# NON-ABELIAN $p$-ADIC RANKIN-SELBERG $L$-FUNCTIONS AND NON-VANISHING OF CENTRAL $L$-VALUES 

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Dedicated to the memory of Claus-Günther Schmidt.


#### Abstract

We construct $p$-adic $L$-functions for torsion classes for $\mathrm{GL}(n+1) \times \operatorname{GL}(n)$ and along the way prove new congruences between special values of Rankin-Selberg $L$ functions for $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ over arbitrary number fields. This allows us to control the behavior of $p$-adic $L$-functions under Tate twists and to prove the existence of nonabelian $p$-adic $L$-functions for Hida families on GL $(n+1) \times \operatorname{GL}(n)$. As an application, we establish generic non-vanishing results for central $L$-values: We give sufficient local conditions for twisted central Rankin-Selberg $L$-values to be generically non-zero.


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## Introduction

Fix a rational prime $p$, a number field $F / \mathbf{Q}$ and define the reductive group

$$
G=\operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}(n+1) \times \operatorname{GL}(n), \quad n \geq 1 .
$$

Consider the Rankin-Selberg $L$-function $L(s, \Pi \hat{\otimes} \Sigma)$ attached to an irreducible cuspidal automorphic representation $\Pi \widehat{\otimes} \Sigma$ of $G(\mathbf{A})$ as studied by Jacquet, Piatetski-Shapiro and Shalika [42, 43, 44].

If $\Pi$ and $\Sigma$ are regular algebraic in the sense of Clozel [6], we expect $L(s, \Pi \widehat{\otimes} \Sigma)$ to agree (up to shift) with the $L$-function of the tensor product of the conjectural irreducible motives $M_{\Pi}$ and $M_{\Sigma}$ attached to $\Pi$ and $\Sigma$. In this context, we expect the special values of $L(s, \Pi \widehat{\otimes} \Sigma)$ to be intricately related to the arithmetic of $M_{\Pi}$ and $M_{\Sigma}$. In particular, when deforming $M_{\Pi}$ and $M_{\Sigma}$ in $p$-adic families, we expect these special values to vary $p$-adic analytically as well.

The aim of this paper is to establish this expected $p$-adic variation of $L$-values in the case when $\Pi$ and $\Sigma$ are nearly ordinary at $p$ in the sense of [34, 36]. In order to do so, we prove new congruences for the special values under consideration.

Abelian $p$-adic interpolation. Our results for abelian deformations are the following. Write $T \subseteq G$ for the standard diagonal maximal torus and $C_{F}\left(p^{\infty}\right)$ for the ray class group of level $p^{\infty}$ of $F$. Let $E / \mathbf{Q}_{p}$ denote a finite extension which contains the fields of rationality of $\Pi$ and $\Sigma$.

Our first main result is (cf. Theorem 6.1 in the text),
Theorem A. Let $\Pi \widehat{\otimes} \Sigma$ be an irreducible regular algebraic cuspidal automorphic representation of $G(\mathbf{A})$ of cohomological weight $\lambda$. Assume the following:
(i) $\lambda$ is balanced (in the sense of eq. (4)).
(ii) $\Pi \widehat{\otimes} \Sigma$ is nearly ordinary at a prime $p$ and $\vartheta: T\left(\mathbf{Q}_{p}\right) \rightarrow \mathbf{C}^{\times}$the corresponding eigenvalue.
Then there are complex periods $\Omega_{ \pm, j} \in \mathbf{C}^{\times}$, indexed by the characters of $\pi_{0}(F \otimes \mathbf{R})^{\times}$ and $j \in \mathbf{Z}$ for which $s_{0}=\frac{1}{2}+j$ is critical for $L(s, \Pi \widehat{\otimes} \Sigma)$, and a unique $p$-adic measure $\mu_{\Pi \widehat{\otimes} \Sigma} \in \mathcal{O}\left[\left[C_{F}\left(p^{\infty}\right)\right]\right]$ with the following property. For every $s_{0}=\frac{1}{2}+j$ critical for $L(s, \Pi \widehat{\otimes} \Sigma)$, for all finite order Hecke characters $\chi$ of $F$ unramified outside $p \infty$ and such that $\chi_{p} \vartheta$ has fully supported constant conductor,

$$
\begin{aligned}
& \quad \int_{C_{F}\left(p^{\infty}\right)} \chi(x) \omega_{F}^{j}(x)\langle x\rangle_{F}^{j} d \mu_{\Pi \widehat{\otimes} \Sigma}(x)= \\
& \mathfrak{N}\left(\mathfrak{f}_{\chi \vartheta}\right)^{j \frac{(n+1) n}{2}-\frac{(n+1) n(n-1)}{6}} \cdot \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} G\left(\chi \vartheta_{\mu, \nu}\right) \cdot \frac{L\left(s_{0}, \Pi \widehat{\otimes} \Sigma \otimes \chi\right)}{\Omega_{(-1)^{j} \operatorname{sgn} \chi, j}} .
\end{aligned}
$$

Hypothesis (i) is imminent also in previous and related work on modular symbols. A condition of this type is imposed upon us also for motivic reasons (cf. Deligne's Conjecture on special values of motivic $L$-functions and its integral refinements; for further details we refer to the discussion following Theorem 6.1). We refer the reader to recent work of Claus-Günter Schmidt [77] for a classification of the possible non-balanced cases whose treatment remains an important open problem.

The notion of fully supported constant conductor is formally defined in section 2.3 in the text. This technical condition is automatically satisfied whenever the ramification
of $\chi$ is deep enough (deeper than the Nebentypus' ramification). For example, if $\Pi$ is spherical at all places $\mathfrak{p} \mid p$, then every $\chi$ ramified at all $\mathfrak{p} \mid p$ satisfies this condition.

The set of Hecke characters with fully supported constant conductor is Zariski dense in the $p$-adic rigid space of all $p$-adic characters. In particular, the interpolation property in Theorem A characterizes the $p$-adic measure $\mu_{\Pi \widehat{\otimes} \Sigma}(x)$ uniquely and confirms the Conjecture of Coates and Perrin-Riou [10, 11] in the cases where the interpolation property is known.

We expect an most identical interpolation property in the case of fully supported but not necessarily constant conductor to hold, whereas in the presence of unramified characters among $\chi \vartheta_{\mu, \nu}$ we should see a modified Euler factor of positive degree as a $p$-adic multiplier as predicted by Coates and Perrin-Riou. For $n=2$, i.e. GL(3) $\times \mathrm{GL}(2)$, this was confirmed in the spherical case in unpublished work of D. Ungemach.

By the construction of $\mu_{\Pi \widehat{\otimes} \Sigma}(x)$, Theorem A has a straightforward generalization to finite slope forms and yields a unique locally analytic distribution on the cyclotomic line $\mathbf{Z}_{p}^{\times}$, provided the slope is bounded above by the number of critical values. However, to keep the exposition and notation simple, we emphasize the nearly ordinary case here.

Theorem A improves the main results of $[75,56,50,51,52]$ in two ways. Firstly, we cover nearly ordinary representations and not only Iwahori spherical representations at places $v \mid p$. Secondly, we construct a single $p$-adic $L$-function interpolating all critical values at once. In the case $n=1$, Theorem A recovers Namikawa's recent construction of abelian $p$-adic $L$-functions for GL(2) over number fields from [67]. In this case, the interpolation formula is known is all cases (cf. loc. cit.).

The non-vanishing of the complex periods $\Omega_{ \pm, j}$ is an important recent result of Sun [84]. Their dependence on $j$ is studied in [54, 24, 27]. However, as of now there is no integral relation between the various $\Omega_{ \pm, j}$ known for $n>1$, although our results suggest that such relations should indeed exist (cf. Remark 6.2 in the text).

Among all critical values, central $L$-values are expected to be arithmetically the most interesting ones. The Conjecture of Birch and Swinnerton-Dyer and its generalizations suggest that the central value of a motivic $L$-function should vanish only for specific arithmetic reasons.

From an automorphic perspective, non-vanishing of central values is believed to be equivalent to the existence of non-zero periods. For example, by Ginzburg-Jiang-Rallis' [19], non-vanishing of the central value of $L(s, \Pi \widehat{\otimes} \Sigma)$ for suitable $\Pi$ and $\Sigma$ is equivalent to non-vanishing of certain period integrals on related groups.

Since $\mu_{\Pi \widehat{\otimes} \Sigma}$ is determined uniquely by the interpolation property for a single critical $s_{0}=\frac{1}{2}+j$, we deduce from Theorem A the following non-vanishing result (Theorem 6.10 and Corollary 6.11 in the text).

Theorem B. Let $\Pi$ and $\Sigma$ be irreducible cuspidal regular algebraic automorphic representations of $\mathrm{GL}_{n+1}\left(\mathbf{A}_{F}\right)$ and $\mathrm{GL}_{n}\left(\mathbf{A}_{F}\right)$ of balanced weight. Assume that $\Pi$ and $\Sigma$ are unitary and that $\Pi$ and $\Sigma$ are nearly ordinary at all $\mathfrak{p} \mid p$ for some prime $p$.

If $s_{0}=\frac{1}{2}$ is critical for $L(s, \Pi \widehat{\otimes} \Sigma)$ and if there exists a second critical value $s_{0} \neq \frac{1}{2}$, then

$$
\begin{equation*}
L\left(\frac{1}{2}, \Pi \widehat{\otimes} \Sigma \otimes \chi\right) \neq 0 \tag{1}
\end{equation*}
$$

generically for $\chi$ varying over all finite order Hecke characters of $F$ unramified outside $p \infty$.

Moreover, the vanishing locus is transversal to the cyclotomic line, i.e. (1) holds for all but finitely many characters of the form $\chi=\chi^{\prime} \circ N_{F / \mathbf{Q}}$ where $\chi^{\prime}$ is a Dirichlet character of p-power conductor.

We may always pass from general regular algebraic automorphic representations $\Pi$ and $\Sigma$ to unitary twists $\Pi^{u}=\Pi \otimes|\cdot|_{\mathbf{A}}^{-w}$ and $\Sigma^{u}=\Sigma \otimes|\cdot|_{\mathbf{A}}^{-w^{\prime}}$ for suitable $w, w^{\prime} \in \frac{1}{2} \mathbf{Z}$.

The existence and location of critical values and the notion of being of balanced weight is entirely governed by the cohomological weights of $\Pi$ and $\Sigma$, i.e. by the infinity types $\Pi_{\infty}$ and $\Sigma_{\infty}$. Likewise, being nearly ordinary at places $\mathfrak{p}$ above $p$ is a local condition on $\Pi_{\mathfrak{p}}$ and $\Sigma_{\mathfrak{p}}$ (cf. section 6.1). Therefore, Theorem B provides sufficient local conditions on $\Pi$ and $\Sigma$ for generic non-vanishing of twists.

In particular, it is easy to see that if $\Pi$ and $\Sigma$ satisfy the hypotheses of Theorem B, then so do $\Pi \otimes \chi^{\prime}$ and $\Sigma \otimes \chi^{\prime \prime}$ for arbitrary finite order Hecke characters $\chi^{\prime}$ and $\chi^{\prime \prime}$ over $F$.

Theorem B implies simultaneous generic non-vanishing for any finite collection of representations $\Pi_{1}, \ldots, \Pi_{r}, \Sigma_{1}, \ldots, \Sigma_{r}$ on $\mathrm{GL}_{n_{1}+1}\left(\mathbf{A}_{F}\right), \ldots, \mathrm{GL}_{n_{r}+1}\left(\mathbf{A}_{F}\right), \mathrm{GL}_{n_{1}}\left(\mathbf{A}_{F}\right), \ldots, \mathrm{GL}_{n_{r}}\left(\mathbf{A}_{F}\right)$, such that the pairs $\left(\Pi_{i}, \Sigma_{i}\right)$ satisfy the hypothesis of Theorem B for the same prime $p$. We may allow the pairs $\left(\Pi_{i}, \Sigma_{i}\right)$ to live over different number fields $F_{i}$, at the cost of obtaining simultaneous non-vanishing only for all but finitely many norm-inflated Dirichlet characters of $p$-power conductor.

Using symmetric power functoriality for GL(2), it is easy to produce automorphic representations $\Pi$ and $\Sigma$ satisfying the hypotheses of Theorems A and B. By [18, 58, 59], the symmetric power functoriality $\mathrm{Sym}^{n}$ from GL(2) to GL $(n+1)$ is known for $n \leq 4$ over arbitrary number fields. Thanks to recent progress by Clozel-Thorne [7, 8, 9], we know that $\operatorname{Sym}^{n}$ exists for $n \leq 8$ for Hilbert modular forms over totally real fields $F$ under mild hypotheses. Symmetric power lifts preserve near ordinarity and regular algebraicity (cf. Theorem 5.3 and Proposition 5.4 in [70] and Theorem 3.2 in [71]). Using base change in solvable extensions [1], we may produce examples over more general number fields.

Iteratively, Theorems A and B imply the existence of $p$-adic meromorphic $L$-functions for symmetric power $L$-functions $L\left(s, \operatorname{Sym}^{n} f\right)$ of non-CM nearly ordinary Hilbert modular cusp forms $f$ over a totally real number field $F / \mathbf{Q}$, which is linearly disjoint from $\mathbf{Q}\left(e^{2 \pi i / 35}\right)$, and $1 \leq n \leq 8$, provided that $f$ is of sufficiently large parallel weight. Our non-vanishing result also allows for the extension of the rationality results in $[70,71]$ to central $L$-values.

We refer to [15] for a detailed discussion of examples and applications and a direct construction of $p$-adic $L$-functions for odd symmetric powers.

The result of Theorem B appears to be new for $n>1$, the case $n=1$ being to Shimura and Rohrlich [81, 72, 73]. The idea to use congruences to deduce non-vanishing of central $L$-values goes back to Greenberg [21, 22] (the author kindly thanks Michael Harris for pointing this out). Non-vanishing is known for non-central critical values by results of Jacquet-Shalika [46] and Shahidi [79], generalizing previous results for non-central $L$ values for $\mathrm{GL}(n)$ in [45]. At the moment, the most general result for central values known appears to be due to Luo [63] (see also Nastasescu's recent thesis [68]). Independently of our work, Sugiyama and Tsuzuki recently established in [83] a non-vanishing result for central values for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ where the representation on the $\mathrm{GL}(2)$ factor arises from a cuspidal Maaß form. For yet another analytic approach to non-vanishing for $\mathrm{GL}(2) \times \mathrm{GL}(2)$, and possibly triple products we refer to [87].

Non-abelian $p$-adic interpolation. Previous work on $p$-adic $L$-functions for GL $(n+$ $1) \times \mathrm{GL}(n)[75,56,50,51,52]$ as well as Theorem A is limited to abelian deformations, i. e. interpolation of twists by finite order Hecke characters. Furthermore, these methods did not apply to torsion classes.

In order to study the variation of the above $p$-adic $L$-functions in Hida families and to treat torsion classes, we first establish in Theorem 4.14 a control theorem for nearly ordinary cohomology of $G$ following ideas of Hida $[28,29,30,31,32,33,34,36]$ and using delicate results on the contribution of the residual spectrum of GL( $n$ ) to the cohomology of arithmetic groups $[46,47,66,17,78,27]$.

Assume $F$ totally real, CM or that Conjecture 4.9 on the existence of Galois representations for torsion classes holds for $\mathrm{GL}(m)$ over $F(m \leq n+1)$ (which is known for $F$ totally real or CM by Scholze's breakthrough [78]). In this situation, we have a notion of non-Eisenstein maximal ideals $\mathfrak{m}$ in Hida's universal nearly ordinary Hecke algebra $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ for $G$, which are characterized by corresponding to residual Galois representations $\bar{\rho}_{\mathfrak{m}}$ which are (semi-simplifications of) tensor products of absolutely irreducible residual representations for $\mathrm{GL}(n+1)$ and $\mathrm{GL}(n)$ respectively. In particular, the residual representation $\bar{\rho}_{\mathfrak{m}}$ attached to $\mathfrak{m}$ itself may be reducible (take a tensor product of symmetric powers of the same 2-dimensional representation for example).

We consider universal nearly ordinary cohomology for $G$

$$
\mathcal{H}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)={\underset{ఙ}{\alpha}}_{\lim _{\text {ord }}} H^{q_{0}+l_{0}}\left(K_{\alpha, \alpha} ; \mathcal{O} / p^{\alpha} \mathcal{O}\right)
$$

which is canonically an $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$-module and may contain torsion classes. Its localization at a (contragredient) non-Eistenstein maximal ideal $\mathfrak{m}^{\vee}$ is expected to be free of rank 1 over $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}} \vee$ (cf. Conjecture 6.4; see Theorem 4.9 in [25] for a partial result towards this conjecture). Independently of this conjecture we show (cf. Theorem 6.3 in the text)

Theorem C. Assume $p \nmid(n+1) n$ and let $F / \mathbf{Q}$ denote a totally real or $C M$ field or otherwise assume the existence of Galois respresentations for torsion classes. Let $\mathfrak{m}$ denote a non-Eisenstein maximal ideal in $\mathbf{h}_{\mathrm{ord}}\left(K_{\infty, \infty} ; \mathcal{O}\right)$. Then there exists an element

$$
L_{p, \mathfrak{m}}^{\text {univ }} \in \mathcal{H}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))}
$$

with the following interpolation property. For every classical point

$$
\xi \in \operatorname{Spec}_{\mathbf{h}_{\mathrm{ord}}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}}(\bar{E})
$$

of balanced weight $\lambda$ and Nebentypus $\vartheta$, such that $s_{0}=\frac{1}{2}$ is critical for $L\left(s, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}\right)$, we have

$$
\begin{aligned}
\Omega_{\xi, p}^{-1} \cdot \xi^{\vee}\left(L_{p, \mathfrak{m}}^{\mathrm{univ}}\right) & =\int_{C_{F}\left(p^{\infty}\right)} d \mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}} \\
& =\mathfrak{N}\left(\mathfrak{f}_{\vartheta}\right)^{\frac{(n+1) n(n-1)}{6}} \cdot \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} G\left(\vartheta_{\mu, \nu}\right) \cdot \frac{L\left(\frac{1}{2}, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}\right)}{\Omega_{\xi}},
\end{aligned}
$$

where the second identity is valid whenever $\vartheta$ has fully supported constant conductor.
Here $\Omega_{\xi, p} \in \mathcal{O}[\xi]^{\times}$is a p-adic period and $\Omega_{\xi} \in \mathbf{C}^{\times}$is a complex period.
By Theorem A, both periods $\Omega_{\xi, p}$ and $\Omega_{\xi}$ may be normalized in such a way that they are invariant under twists of $\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}$ by finite order Hecke characters $\chi$ unramified outside $p$.

Previous results on the variation of special values in Hida families in our setup were limited to the case $n=1$ and $F$ totally real, cf. [14]. Even for GL(2) over a CM field our results are new.

A subtle consequence of Theorem C is that the 'big' $p$-adic $L$-function $L_{p, \mathfrak{m}}^{\text {univ }}$ does not see any possible defects to exact control, although we can currently only establish control up to finite error (cf. Theorem 4.14). This is in line with the following expectation.

Current methods of establishing $\mathcal{R}=\mathbb{T}$ theorems usually establish the freeness and cyclicity of $\mathcal{H}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))}$ over $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}$ as a byproduct. In this situation, we may interpret $L_{p, \mathfrak{m}}^{\text {univ }}$ as an element of the Hecke algebra, and if $\mathcal{R}=\mathbb{T}$ is established for $\mathfrak{m}$, then $L_{p, \mathfrak{m}}^{\text {univ }}$ may also be considered as an element of $\mathcal{R}$ (remark that unless the residual Galois representation $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible, we cannot define $\mathcal{R}$ as a classical deformation ring). Assuming this freeness, we establish with Theorem 6.7 a
refinement of Theorem C, which is in line with general expectations on the existence and properties of genuine $p$-adic $L$-functions (cf. Conjecture 4.2.1 and Question 4.4.1 in [35]).

As an application of Theorem C, we extend Theorem B to cases where the central value is the only critical value (cf. Corollary 6.16 in the text).

Theorem D. Let $p \nmid(n+1) n, F / \mathbf{Q}$ totally real or CM, or assume that Conjecture 4.9 holds for $F$. Let $\mathfrak{m}$ denote a non-Eisenstein maximal ideal in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$.

Assume that $\mathcal{X} \subseteq \operatorname{Spec} \mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}}(\bar{E})$ is an irreducible component containing a classical point $\xi$ of balanced weight such that $L\left(s, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}\right)$ admits at least two critical values.

Assume furthermore that the classical points in $\mathcal{X}$ of balanced cohomological weight $\lambda=\lambda_{n+1} \otimes \lambda_{n}$, are dense and that their L-functions admit $s_{0}=\frac{1}{2}$ as the unique critical value.

Then there exist irreducible regular algebraic cuspidal automorphic representations $\Pi^{\prime}$ and $\Sigma^{\prime}$ of cohomological weights $\lambda_{n+1}$ and $\lambda_{n}$ contributing to $\mathcal{X}$ satisfying

$$
\begin{equation*}
L\left(\frac{1}{2}, \Pi^{\prime} \widehat{\otimes} \Sigma^{\prime}\right) \neq 0 \tag{2}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
L\left(\frac{1}{2}, \Pi^{\prime} \widehat{\otimes} \Sigma^{\prime} \otimes \chi\right) \neq 0 . \tag{3}
\end{equation*}
$$

for all but finitely many finite order Hecke characters $\chi$ in the cyclotomic line.
In fact, (2) holds generically for classical points $\xi$ of weight $\lambda$. By construction, the representation $\Pi^{\prime} \widehat{\otimes} \Sigma^{\prime}$ also contributes to the non-Eisenstein component Spec $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}}$. In particular, it is nearly ordinary at $p$ and unramified outside $p$.

Remark that for $F$ not totally real, we know that there are components which do not contain a Zariski dense subset of classical points, cf. [3] for the case $n=1$, and the same is to be expected for general $n$.

Outline of the paper. In the first section, we recollect fundamental facts on Hecke algebras and $p$-stabilization. Our treatment diverges from $[75,56,50,51,52]$, since $p$ stabilization may not be achieved by the projectors constructed in loc. cit. due to the arbitrary ramification we allow: The known explicit formulae for the valuation of Whittaker functions (cf. [61, 64, 65]) imply the vanishing of the ordinary projection of the essential vectors in these cases.

In section 2, we prove local and global Birch Lemmata for general nearly ordinary automorphic representations in Theorems 2.8 and 2.9.

Section 3 contains the fundamental observations on the level of lattices in rational representations of $G$, which will allow us to prove the congruences which are necessary to establish Theorems A and C.

In section 4, exploiting results on the contribution of the residual spectrum of $\mathrm{GL}(n)$ to the cohomology of arithmetic groups $[46,47,66,17,78,27]$, we establish with Theorem 4.14 a control Theorem for $G$. We build on Hida's fundamental work [28, 29, 30, 31, 32, 33] on the case GL(2) and make extensive use of his results for $\operatorname{SL}(n)$ and (inner forms of) $\mathrm{GL}(n)$ in $[34,36]$.

Section 5 contains our construction of $p$-adic measures, both in the Iwasawa algebra for abelian interpolation and also on Hida's universal nearly ordinary cohomology with torsion coefficients. In this section we use the beautiful formalism of automorphic symbols introduced in [14] in the case of GL(2).

Finally, section 6 applies the results of the previous sections to the construction of abelian and non-abelian $p$-adic $L$-functions for automorphic representations of $G$. It also contains our applications to non-vanishing of central $L$-values. Two local non-vanishing
results are crucial to our approach: At places dividing $p$, we need the results on $p$ stabilization and on the local zeta integrals from sections 1.6 and 2 , and at archimedean places we rely on the non-vanishing results established in [55, 84].

The reader only interested in the construction of abelian $p$-adic $L$-functions or the proof of the non-vanishing result in Theorem B may safely skip sections 4.3 to 4.6 and sections 5.5, 6.2 and 6.4.

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## Notation

Fields and Algebras. In the body of the paper, $F$ denotes a number field, i.e. a finite extension of $\mathbf{Q}_{p}$ or of $\mathbf{Q}$.

In the first case, we write $\mathcal{O} \subseteq F$ for the valuation ring, $\mathfrak{p} \subseteq \mathcal{O}$ for the maximal ideal, and $q=\mathfrak{N}(\mathfrak{p})$ for the cardinality of the residue field $\mathcal{O} / \mathfrak{p}$. We normalize the valuation $|\cdot|$ on $F$ in such a way that $|\varpi|=q^{-1}$ for a uniformizer $\varpi \in \mathfrak{p}$.

In the case of a number field $F / \mathbf{Q}$, let $\mathbf{A}_{F}=\mathbf{A}_{\mathbf{Q}} \otimes_{\mathbf{Q}} F$ denote the ring of adèles over $F$ and abbreviate $\mathbf{A}=\mathbf{A}_{\mathbf{Q}}$. We write $\mathbf{A}_{F}=\mathbf{A}_{F, \infty} \otimes \mathbf{A}_{F}^{(\infty)}$ where $\mathbf{A}_{F}^{(\infty)}$ denotes the ring of finite adeles over $F$. More generally, $\mathbf{A}_{F}^{(p \infty)}$ denotes the ring of finite adeles outside $p$, etc.

The letter $p$ always denotes a prime in $\mathbf{Z} . E / \mathbf{Q}_{p}$ is a finite extension whose valuation ring is also denoted by $\mathcal{O}$ (there is no confusion possible). We let $\bar{E}$ denote an algebraic closure and $\overline{\mathcal{O}} \subseteq \bar{E}$ the corresponding valuation ring.
Groups. If $L$ is an algebraic or topological group, we write $L^{0}$ for the connected component containing the identity, and $\pi_{0}(L)=L / L^{0}$ for the component group. We denote by $L^{\text {der }}=\overline{[L, L]}$ the derived group. We let $B_{n}$ denote the standard Borel subgroup of $\mathrm{GL}_{n}$ of upper triangular matrices, $U_{n}$ denotes its unipotent radical and we choose the subgroup $B_{n}^{-} \subseteq \mathrm{GL}_{n}$ of lower triangular matrices as opposite to $B_{n}$. Then $T_{n}=B_{n} \cap B_{n}^{-}$is the standard diagonal torus.
$W\left(\mathrm{GL}_{n}, T_{n}\right)$ denotes the Weyl group of $\mathrm{GL}_{n}$ with respect to $T_{n}$, realized as the subgroup of permutation matrices in $\mathrm{GL}_{n}$. It is canonically isomorphic to the symmetric group $S_{n}$ on $\{1, \ldots, n\}$ : To each $\omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)$ we associate a permutation $\sigma \in S_{n}$ via the rule

$$
\omega \cdot b_{k}=b_{\sigma^{-1}(k)}
$$

for $1 \leq k \leq n$. Here $b_{k}$ denotes the $k$-th standard basis vector of $\mathbf{Z}^{n}$. Then for any $a=\left(a_{i}\right)_{1 \leq i \leq n} \in A^{n}, A$ a (commutative) ring, we have $\omega a=\left(a_{\sigma(i)}\right)_{1 \leq i \leq n}$. The map $\omega \mapsto \sigma^{-1}$ is an isomorphism $W\left(\mathrm{GL}_{n}, T_{n}\right) \rightarrow S_{n}$. Let $w_{n} \in W\left(\mathrm{GL}_{n}, T_{n}\right)$ denote the longest element. It is explicitly given by

$$
w_{n}=\left(\begin{array}{ll} 
& . \\
1 &
\end{array}\right)
$$

We have the embedding

$$
\begin{gathered}
j_{n} \quad: \mathrm{GL}(n) \rightarrow \operatorname{GL}(n+1), \\
\quad g \mapsto\left(\begin{array}{ll}
g & 0 \\
0 & 1
\end{array}\right)
\end{gathered}
$$

Set

$$
G:=\operatorname{res}_{F / \mathbf{Q}}(\mathrm{GL}(n+1) \times \operatorname{GL}(n))
$$

for a number field $F$. Consistent with the notation for $\mathrm{GL}_{n}$, let $B \subseteq G$ denote the standard upper triangular Borel subgroup with unipotent radical $U, B^{-}$its standard opposite of lower triangular matrices, and $T=\operatorname{res}_{F / \mathbf{Q}} T_{n+1} \times T_{n}$ the diagonal torus. The longest element in the Weyl group $W(G, T)$ with respect to $B$ is denoted

$$
w_{0}:=\left(w_{n+1}, w_{n}\right) \in G(\mathbf{Z})
$$

Dominance for $G$ and $\mathrm{GL}_{n}$ is understood with respect to $B$ and $B_{n}$ respectively.
Write $\Delta=j_{n} \times 1: \operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}(n) \rightarrow G$ for the diagonal embedding and set

$$
H:=\Delta\left(\operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}(n)\right) \subseteq G
$$

for the diagonally embedded copy of $\mathrm{GL}(n)$, which we freely identify with the latter. It comes with a distinguished character

$$
N_{H}:=N_{F / \mathbf{Q}} \circ \operatorname{det}: \quad H \rightarrow \mathrm{GL}_{1}
$$

defined over $\mathbf{Q}$.
Write $X_{\mathbf{Q}}(H)$ for the lattice of $\mathbf{Q}$-rational characters of $H$, which is generated by $N_{H}$. Put

$$
\widetilde{H}:=\bigcap_{\chi \in X_{\mathbf{Q}}(H)} \operatorname{ker} \chi
$$

For a dominant weight $\lambda$ of $G$, write $L_{\lambda, E}$ for the irreducible rational representation of highest weight $\lambda$ defined over some field $E / \mathbf{Q}$. We call $\lambda$ balanced, if

$$
\begin{equation*}
H^{0}\left(\widetilde{H} ; L_{\lambda, E}\right) \neq 0 \tag{4}
\end{equation*}
$$

This is the same to say that there is a non-zero $H$-invariant functional

$$
\eta_{j}: \quad L_{\lambda, E} \rightarrow E_{(j)}:=N_{H}^{\otimes j} \otimes E
$$

for some $j \in \mathbf{Z}$, i.e.

$$
\begin{equation*}
\operatorname{Hom}_{H}\left(L_{\lambda, E}, E_{(j)}\right) \neq 0 \tag{5}
\end{equation*}
$$

We call such a non-zero $\eta_{j}$ admissible for $\lambda$. We refer to section 2.4.5 of [71] for an explicit description of condition (5) in terms of highest weights for GL( $n$ ).

