathcal{P}(W_{n,p,q})^{U_n}$$

is the algebra of all the polynomial functions in $\mathcal{P}(W_{n,p,q})$ which are fixed by the unipotent subgroup U_n of GL_n. Since the torus A_n normalizes U_n, A_n acts on $\mathcal{R}$ via the restriction of the action of GL_n on $\mathcal{P}(W_{n,p,q})$. Moreover, since the action of $\mathrm{GL}_p \times \mathrm{GL}_q$ on $\mathcal{P}(W_{n,p,q})$ commutes with that of GL_n (and so with U_n), $\mathcal{R}$ is also stable under $\mathrm{GL}_p \times \mathrm{GL}_q$, and so it becomes a module for $A_n \times \mathrm{GL}_p \times \mathrm{GL}_q$. We shall restrict this action to the product of toruses $A_n \times A_p \times A_q$. We shall describe an algebra grading for $\mathcal{R}$ and construct a basis for $\mathcal{R}$.

Note that when $n = 1$, U_1 is the trivial subgroup of GL_1, so that

$$\mathcal{R}_{1,p,q} = \mathcal{P}(W_{1,p,q})^{U_1} = \mathcal{P}(W_{1,p,q}).$$

This case will be discussed in §3.8. Except in §3.8, we shall assume that $n \geq 2$.

3.3. Iterated Pieri rule.

Our goal in this subsection is to describe the multiplicity of an irreducible rational representation ρ_n^λ of GL_n in the tensor product $T_n(\alpha, \beta)$ defined in equation (3.1.6). We will later see that this multiplicity is equal to the dimension of a homogeneous component of $\mathcal{R}$.

We first state a version of the Pieri rule that we need.

Definition 3.3.1. (i) If $\lambda = (\lambda_1, ..., \lambda_n)$, $\mu = (\mu_1, ..., \mu_n) \in \Lambda_n^+$ where Λ_n^+ is defined in equation (2.1.2), and

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_n,$$

then we say that μ **interlaces** λ and write $\lambda \sqsubseteq \mu$.

(Remark: For later purposes (see §3.5), we note that the interlacing condition is equivalent to thinking of μ and λ as the values of a function on the rows of the poset

$$\begin{array}{cccccccc} \bullet & \bullet & \bullet & \bullet & \bullet & . & . & . \\ \ \bullet & \bullet & \bullet & \bullet & \bullet & . & . & . \end{array}$$

where each point dominates all points to its right.)

(ii) For $\lambda = (\lambda_1, ..., \lambda_n) \in \Lambda_n^+$, $|\lambda|$ shall denote the integer defined by

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

Note that, although $|\ \ |$ is frequently used to denote absolute value, here $|\lambda|$ may take on negative values. Also, it is additive in λ.

Proposition 3.3.2. (Generalized Pieri Rule for $\mathbf{GL_n}$, [Wa]) *Let $\lambda \in \Lambda_n^+$ and $\alpha \in \mathbb{Z}^+$. Then*

$$\rho_n^\lambda \otimes \rho_n^{(\alpha)} \cong \bigoplus_{\substack{\lambda \sqsubseteq \mu \\ |\lambda|+\alpha=|\mu|}} \rho_n^\mu \quad \textit{and} \quad \rho_n^\lambda \otimes \rho_n^{(\alpha)^*} \cong \bigoplus_{\substack{\mu \sqsubseteq \lambda \\ |\lambda|-\alpha=|\mu|}} \rho_n^\mu.$$

Note that the usual Pieri rule corresponds to the first equation, in the case when $\lambda \in \Lambda_n^{++}$. The second equation follows from the first by writing $\lambda = (\lambda^*)^*$

Notation 3.3.3. For $\alpha = (\alpha_1, ..., \alpha_p) \in (\mathbb{Z}^+)^p$, $\beta = (\beta_1, ..., \beta_q) \in (\mathbb{Z}^+)^q$ and $\lambda \in \Lambda_n^+$, let $S(\lambda, \alpha, \beta)$ be the set of all sequences $\mu = (\mu^{(i)})_{1\le i\le p+q} \in (\Lambda_n^+)^{p+q}$ such that

(i) $\mu^{(1)} = (\alpha_1, 0, ..., 0)$ and $\mu^{(p+q)} = \lambda$.
(ii) $\mu^{(1)} \sqsubseteq \mu^{(2)} \sqsubseteq \cdots \sqsubseteq \mu^{(p)} \sqsupseteq \mu^{(p+1)} \sqsupseteq \mu^{(p+2)} \sqsupseteq \cdots \sqsupseteq \mu^{(p+q)}$.
(iii) $|\mu^{(s-1)}| + \alpha_s = |\mu^{(s)}|$ for $2 \le s \le p$.
(iv) $|\mu^{(p+j-1)}| - \beta_j = |\mu^{(p+j)}|$ for $1 \le j \le q$.

We also let

$$K_{\lambda,\alpha,\beta} = \#(S(\lambda, \alpha, \beta)), \tag{3.3.4}$$

that is, $K_{(\lambda,\alpha,\beta)}$ is the number of elements in $S(\lambda, \alpha, \beta)$.

Proposition 3.3.5. (i) *For* $(\alpha, \beta) \in (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$,

$$T_n(\alpha, \beta) \cong \bigoplus_{\lambda\in\Lambda_n^+} K_{(\lambda,\alpha,\beta)}\rho_n^\lambda.$$

(ii) *If* $\alpha, \alpha' \in (\mathbb{Z}^+)^p$, $\beta, \beta' \in (\mathbb{Z}^+)^q$ *and* $\lambda, \lambda' \in \Lambda_n^+$ *are such that* $K_{(\lambda,\alpha,\beta)} \ne 0$ *and* $K_{(\lambda',\alpha',\beta')} \ne 0$, *then*

$$K_{(\lambda+\lambda',\alpha+\alpha',\beta+\beta')} \ne 0.$$

Proof. Part (i) follows directly from iterating the generalized Pieri rule for GL_n.

For (ii), if we check the four conditions defining $S(\lambda, \alpha, \beta)$ in Notation 3.3.3, we can see that they are compatible with addition, in the sense that, if $\mu = (\mu^{(i)})_{1\le i\le p+q} \in S(\lambda, \alpha, \beta)$ and $\mu' = (\mu'^{(i)})_{1\le i\le p+q} \in S(\lambda', \alpha', \beta')$, then

$$\mu + \mu' = (\mu^{(i)} + \mu'^{(i)})_{1\le i\le p+q} \in S(\lambda + \lambda', \alpha + \alpha', \beta + \beta').$$

□

3.4. **Graded algebra structure of $\mathcal{R}$.** In this subsection, we shall describe a graded algebra structure on $\mathcal{R}$ and determine the dimension of each of its homogeneous components with respect to this grading.

The sets Λ_n^+, $(\mathbb{Z}^+)^p$ and $(\mathbb{Z}^+)^q$ are semigroups with respect to componentwise addition, so their direct product $\Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$ also forms a semigroup. For $(\lambda, \alpha, \beta) \in \Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$, let $\mathcal{R}_{(\lambda,\alpha,\beta)}$ be the subspace of $\mathcal{R}$ defined by

$$\mathcal{R}_{(\lambda,\alpha,\beta)} = \{f \in \mathcal{R} : t.f = \psi_n^\lambda(t_1)\psi_p^\alpha(t_2)\psi_q^\beta(t_3)f \ \forall t = (t_1, t_2, t_3) \in A_n \times A_p \times A_q\},$$

that is, $\mathcal{R}_{(\lambda,\alpha,\beta)}$ is the $\psi_n^\lambda \times \psi_p^\alpha \times \psi_q^\beta$-eigenspace of $A_n \times A_p \times A_q$ in $\mathcal{R}$. Note that the nonzero vectors in $\mathcal{R}_{(\lambda,\alpha,\beta)}$ are highest weight vectors for $\mathrm{GL}_n \times A_p \times A_q$ of weight $\psi_n^\lambda \times \psi_p^\alpha \times \psi_q^\beta$.

Proposition 3.4.1. (i) *For each* $(\lambda, \alpha, \beta) \in \Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$,

$$\dim \mathcal{R}_{(\lambda,\alpha,\beta)} = K_{(\lambda,\alpha,\beta)},$$

where $K_{(\lambda,\alpha,\beta)}$ *is defined in equation* (3.3.4). *Consequently,* $\mathcal{R}_{(\lambda,\alpha,\beta)}$ *is a nonzero subspace of* $\mathcal{R}$ *if and only if* $K_{(\lambda,\alpha,\beta)} \ne 0$.

(ii) *Let* $\Lambda(\mathcal{R})$ *be the subset of* $\Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$ *defined by*

$$\Lambda(\mathcal{R}) = \{(\lambda, \alpha, \beta) \in \Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q : \ K_{(\lambda,\alpha,\beta)} \ne 0\}.$$

Then $\Lambda(\mathcal{R})$ *is a subsemigroup of* $\Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$.

(iii) *The algebra $\mathcal{R}$ is graded by $\Lambda(\mathcal{R})$ and it decomposes into homogeneous components for this grading as*

$$\mathcal{R} = \bigoplus_{(\lambda,\alpha,\beta)\in\Lambda(\mathcal{R})} \mathcal{R}_{(\lambda,\alpha,\beta)}. \tag{3.4.2}$$

Proof. We extract the U_n invariants in the direct sum decomposition of $\mathcal{P}(W_{n,p,q})$ given in equation (3.1.7) and obtain

$$\mathcal{R} = \mathcal{P}(W_{n,p,q})^{U_n} \cong \bigoplus_{(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q} T_n(\alpha,\beta)^{U_n} \otimes \mathbb{C}_{\psi_p^\alpha} \otimes \mathbb{C}_{\psi_q^\beta} \tag{3.4.3}$$

where $T_n(\alpha,\beta)^{U_n}$ is the space of vectors in $T_n(\alpha,\beta)$ fixed by U_n, and it is a module for A_n. So $T_n(\alpha,\beta)^{U_n}$ has a direct sum decomposition

$$T_n(\alpha,\beta)^{U_n} = \bigoplus_{\lambda\in\Lambda_n^+} T_n(\alpha,\beta)_\lambda^{U_n}$$

where for each $\lambda \in \Lambda_n^+$,

$$T_n(\alpha,\beta)_\lambda^{U_n} = \{v \in T_n(\alpha,\beta)^{U_n} : \ t.v = \psi_n^\lambda(t)v \ \forall t \in A_n\}.$$

Using this and equation (3.4.3), we obtain

$$\begin{aligned}
\mathcal{R} &\cong \bigoplus_{(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q} \left(\bigoplus_{\lambda\in\Lambda_n^+} T_n(\alpha,\beta)_\lambda^{U_n} \right) \otimes \mathbb{C}_{\psi_p^\alpha} \otimes \mathbb{C}_{\psi_q^\beta} \\
&\cong \bigoplus_{(\lambda,\alpha,\beta)\in\Lambda_n^+\times(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q} T_n(\alpha,\beta)_\lambda^{U_n} \otimes \mathbb{C}_{\psi_p^\alpha} \otimes \mathbb{C}_{\psi_q^\beta}.
\end{aligned}$$

From this decomposition, we deduce that for each $(\lambda,\alpha,\beta) \in \Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$,

$$\mathcal{R}_{(\lambda,\alpha,\beta)} \cong T_n(\alpha,\beta)_\lambda^{U_n} \otimes \mathbb{C}_{\psi_p^\alpha} \otimes \mathbb{C}_{\psi_q^\beta}, \tag{3.4.4}$$

and so

$$\dim \mathcal{R}_{(\lambda,\alpha,\beta)} = (\dim T_n(\alpha,\beta)_\lambda^{U_n})(\dim \mathbb{C}_{\psi_p^\alpha})(\dim \mathbb{C}_{\psi_q^\beta}) = \dim T_n(\alpha,\beta)_\lambda^{U_n}.$$

Now we use the fact that the nonzero vectors in $T_n(\alpha,\beta)_\lambda^{U_n}$ are the GL_n highest weight vectors of weight ψ^λ in $T_n(\alpha,\beta)$, so that by Part (i) of Proposition 3.3.5,

$$\dim T_n(\alpha,\beta)_\lambda^{U_n} = \dim \mathrm{Hom}_{\mathrm{GL}_n}(\rho_n^\lambda, T_n(\alpha,\beta)) = K_{\lambda,\alpha,\beta}.$$

This proves (i).

(ii) follows from Part (ii) of Proposition 3.3.5.

As for (iii), for each $(\lambda,\alpha,\beta) \in \Lambda(\mathcal{R})$, $\mathcal{R}_{(\lambda,\alpha,\beta)}$ is the $\psi_n^\lambda \times \psi_p^\alpha \times \psi_q^\beta$- isotypic component of $\mathcal{R}$ under $A_n \times A_p \times A_q$, and equation (3.4.2) is the decomposition of $\mathcal{R}$ into a direct sum of isotypic components. Since $A_n \times A_p \times A_q$ acts on $\mathcal{R}$ by algebra automorphisms, this decomposition also defines an algebra grading on $\mathcal{R}$. □

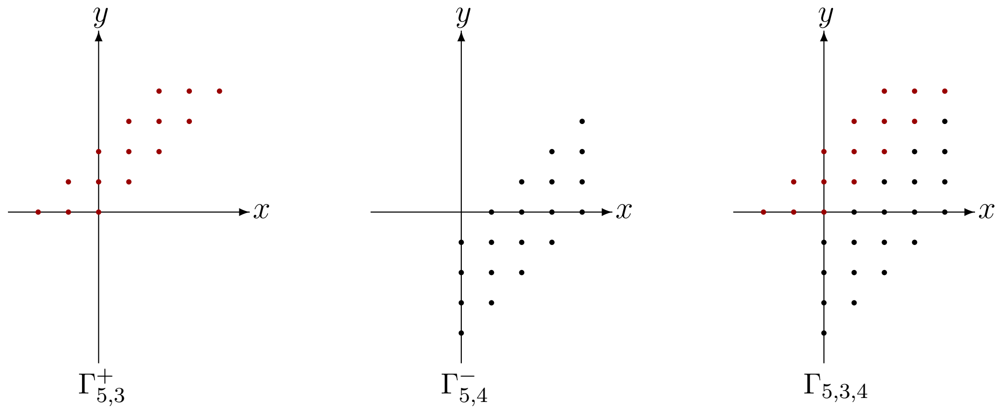


FIGURE 1

3.5. **The signed Hibi cone $\Omega_{n,p,q}$.** In the previous subsection, we saw that the dimension of the homogeneous component $\mathcal{R}_{(\lambda,\alpha,\beta)}$ of $\mathcal{R}$ is equal to the number of elements in the set $S(\lambda,\alpha,\beta)$ of sequences $(\mu^{(i)})_{1\le i\le p+q} \in (\Lambda_n^+)^{p+q}$ which satisfy conditions (i)-(iv) of Notation 3.3.3. We shall show in Proposition 3.5.5 that each of these sequences $(\mu^{(i)})_{1\le i\le p+q}$ can be identified with an integer-valued order preserving function on a finite poset Γ. Moreover, the collection of all functions on Γ that we obtain in this way will be shown to be a semigroup of the type studied in [Wa]: a ***signed Hibi cone***. The structure of such semigroups is well understood, and we shall use it to study the structure of the algebra $\mathcal{R}$. A review on some basic facts about signed Hibi cones is given in the Appendix.

For $1 \le s \le p$ and $1 \le t \le n$, let

$$\gamma_t^{(s)} = \begin{pmatrix} s-p+n-t \\ n-t \end{pmatrix} = (p-s)\begin{pmatrix} -1 \\ 0 \end{pmatrix} + (n-t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{Z}^2.$$

Then the set

$$\Gamma^+_{n,p} := \left\{ \gamma_t^{(s)} \ : \ 1 \le s \le p,\ 1 \le t \le n \right\} \tag{3.5.1}$$

is a parallelogram of integer points in the plane.

Similarly, for $1 \le i \le q$ and $1 \le j \le n$, let

$$\eta_j^{(i)} = \begin{pmatrix} n-j \\ n-i-j \end{pmatrix} = i\begin{pmatrix} 0 \\ -1 \end{pmatrix} + (n-j)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{Z}^2,$$

and the set

$$\Gamma^-_{n,q} := \left\{ \eta_j^{(i)} \ : \ 1 \le i \le q,\ 1 \le j \le n \right\} \tag{3.5.2}$$

is another parallelogram of integer points in the plane.

We now take the union of these two parallelograms of integer points, i.e., we consider the set Γ of points defined by

$$\Gamma = \Gamma_{n,p,q} := \Gamma^+_{n,p} \cup \Gamma^-_{n,q}. \tag{3.5.3}$$

The sets $\Gamma^+_{5,3}$, $\Gamma^-_{5,4}$ and $\Gamma_{(5,3,4)}$ are illustrated in Figure 1.

We let $\mathbb{Z}^2$ be given the usual partial order

$$\begin{pmatrix} a \\ b \end{pmatrix} \leq \begin{pmatrix} c \\ d \end{pmatrix} \text{ if and only if } a \leq c \text{ and } b \leq d,$$

and we consider Γ as a poset with the induced partial ordering. In particular, we have

(i) $\gamma_{t-1}^{(s)} \geq \gamma_t^{(s+1)} \geq \gamma_t^{(s)}$ for $1 \leq s \leq p-1$ and $2 \leq t \leq n$;

(ii) $\eta_j^{(i)} \geq \eta_j^{(i+1)} \geq \eta_{j+1}^{(i)}$ for $0 \leq i \leq q-1$ and $1 \leq j \leq n-1$, where $\eta_t^{(0)} = \gamma_t^{(p)}$ for $1 \leq t \leq n$.