Let $\mathfrak{g}_{\mathbf{Z}}$ denote the Lie algebra of $G$ over $\mathbf{Z}$, i.e. more precisely we take the restriction of scalars $G=\operatorname{res}_{\mathcal{O}_{F} / \mathbf{Z}} \mathrm{GL}(n+1) \times \mathrm{GL}(n)$ of the ring of integers in $F$ to $\mathbf{Z}$ of the standard smooth group scheme $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ over $\mathcal{O}_{F}$ and use the same notation for $B, H, \ldots$, with the similar standard choice of smooth models over $\mathbf{Z}$ in each case. For any $\mathbf{Z}$-algebra $A / \mathbf{Z}$ we set

$$
\mathfrak{g}_{A}:=\mathfrak{g}_{\mathbf{z}} \otimes_{\mathbf{Z}} A
$$

again likewise for the other Lie algebras under consideration. Let $U\left(\mathfrak{g}_{A}\right)$ denote the universal enveloping algebra of $\mathfrak{g}_{A}$ over $A$.
Matrices. We introduce the matrix

$$
h_{n}:=\left(\begin{array}{cccc} 
& & & 1 \\
& w_{n} & & \vdots \\
& & & \vdots \\
0 & \ldots & 0 & 1
\end{array}\right) \in \operatorname{GL}_{n+1}(\mathbf{Z})
$$

For $e=\left(e_{1}, \ldots, e_{n}\right) \in \mathbf{Z}^{n}$ and $a \in A^{\times}$, define the matrix

$$
a^{e}:=\operatorname{diag}\left(a^{e_{1}}, \ldots, a^{e_{n}}\right) \in \operatorname{GL}_{n}(A)
$$

We consider any $\delta \in \mathbf{Z}$ as the constant tuple denoted $(\delta) \in \mathbf{Z}^{n}$. Then $a^{\delta} \cdot a^{e}=a^{(\delta)+e}$.
The character lattice of $T_{n}$ is canonically identified with $\mathbf{Z}^{n}$ in such a way that the dominant weights for $\mathrm{GL}(n)$ with respect to $B_{n}$ are ordered $n$-tuples $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ of decreasing integers

$$
\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{n}
$$

Then the sum $2 \rho_{n}$ of the positive roots of $\operatorname{GL}(n)$ is represented by the tuple

$$
2 \rho_{n}=(n-1, n-3, \ldots, 3-n, 1-n) \in \mathbf{Z}^{n}
$$

For $x \in A^{\times}$define

$$
\begin{gather*}
d_{x}:=\Delta(\operatorname{diag}(x, 1, \ldots, 1)) \in G(A)  \tag{6}\\
t_{x}:=x^{\rho_{n}+((n-1) / 2)}=\operatorname{diag}\left(x^{n}, x^{n-1}, \ldots, x\right) \in \operatorname{GL}_{n}(A) \tag{7}
\end{gather*}
$$

We also consider the latter element as an element of $G(A)$ via the diagonal embedding. With this notation at hand, set

$$
\begin{equation*}
h:=\left(h_{n} j_{n}\left(t_{-1}\right), \mathbf{1}_{n}\right) \in G\left(\mathbf{Z}_{p}\right) \tag{8}
\end{equation*}
$$

## 1. Hecke Algebras

1.1. Hecke pairs. For any Hecke pair $(R, S)$ consider the free $\mathbf{Z}$-module $\mathcal{H}_{\mathbf{Z}}(R, S)$ over the set of all double cosets $R s R$, which naturally embeds into the free $\mathbf{Z}$-module $\mathscr{R}_{\mathbf{Z}}(R, S)$ over the set of the right cosets $s R, s \in S$ :

$$
R s R=\bigsqcup_{i} s_{i} R \mapsto \sum_{i} s_{i} R
$$

Identify $\mathcal{H}_{\mathbf{Z}}(R, S)$ with its image under this embedding. Then $\mathcal{H}_{\mathbf{Z}}(R, S)$ is the $\mathbf{Z}$-module of $R$-invariants under the action

$$
R \times \mathscr{R}_{\mathbf{Z}}(R, S) \rightarrow \mathscr{R}_{\mathbf{Z}}(R, S), \quad(r, s R) \mapsto r s R
$$

It is well known that $\mathcal{H}_{\mathbf{Z}}(R, S)$ admits a structure of an associative $\mathbf{Z}$-algebra with multiplication

$$
\left(\sum_{i} s_{i} R\right) \cdot\left(\sum_{j} t_{j} R\right):=\sum_{i, j} s_{i} t_{j} R
$$

This algebra is unitary if and only if $R \cap S \neq \emptyset$. For any commutative ring $A$ set

$$
\mathcal{H}_{A}(R, S):=\mathcal{H}_{\mathbf{Z}}(R, S) \otimes_{\mathbf{Z}} A
$$

Then $\mathcal{H}_{A}(R, S)$ is an associative $A$-algebra, the Hecke algebra of the pair $(R, S)$ over $A$.
For a locally compact topological group $G$ and a compact open subgroup $K \leq G$, the module $\mathscr{R}_{A}(K, G)$ may be interpreted as the $A$-module of locally constant right $K$ invariant mappings $f: G \rightarrow A$ with compact support and $\mathcal{H}_{A}(K, G)$ is just the submodule of left $K$-invariant mappings. If $A \subseteq \mathbf{C}$, then multiplication is nothing but convolution

$$
\alpha * \beta: x \mapsto \int_{G} \alpha(g) \beta\left(x g^{-1}\right) d g
$$

where $d g$ is the right invariant Haar measure on $G$ which assigns measure 1 to $K$. This integral is eventually a finite sum with integer coefficients. Therefore, this interpretations is valid even without the assumption $A \subseteq \mathbf{C}$.
1.2. $p$-adic Hecke algebras. Let $F / \mathbf{Q}_{p}$ denote a $p$-adic field with integer ring $\mathcal{O}$. Write $\mathfrak{p} \subseteq \mathcal{O}$ for the maximal ideal, $\varpi \in \mathfrak{p}$ for a uniformizer and $q=\mathfrak{N}(\mathfrak{p})$ the norm of $\mathfrak{p}$. For any $\alpha \geq \alpha^{\prime} \geq 0$ write $I_{\alpha^{\prime}, \alpha}^{n} \subseteq \mathrm{GL}_{n}(\mathcal{O})$ for the subgroup of matrices becoming upper triangular modulo $\mathfrak{p}^{\alpha}$ and which lies in $U_{n}\left(\mathcal{O} / \mathfrak{p}^{\alpha^{\prime}}\right)$ when considered modulo $\mathfrak{p}^{\alpha^{\prime}}$. Set $I_{\alpha}^{n}:=I_{0, \alpha}^{n}$.

Recall that a tuple $e=\left(e_{1}, \ldots, e_{n}\right) \in \mathbf{Z}^{n}$ is dominant if

$$
e_{1} \geq e_{2} \geq \cdots \geq e_{n}
$$

Consider the semigroup

$$
\Delta_{F, n}:=T_{n}\left(\mathcal{O}^{\times}\right) \cdot\left\{\varpi^{e} \mid e \in \mathbf{Z}_{\geq 0}^{n} \text { dominant }\right\} \subseteq T_{n}(F)
$$

and define the Hecke algebra

$$
\mathcal{H}_{A}^{n}\left(\alpha^{\prime}, \alpha\right):=\mathcal{H}_{A}\left(I_{\alpha^{\prime}, \alpha}^{n}, I_{\alpha^{\prime}, \alpha}^{n} \Delta_{F, n} I_{\alpha^{\prime}, \alpha}^{n}\right)
$$

Whenever $e$ is dominant, $e_{n} \geq 0$ and $\alpha>0$ we define a Hecke operator

$$
\begin{equation*}
U_{\varpi}^{e}:=I_{\alpha^{\prime}, \alpha}^{n} \varpi^{e} I_{\alpha^{\prime}, \alpha}^{n}=\bigsqcup_{u \in U_{n}(\mathcal{O}) / \varpi^{e} U_{n}(\mathcal{O}) \varpi^{-e}} u \varpi^{e} I_{\alpha^{\prime}, \alpha} \tag{9}
\end{equation*}
$$

which depends on the choice of $\varpi$ whenever $\alpha^{\prime}>0$. It is well known that these operators commute [40, 23, 34, 36]. Moreover, we have the relation

$$
U_{\varpi}^{e} \cdot U_{\varpi}^{e^{\prime}}=U_{\varpi}^{e+e^{\prime}}
$$

for any dominant $e, e^{\prime} \in \mathbf{Z}_{\geq 0}^{n}$. Therefore, writing

$$
\omega_{\nu}:=(\underbrace{1, \ldots, 1}_{\nu}, \underbrace{0, \ldots, 0}_{n-\nu})
$$

for the $\nu$-th fundamental weight with $\nu$ leading 1 's and $n-\nu$ tailing 0 's, the operators

$$
V_{\nu}:=U_{\varpi}^{\omega_{\nu}}, \quad 1 \leq \nu \leq n
$$

generate $\mathcal{H}_{A}^{n}(0, \alpha)$.
Sending $U_{\varpi}^{e} \in \mathcal{H}_{A}^{n}(0, \alpha)$ to $U_{\varpi}^{e} \in \mathcal{H}_{A}^{n}\left(\alpha^{\prime}, \alpha\right)$ defines an inclusion

$$
\mathcal{H}_{A}^{n}(0, \alpha) \subseteq \mathcal{H}_{A}^{n}\left(\alpha^{\prime}, \alpha\right)
$$

depending on the choice of uniformizer $\varpi$. We see that

$$
\begin{equation*}
\mathcal{H}_{A}^{n}\left(\alpha^{\prime}, \alpha\right)=\mathcal{H}_{A}^{n}(0, \alpha)\left[I_{\alpha}^{n} / I_{\alpha^{\prime}, \alpha}^{n}\right]=\mathcal{H}_{A}^{n}(0, \alpha)\left[T_{n}\left(\mathcal{O} / \mathfrak{p}^{\alpha^{\prime}}\right)\right] \tag{10}
\end{equation*}
$$

which is a finitely generated commutative $A$-algebra (cf. [34, 36]).
1.3. Parabolic Hecke algebras. Define

$$
I_{\alpha^{\prime}}^{B_{n}}:=B_{n}(F) \cap I_{\alpha^{\prime}, \alpha}^{n} .
$$

As the notation suggests, this compact open subgroup of $B_{n}(F)$ is independent of $\alpha$. Restriction induces a canonical isomorphism

$$
\begin{equation*}
\mathcal{H}_{A}^{n}\left(\alpha^{\prime}, \alpha\right) \cong \mathcal{H}_{A}\left(I_{\alpha^{\prime}}^{B_{n}}, I_{\alpha^{\prime}}^{B_{n}} \Delta_{F, n} I_{\alpha^{\prime}}^{B_{n}}\right) \tag{11}
\end{equation*}
$$

which on cosets is explicitly given by the map

$$
g I_{\alpha^{\prime}, \alpha}^{n} \mapsto g I_{\alpha^{\prime}}^{B_{n}}
$$

Existence of the Iwasawa decomposition shows that this is well defined. Set

$$
\mathcal{H}_{A}^{B_{n}}\left(\alpha^{\prime}\right):=\mathcal{H}_{A}\left(I_{\alpha^{\prime}}^{B_{n}}, I_{\alpha^{\prime}}^{B_{n}}\left(T_{n}(F) \cap \mathcal{O}^{n}\right) I_{\alpha^{\prime}}^{B_{n}}\right)
$$

Then by (11), $\mathcal{H}_{A}^{n}\left(\alpha^{\prime}, \alpha\right)$ is a subalgebra of $\mathcal{H}_{A}^{B_{n}}\left(\alpha^{\prime}\right)$.
1.4. The $U_{\mathfrak{p}}$-operators. In $\mathcal{H}_{A}^{B_{n}}\left(\alpha^{\prime}\right)$ we have the Hecke operators

$$
\widetilde{U}_{i}:=I_{\alpha^{\prime}}^{B_{n}}\left(\begin{array}{ccc}
\mathbf{1}_{i-1} & 0 & 0 \\
0 & \varpi & 0 \\
0 & 0 & \mathbf{1}_{n-i}
\end{array}\right) I_{\alpha^{\prime}}^{B_{n}}, \quad 1 \leq i \leq n
$$

With (10) we see

$$
\mathcal{H}_{A}^{B_{n}}\left(\alpha^{\prime}\right)=\mathcal{H}_{A}^{B_{n}}(0)\left[T_{n}\left(\mathcal{O} / \mathfrak{p}^{\alpha^{\prime}}\right)\right]
$$

Proposition 1.1. We have for $0 \leq \nu \leq n$,

$$
\begin{equation*}
q^{\frac{\nu(\nu-1)}{2}} \cdot V_{\nu}=\widetilde{U}_{1} \widetilde{U}_{2} \cdots \widetilde{U}_{\nu} \tag{12}
\end{equation*}
$$

Proof. The proof of Lemma 4.1 in [56] remains valid in our setting.
Set

$$
U_{\mathfrak{p}}:=\prod_{\nu=1}^{n-1} V_{\nu}
$$

and

$$
U_{\mathfrak{p}}^{\prime}:=V_{n} \cdot U_{\mathfrak{p}}=\prod_{\nu=1}^{n} V_{\nu}
$$

1.5. Decomposition of Hecke polynomials. Consider the standard Hecke operators

$$
T_{\nu}:=I_{0,0} \varpi^{\omega_{\nu}} I_{0,0} \in \mathcal{H}_{A}^{n}(0,0)
$$

in the spherical Hecke algebra. The reciprocal Hecke polynomial

$$
\begin{equation*}
H_{F}(X):=\sum_{\nu=0}^{n}(-1)^{\nu} q^{\frac{\nu(\nu-1)}{2}} T_{\nu} X^{n-\nu} \in \mathcal{H}_{A}^{n}\left(\alpha^{\prime}, \alpha\right) \tag{13}
\end{equation*}
$$

admits a factorization

$$
\begin{equation*}
H_{F}(X)=\prod_{i=1}^{n}\left(X-\widetilde{U}_{i}\right) \tag{14}
\end{equation*}
$$

cf. [23, Theorem 2]. Although $H_{F}(X)$ is defined via the spherical Hecke algebra, it is relevant for us in the ramified case as well due to its relation to the Hodge polygon (cf. [36]).
1.6. p-stabilization in principal series representations. Let $E / \mathbf{Q}$ denote a field of characteristic 0 . Recall that the norm $|\cdot|$ on $F$ is normalized such that $|\varpi|=q^{-1}$ and consider its values in $E$.

For an admissible representation $(V, \pi)$ of $\mathrm{GL}_{n}(F)$ over $E$ the Jacquet module is defined as

$$
V_{B_{n}}:=V /\left\langle u v-v \mid u \in U_{n}(F), v \in V\right\rangle
$$

This is an admissible representation of $T_{n}(F)$. For a representation $W$ of $B_{n}(F)$ over $E$, define the space

$$
W^{U_{n}(\mathcal{O})}:=\left\{w \in W \mid \forall u \in U_{n}(\mathcal{O}): u w=w\right\}
$$

of invariants. If $W$ is of finite length, then $W^{U_{n}(\mathcal{O})}$ is naturally an $\mathcal{H}_{E}^{B_{n}}\left(\alpha^{\prime}\right)$-module for $\alpha^{\prime} \gg 0$ sufficiently large. The operator

$$
w \mapsto \int_{U_{n}(\mathcal{O})} u w d u
$$

for the normalized Haar measure $d u$ on $U_{n}(\mathcal{O})$ is a projector $W \rightarrow W^{U_{n}(\mathcal{O})}$. Therefore, taking invariants is an exact functor and we have an epimorphism

$$
\begin{equation*}
V^{U_{n}(\mathcal{O})} \rightarrow V_{B_{n}}^{U_{n}(\mathcal{O})} \tag{15}
\end{equation*}
$$

Fix a continuous character $\lambda: T_{n}(F) \rightarrow E^{\times}$, where $E^{\times}$is topologized with the discrete topology. We introduce a modified character

$$
\widetilde{\lambda}: \quad \operatorname{diag}\left(t_{1}, \ldots, t_{n}\right) \mapsto \prod_{i=1}^{n}\left|t_{i}\right|^{n-i} \lambda_{i}\left(t_{i}\right)
$$

where $\lambda_{i}$ denotes the restriction of $\lambda$ to the $i$-th component of $T_{n}(F)$. Considering $\tilde{\lambda}$ as a character of $B_{n}(F)$, define an algebraically induced principal series representation

$$
I_{B_{n}}^{\mathrm{GL}_{n}}(\lambda):=\operatorname{Ind}_{B_{n}(F)}^{\mathrm{GL}_{n}(F)} \widetilde{\lambda}
$$

where $\operatorname{Ind}_{B_{n}(F)}^{\mathrm{GL}_{n}(F)}$ denotes unnormalized algebraic induction in the category of smooth representations (cf. Définition 1.9 in [6]). Then, if $E=\mathbf{C}$,

$$
|\operatorname{det}(\cdot)|^{\frac{1-n}{2}} \otimes I_{B_{n}}^{\mathrm{GL}_{n}}(\lambda)
$$

agrees with the normalized induction of $\lambda$ from $B_{n}(F)$ to $\mathrm{GL}_{n}(F)$.
Recall that the Weyl group $W\left(\mathrm{GL}_{n}, T_{n}\right)$ acts naturally on the set of characters of $T_{n}(F)$ from the right. Then for every $\omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)$,

$$
\left(I_{B_{n}}^{\mathrm{GL}_{n}}(\lambda)\right)^{s s}=\left(I_{B_{n}}^{\mathrm{GL}_{n}}\left(\lambda^{\omega}\right)\right)^{s s}
$$

where the superscript $(-)^{s s}$ denotes semisimplification.
Proposition 1.2 (Hida). Let $E / \mathbf{Q}$ denote a field of characteristic 0 and let $\lambda: T_{n}(F) \rightarrow$ $E^{\times}$denote a continuous character as above. Assume that the characters $\lambda^{\omega}$ for $\omega \in$ $W\left(\mathrm{GL}_{n}, T_{n}\right)$ are pairwise distinct. Then
(i) The Jacquet module of $I_{B_{n}}^{\mathrm{GL}_{n}}(\lambda)$ is a semisimple $T_{n}(F)$-module and as such

$$
\begin{equation*}
\left(I_{B_{n}}^{\mathrm{GL}}(\lambda)\right)_{B_{n}}=\bigoplus_{\omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)} \widetilde{\lambda^{\omega}} \tag{16}
\end{equation*}
$$

(ii) Each $v$ in the $\widetilde{\lambda^{\omega}}$-isotypic component in (16) is a simultaneous eigenvector of $V_{\nu}$, $1 \leq \nu \leq n$, and

$$
\begin{equation*}
V_{\nu} v=q^{-\frac{\nu(\nu-1)}{2}}\left(\lambda^{\omega}\right)\left(\varpi^{\omega_{\nu}}\right) \cdot v \tag{17}
\end{equation*}
$$

Proof. This is a restatement of Proposition 5.4 and Corollary 5.5 in [36], taking into account that Hida works with a different normalization stemming from the right actions considered in loc. cit., where we work with left actions. Hida showed in particular that

$$
\begin{equation*}
\left(\left(I_{B_{n}}^{\mathrm{GL}_{n}}(\lambda)\right)_{B_{n}}\right)^{U_{n}(\mathcal{O})}=\left(I_{B_{n}}^{\mathrm{GL}_{n}}(\lambda)\right)_{B_{n}} \tag{18}
\end{equation*}
$$

Therefore, for $\alpha^{\prime}$ sufficiently large, $\mathcal{H}_{E}^{B_{n}}\left(\alpha^{\prime}\right)$ acts on the Jacquet module canonically.
Let $\pi$ be a generic irreducible admissible representation of $\mathrm{GL}_{n}(F)$ with Whittaker model $\mathscr{W}(\pi, \psi)$ with respect to a generic character $\psi$ of $U_{n}(F)$ trivial on $U_{n}(\mathcal{O})$.

Proposition 1.3. Let $\pi$ be a generic irreducible admissible representation of $\mathrm{GL}_{n}(F)$. Assume that $\pi$ occurs as a subquotient of $I_{B_{n}}^{\mathrm{GL}_{n}}(\lambda)$ for a character $\lambda$ satisfying the condition that $\lambda^{\omega}$ for $\omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)$ are pairwise distinct. Then every simultaneous $V_{\nu^{-}}$eigenvector $W \in \mathscr{W}(\pi, \psi)^{U_{n}(\mathcal{O})}, 1 \leq \nu<n$, with non-zero eigenvalues enjoys the following properties:
(i) There exists a unique $\omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)$ such that for all $1 \leq \nu \leq n$ :

$$
\begin{equation*}
V_{\nu} W=q^{-\frac{\nu(\nu-1)}{2}}\left(\lambda^{\omega}\right)\left(\varpi^{\omega_{\nu}}\right) \cdot W \tag{19}
\end{equation*}
$$

(ii) $W$ lies in a unique line in $\mathscr{W}(\pi, \psi)$ characterized by (19) and

$$
W\left(\mathbf{1}_{n}\right) \neq 0
$$

Proof. By equation (5.4) on page 678 of [36] we know that

$$
\begin{equation*}
W(\pi, \psi)^{U_{n}(\mathcal{O})}=W(\pi, \psi)_{B_{n}}^{U_{n}(\mathcal{O})} \oplus \operatorname{ker}\left(W(\pi, \psi)^{U_{n}(\mathcal{O})} \rightarrow W(\pi, \psi)_{B_{n}}^{U_{n}(\mathcal{O})}\right) \tag{20}
\end{equation*}
$$

and each $V_{\nu}$ acts nilpotently on the second summand on the right hand side. Therefore, by our hypothesis on $W$, this Whittaker vector maps to a non-zero $V_{\nu}$-eigenvector $W_{B_{n}} \in$ $W(\pi, \psi)_{B_{n}}^{U_{n}(\mathcal{O})}$ with same eigenvalue. By the hypothesis on $\lambda$, relation (17) shows that there is a unique $\omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)$ satisfying (i), because distinct elements of the Weyl group yield different sets of eigenvalues. This also implies the uniqueness of $W$ up to a scalar.

This also shows that the Jacquet module of $\pi$ admits $\widetilde{\lambda^{\omega}}$ as a direct summand. Therefore, by Frobenius reciprocity, $\pi$ occurs as a submodule of $I_{B_{n}}^{\mathrm{GL}_{n}}\left(\lambda^{\omega w_{n}}\right)$.

For the second statement in (ii), we may assume $E=\mathbf{C}$ and realize $I_{B_{n}}^{\mathrm{GL}_{n}}\left(\lambda^{\omega w_{n}}\right)$ inside the space of functions

$$
\left\{f: \mathrm{GL}_{n}(F) \rightarrow \mathbf{C} \mid \forall g \in \mathrm{GL}_{n}(F), b \in B_{n}(F): f(b g)=\widetilde{\lambda^{\omega w_{n}}}(b) f(g)\right\}
$$

Remark that for $\alpha \geq 1$, the double coset

$$
\begin{equation*}
B_{n}(F) w_{n} I_{\alpha, \alpha}^{n}=B_{n}(F) w_{n} U_{n}(\mathcal{O}) \tag{21}
\end{equation*}
$$

is independent of $\alpha$. Put

$$
f_{0}: \quad g \mapsto \begin{cases}\widetilde{\lambda^{\omega w_{n}}}(b), & \text { if } g=b w_{n} r, b \in B_{n}(F), r \in U_{n}(\mathcal{O}), \\ 0, & \text { else }\end{cases}
$$

By construction, $f_{0}$ is an element of $I_{B_{n}}^{\mathrm{GL}_{n}}\left(\lambda^{\omega w_{n}}\right)^{U_{n}(\mathcal{O})}$ satisfying

$$
\begin{equation*}
V_{\nu} f_{0}=q^{-\frac{\nu(\nu-1)}{2}} \lambda^{\omega}\left(\varpi^{\omega \nu}\right) \cdot f_{0}, \quad 1 \leq \nu \leq n \tag{22}
\end{equation*}
$$

For every $f \in V$ supported on $B_{n}(F) w_{n} B_{n}(F)$ and $g \in B_{n}(F) w_{n} B_{n}(F)$, consider for the normalized Haar measure $d u$ on $U_{n}(F)$ the integral

$$
\begin{equation*}
W_{f}(g):=\int_{U_{n}(F)} f\left(w_{n} u g\right) \bar{\psi}(u) d u \tag{23}
\end{equation*}
$$

By a well known result of Rodier, this integral converges and extends uniquely to an intertwining operator

$$
I_{B_{n}}^{\mathrm{GL}_{n}}\left(\lambda^{\omega w_{n}}\right) \rightarrow \operatorname{Ind}_{U_{n}(F)}^{\mathrm{GL}_{n}(F)} \psi
$$

cf. Corollary 1.8 in [5]. By (23), the vector $f_{0}$ is sent to a Whittaker vector which evaluates at $\mathbf{1}_{n}$ to

$$
\begin{aligned}
W_{f_{0}}\left(\mathbf{1}_{n}\right) & =\int_{U_{n}(F)} f_{0}\left(w_{n} u\right) \bar{\psi}(u) d u \\
& =\int_{U_{n}(\mathcal{O})} d u \\
& \neq 0
\end{aligned}
$$

because the integrand vanishes for $u \notin U_{n}(\mathcal{O})$ and assumes the value $f_{0}\left(w_{0}\right)=1$ for $u \in U_{n}(\mathcal{O})$.

Now since $\pi$ is a submodule of $I_{B_{n}}^{\mathrm{GL}}\left(\lambda^{\omega w_{n}}\right)$, the uniqueness of Whittaker models for $I_{B_{n}}^{\mathrm{GL}_{n}}\left(\lambda^{\omega w_{n}}\right)$ shows that we have a commutative square


Therefore, $W_{f_{0}}$ maps to a simultaneous eigenvector $W^{\prime}$ of the operators $V_{\nu}$ in $\mathscr{W}(\pi, \psi)$. By the multiplicity one property of the eigenspaces characterized by (22), there is a non-zero scalar $c \in F^{\times}$satisfying

$$
W=c \cdot W^{\prime}
$$

and therefore

$$
W(\mathbf{1})=c \cdot W^{\prime}\left(\mathbf{1}_{n}\right)=c \cdot W_{f_{0}}\left(\mathbf{1}_{n}\right) \neq 0
$$

This concludes the proof.
Remark 1.4. The relation between the eigenvalues of the operators $V_{\nu}$ and the local $L$ function $L(s, \pi)$ attached to $\pi$ as in [20] is the following. Let

$$
L(s, \pi)=\prod_{i=1}^{n} \frac{1}{1-\alpha_{i} q^{-s}}
$$

for $\alpha_{i} \in \mathbf{C}$. By Corollary 3.6 in loc. cit. we know that there is a polynomial $P(X) \in \mathbf{C}[X]$ satisfying $P(1)=1$ and

$$
L(s, \pi)=P\left(q^{-s}\right) \cdot L\left(s, I_{B_{n}}^{\mathrm{GL}_{n}}(\lambda)\right)=\frac{P\left(q^{-s}\right)}{\prod_{\substack{i=1 \\ \lambda_{i}\left(\mathcal{O}^{\times}\right)=1}}^{n}\left(1-\lambda_{i}(\varpi) q^{-\frac{n-1}{2}-s}\right)},
$$

where the product in the denominator on the right hand side runs over all $i$ for which $\lambda_{i}$ is unramified. Hence, $\alpha_{i}$ equals either $\lambda_{i}(\varpi) q^{-\frac{n-1}{2}} \neq 0$ or 0 . Assuming without loss of generality that $\alpha_{1}, \ldots, \alpha_{\ell} \neq 0$ and

$$
\alpha_{\ell+1}=\cdots=\alpha_{n}=0
$$

for some $0 \leq \ell \leq n$, we have $\ell=n$ if and only if $\pi$ is spherical. Furthermore, $\pi$ is Iwahori-spherical if and only if the characters $\lambda_{1}, \ldots, \lambda_{n}$ are unramified.

## 2. A Birch Lemma for $p$-NEARLY ordinary automorphic forms

In this section we generalize the Birch Lemma from [50, 51] to arbitrary p-nearly ordinary forms. The local main result is Theorem 2.8 , whose proof will occupy section 2.1. The global main result is Theorem 2.9.
2.1. The twisted local Zeta integral. We use the notation of [50, Section 2] in the modified setting of [51] with minor modifications. In particular, in this section $F$ denotes a non-archimedean local field with valuation ring $\mathcal{O} \subseteq F$. Fix again a uniformizer $\varpi \in \mathcal{O}$ and write $\mathfrak{p} \subset \mathcal{O}$ for the maximal ideal and $q=\mathfrak{N}(\mathfrak{p})$ as before.

If $\chi: F^{\times} \rightarrow \mathbf{C}^{\times}$is a quasi-character, we write $\mathfrak{f}_{\chi}$ for its conductor and assume that it is generated by $f_{\chi}=\varpi^{e_{\chi}}, e_{\chi} \geq 0$. By abuse of notation, we occasionally write $\chi(g)$ for $\chi(\operatorname{det}(g)), g \in \mathrm{GL}_{n}(F)$.

Fix a non-trivial additive character $\psi: F \rightarrow \mathbf{C}^{\times}$of conductor $\mathcal{O}$. The choice of $\psi$ normalizes the Gauß sum

$$
G(\chi):=\sum_{x+\mathfrak{f}_{\chi} \in\left(\mathcal{O} / \mathfrak{f}_{\chi}\right)^{\times}} \chi\left(x / f_{\chi}\right) \psi\left(x / f_{\chi}\right)=\frac{1}{\mathfrak{N}\left(\mathfrak{f}_{\chi}\right)} \cdot \int_{F^{\times}} \chi(x) \psi(x) d x
$$

where the second identity is only valid for $\mathfrak{f}_{\chi} \neq 1$ and $d x$ denotes the additive Haar measure on $F$ which attaches volume 1 to $\mathcal{O}$.

Implicit in the second identity is the fact that for any $0 \neq g \in \mathcal{O}$, we have with $\mathfrak{h}:=\mathfrak{f} \cap g \mathcal{O}$ the relation

$$
\sum_{x+\mathfrak{h} \in(\mathcal{O} / \mathfrak{h})^{\times}} \chi(x / g) \psi(x / g)= \begin{cases}\mathfrak{N}\left(\mathfrak{h} / \mathfrak{f}_{\chi}\right) \cdot \chi\left(g / f_{\chi}\right) \cdot G(\chi), & \text { if } \mathfrak{f}_{\chi}=\mathcal{O} g  \tag{24}\\ 0, & \text { otherwise }\end{cases}
$$

Extend $\psi$ to $U_{n}(F)$ by the rule

$$
\begin{equation*}
\psi(u):=\prod_{i=1}^{n-1} \psi\left(u_{i i+1}\right) \tag{25}
\end{equation*}
$$

for $u=\left(u_{i j}\right) \in U_{n}(F)$.
The Haar measure $d g$ on $\mathrm{GL}_{n}(F)$ is normalized such that the maximal compact subgroup $\mathrm{GL}_{n}(\mathcal{O})$ has measure 1 .

Denote by $I^{n}$ the Iwahori subgroup of $\mathrm{GL}_{n}(\mathcal{O})$, i.e. the group of matrices $g \in \mathrm{GL}_{n}(\mathcal{O})$ that become upper triangular modulo $\mathfrak{p}$. Fixing another element $f=\varpi^{\alpha} \in \mathcal{O}$ with $\alpha \geq 1$, write $I_{\alpha}^{n}$ for the subgroup of elements of $\mathrm{GL}_{n}(\mathcal{O})$ lying in $B_{n}(\mathcal{O} / \mathfrak{f})$ modulo $\mathfrak{f}:=\mathcal{O} \cdot f$.

All quantities that are defined relative to $f$ in [50] keep their meaning, i.e. the matrices $A_{n}, B_{n}, C_{n}, D_{n}, E_{n}, \phi_{n}$ are all defined with respect to $f=\varpi^{\alpha}$. We recall their definition below.

In particular,

$$
D_{n}:=\operatorname{diag}\left(f^{-(n-1)}, f^{-(n-3)}, \ldots, f^{n-3}, f^{n-1}\right) \in \mathrm{GL}_{n}(F)
$$

and for any $\delta \in \mathbf{Z}$ the definition of the linear form

$$
\lambda_{n}^{\delta}: F^{n \times n} \rightarrow F, \quad g \mapsto \varpi^{-\delta} \cdot b_{n}^{t} \cdot g \cdot \phi_{n}
$$

where

$$
\phi_{n}:=\left(f^{-n}, f^{-(n-1)}, \ldots, f^{-1}\right)^{t}
$$

Again for $\delta \in \mathbf{Z}$ we have

$$
\begin{gathered}
j_{n, \delta}: \mathrm{GL}_{n}(F) \rightarrow \mathrm{GL}_{n+1}(F), \\
g \mapsto\left(\begin{array}{cc}
g & 0 \\
0 & \varpi^{\delta}
\end{array}\right) .
\end{gathered}
$$

Then $j_{n}=j_{n, 0}$.
Deviating slightly from notation in previous works, put

$$
J_{\ell}^{n}:=\operatorname{ker}\left[\mathrm{GL}_{n}(\mathcal{O}) \rightarrow \operatorname{GL}_{n}\left(\mathcal{O} / \mathfrak{f}^{\ell}\right)\right]
$$

Assume $\ell \geq 2 n$. Then

$$
J_{\ell}^{n} \subseteq I_{\alpha}^{n} \cap w_{n} D_{n}^{-1} I_{\alpha}^{n} D_{n} w_{n}
$$

Fix a system $R_{\ell}$ of representatives for $\mathcal{O} / \mathfrak{f}^{\ell}$ and let $R_{\ell}^{\times} \subseteq R_{\ell}$ be a system of representatives of $\left(\mathcal{O} / \mathfrak{f}^{\ell}\right)^{\times}$. To simplify notation in the sequel, we assume that

$$
\begin{equation*}
0, \pm 1, \pm f, \ldots, \pm f^{\ell-1} \in R_{\ell} \tag{26}
\end{equation*}
$$

Set

$$
\Re_{n, \ell}:=\left\{\left(r_{i j}\right) \in I^{n} \mid r_{i j} \in R_{\ell}\right\} .
$$

As in [50] (where $l$ is our $\ell$ ), $\Re_{n, \ell}$ is a system of representatives for $I^{n} / J_{\ell}^{n}$ and as such may be endowed with the natural group structure which is induced by matrix multiplication modulo $\mathfrak{f}^{\ell}$.

Define for any $\omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)$,

$$
\mathfrak{R}_{n, \ell}^{\omega}:=\Re_{n, \ell} \cap \omega^{-1} B_{n}^{-}(\mathcal{O}) \omega
$$

Then $\mathfrak{R}_{n, \ell}^{\omega}$ is a subgroup of $\Re_{n, \ell}$.

We consider the action of the compact torus

$$
T_{n}(\mathcal{O})=\left(\mathcal{O}^{\times}\right)^{n}
$$

on $\mathrm{GL}_{n}(F)$, which for $\gamma=\left(\gamma_{1}, \ldots, \gamma_{n}\right) \in T_{n}(\mathcal{O})$ is given by

$$
{ }^{\gamma} \cdot: \operatorname{GL}_{n}(F) \rightarrow \mathrm{GL}_{n}(F), \quad g \mapsto^{\gamma} g:=g \cdot \operatorname{diag}\left(\gamma_{1}, \ldots, \gamma_{n}\right) .
$$

Then $T_{n}(\mathcal{O})$ acts naturally on the set of representatives $\mathfrak{R}_{n, \ell}^{\omega}$ via its action on the quotient $I^{n} / J_{\ell}^{n}$. This action factors over the finite torus

$$
T_{n}\left(\mathcal{O} / \mathfrak{f}^{\ell}\right)=\left(\mathcal{O}^{\times} /\left(1+\mathfrak{f}^{\ell}\right)\right)^{n}
$$

The action of the latter on $\mathfrak{R}_{n, \ell}^{\omega}$ is faithful. We fix a system of representatives $\mathcal{T}_{n, \ell} \subseteq T_{n}(\mathcal{O})$ for $T_{n}\left(\mathcal{O} / \mathfrak{f}^{\ell}\right)$.

If $\sigma \in S_{n}$ corresponds to $\omega$ and $\sigma(n)=n$, set

$$
\tilde{\mathfrak{R}}_{n, \ell}^{\omega}:=\left\{\left(r_{i j}\right) \in \mathfrak{R}_{n, \ell}^{\omega} \mid r_{n 1}=f^{n-1}, r_{n j}=-f^{n-j}, 2 \leq j \leq n\right\} .
$$

Proposition 2.1 (Proposition 2.4 in [50]). If $\sigma(n)=n$ we have for any $r \in \tilde{\mathfrak{R}}_{n, \ell}^{\omega}$

$$
\#\left(T_{n}(\mathcal{O}) \cdot r \cap \tilde{\mathfrak{R}}_{n, \ell}^{\omega}\right)=\mathfrak{N}(\mathfrak{f})^{\frac{n(n-1)}{2}}
$$

In other words, the orbit of $r$ under the action of $T_{n}(\mathcal{O})$ on $\mathfrak{R}_{n, \ell}^{\omega}$ contains $\mathfrak{N}(\mathfrak{f})^{\frac{n(n-1)}{2}}$ elements of $\tilde{\mathfrak{R}}_{n, \ell}^{\omega}$.

As in [50], define the matrices

$$
\begin{gathered}
A_{n}:=\left(\begin{array}{ccccc}
1 & f^{-1} & 0 & \ldots & 0 \\
0 & 1 & -f^{-1} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & 1 & -f^{-1} \\
0 & \ldots & \ldots & 0 & 1
\end{array}\right) \in I^{n}, \\
B_{n}
\end{gathered}:=\left(\begin{array}{ccccc}
1 & 0 & \ldots & \ldots & 0 \\
f & -1 & \ddots & & \vdots \\
0 & f & -1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & f & -1
\end{array}\right) \in I^{n},
$$

and

$$
C_{n}:=\left(\begin{array}{ccccc}
1 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
f^{n-1} & -f^{n-2} & \ldots & -f & -1
\end{array}\right) \in I^{n},
$$

subject to the convention that $B_{1}=C_{1}=\mathbf{1}_{1}$ and $B_{0}:=\mathbf{1}_{0}$. This guarantees that for all $n \geq 0$ relation (41) below holds.

Define the projection

$$
p: F^{n \times n} \rightarrow F^{n-1 \times n-1}, \quad\left(g_{i j}\right) \mapsto\left(g_{i j}\right)_{1 \leq i, j \leq n-1} .
$$

Proposition 2.2 (Proposition 2.5 in [50]). If $\sigma(n)=n$, we have for $\tilde{\omega}:=p(\omega), \tilde{r}:=p(r)$,

$$
\# \tilde{R}_{n, \ell}^{\omega}=\# \Re_{n-1, \ell}^{\tilde{\omega}}
$$

Furthermore, the projection $p$ induces a bijection

$$
p: \tilde{\mathfrak{R}}_{n, \ell}^{\omega} \rightarrow \mathfrak{R}_{n-1, \ell}^{\tilde{\omega}}
$$

and

$$
\mathrm{GL}_{n-1}(F) \rightarrow \mathrm{GL}_{n}(F), \quad \tilde{g} \mapsto j_{n-1}(\tilde{g}) \cdot C_{n}
$$

induces the inverse of $p$.
Recall the well known decomposition

$$
\begin{equation*}
\mathrm{GL}_{n}(F)=\bigsqcup_{\substack{\omega \in W\left(\mathrm{GL}_{n}, T_{n}\right) \\ e \in \mathbf{Z}^{n}}} U_{n}(F) \varpi^{e} \omega I^{n} \tag{27}
\end{equation*}
$$

by Iwahori-Matsumoto [40], Proposition 2.33, and Satake [74], section 8.2. The definition of $\Re_{n, \ell}^{\omega}$ is justified by the following refinement of (27).

Proposition 2.3 (Proposition 2.2 in [50]). The set $\varpi^{e} \omega \mathfrak{R}_{n, \ell}^{\omega}$ is a system of representatives for the double cosets

$$
U_{n}(F) \varpi^{e} \omega r J_{\ell}^{n}, \quad r \in I^{n}
$$

in $U_{n}(F) \varpi^{e} \omega I^{n}$. Fix an $\ell(e) \geq 2 n$ for any $e \in \mathbf{Z}^{n}$. Then

$$
\mathrm{GL}_{n}(F)=\bigsqcup_{\substack{\left.e \in \mathbb{Z}^{n} \\ \omega \in W \in \mathrm{GL}_{n}, T_{n}\right) \\ r \in \mathfrak{R}_{n, \ell(e)}^{\omega}}} U_{n}(F) \varpi^{e} \omega r J_{\ell(e)}^{n} .
$$

We will also need the following refinement of Proposition 2.3 in [50].
Proposition 2.4. For any $e \in \mathbf{Z}^{n}, \omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)$ and $r \in \mathfrak{R}_{n, \ell}^{\omega}$ the measure

$$
\int_{U_{n}(\mathcal{O}) \varpi^{e} \omega r J_{\ell}^{n}} d g
$$

is independent of $\omega$ and $r$. If $e \in \mathbf{Z}^{n}$ and $\ell>0$, then

$$
\int_{U_{n}(\mathcal{O}) \varpi^{e} \omega r J_{\ell}^{n}} d g=\left[\varpi^{e} U_{n}(\mathcal{O}) \varpi^{-e}: U_{n}(\mathcal{O})\right] \cdot \prod_{\mu=1}^{n}\left(1-\mathfrak{N}(\mathfrak{p})^{-\mu}\right)^{-1} \cdot \mathfrak{N}(\mathfrak{f})^{-\frac{\ell(n+1) n}{2}}
$$

Proof. The first statement and the second in the case $e=0$ follows as in Proposition 2.3 in [50]. For general dominant $e \in \mathbf{Z}^{n}$, observe that

$$
\int_{U_{n}(\mathcal{O}) \varpi^{e} J_{\ell}^{n}} d g=\left[\varpi^{e} U_{n}(\mathcal{O}) \varpi^{-e}: U_{n}(\mathcal{O})\right] \cdot \int_{U_{n}(\mathcal{O}) \omega r J_{\ell}^{n}} d g
$$

whence the claim.
Let $\theta_{1}, \ldots, \theta_{n+1}: F^{\times} \rightarrow \mathbf{C}^{\times}$denote quasi-characters with conductors dividing $f$. These characters give rise to a character

$$
\begin{align*}
& \theta: \quad I_{\alpha}^{n+1} \rightarrow \mathbf{C}^{\times}  \tag{28}\\
&\left(r_{i j}\right)_{i j} \mapsto \prod_{i=1}^{n+1} \theta_{i}\left(r_{i i}\right) .
\end{align*}
$$

Assume furthermore given another set of (finite order) characters $\theta_{1}^{\prime}, \ldots, \theta_{n}^{\prime}$ again with conductors dividing $f$. These give likewise rise to a character

$$
\begin{equation*}
\theta^{\prime}: \quad I_{\alpha}^{n} \rightarrow \mathbf{C}^{\times} \tag{29}
\end{equation*}
$$

Remark that

$$
J_{1}^{n+1} \subseteq \operatorname{ker} \theta, \quad J_{1}^{n} \subseteq \operatorname{ker} \theta^{\prime}
$$

Let $w$ and $v$ denote $\psi$ - resp. $\psi^{-1}$-Whittaker functions on $\mathrm{GL}_{n+1}(F)$ resp. $\mathrm{GL}_{n}(F)$, with the additional property that $w$ and $v$ transform under $I_{\alpha}^{n+1}$ and $I_{\alpha}^{n}$ (from the right) via $\theta$ resp. $\theta^{\prime}$ :

$$
\begin{align*}
\forall g \in \mathrm{GL}_{n+1}(F), r \in I_{\alpha}^{n+1}: & w(g r) & =\theta(r) \cdot w(g),  \tag{30}\\
\forall g \in \mathrm{GL}_{n}(F), r \in I_{\alpha}^{n}: & v(g r) & =\theta^{\prime}(r) \cdot v(g) . \tag{31}
\end{align*}
$$

We need the following statement which generalizes Lemma 2.6 in [50] and Lemma 4.1 in [51].

Lemma 2.5. We have for any $\delta \in \mathbf{Z}$ an identity

$$
\begin{gather*}
w\left(j_{n,-\delta}(g) C_{n+1} \cdot D_{n+1} w_{n+1}\right) v(g)= \\
\psi\left(\lambda_{n}^{\delta}\left(g B_{n}\right)\right) w\left(j_{n,-\delta}\left(g B_{n} \cdot D_{n} w_{n}\right)\right) v\left(g B_{n}\right) \cdot \theta^{\prime}\left(B_{n}^{-1}\right) \tag{32}
\end{gather*}
$$

Proof. The proof proceeds as the proof of Lemma 4.1 in [51], with the following additional observations:

$$
w_{n+1} D_{n+1}^{-1} A_{n+1} D_{n+1} w_{n+1} \in J_{1}^{n+1} \subseteq \operatorname{ker} \theta
$$

and

$$
v(g)=\theta^{\prime}\left(B_{n}^{-1}\right) \cdot v\left(g B_{n}\right)
$$

Let $\chi$ be a quasi-character of $F$. We suppose that the condition

$$
\begin{equation*}
\alpha \geq e_{\chi} \tag{33}
\end{equation*}
$$

is satisfied in all what follows. Let $e \in \mathbf{Z}^{n}, \omega \in W\left(\mathrm{GL}_{n}, T_{n}\right)$,

$$
\begin{equation*}
\ell \geq \max \left\{2 n, n-e_{1} / \alpha, \ldots, n-e_{n} / \alpha\right\} \tag{34}
\end{equation*}
$$

and $\delta \in \mathbf{Z}$.
The decomposition of $\mathfrak{R}_{n, \ell}^{\omega}$ into $T_{n}(\mathcal{O})$-orbits leads to partial sums

$$
\begin{aligned}
Z_{n}(s ; w, v, \delta, e, \omega, r):= & \sum_{\gamma \in \mathcal{T}_{n, \ell}} \psi\left(\lambda_{n}^{\delta}\left(\varpi^{e} \omega^{\gamma} r\right)\right) \cdot w\left(\varpi^{e} \omega^{\gamma} r \cdot D_{n} w_{n}\right) \\
& v\left(\varpi^{e} \omega^{\gamma} r\right) \cdot \chi\left(\varpi^{e} \omega^{\gamma} r\right) \cdot\left|\operatorname{det}\left(\varpi^{e} \omega^{\gamma} r\right)\right|^{s-\frac{1}{2}}
\end{aligned}
$$

for any $r \in \mathfrak{R}_{n, \ell}^{\omega}$ and $s \in \mathbf{C}$.
For any $g \in \mathrm{GL}_{n+1}(F)$ set

$$
\theta^{g}(r):=\theta\left(g r g^{-1}\right)
$$

Lemma 2.6. Let $s \in \mathbf{C}, e \in \mathbf{Z}^{n}, \omega \in W\left(\mathrm{GL}_{n}, T_{n}\right), \ell \in \mathbf{Z}$ subject to (34), and $\delta \in \mathbf{Z}$. Then
(i) $Z_{n}(s ; w, v, \delta, e, \omega, r)$ is independent of the choice of $\mathcal{T}_{n, \ell}$.
(ii) Assume that for all $1 \leq \nu \leq n$ and $\mu=n+1-\nu$,

$$
\begin{equation*}
e_{\chi \theta_{\mu} \theta_{\nu}^{\prime}}>0 \tag{35}
\end{equation*}
$$

is satisfied. Then $Z_{n}(s ; w, v, \delta, e, \omega, r)$ vanishes unless

$$
\begin{equation*}
e_{n}=\delta+\alpha \cdot(n+1-\sigma(n))-e_{\chi \theta_{n+1-\sigma(n)} \theta_{\sigma(n)}^{\prime}} \tag{36}
\end{equation*}
$$

(iii) If conditions (35) and (36) are satisfied, and if the exponent in (35) is independent of $\nu$, then $Z_{n}(s ; w, v, \delta, e, \omega, r)$ vanishes unless $\sigma(n)=n$ and for $1 \leq \nu \leq n$,

$$
\begin{equation*}
\left|r_{n \nu}\right|=\left|f^{n-\nu}\right| \tag{37}
\end{equation*}
$$

(iv) If the hypotheses of (iii) are satisfied, we may assume without loss of generality that

$$
\begin{equation*}
r_{n 1}=f^{n-1}, \quad \text { and } \quad r_{n \nu}=-f^{n-\nu} \text { for } 2 \leq \nu \leq n \tag{38}
\end{equation*}
$$

If additionally, (35) holds for all $1 \leq \nu \leq \mu \leq n$, then

$$
\begin{aligned}
Z_{n}(s ; w, v, \delta, e, \omega, r)= & \chi \theta^{w_{n}} \theta^{\prime}\left(B_{n}\right) \cdot \prod_{\nu=1}^{n} \chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot \mathfrak{N}\left(\mathfrak{f}^{\ell} / \mathfrak{f}_{\chi \theta_{\nu}^{w_{n}}} \theta_{\nu}^{\prime}\right) \cdot G\left(\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\right) \\
& w\left(\varpi^{e} \omega r \cdot D_{n} w_{n}\right) \cdot v\left(\varpi^{e} \omega r\right) \cdot \chi\left(\varpi^{e} \omega r\right) \cdot\left|\operatorname{det}\left(\varpi^{e}\right)\right|^{s-\frac{1}{2}}
\end{aligned}
$$

Proof. The claimed property of $Z_{n}(s ; w, v, \delta, e, \omega, r)$ in (i) is clear by definition.
The relation

$$
\varpi^{e} \omega^{\gamma} r \cdot D_{n} w_{n}=\varpi^{e} \omega r \cdot D_{n} w_{n} \cdot\left(w_{n}^{\gamma} \mathbf{1}_{n} w_{n}\right)
$$

with

$$
w_{n}^{\gamma} \mathbf{1}_{n} w_{n} \in I_{\alpha}^{n}
$$

yields

$$
\begin{aligned}
Z_{n}(s ; w, v, \delta, e, \omega, r)= & \chi \theta^{\prime}(r) \cdot \sum_{\gamma \in \mathcal{T}_{n, \ell}} \chi \theta^{w_{n}} \theta^{\prime}\left({ }^{\gamma} \mathbf{1}_{n}\right) \cdot \psi\left(\lambda_{n}^{0}\left(\varpi^{e-(\delta)} \omega^{\gamma} r\right)\right) . \\
& w\left(\varpi^{e} \omega r \cdot D_{n} w_{n}\right) \cdot v\left(\varpi^{e} \omega\right) \cdot\left|\operatorname{det}\left(\varpi^{e}\right)\right|^{s-\frac{1}{2}} \cdot \chi\left(\varpi^{e} \omega\right) .
\end{aligned}
$$

Unfolding gives

$$
\psi\left(\lambda_{n}\left(\varpi^{e-(\delta)} \omega^{\gamma} r\right)\right)=\prod_{\nu=1}^{n} \psi\left(\varpi^{e_{n}-\delta} f^{\nu-n-1} r_{\sigma(n) \nu} \cdot \gamma_{\nu}\right)
$$

which in turn shows that

$$
\begin{align*}
& \sum_{\gamma \in \mathcal{T}_{n, \ell}} \chi \theta^{w_{n}} \theta^{\prime}\left({ }^{\gamma} \mathbf{1}_{n}\right) \cdot \psi\left(\lambda_{n}\left(\varpi^{e-(\delta)} \omega^{\gamma} r\right)\right) \\
= & \prod_{\nu=1}^{n} \sum_{\gamma_{\nu} \in\left(\mathcal{O} / \mathfrak{f}^{l}\right)^{\times}} \chi \theta_{\nu}^{\prime} \theta_{\nu}^{w_{n}}\left(\gamma_{\nu}\right) \cdot \psi\left(\varpi^{e_{n}-\delta} f^{\nu-n-1} r_{\sigma(n) \nu} \cdot \gamma_{\nu}\right) . \tag{39}
\end{align*}
$$

In the case $\nu=\sigma(n)$, the entry $r_{\sigma(n) \nu}$ is a unit, and hypothesis (35) together with the vanishing relation (24) therefore implies (ii).

Under the hypotheses of (iii), if $\sigma(n) \neq n$, we have $\sigma(n)<n$ and we see with (36) that therefore

$$
e_{n}-\delta-\alpha>-e_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}
$$

for $1 \leq \nu \leq n$ (recall that the right hand side is independent of $\nu$ by our hypothesis). In the case $\nu=n$, we obtain

$$
\left|\varpi^{e_{n}-\delta} f^{n-n-1} r_{\sigma(n) n} \cdot \gamma_{n}\right| \leq\left|\varpi^{e_{n}-\delta} f^{-1}\right|<\left|f_{\chi \theta_{1} \theta_{n}^{\prime}}^{-1}\right|
$$

whence $Z_{n}(s ; w, v, \delta, e, \omega, r)$ vanishes by (24).
In the case $\sigma(n)=n$, by (24) once again any violation of condition (37) implies vanishing of (39). This proves (iii).

In case (iv), we may by (i) and (37) assume without loss of generality that $r_{n} \bullet$ is chosen as in (38). Then by (24), the value of (39) is given by

$$
\chi \theta^{w_{n}} \theta^{\prime}\left(B_{n}\right) \prod_{\nu=1}^{n} \chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot \mathfrak{N}\left(\mathfrak{f}^{\ell} / \mathfrak{f}_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot G\left(\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\right)
$$

and (iv) follows.

Under the hypotheses in statement (iii) of the previous lemma set for $\gamma \in \mathbf{Z}$

$$
d_{n, \gamma}:=(\gamma \cdot(n+1-i))_{1 \leq i \leq n} \in \mathbf{Z}^{n}
$$

For $1 \leq \mu \leq n$ we denote by $w_{\mu} \in \mathrm{GL}(n+1)$ the long Weyl element in $S_{\mu}$ as embedded as antidiagonal matrix in the upper $\mu \times \mu$-block.

Lemma 2.7. Assume that for all $1 \leq \mu \leq n$ and all $1 \leq \nu \leq \mu$ :

$$
e_{\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}}>0
$$

Assume additionally that these exponents are independent of $\mu$ and $\nu$. Let $\mathfrak{f}_{\chi \theta \theta^{\prime}}$ denote the common conductor of $\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}$ and define

$$
\gamma:=\alpha-e_{\chi \theta_{\nu}^{w_{n}}} \theta_{\nu}^{\prime}
$$

Then for all $e \in \mathbf{Z}^{n}, \omega \in W\left(\mathrm{GL}_{n}, T_{n}\right), \ell \geq \max \left\{2 n, n-e_{1} / \alpha, \ldots, n-e_{n} / \alpha\right\}$ and $\delta \in \mathbf{Z}$, we have

$$
\begin{aligned}
& \operatorname{vol}\left(U_{n}(\mathcal{O}) \varpi^{e} J_{\ell}^{n}\right) \cdot \sum_{g \in \varpi^{e} \omega \mathfrak{R}_{n, \ell}^{\omega}} \psi\left(\lambda_{n}^{\delta}(g)\right) \cdot w\left(g \cdot D_{n} w_{n}\right) \cdot v(g) \cdot \chi(\operatorname{det}(g)) \cdot|\operatorname{det}(g)|^{s-\frac{1}{2}} \\
= & \mathfrak{N}\left(\mathfrak{f}_{\chi \theta \theta^{\prime}}\right)^{-\frac{(n+1) n(n-1)}{2}} \cdot \prod_{\mu=1}^{n} \theta^{w_{\mu}}\left(B_{\mu}\right) \cdot \prod_{\nu=1}^{\mu} \mathfrak{N}\left(\mathfrak{f}_{\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}}\right)^{-1} \cdot \chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}}\right) \cdot G\left(\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\right) \\
& w\left(\varpi^{e}\right) v\left(\varpi^{e}\right) \chi\left(\varpi^{e}\right)\left|\varpi^{e}\right|^{s-\frac{1}{2}}
\end{aligned}
$$

if $(e, \omega)=\left(d_{n, \gamma}+(\delta), \mathbf{1}_{n}\right)$ and 0 otherwise.
Proof. We proceed by induction on $n$. If $n=0$, then $W_{0}=\mathrm{GL}_{0}(\mathcal{O})=\left\{\mathbf{1}_{0}\right\}, \mathfrak{R}_{0}^{\omega}=\left\{\mathbf{1}_{0}\right\}$, $\mathbf{Z}^{0}=\{0\}$. The case $\omega \neq \mathbf{1}_{0}$ or $e \neq 0$ actually never occurs. This concludes the case $n=0$. Now let $n \geq 1$ and suppose that the claim is true for $n-1$.

By the constancy of the partial sums on $T_{n}\left(\mathcal{O} / \mathfrak{f}^{\ell}\right)$-orbits, we have (40)
$\sum_{g \in \varpi^{e} \omega \mathfrak{R}_{n, \ell}^{\omega}} \psi\left(\lambda_{n}^{\delta}(g)\right) w\left(g \cdot D_{n} w_{n}\right) v(g) \chi(g)|\operatorname{det}(g)|^{s-\frac{1}{2}}=\mathfrak{N}(\mathfrak{f})^{-\ell n} \cdot \sum_{r \in \mathfrak{R}_{n, \ell}^{\omega}} Z_{n}(s ; w, v, \delta, e, \omega, r)$.

By Lemma 2.6 (iii), we already know that this expression vanishes unless $e_{n}$ is given by (36) and $\sigma(n)=n$, which we henceforth assume. In that case the summands on the right hand side of (40) vanish unless the representatives $r$ satisfy (37).

By Propositions 2.1 and 2.2, the right hand side of (40) therefore simplifies to
$\mathfrak{N}(\mathfrak{f})^{-\frac{n(n-1)}{2}} \cdot \sum_{r \in \tilde{\mathfrak{R}}_{n, \ell}^{\omega}} Z_{n}(s ; w, v, \delta, e, \omega, r)=\mathfrak{N}(\mathfrak{f})^{-\frac{n(n-1)}{2}} \cdot \sum_{\tilde{r} \in \mathfrak{R}_{n-1, \ell}^{p(\omega)}} Z_{n}\left(s ; w, v, \delta, e, \omega, j_{n-1,0}(\tilde{r}) C_{n}\right)$.
Statement (iv) in Lemma 2.6 together with Lemma 2.5 yields with the abbreviations $\tilde{\omega}:=p(\omega)$ and $\tilde{e}:=\left(e_{\nu}\right)_{1 \leq \nu \leq n-1}$, and the relation

$$
\begin{equation*}
j_{n-1,0}\left(B_{n-1}^{-1}\right) C_{n}=B_{n}^{-1} \tag{41}
\end{equation*}
$$

the identity

$$
\begin{aligned}
& Z_{n}\left(s ; w, v, \delta, e, \omega, j_{n-1,0}(\tilde{r}) C_{n}\right) \\
& =\left|\varpi^{e_{n}}\right|^{s-\frac{1}{2}} \cdot \chi\left(\varpi^{e_{n}}\right) \cdot \chi \theta^{w_{n}} \theta^{\prime}\left(B_{n}\right) \cdot \prod_{\nu=1}^{n} \chi \theta_{n+1-\nu} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot \mathfrak{N}\left(\mathfrak{f}^{\ell} / \mathfrak{f}_{\chi \theta_{\nu}^{w_{n}}} \theta_{\nu}^{\prime}\right) \cdot G\left(\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\right) \text {. } \\
& w\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r}\right) C_{n} \cdot D_{n} w_{n}\right) \cdot v\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r}\right) C_{n}\right) \cdot \chi\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r}\right) C_{n}\right) \cdot\left|\varpi^{\tilde{e}}\right|^{s-\frac{1}{2}} \\
& =\prod_{\nu=1}^{n} \chi \theta_{n+1-\nu} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot \mathfrak{N}\left(\mathfrak{f}^{\ell} / \mathfrak{f}_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot G\left(\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\right) \text {. } \\
& \left|\varpi^{e_{n}}\right|^{s-\frac{1}{2}} \cdot \chi\left(\varpi^{e_{n}}\right) \cdot \chi \theta^{\prime}\left(C_{n}\right) \cdot \chi \theta^{w_{n}} \theta^{\prime}\left(B_{n}\right) \text {. } \\
& w\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r}\right) C_{n} \cdot D_{n} w_{n}\right) \cdot v\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r}\right)\right) \cdot \chi\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r}\right)\right) \cdot\left|\varpi^{\tilde{l}}\right|^{s-\frac{1}{2}} \\
& =\prod_{\nu=1}^{n} \chi \theta_{n+1-\nu} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot \mathfrak{N}\left(\mathfrak{f}^{\ell} / \mathfrak{f}_{\chi \theta_{\nu}^{w_{n}}} \theta_{\nu}^{\prime}\right) \cdot G\left(\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\right) \text {. } \\
& \left|\varpi^{e_{n}}\right|^{s-\frac{1}{2}} \cdot \chi\left(\varpi^{e_{n}}\right) \cdot \chi \theta^{\prime}\left(j_{n-1,0}\left(B_{n-1}^{-1}\right) C_{n}\right) \cdot \chi \theta^{w_{n}} \theta^{\prime}\left(B_{n}\right) \text {. } \\
& \psi\left(\lambda_{n-1}^{-e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right) \cdot w\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1} \cdot D_{n-1} w_{n-1}\right)\right) \cdot v\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right)\right)\right. \text {. } \\
& \chi\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right)\right) \cdot\left|\varpi^{\tilde{e}}\right|^{s-\frac{1}{2}} \\
& =\prod_{\nu=1}^{n} \chi \theta_{n+1-\nu} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot \mathfrak{N}\left(\mathfrak{f}^{\ell} / \mathfrak{f}_{\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}}\right) \cdot G\left(\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\right) \text {. } \\
& \left|\varpi^{e_{n}}\right|^{s-\frac{1}{2}} \cdot \chi\left(\varpi^{e_{n}}\right) \cdot \theta^{w_{n}}\left(B_{n}\right) . \\
& \psi\left(\lambda_{n-1}^{-e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right) \cdot w\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1} \cdot D_{n-1} w_{n-1}\right)\right) \cdot v\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right)\right)\right. \text {. } \\
& \chi\left(j_{n, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right)\right) \cdot\left|\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right|^{s-\frac{1}{2}} \text {. }
\end{aligned}
$$

Right multiplication by $B_{n-1} \in I_{n-1}$ permutes the double cosets $U_{n-1}(F) \varpi^{\tilde{e}} \tilde{r} J_{n-1, \ell}$, but leaves the double cosets $U_{n-1}(F) \varpi^{\tilde{e}} I_{n-1}$ invariant, whence induces a permutation of the system of representatives $\mathfrak{R}_{n-1, \ell}^{\tilde{\omega}}$.

Therefore, summing over all $\tilde{r} \in \mathfrak{R}_{n-1, \ell}^{\tilde{\omega}}$, gives

$$
\begin{aligned}
& \sum_{\tilde{r} \in \tilde{\mathfrak{R}} \tilde{\omega}} \psi\left(\lambda_{n-1, \ell}^{-e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right)\right) \cdot \\
& w\left(j_{n-1, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1} \cdot D_{n-1} w_{n-1}\right)\right) \cdot v\left(j_{n-1, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right)\right) \cdot \\
& =\sum_{\tilde{r} \in \tilde{\mathfrak{R}}_{n-1, \ell}} \psi\left(\left.\operatorname{det}^{\tilde{\omega}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right)\right|^{s-\frac{1}{2}} \cdot \chi\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r} B_{n-1}\right)\right. \\
& \quad w\left(j_{n-1, e_{n}}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{\omega} \tilde{r}\right)\right) \cdot \\
& \quad\left|\operatorname{det}\left(\varpi^{\tilde{e}} \tilde{\omega} \tilde{r}\right)\right|^{s-\frac{1}{2}} \cdot \chi\left(\varpi_{n-1} \tilde{e} \tilde{\omega} \tilde{r}\right) .
\end{aligned}
$$

By the induction hypothesis this expression vanishes unless

$$
\begin{equation*}
(\tilde{e}, \tilde{\omega})=\left(d_{n-1, e_{n}}+\left(e_{n}\right), \mathbf{1}_{n-1}\right) \tag{43}
\end{equation*}
$$

With (36) we see that $e_{n}=\gamma+\delta$, whence

$$
\begin{aligned}
\left(d_{n-1, \gamma}+\left(e_{n}\right)\right)_{i} & =\gamma \cdot(n+1-i)+\delta \\
& =\left(d_{n, \gamma}+(\delta)\right)_{i}
\end{aligned}
$$

Therefore, condition (43) is equivalent to

$$
(e, \omega)=\left(d_{n, \gamma}+(\delta), \mathbf{1}_{n}\right)
$$

Under condition (43), the induction hypothesis shows with Proposition 2.4 that (42) takes the value

$$
\begin{aligned}
& \mathfrak{N}(\mathfrak{f})^{\frac{\ln (n-1)}{2}-\frac{n(n-1)(n-2)}{6}} \cdot \prod_{\mu=1}^{n-1} \prod_{\nu=1}^{\mu} \mathfrak{N}\left(\mathfrak{f}_{\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}}\right)^{-1} \\
& \prod_{\mu=1}^{n-1} \prod_{\nu=1}^{\mu} \chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{\mu}}} \theta_{\nu}^{\prime}\right) \cdot G\left(\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\right) \\
& \prod_{\mu=1}^{n-1} \theta^{w_{\mu}}\left(B_{\mu}\right) \cdot w\left(\varpi^{e}\right) v\left(\varpi^{e}\right) \chi\left(\varpi^{\tilde{e}}\right)\left|\varpi^{\tilde{e}}\right|^{s-\frac{1}{2}}
\end{aligned}
$$

Whence (40) is given by

$$
\begin{aligned}
& \mathfrak{N}(\mathfrak{f})^{\frac{\ell n(n-1)}{2}+\ell n-\left(\frac{n(n-1)(n-2)}{6}+\frac{n(n-1)}{2}\right)} \\
& \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} \mathfrak{N}\left(\mathfrak{f}_{\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}}\right)^{-1} \cdot \chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}}\right) \cdot G\left(\chi \theta_{\nu}^{w_{n}} \theta_{\nu}^{\prime}\right) \\
& \prod_{\mu=1}^{n} \theta^{w_{\mu}}\left(B_{\mu}\right) \cdot w\left(\varpi^{e}\right) \cdot v\left(\varpi^{e}\right) \cdot \chi\left(\varpi^{e}\right) \cdot\left|\varpi^{e}\right|^{s-\frac{1}{2}}
\end{aligned}
$$

and the claim follows by another application of Proposition 2.4.