An alternative way to visualize the partial ordering in Γ is to arrange its elements as in a Gelfand-Tsetlin pattern, as illustrated below. In this pattern, each element dominates the two elements immediately to its right, in the rows just above and below it. In other words, the elements along any diagonal decrease to the right. (This also implies that they decrease to the right along any row. Thus, any two adjacent rows constitute a poset such as illustrated in the remark after Definition 3.3.1, and the sequences in these rows are interlaced.)

$$\begin{array}{ccccccccccc}
 & & & \gamma_1^{(1)} & & \gamma_2^{(1)} & & \cdots & & \gamma_n^{(1)} & \\
 & & \gamma_1^{(2)} & & \gamma_2^{(2)} & & \cdots & & \gamma_n^{(2)} & & \\
 & \iddots & & \iddots & & & & \iddots & & & \\
\gamma_1^{(p)} & & \gamma_2^{(p)} & & \cdots & & \gamma_n^{(p)} & & & & \\
 & \eta_1^{(1)} & & \eta_2^{(1)} & & \cdots & & \eta_n^{(1)} & & & \\
 & & \eta_1^{(2)} & & \eta_2^{(2)} & & \cdots & & \eta_n^{(2)} & & \\
 & & & \ddots & & \ddots & & & & \ddots & \\
 & & & & \eta_1^{(q)} & & \eta_2^{(q)} & & \cdots & & \eta_n^{(q)}.
\end{array} \tag{3.5.4}$$

Next, we consider the signed Hibi cone (see Appendix or [Wa])

$$\mathbb{Z}^{\Gamma,\geq} := \{f : \Gamma \to \mathbb{Z} |\ f \text{ is order preserving}\}$$

and let

$$\Omega = \Omega_{n,p,q} = \left\{ f \in \mathbb{Z}^{\Gamma,\geq} : f(\gamma_1^{(1)}) \geq 0,\ f(\gamma_2^{(1)}) = \cdots = f(\gamma_n^{(1)}) = 0 \right\}.$$

Then

$$\Omega = \Omega_{A,B}(\Gamma) = \{f \in \mathbb{Z}^{\Gamma,\geq} :\ f(A) \geq 0,\ f(B) \leq 0\}.$$

where

$$A = \{\gamma_n^{(1)}\}, \quad B = \{\gamma_2^{(1)}\}.$$

So Ω is also a signed Hibi cone (see Definition A1(v) of the Appendix). Observe that for $f \in \mathbb{Z}^{\Gamma,\geq}$, $f \in \Omega$ if and only if f takes nonnegative values on all the elements of Γ which are on the diagonal $\gamma_k^{(k)}$ $(1 \leq k \leq \min(n,p))$, and 0 on all elements above this diagonal.

Now we want to define certain subsets of Ω based on some boundary conditions. For $f \in \mathbb{Z}^{\Gamma,\geq}$, let

$$f^{(s)}(\gamma) = (f(\gamma_1^{(s)}), \ldots, f(\gamma_n^{(s)})) \qquad \text{for} \qquad 1 \leq s \leq p,$$
$$f^{(t)}(\eta) = (f(\eta_1^{(t)}), \ldots, f(\eta_n^{(t)})) \qquad \text{for} \qquad 1 \leq t \leq q.$$

Then $f^{(s)}(\gamma), f^{(t)}(\eta) \in \Lambda_n^+$ for $1 \leq s \leq p$, $1 \leq t \leq q$. We also let

$$\text{wt}(f) = (\omega_1, \omega_2)$$

where

$$\begin{aligned}\omega_1 &= (|f^{(1)}(\gamma)|, |f^{(2)}(\gamma)| - |f^{(1)}(\gamma)|, \cdots, |f^{(p)}(\gamma)| - |f^{(p-1)}(\gamma)|),\\ \omega_2 &= (|f^{(1)}(\eta)| - |f^{(p)}(\gamma)|, |f^{(2)}(\eta)| - |f^{(1)}(\eta)|, \cdots, |f^{(q)}(\eta)| - |f^{(q-1)}(\eta)|),\end{aligned}$$

and for $\nu = (\nu_1, ..., \nu_n) \in \Lambda_n^+$, $|\nu| = \nu_1 + \cdots + \nu_n$, as defined in Definition (3.3.1) (ii). For $(\lambda, \alpha, \beta) \in \Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$, we let

$$\Omega_{(\lambda,\alpha,\beta)} = \{f \in \Omega_{n,p,q} : f^{(q)}(\eta) = \lambda, \mathrm{wt}(f) = (\alpha, -\beta)\}.$$

Proposition 3.5.5. (i) *The signed Hibi cone Ω can be written as a disjoint union*

$$\Omega = \bigcup_{(\lambda,\alpha,\beta)\in\Lambda_n^+\times(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q} \Omega_{(\lambda,\alpha,\beta)}.$$

(ii) *For $(\lambda, \alpha, \beta) \in \Lambda_n^+ \times (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$, the cardinality of $\Omega_{(\lambda,\alpha,\beta)}$ is given by*

$$\#(\Omega_{(\lambda,\alpha,\beta)}) = K_{(\lambda,\alpha,\beta)}$$

where $K_{(\lambda,\alpha,\beta)}$ is defined in Proposition 3.3.5.

Proof. (i) It is clear that the union is disjoint, so it remains to show that every element of Ω is contained in some $\Omega_{(\lambda,\alpha,\beta)}$.

Let $f \in \Omega$. Since f is order-preserving,

$$f^{(1)}(\gamma) \sqsubseteq f^{(2)}(\gamma) \sqsubseteq \ldots \sqsubseteq f^{(p)}(\gamma) \sqsupseteq f^{(1)}(\eta) \sqsupseteq f^{(2)}(\eta) \sqsupseteq \ldots \sqsupseteq f^{(q)}(\eta)$$

so that

$$|f^{(1)}(\gamma)| \le |f^{(2)}(\gamma)| \le \ldots \le |f^{(p)}(\gamma)| \ge |f^{(1)}(\eta)| \ge |f^{(2)}(\eta)| \ge \ldots \ge |f^{(q)}(\eta)|.$$

We now let

$$\lambda = f^{(q)}(\eta), \quad \alpha = (\alpha_1, ..., \alpha_p) \quad \text{and} \quad \beta = (\beta_1, ..., \beta_q)$$

where

$$\alpha_i = \begin{cases} |f^{(1)}(\gamma)| & i = 1 \\ |f^{(i)}(\gamma)| - |f^{(i-1)}(\gamma)| & 2 \le i \le p, \end{cases}$$

and

$$\beta_j = \begin{cases} -(|f^{(1)}(\eta)| - |f^{(p)}(\gamma)|) & j = 1 \\ -(|f^{(j)}(\eta)| - |f^{(j-1)}(\eta)|) & 2 \le j \le q. \end{cases}$$

Then $\mathrm{wt}(f) = (\alpha, -\beta)$. So $f \in \Omega_{(\lambda,\alpha,\beta)}$.

(ii) For $f \in \Omega_{(\lambda,\alpha,\beta)}$, define $\mu(f) = (\mu^{(i)})_{1\le i\le p+q}$ by

$$\mu^{(s)} = f^{(s)}(\gamma),\ \mu^{(p+t)} = f^{(t)}(\eta) \quad \text{for } 1 \le s \le p,\ 1 \le t \le q,$$

and we claim that $\mu(f) \in S(\lambda, \alpha, \beta)$, that is, $\mu(f)$ satisfies Conditions (i)-(iv) given in Notation 3.3.3. Note that

$$0 \ge f(\gamma_2^{(1)}) \ge f(\gamma_3^{(1)}) \ge \cdots \ge f(\gamma_n^{(1)}) \ge 0,$$

so

$$f(\gamma_2^{(1)}) = f(\gamma_3^{(1)}) = \cdots = f(\gamma_n^{(1)}) = 0,$$

and

$$\mu^{(1)} = (f(\gamma_1^{(1)}), 0, ..., 0).$$

So $\mu(f)$ satisfies condition (i) in Notation 3.3.3. The remaining conditions can be checked similarly as in Part (i).

We can now define a map from $\Omega_{(\lambda,\alpha,\beta)}$ to the set of sequences in $(\Lambda_n^+)^{p+q}$ satisfying conditions (i)-(iv) of Proposition 3.3.5 by $f \to \mu(f)$. It is easy to check that this map is bijective. □

Proposition 3.5.5 suggests that the element of $\Omega_{(\lambda,\alpha,\beta)}$ may be used to label the elements of a basis for $\mathcal{R}_{(\lambda,\alpha,\beta)}$. So the elements of the signed Hibi cone Ω may be used to label the elements of a basis for the algebra $\mathcal{R}$.

Our next step is to determine the generators of the semigroup Ω using the general results in the Appendix. Recall that the signed Hibi cone $\Omega = \Omega_{A,B}(\Gamma)$ is determined by the subsets $A = \{\gamma_n^{(1)}\}$ and $B = \{\gamma_2^{(1)}\}$ of Γ. Let P_A be the smallest increasing subset of Γ which contains A, and let N_B be the smallest decreasing subset of Γ which contains B. Let $\Gamma^+ = \Gamma \backslash P_A$ and $\Gamma^- = \Gamma \backslash N_B$. We shall regard Γ^+ and Γ^- as subposets of Γ. The following lemma gives an explicit description of Γ^+ and Γ^-.

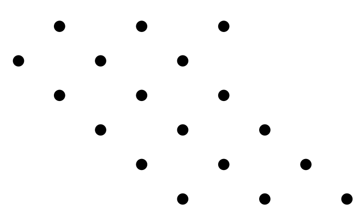
Lemma 3.5.6. *We have*

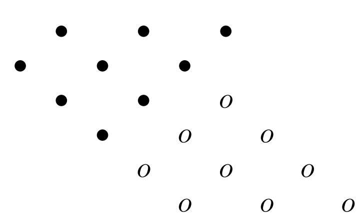
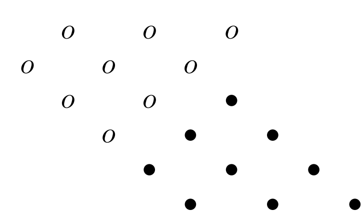
$$\Gamma^+ = \left\{ \gamma_t^{(s)}, \eta_j^{(i)} : 1 \le s \le p,\ 1 \le t \le \min(s,n),\ 0 \le i \le q,\ 1 \le j \le \min(p,n) \right\}$$

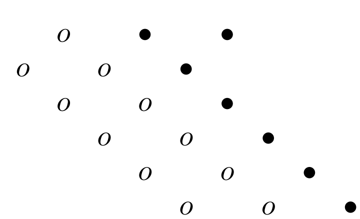
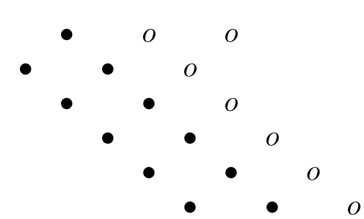
and

$$\Gamma^- = \left\{ \eta_t^{(s)} : 1 \le s \le q,\ \max(1, n+1-s) \le t \le n \right\}.$$

We omit the proof of Lemma 3.5.6. Instead, we give the example for $n = 3$, $p = 2$ and $q = 4$ below where the elements of the posets are arranged according to the convention of a Gelfand-Tsetlin pattern (see equation (3.5.4)). The elements of the posets are represented by black circles in the following diagrams:

$\Gamma =$ 

$P_A =$  , $\Gamma^- =$ 

$N_B =$  , $\Gamma^+ =$ 

Definition 3.5.7. (a) For a finite set $H \subseteq \mathbb{Z}$ and an integer s, let

$$H_{\leq s} = \{h \in H : h \leq s\}.$$

(b) For $I \subseteq \{1, ..., p\}$ and $J \subseteq \{1, ..., q\}$ such that $I \cup J \neq \emptyset$ and $n \geq \#(I) \geq \#(J)$, let

$$A(I, J) = \left\{\gamma_t^{(s)} : 1 \leq t \leq a_s, 1 \leq s \leq p\right\} \cup \left\{\eta_j^{(i)} : 1 \leq j \leq a_i', 0 \leq i \leq q\right\},$$

where $a_s = \#(I_{\leq s})$ and $a_i' = \#(I) - \#(J_{\leq i})$.

(c) For $L \subseteq \{1, ..., q\}$ such that $n \geq \#(L) \geq 1$, let

$$B(L) = \{\eta_t^{(s)} : b_s \leq t \leq n \text{ for } 1 \leq s \leq q\},$$

where $b_s = n + 1 - \#(L_{\leq s})$.

The following lemma states that all the increasing subsets of Γ^+ are of the form $A(I, J)$ and all the decreasing subsets of Γ^- are of the form $B(L)$. We omit its proof because the arguments are standard. For example, it is similar to the proof of Lemma 5.2 of [KL1].

Lemma 3.5.8. (i) *The collection of all increasing subsets of* Γ^+ *is given by*

$$J^*(\Gamma^+, \geq) = \{A(I, J) : I \subseteq \{1, ..., p\},\ J \subseteq \{1, ..., q\},\ n \geq \#(I) \geq \#(J),\ \#(I) + \#(J) \geq 1\}.$$

(ii) *The collection of all decreasing subsets of* Γ^- *is given by*

$$J_*(\Gamma^-, \geq) = \{B(L) : L \subseteq \{1, 2, ..., q\},\ 1 \leq \#(L) \leq n\}.$$

Following the general results in the Appendix, we let $\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-$ where

$$\mathcal{G}^+ = \{\chi_A : A \in J^*(\Gamma^+, \geq)\} \quad \text{and} \quad \mathcal{G}^- = \{-\chi_B : B \in J_*(\Gamma^-, \geq)\},$$

and define a partial ordering $\succeq$ on $\mathcal{G}$ as follows:

(i) For $A(I_1, J_1), A(I_2, J_2) \in J^*(\Gamma^+, \geq)$, $\chi_{A(I_1,J_1)} \preceq \chi_{A(I_2,J_2)}$ if and only if $A(I_1, J_1) \subseteq A(I_2, J_2)$.
(ii) For $B(L_1), B(L_2) \in J_*(\Gamma^-, \geq)$, $-\chi_{B(L_1)} \preceq -\chi_{B(L_2)}$ if and only if $B(L_1) \supseteq B(L_2)$;
(iii) For $A(I, J) \in J^*(\Gamma^+, \geq)$ and $B_L \in J_*(\Gamma^-, \geq)$, $\chi_{A(I,J)} \preceq -\chi_{B(L)}$ if and only if $A(I, J) \cap B(L) = \emptyset$.

Proposition 3.5.9. *The semigroup* Ω *is generated by* $\mathcal{G}$. *Moreover, each nonzero element* f *of* Ω *can be expressed uniquely as*

$$f = \sum_{i=1}^{s} a_i \chi_{A(I_i,J_i)} + \sum_{j=1}^{t} b_j(-\chi_{B(L_j)}),$$

where $a_1, \ldots, a_s, b_1, \ldots, b_t$ *are positive integers and*

$$\chi_{A(I_1,J_1)} \prec \cdots \prec \chi_{A(I_s,J_s)} \prec -\chi_{B(L_t)} \prec \cdots \prec -\chi_{B(L_1)}$$

is a chain in $\mathcal{G}$.

Proof. This follows from applying Theorem A3 to Ω. □

3.6. **A basis for $\mathcal{R}$.** We shall denote a typical element of $W_{n,p,q}$ by

$$(X,Y) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} & y_{11} & y_{12} & \cdots & y_{1q} \\ x_{21} & x_{22} & \cdots & x_{2p} & y_{21} & y_{22} & \cdots & y_{2q} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} & y_{n1} & y_{n2} & \cdots & y_{nq} \end{pmatrix}, \tag{3.6.1}$$

so that the algebra $\mathcal{P}(W_{n,p,q})$ can be regarded as a polynomial algebra on the variables $\{x_{ai}, y_{aj} : 1 \le a \le n, 1 \le i \le p, 1 \le j \le q\}$ and we write

$$\mathcal{P}(W_{n,p,q}) = \mathbb{C}[X,Y]. \tag{3.6.2}$$

For $1 \le i \le p$ and $1 \le j \le q$, let

$$r_{ij} = \sum_{a=1}^{n} x_{ai} y_{aj}.$$

Then $\{r_{ij} : 1 \le i \le p, 1 \le j \le q\}$ generates the algebra $\mathcal{P}(W_{n,p,q})^{\mathrm{GL}_n}$ of GL_n invariants in $\mathcal{P}(W_{n,p,q})$ ([Ho4]).

Recall that by Proposition 3.5.9, the signed Hibi cone Ω is generated by $\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-$, and the elements of $\mathcal{G}$ are in bijective correspondence with $J^*(\Gamma^+) \cup J_*(\Gamma^-)$. We now attach a polynomial function to each element of $J^*(\Gamma^+) \cup J_*(\Gamma^-)$.

Definition 3.6.3. (a) For $n \ge b \ge c$, $I = \{i_1 < i_2 < \cdots < i_b\} \subseteq \{1, ..., p\}$ and $J = \{j_1 < j_2 < \cdots < j_c\} \subseteq \{1, ..., q\}$, let

$$v_{A(I,J)} = \begin{vmatrix} x_{1,i_1} & x_{1,i_2} & \cdots & x_{1,i_b} \\ \vdots & \vdots & & \vdots \\ x_{d,i_1} & x_{d,i_2} & \cdots & x_{d,i_b} \\ r_{i_1,j_c} & r_{i_2,j_c} & \cdots & r_{i_b,j_c} \\ \vdots & \vdots & & \vdots \\ r_{i_1,j_1} & r_{i_2,j_1} & \cdots & r_{i_b,j_1} \end{vmatrix}$$

where $d = b - c$.

(b) For $1 \le k \le n$ and $L = \{l_1 < l_2 < \cdots < l_k\} \subseteq \{1, ..., q\}$, let

$$v_{B(L)} = \begin{vmatrix} y_{h,l_k} & y_{h,l_{k-1}} & \cdots & y_{h,l_1} \\ y_{h+1,l_k} & y_{h+1,l_{k-1}} & \cdots & y_{h+1,l_1} \\ \vdots & \vdots & & \vdots \\ y_{n,l_k} & y_{n,l_{k-1}} & \cdots & y_{n,l_1} \end{vmatrix}$$

where $h = n - k + 1$.

Next, for each nonzero element f of Ω, we shall define a polynomial v_f and a monomial m_f in $\mathcal{P}(W_{n,p,q})$ as follows:

Definition 3.6.4. Let f be a nonzero element of Ω and let

$$f = \sum_{s=1}^{h} a_s \chi_{A(I_s,J_s)} + \sum_{t=1}^{k} b_t(-\chi_{B(L_t)}),$$

be the unique expression of f given in Proposition 3.5.9. Define the polynomial v_f and the monomial m_f in $\mathcal{P}(W_{n,p,q})$ by

$$v_f := \prod_{s=1}^{h} v_{A(I_s,J_s)}^{a_s} \prod_{t=1}^{k} v_{B(L_t)}^{b_t}$$

and

$$m_f := \prod_{\substack{1\le s\le n\\ 1\le t\le p}} x_{st}^{f(\gamma_s^{(t)})-f(\gamma_s^{(t-1)})} \prod_{\substack{1\le i\le n\\ 1\le j\le q}} y_{ij}^{f(\eta_i^{(j-1)})-f(\eta_i^{(j)})}.$$

We now define an ordering on the monomials in $\mathcal{P}(W_{n,p,q})$.