Recall the definition of the matrix $t_{x}$ in (7).
Theorem 2.8 (Local Birch Lemma). Let $w$ and $v$ be $\psi$ - (resp. $\psi^{-1}$-) Whittaker functions on $\mathrm{GL}_{n+1}(F)$ resp. $\mathrm{GL}_{n}(F)$, which satisfy relations (30) and (31). Assume furthermore that $\chi: F^{\times} \rightarrow \mathbf{C}^{\times}$is a quasi-character with the property that for all $1 \leq \mu \leq n$ and all $1 \leq \nu \leq \mu$ the conductors of $\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}$ are non-trivial, all agree, and are generated by an element $f_{\chi \theta \theta^{\prime}}=\varpi^{e}{ }^{\boldsymbol{\chi} \theta \theta^{\prime}}$. Set $\mathfrak{f}_{\chi \theta \theta^{\prime}}:=\mathcal{O} f_{\chi \theta \theta^{\prime}}$. Then for every $s \in \mathbf{C}$,

$$
\begin{aligned}
& \int_{U_{n}(F) \backslash \mathrm{GL}_{n}(F)} w\left(j_{n, 0}(g) \cdot h_{n} \cdot j_{n}\left(t_{-f}\right)\right) v\left(g \cdot t_{f}\right) \chi(\operatorname{det}(g))|\operatorname{det}(g)|^{s-\frac{1}{2}} d g \\
= & \prod_{\mu=1}^{n}\left(1-q^{-\mu}\right)^{-1} \cdot \mathfrak{N}\left(\mathfrak{f}_{\chi \theta \theta^{\prime}}\right)^{-\frac{(n+2)(n+1) n}{6}} \cdot\left|t_{f_{\chi \theta \theta^{\prime}}}\right|^{\frac{1}{2}-s} \cdot \\
& \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu}\left[\theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\left(f_{\chi \theta \theta^{\prime}}\right) \cdot G\left(\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\right)\right] \cdot w\left(j_{n, 0}\left(t_{f f_{\chi \theta \theta^{\prime}}^{-1}}\right)\right) \cdot v\left(t_{f f_{\chi \theta \theta^{\prime}}^{-1}}\right)
\end{aligned}
$$

We emphasize that there are modifications in the formulation of Theorem 2.8 compared to previous statements in $[76,50,51,52]$. In particular, the matrix

$$
h_{n} \cdot j_{n}\left(t_{(-1)}\right)
$$

plays the role of the matrix $h_{n}$ in loc. cit.

Proof. Introduce the matrix

$$
\tilde{E}_{n+1}:=\left(\begin{array}{ccccc}
1 & -f & -f^{2} & \cdots & -f^{n} \\
0 & 1 & f & \cdots & f^{n-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & f \\
0 & \cdots & \cdots & 0 & 1
\end{array}\right) \in J_{n+1,1} .
$$

In the notation of [50] this matrix agrees with $w_{n+1} E_{n+1}$. A direct computation shows

$$
\begin{equation*}
h_{n} \cdot j_{n}\left(t_{f}\right)=j_{n}\left(t_{f}\right) \cdot B_{n+1}^{-1} \cdot D_{n+1} \cdot w_{n+1} \cdot \tilde{E}_{n+1} \tag{44}
\end{equation*}
$$

We deduce the relation

$$
w\left(j_{n}(g) \cdot h_{n} \cdot j_{n}\left(t_{-f}\right)\right)=\theta\left(t_{-1}\right) w\left(j_{n}(g) \cdot j_{n}\left(t_{f}\right) \cdot B_{n+1}^{-1} \cdot D_{n+1} w_{n+1}\right)
$$

Together with (41) and the right invariance of the Haar measure $d g$, we obtain

$$
\begin{aligned}
& \int w\left(j_{n}(g) \cdot h_{n} \cdot j_{n}\left(t_{f}\right)\right) v\left(g \cdot t_{f}\right) \chi(g)|g|^{s-\frac{1}{2}} d g \\
= & \chi\left(t_{f}^{-1}\right)\left|t_{f}\right|^{\frac{1}{2}-s} \int w\left(j_{n}\left(g B_{n}^{-1}\right) \cdot C_{n+1} D_{n+1} w_{n+1}\right) v(g) \chi(g)|g|^{s-\frac{1}{2}} d g \\
= & \chi\left(t_{f}^{-1}\right)\left|t_{f}\right|^{\frac{1}{2}-s} \int w\left(j_{n}(g) \cdot C_{n+1} D_{n+1} w_{n+1}\right) v\left(g B_{n}\right) \chi\left(g B_{n}\right)\left|g B_{n}\right|^{s-\frac{1}{2}} d g \\
= & \chi \theta^{\prime}\left(B_{n}\right) \chi\left(t_{f}^{-1}\right)\left|t_{f}\right|^{\frac{1}{2}-s} \int w\left(j_{n}(g) \cdot C_{n+1} D_{n+1} w_{n+1}\right) v(g) \chi(g)|g|^{s-\frac{1}{2}} d g .
\end{aligned}
$$

At this point, invoke Lemma 2.5 once again to obtain the expression

$$
\begin{aligned}
& \chi\left(B_{n}\right) \chi\left(t_{f}^{-1}\right)\left|t_{f}\right|^{\frac{1}{2}-s} \int \psi\left(\lambda_{n}^{0}\left(g B_{n}\right)\right) w\left(j_{n}\left(g B_{n} \cdot D_{n} w_{n}\right)\right) v\left(g B_{n}\right) \chi(g)|g|^{s-\frac{1}{2}} d g \\
= & \chi\left(t_{f}^{-1}\right)\left|t_{f}\right|^{\frac{1}{2}-s} \int \psi\left(\lambda_{n}^{0}(g)\right) w\left(j_{n}\left(g \cdot D_{n} w_{n}\right)\right) v(g) \chi(g)|g|^{s-\frac{1}{2}} d g
\end{aligned}
$$

By Lemma 2.7, we may evaluate the latter integral explicitly as

$$
\begin{aligned}
& \int \psi\left(\lambda_{n}^{0}(g)\right) w\left(j_{n}\left(g \cdot D_{n} w_{n}\right)\right) v(g) \chi(g)|g|^{s-\frac{1}{2}} d g \\
= & \operatorname{vol}\left(U_{n}(\mathcal{O}) \varpi^{e} J_{\ell}^{n}\right) \cdot \sum_{g \in w^{e} \omega \Re_{n, \ell}^{\omega}} \psi\left(\lambda_{n}^{\delta}(g)\right) \cdot w\left(g \cdot D_{n} w_{n}\right) \cdot v(g) \cdot \chi(g) \cdot|g|^{s-\frac{1}{2}} \\
= & \mathfrak{N}\left(f_{\chi \theta \theta^{\prime}}\right)^{-\frac{(n+1) n(n-1)}{6}} \cdot \prod_{\mu=1}^{n} \theta^{w_{\mu}}\left(B_{\mu}\right) \cdot \prod_{\nu=1}^{\mu} \mathfrak{N}\left(f_{\chi \theta_{\nu}^{w_{\mu}}}\right)^{-1} \cdot \chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\left(f_{\chi \theta_{\nu}^{\theta_{\mu}} \theta_{\nu}^{\prime}}\right) \cdot G\left(\chi \theta_{\nu}^{w_{\mu}} \theta_{\nu}^{\prime}\right) \cdot \\
& w\left(t_{f f_{\chi \theta \theta^{\prime}}^{-1}}\right) v\left(t_{f f_{\chi \theta \theta^{\prime}}^{-1}}^{-1}\right) \chi\left(t_{f f_{\chi \theta^{\prime}}^{-1}}\right)\left|t_{f f_{\chi \theta \theta^{\prime}}^{-1}}\right|^{s-\frac{1}{2}} .
\end{aligned}
$$

Finally, remark that

$$
\theta\left(t_{(-1)}\right)=\prod_{\mu=1}^{n} \theta^{w_{\mu}}\left(B_{\mu}\right)
$$

and the claim follows.
2.2. The generically nearly ordinary $p$-adic Hecke algebra. Fix a number field $F / \mathbf{Q}$ with ring of integers $\mathcal{O}_{F}$ and a rational prime $p$. Put

$$
F_{p}:=F \otimes_{\mathbf{Q}} \mathbf{Q}_{p}, \quad \text { and } \quad \mathcal{O}_{p}:=\mathcal{O}_{F} \otimes_{\mathbf{Z}} \mathbf{Z}_{p}
$$

Let

$$
I_{\alpha}:=\left\{r \in G\left(\mathbf{Z}_{p}\right) \mid r\left(\bmod p^{\alpha}\right) \in B\left(\mathbf{Z}_{p} / p^{\alpha} \mathbf{Z}_{p}\right)\right\}
$$

$$
I_{\alpha^{\prime}, \alpha}:=\left\{r \in I_{\alpha} \mid r\left(\bmod p^{\alpha^{\prime}}\right) \in U\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right)\right\}
$$

for $\alpha \geq \alpha^{\prime} \geq 0$.
Define the semigroup

$$
\Delta_{G}:=\prod_{v \mid p} \Delta_{F_{v}, n+1} \times \Delta_{F_{v}, n} \subseteq T\left(\mathbf{Q}_{p}\right)
$$

and the corresponding Hecke algebra

$$
\mathcal{H}_{A}\left(\alpha^{\prime}, \alpha\right):=\mathcal{H}_{A}\left(I_{\alpha^{\prime}, \alpha}, I_{\alpha^{\prime}, \alpha} \Delta_{G} I_{\alpha^{\prime}, \alpha}\right)
$$

Then $\mathcal{H}_{A}(\alpha ;, \alpha)$ is the product of the products of the local Hecke algebras

$$
\mathcal{H}_{A}^{n+1}\left(v_{\mathfrak{p}}(p) \alpha^{\prime}, v_{\mathfrak{p}}(p) \alpha\right) \quad \text { and } \quad \mathcal{H}_{A}^{n}\left(v_{\mathfrak{p}}(p) \alpha^{\prime}, v_{\mathfrak{p}}(p) \alpha\right)
$$

introduced in section 1.2. Therefore, (10) translates to

$$
\begin{equation*}
\mathcal{H}_{A}\left(\alpha^{\prime}, \alpha\right)=\mathcal{H}_{A}(0, \alpha)\left[I_{\alpha} / I_{\alpha^{\prime}, \alpha}\right]=\mathcal{H}_{A}(0, \alpha)\left[T_{n}\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right)\right] \tag{45}
\end{equation*}
$$

This is a finitely generated commutative $A$-algebra. By (45), there is a canonical map

$$
\begin{gathered}
\langle\cdot\rangle: \quad T_{n}\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right) \rightarrow \mathcal{H}_{A}\left(\alpha^{\prime}, \alpha\right), \\
t
\end{gathered}
$$

Furthermore, we have a canonical isomorphism

$$
U: \quad A\left[\Delta_{G}\right] \cong \mathcal{H}_{A}(0, \alpha)
$$

given by

$$
\Delta_{G} \ni \delta \mapsto I_{\alpha} \delta I_{\alpha}=: U(\delta)
$$

We define the generic nearly ordinary p-adic Hecke algebra as the localization

$$
\mathcal{H}_{A}^{\text {ord }}\left(\alpha^{\prime}, \alpha\right):=\mathcal{H}_{A}\left(\alpha^{\prime}, \alpha\right)\left[U(\delta)^{-1} \mid \delta \in \Delta_{G}\right]
$$

If $\left\langle\Delta_{G}\right\rangle$ denotes the group generated by $\Delta_{G}$, we obtain a canonical isomorphism

$$
A\left[\left\langle\Delta_{G}\right\rangle\right] \cong \mathcal{H}_{A}^{\mathrm{ord}}(0, \alpha)
$$

extending $U$ and (45) extends to

$$
\begin{equation*}
\mathcal{H}_{A}^{\mathrm{ord}}\left(\alpha^{\prime}, \alpha\right)=\mathcal{H}_{A}(0, \alpha)^{\mathrm{ord}}\left[T_{n}\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right)\right] \tag{46}
\end{equation*}
$$

If $\mathcal{M}$ is an $\mathcal{H}_{A}\left(\alpha^{\prime}, \alpha\right)$-module, then the Hecke module structure on $\mathcal{M}$ extends to $\mathcal{H}_{A}^{\text {ord }}\left(\alpha^{\prime}, \alpha\right)$ if and only if the operators $U(\delta), \delta \in \Delta_{G}$, act invertibly on $\mathcal{M}$. Since $\Delta_{G}$ is finitely generated, it suffices to check this for an arbitrary finite set of generators of $\Delta_{G}$, or a product thereof.

For any character $\vartheta: T\left(\mathbf{Q}_{p}\right) \rightarrow A^{\times}$whose restriction to $T\left(\mathbf{Z}_{p}\right)$ factors over $T\left(\mathbf{Z}_{p} / p^{\alpha} \mathbf{Z}_{p}\right)$, we consider $\left.\vartheta\right|_{T\left(\mathbf{Z}_{p}\right)}$ as an algebra homomorphism $A\left[T\left(\mathbf{Z}_{p} / p^{\alpha} \mathbf{Z}_{p}\right)\right] \rightarrow A$. By (45), $\vartheta$ then extends to an $A$-algebra homorphism

$$
\vartheta: \quad \mathcal{H}_{A}(\alpha, \alpha) \rightarrow A
$$

by setting

$$
U(\delta) \mapsto \vartheta(\delta), \quad \delta \in \Delta_{G}
$$

The values $\vartheta(\delta)$ being invertible, this extends uniquely to an $A$-algebra homomorphism

$$
\vartheta: \quad \mathcal{H}_{A}^{\text {ord }}(\alpha, \alpha) \rightarrow A .
$$

In order to streamline notation in the sequel, set for $\underline{\varepsilon}=\left(\varepsilon_{\varpi}\right)_{\varpi \mid p}, e_{\varpi} \in \mathbf{Z}$,

$$
\begin{equation*}
p^{\varepsilon}:=\prod_{\varpi \mid p} \varpi^{\varepsilon_{\varpi}}, \tag{47}
\end{equation*}
$$

and define for $\alpha \geq \alpha^{\prime} \geq 0, \alpha>0$, the Hecke operator

$$
U_{p \S}=I_{\alpha^{\prime}, \alpha} \Delta\left(t_{p \underline{\varepsilon}}\right) I_{\alpha^{\prime}, \alpha}=\bigsqcup_{u \in U\left(\mathbf{Z}_{p}\right) / \Delta\left(t_{p \S}\right) U\left(\mathbf{Z}_{p}\right) \Delta\left(t_{p \underline{\varrho}}^{-1}\right)} u \Delta\left(t_{p \S}\right) I_{\alpha^{\prime}, \alpha} \in \mathcal{H}_{A}\left(\alpha^{\prime}, \alpha\right)
$$

Put

$$
\begin{equation*}
U_{p}:=I_{\alpha^{\prime}, \alpha} t_{p} I_{\alpha^{\prime}, \alpha}=\prod_{\mathfrak{p} \mid p}\left(U_{\mathfrak{p}} \otimes U_{\mathfrak{p}}^{\prime}\right)^{v_{\mathfrak{p}}(p)} \in \mathcal{H}_{A}\left(\alpha^{\prime}, \alpha\right) \tag{48}
\end{equation*}
$$

2.3. The twisted global Zeta integral. Recall from the previous section that $F / \mathbf{Q}$ is a number field with ring of integers $\mathcal{O}_{F}$ and $p$ denotes a rational prime. Fix a non-trivial additive character $\psi: F \backslash \mathbf{A}_{F} \rightarrow \mathbf{C}^{\times}$with local factors $\psi_{v}: F_{v}^{\times} \rightarrow \mathbf{C}^{\times}$. Put

$$
\psi_{p}:=\underset{\mathfrak{p} \mid p}{\otimes} \psi_{\mathfrak{p}}: \quad F_{p} \rightarrow \mathbf{C}^{\times}
$$

and assume without loss of generality that $\psi_{p}$ is of conductor $\mathcal{O}_{p}$.
Let $\Pi$ and $\Sigma$ be irreducible cuspidal automorphic representations of $\mathrm{GL}_{n+1}\left(\mathbf{A}_{F}\right)$ and $\mathrm{GL}_{n}\left(\mathbf{A}_{F}\right)$ respectively, which we also consider as an automorphic representation $\Pi \widehat{\otimes} \Sigma$ of $G(\mathbf{A})$. Let $S_{\infty}$ denote the set of infinite places of $F$ and $S_{\Pi \widehat{\otimes} \Sigma}$ the set of finite places where $\Pi$ or $\Sigma$ ramifies.

We make free use of the theory of Rankin-Selberg $L$-function $L(s, \Pi \widehat{\otimes} \Sigma)$ as developed in $[42,43,44,49,12,13]$.

At any archimedean places $v \in S_{\infty}$, we consider the smooth models $\Pi_{v}$ and $\Sigma_{v}$, i.e. these are smooth Fréchet representations of $\mathrm{GL}_{n+1}\left(F_{v}\right)$ and $\mathrm{GL}_{n}\left(F_{v}\right)$ of moderate growth and finite length. They agree with the Casselman-Wallach completions of the subspaces of $K$-finite vectors. We refer the reader to [4, 2] for the notion of Casselman-Wallach completion. We also write $\Pi_{v} \widehat{\otimes} \Sigma_{v}$ for the completed projective tensor product. This is a Casselman-Wallach representation of $G\left(F_{v}\right)$.

Recall that $\Pi$ and $\Sigma$ are always generic [80]. At any place $v$ of $F$, consider for any vector

$$
W_{v} \in \mathscr{W}\left(\Pi_{v} \otimes \Sigma_{v}, \psi_{v} \otimes \psi_{v}^{-1}\right)=\mathscr{W}\left(\Pi_{v}, \psi_{v}\right) \otimes \mathscr{W}\left(\Sigma_{v}, \psi_{v}^{-1}\right)
$$

in the local Whittaker model of $\Pi_{v} \otimes \Sigma_{v}$ (or of $\Pi_{v} \widehat{\otimes} \Sigma_{v}$ for $\left.v \mid \infty\right)$ the local zeta integral

$$
\Psi_{v}\left(s, W_{v}\right):=\int_{U_{n}\left(F_{v}\right) \backslash \mathrm{GL}_{n}\left(F_{v}\right)} W_{v}\left(\Delta\left(g_{v}\right)\right)\left|g_{v}\right|^{s-\frac{1}{2}} d g_{v}
$$

as in [44].
At archimedean places $v$, we know that for any $W_{v}$ the local zeta integral $\Psi_{v}\left(s, W_{v}\right)$ satisfies an identity

$$
\Psi_{v}\left(s, W_{v}\right)=\Omega_{v}\left(s, W_{v}\right) \cdot L\left(s, \Pi_{v} \widehat{\otimes} \Sigma_{v}\right)
$$

for a function $\Omega_{v}\left(s, W_{v}\right)$ holomorphic in $s$. For $K$-finite $W_{v}$, it is known that $\Omega_{v}\left(s, W_{v}\right)$ is a polynomial. Moreover, there is a good $K$-finite test vector $W_{v}^{0}$ trivializing $\Omega_{v}\left(s, W_{v}\right)$ [13, 41].

At all finite places $v$, we fix a good tensor $W_{v}^{0} \in \mathscr{W}\left(\Pi_{v}, \psi_{v}\right) \otimes \mathscr{W}\left(\Sigma_{v}, \psi_{v}^{-1}\right)$ with the property that

$$
L\left(s, \Pi_{v} \otimes \Sigma_{v}\right)=\Psi\left(s, W_{v}^{0}\right)
$$

We suppose that $W_{v}^{0}=W_{n+1, v}^{0} \otimes W_{n, v}^{0}$ for spherical Whittaker functions $W_{n+1, v}^{0}$ and $W_{n_{v}}^{0}$ at all places $v \notin S_{\Pi \widehat{\otimes} \Sigma} \cup S_{\infty}$. By Shintani's explicit formula [82], for $v \notin S_{\Pi \widehat{\otimes} \Sigma} \cup S_{\infty}$, the local $L$-function is given explicitly by

$$
L\left(s, \Pi_{v} \otimes \Sigma_{v}\right)=\operatorname{det}\left(\mathbf{1}_{(n+1) n}-q_{v}^{-s}\left(A_{\Pi_{v}} \otimes A_{\Sigma_{v}}\right)\right)^{-1}
$$

where $A_{\pi_{\mathfrak{q}}}$ and $A_{\sigma_{\mathfrak{q}}}$ denote the corresponding Frobenius-Hecke parameters (Satake parameters) and $q_{v}=\mathfrak{N}\left(\mathfrak{p}_{v}\right)$.

Define the good test vector

$$
W^{0}:=\underset{v}{\otimes} W_{v}^{0} \in \mathscr{W}\left(\Pi \widehat{\otimes} \Sigma, \psi \otimes \psi^{-1}\right)
$$

Then for any other test vector

$$
W \in \mathscr{W}\left(\Pi \widehat{\otimes} \Sigma, \psi \otimes \psi^{-1}\right)
$$

which agrees with $W^{0}$ at all places outside a finite set $S_{W}$, the global zeta integral

$$
\Psi(s, W):=\int_{U_{n}\left(\mathbf{A}_{F}\right) \backslash \mathrm{GL}_{n}\left(\mathbf{A}_{F}\right)} W(\Delta(g))|g|^{s-\frac{1}{2}} d g
$$

is well defined: $\Psi(s, W)$ converges absolutely for $\operatorname{Re}(s) \gg 0$ and has an Euler product decomposition

$$
\Psi(s, W)=\prod_{v} \Psi_{v}\left(s, W_{v}\right)=\prod_{v \in S_{W}} \Omega_{v}\left(s, W_{v}\right) \cdot \prod_{v} L\left(s, \Pi_{v} \widehat{\otimes} \Sigma_{v}\right) .
$$

Therefore,

$$
\begin{equation*}
\Psi(s, W)=\Omega(s, W) \cdot L\left(s, \Pi_{v} \widehat{\otimes} \Sigma_{v}\right) \tag{49}
\end{equation*}
$$

for a function

$$
\Omega(s, W):=\prod_{v} \Omega_{v}\left(s, W_{v}\right)=\prod_{v \in S_{W}} \Omega_{v}\left(s, W_{v}\right)
$$

holomorphic in $s$. Moreover, $\Omega(s, W)$ is a polynomial in $s$ and $q_{v}^{-s}, v \in S_{W}$ finite, whenever $W$ is $K$-finite.

By Fourier transform, we may associate to $W$ an automorphic form $\phi_{W}$ on $G(\mathbf{A})$. Then we have an identity

$$
\begin{equation*}
\Psi(s, W)=\Phi\left(s, \phi_{W}\right):=\int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \phi_{W}(g)|\operatorname{det}(g)|^{s-\frac{1}{2}} d g \tag{50}
\end{equation*}
$$

for $\operatorname{Re}(s) \gg 0$, where the right hand side converges absolutely for all $s \in \mathbf{C}$ and defines an analytic function in $s$. Then (49) extends by holomorphic continuation to the identity

$$
\begin{equation*}
\Phi\left(s, \phi_{W}\right)=\Omega(s, W) \cdot L\left(s, \Pi_{v} \widehat{\otimes} \Sigma_{v}\right) \tag{51}
\end{equation*}
$$

which is valid for all $s \in \mathbf{C}$.
For any quasi-character $\chi: F^{\times} \backslash \mathbf{A}_{F}^{\times} \rightarrow \mathbf{C}^{\times}$we identify

$$
L(s, \Pi \widehat{\otimes} \Sigma \otimes \chi):=L(s, \Pi \widehat{\otimes}(\Sigma \otimes \chi))=L(s,(\Pi \otimes \chi) \widehat{\otimes} \Sigma)
$$

Then the twisted Whittaker function

$$
W_{\chi}: \quad G(\mathbf{A}) \rightarrow \mathbf{C}, \quad\left(g_{1}, g_{2}\right) \mapsto \chi\left(\operatorname{det}\left(g_{2}\right)\right) W\left(g_{1}, g_{2}\right)
$$

is an element of $\mathscr{W}\left(\Pi \widehat{\otimes}(\Sigma \otimes \chi), \psi \otimes \psi^{-1}\right)$. At all places outside $S_{W} \cup\left(S_{\Pi \widehat{\otimes} \Sigma} \cap S_{\chi}\right)$ the function $W_{\chi}$ coincides with a good test vector for the twisted $L$-function.

For a finite set of places $S$ of $F$ we write $L^{S}(s, \Pi \widehat{\otimes} \Sigma)$ for the partial $L$-functions where the Euler factors at places in $S$ have been removed. Likewise, $\Omega^{S}(s, W)$ denotes the correction factor with the factors for places in $S$ removed.

For a quasi-character $\vartheta: T\left(\mathbf{Q}_{p}\right) \rightarrow \mathbf{C}^{\times}$, write $\vartheta_{\mu, \nu}: F_{p}^{\times} \rightarrow \mathbf{C}^{\times}$for the respective $(\mu, \nu)$-component according to the decomposition $T=T_{n+1} \times T_{n}$, where $1 \leq \mu \leq n+1$, $1 \leq \nu \leq n$.

If $\chi_{p}: F_{p}^{\times} \rightarrow \mathbf{C}^{\times}$is another quasi-character, consider $\chi_{p}$ as a quasi-character of $T\left(\mathbf{Q}_{p}\right)$ via pullback along the determinant in the second factor. We say that the resulting character $\chi_{p} \vartheta$ has constant conductor, if:
(C) For all $1 \leq \nu \leq \mu \leq n$ the conductors of $\chi_{p} \vartheta_{\mu, \nu}$ all agree.

We say furthermore that $\chi_{p} \vartheta$ has fully supported conductor, if
(F) For all $1 \leq \nu \leq \mu \leq n$, the conductors of the local characters $\chi_{p} \vartheta_{\mu, \nu}$ are supported at all $\mathfrak{p} \mid p$.

For a character satisfying (C) and (F), let

$$
\begin{equation*}
f_{\chi \vartheta}=p^{\underline{\varepsilon}} \chi \vartheta \in \mathcal{O}_{p} \tag{52}
\end{equation*}
$$

denote a generator of the conductor $\mathfrak{f}_{\chi \vartheta}$ of any $\chi_{p} \vartheta_{\mu, \nu}$ for $\nu \leq \mu$.
Recall that for $\alpha>0$ with the property that $f=p^{\alpha}$ is divisible by $f_{\vartheta}$ the quasi-character $\vartheta$ extends to an algebra homomorphism

$$
\vartheta: \quad \mathcal{H}_{\mathbf{C}}(\alpha, \alpha) \rightarrow \mathbf{C} .
$$

Recall the definition of the matrix $h$ in (8). We consider $h$ as a diagonally (with respect to the places above $p$ ) embedded element in $G\left(\mathbf{Z}_{p}\right)$ and identify for any $f \in F_{p}$ the element $t_{f} \in \mathrm{GL}_{n}\left(F_{p}\right)$ with its image under $\Delta \circ\left(j_{n} \times \mathbf{1}\right)$ in $G\left(\mathbf{Q}_{p}\right)$. Let $S(p)$ denote the set of places of $F$ dividing $p$.

Theorem 2.9 (Global Birch Lemma). Let $W \in \mathscr{W}\left(\Pi \widehat{\otimes} \Sigma, \psi \otimes \psi^{-1}\right)$ be a Whittaker function with the following properties:
(i) $W$ admits a factorization $W=W_{p} \otimes W^{S(p)}$ for a local Whittaker function $W_{p}$ at $p$ and a Whittaker function $W^{S(p)}$ outside $p$.
(ii) $W_{p}$ is right invariant under $I_{\alpha, \alpha}$ for some $\alpha \geq 1$.
(iii) $W_{p}$ is an eigenvector for $\mathcal{H}_{\mathbf{C}}(\alpha, \alpha)$ for a quasi-character $\vartheta: T\left(\mathbf{Q}_{p}\right) \rightarrow \mathbf{C}^{\times}$.

Let $\chi: F^{\times} \backslash \mathbf{A}_{F}^{\times} \rightarrow \mathbf{C}^{\times}$be a finite order character such that $\chi_{p} \vartheta$ has fully supported constant conductor $\mathfrak{f}_{\chi \theta}$ dividing $f=p^{\alpha}$ for an $\alpha>0$ which is sufficiently large satisfying (ii).

Then for every $s \in \mathbf{C}$,

$$
\begin{aligned}
& \int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \phi_{W}\left(g \cdot h t_{f}\right) \chi(\operatorname{det}(g))|\operatorname{det}(g)|^{s-\frac{1}{2}} d g \\
= & \Omega^{S(p)}\left(s, W_{\chi}\right) \cdot \delta\left(W_{p}\right) \cdot \mathfrak{N}(\mathfrak{f})^{-\frac{(n+2)(n+1) n+(n+1) n(n-1)}{6}} \cdot \mathfrak{N}\left(\mathfrak{f}_{\chi \vartheta}\right)^{-\frac{(n+1) n(n-1)}{6}} \cdot\left|t_{f_{\chi \theta}}\right|^{\frac{1}{2}-s} . \\
& \vartheta\left(t_{p}^{\alpha}\right) \cdot \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} G\left(\chi \vartheta_{\mu, \nu}\right) \cdot L(s, \Pi \widehat{\otimes} \Sigma \otimes \chi),
\end{aligned}
$$

where

$$
\delta\left(W_{p}\right):=W_{p}\left(\mathbf{1}_{n}\right) \cdot \prod_{\mu=1}^{n} \prod_{v \mid p}\left(1-q_{v}^{-\mu}\right)^{-1}
$$

Remark 2.10. By the discussion preceeding Theorem 2.9,

$$
\Omega^{S(p)}\left(s, W_{\chi}\right)=\Omega_{\infty}\left(s, W_{\chi, \infty}\right) \cdot \Omega_{S_{W} \cup S_{\chi}}^{S_{\infty} \cup S(p)}(s, W)
$$

In particular, if $S_{W} \cup S_{\chi} \subseteq S_{\infty} \cup S(p)$, then

$$
\begin{equation*}
\Omega^{S(p)}\left(s, W_{\chi}\right)=\Omega_{\infty}\left(s, W_{\chi, \infty}\right) \tag{53}
\end{equation*}
$$

Proof of Theorem 2.9. For $\operatorname{Re}(s) \gg 0$ we have by (50) and identity

$$
\begin{aligned}
& \int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \phi_{W}\left(g \cdot h t_{f}\right) \chi(g)|g|^{s-\frac{1}{2}} d g \\
= & \chi\left(t_{f}\right)\left|t_{f}\right|^{s-\frac{1}{2}} \cdot \Phi\left(s, \phi_{W_{\chi}}\left(-\cdot h t_{f}\right)\right) \\
= & \chi\left(t_{f}\right)\left|t_{f}\right|^{s-\frac{1}{2}} \cdot \Psi\left(s, W_{\chi}\left(-\cdot h t_{f}\right)\right) \\
= & \chi\left(t_{f}\right)\left|t_{f}\right|^{s-\frac{1}{2}} \cdot \Psi_{p}\left(s, W_{\chi, p}\left(-\cdot h t_{f}\right)\right) \cdot \Omega^{S(p)}\left(s, W_{\chi}\right) \cdot L(s, \Pi \widehat{\otimes} \Sigma \otimes \chi) .
\end{aligned}
$$

By holomorphic continuation, this identity extends to all $s \in \mathbf{C}$ and the value of the expression

$$
\begin{aligned}
& \chi\left(t_{f}\right)\left|t_{f}\right|^{s-\frac{1}{2}} \cdot \Psi_{p}\left(s, W_{\chi, p}\left(-\cdot h t_{f}\right)\right) \\
= & \prod_{v \mid p} \int_{U_{n}\left(F_{v}\right) \backslash \mathrm{GL}_{n}\left(F_{v}\right)} W_{v}\left(\Delta\left(g_{v}\right) \cdot h_{v} t_{f_{v}}\right) \cdot \chi_{v}\left(g_{v}\right)\left|g_{v}\right|^{s-\frac{1}{2}} d g
\end{aligned}
$$

is given by the Local Birch Lemma (Theorem 2.8). Therefore, the global zeta integral evaluates to

$$
\begin{aligned}
& \Omega^{S(p)}\left(s, W_{\chi}\right) \cdot \prod_{\mu=1}^{n} \prod_{v \mid p}\left(1-q_{v}^{-\mu}\right)^{-1} \cdot \mathfrak{N}\left(\mathfrak{f}_{\chi \vartheta}\right)^{-\frac{(n+2)(n+1) n}{6}} \cdot\left|t_{f_{\chi \theta}}\right|^{\frac{1}{2}-s} \\
& W_{p}\left(t_{f f_{\chi \vartheta}^{-1}}\right) \cdot \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu}\left[\vartheta_{\mu+1-\nu, \nu}\left(f_{\chi \vartheta}\right) \cdot G\left(\chi \vartheta_{\mu+1-\nu, \nu}\right)\right] \cdot L(s, \Pi \widehat{\otimes}(\Sigma \otimes \chi))
\end{aligned}
$$

Recall that $\psi_{p}\left(\mathcal{O}_{p}\right)=1$ and $f_{\chi \theta}=p^{\varepsilon} \chi^{\chi \theta}$, whence

$$
\begin{aligned}
W_{p}\left(t_{f f_{\chi \vartheta}^{-1}}\right) & =W_{p}\left(t_{f p^{-\underline{\varepsilon}_{\chi \vartheta}}}\right) \\
& =\left[U\left(\mathbf{Z}_{p}\right): t_{f p^{-\varepsilon_{\chi}}} U\left(\mathbf{Z}_{p}\right) t_{f p^{-\underline{\varepsilon}_{\chi \vartheta}}}^{-1}\right]^{-1} \cdot U_{f p^{-\varepsilon_{\chi \vartheta}}} W_{p}(\mathbf{1}) \\
& =\mathfrak{N}\left(\mathfrak{f}_{\chi \vartheta} / \mathfrak{f}\right)^{\frac{(n+2)(n+1) n+(n+1) n(n-1)}{6}} \cdot \vartheta\left(t_{\left.f_{\chi \vartheta}^{-1}\right) \cdot \vartheta\left(t_{f}\right) \cdot W_{p}(\mathbf{1}) .} .\right.
\end{aligned}
$$

The observation

$$
\vartheta\left(t_{f_{\chi \vartheta}}\right)=\prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} \vartheta_{\mu+1-\nu, \nu}\left(f_{\chi \vartheta}\right)
$$

concludes the proof.

## 3. $p$-ADIC LATTICES

In this section we establish a general relation between lattices which will translate into congruences essential to establish the independence of the weight of the non-abelian $p$-adic $L$-function we construct.
3.1. Integral algebras. As before, we consider the element $t_{p} \in \mathrm{GL}_{n}\left(F_{p}\right)$ as a diagonally embedded element $t_{p} \in G\left(\mathbf{Q}_{p}\right)$. For any non-negative integer $\alpha \geq 0$ consider the element

$$
g_{\alpha}:=h \cdot t_{p}^{\alpha} \in G\left(\mathbf{Q}_{p}\right)
$$

and set for any subgroup $L \subseteq G$,

$$
L^{\alpha}:=g_{\alpha} L g_{\alpha}^{-1}
$$

and likewise for sub Lie algebras of $\mathfrak{g}$ with respect to the adjoint action on $\mathfrak{g}$.
Lemma 3.1. For any $\alpha \geq 0$, the subgroups $H$ and $B^{-, \alpha}$ are transversal, i.e.

$$
\begin{equation*}
\mathfrak{g}_{E}=\mathfrak{h}_{E} \oplus \mathfrak{b}_{E}^{-, \alpha} \tag{54}
\end{equation*}
$$

Furthermore, we have $\mathfrak{b}_{E}^{-, \alpha}=\mathfrak{b}_{E}^{-, 0}$, and likewise for $B$ replacing $B^{-}$.
Proof. Since $t_{p}$ normalizes $B^{-}$over fields, we are reduced to the case $\alpha=0$ and $E=\mathbf{Q}_{p}$. Counting dimensions shows that (54) is equivalent to

$$
\mathfrak{h}_{\mathbf{Q}_{p}} h \cap h \mathfrak{b}_{\mathbf{Q}_{p}}^{-}=0
$$

which in turn is an easy excercise in linear algebra.
By the Poincaré-Birkhoff-Witt Theorem we obtain

Corollary 3.2. For any $\alpha \geq 0$ we have a canonical isomorphism

$$
\begin{equation*}
U\left(\mathfrak{g}_{E}\right)=U\left(\mathfrak{h}_{E}\right) \otimes_{E} U\left(\mathfrak{b}_{E}^{-, \alpha}\right) \tag{55}
\end{equation*}
$$

Lemma 3.3. For any $\alpha \geq 0$ we have

$$
\begin{equation*}
U\left(\mathfrak{u}_{\mathcal{O}}^{\alpha}\right) \subseteq\left(\mathcal{O}+p^{\alpha} U\left(\mathfrak{h}_{\mathcal{O}}\right)\right) \otimes_{\mathcal{O}}\left(\mathcal{O}+p^{\alpha} U\left(\mathfrak{b}_{\mathcal{O}}^{-, 0}\right)\right) \tag{56}
\end{equation*}
$$

as subspaces of (55).
Proof. In the case $\alpha=0$, it suffices to remark that $h \in G\left(\mathbf{Z}_{p}\right)$ and therefore $g_{0} \in G(\mathcal{O})$, whence

$$
\begin{equation*}
\mathfrak{g}_{\mathcal{O}}=\mathfrak{h}_{\mathcal{O}} \oplus \mathfrak{b}_{\mathcal{O}}^{-, 0} \tag{57}
\end{equation*}
$$

by (54). For $\alpha>0$, observe

$$
\begin{equation*}
t_{p}^{\alpha} \mathfrak{u}_{\mathcal{O}} t_{p}^{-\alpha} \subseteq p^{\alpha} \mathfrak{u}_{\mathcal{O}} \tag{58}
\end{equation*}
$$

which implies

$$
\mathfrak{u}_{\mathcal{O}}^{\alpha} \subseteq p^{\alpha} \mathfrak{u}_{\mathcal{O}}^{0}
$$

Therefore,

$$
U\left(\mathfrak{u}_{\mathcal{O}}^{\alpha}\right) \subseteq \mathcal{O}+p^{\alpha} U\left(\mathfrak{u}_{\mathcal{O}}^{0}\right)
$$

and the claim follows with (57).
3.2. $p$-integral structures on rational representations. Consider a finite extension $E / \mathbf{Q}_{p}$ and let $L_{\lambda, E}$ denote a rational representation of $G$ of $B$-highest weight $\lambda$ defined over $E$. Write $w_{0}=\left(w_{n+1}, w_{n}\right) \in W(G, T)$ for the longest element in the Weyl group of $G$ with respect to the torus $T$ corresponding to the diagonal matrices in $G$ and the positive system $\Delta^{+}$given by our choice of $\mathfrak{b}$. The (algebraic) differential of $\lambda$ is a canonical $E$-valued character of $\mathfrak{b}$ (trivial on the radical), which in turn gives rise to a character $\lambda^{w_{0}}$ of $\mathfrak{b}_{E}^{-}=w_{0} \mathfrak{b}_{E} w_{0}$. It extends uniquely to a character of $U\left(\mathfrak{b}_{E}^{-}\right)$that we denote the same. We pull it back to a character $\left(\lambda^{w_{0}}\right)^{\alpha}$ of $\mathfrak{b}_{E}^{-, \alpha}$. Fix a highest weight vector $v_{0} \in L_{\lambda, E}$ once and for all. Then $g_{\alpha} \cdot v_{0}$ is a $B_{E}^{-, \alpha}$-highest weight vector of weight $\lambda^{\alpha}$.

Then any $t \in T\left(\mathbf{Q}_{p}\right)$ acts on $v_{0}$ via the scalar $\lambda^{w_{0}}(t) \in E^{\times}$. Renormalize its action on $V_{\lambda, E}$ by defining

$$
\begin{equation*}
t \bullet v:=\left(-\lambda^{w_{0}}\right)(t) \cdot(t v), \quad v \in V_{\lambda, E} \tag{59}
\end{equation*}
$$

Inside $L_{\lambda, E}$ consider the $G(\mathcal{O})$-lattice $L_{\lambda, \mathcal{O}}$ generated by $v_{0} \in L_{\lambda, E}$. Then

$$
\begin{equation*}
L_{\lambda, \mathcal{O}}=U\left(\mathfrak{u}_{\mathcal{O}}\right) \cdot v_{0} \tag{60}
\end{equation*}
$$

In particular, by (58) and $t_{p} \bullet v_{0}=v_{0}$ the lattice $L_{\lambda, \mathcal{O}}$ is stable under the renormalized action of $t_{p}$. Recall the definition of $d_{x}$ in (6) and define the lattices

$$
\begin{aligned}
L_{\lambda, \mathcal{O}}^{x, \alpha} & :=d_{x} h \cdot\left(t_{p}^{\alpha} \bullet L_{\lambda, \mathcal{O}}\right) \\
& =\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\alpha}\right) \cdot d_{x} g_{\alpha} \cdot L_{\lambda, \mathcal{O}}
\end{aligned}
$$

Recall that $X_{\mathbf{Q}}(H) \cong \mathbf{Z}$ is generated by $N_{H}: H \rightarrow \mathrm{GL}_{1}$, and identify the $\mathbf{Q}$-rational characters of $H$ likewise with the $H$-representations $\mathbf{Q}_{(j)}$ for $j \in \mathbf{Z}$. Again, $A_{(j)}=$ $A \otimes_{\mathbf{Q}} \mathbf{Q}_{(j)}$ for any $\mathbf{Q}$-algebra $A$.

Proposition 3.4. For all $x \in \mathcal{O}^{\times}, \alpha \geq 0$ and $v \in L_{\lambda, \mathcal{O}}^{x, \alpha}$, there is a constant $\Omega_{p}^{\alpha, v} \in \mathcal{O}$ with the following property: For every non-zero $H$-invariant functional

$$
\eta_{j}: \quad L_{\lambda, E} \rightarrow E_{(j)}
$$

we have the congruence

$$
\begin{equation*}
\eta_{j}(v) \equiv N_{F / \mathbf{Q}}(x)^{j} \cdot \Omega_{p}^{\alpha, v} \cdot \eta_{j}\left(g_{0} v_{0}\right) \quad\left(\bmod \mathcal{O} \cdot p^{\alpha} \eta_{j}\left(g_{0} v_{0}\right)\right) \tag{61}
\end{equation*}
$$

with

$$
\begin{equation*}
\eta_{j}\left(g_{0} v_{0}\right) \neq 0 . \tag{62}
\end{equation*}
$$

Furthermore, if $v=d_{x} \cdot h \cdot v_{0}$ we have

$$
\begin{equation*}
\Omega_{p}^{\alpha, v}=1 . \tag{63}
\end{equation*}
$$

Proof. Observe that for any $v \in L_{\lambda, \mathcal{O}}^{x, \alpha}$,

$$
L_{\lambda, \mathcal{O}}^{x, \alpha}=\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\alpha}\right) \cdot d_{x} \cdot U\left(\mathfrak{u}_{\mathcal{O}}^{\alpha}\right) \cdot g_{\alpha} v_{0} .
$$

In particular, we find a $u \in U\left(\mathfrak{u}_{\mathcal{O}}^{\alpha}\right)$ with the property that

$$
v=\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\alpha}\right) \cdot d_{x} \cdot u \cdot g_{\alpha} v_{0} .
$$

According to Lemma 3.3, applying the decomposition (56) to $u$, we find

$$
r=r_{0}+p^{\alpha} r_{1} \in \mathcal{O}+p^{\alpha} U\left(\mathfrak{h}_{\mathcal{O}}\right)
$$

and

$$
s=s_{0}+p^{\alpha} s_{1} \in \mathcal{O}+p^{\alpha} U\left(\mathfrak{b}_{\mathcal{O}}^{-, 0}\right)
$$

satisfying the relation $u=r s$. Therefore,

$$
\begin{aligned}
\eta_{j}(v) & =\eta_{j}\left(d_{x} \cdot r s \cdot g_{0} \cdot\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\alpha}\right) \cdot t_{p}^{\alpha} v_{0}\right) \\
& =\eta_{j}\left(d_{x} \cdot r s \cdot g_{0} v_{0}\right) \\
& =\left(s_{0}+p^{\alpha} s_{1}\right) \cdot\left(r_{0}+p^{\alpha} r_{1}\right) \cdot \Delta(\operatorname{diag}(x, 1, \ldots, 1)) \cdot \eta_{j}\left(g_{0} v_{0}\right) \\
& \equiv N_{H}(\operatorname{diag}(x, 1, \ldots, 1))^{j} \cdot r_{0} s_{0} \cdot \eta_{j}\left(g_{0} v_{0}\right) \quad\left(\bmod \mathcal{O} \cdot p^{\alpha} \eta_{j}\left(g_{0} v_{0}\right)\right) .
\end{aligned}
$$

This proves the first claim. The non-vanishing statement (62) is an immediate consequence of Corollary 3.2.

## 4. Universal $p$-Ordinary cohomology

For a compact open subgroup $K \subseteq G\left(\mathbf{A}^{(\infty)}\right)$ we consider the locally symmetric space

$$
\mathscr{X}(K):=G(\mathbf{Q}) \backslash G(\mathbf{A}) / K \cdot \widetilde{K}_{\infty},
$$

where

$$
\widetilde{K}_{\infty}:=Z^{s}(\mathbf{R})^{0} K_{\infty}^{0} \subseteq G(\mathbf{R})
$$

with $Z^{\text {s }} \subseteq G$ the maximal $\mathbf{Q}$-split torus in the center $Z \subseteq G$ and $K_{\infty} \subseteq G(\mathbf{R})$ a standard maximal compact subgroup. If $K=K_{n+1} \times K_{n}$ with compact open subgroups $K_{m} \subseteq \mathrm{GL}_{m}\left(\mathbf{A}_{F}^{(\infty)}\right)$, we have

$$
\begin{equation*}
\mathscr{X}(K)=\mathscr{X}_{n+1}\left(K_{n+1}\right) \times \mathscr{X}_{n}\left(K_{n}\right) \tag{64}
\end{equation*}
$$

with

$$
\begin{gathered}
\mathscr{X}_{m}\left(K_{m}\right):=G L_{m}(F) \backslash \mathrm{GL}_{m}\left(\mathbf{A}_{F}\right) / K_{m} \cdot \widetilde{K}_{\infty}^{m}, \\
\widetilde{K}_{\infty}^{m}:=Z_{m}^{\mathrm{s}}(F \otimes \mathbf{R})^{0} K_{\infty}^{m, 0} \subseteq \mathrm{GL}_{m}(F \otimes \mathbf{R}),
\end{gathered}
$$

$Z_{m}^{\mathrm{s}} \subseteq \operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}_{m}$ the $\mathbf{Q}$-split center and $\widetilde{K}_{\infty}^{m} \subseteq \mathrm{GL}_{n}(F \otimes \mathbf{R})$ the corresponding standard maximal compact subgroup.
4.1. Arithmetic subgroups. We call $K$ torsion free or neat if for all $g \in G\left(\mathbf{A}^{(\infty)}\right)$ the arithmetic group

$$
\Gamma_{g}:=G(\mathbf{Q}) \cap g K g^{-1}
$$

is torsion free resp. neat. Each neat $K$ is torsion-free and has the property that the arithmetic subgroups

$$
\Gamma_{g} \subseteq G(\mathbf{R})^{0}
$$

contain only totally positive elements. Every $K$ contains a neat $K$ of finite index.
Each rational representation $L_{\lambda, E}$ gives rise to a sheaf $\underline{L}_{\lambda, E}$ on $\mathscr{X}(K)$.
For $K$ neat, $\mathscr{X}(K)$ is a manifold and the sheaf cohomology of $\underline{L}_{\lambda, E}$ is a sum of the cohomologies of the arithmetic subgroups $\Gamma_{g}$ corresponding to $K$. The sheaf $\underline{L}_{\lambda, E}$ is non-trivial, provided that the algebraicity condition

$$
\begin{equation*}
H^{0}\left(Z(\mathbf{Q}) \cap K ; L_{\lambda, E}\right)=L_{\lambda, E} \tag{65}
\end{equation*}
$$

is satisfied, i.e. the centers of the arithmetic groups $\Gamma_{g}$ act trivially on $L_{\lambda, E}$. Condition (65) only depends on the Zariski closure of $\Gamma_{g}$ in $G$ and therefore is independent of $K$ if $K$ is sufficiently small.

A dominant weight $\lambda \in X_{\mathbf{C}}\left(\operatorname{res}_{F / \mathbf{Q}} T_{m}\right)$ of $\operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}_{m}$ corresponds to a tuple $\lambda=$ $\left(\lambda_{\tau, i}\right)_{\tau: F \rightarrow \mathbf{C}, 1 \leq i \leq m}$ of dominant weights $\left(\lambda_{\tau, 1}, \ldots, \lambda_{\tau, m}\right) \in \mathbf{Z}^{m}, \tau: F \rightarrow \mathbf{C}$ running through all field embeddings. Complex conjugation canonically acts on the set of dominant weights $\lambda$ via its action on the embeddings $\tau: F \rightarrow \mathbf{C}$, sending $\tau$ to $\tau^{\mathrm{c}}$, the postcomposition of $\tau$ with complex conjugation. Write $\lambda^{c}$ for the complex conjugate weight attached to $\lambda$. We say that $\lambda$ is essentially conjugate self-dual over $\mathbf{Q}$ if

$$
\lambda=\lambda^{\mathrm{V}, \mathrm{c}}+(w)
$$

for some $w \in \mathbf{Z}$. This is the same to say that

$$
L_{\lambda, \mathbf{C}} \cong L_{\lambda, \mathbf{C}}^{\vee, \mathbf{c}} \otimes\left(N_{F / \mathbf{Q}} \circ \operatorname{det}\right)^{\otimes w} .
$$

The absolute Galois group $\operatorname{Gal}(\overline{\mathbf{Q}} / \mathbf{Q})$ of $\mathbf{Q}$ also acts on the dominant weights $\lambda$ via its action on $\operatorname{res}_{F / \mathbf{Q}} T_{m}$, which permutes the entries in each tuple $\left(\lambda_{\tau, i}\right)_{\tau: F \rightarrow \mathbf{C}}, 1 \leq i \leq m$ via precomposition with $\tau$.

We say that $\lambda$ is arithmetic or strongly pure (in the terminology of [71]), if

$$
\lambda^{\sigma}=\left(\lambda^{\mathrm{V}, \mathrm{c}}\right)^{\sigma}+\left(w_{\lambda}\right)
$$

for all $\sigma \in \operatorname{Gal}(\overline{\mathbf{Q}} / \mathbf{Q})$ and $w_{\lambda} \in \mathbf{Z}$ independent of $\sigma$. We call $w_{\lambda}$ the (purity) weight of $\lambda$. In other words,

$$
\lambda_{\tau, i}+\lambda_{\tau^{\mathrm{c}}, m+1-i}=w_{\lambda},
$$

for all $\tau$ and all $i$.
We adopt the same terminology for dominant weights $\lambda=\lambda_{n+1} \otimes \lambda_{n}$ of $G$, that we call arithmetic if $\lambda_{n+1}$ and $\lambda_{n}$ both are arithmetic. For such a $\lambda$, define

$$
w_{\lambda}:=w_{\lambda_{n+1}}+w_{\lambda_{n}} .
$$

Cuspidal cohomology

$$
H_{\text {cusp }}^{\bullet}\left(\mathscr{X}(K) ; \underline{L}_{\lambda, \mathbf{C}}\right)
$$

vanishes if $\lambda$ is not arithmetic (use the Künneth Theorem to reduce to Clozel's 'Lemme de pureté' for GL( $n$ ) in [6]). Put

$$
l_{0}:=\operatorname{rk} G(\mathbf{R})-\operatorname{rk} \widetilde{K}_{\infty},
$$

and

$$
q_{0}:=\frac{\operatorname{dim} \mathscr{X}(K)-l_{0}}{2} .
$$

Then $q_{0}$ is an integer, which is known as the bottom degree of $G$, because cuspidal cohomology vanishes in degree $q<q_{0}$ and if it is non-zero, then it is non-zero precisely in degrees $q_{0} \leq q \leq q_{0}+l_{0}, q_{0}+l_{0}$ being the top degree.
4.2. Nearly ordinary cohomology. Recall the compact open subgroups

$$
I_{\alpha}=\prod_{\mathfrak{p} \mid p} I_{v_{\mathfrak{p}}(p) \alpha}^{n+1} \times I_{v_{\mathfrak{p}}(p) \alpha}^{n}
$$

and

$$
I_{\alpha^{\prime}, \alpha}=\prod_{\mathfrak{p} \mid p} I_{v_{\mathfrak{p}}(p) \alpha^{\prime}, v_{\mathfrak{p}}(p) \alpha}^{n+1} \times I_{v_{\mathfrak{p}}(p) \alpha^{\prime}, v_{\mathfrak{p}}(p) \alpha}^{n}
$$

of $G\left(\mathbf{Q}_{p}\right)$ from section 2.2 and the corresponding Hecke algebras $\mathcal{H}_{A}(\alpha)$ and $\mathcal{H}_{A}\left(\alpha^{\prime}, \alpha\right)$ for $\alpha \geq \alpha^{\prime} \geq 0, \alpha>0$, which contain the distinguished Hecke operator $U_{p}$ defined in (48).

Consider any family of compact open subgroups $K_{\alpha^{\prime}, \alpha} \subseteq G\left(\mathbf{A}^{(\infty)}\right), \alpha \geq \alpha^{\prime} \geq 0$ and $\alpha>0$, which admits a decomposition $K_{\alpha^{\prime}, \alpha}=I_{\alpha^{\prime}, \alpha} \times K^{(p)}$ with $K^{(p)}$ a compact open outside $p$, trivial at $p$ and independent of $\alpha^{\prime}, \alpha$. Define $\alpha_{0}^{K}$ to be the minimal $\alpha_{0}>0$ such that $K_{0, \alpha_{0}}$ is neat.

The cohomology

$$
H_{?}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, E}\right)
$$

for $? \in\{-, \mathrm{c},!\}$ is naturally a module over the Hecke algebra of level $K_{\alpha^{\prime}, \alpha}$. At $p$ we renormalize the action of $U_{p}$ by multiplication by the scalar $\lambda^{\vee}\left(t_{p}\right)=\left(-\lambda^{w_{0}}\right)\left(t_{p}\right)$. Likewise, we may renormalize every Hecke operator in $\mathcal{H}_{\mathcal{O}}(0, \alpha) \subseteq \mathcal{H}_{\mathcal{O}}\left(\alpha^{\prime}, \alpha\right)$ via (59). Then $\mathcal{H}_{\mathcal{O}}(0, \alpha)$ acts on cohomology

$$
\begin{equation*}
H_{?}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}\right) \tag{66}
\end{equation*}
$$

with $p$-integral coefficients $p$-optimally. Attached to this action is an ordinary projector $e_{p}$, which projects onto the subspace

$$
\begin{equation*}
H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}\right):=e_{p} H_{?}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}\right) \tag{67}
\end{equation*}
$$

of (66) on which $U_{p}$ acts invertibly. More generally, we will consider the spaces

$$
H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, A}\right)
$$

for

$$
\begin{equation*}
A \in\left\{\mathcal{O} / p^{\alpha} \mathcal{O}, p^{-\alpha} \mathcal{O} / \mathcal{O}, \mathcal{O}, K / \mathcal{O}\right\} \tag{68}
\end{equation*}
$$

where

$$
L_{\lambda, A}=L_{\lambda, \mathcal{O}} \otimes_{\mathcal{O}} A
$$

Since $U_{p}$ is a product of the local operators $V_{\nu} \otimes 1,1 \otimes V_{\nu}, 1 \leq \nu \leq n$, and since $V_{n+1} \otimes 1$ acts invertibly as well (for every place $v \mid p$ ) from section 1.2 , the action of $\mathcal{H}_{\mathcal{O}}\left(\alpha^{\prime}, \alpha\right)$ on these nearly ordinary cohomology naturally extends to an action of $\mathcal{H}_{\mathcal{O}}^{\text {ord }}\left(\alpha^{\prime}, \alpha\right)$ and

$$
H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{A}\right)=\mathcal{H}_{\mathcal{O}}^{\text {ord }}\left(\alpha^{\prime}, \alpha\right) \otimes_{\mathcal{H}_{\mathcal{O}}\left(\alpha^{\prime}, \alpha\right)} H_{?}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{A}\right)
$$

as Hecke modules for $A \in\left\{\mathcal{O} / p^{\beta} \mathcal{O}, p^{-\beta} \mathcal{O} / \mathcal{O}, \mathcal{O}, E / \mathcal{O}\right\}$. Put

$$
H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{E}\right):=H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{\mathcal{O}}\right) \otimes_{\mathcal{O}} E .
$$

Then in all cases, passing to ordinary parts is an exact functor.
Proposition 4.1. For $A$ as in (68), any dominant weight $\lambda$, and every $\alpha \geq \alpha^{\prime} \geq 0$ with $\alpha \geq \alpha_{0}^{K}$, we have for every $\alpha^{\prime \prime} \geq \alpha$ a canonical isomorphism

$$
H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, A}\right)=H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha^{\prime \prime}}\right) ; \underline{L}_{\lambda, A}\right)
$$

of Hecke modules.
Proof. The proof proceeds as the proof of the isomorphism (4.7c) on p. 445 of [34], using the explicit left coset decomposition of the Hecke operators $T_{1}, T_{2}, T_{3}$ of loc. cit., adapted to right coset decompositions we are working with.

Remark 4.2. As in [34], Proposition 4.1 holds for more general coefficient sheaves, in particular for the ones considered in the proof of the Control Theorem (Theorem 4.14) below.
4.3. Independence of weight. Write $\mathcal{O}\left[\lambda^{w_{0}}\right]$ for the $\mathcal{O}$-module of rank 1 on which $B^{-}$ acts via $\lambda^{w_{0}}$. We assume that we are given a fixed generator $1 \in \mathcal{O}\left[\lambda^{w_{0}}\right]$, which we use to identify this space with $\mathcal{O}$. Consider the inclusion

$$
i: \quad \mathcal{O}\left[\lambda^{w_{0}}\right] \rightarrow L_{\lambda, \mathcal{O}}, \quad c \mapsto c \cdot v_{0},
$$

and the projection

$$
p: \quad L_{\lambda, \mathcal{O}} \rightarrow \mathcal{O}\left[\lambda^{w_{0}}\right],
$$

which projects $T$-equivariantly onto the lowest weight space. By our identification of $\mathcal{O}\left[\lambda^{w_{0}}\right]$ with $\mathcal{O}$, we obtain maps

$$
i_{\lambda}: \mathcal{O} \rightarrow L_{\lambda, \mathcal{O}},
$$

and

$$
p_{\lambda}: \quad L_{\lambda, \mathcal{O}} \rightarrow \mathcal{O} .
$$

Theorem 4.3. For $\alpha \geq \alpha^{\prime} \geq 0, \alpha \geq \alpha_{0}^{K}$, and $A \in\left\{\mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}, p^{-\alpha^{\prime}} \mathcal{O} / \mathcal{O}\right\}$ the maps $i_{\lambda}$ and $p_{\lambda}$ induce isomorphisms

$$
\begin{equation*}
\iota_{\lambda}: \quad H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{A}\right) \rightarrow H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, A}\right) \tag{69}
\end{equation*}
$$

and

$$
\begin{equation*}
\pi_{\lambda}: \quad H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, A}\right) \rightarrow H_{?, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{A}\right), \tag{70}
\end{equation*}
$$

which are inverses of each other, Hecke-equivariant outside $p$, and satisfy

$$
\begin{equation*}
T \circ \iota_{\lambda}=\iota_{\lambda} \circ T, \quad \text { and } \quad T \circ \pi_{\lambda}=\pi_{\lambda} \circ T, \tag{71}
\end{equation*}
$$

for $T \in \mathcal{H}_{\mathcal{O}}(0, \alpha)$ and for every $t \in T\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right)$,

$$
\begin{equation*}
\langle t\rangle \circ \iota_{\lambda}=\iota_{\lambda} \circ \lambda^{w_{0}}(t)\langle t\rangle, \quad \text { and } \quad \lambda^{w_{0}}(t)\langle t\rangle \circ \pi_{\lambda}=\pi_{\lambda} \circ\langle t\rangle, \tag{72}
\end{equation*}
$$

Proof. We discuss the case $A=\mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}$, the other case follow similarly. Consider the short exact sequences

$$
\begin{gather*}
0 \rightarrow \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O} \rightarrow L_{\lambda, \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}} \rightarrow \operatorname{coker} i_{\lambda} \otimes \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O} \rightarrow 0  \tag{73}\\
0 \rightarrow \operatorname{ker} p_{\lambda} \otimes \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O} \rightarrow L_{\lambda, \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}} \rightarrow \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O} \rightarrow 0
\end{gather*}
$$

We have

$$
\left[I_{\alpha^{\prime}, \alpha} t_{p} I_{\alpha^{\prime}, \alpha}\right]=\bigsqcup_{u \in U(\mathcal{O}) / t_{p} U(\mathcal{O}) t_{p}^{-1}} u t_{p} I_{\alpha, \alpha}
$$

which shows that

$$
H_{?}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{\operatorname{coker} i_{\lambda} \otimes \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}}\right) .
$$

is a $U_{p}$-module, and likewise for $\operatorname{ker} p_{\lambda} \otimes \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}$.
By construction, $\left(-\lambda^{w_{0}}\right)\left(t_{p}\right) \cdot t_{p}$ acts nilpotently on both coker $i_{\lambda} \otimes \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}$ and ker $p_{\lambda} \otimes$ $\mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}$. Therefore, $U_{p}$ acts nilpotently on

$$
H_{?}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{\operatorname{coker} i_{\lambda} \otimes \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}}\right) .
$$

This shows that

Since projection to the ordinary part is an exact functor, the long exact sequence attached to (73) implies that $i_{\lambda}$ must be an isomorphism of $\mathcal{O}$-modules on the ordinary part. The same argument shows that $p_{\lambda}$ induces the inverse isomorphism.

We introduce for every dominant weight $\lambda$ the notation

$$
L_{\lambda^{\vee}, E}^{\circ}:=\operatorname{Hom}_{E}\left(L_{\lambda, E}, E\right)
$$

for the contragredient. It comes with a canonical perfect pairing

$$
\langle-,-\rangle: \quad L_{\lambda, E} \otimes_{E} L_{\lambda^{\vee}, E}^{\circ} \rightarrow E .
$$

Define the dual lattice for $L_{\lambda, \mathcal{O}}$ accordingly as

$$
L_{\lambda^{\vee}, \mathcal{O}}^{\circ}:=\left\{f \in L_{\lambda^{\vee}, E}^{\circ} \mid f\left(L_{\lambda, \mathcal{O}}\right) \subseteq \mathcal{O}\right\}
$$

Then it is easy to check that

$$
\langle-,-\rangle: \quad L_{\lambda, \mathcal{O}} \otimes_{\mathcal{O}} L_{\lambda^{\vee}, \mathcal{O}}^{\circ} \rightarrow \mathcal{O}
$$

is still a perfect pairing: As a sublattice of $L_{\lambda^{\vee}, E}^{\circ}$ is generated by a $B$-highest weight vector $v_{0}^{\circ}$ dual to $v_{0}$. In particular, $L_{\lambda^{\vee}, \mathcal{O}}^{\circ}$ is canonically identified with the $\mathcal{O}$-dual of $L_{\lambda, \mathcal{O}}$.

Along the same lines we see that $L_{\lambda \vee, \mathcal{O}}^{\circ}$ is also canonically identified with the Pontryagin dual of $L_{\lambda, E / \mathcal{O}}$ and likewise for finite torsion coefficients.

With this notation at hand, define the universal nearly ordinary cohomology with torsion coeffients as

$$
\mathcal{H}_{\mathrm{ord}}^{\bullet}\left(K_{\infty, \infty} ; \lambda, E / \mathcal{O}\right):=\underset{\alpha, \alpha^{\prime}}{\lim _{\mathrm{ord}}} H^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, p^{-\alpha^{\prime}} \mathcal{O} / \mathcal{O}}\right)
$$

Likewise, we obtain with respect to the transfer maps its Pontryagin dual

$$
\mathcal{H}_{\mathrm{c}, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; \lambda^{\vee}, \mathcal{O}\right):=\lim _{\alpha, \alpha^{\prime}} H_{\mathrm{c}, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda^{\vee}, \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}}^{\circ}\right)
$$

We also consider the mutually Pontryagin dual nearly ordinary cohomology groups of the form

$$
\begin{gathered}
\mathcal{H}_{\text {ord }}^{\bullet}\left(K_{\infty, \infty} ; \lambda, \mathcal{O}\right):={\underset{\alpha, \alpha^{\prime}}{\lim } H_{\mathrm{ord}}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O} / p^{\alpha^{\prime}} \mathcal{O}}^{\circ}\right)}^{\mathcal{H}_{\mathrm{c}, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; \lambda^{\vee}, E / \mathcal{O}\right):=\underset{\alpha, \alpha^{\prime}}{\lim } H_{\mathrm{c}, \text { ord }}^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ;{\underline{L_{\lambda}, p^{-\alpha^{\prime}} \mathcal{O} / \mathcal{O}}}\right)} .
\end{gathered}
$$

We define analogously the maps $\iota_{\lambda \vee}^{\circ}$ and $p_{\lambda \vee}^{\circ}$ as the Pontryagin duals of the maps $p_{\lambda}$ and $\iota_{\lambda}$ respectively.