Definition 3.6.5. Let τ_1 be the graded lexicographic order ([CLO]) with respect to the following ordering on the variables:

(a) $x_{ab} > x_{cd}$ iff either (i) $b < d$ or (ii) $b = d$ and $a < c$.
(b) Similarly, $y_{ab} > y_{cd}$ iff either (i) $b < d$ or (ii) $b = d$ and $a > c$.
(c) Finally, $x_{ab} > y_{cd}$ for all pairs (a, b) and (c, d) of indices.

That is,

$$x_{11} > x_{21} > \cdots > x_{n1} > x_{12} > \cdots > x_{np}$$
$$> y_{n1} > y_{(n-1)1} > \cdots > y_{11} > y_{n2} > \cdots > y_{1q}.$$

So the x variables decrease down each column, and all entries in a given column dominate all entries in columns to the right. For the y variables, the variables increase down each column, and as with the xs, the variable in a given column dominate the variables in all columns to the right.

If p is a nonzero polynomial function in $\mathcal{P}(W_{n,p,q})$, we shall denote its leading monomial with respect to τ_1 by $\mathrm{LM}_{\tau_1}(p)$.

Lemma 3.6.6. *If f is a nonzero element of Ω, then*

$$\mathrm{LM}_{\tau_1}(v_f) = m_f.$$

Proof. This is similar to the proof of Theorem 6.3.1 and Theorem 6.3.2 of [Wa]. □

Theorem 3.6.7. (i) *For $(\lambda, \alpha, \beta) \in \Lambda(\mathcal{R})$, the set*

$$\mathcal{B}_{(\lambda,\alpha,\beta)} := \{v_f : \ f \in \Omega_{(\lambda,\alpha,\beta)}\}$$

is a basis for $\mathcal{R}_{(\lambda,\alpha,\beta)}$.

(ii) *The set*

$$\mathcal{B}_{\mathcal{R}} := \bigcup_{(\lambda,\alpha,\beta)\in\Lambda(\mathcal{R})} \mathcal{B}_{(\lambda,\alpha,\beta)}$$

is a basis for $\mathcal{R}$.

Proof. Fix $(\lambda, \alpha, \beta) \in \Lambda(\mathcal{R})$. It is easy to verify that $\mathcal{B}_{(\lambda,\alpha,\beta)} \subseteq \mathcal{R}_{(\lambda,\alpha,\beta)}$. Moreover, by Lemma 3.6.6, the polynomials v_f in $\mathcal{B}_{(\lambda,\alpha,\beta)}$ have distinct leading monomials with respect to the monomial ordering τ_1. So $\mathcal{B}_{(\lambda,\alpha,\beta)}$ is linearly independent. Since we also have

$$\dim \mathcal{R}_{(\lambda,\alpha,\beta)} = \#(\mathcal{B}_{(\lambda,\alpha,\beta)}) = K_{(\lambda,\alpha,\beta)},$$

$\mathcal{B}_{(\lambda,\alpha,\beta)}$ is a basis for $\mathcal{R}_{(\lambda,\alpha,\beta)}$. This proves (i).

Part (ii) follows from (i) and equation (3.4.2). □

Remark 3.6.8. The algebra $\mathcal{R}$ has some properties which are similar to the algebras we have studied in [HKL, KL1, Wa]. We summarize these properties below:

(a) Let

$$\tilde{\mathcal{G}} = \{v_f : \ f \in \mathcal{G}\}.$$

So each element v_f of $\tilde{\mathcal{G}}$ is either of the form $v_{A(I,J)}$ or $v_{B(L)}$. We define a partial ordering $\succeq$ on $\tilde{\mathcal{G}}$ as follows: For $v_{f_1}, v_{f_2} \in \tilde{\mathcal{G}}$,

$$v_{f_1} \succeq v_{f_2} \quad \text{if and only if} \quad f_1 \succeq f_2 \text{ in } \mathcal{G}.$$

Then for each nonzero element f of Ω,

$$v_f = v_{f_1} \cdots v_{f_s}$$

for some $v_{f_1}, ..., v_{f_s} \in \tilde{\mathcal{G}}$ such that $v_{f_1} \preceq \cdots \preceq v_{f_s}$. In other words, v_f is a ***standard monomial*** on $\tilde{\mathcal{G}}$. In this case, we say that $\mathcal{R}$ has a ***standard monomial theory*** with respect to $\tilde{\mathcal{G}}$.

(b) The subalgebra of $\mathcal{P}(W_{n,p,q})$ generated by the set of leading monomials

$$\mathrm{LM}_{\tau_1}(\mathcal{R}) = \{\mathrm{LM}_{\tau_1}(h) : \ h \in \mathcal{R}\backslash\{0\}\}$$

is called the ***initial algebra*** of $\mathcal{R}$. It is easy to check that $\mathrm{LM}_{\tau_1}(\mathcal{R})$ is a semigroup with respect to the product of monomials, and it is isomorphic to Ω. Hence the initial algebra of $\mathcal{R}$ is isomorphic to the semigroup algebra $\mathbb{C}[\Omega]$ on Ω. Since Ω is a signed Hibi cone, we call $\mathbb{C}[\Omega]$ a ***signed Hibi ring***.

(c) Since the semigroup Ω is finitely generated, by a general result of [CHV], there exists a flat one-parameter family of complex algebras with general fiber $\mathcal{R}$ and special fiber $\mathbb{C}[\Omega]$.

3.7. **Bases for highest weight modules.** In this subsection, we shall show that $\mathcal{R}$ is a direct sum of a family of irreducible representations of the complex general linear algebra $\mathfrak{gl}_{p+q}$, and we use the results of the preceding sections to obtain a basis for each of these representations.

Let $\mathcal{PD}(W_{n,p,q})$ be the algebra of all polynomial-coefficient differential operators on $W_{n,p,q}$ ([Ho4]). Consider the following differential operators in $\mathcal{PD}(W_{n,p,q})$:

$$r_{ab} = \sum_{i=1}^{n} x_{ia}y_{ib}, \quad \Delta_{ab} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_{ia}\partial y_{ib}} \quad (1 \le a \le p, 1 \le b \le q)$$

$$E_{ij}^{(x)} = \sum_{s=1}^{n} x_{si}\frac{\partial}{\partial x_{sj}} \quad (1 \le i,j \le p), \quad E_{kl}^{(y)} = \sum_{s=1}^{n} y_{sk}\frac{\partial}{\partial y_{sl}} \quad (1 \le k,l \le q).$$

Let $\mathfrak{g}'$ be the Lie algebra spanned by the set of differential operators

$$\{r_{ab}, \Delta_{ab} : \ 1 \le a \le p, 1 \le b \le q\} \cup \left\{E_{ij}^{(x)} + \frac{n}{2}\delta_{ij} : \ 1 \le i,j \le p\right\} \cup \left\{E_{ij}^{(y)} + \frac{n}{2}\delta_{ij} : \ 1 \le i,j \le q\right\}.$$

Let $\mathfrak{gl}_{p+q} = \mathfrak{gl}_{p+q}(\mathbb{C})$ be the complex general linear algebra. For $1 \le i,j \le p+q$, let e_{ij} be the element of $\mathfrak{gl}_{p+q}$ with 1 at its (i,j)-th entry and 0 elsewhere. Let $\omega : \mathfrak{gl}_{p+q} \to \mathfrak{g}'$ be the linear map such that

$$\begin{aligned}
\omega(e_{i,j}) &= -E^{(x)}_{ji} - \frac{n}{2}\delta_{ij} \quad (1 \le i, j \le p) \\
\omega(e_{p+i,p+j}) &= E^{(y)}_{ij} + \frac{n}{2}\delta_{ij} \quad (1 \le i, j \le q) \\
\omega(e_{i,p+j}) &= \Delta_{ij} \quad (1 \le i \le p, 1 \le j \le q) \\
\omega(e_{p+i,j}) &= -r_{ji} \quad (1 \le i \le q, 1 \le j \le p).
\end{aligned} \tag{3.7.1}$$

Then ω is a Lie algebra isomorphism, and so it defines a representation of $\mathfrak{gl}_{p+q}$ on $\mathcal{P}(W_{n,p,q})$.

The action by $\mathfrak{gl}_{p+q}$ on $\mathcal{P}(W_{n,p,q})$ commutes with that by GL_n. Recall from equation (1.0.1) that under the joint action by $\mathrm{GL}_n \times \mathfrak{gl}_{p+q}$, $\mathcal{P}(W_{n,p,q})$ can be decomposed as

$$\mathcal{P}(W_{n,p,q}) = \bigoplus_{\lambda \in \Lambda^+_n(p,q)} \rho^\lambda_n \otimes \pi^\lambda_{p,q}$$

where

(i) $\Lambda^+_n(p,q)$ is the set of all $\lambda = (D,E) \in \Lambda^+_n$ such that $\mathbf{r}(D) \le p$, $\mathbf{r}(E) \le q$ and $\mathbf{r}(D) + \mathbf{r}(E) \le n$; and

(ii) $\pi^\lambda_{p,q}$ is an irreducible infinite dimensional representation of $\mathfrak{gl}_{p+q}$. (In fact, $\pi^\lambda_{p,q}$ is a highest weight module of $\mathfrak{gl}_{p+q}$.)

By extracting the U_n invariants in the direct sum decomposition of $\mathcal{P}(W_{n,p,q})$ given above, we see that the algebra $\mathcal{R}$ is a module for $A_n \times \mathfrak{gl}_{p+q}$, and that it decomposes as a direct sum

$$\mathcal{R} = \mathcal{P}(W_{n,p,q})^{U_n} = \bigoplus_{\lambda \in \Lambda^+_n(p,q)} (\rho^\lambda_n)^{U_n} \otimes \pi^\lambda_{p,q}. \tag{3.7.2}$$

For each $\lambda \in \Lambda^+_n(p,q)$, $(\rho^\lambda_n)^{U_n} \otimes \pi^\lambda_{p,q}$ is the ψ^λ_n-eigenspace of A_n in $\mathcal{R}$. On the other hand, since $\dim(\rho^\lambda_n)^{U_n} = 1$,

$$(\rho^\lambda_n)^{U_n} \otimes \pi^\lambda_{p,q} \cong \pi^\lambda_{p,q}. \tag{3.7.3}$$

So $\pi^\lambda_{p,q}$ can be realized as the ψ^λ_n-eigenspace of A_n in $\mathcal{R}$.

Next we let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{gl}_{p+q}$ consisting of all diagonal matrices in $\mathfrak{gl}_{p+q}$. For $(\alpha,\beta) \in (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q$, let $\phi_{\alpha,\beta} : \mathfrak{h} \to \mathbb{C}$ be defined by

$$\phi_{\alpha,\beta}(h) = -\sum_{i=1}^{p}(\alpha_i + \frac{n}{2})h_i + \sum_{j=1}^{q}(\beta_j + \frac{n}{2})h_j \quad \text{for } h = \textstyle\sum_{i=1}^{p} h_i e_{ii} \in \mathfrak{h}, \tag{3.7.4}$$

and let

$$(\pi^\lambda_{p,q})_{(\alpha,\beta)} = \left\{ v \in \pi^\lambda_{p,q} : \ h.v = \phi_{\alpha,\beta}(h)v \ \forall h \in \mathfrak{h} \right\}.$$

Theorem 3.7.5. *Let $\lambda \in \Lambda^+_n(p,q)$, and let*

$$\Lambda(\pi^\lambda_{p,q}) = \{(\alpha,\beta) \in (\mathbb{Z}^+)^p \times (\mathbb{Z}^+)^q : \ K_{(\lambda,\alpha,\beta)} \ne 0\}.$$

(i) *The representation $\pi^\lambda_{p,q}$ of $\mathfrak{gl}_{p+q}$ is decomposed into weight spaces as*

$$\pi^\lambda_{p,q} = \bigoplus_{(\alpha,\beta) \in \Lambda(\pi^\lambda_{p,q})} (\pi^\lambda_{p,q})_{\alpha,\beta}.$$

(ii) *For each $(\alpha,\beta) \in \Lambda(\pi^\lambda_{p,q})$, the set $\mathcal{B}_{(\lambda,\alpha,\beta)}$ defined in Theorem 3.6.7 is a basis for $(\pi^\lambda_{p,q})_{\alpha,\beta}$.*

(iii) *The set*

$$\mathcal{B}_\lambda = \bigcup_{(\alpha,\beta)\in(\mathbb{Z}^+)^p\times(\mathbb{Z}^+)^q} \mathcal{B}_{(\lambda,\alpha,\beta)}$$

is a basis for $\pi_{p,q}^\lambda$.

Proof. We have

$$(\pi_{p,q}^\lambda)_{\alpha,\beta} \cong (\rho_n^\lambda)^{U_n} \otimes (\pi_{p,q}^\lambda)_{\alpha,\beta} = \mathcal{R}_{(\lambda,\alpha,\beta)}. \tag{3.7.6}$$

The theorem follows from this and Theorem 3.6.7. □

Remark 3.7.7. It is well known that every irreducible polynomial representation of GL_n has a standard monomial basis ([Kim, Lee]), and it is proved in [KL2] that every irreducible rational representation of GL_n also has a standard monomial basis. Theorem 3.7.5 can be viewed as an extension of this result to the representations $\pi_{p,q}^\lambda$ of $\mathfrak{gl}_{p+q}$.

We now present two examples.

Example 3.7.8. Let $n \geq 2$ and $p = q = 1$. Then

$$W_{n,1,1} = \left\{ \begin{pmatrix} x_{11} & y_{11} \\ \vdots & \vdots \\ x_{n1} & y_{n1} \end{pmatrix} : \ x_{ij}, y_{ij} \in \mathbb{C}\ \forall i,j \right\}, \text{ and } r_{11} = x_{11}y_{11} + \cdots + x_{n1}y_{n1}.$$

In this case, $\mathcal{R}$ has the following decomposition as an $A_n \times \mathfrak{gl}_2$ module:

$$\mathcal{R} = \mathcal{P}(W_{n,1,1})^{U_n} = \bigoplus_{a,b\in\mathbb{Z}_{\geq 0}} \left(\rho_n^{(a,0,\dots,0,-b)}\right)^{U_n} \otimes \pi_{1,1}^{(a,0,\dots,0,-b)}.$$

Moreover, the algebra $\mathcal{R}$ is freely generated by x_{11}, r_{11} and y_{n1}. For $a, b \in \mathbb{Z}^+$, the set

$$\mathcal{B}_{(a,0,\dots,0,-b)} = \left\{ x_{11}^a y_{n1}^b r_{11}^c : \ c \in \mathbb{Z}^+ \right\}.$$

is a basis for $\pi_{1,1}^{(a,0,\dots,0,-b)}$.

Example 3.7.9. Let $n = 2$, $p = 1$ and $q = 2$. Then

$$W_{2,1,2} = \left\{ \begin{pmatrix} x_{11} & y_{11} & y_{12} \\ x_{21} & y_{21} & y_{32} \end{pmatrix} : \ x_{ij}, y_{ij} \in \mathbb{C}\ \forall i,j \right\}, \text{ and } \left\{ \begin{array}{l} r_{11} = x_{11}y_{11} + x_{21}y_{21}, \\ r_{12} = x_{11}y_{12} + x_{21}y_{22}. \end{array} \right.$$

The algebra $\mathcal{R} = \mathcal{P}(W_{2,1,2})^{U_2}$ has 6 generators, and the partial ordering on these generator is indicated in the following Hasse diagram:

$$\begin{array}{ccccc}
 & & y_{22} & & \\
 & & | & & \\
 & & y_{21} & & \\
 & / & & \backslash & \\
x_{11} & & & & \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} \\
 & \backslash & & / & \\
 & & r_{12} & & \\
 & & | & & \\
 & & r_{11} & &
\end{array}$$

Moreover, $\mathcal{R}$ has the following decomposition as an $A_2 \times \mathfrak{gl}_3$ module:

$$\mathcal{R} = \left\{ \bigoplus_{a,b\geq 0} \left(\rho_2^{(a,-b)}\right)^{U_2} \otimes \pi_{1,2}^{(a,-b)} \right\} \oplus \left\{ \bigoplus_{b\geq a\geq 0} \left(\rho_n^{(-a,-b)}\right)^{U_2} \otimes \pi_{1,2}^{(-a,-b)} \right\}.$$

For $a, b \in \mathbb{Z}^+$,

$$\mathcal{B}_{(a,-b)} = \left\{ x_{11}^a y_{21}^c y_{22}^d r_{11}^e r_{12}^f : c, d, e, f \in \mathbb{Z}^+, c + d = b \right\}$$

is a basis for $\pi_{1,2}^{(a,-b)}$, and

$$\mathcal{B}_{(-a,-b)} = \left\{ y_{21}^c y_{22}^d \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}^a r_{11}^e r_{12}^f : c, d, e, f \in \mathbb{Z}^+,\ c + d = b - a \right\}$$

is a basis for $\pi_{1,2}^{(-a,-b)}$,

3.8. **The case $n = 1$.** In this subsection, we assume that $n = 1$, $p, q \geq 1$. Then

$$W_{1,p,q} = \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1p} & y_{11} & \cdots & y_{1q} \end{pmatrix} \right\}, \text{ and } r_{ij} = x_{1i} y_{1j} \quad (1 \leq i \leq p, 1 \leq j \leq q).$$

Since U_1 is the trivial group,

$$\mathcal{R} = \mathcal{P}(W_{1,p,q})^{U_1} = \mathcal{P}(W_{1,p,q}) = \mathbb{C}[x_{11}, ..., x_{1p}, y_{11}, ..., y_{1q}].$$

Thus $\mathcal{R}$ is freely generated by $x_{11}, ..., x_{1p}, y_{11}, ..., y_{1q}$. Moreover,

$$\mathcal{R} = \bigoplus_{a\in\mathbb{Z}} \rho_1^{(a)} \otimes \pi_{p,q}^{(a)}$$

For $\alpha = (\alpha_1, ..., \alpha_p) \in (\mathbb{Z}^+)^p$ and $\beta = (\beta_1, ..., \beta_q) \in (\mathbb{Z}^+)^q$,

$$x^\alpha y^\beta = x_{11}^{\alpha_1} \cdots x_{1p}^{\alpha_p} \cdots y_{11}^{\beta_1} \cdots x_{1q}^{\beta_q} \in \pi_{p,q}^{(a)} \quad \text{if and only if} \quad a = |\alpha| - |\beta|.$$

So for each $a \in \mathbb{Z}$, the set

$$\mathcal{B}_{(a)} = \{x^\alpha y^\beta :\ \alpha \in (\mathbb{Z}^+)^p,\ \beta \in (\mathbb{Z}^+)^q,\ |\alpha| - |\beta| = a\}.$$

is a basis for $\pi_{p,q}^{(a)}$.

4. A basis for $\mathcal{Q} = \mathcal{P}(W_{n,p,q})^{U_n\times U_p\times U_q}$

4.1. **The algebra $\mathcal{Q}$.** Recall that the group $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ acts on the polynomial algebra $\mathcal{P}(W_{n,p,q})$ by algebra automorphisms via formula (3.1.1), and we construct a basis for the algebra $\mathcal{R} = \mathcal{P}(W_{n,p,q})^{U_n}$ of U_n invariants in $\mathcal{P}(W_{n,p,q})$ in Section 3. In this section, we consider the algebra

$$\mathcal{Q} = \mathcal{Q}_{n,p,q} := \mathcal{P}(W_{n,p,q})^{U_n\times U_p\times U_q} = \left(\mathcal{P}(W_{n,p,q})^{U_n}\right)^{U_p\times U_q} = \mathcal{R}^{U_p\times U_q}.$$

That is, $\mathcal{Q}$ is the algebra of $U_n \times U_p \times U_q$ invariants in $\mathcal{P}(W_{n,p,q})$, and also the algebra of $U_p \times U_q$ invariants in $\mathcal{R}$. So information on the structure of $\mathcal{Q}$ provides a further refinement of the understanding of $\mathcal{R}$. The algebra $\mathcal{Q}$ is a module for $A_n \times A_p \times A_q$. Like the algebra $\mathcal{R}$, $\mathcal{Q}$ is also a ***reciprocity algebra*** in the sense of [HTW2]. Our main goal in this section is to construct a basis for $\mathcal{Q}$. Using this basis, we also obtain a basis for the highest weight vectors in $\pi_{p,q}^\lambda$ (defined in Section 3.7) with respect to the maximal compact subgroup of $\widetilde{\mathbf{U}}_{p,q}$.

4.2. **The Littlewood-Richardson rule.** We shall describe an algebra grading for $\mathcal{Q}$ by a semigroup $\Lambda(\mathcal{Q})$ defined in §4.3. In order to describe $\Lambda(\mathcal{Q})$, we need an extension of the Littlewood-Richardson rule. This subsection is devoted to establishing this result.

Let us first review the Littlewood-Richardson rule which describes the multiplicities in the tensor product of two irreducible polynomial representations of GL_n. Let $D = (\mu_1, ..., \mu_n)$ and $F = (\lambda_1, ..., \lambda_n)$ be Young diagrams such that D sits inside F, that is, $\mu_i \leq \lambda_i$ for $1 \leq i \leq n$. By removing all boxes belonging to D, we obtain the ***skew diagram*** F/D. If we put a positive number in each box of F/D, then it becomes a ***skew tableau*** and we say that the ***shape*** of this skew tableau is F/D. If the entries of this skew tableau are taken from $\{1, 2, ..., m\}$, and μ_j of them are j for $1 \leq j \leq m$, then we say the ***content*** of this skew tableau is $E = (\mu_1, ..., \mu_m)$. If T is a skew tableau, then the ***word*** of T is the sequence $w(T)$ of positive integers obtained by reading the entries of T from top to bottom, and right to left in each row. For example,

$$T = \begin{array}{|c|c|c|c|c|c|} \hline & & & 1 & 1 & 2 \\ \hline & & 2 & 3 & & \\ \hline & 1 & 3 & 4 & & \\ \hline 1 & 2 & & & & \\ \hline \end{array}$$

is a skew tableau of shape F/D and content E where $D = (3, 2, 1)$, $F = (6, 4, 4, 2)$ and $E = (4, 3, 2, 1)$, and its word is given by

$$w(T) = (2, 1, 1, 3, 2, 4, 3, 1, 2, 1).$$

Definition 4.2.1. A ***Littlewood-Richardson*** (LR) tableau is a skew tableau T with the following properties:

(i) It is semistandard, that is, the numbers in each row of T weakly increase from left-to-right, and the numbers in each column of T strictly increase from top-to-bottom.

(ii) It satisfies the Yamanouchi word condition, that is, for each positive integer j, starting from the first entry of $w(T)$ to any place in $w(T)$, there are at least as many js as $(j + 1)$s.

Notation 4.2.2. (a) For Young diagrams D, E and F, we denote the set of all LR tableaux of shape F/D and content E by $\mathrm{LRTab}(F, D, E)$.

(b) For $\lambda, \mu, \nu \in \Lambda_n^+$, let $c_{\mu,\nu}^{\lambda}$ denote the multiplicity of ρ_n^{λ} in the tensor product $\rho_n^{\mu} \otimes \rho_n^{\nu}$, that is,

$$c_{\mu,\nu}^{\lambda} = \dim \mathrm{Hom}_{\mathrm{GL}_n}(\rho_n^{\lambda}, \rho_n^{\mu} \otimes \rho_n^{\nu}).$$

Theorem 4.2.3. (The Littlewood-Richardson (LR) rule) ([Fu, HL2]) *If D,E and F are Young diagrams with at most n rows, then the multiplicity $c_{D,E}^F$ of ρ_n^F in the tensor product $\rho_n^D \otimes \rho_n^E$ is equal the number of LR tableaux of shape F/D and content E, that is,*

$$c_{D,E}^F = \#(\mathrm{LRTab}(F, D, E)).$$

To describe our algebra $\mathcal{Q}$, we need to extend the LR Rule to not-necessarily polynomial representations. In some sense, this is a simple process, since it only requires twisting representations by powers of the determinant character. However, we also want to adapt the resulting description to the context of signed Hibi cones. An important ingredient in doing this is the description of LR tableaux in terms of ***Littlewood-Richardson triangles***, as given in [PV].

Definition 4.2.4. A ***Littlewood-Richardson (LR) triangle*** ([PV]) of size n is a triangular array of real numbers

$$A = [a_{ij}]_{0\le i\le j\le n} = \left(\begin{array}{ccccccccccc} & & & & & a_{00} & & & & & \\ & & & & a_{01} & & a_{11} & & & & \\ & & & a_{02} & & a_{12} & & a_{22} & & & \\ & & a_{03} & & a_{13} & & a_{23} & & a_{33} & & \\ & \cdot^{\cdot^{\cdot}} & & \dots & & \dots & & \dots & & \ddots & \\ a_{0n} & & a_{1n} & & \dots & & \dots & & \dots & & a_{nn} \end{array}\right)$$

which satisfies the following conditions:

(P) $a_{00} = 0$ and $a_{ij} \ge 0$ for all $1 \le i < j \le n$.
(CS) $\sum_{p=0}^{i-1} a_{pj} \ge \sum_{p=0}^{i} a_{p,j+1}$ for all $1 \le i \le j+1 \le n$.
(LR) $\sum_{q=i}^{j} a_{iq} \ge \sum_{q=i+1}^{j+1} a_{i+1,q}$ for all $1 \le i \le j < n$.

Remark 4.2.5. Note that, although all the entries below the top diagonal on either side of the triangle are stipulated to be non-negative in condition (P), these top entries could possibly be negative. In the standard LR rule for polynomial representations, they will be non-negative, but for the general case, they may take negative values.

If we identify each LR triangle $A = [a_{ij}]_{0\le i\le j\le n}$ with the point

$$(a_{01}, a_{11}, a_{02}, a_{12}, a_{22}, ..., a_{nn})$$

of $\mathbb{R}^N$ where $N = n(n+3)/2$, then the three conditions defining LR triangles translate to linear inequalities on $\mathbb{R}^N$, so the LR triangles with integer entries constitute the integer points in a rational polyhedral cone in $\mathbb{R}^N$, which we will call the ***Littlewood-Richardson (LR) cone***. It is a ***lattice cone*** in the sense of [Ho5]. In particular, it is a semigroup with respect to vector addition.

Notation 4.2.6. We shall denote the semigroup of all LR triangles of size n with integer entries by $\mathcal{LR}_n$.

Definition 4.2.7. If $A = [a_{ij}]_{0\le i\le j\le n} \in \mathcal{LR}_n$, define $\mu = (\mu_1, ..., \mu_n)$, $\nu = (\nu_1, ..., \nu_n)$ and $\lambda = (\lambda_1, ...\lambda_n)$ by

$$\lambda_j = \sum_{p=0}^{j} a_{pj}, \quad \mu_j = a_{0j}, \quad \nu_j = \sum_{q=j}^{n} a_{jq}, \qquad 1 \le j \le n. \tag{4.2.8}$$

Then $\lambda, \mu, \nu \in \Lambda_n^+$, and we say A is of type (λ, μ, ν). The set of all LR triangles in $\mathcal{LR}_n$ of type (λ, μ, ν) will be denoted by $\mathcal{LR}_n(\lambda, \mu, \nu)$

Remark 4.2.9. Note that in Definition 4.2.7, μ can be read off from the entries of the top left diagonal of the LR triangle (except a_{00}), ν is given by the sums along the leftward descending diagonals, and λ is given by the sums along the rows.

Lemma 4.2.10. ([PV])**(LR triangle associated with LR tableau)** *Let D, E and F be Young diagrams with at most n rows.*

(i) *Let T be an LR tableau of shape F/D and content E and let $A_T = [a_{ij}]_{0\le i\le j\le n}$ be defined as follows:*
 (a) $a_{00} = 0$, $a_{0j} = \mu_j$ *for* $1 \le j \le n$ *where* $D = (\mu_1, ..., \mu_n)$, *and*
 (b) a_{ij} *is equal to the number of* i*'s in row* j *of* T *for* $1 \le i \le j \le n$.
 Then A_T is a LR triangle of size n and of type (F, D, E).

(ii) *The map* $\mathrm{LRTab}(F, D, E) \to \mathcal{LR}_n(F, D, E)$ *given by*

$$T \to A_T \tag{4.2.11}$$

is a bijection. Consequently we have

$$\#\left(\mathcal{LR}_n(F, D, E)\right) = c^F_{D,E}. \tag{4.2.12}$$

Theorem 4.2.13. (The Generalized Littlewood-Richardson Rule) *If $\lambda, \mu, \nu \in \Lambda^+_n$, then the multiplicity $c^\lambda_{\mu,\nu}$ of ρ^λ_n in the tensor product $\rho^\mu_n \otimes \rho^\nu_n$ is equal to the number of elements in $\mathcal{LR}_n(\lambda, \mu, \nu)$, that is,*

$$c^\lambda_{\mu,\nu} = \#\left(\mathcal{LR}_n(\lambda, \mu, \nu)\right).$$

We will prove this theorem after establishing Lemma 4.2.14. It will be seen that the generalized LR rule follows from the standard LR rule, by means of twisting with powers of the determinant character, and that the formulation in terms of LR triangles makes the rule compatible with such twisting.

Lemma 4.2.14. *Let $\lambda, \mu, \nu \in \Lambda^+_n$.*

(i) *Let $A = [a_{ij}]_{0\le i\le j\le n} \in \mathcal{LR}_n(\lambda, \mu, \nu)$ and $p, q \in \mathbb{Z}$, and define the triangular arrays $\Phi^{(p)}_{\lambda,\mu,\nu}(A) = B = [b_{ij}]_{0\le i\le j\le n}$ and $\Psi^{(q)}_{\lambda,\mu,\nu}(A) = C = [c_{ij}]_{0\le i\le j\le n}$ by*

$$b_{00} = c_{00} = 0,$$

$$b_{ij} = \begin{cases} a_{0j} + p & i = 0 \\ a_{ij} & i > 0, \end{cases} \qquad c_{ij} = \begin{cases} a_{jj} + q & i = j \\ a_{ij} & i < j \end{cases} \qquad (0 \le i \le j \le n,\ j \ne 0).$$

Then $\Phi^{(p)}_{\lambda,\mu,\nu}(A) \in \mathcal{LR}_n(\lambda + p\mathbf{1}_n, \mu + p\mathbf{1}_n, \nu)$ and $\Psi^{(q)}_{\lambda,\mu,\nu}(A) \in \mathcal{LR}_n(\lambda + q\mathbf{1}_n, \mu, \nu + q\mathbf{1}_n)$, where

$$\mathbf{1}_n = (\overbrace{1, 1, ..., 1}^{n}). \tag{4.2.15}$$

(ii) *For $p, q \in \mathbb{Z}$, the maps*

$$\Phi^{(p)}_{\lambda,\mu,\nu} : \mathcal{LR}_n(\lambda, \mu, \nu) \to \mathcal{LR}_n(\lambda + p\mathbf{1}_n, \mu + p\mathbf{1}_n, \nu)$$

and

$$\Psi^{(q)}_{\lambda,\mu,\nu} : \mathcal{LR}_n(\lambda, \mu, \nu) \to \mathcal{LR}_n(\lambda + q\mathbf{1}_n, \mu, \nu + q\mathbf{1}_n)$$

are bijections.

(iii) *For $p, q \in \mathbb{Z}$,*

$$\#\left((\mathcal{LR}_n(\lambda + (p+q)\mathbf{1}_n, \mu + p\mathbf{1}_n, \nu + q\mathbf{1}_n)\right) = \#\left(\mathcal{LR}_n(\lambda, \mu, \nu)\right).$$

Proof. Note that the change from A to B amounts to adding a fixed number p to all the top left diagonal entries (except for a_{00}, which always remains 0). This will also add p to each of the row sums, and it is easily checked to preserve the three conditions defining LR triangles. Similarly, adding q to each of the top right diagonal entries below a_{00} will preserve the three conditions and will add q to each row sum. This establishes statements (i) and (ii). Statement (iii) results from the combination of statements (i) and (ii). □

Proof of the generalized LR rule: For any integer m, we shall write the irreducible representation $\rho_n^{m\mathbf{1}_n}$ as $\det_n^m$ as it corresponds to the character $g \to (\det g)^m$ of GL_n.

Choose nonnegative integers p and q such that $M = \mu + p\mathbf{1}_n$ and $N = \nu + q\mathbf{1}_n$ are Young diagrams. Then

$$\begin{aligned} c_{\mu,\nu}^{\lambda} &= \dim \mathrm{Hom}_{\mathrm{GL}_n}(\rho_n^{\lambda}, \rho_n^{\mu} \otimes \rho_n^{\nu}) \\ &= \dim \mathrm{Hom}_{\mathrm{GL}_n}(\rho_n^{\lambda}, (\rho_n^{M} \otimes \det{}_n^{-p}) \otimes (\rho_n^{N} \otimes \det{}_n^{-q})) \\ &= \dim \mathrm{Hom}_{\mathrm{GL}_n}(\rho_n^{\lambda}, \det{}_n^{-p-q} \otimes (\rho_n^{M} \otimes \rho_n^{N})) \\ &= \dim \mathrm{Hom}_{\mathrm{GL}_n}(\det{}_n^{p+q} \otimes \rho_n^{\lambda}, \rho_n^{M} \otimes \rho_n^{N}) \\ &= \dim \mathrm{Hom}_{\mathrm{GL}_n}(\rho_n^{F}, \rho_n^{M} \otimes \rho_n^{N}) \\ &= c_{M,N}^{F} \end{aligned}$$

where $F = \lambda + (p+q)\mathbf{1}_n$. Note that, since M and N are Young diagrams, in order for $c_{M,N}^{\lambda+(p+q)\mathbf{1}_n}$ to be non-zero, $\lambda + (p+q)\mathbf{1}$ must also define a Young diagram, by the standard LR rule. It follows from this, equation (4.2.12) and Part (iii) of Lemma 4.2.14 that

$$\begin{aligned} c_{\mu,\nu}^{\lambda} &= \#\left(\mathcal{LR}_n(F, M, N)\right) \\ &= \#\left(\mathcal{LR}_n(F - (p+q)\mathbf{1}_n, M - p\mathbf{1}_n, N - q\mathbf{1}_n)\right) \\ &= \#\left((\mathcal{LR}_n(\lambda, \mu, \nu)\right). \ \square \end{aligned}$$

4.3. **The graded algebra structure of $\mathcal{Q}$.** In this subsection, we shall define an algebra grading for $\mathcal{Q}$ and describe its homogeneous components.

Let

$$\Lambda_p^{++}(n) = \{\lambda = (\lambda_1, ..., \lambda_p) \in \Lambda_p^{+} : \lambda_{\min(n,p)} \geq 0 = \lambda_{\min(n,p)+1}\}$$

and

$$\Lambda_q^{++}(n) = \{\lambda = (\lambda_1, ..., \lambda_q) \in \Lambda_q^{+} : \lambda_{\min(n,q)} \geq 0 = \lambda_{\min(n,q)+1}\}.$$

Both $\Lambda_p^{++}(n)$ and $\Lambda_q^{++}(n)$ are semigroups with respect to componentwise addition and they can be identified with the sets of all Young diagrams with at most $\min(n,p)$ rows and $\min(n,q)$ rows, respectively. So the direct product $\Lambda_n^{+} \times \Lambda_p^{++}(n) \times \Lambda_q^{++}(n)$ also forms a semigroup. For each $(\lambda, G, H) \in \Lambda_n^{+} \times \Lambda_p^{++}(n) \times \Lambda_q^{++}(n)$, let $\mathcal{Q}_{\lambda,G,H}$ be the subspace of $\mathcal{Q}$ defined by

$$\mathcal{Q}_{\lambda,G,H} = \{f \in \mathcal{Q} : t.f = \psi_n^{\lambda}(t_1)\psi_p^{G}(t_2)\psi_q^{H}(t_3)f \ \forall t = (t_1, t_2, t_3) \in A_n \times A_p \times A_q\}, \quad (4.3.1)$$

that is, $\mathcal{Q}_{(\lambda,G,H)}$ is the $\psi_n^{\lambda} \times \psi_p^{G} \times \psi_q^{H}$-eigenspace of $A_n \times A_p \times A_q$ in $\mathcal{Q}$. Note that the nonzero vectors in $\mathcal{Q}_{(\lambda,G,H)}$ are highest weight vectors for $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ of weight $\psi_n^{\lambda} \times \psi_p^{G} \times \psi_q^{H}$.

Proposition 4.3.2. (i) *For each* $(\lambda, G, H) \in \Lambda_n^+ \times \Lambda_p^{++}(n) \times \Lambda_q^{++}(n)$,

$$\dim \mathcal{Q}_{\lambda,G,H} = c^{\lambda}_{H^*,G},$$

where $c^{\lambda}_{H^*,G}$ *is defined in Notation 4.2.2 (b). Consequently,* $\mathcal{Q}_{\lambda,G,H}$ *is a nonzero subspace of* $\mathcal{Q}$ *if and only if* $c^{\lambda}_{H^*,G} \neq 0$.