Then Theorem 4.3 shows
Corollary 4.4. The map $\pi_{\lambda}$ induces an isomorphism

$$
\pi_{\lambda}: \quad \mathcal{H}_{?, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; \lambda, E / \mathcal{O}\right) \rightarrow \mathcal{H}_{?, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; 0, E / \mathcal{O}\right)=: \mathcal{H}_{?, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; E / \mathcal{O}\right)
$$

respecting the actions of $\mathcal{H}_{\mathcal{O}}(0, \alpha)$ and of the Hecke operators outside p, and for every $t \in T\left(\mathbf{Z}_{p}\right)$,

$$
\lambda^{w_{0}}(t)\langle t\rangle \circ \pi_{\lambda}=\pi_{\lambda} \circ\langle t\rangle
$$

Likewise, the map $\pi_{\lambda}^{\circ}$ induces an isomorphism

$$
\pi_{\lambda}^{\circ}: \quad \mathcal{H}_{?, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; \lambda, \mathcal{O}\right) \rightarrow \mathcal{H}_{?, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; 0, \mathcal{O}\right)=: \mathcal{H}_{?, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; \mathcal{O}\right)
$$

respecting the actions of $\mathcal{H}_{\mathcal{O}}(0, \alpha)$ and of the Hecke operators outside $p$, and for every $t \in T\left(\mathbf{Z}_{p}\right)$,

$$
\lambda(t)\langle t\rangle \circ \pi_{\lambda}^{\circ}=\pi_{\lambda}^{\circ} \circ\langle t\rangle
$$

### 4.4. The universal nearly ordinary Hecke algebra. Put

$$
\begin{gathered}
\Lambda:=\mathcal{O}\left[\left[T\left(\mathbf{Z}_{p}\right)\right]\right]={\underset{\zeta}{\overleftarrow{\alpha}}}_{\lim } \mathcal{O}\left[T\left(\mathbf{Z}_{p} / p^{\alpha} \mathbf{Z}_{p}\right)\right] . \\
\Lambda^{\circ}:=\mathcal{O}\left[\left[T\left(\mathbf{Z}_{p}\right) / T\left(\mathbf{Z}_{p}\right)_{\text {tors }}\right]\right]
\end{gathered}
$$

Then $\Lambda$ is a complete Noetherian semi-local ring with each local factor isomorphic to $\Lambda^{\circ}$ and each $\mathcal{H}_{\mathcal{O}}\left(\alpha^{\prime}, \alpha\right)$ carries a canonical $\Lambda$-module structure. Following Hida, define the universal nearly ordinary Hecke algebra $\mathbf{h}_{\text {?,ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ as the $\Lambda$-subalgebra generated by the image of the canonical map

$$
\left[\underset{\alpha, \alpha^{\prime}}{\lim _{\mathcal{O}}} \mathcal{H}_{\mathcal{O}}\left(\alpha^{\prime}, \alpha\right)\right] \otimes \mathcal{O} \mathcal{H}_{\mathcal{O}}\left(K^{S(K)}, G\left(\mathbf{A}^{S(K)}\right)\right) \rightarrow \operatorname{End}_{\mathcal{O}}\left(\mathcal{H}_{?, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; E / \mathcal{O}\right)\right)
$$

where $S(K)$ denotes the minimal finite set of places of $\mathbf{Q}$ containing $p$ and $\infty$ and for which there is a factorization

$$
K_{\alpha^{\prime}, \alpha}=L_{S(K)} \times \prod_{v \notin S(K), v \ngtr \infty} G\left(\mathbf{Z}_{v}\right)
$$

for some compact open subgroup

$$
L_{S(K)} \subseteq \prod_{v \in S(K), v \nmid \infty} G\left(\mathbf{Q}_{v}\right)
$$

Since $\mathcal{H}_{\mathcal{O}}\left(K^{S(K)}, G\left(\mathbf{A}^{S(K)}\right)\right)$ is by definition a product of spherical Hecke algebras, $\mathbf{h}_{\text {?,ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ is commutative. By Corollary 4.4, $\mathcal{H}_{\text {?, ord }}^{\bullet}\left(K_{\infty, \infty} ; \lambda, E / \mathcal{O}\right)$ is an $\mathbf{h}_{\text {?, ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$-module, which differs from $\mathcal{H}_{?, \text { ord }}^{\bullet}\left(K_{\infty, \infty} ; E / \mathcal{O}\right)$ only in terms of the $\Lambda$-action.
4.5. Galois representations. In this section, assume that $S(K)=\{p, \infty\}$, i.e. $K^{(p)}=$ $G\left(\widehat{\mathbf{Z}}^{(p)}\right)$ is the product of the standard maximal compact open subgroup over all primes $\ell \neq p$ and

$$
K_{\alpha^{\prime}, \alpha}=K^{(p)} \times I_{\alpha^{\prime}, \alpha} .
$$

Then the Hecke algebra of interest is
$\mathcal{H}_{\mathcal{O}}\left(K_{\alpha^{\prime}, \alpha}, G\left(\mathbf{A}^{(p \infty)}\right) \times \Delta_{G}\right)=\otimes_{v \nmid p \infty}^{\prime}\left(\mathcal{H}_{\mathcal{O}}\left(\mathrm{GL}_{n+1}\left(\mathcal{O}_{v}\right), \mathrm{GL}_{n+1}\left(F_{v}\right)\right) \otimes \mathcal{H}_{\mathcal{O}}\left(\mathrm{GL}_{n}\left(\mathcal{O}_{v}\right), \mathrm{GL}_{n}\left(F_{v}\right)\right)\right) \otimes \mathcal{H}_{\mathcal{O}}\left(\alpha^{\prime}, \alpha\right)$.
In $\mathcal{H}_{\mathcal{O}}\left(\mathrm{GL}_{n}\left(\mathcal{O}_{v}\right), \mathrm{GL}_{n}\left(F_{v}\right)\right)$, we find the standard Hecke operators

$$
T_{v, \nu}=\mathrm{GL}_{n}\left(\mathcal{O}_{v}\right) \varpi_{v}^{\omega_{\nu}} \mathrm{GL}_{n}\left(\mathcal{O}_{v}\right), \quad 1 \leq \nu \leq n
$$

Consider as in (13) the spherical (reciprocal) Hecke polynomial

$$
H_{F_{v}, n}(X):=\sum_{\nu=0}^{n}(-1)^{\nu} q_{v}^{\frac{\nu(\nu-1)}{2}} T_{\nu} X^{n-\nu} \in \mathcal{H}_{A}^{n}\left(\alpha^{\prime}, \alpha\right)
$$

Recall that $H_{F_{v}, n}(X)$ admits a factorization

$$
H_{F_{v}, n}(X)=\prod_{i=1}^{n}\left(X-\widetilde{U}_{i}\right)
$$

for $\widetilde{U}_{i}$ in the parabolic Hecke algebra at $v$. Define the Hecke polynomial
$H_{v}(X):=\prod_{i=1}^{n+1} \prod_{j=1}^{n}\left(X-\widetilde{U}_{i} \otimes \widetilde{U}_{j}\right) \in\left(\mathcal{H}_{\mathcal{O}}\left(\mathrm{GL}_{n+1}\left(\mathcal{O}_{v}\right), \mathrm{GL}_{n+1}\left(F_{v}\right)\right) \otimes \mathcal{H}_{\mathcal{O}}\left(\mathrm{GL}_{n}\left(\mathcal{O}_{v}\right), \mathrm{GL}_{n}\left(F_{v}\right)\right)\right)[X]$.
This is the spherical Hecke polynomial for $\mathrm{GL}_{n+1} \times \mathrm{GL}_{n}$ over $F_{v}$.
For $1 \leq \mu \leq n+1$ and $1 \leq \nu \leq n$, the operators $T_{v, \mu} \otimes 1$ and $1 \otimes T_{v, \nu}$ act on

$$
\sum_{q} H_{?, \text { ord }}^{q}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, A}\right)
$$

for $? \in\{-, \mathrm{c},!\}$ and $A \in\left\{\mathcal{O}, E, E / \mathcal{O}, p^{-\beta} \mathcal{O} / \mathcal{O}, \mathcal{O} / p^{\beta} \mathcal{O}\right\}$.
The operators $T_{v, \mu} \otimes T_{v, \nu}$ for $v \notin S(p)$ together with the image of $\mathcal{H}_{\mathcal{O}}\left(\alpha^{\prime}, \alpha\right)$ generate the nearly ordinary Hecke algebra $\mathrm{h}_{\text {?,ord }}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)$ over $\mathcal{O}\left[T\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right]\right.$. Passing to the projective limit over $\alpha^{\prime}$, $\alpha$, we obtain the universal nearly ordinary Hecke algebra $\mathbf{h}_{\text {? }, \operatorname{ord}}\left(K_{\infty, \infty} ; \lambda, \mathcal{O}\right)$ over $\Lambda$. Recall for $\lambda=0$, define

$$
\mathbf{h}_{?, \text { ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right):=\mathbf{h}_{?, \text { ord }}\left(K_{\infty, \infty} ; 0, \mathcal{O}\right)
$$

Finally, for any $q \in \mathbf{Z}$, let $\mathrm{h}_{\text {?,ord }}^{q}\left(K_{\alpha, \alpha} ; \lambda, \mathcal{O}\right)$ denote the image of the canonical map

$$
\mathrm{h}_{?, \text { ord }}\left(K_{\alpha, \alpha} ; \lambda, \mathcal{O}\right) \rightarrow \operatorname{End}_{\mathcal{O}} H_{?, \text { ord }}^{q}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, A}\right) .
$$

We adopt the same notation for univeral nearly ordinary Hecke algebras.
Write $\lambda=\lambda_{n+1} \otimes \lambda_{n}$ for dominant weights $\lambda_{m}$ on $\operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}_{m}$. Then according to (64), we have a Künneth spectral sequence

$$
\begin{aligned}
& E_{p q}^{2}:=\bigoplus_{q_{n+1}+q_{n}=q} \operatorname{Tor}_{-p}^{\mathcal{O}}\left(H_{?}^{q_{n+1}}\left(\mathscr{X}_{n+1}\left(K_{\alpha, \alpha}^{n+1}\right) ; \underline{L}_{\lambda_{n+1}, E / \mathcal{O}}\right), H_{?}^{q_{n}}\left(\mathscr{X}_{n}\left(K_{\alpha, \alpha}^{n}\right) ; \underline{L}_{\lambda_{n}, E / \mathcal{O}}\right)\right) \\
& \Longrightarrow H_{?}^{p+q}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right) .
\end{aligned}
$$

Since $\mathcal{O}$ is a principal ideal domain, Tor. vanishes in degrees $\geq 2$, whence we deduce a short exact sequence

$$
\begin{gathered}
0 \rightarrow \bigoplus_{q_{n+1}+q_{n}=q} H_{?}^{q_{n+1}}\left(\mathscr{X}_{n+1}\left(K_{\alpha, \alpha}^{n+1}\right) ; \underline{L}_{\lambda_{n+1}, E / \mathcal{O}}\right) \otimes H_{?}^{q_{n}}\left(\mathscr{X}_{n}\left(K_{\alpha, \alpha}^{n}\right) ; \underline{L}_{\lambda_{n}, E / \mathcal{O}}\right) \rightarrow \\
H_{?}^{p+q}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right) \rightarrow \bigoplus_{q_{n+1}+q_{n}=q+1} \operatorname{Tor}_{1}^{\mathcal{O}}\left(H_{?}^{q_{n+1}}\left(\mathscr{X}_{n+1}\left(K_{\alpha, \alpha}^{n+1}\right) ; \underline{L}_{\lambda_{n+1}, E / \mathcal{O}}\right), H_{?}^{q_{n}}\left(\mathscr{X}_{n}\left(K_{\alpha, \alpha}^{n}\right) ; \underline{L}_{\lambda_{n}, E / \mathcal{O}}\right)\right) \rightarrow 0 .
\end{gathered}
$$

In particular, the edge morphism of above spectral sequence provides us with a canonical monomorphism
$\bigoplus_{q_{n+1}+q_{n}=q} \underset{\alpha}{\lim } H_{?}^{q_{n+1}}\left(\mathscr{X}_{n+1}\left(K_{\alpha, \alpha}^{n+1}\right) ; \underline{L}_{\lambda_{n+1}, E / \mathcal{O}}\right) \otimes \underset{\alpha}{\lim } H_{?}^{q_{n}}\left(\mathscr{X}_{n}\left(K_{\alpha, \alpha}^{n}\right) ; \underline{L}_{\lambda_{n}, E / \mathcal{O}}\right) \rightarrow \underset{\alpha}{\lim } H_{?}^{q}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right)$,
and likewise for ordinary cohomology.
For any field $k$ which is either (an extension of) the residue field of $\mathcal{O}$, or the field $E$, the Künneth spectral sequence degenerates and induces a canonical isomorphism
$\bigoplus_{q_{n+1}+q_{n}=q} \underset{\alpha}{\lim } H^{q_{n+1}}\left(\mathscr{X}_{n+1}\left(K_{\alpha, \alpha}^{n+1}\right) ; \underline{L}_{\lambda_{n+1}, k}\right) \otimes \underset{\alpha}{\lim } H^{q_{n}}\left(\mathscr{X}_{n}\left(K_{\alpha, \alpha}^{n}\right) ; \underline{L}_{\lambda_{n}, k}\right) \cong \underset{\alpha}{\lim _{\vec{~}}} H^{q}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, k}\right)$.
In particular, we obtain a canonical isomorphism

$$
\begin{equation*}
\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty}^{n+1} ; k\right) \otimes \mathbf{h}_{\text {ord }}\left(K_{\infty, \infty}^{n} ; k\right) \cong \mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; k\right), \tag{75}
\end{equation*}
$$

of $\Lambda$-algebras. Here $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty}^{m} ; k\right)$ is the Hecke algebra generated by $T_{v, m}, v \notin S(p)$, and $\prod_{v \mid p} \mathcal{H}_{k}^{m}(\alpha, \alpha)$ acting on

$$
\underset{\alpha}{\lim } \sum_{q_{m}} H^{q_{m}}\left(\mathscr{X}_{m}\left(K_{\alpha, \alpha}^{m}\right) ; \underline{L}_{\lambda_{m}, k}\right)
$$

over the corresponding Iwasawa algebra $\mathcal{O}\left[\left[T_{m}\left(\mathcal{O}_{p}\right)\right]\right]$.
Let $\mathfrak{m}$ denote a maximal ideal in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ with residue field $k$. Write $S$ for the set of finite places of $F$ containing the places above $p$ and the places which ramify in $F / \mathbf{Q}$.

Assume that (after possibly enlarging $k$ ) the following condition is satisfied:
(i) There exists a continuous semisimple Galois representation

$$
\bar{\rho}_{\mathfrak{m}}: \quad \operatorname{Gal}(\overline{\mathbf{Q}} / F) \rightarrow \mathrm{GL}_{(n+1) n}(k),
$$

such that for every finite place $v \notin S$ the image $\bar{\rho}_{\mathfrak{m}}\left(\operatorname{Frob}_{v}\right)$ of the geometric Frobenius element has characteristic polynomial

$$
H_{v}(X) \in\left(\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right) / \mathfrak{m}\right)[X] .
$$

By the Chebotarev density theorem, property (i) characterizes the Galois representation in question uniquely up to isomorphism, provided it exists. According to (75), we find (after possibly enlarging $k$ once again) maximal ideals $\mathfrak{m}_{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty}^{m} ; \mathcal{O}\right), m \in\{n, n+1\}$, such that condition (i) amounts to
(i') There exist continuous semisimple Galois representations

$$
\bar{\rho}_{\mathfrak{m}_{m}}: \quad \operatorname{Gal}(\overline{\mathbf{Q}} / F) \rightarrow \mathrm{GL}_{m}(k)
$$

such that for every finite place $v \notin S$ the image $\bar{\rho}_{\mathfrak{m}_{m}}\left(\operatorname{Frob}_{v}\right)$ of the geometric Frobenius element has characteristic polynomial

$$
H_{F_{v}, m}(X) \in\left(\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty}^{m} ; \mathcal{O}\right) / \mathfrak{m}_{m}\right)[X]
$$

for $m \in\{n, n+1\}$.
Remark 4.5. Condition (i') implies

$$
\bar{\rho}_{\mathfrak{m}}=\left(\bar{\rho}_{\mathfrak{m}_{n+1}} \otimes \bar{\rho}_{\mathfrak{m}_{n}}\right)^{\mathrm{ss}}
$$

Remark 4.6. Condition (i') implies that action of $\mathcal{O}\left[\left[Z\left(\mathbf{Z}_{p}\right)\right]\right] \subseteq \Lambda$ is compatible with the determinant of $\bar{\rho}_{\mathfrak{m}}$ in the following sense. On the one hand, we know that for $v \notin S$,

$$
T_{v, n+1} \otimes 1 \equiv \chi_{\mathrm{cyc}}^{\otimes \frac{(n+1) n}{2}} \otimes \operatorname{det} \bar{\rho}_{\mathfrak{m}_{n+1}}\left(\operatorname{Frob}_{v}\right) \quad(\bmod \mathfrak{m})
$$

and

$$
1 \otimes T_{v, n} \equiv \chi_{\mathrm{cyc}}^{\otimes \frac{n(n-1)}{2}} \otimes \operatorname{det} \bar{\rho}_{\mathfrak{m}_{n}}\left(\operatorname{Frob}_{v}\right) \quad(\bmod \mathfrak{m})
$$

On the other hand, suitable powers of these Hecke operators away from $p$ may be considered as elements of $\Lambda$ (as diamond operators).

Definition 4.7. A maximal ideal $\mathfrak{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ is called non-Eisenstein, if (i') is satisfied and
(ii) Each $\bar{\rho}_{\mathfrak{m}_{m}}$ is absolutely irreducible.

Remark 4.8. For $\mathfrak{m}$ non-Eisenstein, $\bar{\rho}_{\mathfrak{m}}$ may be reducible (when considering the tensor product of two symmetric powers of the same two-dimensional Galois representation for example).

Conjecture 4.9. For each $m \geq 1$ and each maximal ideal $\mathfrak{m}_{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty}^{m} ; \mathcal{O}\right)$, there exists a Galois representation $\bar{\rho}_{\mathfrak{m}}$ as in ( $i^{\prime}$ ).

For $F$ totally real or a CM field, results of Scholze [78, Corollary 5.4.3] imply the existence of $\bar{\rho}_{\mathfrak{m}_{m}}$ for all maximal ideals $\mathfrak{m}_{m}$, which also ensures the existence of $\bar{\rho}_{\mathfrak{m}}$. It is expected that the representation $\bar{\rho}_{\mathfrak{m}_{m}}$ lifts to a representation over the localization $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty}^{m} ; \mathcal{O}\right)_{\mathfrak{m}_{m}}$, which is known by Corollary 5.4.4 in loc. cit. modulo a nilpotent ideal of bounded exponent (see also Theorem 5.13 in [69]).

Theorem 4.10. Assume that $F$ is totally real, CM or that Conjecture 4.9 holds over $F$. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal in $\mathbf{h}_{\mathrm{ord}}\left(K_{\infty, \infty} ; \mathcal{O}\right)$. Assume that the residue field $k$ of $\mathfrak{m}$ embeds into the residue field of $E$. Then for every $E$-rational dominant weight $\lambda$ of $G$, we have an identity

$$
\begin{equation*}
H_{\mathrm{c}}^{q}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, E}\right)_{\mathfrak{m}}=H^{q}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, E}\right)_{\mathfrak{m}} \tag{76}
\end{equation*}
$$

of localizations at $\mathfrak{m}$ and if this space is non-zero, then $q_{0} \leq q \leq q_{0}+l_{0}$. Furthermore, every absolutely irreducible constituent of (76) is cuspidal.

Proof. By (64) and (74), the first statement is a consequence of Theorem 4.2 in [69] (c.f. the proof of Theorem 6.23 in [57] for a sketch of the argument in [69]). For regular $\lambda$ we may invoke Proposition 4.2 in [62], which says that every automorphic representation contributing to (76) is essentially tempered at infinity (mod center), whence by [88] the localized cohomology vanishes outside the cuspidal range.

For general weights $\lambda$ observe that since we are working with a product of general linear groups, the Künneth formula reduces us to a single copy of $\mathrm{GL}(n)$. In this situation, we remark that every irreducible constituent of (76) is strongly inner in the sense of [27], and therefore by the classification of the residual spectrum of [66] combined with the classification result of Jacquet-Shalika in [46, 47] implies that every constituent of (76) is in fact cuspidal. We refer the reader to section 4.3 in Harder-Raghuram [27] for the details of this argument and also to Proposition 4.1 in [17] for an argument along the same lines.

Remark 4.11. By [89], every automorphic representation contributing to (76) is in fact cuspidal.

Remark 4.12. The vanishing in degrees $<q_{0}$ is also implied by [62].
4.6. The Control Theorem. At every place $v \notin S$, the parabolic Hecke algebra $\mathcal{H}_{\mathcal{O}}^{B_{n+1}\left(F_{v}\right) \times B_{n}\left(F_{v}\right)}\left(\alpha^{\prime}, \alpha\right)$ at $v$ naturally acts on the cohomology

$$
\begin{equation*}
H^{\bullet}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha} \cap G^{\operatorname{der}}\left(\mathbf{A}^{(\infty)}\right)\right) ; \underline{L}_{\lambda, A}\right) \tag{77}
\end{equation*}
$$

The canonical inclusion

$$
\mathcal{H}_{\mathcal{O}}\left(\operatorname{GL}_{n+1}\left(\mathcal{O}_{v}\right) \times \mathrm{GL}_{n}\left(\mathcal{O}_{v}\right), \mathrm{GL}_{n+1}\left(F_{v}\right) \times \mathrm{GL}_{n}\left(F_{v}\right)\right) \rightarrow \mathcal{H}_{\mathcal{O}}^{B_{n+1}\left(F_{v}\right) \times B_{n}\left(F_{v}\right)}\left(\alpha^{\prime}, \alpha\right)
$$

of the spherical Hecke algebra at $v$ of $G$ into the parabolic Hecke algebra induces a canonical action of the spherical Hecke algebra on (77). Likewise, $\mathcal{H}_{\mathcal{O}}\left(\alpha^{\prime}, \alpha\right)$ acts on (77) as well. This remains valid for the more general coefficient systems considered below.

We call a character $\eta: T\left(\mathbf{Q}_{p}\right) \rightarrow E^{\times}$locally algebraic if there is an $E$-rational algebraic character $\lambda: T \rightarrow \mathrm{GL}_{1}$ and a finite order character $\vartheta: T\left(\mathbf{Q}_{p}\right) \rightarrow E^{\times}$with

$$
\eta=\lambda^{w_{0}} \vartheta
$$

Furthermore, $\eta$ is dominant if $\lambda$ is a dominant $E$-rational character of $T$. Write $P_{\lambda^{w_{0}} \vartheta} \subseteq$ $\Lambda=\mathcal{O}\left[\left[T\left(\mathbf{Z}_{p}\right)\right]\right]$ for the kernel of the algebra homomorphism $\lambda^{w_{0}} \vartheta: \Lambda \rightarrow E$ induced by $\lambda^{w_{0}} \vartheta$.

Proposition 4.13. The set

$$
\mathscr{X}_{\text {reg,bal }}^{0}:=\left\{P_{\lambda^{w_{0}}} \mid \lambda \in X_{\mathbf{C}}(T) \text { regular dominant, } \eta_{0} \text { admissible for } \lambda\right\}
$$

of regular dominant arithmetic points for which $\eta_{0}$ is admissible is Zariski dense in Spec $E\left[\left[T\left(\mathbf{Z}_{p}\right)\right]\right]$.
Proof. Write $\lambda=\lambda_{n+1} \otimes \lambda_{n}$ with regular dominant weights $\lambda_{n+1}$ and $\lambda_{n}$ of $\operatorname{res}_{F / \mathbf{Q}} T_{n+1}$ and $\operatorname{res}_{F / \mathbf{Q}} T_{n}$, where $\lambda_{m}=\left(\lambda_{m, \tau, i}\right)_{\tau: F \rightarrow \mathbf{C}, 1 \leq i \leq m}$ with $\lambda_{m, \tau, i} \in \mathbf{Z}$ and

$$
\lambda_{m, \tau, 1} \geq \lambda_{m, \tau, 2} \geq \cdots \geq \lambda_{m, \tau, m}
$$

regularity meaning that these inequalities are all strict. Then $\eta_{0}$ is admissible for $\lambda$ if and only if

$$
\lambda_{n+1, \tau, 1} \geq-\lambda_{n, \tau, n} \geq \lambda_{n+1, \tau, 2} \geq \cdots \geq-\lambda_{n, \tau, 1} \geq \lambda_{n+1, \tau, n+1}
$$

for all embeddings $\tau: F \rightarrow \mathbf{C}$. Therefore, considering $T$ as a maximal torus in $\operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}_{(n+1) n}$, the semigroup of regular dominant $\lambda$ for which $\eta_{0}$ is admissible is in canonical bijection with a subset of weights of $T$ which are dominant for a suitable choice of Borel in $\operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}_{(n+1) n}$. In fact, the subset of weights we obtain contains all regular dominant weights of this larger general group. The prime ideals corresponding to the latter set are visibly Zariski dense in Spec $E\left[\left[T\left(\mathbf{Z}_{p}\right)\right]\right]$.

For any $\Lambda$-module $\mathcal{M}$, set

$$
\mathcal{M}[\eta]:=\left\{m \in \mathcal{M} \mid \forall x \in T\left(\mathbf{Z}_{p}\right): x \cdot m=\eta(x) \cdot m\right\}
$$

Theorem 4.14. Let $F / \mathbf{Q}$ be an arbitrary number field. Assume $p \nmid(n+1) n$. Then for any E-valued locally algebraic character $\eta=\lambda^{w_{0}} \vartheta$ of $T\left(\mathbf{Q}_{p}\right)$ such that $\vartheta$ factors over $T\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right), \alpha \geq \alpha^{\prime} \geq 0, \alpha \geq \alpha_{0}^{K}$, and $\lambda$ regular dominant, the canonical map

$$
H_{\mathrm{ord}}^{q_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right)[\vartheta] \rightarrow \mathcal{H}_{\text {ord }}^{q_{0}}\left(K_{\infty, \infty} ; E / \mathcal{O}\right)\left[\lambda^{w_{0}} \vartheta\right]
$$

has finite kernel and finite cokernel.
Assume $F / \mathbf{Q}$ is totally real, CM or that Conjecture 4.9 applies over $F$. Let $\mathfrak{m}$ denote a non-Eisenstein maximal ideal in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$. Assume that the residue field $k$ of $\mathfrak{m}$ embeds into the residue field of $E$. Then for every E-rational dominant weight $\lambda$ of $G$, the canonical map

$$
H_{\mathrm{c}, \text { ord }}^{q_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right)_{\mathfrak{m}}[\vartheta] \rightarrow \mathcal{H}_{\mathrm{c}, \text { ord }}^{q_{0}}\left(K_{\infty, \infty} ; E / \mathcal{O}\right)_{\mathfrak{m}}\left[\lambda^{w_{0}} \vartheta\right]
$$

has finite kernel and finite cokernel.
Proof. Given the general vanishing results of Li-Schwermer in [62] (for the first statement) and the vanishing statement in Theorem 4.10 (for the second statement), the proof proceeds along the lines of Hida's proof of Theorem 6.2 in $[34,36]$, adapted to split GL $(n)$, and working with $G$ instead of a single copy of the general linear group.

To be more specific, recall that $U \subseteq G$ denotes the unipotent radical of the standard upper triangular Borel. Put

$$
I_{0, \alpha}^{\circ}:=I_{0, \alpha} \cap G^{\operatorname{der}}\left(\mathbf{Z}_{p}\right)
$$

and furnish

$$
Y_{\alpha}:=I_{0, \alpha}^{\circ} / U\left(\mathbf{Z}_{p}\right)
$$

with the right action of the semigroup

$$
I_{\alpha^{\prime}, \alpha} \Delta_{G} I_{\alpha^{\prime}, \alpha}=I_{\alpha^{\prime}, \alpha}^{\circ} \Delta_{G} I_{\alpha^{\prime}, \alpha}^{\circ}
$$

defined by Hida (cf. section 3 in [34] and p. 682 of [36], taking into account that Hida considers right actions of Hecke operators where we consider left actions, which results in the opposite dominance condition in loc. cit.).

Following Hida, define for $A \in\{\mathcal{O}, E, E / \mathcal{O}\}$,

$$
\mathcal{C}_{\alpha}(A):=\left\{\phi: Y_{\alpha} \rightarrow A \mid \phi \text { continuous }\right\}=\operatorname{ind}_{\left(B \cap G^{\mathrm{der}}\right)\left(\mathbf{Z}_{p}\right)}^{I_{0, \alpha} \cap G^{\mathrm{der}}\left(\mathbf{Z}_{p}\right)} A
$$

which carries an action of $I_{\alpha^{\prime}, \alpha} \Delta_{G} I_{\alpha^{\prime}, \alpha}$ by right translation.
Hida modified the action of the semigroup $I_{\alpha, \alpha} \Delta_{G} I_{\alpha, \alpha}$ on $L_{\lambda, E / \mathcal{O}}$ by twisting the action of $T\left(\mathbf{Z}_{p}\right)$ by the character $\vartheta$ (cf. page 684 in [36]). The resulting representation is denoted $L_{\lambda \otimes \vartheta, E / \mathcal{O}}$. Put

$$
I_{\alpha^{\prime}, \alpha}^{\circ}:=I_{\alpha^{\prime}, \alpha} \cap G^{\operatorname{der}}\left(\mathcal{O}_{p}\right)
$$

and

$$
K_{\bullet, \bullet}^{\circ}:=K_{\bullet, \bullet} \cap G^{\operatorname{der}}\left(\mathbf{A}^{(\infty)}\right)
$$

We observe first that Proposition 4.1 and Theorem 4.1 in [34] are valid for GL( $n$ ) both for cohomology with and without compact supports (without the need of localization). These results also extend to $G$ without modification in both cases. Hence Proposition 5.1 of [34] remains valid for $G$ in both cases as well.

Given this, Theorem 5.1 of [34] and its proof remain valid without modification for nearly ordinary cohomology of $G$ with and without compact supports. This shows that there is a canonical isomorphism of $I_{\alpha^{\prime}, \alpha}^{\circ} \Delta_{G} I_{\alpha^{\prime}, \alpha}^{\circ} \times G\left(\mathbf{A}^{S}\right)$-modules

$$
\begin{equation*}
\iota_{\lambda}: \quad \mathcal{H}_{?, \text { ord }}^{q}\left(\mathscr{X}\left(K_{\infty, \infty}^{\circ}\right) ; \underline{L}_{\lambda \otimes \vartheta, E / \mathcal{O}}\right) \cong H_{?, \text { ord }}^{q}\left(\mathscr{X}\left(K_{0,1}^{\circ}\right) ; \mathscr{\mathscr { C }}_{1}(E / \mathcal{O})\right) \tag{78}
\end{equation*}
$$

for all degrees $q \in \mathbf{Z}$, satisfying

$$
\iota_{\lambda}\left(\lambda^{w_{0}}(t)\langle t\rangle(-)\right)=\langle t\rangle \iota_{\lambda}(-)
$$

for any $t \in T\left(\mathbf{Z}_{p}\right)$ and $? \in\{-, \mathrm{c}\}$ (cf. also (6.8) in [36]). This isomorphism is compatible with the isomorphisms constructed in Theorem 4.3.

By [62], regularity of $\lambda$ implies vanishing

$$
H^{q}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}^{\circ}\right) ; \underline{L}_{\lambda \otimes \vartheta, \mathbf{C}}\right)=0, \quad q<q_{0}
$$

for all $\alpha \geq \alpha^{\prime} \geq 0, \alpha \geq \alpha_{0}^{K}$. In the case of compactly supported cohomology localized at the non-Eisenstein maximal ideal $\mathfrak{m}$ we appeal to Theorem 4.10 to deduce the corresponding vanishing statement in degrees $q<q_{0}$ for general dominant $\lambda$.

Therefore, Theorem 5.2 and its proof show with with Lemma 5.1 of [34] that for $\alpha \geq \alpha^{\prime}$, $\alpha \geq \alpha_{0}^{K}$ such that $\vartheta$ factors over $T\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right)$, (78) induces an canonical map

$$
\iota_{\lambda}^{\vartheta}: \quad H_{\mathrm{ord}}^{q_{0}}\left(K_{\alpha^{\prime}, \alpha}^{\circ} ; L_{\lambda \otimes \vartheta, E / \mathcal{O}}\right) \rightarrow H_{\mathrm{ord}}^{q_{0}}\left(\mathscr{X}\left(K_{0,1}^{\circ}\right) ; \mathscr{C}_{1}(E / \mathcal{O})\right)\left[\lambda^{w_{0}} \vartheta\right]
$$

with finite kernel and finite cokernel provided $\lambda$ is regular, see also (6.9) in [36].
Again along the same lines, the canonical map

$$
\iota_{\lambda}^{\vartheta}: \quad H_{\mathrm{c}, \text { ord }}^{q_{0}}\left(K_{\alpha^{\prime}, \alpha}^{\circ} ; L_{\lambda \otimes \vartheta, E / \mathcal{O}}\right)_{\mathfrak{m}} \rightarrow H_{\mathrm{c}, \text { ord }}^{q_{0}}\left(\mathscr{X}\left(K_{0,1}^{\circ}\right) ; \mathscr{\mathscr { C }}_{1}(E / \mathcal{O})\right)_{\mathfrak{m}}\left[\lambda^{w_{0}} \vartheta\right]
$$

has finite kernel and finite cokernel for general dominant $\lambda$.
So far all arguments are valid without any assumption on $p$.
The rest of the argument then proceeds as the proof of Theorem 6.1 in section 6.3 of [36] without modifications.

Corollary 4.15. Under the hypotheses of Theorem 4.14, consider the algebra homomorphism

$$
(\lambda \vartheta: \quad \Lambda \rightarrow \mathcal{O}) \in \operatorname{Spec}(\Lambda)
$$

If $1+p^{\alpha^{\prime}} \mathcal{O}$ lies in the kernel of $\vartheta$, there is a canonical map

$$
\mathcal{H}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right) \otimes_{\Lambda, \lambda \vartheta} \mathcal{O} \rightarrow H_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}^{\circ}\right) \otimes_{\mathcal{O}\left[T\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{z}_{p}\right)\right], \vartheta} \mathcal{O}
$$

with finite kernel and cokernel whenever $\lambda$ is regular dominant. Pulling back $\lambda \vartheta$ to $\Lambda^{\circ}$, we obtain for general (not necessarily regular) dominant $\lambda$, a canonical map

$$
\mathcal{H}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}} \otimes_{\Lambda^{\circ}, \lambda \vartheta} \mathcal{O} \rightarrow H_{\text {ord }}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}^{\circ}\right)_{\mathfrak{m}} \otimes_{\Lambda^{\circ}, \vartheta} \mathcal{O}
$$

with finite kernel and cokernel.
Proof. It suffices to observe that by Poincaré-Pontryagin duality, the Pontryagin dual of $\mathcal{H}_{\text {ord }}^{q_{0}}\left(K_{\infty, \infty} ; E / \mathcal{O}\right)$ is $\mathcal{H}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)$, and similarly in the second case (the condition of being non-Eisenstein is stable under taking contragredients).

Corollary 4.16. Write $P_{\lambda \vartheta}$ for the kernel of $\lambda \vartheta \in \operatorname{Spec}(\Lambda)$ and $P_{\vartheta}$ for that of $\vartheta \in$ $\operatorname{Spec}\left(\mathcal{O}\left[T\left(\mathbf{Z}_{p} / p^{\alpha^{\prime}} \mathbf{Z}_{p}\right)\right]\right)$. Then we have a canonical isogeny

$$
\mathbf{h}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right) / P_{\lambda \vartheta} \mathbf{h}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right) \rightarrow \mathrm{h}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right) / P_{\vartheta} \mathrm{h}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)
$$

for regular dominant $\lambda$ and pulling back $\lambda \vartheta$ to $\Lambda^{\circ}$, we have a canonical isogeny
$\mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}} / P_{\lambda \vartheta}^{\circ} \mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}} \rightarrow \mathrm{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)_{\mathfrak{m}} / P_{\vartheta}^{\circ} \mathrm{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)_{\mathfrak{m}}$ with $P_{\lambda \vartheta}^{\circ}=P_{\lambda \vartheta} \cap \Lambda_{0}$ (likewise for $P_{\vartheta}^{\circ}$ ) for general dominant $\lambda$.

Corollary 4.17. The universal nearly ordinary Hecke algebra $\mathbf{h}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ is finite over $\Lambda$. Likewise, $\mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}}$ is finite over $\Lambda^{\circ}$.
Corollary 4.18. Assume $n>2$ or $F$ not totally real and $p \nmid(n+1) n$. Then $\mathcal{H}_{\text {ord }}^{q_{0}}\left(K_{\infty, \infty} ; E / \mathcal{O}\right)$ is a cotorsion $\Lambda$-module, i.e. its Pontryagin dual $\mathcal{H}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ is a torsion $\Lambda$-module. The same applies in the second case.

Proof. The claim follows from the existence of non-arithmetic regular dominant weights $\lambda$ as observed on p. 690 in [36].

The Krull dimension of $\mathbf{h}^{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ is closely related to the Leopoldt Conjecture, cf. Conjecture 1.1 in [36], see also [57].

The Hecke module $\mathcal{H}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ is always a faithful $\mathbf{h}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)$-module by definition. The freeness of universal nearly ordinary cohomology over the universal Hecke algebra turns out to be related to the Leopoldt Conjecture, cf. Theorem 4.9 in [25].

In fact, Theorems 4.10 and 4.14 suggest that the same should apply to $\mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}}$.

## 5. Cohomological construction of $p$-Adic measures

We begin by recalling the construction of abelian $p$-adic $L$-functions from [50, 51, 52].
5.1. The modular symbol. Following the formalism from Sections 5.1 and 6.4 in [52] we define for any finite adèle $g \in G\left(\mathbf{A}^{(\infty)}\right)$ and any $\mathcal{O}$-submodule $L_{\mathcal{O}}$ of $L_{\lambda, E}$ the translated lattice

$$
g L_{\mathcal{O}}:=L_{\lambda, E} \cap g \cdot\left(L_{\mathcal{O}} \otimes_{\mathbf{Z}} \widehat{\mathbf{Z}}\right)
$$

 $\mathscr{X}(K)$ and as in loc. cit. we have a canonical morphism

$$
T_{g}: \quad t_{g}^{*} \underline{L}_{\mathcal{O}} \rightarrow \underline{g L_{\mathcal{O}}}
$$

of sheaves on $\mathscr{X}\left(g K g^{-1}\right)$. This morphism allows us to define a normalized pull back operator $t_{g}^{\lambda}$, sending sections of the sheaf $\underline{L}_{\mathcal{O}}$ over an open $U \subseteq \mathscr{X}(K)$ to sections of $\underline{g L_{\mathcal{O}}}$ over $U g^{-1} \subseteq \mathscr{X}\left(g K g^{-1}\right)$.

We remark that for $g_{1}, g_{2} \in G\left(\mathbf{A}^{(\infty)}\right)$ there is an identity $\left(g_{1} g_{2}\right) L_{\mathcal{O}}=g_{1}\left(g_{2} L_{\mathcal{O}}\right)$, and the construction of the translated lattice $g L_{\mathcal{O}}$ is functorial in $L_{\mathcal{O}}$. Furthermore, the translated lattice always comes with a canonical map $g L_{\mathcal{O}} \rightarrow L_{\lambda, E}$ and we have $g L_{\lambda, E}=L_{\lambda, E}$ for all $g \in G\left(\mathbf{A}^{(\infty)}\right)$.

As before, let $\eta_{j}: L_{\lambda, E} \rightarrow E_{(j)}$ denote a non-zero $H$-equivariant functional. We fix once and for all an isomorphism $E_{(j)} \cong E$, which induces isomorphisms $A_{(j)} \cong A$. This allows us to identify $A_{(j)}$ and $A$ in the sequel. From this identification and the non-vanishing statement (62) in Proposition 3.4, we deduce by restriction a $p$-adically normalized $p$ adically optimal functional

$$
\eta_{j, A}: \quad L_{\lambda, A}^{1,0} \rightarrow A_{(j)}
$$

for $A=\mathcal{O}$ and $A=E$, given by

$$
v \mapsto \frac{\eta_{j}(v)}{\eta_{j}\left(g_{0} v_{0}\right)}
$$

Then $\eta_{j, A}$ is independent of the choice of $\eta_{j}$ and also independent of the identification $A_{(j)}=A$.

The codomain of $\eta_{j, A}$ gives rise to a sheaf $\underline{A}_{(j)}$ on the locally symmetric space

$$
\mathscr{Y}(L):=H(\mathbf{Q}) \backslash H(\mathbf{A}) / L \cdot K_{\infty}^{\prime}{ }^{0}
$$

where $L \subseteq H\left(\mathbf{A}^{(\infty)}\right)$ is any compact open subgroup and

$$
K_{\infty}^{\prime}:=H(\mathbf{R}) \cap \widetilde{K}_{\infty}
$$

happens to be a standard maximal compact subgroup. The numerical coincidence we exploit is

$$
\operatorname{dim} \mathscr{Y}:=\operatorname{dim} \mathscr{Y}(L)=q_{0}
$$

which over $\mathbf{Q}$ was first observed in [56], and over general base fields $F$ in [50]. Then

$$
\operatorname{dim} \mathscr{X}-\operatorname{dim} \mathscr{Y}=q_{0}+l_{0}
$$

is the top degree for $G$.

By strong approximation for $\operatorname{SL}(n)$, the connected components of $\mathscr{Y}(L)$ are parametrized by elements in the class group

$$
C(L):=F^{\times} \backslash \mathbf{A}_{F}^{\times} / \operatorname{det}(L) F_{\infty}^{0}
$$

We write $\mathscr{Y}(L)[x]$ for the component mapping to $x \in C(\operatorname{det}(L))$ under the determinant.
We fix once and for all a system of fundamental classes as in [52, Section 5.3]. Then, for each $L$ neat and each $x \in C(L)$, we have compatible isomorphisms

$$
\int_{\mathscr{Y}(L)[x]}: \quad H_{\mathrm{c}}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{Y}(L)_{x} ; \underline{A}_{(j)}\right) \rightarrow A_{(j)}
$$

Whenever $L \subseteq K$, the inclusion $H \rightarrow G$ induces a proper map

$$
i: \quad \mathscr{Y}(L) \rightarrow \mathscr{X}(K) .
$$

We need the following generalization of Proposition 3.4 in [76].
Proposition 5.1. For any $\beta \geq \alpha>0$, the compact open subgroup

$$
\mathfrak{I}_{\beta}^{n}:=H\left(\mathbf{Q}_{p}\right) \cap g_{\beta} I_{\alpha} g_{\beta}^{-1}
$$

of $H\left(\mathbf{Z}_{p}\right)$ is independent of $\alpha$. It satisfies

$$
\left(H\left(\mathbf{Z}_{p}\right): \mathfrak{I}_{\beta}^{n}\right)=\prod_{v \mid p} \prod_{\mu=1}^{n}\left(1-q_{v}^{-\mu}\right)^{-1} \cdot p^{\beta \frac{(n+2)(n+1) n+(n+1) n(n-1)}{6}}
$$

and

$$
\operatorname{det} \mathfrak{I}_{\beta}^{n}=1+p^{\beta} \mathcal{O}_{p}
$$

Proof. Set

$$
I_{\alpha, p}^{n, 1}:=\left\{r \in \mathrm{GL}_{n}\left(\mathcal{O}_{p}\right) \mid r \quad\left(\bmod p^{\alpha}\right) \in U_{n}\left(\mathcal{O}_{p}\right)\right\}
$$

We denote the opposite group constructed with $U_{n}^{-}$by $I_{\alpha, p}^{n, 1-}$ and write $\sim$ for an identity of subgroups of $H\left(\mathbf{Z}_{p}\right)$ up to conjugation. Then

$$
\begin{aligned}
g_{\beta} I_{\alpha} g_{\beta}^{-1} \cap H\left(\mathbf{Q}_{p}\right) & =h_{n} t_{p}^{\beta} I_{\alpha, p}^{n+1,1} t_{p}^{-\beta} h_{n}^{-1} \cap t_{p}^{\beta} I_{\alpha, p}^{n, 1} t_{p}^{-\beta} \\
& \sim w_{n} h_{n} t_{p}^{\beta} I_{\alpha, p}^{n+1,1} t_{p}^{-\beta}\left(w_{n} h_{n}\right)^{-1} \cap w_{n} t_{p}^{\beta} I_{\alpha, p}^{n, 1}\left(w_{n} t\right)^{-\beta} \\
& =w_{n} h_{n} t_{p}^{\beta} I_{\alpha, p}^{n+1,1} t_{p}^{-\beta}\left(w_{n} h_{n}\right)^{-1} \cap w_{n} t_{p}^{\beta} w_{n} I_{\alpha, p}^{n, 1-}\left(w_{n} t w_{n}\right)^{-\beta} \\
& =w_{n} h_{n} t_{p}^{\beta} I_{\alpha, p}^{n+1,1} t_{p}^{-\beta}\left(w_{n} h_{n}\right)^{-1} \cap t_{p}^{-\beta} I_{\alpha, p}^{n, 1-} t_{p}^{\beta}
\end{aligned}
$$

Now

$$
w_{n} h_{n}=\left(\begin{array}{cccc} 
& & & 1 \\
& \mathbf{1}_{n} & & \vdots \\
& & & \vdots \\
0 & \ldots & 0 & 1
\end{array}\right)
$$

and

$$
\left(w_{n} h_{n}\right)^{-1}=\left(\begin{array}{cccc} 
& & & -1 \\
& \mathbf{1}_{n} & & \vdots \\
& & & -1 \\
0 & \ldots & 0 & 1
\end{array}\right)
$$

For $r \in I_{\alpha, p}^{n, 1}$ we have

$$
\left(w_{n} h_{n} t_{p}^{\beta} r t_{p}^{-\beta}\left(w_{n} h_{n}\right)^{-1}\right)_{i j}=
$$

$$
\begin{cases}p^{\beta(j-i)} r_{i j}+p^{\beta(n+1-i)} r_{i n+1} & 1 \leq i, j \leq n \\ p^{\beta(j-(n+1))} r_{n+1 j} & i=n+1,1 \leq j \leq n \\ p^{\beta(n+1-i)} r_{i n+1}+r_{n+1 n+1}-\sum_{j=1}^{n}\left(p^{\beta(j-(n+1))} r_{n+1 j}+p^{\beta(j-i)} r_{i j}\right) & 1 \leq i \leq n, j=n+1 \\ r_{n+1 n+1}-\sum_{j=1}^{n} p^{\beta(j-(n+1))} r_{n+1 j} & i=j=n+1\end{cases}
$$

The condition that this be an element of $H\left(\mathbf{Q}_{p}\right)$ is equivalent to the conditions

$$
\begin{align*}
r_{n+1 j} & =0, & & 1 \leq j \leq n, \\
r_{n+1 n+1} & =1, & & \\
p^{\beta(n+1-i)} r_{i n+1}+1-\sum_{j=1}^{n} p^{\beta(j-i)} r_{i j} & =0, & & 1 \leq i \leq n . \tag{79}
\end{align*}
$$

Therefore, $w_{n} h_{n} t_{p}^{\beta} r t_{p}^{-\beta}\left(w_{n} h_{n}\right)^{-1}$ lies in $t_{p}^{-\beta} I_{\alpha, p}^{n, 1-} t_{p}^{\beta}$ if and only if (79) is satisfied and

$$
\begin{array}{ll}
r_{i j}+p^{\beta(n+1-j)} r_{i n+1} \in p^{\alpha+\beta(2 i-2 j)} \mathcal{O}, & 1 \leq i<j \leq n, \\
r_{i i}+p^{\beta(n+1-i)} r_{i n+1} \in 1+p^{\alpha} \mathcal{O}, & 1 \leq i \leq n \\
r_{i j}+p^{\beta(n+1-j)} r_{i n+1} \in p^{\beta(2 i-2 j)} \mathcal{O}, & 1 \leq j<i \leq n \tag{82}
\end{array}
$$

Conditions (80) and (81) are automatic because $\beta \geq \alpha$.
Condition (79) is equivalent to

$$
r_{i i}=1+\sum_{j=1}^{i-1} p^{\beta(j-i)} r_{i j}+\sum_{j=i+1}^{n} p^{\beta(j-i)} r_{i j}+p^{\beta(n+1-i)} r_{i n+1}
$$

which in turn is equivalent to

$$
\begin{equation*}
r_{i i}=1+\sum_{j=1}^{i-1} p^{\beta(j-i)}\left(r_{i j}+p^{\beta(n+1-j)} r_{i n+1}\right)+\sum_{j=i+1}^{n} p^{\beta(j-i)} r_{i j}-(i-2) \cdot p^{\beta(n+1-i)} r_{i n+1} \tag{83}
\end{equation*}
$$

for $1 \leq i \leq n$. The last two summands in (83) lie in $p^{\beta} \mathcal{O}$, and by (82), the summands of the frist sum on the right hand side lies in $p^{\beta(i-j)} \mathcal{O} \subseteq p^{\beta} \mathcal{O}$. This readily implies

$$
\begin{equation*}
r_{i i} \in 1+p^{\beta} \mathcal{O}, \quad 1 \leq i \leq n \tag{84}
\end{equation*}
$$

Reversing this argument shows that the diagonal $\left(r_{i i}\right)_{1 \leq i \leq n}$ may assume any value in $\left(1+p^{\beta} \mathcal{O}\right)^{n}$. With our previous computation, this shows

$$
\begin{aligned}
\operatorname{det}\left(w_{n} h_{n} t_{p}^{\beta} r t_{p}^{-\beta}\left(w_{n} h_{n}\right)^{-1}\right)_{1 \leq i, j \leq n} & =\operatorname{det}\left(r_{i j}+p^{\beta(n+1-j)} r_{i n+1}\right)_{1 \leq i, j \leq n} \\
& \equiv \prod_{i=1}^{n} r_{i i} \quad\left(\bmod p^{\beta} \mathcal{O}\right) \\
& \equiv 1 \quad\left(\bmod p^{\beta} \mathcal{O}\right),
\end{aligned}
$$

and the determinant maps $\mathfrak{I}_{\beta}^{n}$ surjectively onto $1+p^{\beta} \mathcal{O}$ as claimed.
The same computation shows that

$$
w_{n} h_{n} t_{p}^{\beta} I_{\alpha, p}^{n+1,1} t_{p}^{-\beta}\left(w_{n} h_{n}\right)^{-1} \cap t_{p}^{-\beta} I_{\alpha, p}^{n, 1-} t_{p}^{\beta}=I_{\beta, p}^{n} \cap t_{p}^{-\beta} I_{\beta, p}^{n,-} t_{p}^{\beta}
$$

which consists of matrices $s \in \mathrm{GL}_{n}(\mathcal{O})$ with entries

$$
s_{i j} \in \begin{cases}p^{\beta(j-i)} \mathcal{O}, & i<j \\ 1+p^{\beta} \mathcal{O}, & i=j \\ p^{\beta(i-j)} \mathcal{O}, & i>j\end{cases}
$$

With this explicit description of the intersection the computation of the index is straightforward.

Put

$$
C\left(p^{\beta}\right):=F^{\times} \backslash \mathbf{A}_{F}^{\times} /\left(1+p^{\beta} \mathcal{O}\right) \cdot \operatorname{det}\left(K^{(p)} \cap H\left(\mathbf{A}^{(p \infty)}\right)\right)=C\left(g_{\beta} K_{\alpha^{\prime}, \alpha} g_{\beta}^{-1} \cap H\left(\mathbf{A}^{(\infty)}\right)\right)
$$

where the second identity is a consequence of Proposition 5.1.
For any $\beta>0$ and $x \in C\left(p^{\beta}\right)$ we consider the modular symbol

$$
\mathscr{P}_{A, x, \beta}^{\lambda, j}: \quad H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L_{\lambda, A}}\right) \rightarrow A_{(j)}
$$

explicitly defined as

$$
\phi \mapsto \int_{\mathscr{Y}\left(g_{\beta} K_{\alpha^{\prime}, \alpha} g_{\beta}^{-1} \cap H\left(\mathbf{A}^{(\infty)}\right)\right)[x]} \eta_{j, A} i^{*}\left[\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\beta}\right) \cdot t_{g_{\beta}}^{\lambda}\right]\left(U_{p}^{-\beta} \phi\right)
$$

By (60), we know that $L_{\lambda, A}^{x, \beta} \subseteq L_{\lambda, A}^{1,0}$ and hence

$$
\begin{equation*}
\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\beta}\right) \cdot t_{g_{\beta}}^{\lambda}\left(U_{p}^{-\beta} \phi\right) \in L_{\lambda, A}^{x, \beta} \subseteq L_{\lambda, A}^{1,0} \tag{85}
\end{equation*}
$$

whence $\mathscr{P}_{A, x, \beta}^{\lambda, j}$ is indeed well defined.
Then for any cohomology class $\phi$ as above, we obtain an element

$$
\mu_{A, \beta}^{\lambda, j}(\phi):=\sum_{x \in C\left(p^{\beta}\right)} \mathscr{P}_{A, x, \beta}^{\lambda, j}(\phi) \cdot x \in A_{(j)}\left[C\left(p^{\beta}\right)\right]:=A_{(j)} \otimes \mathcal{O} \mathcal{O}\left[C\left(p^{\beta}\right)\right] .
$$

Here the right hand side denotes the tensor product of $A_{(j)}$ with the group ring of the finite ray class group $C\left(p^{\beta}\right)$ over $A$.
5.2. The distribution relation. To establish the distribution relation for $\mu_{A, \beta}^{\lambda, j}$ we follow the argument in section 6.6 of [51].

The following generalizes Lemma 6.5 in [51] and Lemma 6.1 in [52].
Lemma 5.2. Let $u \in U\left(\mathcal{O}_{p}\right)$. Then for every $\beta>0$ :
(i) There exists $k_{u}=\left(k_{u}^{\prime}, k_{u}^{\prime \prime}\right) \in I_{\alpha, \alpha}$ satisfying

$$
\begin{equation*}
h t_{p}^{\beta} \cdot u t_{p}=\Delta\left(k_{u}^{\prime \prime}\right)^{-1} \cdot h t_{p}^{\beta+1} \cdot k_{u} \tag{86}
\end{equation*}
$$

(ii) For every $k_{u}=\left(k_{u}^{\prime}, k_{u}^{\prime \prime}\right) \in I_{\alpha, \alpha}$ satisfying (86) the residue class of the determinant

$$
\operatorname{det} k_{u}^{\prime} \equiv \operatorname{det} k_{u}^{\prime \prime} \quad\left(\bmod p^{\beta+1}\right)
$$

is uniquely determined by $u \in U\left(\mathcal{O}_{p}\right) / t_{p} U\left(\mathcal{O}_{p}\right) t_{p}^{-1}$ and lies in $1+p^{\beta} \mathcal{O}_{p}$.
(iii) The map

$$
\begin{aligned}
U\left(\mathcal{O}_{p}\right) / t_{p} U\left(\mathcal{O}_{p}\right) t_{p}^{-1} & \rightarrow\left(1+p^{\beta} \mathcal{O}_{p}\right) /\left(1+p^{\beta+1} \mathcal{O}_{p}\right) \\
u & \mapsto \operatorname{det} k_{u}^{\prime}
\end{aligned}
$$

is a surjective group homomorphism.
Proof. The proof proceeds as the proof of Lemma 6.5 in [51] with the following additional observations:

$$
\begin{aligned}
h t_{p}^{\beta} \cdot u t_{p} & =\left(h_{n} j_{n}\left(t_{-1}\right), \mathbf{1}_{n}\right) \cdot t_{p}^{\beta} u t_{p} \\
& =\left(h_{n}, \mathbf{1}_{n}\right) \cdot t_{p}^{\beta}\left(j_{n}\left(t_{-1}\right), \mathbf{1}_{n}\right) u t_{p} \\
& =\left(h_{n}, \mathbf{1}_{n}\right) \cdot t_{p}^{\beta} u^{\left(j_{n}\left(t_{-1}\right), \mathbf{1}_{n}\right)} t_{p} \cdot\left(j_{n}\left(t_{-1}\right), \mathbf{1}_{n}\right),
\end{aligned}
$$

and likewise

$$
h t_{p}^{\beta+1}=\left(h_{n}, \mathbf{1}_{n}\right) \cdot t_{p}^{\beta+1} \cdot\left(j_{n}\left(t_{-1}\right), \mathbf{1}_{n}\right)
$$

Therefore, the statement reduces to the same statement with $\left(h_{n}, \mathbf{1}_{n}\right)$ replacing $h$, which is treated in loc. cit. for the compact open $I_{0, \alpha}$. Hence it suffices to remark that the elements $k_{u, w}$ and $k_{u, w}^{\prime}$ constructed in said proof lie in $I_{\alpha, \alpha}^{n+1}$ and $I_{\alpha, \alpha}^{n}$ respectively.

For any $\beta \geq \beta^{\prime}>0$ the a canonical projection

$$
C\left(p^{\beta}\right) \rightarrow C\left(p^{\beta^{\prime}}\right)
$$

induces an $\mathcal{O}$-linear epimorphism

$$
\operatorname{res}_{\beta^{\prime}}^{\beta}: \quad A_{(j)}\left[C\left(p^{\beta}\right)\right] \rightarrow A_{(j)}\left[C\left(p^{\beta^{\prime}}\right)\right] .
$$

Proposition 5.3. For any cohomology class $\phi$ and any $\beta \geq \beta^{\prime}>0$ we have the distribution relation

$$
\operatorname{res}_{\beta}^{\beta^{\prime}}\left(\mu_{A, \beta^{\prime}}^{\lambda, j}(\phi)\right)=\mu_{A, \beta}^{\lambda, j}(\phi) .
$$

Proof. It suffices to treat the case $\beta=\beta^{\prime}+1$. In this case, the proof proceeds as the proof of Theorem 6.1 in [52], taking into account that equation (18) in loc. cit. remains valid by Proposition 5.1, and Lemma 5.2 and replaces Lemma 6.1 there.

By Proposition 5.3, we have a projective system $\left(\mu_{A, \beta}^{\lambda, j}(\phi)\right)_{\beta}$. Put

$$
C_{F}\left(p^{\infty}\right):={\underset{\beta}{\beta}}_{\lim _{F}} C_{F}\left(p^{\beta}\right)
$$

and

$$
\mu_{A}^{\lambda, j}(\phi):={\underset{\check{\beta}}{\beta}}^{\lim _{A, \beta}} \mu^{\lambda, j}(\phi) .
$$

Thus we obtain an $\mathcal{O}$-linear map

$$
\mu_{A}^{\lambda, j}: \quad H_{\mathrm{c}, \mathrm{ord}}^{\operatorname{dim} \mathscr{y}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, A}\right) \rightarrow A_{(j)}\left[\left[C_{F}\left(p^{\infty}\right)\right]\right] .
$$

5.3. $p$-adic character varieties. Recall that

We have

$$
C_{F}\left(p^{\infty}\right)=\Delta \times \mathbf{Z}_{p}^{r_{F}},
$$

for a finite group $\Delta$ and an integers $r_{F}>0$.
The norm map $N_{F / \mathbf{Q}}: F \rightarrow \mathbf{Q}$ induces a morphism

$$
N_{F / \mathbf{Q}}: \quad C_{F}\left(p^{\infty}\right) \rightarrow C_{\mathbf{Q}}\left(p^{\infty}\right)=\mathbf{Z}_{p}^{\times}
$$

with image of finite index. The decomposition

$$
\begin{equation*}
C_{\mathbf{Q}}\left(p^{\infty}\right)=\mu_{p-1} \times\left(1+p \mathbf{Z}_{p}\right) \tag{87}
\end{equation*}
$$

gives rise to two distinguished characters of $C_{F}\left(p^{\infty}\right)$ as follows. Write $\pi_{i}, i \in\{1,2\}$, for the projection onto the $i$-th factor of the right hand side in (87). Set

$$
\omega_{F}:=\pi_{1} \circ N_{F / \mathbf{Q}}: \quad C_{F}\left(p^{\infty}\right) \rightarrow \mathbf{Q}\left[\mu_{p-1}\right],
$$

where $\mu_{p-1}$ denotes the group of ( $p-1$ )-st roots of unity and the group $\mu_{2}=\{ \pm 1\}$ for $p=2$. Put

$$
\langle-\rangle_{F}:=\pi_{2} \circ N_{F / \mathbf{Q}}: \quad C_{F}\left(p^{\infty}\right) \rightarrow \mathbf{C}_{p}^{\times},
$$

where $\mathbf{C}_{p}$ denotes the completion of an algebraic closure of $\mathbf{Q}_{p}$. Here we implicitly embedded $1+p \mathbf{Z}_{p}$ into $\mathbf{C}_{p}^{\times}$canonically.

Let

$$
\mathscr{X}_{F}:=\operatorname{Hom}_{\mathrm{cts}}\left(C_{F}\left(p^{\infty}\right), \mathbf{C}_{p}^{\times}\right)
$$

denote the rigid analytic variety of continuous $p$-adic characters of $C_{F}\left(p^{\infty}\right)$.
The space $\mathscr{X}_{F}$ is an equidimensional rigid analytic variety of dimension $r_{F}$ with $\# \Delta$ irreducible components:

$$
\mathscr{X}_{F}=\bigsqcup_{\chi_{0} \in \widehat{\Delta}} \chi_{0} \cdot\left(1+\mathfrak{m}_{\mathbf{C}_{p}}\right)^{r_{F}}
$$

Inside this space, we have the Zariski dense set

$$
\mathscr{X}_{F}^{0}:=\left\{\chi \in \mathscr{X}_{F} \mid \chi \text { of finite order }\right\}
$$

of finite order characters.
Since $r_{\mathbf{Q}}=1$, there is canonical projection

$$
C_{\mathbf{Q}}\left(p^{\infty}\right)=\Delta \times \mathbf{Z}_{p} \rightarrow \mathbf{Z}_{p}:=C_{\mathbf{Q}}^{\mathrm{cyc}}\left(p^{\infty}\right)
$$

Correspondingly, the cyclotomic line is defined as

$$
\mathscr{X}_{\mathbf{Q}}^{\mathrm{cyc}}:=\operatorname{Hom}_{\mathrm{cts}}\left(C_{\mathbf{Q}}^{\mathrm{cyc}}\left(p^{\infty}\right), \mathbf{C}_{p}^{\times}\right) \subseteq \mathscr{X}_{\mathbf{Q}}
$$

Finite order characters of $C_{\mathbf{Q}}^{\text {cyc }}\left(p^{\infty}\right)$ are precisely the Dirichlet characters of $p$-power order which are unramified outside $p \infty$. These are Zariski dense in $\mathscr{X}_{\mathbf{Q}}^{\text {cyc }}$.

The norm induces a commutative diagram


We call the image

$$
\mathscr{X}_{F}^{\mathrm{cyc}}:=N_{F / \mathbf{Q}}^{*}\left(\mathscr{X}_{\mathbf{Q}}^{\mathrm{cyc}}\right) \subseteq \mathscr{X}_{F}
$$

the cyclotomic line over $F$. It has a canonical (topological) generator $\langle\cdot\rangle_{F}$, the cyclotomic character of $F$. All other characters in the cyclotomic line are of the form $\langle\cdot\rangle_{F}^{S}$ for some $s \in \mathbf{Z}_{p}$.