(ii) *Let* $\Lambda(\mathcal{Q})$ *be the subset of* $\Lambda_n^+ \times \Lambda_p^{++}(n) \times \Lambda_q^{++}(n)$ *defined by*

$$\Lambda(\mathcal{Q}) = \{(\lambda, G, H) \in \Lambda_n^+ \times \Lambda_p^{++}(n) \times \Lambda_q^{++}(n) : \ c^{\lambda}_{H^*,G} \neq 0\}. \tag{4.3.3}$$

Then $\Lambda(\mathcal{Q})$ *is a subsemigroup of* $\Lambda_n^+ \times \Lambda_p^{++}(n) \times \Lambda_q^{++}(n)$.

(iii) *The algebra* $\mathcal{Q}$ *is graded by* $\Lambda(\mathcal{Q})$*, and can be decomposed into homogeneous components for this grading as*

$$\mathcal{Q} = \bigoplus_{(\lambda,G,H)\in\Lambda(\mathcal{Q})} \mathcal{Q}_{\lambda,G,H}. \tag{4.3.4}$$

Proof. Let $\mathrm{GL}_n \times \mathrm{GL}_p$ act on $\mathcal{P}(\mathrm{M}_{np})$ by $\tau_{n,p}$ as defined in formula (2.2.2), and let $\mathrm{GL}_n \times \mathrm{GL}_q$ act on $\mathcal{P}(\mathrm{M}_{nq})$ by $\tau'_{n,q}$ as defined in formula (2.2.6). Then the direct product of groups

$$(\mathrm{GL}_n \times \mathrm{GL}_n) \times \mathrm{GL}_p \times \mathrm{GL}_q \cong (\mathrm{GL}_n \times \mathrm{GL}_p) \times (\mathrm{GL}_n \times \mathrm{GL}_q)$$

acts on

$$\mathcal{P}(W_{n,p,q}) = \mathcal{P}(\mathrm{M}_{np} \oplus \mathrm{M}_{nq}) \cong \mathcal{P}(\mathrm{M}_{np}) \otimes \mathcal{P}(\mathrm{M}_{nq}). \tag{4.3.5}$$

by $\tau_{n,p} \otimes \tau'_{n,q}$. The restriction of $\tau_{n,p} \otimes \tau'_{n,q}$ to $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ coincides with the action (3.1.1), where GL_n is identified with the diagonal subgroup $\Delta(\mathrm{GL}_n) = \{(g, g) : g \in \mathrm{GL}_n\}$ of $\mathrm{GL}_n \times \mathrm{GL}_n$. So by Theorem 2.2.3 and Theorem 2.2.7, the algebra $\mathcal{P}(W_{n,p,q})$ admits the following decomposition as a $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ module

$$\begin{aligned}
\mathcal{P}(W_{n,p,q}) &\cong \mathcal{P}(\mathrm{M}_{np}) \otimes \mathcal{P}(\mathrm{M}_{nq}) \\
&\cong \left(\bigoplus_{G\in\Lambda_p^{++}(n)} \rho_n^G \otimes \rho_p^G\right) \otimes \left(\bigoplus_{H\in\Lambda_q^{++}(n)} \rho_n^{H^*} \otimes \rho_n^H\right) \\
&\cong \bigoplus_{(G,H)\in\Lambda_p^{++}(n)\times\Lambda_q^{++}(n)} \left(\rho_n^G \otimes \rho_n^{H^*}\right) \otimes \rho_p^G \otimes \rho_q^H.
\end{aligned} \tag{4.3.6}$$

Then by extracting the $U_n \times U_p \times U_q$-invariants from $\mathcal{P}(W_{n,p,q})$, we obtain

$$\mathcal{Q} = \mathcal{P}(W_{n,p,q})^{U_n\times U_p\times U_q} \cong \bigoplus_{(G,H)\in\Lambda_p^{++}(n)\times\Lambda_q^{++}(n)} \left(\rho_n^G \otimes \rho_n^{H*}\right)^{U_n} \otimes \left(\rho_p^G\right)^{U_p} \otimes \left(\rho_q^H\right)^{U_q}. \tag{4.3.7}$$

For each $(G, H) \in \Lambda_p^{++}(n) \times \Lambda_q^{++}(n)$, the space $\left(\rho_n^G \otimes \rho_n^{H*}\right)^{U_n}$ of vectors in $\rho_n^G \otimes \rho_n^{H*}$ fixed by U_n is a module for A_n, and so it can be decomposed as

$$\left(\rho_n^G \otimes \rho_n^{H*}\right)^{U_n} = \bigoplus_{\lambda\in\Lambda_n^+} \left(\rho_n^G \otimes \rho_n^{H*}\right)^{U_n}_{\lambda}$$

where for each $\lambda \in \Lambda_n^+$,

$$\left(\rho_n^G \otimes \rho_n^{H*}\right)^{U_n}_{\lambda} = \left\{v \in \left(\rho_n^G \otimes \rho_n^{H*}\right)^{U_n} : \ t.v = \psi_n^{\lambda}(t)v \ \forall t \in A_n\right\}.$$

Using this and equation (4.3.7), we obtain

$$\mathcal{Q} = \mathcal{P}(W_{n,p,q})^{U_n \times U_p \times U_q} \cong \bigoplus_{(\lambda,G,H)\in\Lambda_n^+\times\Lambda_p^{++}(n)\times\Lambda_q^{++}(n)} \left(\rho_n^G \otimes \rho_n^{H*}\right)_\lambda^{U_n} \otimes \left(\rho_p^G\right)^{U_p} \otimes \left(\rho_q^H\right)^{U_q}. \tag{4.3.8}$$

We deduce from this decomposition that for each $(\lambda, G, H) \in \Lambda_n^+ \times \Lambda_p^{++}(n) \times \Lambda_q^{++}(n)$,

$$\mathcal{Q}_{\lambda,G,H} \cong \left(\rho_n^G \otimes \rho_n^{H*}\right)_\lambda^{U_n} \otimes \left(\rho_p^G\right)^{U_p} \otimes \left(\rho_q^H\right)^{U_q},$$

and so

$$\begin{aligned} \dim \mathcal{Q}_{\lambda,G,H} &= \left(\dim \left(\rho_n^G \otimes \rho_n^{H*}\right)_\lambda^{U_n}\right)\left(\dim \left(\rho_p^G\right)^{U_p}\right)\left(\dim \left(\rho_q^H\right)^{U_q}\right) \\ &= \dim \left(\rho_n^G \otimes \rho_n^{H*}\right)_\lambda^{U_n}. \end{aligned}$$

Note that the nonzero vectors in $\left(\rho_n^G \otimes \rho_n^{H*}\right)_\lambda^{U_n}$ are GL_n weight vectors of weight ψ_n^λ fixed by the unipotent group U_n, so they are the GL_n highest weight vectors of weight ψ_n^λ in the tensor product $\rho_n^G \otimes \rho_n^{H*} \cong \rho_n^{H*} \otimes \rho_n^G$. Therefore,

$$\dim \mathcal{Q}_{\lambda,G,H} = \dim \mathrm{Hom}_{\mathrm{GL}_n}(\rho_n^\lambda, \rho_n^{H*} \otimes \rho_n^G) = c_{H^*,G}^\lambda.$$

This proves (i).

For(ii), we let $(\lambda, G, H), (\lambda', G', H') \in \Lambda(\mathcal{Q})$, i.e. $c_{H^*,G}^\lambda \neq 0$ and $c_{(H')^*,G'}^{\lambda'} \neq 0$. Then there exist $A \in \mathcal{LR}_n(\lambda, H^*, G)$ and $A' \in \mathcal{LR}_n(\lambda', (H')^*, G')$. By equation (4.2.8), we have

$$A + A' \in \mathcal{LR}_n(\lambda + \lambda', (H + H')^*, G + G'),$$

and so $c_{(H+H')^*,G+G'}^{\lambda+\lambda'} \neq 0$. Hence, $(\lambda, G, H) + (\lambda', G', H') = (\lambda+\lambda', G+G', H+H') \in \Lambda(\mathcal{Q})$.

As for (iii), for each $(\lambda, G, H) \in \Lambda(\mathcal{Q})$, $\mathcal{Q}_{\lambda,G,H}$ is the $\psi_n^\lambda \times \psi_p^G \times \psi_q^H$- isotypic component of $\mathcal{Q}$ under $A_n \times A_p \times A_q$, and equation (4.3.4) is the decomposition of $\mathcal{R}$ into a direct sum of isotypic components. Since $A_n \times A_p \times A_q$ acts on $\mathcal{Q}$ by algebra automorphisms, this decomposition also defines an algebra grading on $\mathcal{Q}$.

□

4.4. **The highest weight vector $\Delta_{\lambda,G,H}$.** Our main goal is to find a basis for the algebra $\mathcal{Q}$. Since $\mathcal{Q}$ is graded by $\Lambda(\mathcal{Q})$, it suffices to construct a basis for each of its homogeneous components with respect to this grading. In this and the next subsection, we fix $(\lambda, G, H) \in \Lambda(\mathcal{Q})$ and shall construct a basis for $\mathcal{Q}_{\lambda,G,H}$.

Recall that we denote a typical element of $W_{n,p,q}$ by (X, Y) (see equation (3.6.1)) and write $\mathcal{P}(W_{n,p,q}) = \mathbb{C}[X, Y]$ (see equation (3.6.2)). Let $\beta = (\beta_{ij})_{1\leq i\leq r, 1\leq j\leq k+\ell}$ be a system of indeterminates where r, k and ℓ are defined in Lemma 4.4.1 below, and let $\mathbb{C}[\beta]$ be the polynomial algebra on the variables β_{ij}. Then we form the tensor product

$$\mathbb{C}[X, Y, \beta] \cong \mathbb{C}[X, Y] \otimes \mathbb{C}[\beta],$$

and extend the action of $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ on $\mathcal{P}(W_{n,p,q})$ to $\mathbb{C}[X, Y, \beta]$ by letting $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ act trivially on $\mathbb{C}[\beta]$. In this subsection, we shall define a $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ highest weight vector $\Delta_{\lambda,G,H}$ in $\mathbb{C}[X, Y, \beta]$ of weight $\psi_n^\lambda \times \psi_p^G \times \psi_q^H$. In the next subsection, we shall use $\Delta_{\lambda,G,H}$ to obtain a basis for $\mathcal{Q}_{\lambda,G,H}$.

The highest weight vector $\Delta_{\lambda,G,H}$ will be the determinant of a matrix $Z = Z_{\lambda,G,H}$ and Lemma 4.4.1 below lays the groundwork for the construction of this matrix. Recall that the ***conjugate diagram*** of a Young diagram L is the Young diagram L^t obtained by flipping L

over its main diagonal from upper left to lower right ([Fu]). For example, if $L = (6,4,4,2)$, then $L^t = (4,4,3,3,1,1)$. Note that the row lengths of L^t are the column lengths of L, and the length of the first row of L is equal to the depth of L^t.

Lemma 4.4.1. *Let D and E be the Young diagrams such that $\lambda = (D,E)$ (see equation (2.1.5)), and suppose that*

$$\lambda = (\lambda_1, ..., \lambda_n),\ \ D = (d_1, ..., d_n), E = (e_1, ..., e_n),\quad D^t = (d'_1, ..., d'_\ell),\quad E^t = (e'_1, ..., e'_t) \tag{4.4.2}$$

where D^t and E^t have depths ℓ and t respectively. (So the entries $\lambda_1, \lambda_2, ...$ of λ will be $d_1, d_2, ...$ up to a certain point, followed by possibly some zeros, and after which they will be the negative of the $e_1, e_2, ...,$ in the opposite order. In addition, $\ell = d_1$ and $t = e_1$.)

We also let

$$H = (h_1, ..., h_n),\quad H^t = (h'_1, ..., h'_k),\quad G^t = (g'_1, ..., g'_r) \tag{4.4.3}$$

where H^t and G^t have depths k and r respectively. (So $k = h_1$.)

(i) *Recall from equation (2.1.3) that*

$$H^* = (-h_n, -h_{n-1}, ..., -h_1).$$

Let

$$\widehat{H} = H^* + h_1\mathbf{1}_n = (h_1 - h_n, h_1 - h_{n-1}, ..., h_1 - h_2, 0), \tag{4.4.4}$$

and

$$F = \lambda + h_1\mathbf{1}_n = (\lambda_1 + h_1, \lambda_2 + h_1, ..., \lambda_n + h_1). \tag{4.4.5}$$

Then $\widehat{H}$ and F are Young diagrams, i.e. $\widehat{H}, F \in \Lambda_n^{++}$.

(ii) *The representation ρ_n^F occurs in the tensor product $\rho_n^{\widehat{H}} \otimes \rho_n^G$ with multiplicity*

$$c^F_{\widehat{H},G} = c^\lambda_{H^*,G}.$$

(iii) *$e_j \le h_j$ for $1 \le j \le n$, i.e. $E \subseteq H$.*

(iv) *$\mathbf{r}(E^t) = t \le \mathbf{r}(H^t) = k$ and $F^t = (n - e'_k, n - e'_{k-1}, ..., n - e'_1, d'_1, d'_2, ..., d'_\ell)$.*

(v) *$\widehat{H}^t = (n - h'_k, n - h'_{k-1}, ..., n - h'_1)$.*

Proof. (i) and (ii): $\widehat{H}$ is clearly a Young diagram. Note that

$$\det{}_n^{h_1} \otimes (\rho_n^{H^*} \otimes \rho_n^G) \cong (\det{}_n^{h_1} \otimes \rho_n^{H^*}) \otimes \rho_n^G \cong \rho_n^{\widehat{H}} \otimes \rho_n^G \text{ and } \det{}_n^{h_1} \otimes\ \rho_n^\lambda \simeq \rho_n^F.$$

Since ρ_n^λ occurs in the tensor product $\rho_n^{H^*} \otimes \rho_n^G$ with multiplicity $c^\lambda_{H^*,G}$, by tensoring both ρ_n^λ and $\rho_n^{H^*} \otimes \rho_n^G$ with $\det_n^{h_1}$, we see that ρ_n^F occurs in $\rho_n^{\widehat{H}} \otimes \rho_n^G$ with the same multiplicity. In particular, since ρ_n^F is a constituent of the tensor product of two irreducible polynomial representations, it is also an irreducible polynomial representation by the Littlewood-Richardson rule. So F is a Young diagram.

(iii) Since ρ_n^F occurs in $\rho_n^{\hat{H}} \otimes \rho_n^G$, each row of $\widehat{H}$ is at most as long as the corresponding row of F. Thus for each $1 \le i \le n$,

$$h_1 - h_{n+1-i} \le \lambda_i + h_1 = d_i - e_{n+1-i} + h_1$$

which gives

$$e_{n+1-i} \le d_i + h_{n+1-i}.$$

If $e_{n+1-i} = 0$, then clearly $e_{n+1-i} \le h_{n+1-i}$. If $e_{n+1-i} > 0$, then since $\mathbf{r}(D) + \mathbf{r}(E) \le n$, we have $d_i = 0$, and so $e_{n+1-i} \le h_{n+1-i}$.

(iv) First note that $\mathbf{r}(E^t) = t = e_1$ and $\mathbf{r}(H^t) = k = h_1$. By (iii), $t \le k$. If $t < k$, then we set $e_j = 0$ for $t < j \le k$. So we can write

$$E^t = (e'_1, ..., e'_k).$$

Now $\lambda = (D, E)$, so

$$\begin{aligned} F &= \lambda + h_1\mathbf{1}_n = D + E^* + k\mathbf{1}_n = \sum_{i=1}^{\ell} \mathbf{1}_{d'_i} + (\sum_{j=1}^{k} \mathbf{1}_{e'_j})^* + k\mathbf{1}_n \\ &= \sum_{i=1}^{\ell} \mathbf{1}_{d'_i} + \sum_{j=1}^{k} (\mathbf{1}_{e'_j})^* + k\mathbf{1}_n = \sum_{i=1}^{\ell} \mathbf{1}_{d'_i} + \sum_{j=1}^{k} \{(\mathbf{1}_{e'_j})^* + \mathbf{1}_n\} = \sum_{i=1}^{\ell} \mathbf{1}_{d'_i} + \sum_{j=1}^{k} \mathbf{1}_{n-e'_j}. \end{aligned}$$

From this we see that the row lengths of F^t are d'_i for $1 \le i \le \ell$ and $n - e'_j$ for $1 \le j \le k$, and we need to arrange them in decreasing order. Since $d'_1 + e'_1 = \mathbf{r}(D) + \mathbf{r}(E) \le n$, $d'_1 \le n - e'_1$. Hence

$$n - e'_k \ge n - e'_{k-1} \ge \cdots \ge n - e'_1 \ge d'_1 \ge d'_2 \ge \cdots \ge d'_\ell.$$

(v) We can write

$$\widehat{H} = H^* + h_1\mathbf{1}_n = \sum_{i=1}^{k} (\mathbf{1}_{h'_i})^* + k\mathbf{1}_n = \sum_{i=1}^{k} \{(\mathbf{1}_{h'_i})^* + \mathbf{1}_n\} = \sum_{i=1}^{k} \mathbf{1}_{n-h'_i}.$$

Hence $\widehat{H}^t = (n - h'_k, n - h'_{k-1}, ..., n - h'_1)$.

□

We now define the matrix $Z_{\lambda,G,H}$. For $1 \le a \le n$, $1 \le b \le p$, $1 \le c \le q$, let

$$X_{ab} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1b} \\ x_{21} & x_{22} & \cdots & x_{2b} \\ \vdots & \vdots & & \vdots \\ x_{a1} & x_{a2} & \cdots & x_{ab} \end{pmatrix}, \quad Y_{ac} = \begin{pmatrix} y_{n1} & y_{n2} & \cdots & y_{nc} \\ y_{(n-1)1} & y_{(n-1)2} & \cdots & y_{(n-1)c} \\ \vdots & \vdots & & \vdots \\ y_{(n+1-a)1} & y_{(n+1-a)2} & \cdots & y_{(n+1-a)c} \end{pmatrix},$$

$$R_{bc} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1c} \\ r_{21} & r_{22} & \cdots & r_{2c} \\ \vdots & \vdots & & \vdots \\ r_{b1} & r_{b2} & \cdots & r_{bc} \end{pmatrix},$$

Let $Z_{\lambda,G,H}(X, Y, \beta)$ be the following matrix

$$\left(\begin{array}{ccccc|ccc} \beta_{11}R_{g'_1,h'_k} & \cdots & \beta_{1(k-2)}R_{g'_1,h'_3} & \beta_{1(k-1)}R_{g'_1,h'_2} & \beta_{1k}R_{g'_1,h'_1} & \beta_{1(k+1)}(X_{d'_1,g'_1})^t & \cdots & \beta_{1(k+\ell)}(X_{d'_\ell,g'_1})^t \\ \beta_{21}R_{g'_2,h'_k} & \cdots & \beta_{2(k-2)}R_{g'_2,h'_3} & \beta_{2(k-1)}R_{g'_2,h'_2} & \beta_{2k}R_{g'_2,h'_1} & \beta_{2(k+1)}(X_{d'_1,g'_2})^t & \cdots & \beta_{2(k+\ell)}(X_{d'_\ell,g'_2})^t \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{r1}R_{g'_r,h'_k} & \cdots & \beta_{r(k-2)}R_{g'_r,h'_3} & \beta_{r(k-1)}R_{g'_r,h'_2} & \beta_{rk}R_{g'_r,h'_1} & \beta_{r(k+1)}(X_{d'_1,g'_r})^t & \cdots & \beta_{r(k+\ell)}(X_{d'_\ell,g'_r})^t \\ \hline 0 & \cdots & 0 & 0 & Y_{e'_1,h'_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & Y_{e'_2,h'_2} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & Y_{e'_3,h'_3} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & ⋰ & \vdots & \vdots & \vdots & \vdots & & \vdots \\ & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right).$$

Here, each 0 represents the zero matrix of appropriate size. More precisely, the matrix $Z_{\lambda,G,H}(X,Y,\beta)$ is partitioned into $(r+t)\times(k+\ell)$ submatrices as

$$Z_{\lambda,G,H}(X,Y,\beta)=\begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1(k+\ell)} \\ Z_{21} & Z_{22} & \cdots & Z_{2(k+\ell)} \\ \vdots & \vdots & & \vdots \\ Z_{(r+t)1} & Z_{(r+t)2} & \cdots & Z_{(r+t)(k+\ell)} \end{pmatrix},$$

where

$$Z_{ij}=\begin{cases} \beta_{ij}R_{g'_i,h'_{k+1-j}} & 1\le i\le r, 1\le j\le k \\ \beta_{ij}X^t_{d'_{j-k},g'_i} & 1\le i\le r, k+1\le j\le k+\ell \\ Y_{e'_{i-r},h'_{i-r}} & r+1\le i\le r+t, j=k+r+1-i \\ 0 & \text{otherwise.} \end{cases}$$

Terminology. For each $1\le i\le r+t$, we call the submatrix

$$\begin{pmatrix} Z_{i1} & Z_{i2} & \cdots & Z_{i(k+\ell)} \end{pmatrix}$$

of $Z_{\lambda,G,H}(X,Y,\beta)$ the ***i-th super row*** of $Z_{\lambda,G,H}(X,Y,\beta)$.

Note that by Part (iv) of Lemma 4.4.1, $t=\mathbf{r}(E^t)=e_1\le h_1=\mathbf{r}(H^t)=k$. So $k+1-t\ge 1$ and $Z_{(r+t)(k+1-t)}=Y_{e'_t,h'_t}$ appears as a block in the lowest super row of $Z_{\lambda,G,H}(X,Y,\beta)$. Hence $Z_{\lambda,G,H}(X,Y,\beta)$ does not have any zero row.

Lemma 4.4.6. *$Z_{\lambda,G,H}(X,Y,\beta)$ is a square matrix.*

Proof. Note that for a Young diagram L, we have $|L|=|L^t|$ since L and L^t have the same number of boxes. So the number of rows in $Z_{\lambda,G,H}(X,Y,\beta)$ is $|G^t|+|E^t|=|G|+|E|$ and the number of columns is $|H^t|+|D^t|=|H|+|D|$. By Part (ii) of Lemma 4.4.1, ρ_n^F occurs in $\rho^{\hat{H}}\otimes\rho_n^G$. So $|F|=|G|+|\hat{H}|$. Now $|F|=|\lambda+h_1\mathbf{1}_n|=|D|-|E|+nh_1$ and $|\hat{H}|=nh_1-|H|$. So

$$|D|-|E|+nh_1=|G|+nh_1-|H|$$

which gives

$$|H|+|D|=|G|+|E|. \ \square$$

We now let $\Delta(X,Y,\beta)$ be the determinant of the square matrix $Z_{\lambda,G,H}(X,Y,\beta)$, that is,

$$\Delta(X,Y,\beta)=\Delta_{\lambda,G,H}(X,Y,\beta):=\det Z_{\lambda,G,H}(X,Y,\beta). \tag{4.4.7}$$

It can be written in the form

$$\Delta(X,Y,\beta)=\sum_{(M,N,L)} t_{(M,N,L)}\beta^M X^N Y^L. \tag{4.4.8}$$

where $M=(m_{ij})$, $N=(n_{ij})$ and $L=(l_{ij})$ are matrices of nonnegative integers of size $r\times(k+\ell)$, $n\times p$ and $n\times q$ respectively, $t_{(M,N,L)}\in\mathbb{C}$ and

$$\beta^M=\prod_{ij}\beta_{ij}^{m_{ij}},\ X^N=\prod_{ij}x_{ij}^{n_{ij}},\ Y^L=\prod_{ij}y_{ij}^{l_{ij}}.$$

It can also be written as

$$\Delta(X,Y,\beta)=\sum_M p_M(X,Y)\beta^M \tag{4.4.9}$$

where for each $r\times(k+\ell)$ matrix M which appears in the sum, $p_M\in\mathcal{P}(W_{n,p,q})=\mathbb{C}[X,Y]$.

Lemma 4.4.10. (i) *The polynomial* $\Delta = \Delta_{\lambda,G,H}$ *in* $\mathbb{C}[X,Y,\beta]$ *is a* $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$ *highest weight vector of weight* $\psi_n^\lambda \times \psi_p^G \times \psi_q^H$ *in* $\mathbb{C}[X,Y,\beta]$.

(ii) *For each* $r \times (k+\ell)$ *matrix* M *which appears in the sum* (4.4.9), $p_M \in \mathcal{Q}_{\lambda,G,H}$.

Proof. Part (i) follows from a routine verification, and part (ii) is essentially the consequence of the fact that the βs are not affected by the action of of $\mathrm{GL}_n \times \mathrm{GL}_p \times \mathrm{GL}_q$, and since Δ is identically a highest weight vector, each coefficient of it as a function of the βs will likewise be a highest weight vector. □

Part (ii) of Lemma 4.4.10 provides a way to produce elements of $\mathcal{Q}_{\lambda,G,H}$. In the next subsection, we will show that for an appropriate set of $r\times(k+\ell)$ matrices M, the corresponding polynomials p_M forms a basis for $\mathcal{Q}_{\lambda,G,H}$.

4.5. **A basis for $\mathcal{Q}$.** This section and the next are heavily intertwined with [HTW1], and the main result obtained is a variant of the result of [HTW1]. We continue to assume that $(\lambda, G, H) \in \Lambda(\mathcal{Q})$ and will use the notation as defined in Lemma 4.4.1. By Proposition 4.3.2 (i) and Theorem 4.2.13,

$$\dim \mathcal{Q}_{\lambda,G,H} = c^\lambda_{H^*,G} = \#\left(\mathcal{LR}_n(\lambda, H^*, G)\right).$$

This suggests that the set $\mathcal{LR}_n(\lambda, H^*, G)$ may be used to label the elements of a basis for $\mathcal{Q}_{\lambda,G,H}$.

Let $A = [a_{ij}] \in \mathcal{LR}_n(\lambda, H^*, G)$, and consider the LR triangle $\widehat{A} = \Phi_{\mu_1}(A) = (\widehat{a}_{ij})$ where

$$\widehat{a}_{ij} = \begin{cases} a_{i0} + h_1 & j = 0 \\ a_{ij} & j \geq 1. \end{cases} \tag{4.5.1}$$

By Part (i) of Lemma 4.2.14, $\widehat{A} \in \mathcal{LR}_n(F, \widehat{H}, G)$ where $\widehat{H}$ and F are Young diagrams defined in equations (4.4.4) and (4.4.5) respectively. By Lemma 4.2.10, there is a unique LR tableau $T_{\widehat{A}}$ of shape $F/\widehat{H}$ and content G such that $\widehat{A} = A_{T_{\widehat{A}}}$. We shall use $T_{\widehat{A}}$ to define a $r \times (k + r)$ matrix $M(T_{\widehat{A}})$ of nonnegative integers.

The ***banal tableau*** of shape G is the tableau $BT(G)$ obtained by filling each column of G from top to bottom with consecutive positive integers starting from 1. For example, if $G = (4, 3, 2, 1)$, then

$$BT(G) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}\ .$$

There is a mapping from the cells in the LR tableau $T_{\widehat{A}}$ to the cells of the banal tableau $BT(G)$ defined in §2.3 of [HTW1]. This mapping is "content preserving" in the sense that each cell of $T_{\widehat{A}}$ is mapped to a cell of $BT(G)$ with the same value, and it can be visualized as the process of successively removing the "vertical skew strips" from $T_{\widehat{A}}$ and reassembling them into the columns of $BT(G)$. In view of this, this mapping is called the ***standard peeling of*** $T_{\widehat{A}}$. We refer the reader to §2.3 of [HTW1] for a precise description of this mapping.

We now consider the inverse mapping of the standard peeling, which can be visualized as the filling of the skew diagram $F/\widehat{H}$ by the contents of $BT(G)$. We number the columns of F and G consecutively from left-to-right. For $1 \leq j \leq r$, the entries of the jth column of $BE(G)$ will be distributed to the various columns of F to form the tableau $T_{\widehat{A}}$, and we let

$$m_{ij} = \text{number of entries in the } j\text{th column of } BT(G) \text{ get put in the } i\text{th column of } F$$

for $1 \leq i \leq k+\ell$. We form $(k+\ell) \times r$ matrix of nonnegative integers

$$M(T_{\widehat{A}}) = (m_{ij}). \tag{4.5.2}$$

Definition 4.5.3. For each $A = [a_{ij}] \in \mathcal{LR}_n(\lambda, H^*, G)$, we define $\xi_A \in \mathcal{Q}_{\lambda,G,H}$ and the monomial m_A in $\mathcal{P}(W_{n,p,q})$ by

$$\xi_A = p_{M(T_{\widehat{A}})^t} \tag{4.5.4}$$

and

$$m_A = \left(\prod_{i=1}^{n} y_{i(n+1-i)}^{-a_{0i}}\right)\left(\prod_{1\leq i\leq j\leq n} x_{ij}^{a_{ji}}\right). \tag{4.5.5}$$

That is, ξ_A is the polynomial p_M defined in (ii) of Lemma 4.4.10 with $M = M(T_{\widehat{A}})^t$, the transpose of the matrix $M(T_{\widehat{A}})$.

Next, we define a monomial ordering τ_2 on the set of monomials in $\mathcal{P}(W_{n,p,q})$ as follows:

Definition 4.5.6. Let τ_2 be the graded lexicographic order ([CLO]) with respect to the following ordering on the variables:

(a) $x_{ab} > x_{cd}$ iff either (i) $b < d$ or (ii) $b = d$ and $a < c$.
(b) $y_{ab} > y_{cd}$ iff either (i) $b < d$ or (ii) $b = d$ and $a > c$.
(c) $x_{ab} < y_{cd}$ for all pairs (a,b) and (c,d) of indices.

That is,

$$\begin{aligned} y_{n1} > y_{(n-1)1} > \cdots > y_{11} > y_{n2} > \cdots > y_{1q} \\ > x_{11} > x_{21} > \cdots > x_{n1} > x_{12} > \cdots > x_{np}. \end{aligned}$$

If p is a nonzero polynomial function in $\mathcal{P}(W_{n,p,q})$, we shall denote its leading monomial with respect to τ_2 by $\mathrm{LM}_{\tau_2}(p)$.

Note that the monomial ordering τ_2 is identical to the monomial ordering τ_1 defined in Definition 3.6.5 on the set of x variables, and also on the set of y variables. The only difference is that under τ_2 the y variables are now taken to be larger than the x variables, whereas it was the opposite for τ_1.

Lemma 4.5.7. *For each $A = [a_{ij}] \in \mathcal{LR}_n(\lambda, H^*, G)$,*

$$\mathrm{LM}_{\tau_2}(\xi_A) = m_A.$$

A proof of this lemma is given in §4.6.

Theorem 4.5.8. (i) *For each $(\lambda, G, H) \in \Lambda(\mathcal{Q})$, the set*

$$\mathbf{B}_{(\lambda,G,H)} = \{\xi_A : A \in \mathcal{LR}_n(\lambda, H^*, G)\}$$

is a basis for $\mathcal{Q}_{\lambda,G,H}$.

(ii) *The set*

$$\mathbf{B} = \bigcup_{(\lambda,G,H)\in\Lambda(\mathcal{Q})} \mathbf{B}_{(\lambda,G,H)}$$

is a basis for $\mathcal{Q}$.

Proof. By Lemma 4.5.7, the polynomials ξ_A in $\mathbf{B}_{(\lambda,G,H)}$ have distinct leading monomials with respect to the monomial ordering τ_2. So $\mathbf{B}_{(\lambda,G,H)}$ is a linearly independent subset of $\mathcal{Q}_{\lambda,G,H}$. Since we also have

$$\dim \mathcal{Q}_{\lambda,G,H} = \#(\mathbf{B}_{(\lambda,G,H)}) = c^{\lambda}_{H^*,G},$$

$\mathbf{B}_{(\lambda,G,H)}$ is a basis for $\mathcal{Q}_{\lambda,G,H}$. This proves (i).

Part (ii) follows from (i) and equation (4.3.4). □

Remark 4.5.9. (i) Let

$$\mathrm{LM}_{\tau_2}(\mathcal{Q}) = \{\mathrm{LM}_{\tau_2}(h) : \ h \in \mathcal{Q}\backslash\{0\}\}$$

be the set of the leading monomials of elements in $\mathcal{Q}$. It forms a semigroup with respect to the product of monomials.

(ii) Let

$$\mathcal{LR}_{\mathcal{Q}} = \bigcup_{(\lambda,G,H)\in\Lambda(\mathcal{Q})} \mathcal{LR}_n(\lambda, H^*, G).$$

Then it can be verified that

$$\mathrm{LM}_{\tau_2}(\mathcal{Q}) = \{m_A : \ A \in \mathcal{LR}_{\mathcal{Q}}\}.$$

Moreover, the map $\mathrm{LM}_{\tau_2}(\mathcal{Q}) \to \mathcal{LR}_n$ given by

$$m_A \to A$$

is a semigroup isomorphism onto $\mathcal{LR}_{\mathcal{Q}}$. In particular, $\mathcal{LR}_{\mathcal{Q}}$ is a subsemigroup of $\mathcal{LR}_n$.

(iii) The initial algebra of $\mathcal{Q}$ is the subalgebra of $\mathcal{P}(W_{n,p,q})$ generated by $\mathrm{LM}_{\tau_2}(\mathcal{Q})$. By (ii), it is isomorphic to the semigroup algebra $\mathbb{C}[\mathcal{LR}_{\mathcal{Q}}]$ on $\mathcal{LR}_{\mathcal{Q}}$.

(iv) Since the semigroup $\mathcal{LR}_{\mathcal{Q}}$ is finitely generated, by a general result of [CHV], there exists a flat one-parameter family of complex algebras with general fibre $\mathcal{Q}$ and special fibre $\mathbb{C}[\mathcal{LR}_{\mathcal{Q}}]$.

Next, we recall that the Lie algebra $\mathfrak{gl}_{p+q}$ acts on $\mathcal{P}(W_{n,p,q})$ by formula (3.7.1). Let $\mathfrak{k}$ and $\mathfrak{b}$ be the subpsaces of $\mathfrak{gl}_{p+q}$ defined by

$$\mathfrak{k} = \mathrm{Span}\{e_{ij} : 1 \le i, j \le p\} \cup \{e_{p+i,p+j} : \ 1 \le i, j \le q\},$$

$$\mathfrak{b} = \mathrm{Span}\{e_{ij} : 1 \le i \le j \le p\} \cup \{e_{p+i,p+j} : \ 1 \le j \le i \le q\}.$$

Then $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{gl}_{p+q}$, $\mathfrak{k} \cong \mathfrak{gl}_p \oplus \mathfrak{gl}_q$, and $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{k}$.

Corollary 4.5.10. *Let $\lambda \in \Lambda^+_n(p,q)$ and consider the irreducible representation $\pi^{\lambda}_{p,q}$ of $\mathfrak{gl}_{p+q}$ realized as a subspace of $\mathcal{P}(W_{n,p,q})$ as in equation (3.7.3). Then for $(G,H) \in \Lambda^{++}_p(n) \times \Lambda^{++}_q(n)$ such that $c^{\lambda}_{H^*,G} \neq 0$, $\mathcal{Q}_{\lambda,G,H}$ is the space of all $\mathfrak{k}$ highest weight vectors with respect to the Borel subalgebra $\mathfrak{b}$ of weight $\phi_{G,H}$ (defined in equation (3.7.4)), and $\mathbf{B}_{(\lambda,G,H)}$ is a basis for $\mathcal{Q}_{\lambda,G,H}$.*

4.6. **Proof of Lemma 4.5.7.** Let $\widetilde{Z} = \widetilde{Z}_{\lambda,G,H}$ be the following matrix

$$\left(\begin{array}{cccc|ccc|cccc}
\beta_{11}R_{g'_1,h'_k} & \cdots & \beta_{1(k-1)}R_{g'_1,h'_2} & \beta_{1k}R_{g'_1,h'_1} & \beta_{1(k+1)}(X_{d'_1,g'_1})^t & \cdots & \beta_{1(k+\ell)}(X_{d'_\ell,g'_1})^t & 0 & \cdots & 0 & 0 \\
\beta_{21}R_{g'_2,h'_k} & \cdots & \beta_{2(k-1)}R_{g'_2,h'_2} & \beta_{2k}R_{g'_2,h'_1} & \beta_{2(k+1)}(X_{d'_1,g'_2})^t & \cdots & \beta_{2(k+\ell)}(X_{d'_\ell,g'_2})^t & 0 & \cdots & 0 & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
\beta_{r1}R_{g'_r,h'_k} & \cdots & \beta_{r(k-1)}R_{g'_r,h'_2} & \beta_{rk}R_{g'_r,h'_1} & \beta_{r(k+1)}(X_{d'_1,g'_r})^t & \cdots & \beta_{r(k+\ell)}(X_{d'_\ell,g'_r})^t & 0 & \cdots & 0 & 0 \\
\hline
0 & \cdots & 0 & Y_{e'_1,h'_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & Y_{e'_2,h'_2} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \iddots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & \cdots & 0 & \widehat{Y}_1 & 0 & \cdots & 0 & \widehat{I}_1 & 0 & 0 & 0 \\
0 & \cdots & \widehat{Y}_2 & 0 & 0 & \cdots & 0 & 0 & \widehat{I}_2 & 0 & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\
\widehat{Y}_k & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \widehat{I}_k
\end{array}\right), \tag{4.6.1}$$

where for each $1 \le i \le k$,

$$\widehat{Y}_i = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1h'_i} \\ y_{21} & y_{22} & \cdots & y_{2h'_i} \\ \vdots & \vdots & & \vdots \\ y_{(n-e'_i)1} & y_{(n-e'_i)2} & \cdots & y_{(n-e'_i)h'_i} \end{pmatrix}$$

and $\widehat{I}_i$ is the identity matrix of size $n - e'_i$. Now note that the submatrix formed by the blocks

$$\begin{pmatrix} \widehat{I}_1 & 0 & 0 & 0 \\ 0 & \widehat{I}_2 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \widehat{I}_k \end{pmatrix}$$

at the lower right corner of $\widetilde{Z}$ is the $N \times N$ identity matrix I_N where $N = kn - \sum_{i=1}^k e'_i = kn - |E|$. So the matrix $\widetilde{Z}$ is of the form

$$\widetilde{Z} = \begin{pmatrix} Z & 0 \\ * & I_N \end{pmatrix}.$$

It follows that

$$\det \widetilde{Z} = \det Z \det I_N = \det Z = \Delta.$$

The reason why we replace Z with $\tilde{Z}$ is that, in $\tilde{Z}$, we can use the blocks of Y-variables in the lower two leftmost sectors of $\tilde{Z}$ to construct row operations that eliminate the R-variables in the upper left sector. We will describe how to do this in each supercolumn.

By equation (4.6.1), $\widetilde{Z}$ is partitioned into $(2k+\ell) \times (2k+\ell)$ submatrices $\widetilde{Z}_{ij}$ $(1 \le i, j \le 2k+\ell)$, i.e.,

$$\widetilde{Z} = \begin{pmatrix} \widetilde{Z}_{11} & \widetilde{Z}_{12} & \cdots & \widetilde{Z}_{1(2k+\ell)} \\ \widetilde{Z}_{21} & \widetilde{Z}_{22} & \cdots & \widetilde{Z}_{2(2k+\ell)} \\ \vdots & \vdots & & \vdots \\ \widetilde{Z}_{(2k+\ell)1} & \widetilde{Z}_{(2k+\ell)2} & \cdots & \widetilde{Z}_{(2k+\ell)(2k+\ell)} \end{pmatrix}.$$

We call $\widetilde{Z}_{ij}$ *the* (i,j)*th block* of $\widetilde{Z}$. For example, $\widetilde{Z}_{ij} = \beta_{ij} R_{g'_i, h'_{k+1-i}}$ for $1 \le i \le r$ and $1 \le j \le k$.

Next, for each $1 \le i \le 2k+\ell$ and each $1 \le j \le 2k+\ell$, we call the submatrices of $\widetilde{Z}$

$$(\widetilde{Z}_{i1}\ \widetilde{Z}_{i2}\ \cdots \widetilde{Z}_{i(2k+\ell)}) \quad \text{and} \quad \begin{pmatrix} \widetilde{Z}_{1j} \\ \widetilde{Z}_{2j} \\ \vdots \\ \widetilde{Z}_{(2k+\ell)j} \end{pmatrix}$$

the i*th superrow* of $\widetilde{Z}$ and the j*th supercolumn* of $\widetilde{Z}$, respectively.

We now fix $1 \le i \le r$ and $1 \le j \le k$, and consider the submatrices $\widetilde{Z}_{i(k-j+1)} = \beta_{i(k-j+1)} R_{g'_i, h'_j}$, $\widetilde{Z}_{(r+i)(k-j+1)} = Y_{e'_j, h'_j}$, $\widetilde{Z}_{(r+t+i)(k+1-j)} = \hat{Y}_j$, in the $(k+1-j)$th supercolumn:

$$\left(\begin{array}{ccccccc|ccc|ccccccc}
\ast & \cdots & \ast & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \ast & \cdots & \ast \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\ast & \cdots & \ast & \beta_{i(k-j+1)} R_{g'_i, h'_j} & \ast & \cdots & \ast & \ast & \cdots & \ast & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\ast & \cdots & \ast & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \ast & \cdots & \ast \\
\hline
\ast & \cdots & \ast & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \ast & \cdots & \ast \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & Y_{e'_j, h'_j} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\ast & \cdots & \ast & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \ast & \cdots & \ast \\
\hline
\ast & \cdots & \ast & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \ast & \cdots & \ast \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & \hat{Y}_j & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \hat{I}_j & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\ast & \cdots & \ast & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \cdots & \ast & \ast & \ast & \cdots & \ast \\
\hline
\end{array}\right)$$

We will perform elementary row operations whch use the variables in the submatrices $Y_{e'_j, h'_j}$ and $\hat{Y}_j$ to eliminate the variables in the submatrix $\beta_{i(k-j+1)} R_{g'_i, h'_j}$. These operations only affect the entries in 6 blocks of $\widetilde{Z}$. To simplify notation, we consider the matrix B_{ij} which is formed by these submatrices as follows:

$$B_{ij} = \left(\begin{array}{c|c} \beta R_{g'_i,h'_j} & 0 \\ \hline Y_{e'_j,h'_j} & 0 \\ \hline \widehat{Y}_j & \widehat{I}_j \end{array}\right) = \left(\begin{array}{cccc|cccc} \beta r_{11} & \beta r_{12} & \cdots & \beta r_{1a} & 0 & 0 & \cdots & 0 \\ \beta r_{21} & \beta r_{22} & \cdots & \beta r_{2a} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \beta r_{c1} & \beta r_{c2} & \cdots & \beta r_{ca} & 0 & 0 & \cdots & 0 \\ \hline y_{n1} & y_{n2} & \cdots & y_{na} & 0 & 0 & \cdots & 0 \\ y_{(n-1)1} & y_{(n-1)2} & \cdots & y_{(n-1)a} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ y_{(b+1)1} & y_{(b+1)2} & \cdots & y_{(b+1)a} & 0 & 0 & \cdots & 0 \\ \hline y_{11} & y_{12} & \cdots & y_{1a} & 1 & 0 & \cdots & 0 \\ y_{21} & y_{22} & \cdots & y_{2a} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{b1} & y_{b2} & \cdots & y_{ba} & 0 & 0 & \cdots & 1 \end{array}\right),$$

where

$$a = h'_j,\ b = n - e'_j,\ c = g'_i,\ \beta = \beta_{i(k-j+1)}.$$

We now fix $1 \leq u \leq c$. The uth row of the matrix $\beta R_{g'_i,h'_j}$ is

$$(\beta r_{u1}, \beta r_{u2}, ..., \beta r_{ua}),$$

where $r_{uv} = \sum_{s=1}^{n} x_{su} y_{sv}$. We use row operations to reduce B_{ij} to

$$B'_{ij} = \left(\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & -\beta x_{11} & -\beta x_{21} & \cdots & -\beta x_{b1} \\ 0 & 0 & \cdots & 0 & -\beta x_{12} & -\beta x_{22} & \cdots & -\beta x_{b2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & -\beta x_{1c} & -\beta x_{2c} & \cdots & -\beta x_{bc} \\ \hline y_{n1} & y_{n2} & \cdots & y_{na} & 0 & 0 & \cdots & 0 \\ y_{(n-1)1} & y_{(n-1)2} & \cdots & y_{(n-1)a} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ y_{(b+1)1} & y_{(b+1)2} & \cdots & y_{(b+1)a} & 0 & 0 & \cdots & 0 \\ \hline y_{11} & y_{12} & \cdots & y_{1a} & 1 & 0 & \cdots & 0 \\ y_{21} & y_{22} & \cdots & y_{2a} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ y_{b1} & y_{b2} & \cdots & y_{ba} & 0 & 0 & \cdots & 1 \end{array}\right)$$

$$= \left(\begin{array}{c|c} 0 & -\beta_{i(k-j+1)} X^t_{n-e'_j,g'_i} \\ \hline Y_{e'_{k+1-j},h'_{k+1-j}} & 0 \\ \hline \widehat{Y}_{k+1-j} & \widehat{I}_{k+1-j} \end{array}\right). \tag{4.6.2}$$

When the above elimination process has been completed for all $1 \le i \le r$ and $1 \le j \le k$, the matrix $\widetilde{Z}$ will be transformed to the following matrix $\widetilde{Z}_1$:

$$\left(\begin{array}{cccc|ccc|ccc}
0 & \cdots & 0 & 0 & \beta_{1(k+1)}(X_{d'_1,g'_1})^t & \cdots & \beta_{1(k+\ell)}(X_{d'_\ell,g'_1})^t & -\beta_{1k}(X_{n-e'_1,g'_1})^t & \cdots & -\beta_{11}(X_{n-e'_k,g'_1})^t \\
0 & \cdots & 0 & 0 & \beta_{2(k+1)}(X_{d'_1,g'_2})^t & \cdots & \beta_{2(k+\ell)}(X_{d'_\ell,g'_2})^t & -\beta_{2k}(X_{n-e'_1,g'_2})^t & \cdots & -\beta_{21}(X_{n-e'_k,g'_2})^t \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \beta_{r(k+1)}(X_{d'_1,g'_r})^t & \cdots & \beta_{r(k+\ell)}(X_{d'_\ell,g'_r})^t & -\beta_{rk}X^t_{n-e'_1,g'_r} & \cdots & -\beta_{r1}(X_{n-e'_k,g'_r})^t \\
\hline
0 & \cdots & 0 & Y_{e'_1,h'_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & Y_{e'_2,h'_2} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \iddots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & 0 & \widehat{Y}_1 & 0 & \cdots & 0 & \widehat{I}_1 & \cdots & 0 \\
0 & \cdots & \widehat{Y}_2 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & & \ddots & \\
\widehat{Y}_k & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \widehat{I}_k
\end{array}\right), \tag{4.6.3}$$

We now identify the largest monomial in Y which occurs in $\det Z = \det \widetilde{Z}_1$. To do this, we examine the matrix B'_{ij} given in equation (4.6.2). As all the entries in the $(1,1)$th and the $(2,2)$ block of B'_{ij} are 0, we need to pick $e'_j = n - b$ entries from the $(2,1)$th block. Observe that the largest monomial in the Y variables which we can form from the $(2,1)$th block is $y_{n1}y_{(n-1)2}\cdots y_{(b+1)(n-b)}$. We now remove the rows and columns of B'_{ij} which contain the entries $y_{n1}, y_{(n-1)2}, \ldots, y_{(b+1)(n-b)}$. Then the resulting matrix is given by

$$B''_{ij} = \left(\begin{array}{cccc|cccc}
0 & 0 & \cdots & 0 & -\beta x_{11} & -\beta x_{21} & \cdots & -\beta x_{b1} \\
0 & 0 & \cdots & 0 & -\beta x_{12} & -\beta x_{22} & \cdots & -\beta x_{b2} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & -\beta x_{1c} & -\beta x_{2c} & \cdots & -\beta x_{bc} \\
\hline
y_{1(n-b+1)} & y_{1(n-b+2)} & \cdots & y_{1a} & 1 & 0 & \cdots & 0 \\
y_{2(n-b+1)} & y_{2(n-b+2)} & \cdots & y_{2a} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
y_{b(n-b+1)} & y_{b(n-b+2)} & \cdots & y_{ba} & 0 & 0 & \cdots & 1
\end{array}\right)$$

Next, we need to choose another $a - b$ entries from the $(2,1)$ block of B''_{ij}. The largest monomial in y is $y_{b(n-b+1)}y_{(b-1)(n-b+2)}\cdots y_{(n-a+1)a}$. This means that we have chosen the monomial $Y^{L(\mathbf{1}_{h_j})}$ defined by

$$Y^{L(\mathbf{1}_{h_j})} = Y^{L(\mathbf{1}_a)} = \prod_{s=1}^{a} y_{(n-s+1)s} = y_{n1}y_{(n-1)2}\cdots y_{(n-a+1)a}$$

from the first supercolumn of B'_{ij}, and the matrix B''_{ij} is reduced to

$$B'''_{ij} = \left(\begin{array}{ccccccc} -\beta x_{11} & -\beta x_{21} & \cdots & -\beta x_{(n-a)1} & -\beta x_{(n-a+1)1} & \cdots & -\beta x_{b1} \\ -\beta x_{12} & -\beta x_{22} & \cdots & -\beta x_{(n-a)2} & -\beta x_{(n-a+1)2} & \cdots & -\beta x_{b2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \beta x_{1c} & -\beta x_{2c} & \cdots & -\beta x_{(n-a)c} & -\beta x_{(n-a+1)c} & \cdots & -\beta x_{bc} \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{array} \right).$$

We then do a row expansion along each of the lowest $(n-a)$ rows which contains 1, and this process eliminates the first $(n-a)$ columns and the lowest $(n-a)$ rows from B'''_{ij}. For the purpose of computing leading monomial, we may remove all the minus signs. So the final matrix which we obtain is

$$\begin{pmatrix} \beta x_{(n-a+1)1} & \beta x_{(n-a+2)1} & \cdots & \beta x_{b1} \\ \beta x_{(n-a+1)2} & \beta x_{(n-a+2)2} & \cdots & \beta x_{b2} \\ \vdots & \vdots & & \vdots \\ \beta x_{(n-a+1)c} & \beta x_{(n-a+2)c} & \cdots & \beta x_{bc} \end{pmatrix}$$

and we denote this matrix by

$$\beta(X_{(n-a,b],c})^t = \beta_{i(k+1-j)}(X_{(n-h'_j,n-e'_j],g'_i})^t. \tag{4.6.4}$$

Here, the notation $(n-a,b]$ is intended to mean that the matrix $\beta(X_{(n-a,b],c})^t$ is obtained by removing the first $n-a$ columns from the matrix $\beta(X_{b,c})^t$.

For each $1 \leq j \leq k$, we choose the monomial $Y^{L(\mathbf{1}_{h_j})}$ from the $(k+1-j)$th supercolumn. Thus the monomial in Y which we have selected is

$$Y^{L(H)} = \prod_{j=1}^{k} Y^{L(\mathbf{1}_{h_j})} = \prod_{i=1}^{n} y_{n+1-i,i}^{-a_{0i}}. \tag{4.6.5}$$

We now write the polynomial Δ as

$$\Delta(X,Y,\beta) = \sum_L q_L(X,\beta) Y^L,$$

where for each matrix L in the sum, q_L is a polynomial in the variables (X,β). The polynomial q_L can be written as

$$q_L(X,\beta) = \sum_M \delta_{L,M}(X)\beta^M$$

where for each matrix M in the sum, $\delta_{L,M}$ is a polynomial in X. Then

$$\Delta(X,Y,\beta) = \sum_L \left(\sum_M \delta_{L,M}(X)\beta^M \right) Y^L$$

From this, we see that when $M = M(T_{\widehat{A}})^t$,

$$\xi_A = p_{M(T_{\widehat{A}})^t}(X,Y) = \sum_L \delta_{L,M(T_{\widehat{A}})^t}(X)\beta^{M(T_{\widehat{A}})^t} Y^L.$$

Since $Y^{L(H)} \geq Y^L$ for all L which appears in the sum,

$$\mathrm{LM}_{\tau_2}(\xi_A) = Y^{L(H)}\mathrm{LM}_{\tau_2}(\delta_{L(H),M(T_{\widehat{A}})^t}). \tag{4.6.6}$$

We now consider the polynomial $q_{L(H)}$. By equations (4.6.3) and (4.6.4), we see that $q_{L(H)}(X,\beta)$ is given by

$$\pm\left|\begin{array}{ccc|ccc} \beta_{1(k+1)}(X_{d'_1,g'_1})^t & \cdots & \beta_{1(k+\ell)}(X_{d'_\ell,g'_1})^t & \beta_{1k}(X_{(n-h'_1,n-e'_1],g'_1})^t & \cdots & \beta_{11}(X_{(n-h'_k,n-e'_k],g'_1})^t \\ \beta_{2(k+1)}(X_{d'_2,g'_2})^t & \cdots & \beta_{2(k+\ell)}(X_{d'_\ell,g'_2})^t & \beta_{2k}(X_{(n-h'_1,n-e'_1],g'_2})^t & \cdots & \beta_{21}(X_{(n-h'_k,n-e'_k],g'_2})^t \\ \vdots & & \vdots & \vdots & & \vdots \\ \beta_{r(k+1)}(X_{d_1,g_r})^t & \cdots & \beta_{r(k+\ell)}(X_{d'_\ell,g'_r})^t & \beta_{rk}(X_{(n-h'_1,n-e'_1],g'_r})^t & \cdots & \beta_{r1}(X_{(n-h'_k,n-e'_k],g'_r})^t \end{array}\right|$$

$$= \pm\left|\begin{array}{cccc} \beta_{11}X_{(n-h'_k,n-e'_k],g'_1} & \beta_{21}X_{(n-h'_k,n-e'_k],g'_2} & \cdots & \beta_{r1}X_{(n-h'_k,n-e'_k],g'_r} \\ \vdots & \vdots & & \vdots \\ \beta_{1k}X_{(n-h'_1,n-e'_1],g'_1} & \beta_{2k}X_{(n-h'_1,n-e'_1],g'_2} & \cdots & \beta_{rk}X_{(n-h'_1,n-e'_1],g'_r} \\ \hline \beta_{1(k+1)}X_{d'_1,g'_1} & \beta_{2(k+1)}X_{d'_1,g'_2} & \cdots & \beta_{r(k+1)}X_{d'_1,g'_r} \\ \vdots & \vdots & & \vdots \\ \beta_{1(k+\ell)}X_{d'_\ell,g'_1} & \beta_{2(k+\ell)}X_{d'_\ell,g'_2} & \cdots & \beta_{r(k+\ell)}X_{d'_\ell,g'_r} \end{array}\right|,$$

where we denote the determinant of a matrix A by $|A|$. By (iv) and (v) of Lemma 4.4.1, we have $F^t = (n-e'_k,...,n-e'_1,d'_1,...,d'_\ell)$ and $\widehat{H}^t = (n-h'_k,...,n-h'_1)$. We now note that by making an appropriate change of notation, the polynomials $q_{L(H)}$ and $\delta_{L(H),M(T_{\widehat{A}})^t)}$ are respectively identified with the polynomials $\det\widetilde{Y_0}$ and $\delta_{T,Y}$ defined in §3.2 of [HTW1]. So by Lemma 3.2 of [HTW1],

$$\mathrm{LM}_{\tau_2}(\delta_{L(H),M(T_{\widehat{A}})^t}) = \prod_{1\leq i\leq j\leq n} x_{ji}^{a_{ij}}.$$

Hence, by equations (4.6.5) and (4.6.6), we obtain

$$\mathrm{LM}_{\tau_2}(\xi_A) = \left(\prod_{i=1}^{n} y_{n+1-i,i}^{-a_{0i}}\right)\left(\prod_{1\leq i\leq j\leq n} x_{ji}^{a_{ij}}\right) = m_A. \ \square$$

4.7. **An example.** In this subsection, we shall illustrate the proof of Lemma 4.5.7 with an example. Let $n=4$, $G=H=(2,1,1,0)$ and $\lambda=(1,0,0,-1)$. Then $\rho_n^\lambda = \rho_n^{(1,0,0,-1)}$ occurs in the tensor product $\rho_n^{H^*}\otimes\rho_n^G = \rho_n^{(0,-1,-1,-2)}\otimes\rho_n^{(2,1,1,0)}$ with multiplicity 2.

(i) There are two LR triangles A_1, A_2 in $\mathcal{LR}_n(\lambda,H^*,G)$ which are given by

$$A_1 = \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 1 & & & \\ & & -1 & & 0 & & 1 & & \\ & -1 & & 0 & & 0 & & 1 & \\ -2 & & 1 & & 0 & & 0 & & 0 \end{array}, \quad A_2 = \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 1 & & & \\ & & -1 & & 1 & & 0 & & \\ & -1 & & 0 & & 1 & & 0 & \\ -2 & & 0 & & 0 & & 1 & & 0 \end{array}.$$

(ii) The monomial m_{A_1} and m_{A_2} as defined in equation (4.5.5) are given respectively by

$$m_{A_2} = y_{41}^2 y_{32} y_{23} x_{11} x_{22} x_{33} x_{41} \quad\text{and}\quad m_{A_2} = y_{41}^2 y_{32} y_{23} x_{11} x_{21} x_{32} x_{43}.$$

(iii) The LR triangles $\widehat{A}_1$ and $\widehat{A}_2$ defined in equation (4.5.1) are given respectively by

$$\widehat{A}_1 = \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 2 & & 1 & & & \\ & & 1 & & 0 & & 1 & & \\ & 1 & & 0 & & 0 & & 1 & \\ 0 & & 1 & & 0 & & 0 & & 0 \end{array}, \quad \widehat{A}_2 = \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 2 & & 1 & & & \\ & & 1 & & 1 & & 0 & & \\ & 1 & & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 1 & & 0 \end{array}.$$

(iv) The LR tableaux $T_{\widehat{A}_1}$ and $T_{\widehat{A}_1}$ which corresponds to $\widehat{A}_1$ and $\widehat{A}_2$ are given respectively by

$$T_{\widehat{A}_1} = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad T_{\widehat{A}_2} = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

(v) The matrices $M(T_{\widehat{A}_1})$ and $M(T_{\widehat{A}_2})$ as defined in equation (4.5.2) are given respectively by

$$M(T_{\widehat{A}_1}) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad M(T_{\widehat{A}_2}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and so

$$\beta^{M(T_{\widehat{A}_1})^t} = \beta_{21}\beta_{12}^2\beta_{13}, \quad \beta^{M(T_{\widehat{A}_2})^t} = \beta_{11}\beta_{12}\beta_{22}\beta_{13}.$$

Now by the definition of $\Delta = \Delta_{(\lambda,G,H)}$ given in equation (4.4.7), we have

$$\Delta = \left| \begin{array}{c|ccc|c} \beta_{11}r_{11} & \beta_{12}r_{11} & \beta_{12}r_{12} & \beta_{12}r_{13} & \beta_{13}x_{11} \\ \beta_{11}r_{21} & \beta_{12}r_{21} & \beta_{12}r_{22} & \beta_{12}r_{23} & \beta_{13}x_{12} \\ \beta_{11}r_{31} & \beta_{12}r_{31} & \beta_{12}r_{32} & \beta_{12}r_{33} & \beta_{13}x_{13} \\ \hline \beta_{21}r_{11} & \beta_{22}r_{11} & \beta_{22}r_{12} & \beta_{22}r_{13} & \beta_{23}x_{11} \\ \hline 0 & y_{41} & y_{42} & y_{43} & 0 \end{array} \right|$$

Then

$$\Delta = \left| \begin{array}{c|ccc|c|ccc|cccc} \beta_{11}r_{11} & \beta_{12}r_{11} & \beta_{12}r_{12} & \beta_{12}r_{13} & \beta_{13}x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11}r_{21} & \beta_{12}r_{21} & \beta_{12}r_{22} & \beta_{12}r_{23} & \beta_{13}x_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \beta_{11}r_{31} & \beta_{12}r_{31} & \beta_{12}r_{32} & \beta_{12}r_{33} & \beta_{13}x_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \beta_{21}r_{11} & \beta_{22}r_{11} & \beta_{22}r_{12} & \beta_{22}r_{13} & \beta_{23}x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & y_{41} & y_{42} & y_{43} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & y_{11} & y_{12} & y_{13} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{21} & y_{22} & y_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_{31} & y_{32} & y_{33} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline y_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ y_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ y_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ y_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right|$$

$$
= \left|\begin{array}{c|ccc|c|ccc|cccc}
0 & 0 & 0 & 0 & \beta_{13}x_{11} & -\beta_{12}x_{11} & -\beta_{12}x_{21} & -\beta_{12}x_{31} & -\beta_{11}x_{11} & -\beta_{11}x_{21} & -\beta_{11}x_{31} & -\beta_{11}x_{41} \\
0 & 0 & 0 & 0 & \beta_{13}x_{12} & -\beta_{12}x_{12} & -\beta_{12}x_{22} & -\beta_{12}x_{32} & -\beta_{11}x_{12} & -\beta_{11}x_{22} & -\beta_{11}x_{32} & -\beta_{11}x_{42} \\
0 & 0 & 0 & 0 & \beta_{13}x_{13} & -\beta_{12}x_{13} & -\beta_{12}x_{23} & -\beta_{12}x_{33} & -\beta_{11}x_{13} & -\beta_{11}x_{23} & -\beta_{11}x_{33} & -\beta_{11}x_{43} \\
\hline
0 & 0 & 0 & 0 & \beta_{23}x_{11} & -\beta_{22}x_{11} & -\beta_{22}x_{21} & -\beta_{22}x_{31} & -\beta_{21}x_{11} & -\beta_{21}x_{21} & -\beta_{21}x_{31} & -\beta_{21}x_{41} \\
\hline
0 & y_{41} & y_{42} & y_{43} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & y_{11} & y_{12} & y_{13} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & y_{21} & y_{22} & y_{23} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & y_{31} & y_{32} & y_{33} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
y_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
y_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
y_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
y_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right|
$$

From this, we observe that the largest monomial in Y which occur in Δ is $Y^{L(H)} = y_{41}^2 y_{32} y_{23}$. By removing the rows and columns of the matrix which contains the entries which made up $Y^{L(H)}$, we obtain

$$
\begin{aligned}
q_{L(H)}(X,\beta) &= \left|\begin{array}{c|ccc|cccc}
\beta_{13}x_{11} & -\beta_{12}x_{11} & -\beta_{12}x_{21} & -\beta_{12}x_{31} & -\beta_{11}x_{11} & -\beta_{11}x_{21} & -\beta_{11}x_{31} & -\beta_{11}x_{41} \\
\beta_{13}x_{12} & -\beta_{12}x_{12} & -\beta_{12}x_{22} & -\beta_{12}x_{32} & -\beta_{11}x_{12} & -\beta_{11}x_{22} & -\beta_{11}x_{32} & -\beta_{11}x_{42} \\
\beta_{13}x_{13} & -\beta_{12}x_{13} & -\beta_{12}x_{23} & -\beta_{12}x_{33} & -\beta_{11}x_{13} & -\beta_{11}x_{23} & -\beta_{11}x_{33} & -\beta_{11}x_{43} \\
\hline
\beta_{23}x_{11} & -\beta_{22}x_{11} & -\beta_{22}x_{21} & -\beta_{22}x_{31} & -\beta_{21}x_{11} & -\beta_{11}x_{21} & -\beta_{21}x_{31} & -\beta_{21}x_{41} \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right| \\
&= \left|\begin{array}{c|cc|c}
\beta_{13}x_{11} & -\beta_{12}x_{21} & -\beta_{12}x_{31} & -\beta_{11}x_{41} \\
\beta_{13}x_{12} & -\beta_{12}x_{22} & -\beta_{12}x_{32} & -\beta_{11}x_{42} \\
\beta_{13}x_{13} & -\beta_{12}x_{23} & -\beta_{12}x_{33} & -\beta_{11}x_{43} \\
\hline
\beta_{23}x_{11} & -\beta_{22}x_{21} & -\beta_{22}x_{31} & -\beta_{21}x_{41}
\end{array}\right| = -\left|\begin{array}{c|cc|c}
\beta_{13}x_{11} & \beta_{12}x_{21} & \beta_{12}x_{31} & \beta_{11}x_{41} \\
\beta_{13}x_{12} & \beta_{12}x_{22} & \beta_{12}x_{32} & \beta_{11}x_{42} \\
\beta_{13}x_{13} & \beta_{12}x_{23} & \beta_{12}x_{33} & \beta_{11}x_{43} \\
\hline
\beta_{23}x_{11} & \beta_{22}x_{21} & \beta_{22}x_{31} & \beta_{21}x_{41}
\end{array}\right| \\
&= -\left|\begin{array}{c|cc|c}
\beta_{11}x_{41} & \beta_{12}x_{21} & \beta_{12}x_{31} & \beta_{13}x_{11} \\
\beta_{11}x_{42} & \beta_{12}x_{22} & \beta_{12}x_{32} & \beta_{13}x_{12} \\
\beta_{11}x_{43} & \beta_{12}x_{23} & \beta_{12}x_{33} & \beta_{13}x_{13} \\
\hline
\beta_{21}x_{41} & \beta_{22}x_{21} & \beta_{22}x_{31} & \beta_{23}x_{11}
\end{array}\right| = -\left|\begin{array}{ccc|c}
\beta_{11}x_{41} & \beta_{11}x_{42} & \beta_{11}x_{43} & \beta_{21}x_{41} \\
\hline
\beta_{12}x_{21} & \beta_{12}x_{22} & \beta_{12}x_{23} & \beta_{22}x_{21} \\
\beta_{12}x_{31} & \beta_{12}x_{32} & \beta_{12}x_{33} & \beta_{22}x_{31} \\
\hline
\beta_{13}x_{11} & \beta_{13}x_{12} & \beta_{13}x_{13} & \beta_{23}x_{11}
\end{array}\right| \\
&= x_{41}\begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{11} & x_{12} & x_{13} \end{vmatrix}\beta_{21}\beta_{12}^2\beta_{13} - \begin{vmatrix} x_{41} & x_{42} & x_{43} & 0 \\ x_{21} & x_{22} & x_{23} & x_{21} \\ x_{31} & x_{32} & x_{33} & x_{31} \\ x_{11} & x_{12} & x_{13} & 0 \end{vmatrix}\beta_{11}\beta_{12}\beta_{22}\beta_{13} \\
&\quad - x_{11}\begin{vmatrix} x_{41} & x_{42} & x_{43} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}\beta_{11}\beta_{12}^2\beta_{23}.
\end{aligned}
$$

Hence,

$$
\delta_1 = \delta_{L(H),M(T_{\widehat{A}_1})^t} = x_{41}\begin{vmatrix} x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \\ x_{11} & x_{12} & x_{13} \end{vmatrix}, \quad \delta_2 = \delta_{L(H),M(T_{\widehat{A}_2})^t} = -\begin{vmatrix} x_{41} & x_{42} & x_{43} & 0 \\ x_{21} & x_{22} & x_{23} & x_{21} \\ x_{31} & x_{32} & x_{33} & x_{31} \\ x_{11} & x_{12} & x_{13} & 0 \end{vmatrix}.
$$

By observation,

$$
\mathrm{LM}_{\tau_1}(\delta_1) = x_{11}x_{22}x_{33}x_{41}, \quad \mathrm{LM}_{\tau_2}(\delta_2) = x_{11}x_{21}x_{32}x_{43}.
$$

So

$$
\mathrm{LM}_{\tau_2}(\xi_{A_1}) = Y^{L(H)}\mathrm{LM}_{\tau_2}(\delta_1) = (y_{41}^2 y_{32} y_{23})(x_{11}x_{22}x_{33}x_{41}) = m_{A_1}
$$

and

$$
\mathrm{LM}_{\tau_2}(\xi_{A_2}) = Y^{L(H)}\mathrm{LM}_{\tau_2}(\delta_2) = (y_{41}^2 y_{32} y_{23})(x_{11}x_{21}x_{32}x_{43}) = m_{A_2}.
$$

Appendix : General theory of signed Hibi cones

Recall that a poset $(P, \geq)$ is a nonempty set P together a partial ordering $\geq$ ([Stan]). If $x, y \in P$, then we write $x > y$ if $x \geq y$ and $x \neq y$. We also write $x \leq y$ if $y \geq x$, and write $x < y$ if $y > x$.

Definition A1. Let $(P, \geq)$ be a finite poset.

(i) A subset S of P is called ***increasing*** if
$$x \in S,\ y \in P,\ y \geq x \Longrightarrow y \in S.$$
The collection of all increasing subsets of P is denoted by $J^*(P, \geq)$.

(ii) A subset S of P is called ***decreasing*** if
$$x \in S,\ y \in P,\ x \geq y \Longrightarrow y \in S.$$
The collection of all decreasing subsets of P is denoted by $J_*(P, \geq)$.

(iii) A map $f : P \to \mathbb{Z}$ is called ***order preserving*** if
$$x, y \in P,\ x \geq y \Longrightarrow f(x) \geq f(y).$$

(iv) Let
$$\mathbb{Z}^{P,\geq} = \{f : P \to \mathbb{Z} |\ f \text{ is order preserving}\}.$$
Then $\mathbb{Z}^{P,\geq}$ is a semigroup with respect to addition of functions.

(v) For $A, B \subseteq P$, let
$$\Omega_{A,B}(P) = \{f \in \mathbb{Z}^{P,\geq} :\ f(A) \geq 0,\ f(B) \leq 0\}. \tag{4.7.1}$$
Then $\Omega_{A,B}(P)$ is a subsemigroup of $\mathbb{Z}^{P,\geq}$, and it is called a ***signed Hibi cone***.

Remark A2.

(i) If we take $A = B = \emptyset$, then $\Omega_{\emptyset,\emptyset}(P) = \mathbb{Z}^{P,\geq}$. So $\mathbb{Z}^{P,\geq}$ is a signed Hibi cone.

(ii) If we take $A = P$ and $B = \emptyset$, and let
$$(\mathbb{Z}^+)^{P,\geq} := \Omega_{P,\emptyset}(P) = \{f : P \to \mathbb{Z}^+ |\ f \text{ is order preserving}\}.$$
The semigroup $(\mathbb{Z}^+)^{P,\geq}$ is called a ***Hibi cone*** ([Ho5, Hi]). So the notion of a signed Hibi cone generalizes a Hibi cone.

(iii) If $f \in \mathbb{Z}^{P,\geq}$ is such that $f(A) \geq 0$, then it takes nonnegative values on the increasing subset of P generated by A. So there is no loss in generality in requiring that A is increasing. Likewise, B may as well be required to be decreasing. Note also that any function in $\mathbb{Z}^{P,\geq}$ will vanish on $A \cap B$.

We now fix $A, B \subseteq P$ and will describe the semigroup structure of the signed Hibi cone $\Omega_{A,B}(P)$. For any subset S of P, the **indicator function** of S is the map $\chi_S : P \to \{0, 1\}$ defined by
$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S. \end{cases}$$
We shall identify a collection of subsets of P such that the indicator functions of these subsets generate the semigroup $\Omega_{A,B}(P)$. Let P_A be the smallest increasing subset of P which contains A, and let N_B be the smallest decreasing subset of P which contains B. Let
$$P^+ := P \backslash N_B, \quad P^- := P \backslash P_A,$$
and we regard P^+ and P^- as subposets of P. Next, we let
$$\mathcal{G}^+_{A,B} = \{\chi_L : L \in J^*(P^+, \geq)\}, \quad \mathcal{G}^-_{A,B} = \{-\chi_M : M \in J_*(P^-, \geq)\}$$

and

$$\mathcal{G}_{A,B} := \mathcal{G}^+_{A,B} \cup \mathcal{G}^-_{A,B}.$$

We define a partial ordering $\preceq$ on $\mathcal{G}_{A,B}$ as follows:

(i) If $L_1, L_2 \in J^*(P^+, \geq)$, then $\chi_{L_1} \preceq \chi_{L_2}$ if and only if $L_1 \subseteq L_2$;
(ii) If $M_1, M_2 \in J_*(P^-, \geq)$, then $-\chi_{M_1} \preceq -\chi_{M_2}$ if and only if $M_1 \supseteq M_2$.
(iii) If $L \in J^*(P^+, \geq)$ and $M \in J_*(P^-, \geq)$, then $\chi_L \preceq -\chi_M$ if and only if $L \cap M = \emptyset$.

Theorem A3. ([Wa]) *Let $(P, \geq)$ be a finite poset and let $A, B \subseteq P$. Then the semigroup $\Omega_{A,B}(P)$ is generated by $\mathcal{G}_{A,B}$. Moreover, each nonzero element f of $\Omega_{A,B}(P)$ can be expressed uniquely as*

$$f = \sum_{i=1}^{s} a_i \chi_{L_i} + \sum_{j=1}^{t} b_j(-\chi_{M_j}),$$

where $L_1, ..., L_s \in J^*(P^+, \geq)$, $M_1, ..., M_t \in J_*(P^-, \geq)$,

$$\chi_{P_1} \prec \cdots \prec \chi_{P_s} \prec -\chi_{Q_1} \prec \cdots \prec -\chi_{Q_t}$$

is a chain in $\mathcal{G}_{A,B}$ and $a_1, ..., a_s$, $b_1, ..., b_t$ are positive integers.

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