Remark that the set

$$
N_{F / \mathbf{Q}}^{*}\left(\mathscr{X}_{\mathbf{Q}}^{\mathrm{cyc}, 0}\right) \subseteq \mathscr{X}_{F}^{0}
$$

of norm-inflated Dirichlet characters of $p$-power order is Zariski dense in $\mathscr{X}_{F}^{c y c}$.
5.4. $p$-adic Tate twists. Recall that we identified $A_{(j)}$ and $A$. This allows us to identify the modules $A_{(j)}$ for varying $j$.
Theorem 5.4. Assume that two non-zero $H$-linear functionals

$$
\eta_{j_{i}}: \quad L_{\lambda, E} \rightarrow E_{\left(j_{i}\right)}, \quad i \in\{1,2\},
$$

are given. Then we have for every

$$
\phi \in H_{\mathrm{c}, \mathrm{ord}}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}\right)
$$

an identity of measures

$$
\omega_{F}^{j_{2}}(x)\langle x\rangle_{F}^{j_{2}} \mu_{\mathcal{O}}^{\lambda, j_{1}}(\phi)(x)=\omega_{F}^{j_{1}}(x)\langle x\rangle_{F}^{j_{1}} d \mu_{\mathcal{O}}^{\lambda, j_{2}}(\phi)(x),
$$

on $C_{F}\left(p^{\infty}\right)$.
Proof. By construction, we have for all $\beta>0$ and all $x \in C_{F}\left(p^{\beta}\right)$,

$$
\mu_{\mathcal{O}, \beta}^{\lambda, j_{i}}(\phi)(x)=\eta_{j_{1}, \mathcal{O}}(v)=\frac{\eta_{j_{i}}(v)}{\eta_{j_{i}}\left(g_{0} v_{0}\right)} \in \mathcal{O}_{\left(j_{i}\right)}
$$

for some vector $v \in L_{\lambda, \mathcal{O}}^{x, \beta}$ independent of $i \in\{1,2\}$. Two applications of the congruence (61) in Proposition 3.4 show:

$$
\begin{array}{rlrl}
N_{F / \mathbf{Q}}(x)^{j_{2}} \cdot \mu_{\mathcal{O}, \beta}^{\lambda, j_{1}}(\phi)(x) & =N_{F / \mathbf{Q}}(x)^{j_{2}} \cdot \eta_{j_{1}, \mathcal{O}}(v) & \\
& \equiv N_{F / \mathbf{Q}}(x)^{j_{1}+j_{2}} \cdot \Omega_{p}^{x, \beta} & \left(\bmod p^{\beta} \mathcal{O}\right) \\
& \equiv N_{F / \mathbf{Q}}(x)^{j_{1}} \cdot \eta_{j_{2}, \mathcal{O}}(v) & \left(\bmod p^{\beta} \mathcal{O}\right) \\
& =N_{F / \mathbf{Q}}(x)^{j_{1}} \cdot \mu_{\mathcal{O}, \beta}^{\lambda, j_{2}}(\phi)(x)
\end{array}
$$

This proves the claim.
5.5. Modular symbols in $p$-adic families. We restrict our attention to the nearly ordinary case. For any $\alpha \geq 0$, the normalized projections $\eta_{j, K}$ and $\eta_{j, \mathcal{O}}$ induce a canonical projection

$$
\eta_{j, p^{-\alpha} \mathcal{O} / \mathcal{O}}: \quad L_{\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}} \rightarrow p^{-\alpha} \mathcal{O} / \mathcal{O}
$$

Given $\beta \geq \alpha \geq \alpha_{0}^{K}$ and $x \in C\left(p^{\beta}\right)$, relation (85) shows that we may consider the modular symbol $\mathscr{P}_{A, x, \beta}^{\lambda, j}$ for the case $A=p^{-\alpha} \mathcal{O} / \mathcal{O}$, i.e. we have

$$
\mathscr{P}_{\alpha, x, \beta}^{\lambda, j}: \quad H_{\mathrm{c}, o r d} \operatorname{dim}^{\mathscr{y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}}\right) \rightarrow\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(j)},
$$

given by

$$
\phi \mapsto \int_{\mathscr{Y}\left(g_{\beta} K_{\alpha, \alpha} g_{\beta}^{-1} \cap H\left(\mathbf{A}^{(\infty)}\right)[x]\right.} \eta_{j, p^{-\alpha} \mathcal{O} / \mathcal{O}^{*}} i^{*}\left[\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\beta}\right) \cdot t_{g_{\beta}}^{\lambda}\right]\left(U_{p}^{-\beta} \phi\right) .
$$

We emphasize that here both the $\mathcal{O}$-module $p^{-\alpha} \mathcal{O} / \mathcal{O}$ and the level $K_{\alpha, \alpha}$ depend on $\alpha$.
As before, we obtain elements

$$
\mu_{\alpha, \beta}^{\lambda, j}(\phi):=\sum_{x \in C\left(p^{\beta}\right)} \mathscr{P}_{\alpha, x, \beta}^{\lambda, j}(\phi) \cdot x \in\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(j)}\left[C\left(p^{\beta}\right)\right],
$$

which in light of Proposition 5.3 satisfy for any $\beta \geq \beta^{\prime} \geq \alpha>\alpha_{0}^{K}$ the distribution relation,

$$
\begin{equation*}
\operatorname{res}_{\beta^{\prime}}^{\beta}\left(\mu_{\alpha, \beta}^{\lambda, j}(\phi)\right)=\mu_{\alpha, \beta^{\prime}}^{\lambda, j}(\phi) . \tag{88}
\end{equation*}
$$

Therefore, we obtain for each $\alpha>\alpha_{0}^{K}$ an $\mathcal{O}$-linear map

$$
\mu_{\alpha}^{\lambda, j}: \quad H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}}\right) \rightarrow\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(j)}\left[\left[C\left(p^{\infty}\right)\right]\right] .
$$

By construction, we have for all $\alpha \geq \alpha^{\prime} \geq \alpha_{0}^{K}$, a commutative square

$$
\begin{gathered}
H_{\mathrm{c}, \mathrm{ord}}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha^{\prime}}\right) ; \underline{L}_{\lambda, p^{-\alpha^{\prime}} \mathcal{O} / \mathcal{O}}\right) \xrightarrow{\mu_{\alpha^{\prime}}^{\lambda, j}}\left(p^{-\alpha^{\prime}} \mathcal{O} / \mathcal{O}\right)_{(j)}\left[\left[C\left(p^{\infty}\right)\right]\right] \\
\downarrow \\
H_{\mathrm{c}, \mathrm{ord}}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}}\right) \xrightarrow{\mu_{\alpha^{\lambda, j}}^{\lambda^{\prime}}\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(j)}\left[\left[C\left(p^{\infty}\right)\right]\right] .}
\end{gathered}
$$

This allows us to pass to the direct limit to obtain a map

$$
\mu^{\lambda, j}: \quad \mathcal{H}_{\mathrm{c}, o \mathrm{rd}}^{\operatorname{dim} \mathscr{Y}}\left(K_{\infty, \infty} ; \underline{L}_{\lambda, E / \mathcal{O}}\right) \rightarrow(E / \mathcal{O})_{(j)}\left[\left[C\left(p^{\infty}\right)\right]\right] .
$$

Theorem 5.5 (Independence of weight). For any $\lambda$ for which $\eta_{0}$ is admissible we have a commuting square

$$
\left.\begin{array}{cl}
\mathcal{H}_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(K_{\infty, \infty} ; \underline{L}_{\lambda, K / \mathcal{O}}\right) & \xrightarrow{\mu^{\lambda, 0}}(K / \mathcal{O})_{(0)}\left[\left[C\left(p^{\infty}\right)\right]\right] \\
\pi_{\lambda} \downarrow
\end{array}\right]
$$

where the map $\pi_{\lambda}$ is the isomorphism from Corollary 4.4.
Proof. By construction of $\mu^{\lambda, 0}$ and $\mu^{0,0}$ as the inductive limits of the maps $\mu_{\alpha}^{\lambda, 0}$ and $\mu_{\alpha}^{0,0}$, the claim is equivalent to the commutativity of the diagram

$$
\begin{gathered}
H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\left.\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}\right)} \xrightarrow{\mu_{\alpha}^{\lambda, 0}}\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(0)}\left[\left[C\left(p^{\infty}\right)\right]\right]\right. \\
\pi_{\lambda} \downarrow \\
H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{\left.p^{-\alpha} \mathcal{O} / \mathcal{O}\right)} \xrightarrow{\mu_{\alpha}^{0,0}}\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(0)}\left[\left[C\left(p^{\infty}\right)\right]\right]\right.
\end{gathered}
$$

for each $\alpha \geq \alpha_{0}^{K}$.
On the one hand, this reduces us to the commutativity of

$$
\begin{gathered}
H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}}\right) \xrightarrow{\mu_{\alpha, \beta}^{\lambda, 0}}\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(0)}\left[C\left(p^{\beta}\right)\right] \\
\pi_{\lambda} \downarrow \\
H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{p}^{-\alpha} \mathcal{O} / \mathcal{O}\right) \xrightarrow{\mu_{\alpha, \beta}^{0,0}}\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(0)}\left[C\left(p^{\beta}\right)\right]
\end{gathered}
$$

for all $\beta \geq \alpha \geq \alpha_{0}^{K}$. And on the other hand, the commutativity of this diagram is by Theorem 4.3 equivalent to the identity

$$
\begin{equation*}
\mathscr{P}_{\alpha, x, \beta}^{\lambda, 0}\left(\iota_{\lambda}(\phi)\right)=\mathscr{P}_{\alpha, x, \beta}^{0,0}(\phi) \tag{89}
\end{equation*}
$$

for all $\beta \geq \alpha \geq \alpha_{0}^{K}$ and $\phi$.
To this point, we observe first, that the congruence (61) and the relation (63) in Proposition 3.4 , imply the identity

$$
\eta_{p^{-\alpha} \mathcal{O} / \mathcal{O}, 0}\left(\left(-\lambda^{w_{0}}\right)\left(t_{\beta}\right) \cdot g_{\beta}\left(p^{-\alpha} v_{0}\right)\right)=\eta_{p^{-\alpha} \mathcal{O} / \mathcal{O}, 0}\left(p^{-\alpha} g_{0} v_{0}\right) \in p^{-\alpha} \mathcal{O} / \mathcal{O}
$$

By the definition of $i_{\lambda}: p^{-\alpha} \mathcal{O} / \mathcal{O} \rightarrow L_{\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}}$, this implies that the following diagram of sheaves on $\mathscr{Y}\left(g_{\beta} K_{\alpha^{\prime}, \alpha} g_{\beta}^{-1} \cap H\left(\mathbf{A}^{(\infty)}\right)\right.$

$$
\begin{align*}
i^{*}\left[\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\beta}\right) \cdot t_{g_{\beta}}^{\lambda}\right] \underline{L}_{\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}} & \xrightarrow{\eta_{0, p^{-\alpha} \mathcal{O} / \mathcal{O}}} \\
i_{\lambda} \uparrow & \frac{p^{-\alpha} \mathcal{O} / \mathcal{O}}{\|}  \tag{90}\\
i^{*} t_{g_{\beta}}^{0} \xrightarrow{p^{-\alpha} \mathcal{O} / \mathcal{O}} & \xrightarrow{\eta_{0, p^{-\alpha} \mathcal{O} / \mathcal{O}}} \xrightarrow{p^{-\alpha} \mathcal{O} / \mathcal{O}}
\end{align*}
$$

commutes. Using the commutativy of (90) and the Hecke equivariance of $\iota_{\lambda}$, we obtain

$$
\begin{aligned}
& \eta_{0, p^{-\alpha} \mathcal{O} / \mathcal{O}}\left(i^{*}\left[\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\beta}\right) \cdot t_{g_{\beta}}^{\lambda}\right]\left(U_{p}^{-\beta} \iota_{\lambda}(\phi)\right)\right) \\
= & \eta_{0, p^{-\alpha} \mathcal{O} / \mathcal{O}}\left(i^{*}\left[\left(-\lambda^{w_{0}}\right)\left(t_{p}^{\beta}\right) \cdot t_{g_{\beta}}^{\lambda}\right]\left(\iota_{\lambda}\left(U_{p}^{-\beta} \phi\right)\right)\right) \\
= & \eta_{0, p^{-\alpha} \mathcal{O} / \mathcal{O}}\left(i^{*} t_{g_{\beta}}^{0}\left(U_{p}^{-\beta} \phi\right)\right),
\end{aligned}
$$

which proves (89).
We fix for all $\alpha \geq \alpha_{0}^{K}$ compatibly isomorphisms

$$
\begin{equation*}
H_{\mathrm{c}}^{\operatorname{dim} \mathscr{X}}\left(K_{\alpha, \alpha} ; \underline{\mathbf{Q}_{p} / \mathbf{Z}_{p}}\right) \cong \mathbf{Q}_{p} / \mathbf{Z}_{p} \tag{91}
\end{equation*}
$$

Such a compatible choice is unique up to scalars in $\mathbf{Z}_{p}^{\times}$and we are free to make it compatible with our choice of Haar measure on $G(\mathbf{A})$. By Poincaré-Pontryagin duality, our choice of (91) allows us to consider $\mu^{\lambda, 0}$ as an element of
$\operatorname{Hom}_{\mathcal{O}}\left(\mathcal{H}_{\mathrm{c}, \mathrm{ord}}^{\operatorname{dim} \mathscr{Y}}\left(K_{\infty, \infty} ; L_{\lambda, E / \mathcal{O}}\right), E / \mathcal{O}\right) \widehat{\otimes}_{\mathcal{O}} \mathcal{O}\left[\left[C\left(p^{\infty}\right)\right]\right]=\mathcal{H}_{\mathrm{ord}}^{\operatorname{dim} \mathscr{X}-\operatorname{dim} \mathscr{Y}}\left(K_{\infty, \infty} ; L_{\lambda^{\vee}, \mathcal{O}}\right)\left[\left[C\left(p^{\infty}\right)\right]\right]$,
where as before (again via transfer maps)

Since the interpretation of $\mu^{\lambda, 0}$ as an element of this space depends on the choice of isomorphism in (91), we introduce an ambiguity in the form of a $p$-adic period in $\mathbf{Z}_{p}^{\times}$.

We remark that while $\operatorname{dim} \mathscr{Y}=q_{0}$ is the bottom degree, $\operatorname{dim} \mathscr{X}-\operatorname{dim} \mathscr{Y}=q_{0}+l_{0}$ is the top degree for $G$.

There is inherent redundancy in this construction: The ambient space containing $\mu^{\lambda, 0}$ admits two canonical $\mathcal{O}\left[\left[C\left(p^{\infty}\right)\right]\right]$-module structures. To understand their interrelation, consider the map

$$
\begin{aligned}
L_{p}^{\lambda, j}: \quad \mathcal{H}_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}( & \left.K_{\infty, \infty} ; L_{\lambda, E / \mathcal{O}}\right) \rightarrow(E / \mathcal{O})_{(j)} \\
\phi & \mapsto \int_{C\left(p^{\infty}\right)} 1 d \mu^{\lambda, j}(\phi)
\end{aligned}
$$

Then by the distribution property, for $\beta \gg 0$,

$$
L_{p}^{\lambda, j}(\phi)=\sum_{x \in C\left(p^{\beta}\right)} \underset{\alpha}{\lim } \mathscr{P}_{\alpha, x, \beta}^{\lambda, j}(\phi) \in(E / \mathcal{O})_{(j)} .
$$

By definition, we may identify $\mu^{\lambda, j}$ and $\mu^{\lambda+(j), 0}$, whence also $L_{p}^{\lambda, j}$ and $L_{p}^{\lambda+(j), 0}$. Therefore, we may and do assume $j=0$ in the sequel. As before,

$$
\operatorname{Hom}_{\mathcal{O}}\left(\mathcal{H}_{\mathrm{c}, \mathrm{ord}}^{\operatorname{dim} \mathscr{Y}}\left(K_{\infty, \infty} ; L_{\lambda, E / \mathcal{O}}\right), E / \mathcal{O}\right)=\mathcal{H}_{\mathrm{ord}}^{\operatorname{dim}} \mathscr{X}-\operatorname{dim} \mathscr{Y}\left(K_{\infty, \infty} ; L_{\lambda \vee, \mathcal{O}}\right)
$$

whence if $\eta_{0}$ is admissible for $\lambda$,

$$
L_{p}^{\lambda, 0} \in \mathcal{H}_{\mathrm{ord}}^{\operatorname{dim} \mathscr{X}-\operatorname{dim} \mathscr{Y}}\left(K_{\infty, \infty} ; L_{\lambda^{\vee}, \mathcal{O}}\right)
$$

Theorem 5.5 implies
Corollary 5.6 (Independence of weight). For any $\lambda$ for which $\eta_{0}$ is admissible,

$$
\pi_{\lambda \vee}^{\circ}\left(L_{p}^{\lambda, 0}\right)=L_{p}^{0,0}
$$

Consider the diagram

$$
\begin{array}{cc}
H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right) \xrightarrow{\mu_{E / \mathcal{O}}^{\lambda, 0}} E / \mathcal{O}\left[\left[C_{F}\left(p^{\infty}\right)\right]\right] \\
\downarrow & \downarrow \mu \mapsto \int_{C_{F}\left(p^{\infty}\right)} d \mu  \tag{92}\\
\mathcal{H}_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(K_{\infty, \infty} ; \underline{L}_{\lambda, E / \mathcal{O}}\right) & \xrightarrow{L_{p}^{\lambda, 0}}
\end{array}
$$

which commutes by construction. We consider the composition

$$
\phi \mapsto \int_{C_{F}\left(p^{\infty}\right)} d \mu_{E / \mathcal{O}}^{\lambda, 0}(\phi)
$$

as an element $L_{p, \alpha}^{\lambda, 0}$ of

$$
H_{\mathrm{ord}}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda^{\vee}, \mathcal{O}}^{\circ}\right)=\operatorname{Hom}_{\mathcal{O}}\left(H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right), E / \mathcal{O}\right)
$$

Then by the commutativity of (92), we obtain
Lemma 5.7. The canonical map

$$
\mathcal{H}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \underline{L}_{\lambda^{\vee}, \mathcal{O}}\right) \rightarrow H_{\mathrm{ord}}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda^{\vee}, \mathcal{O}}^{\circ}\right)
$$

maps $L_{p}^{\lambda, 0}$ to $L_{p, \alpha}^{\lambda, 0}$.
Remark 5.8. We may reconstruct the full measure

$$
\mu_{E / \mathcal{O}}^{\lambda, 0} \in H_{\text {ord }}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \mathcal{O}\right)\left[\left[C_{F}\left(p^{\infty}\right)\right]\right]
$$

from $L_{p}^{0,0}$ as follows. Let $\chi$ be an $\mathcal{O}$-valued character with conductor dividing $p^{\beta}$. After possibly passing to a larger $\alpha$, we may assume $\alpha \geq \beta$ and consider $\chi$ as a function on $C\left(\operatorname{det}\left(K_{\alpha, \alpha}\right)\right)$, which pulls back to a locally constant function on $\mathscr{X}\left(K_{\alpha, \alpha}\right)$. Therefore, $\chi$ gives rise to a degree zero cohomology class

$$
[\chi] \in H^{0}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \mathcal{O}\right)
$$

Then the evaluation of $\mu_{E / \mathcal{O}}(\phi)$ at the character $\chi$ for

$$
\phi \in H_{\mathrm{c}, \mathrm{ord}}^{q_{0}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right)
$$

is given by

$$
\begin{aligned}
L_{p}^{0,0}\left(\pi_{\lambda}(\phi \cup[\chi])\right) & =L_{p}^{\lambda, 0}(\phi \cup[\chi]) \\
& =L_{p, \alpha}^{\lambda, 0}(\phi \cup[\chi]) \\
& =\int_{C_{F}\left(p^{\infty}\right)} d \mu_{E / \mathcal{O}}^{\lambda, 0}(\phi \cup[\chi]) \\
& =\int_{C_{F}\left(p^{\infty}\right)} \chi d \mu_{E / \mathcal{O}}^{\lambda, 0}(\phi) .
\end{aligned}
$$

By Theorem 4.10, together with Corollary 5.6 and the Control Theorem in its dual form in Corollary 4.15 we obtain

Theorem 5.9. Assume $p \nmid(n+1) n$ and that $F$ is either totally real, $C M$, or that conjecture 4.9 holds for $F$. Fix a non-Eisenstein maximal ideal $\mathfrak{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty}, \mathcal{O}\right)$ and write $\mathfrak{m}^{\vee}$ for the contragredient maximal ideal with contragredient residual Galois representation. Then for every locally algebraic $\mathcal{O}$-valued character $\lambda^{\vee} \vartheta^{-1}$ of $T\left(\mathbf{Z}_{p}\right)$ with dominant $\lambda$ for which $\eta_{0}$ is admissible, and $\alpha \geq \alpha_{0}^{K}$ satisfying $\mathfrak{f}_{\vartheta} \mid p^{\alpha}$, the canonical map

$$
\mathcal{H}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee} \otimes_{\Lambda} \Lambda^{\circ} / P_{\lambda^{\vee} \vee}^{\circ} \rightarrow H_{\mathrm{ord}}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda^{\vee}, \mathcal{O}}^{\circ}\right)_{\mathfrak{m} \vee} \otimes_{\Lambda^{\circ}, \vartheta^{-1}} \mathcal{O}
$$

$\operatorname{maps} L_{p}^{0,0} \bmod P_{\lambda^{\vee} \vartheta^{-1}}^{\circ}$ to $L_{p, \alpha}^{\lambda, 0} \bmod P_{\vartheta^{-1}}^{\circ}$.

## 6. $p$-ADIC $L$-FUNCTIONS

6.1. Abelian $p$-adic $L$-functions for automorphic representations. Recall that $F / \mathbf{Q}$ denotes a number field and $G=\operatorname{res}_{F / \mathbf{Q}} \mathrm{GL}(n+1) \times \mathrm{GL}(n)$ as before. For any regular algebraic cuspidal automorphic representation $\Pi \widehat{\otimes} \Sigma$ of $G(\mathbf{A})$ of cohomological weight $\lambda$, the action of the finite Hecke algebra on $\Pi \widehat{\otimes} \Sigma$ is defined over the field of rationality $\mathbf{Q}(\Pi, \Sigma) / \mathbf{Q}$ of $\Pi$ and $\Sigma$, which is a number field by the work of Clozel [6] (cf. [53] for a globalization of this result). We fix embeddings $\mathbf{Q}(\Pi, \Sigma) \rightarrow \mathbf{C}$ and $\mathbf{Q}(\Pi, \Sigma) \rightarrow E$ where $E / \mathbf{Q}_{p}$ is sufficiently large. This allows us to refer to $p$-adic absolute values of eigenvalues of Hecke operators acting on $\Pi \widehat{\otimes} \Sigma$.

We call $\Pi \widehat{\otimes} \Sigma$ nearly ordinary at a rational prime $p$ if for some $\alpha \gg 0$ there is a $\phi \in \Pi \widehat{\otimes} \Sigma$ which is an eigenvector of $\mathcal{H}_{\mathbf{Q}(\Pi)}(\alpha, \alpha)$ with eigenvalue $\vartheta: T\left(\mathbf{Q}_{p}\right) \rightarrow \mathbf{Q}(\Pi)^{\times}$satisfying

$$
\begin{equation*}
\left|\lambda^{\vee}\left(t_{p}\right) \vartheta\left(t_{p}\right)\right|_{p}=1 \tag{93}
\end{equation*}
$$

for the normalized absolute value $|\cdot|_{p}$ on $E$. This is the same to say that both $\Pi$ and $\Sigma$ are nearly ordinary at $p$ (for the standard Borel subgroups $B_{n+1}$ and $B_{n}$ ) in the sense of [36]. We will see in the course of the proof of Theorem 6.1 below that in the nearly ordinary case, $\vartheta$ is then uniquely determined by $\Pi \widehat{\otimes} \Sigma$.

We fix embeddings

$$
i_{\infty}: \quad \mathbf{Q}(\Pi, \Sigma) \rightarrow \mathbf{C},
$$

and

$$
i_{p}: \quad \mathbf{Q}(\Pi, \Sigma) \rightarrow E
$$

Once appropriately normalized, the special values of the $L$-function $L(s, \Pi \widehat{\otimes} \Sigma)$ lie in $\mathbf{Q}(\Pi, \Sigma)$ and hence also in $E$ provided that $\lambda$ is balanced in the sense of (4), cf. [71,54]. In the following we implicitly assume that the a priori complex valued $p$-adic Nebentypus $\vartheta$ of $\Pi \widehat{\otimes} \Sigma$ also takes values in $E$.

Theorem 6.1. Let $\Pi \widehat{\otimes} \Sigma$ be an irreducible regular algebraic cuspidal automorphic representation of $G(\mathbf{A})$ of cohomological weight $\lambda$. Assume the following:
(i) $\lambda$ is balanced.
(ii) $\Pi \widehat{\otimes} \Sigma$ is nearly ordinary at a prime $p$ and $\vartheta: T\left(\mathbf{Q}_{p}\right) \rightarrow \mathbf{C}^{\times}$the corresponding Nebentypus.
Then there are complex periods $\Omega_{ \pm, j} \in \mathbf{C}^{\times}$, indexed by the characters of $\pi_{0}\left(F_{\infty}^{\times}\right)$and $j \in \mathbf{Z}$ for which (5) is non-zero, and a unique p-adic measure $\mu_{\Pi \widehat{\otimes} \Sigma} \in \mathcal{O}\left[\left[C_{F}\left(p^{\infty}\right)\right]\right]$ with the following property. For every $s_{0}=\frac{1}{2}+j$ critical for $L(s, \Pi \widehat{\otimes} \Sigma)$, for all finite order Hecke characters $\chi$ of $F$ unramified outside $p \infty$ and such that $\chi_{p} \vartheta$ has fully supported constant conductor $\mathfrak{f}_{\chi \vartheta}$,

$$
\begin{aligned}
& \int_{C_{F}\left(p^{\infty}\right)} \chi(x) \omega_{F}^{j}(x)\langle x\rangle_{F}^{j} d \mu_{\Pi \widehat{\otimes} \Sigma}(x)= \\
& \mathfrak{N}\left(\mathfrak{f}_{\chi \vartheta}\right)^{j \frac{(n+1) n}{2}-\frac{(n+1) n(n-1)}{6}} \cdot \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} G\left(\chi \vartheta_{\mu, \nu}\right) \cdot \frac{L(s, \Pi \widehat{\otimes} \Sigma \otimes \chi)}{\Omega_{(-1)^{j} \operatorname{sgn} \chi, j}} .
\end{aligned}
$$

Furthermore, $\mu_{\Pi \widehat{\otimes} \Sigma}$ is determined uniquely by the interpolation property for a single critical $s_{0}=\frac{1}{2}+j$.

Previously, Schwab settled the 'Manin congruences' for $n=2, \lambda_{3}=(2,1,0)$ and $\lambda_{2}=$ $(1,0)$ in her Diploma thesis [85], while the case $n=1$ was treated by Namikawa in [67].

The automorphic representations $\Pi \widehat{\otimes} \Sigma$ of $G$ we consider are topological tensor products of regular algebraic cuspidal automorphic representations $\Pi$ and $\Sigma$ of $\mathrm{GL}(n+1)$ and GL( $n$ ) respectively. Motivically, the $L$-functions we study therefore conjecturally are $L$-functions of tensor products of irreducible pure motives of dimensions $n+1$ and $n$. In this context, condition (i) (i.e. condition (4)) translates into a certain condition on the Hodge-Tate weights of those motives and in light of Deligne's Conjecture on special values of motivic $L$-functions and its integral refinements, the complex periods and other invariants in the interpolation formulae of the $p$-adic $L$-functions we construct should depend on the relative position of these Hodge-Tate weights in general (cf. [10, 11]). Therefore, it is not surprising that we need to restrict our treatment to a certain class of relative positions when working with a single modular symbol in a fixed cohomological degree, which applies to the case at hand.

Proof. As explained in section 2.3, we may choose at every finite place $v \nmid p$ of $F$ a good test vector

$$
W_{v}^{0} \in \mathscr{W}\left(\Pi_{v} \otimes \Sigma_{v}, \psi_{v} \otimes \psi_{v}^{-1}\right)
$$

which is the product of two normalized spherical Whittaker functions whenever $\Pi_{v}$ and $\Sigma_{v}$ are unramified. At $p$ we choose an eigenvector

$$
W_{p} \in \mathscr{W}\left(\Pi_{p} \otimes \Sigma_{p}, \psi_{p} \otimes \psi_{p}^{-1}\right)
$$

for $\mathcal{H}_{\mathbf{C}}(\alpha, \alpha)$ with eigenvalue $\vartheta$. From the proof of Proposition 6.4 in [36] we know that at each $v \mid p$ the representations $\Pi_{v}$ and $\Sigma_{v}$ are both subquotients of a principal series representation as considered in Proposition 1.3, and that $\vartheta$ is uniquely determined by $\Pi_{p}$ and $\Sigma_{p}$ and the ordinarity condition. Furthermore, $W_{p}$ lies in a unique line and

$$
W_{p}(\mathbf{1}) \neq 0
$$

Recall that $\mathfrak{g}$ and $\mathfrak{k}$ denote the complexified Lie algebras of $G(\mathbf{R})$ and the standard maximal compact $K_{\infty} \subseteq G(\mathbf{R})$ respectively. We set $\mathfrak{g k}:=\mathfrak{z}+\mathfrak{k}$ where $\mathfrak{z}$ is the complexified Lie algebra of the center $Z(\mathbf{R}) \subseteq G(\mathbf{R})$. Then for every character $\varepsilon: \pi_{0}(G(\mathbf{R})) \rightarrow \mathbf{C}^{\times}$, the $\varepsilon$-eigenspace
$H^{\operatorname{dim} \mathscr{Y}}\left(\mathfrak{g}, \mathfrak{g k} ; \mathscr{W}\left(\Pi_{\infty} \widehat{\otimes} \Sigma_{\infty}, \psi_{\infty} \otimes \psi_{\infty}^{-1}\right) \otimes L_{\lambda, \mathbf{C}}\right)_{\varepsilon}=\left(\bigwedge^{\operatorname{dim} \mathscr{Y}}(\mathfrak{g} / \mathfrak{g k})^{*} \otimes \Pi_{\infty} \widehat{\otimes} \Sigma_{\infty} \otimes L_{\lambda, \mathbf{C}}\right)_{\varepsilon}^{K_{\infty}^{0}}$
in cohomology is at most one-dimensional. Furthermore, it is non-trivial if and only if the restriction of $\varepsilon$ to $\pi_{0}\left(\mathrm{GL}_{m}(F \otimes \mathbf{R})\right)$ where $m \in\{n+1, n\}$ is odd, agrees with the restriction of the product of the central characters $\omega_{\Pi_{\infty} \hat{\otimes} \Sigma_{\infty}} \omega_{L_{\lambda, \mathbf{C}}}$ restricted to the subgroup $\left\{ \pm \mathbf{1}_{m}\right\} \subseteq \mathrm{GL}_{m}(F \otimes \mathbf{R})$.

For each such $\varepsilon$ we pick a generator $\varphi_{\varepsilon, \infty}$ in (94) and fix a non-zero

$$
\eta_{j} \in \operatorname{Hom}_{H}\left(V_{\lambda, \mathbf{C}}, \mathbf{C}_{(j)}\right),
$$

which exists by [55], Theorem 2.3 and [71], Theorem 2.21. Then each $\varphi_{\varepsilon, \infty}$ projects under the map

$$
\begin{aligned}
\operatorname{res}_{H}^{G} \otimes \mathbf{1} \otimes \eta_{j, \mathbf{C}}: & \bigwedge^{\operatorname{dim} \mathscr{Y}}(\mathfrak{g} / \mathfrak{g k})^{*} \otimes \mathscr{W}\left(\Pi_{\infty} \widehat{\otimes} \Sigma_{\infty}, \psi_{\infty} \otimes \psi_{\infty}^{-1}\right) \otimes L_{\lambda, \mathbf{C}} \rightarrow \\
& \bigwedge_{\operatorname{dim} \mathscr{Y}}\left(\mathfrak{h} /\left(\mathfrak{g k}^{\mathfrak{k}} \mathfrak{h}\right)\right)^{*} \otimes \mathscr{W}\left(\Pi_{\infty} \widehat{\otimes} \Sigma_{\infty}, \psi_{\infty} \otimes \psi_{\infty}^{-1}\right) \otimes L_{\lambda, \mathbf{C}}
\end{aligned}
$$

to a vector of the form

$$
\operatorname{res}_{H}^{G} \otimes \mathbf{1} \otimes \eta_{j, \mathbf{C}}\left(\varphi_{\varepsilon, \infty}\right)=\omega_{\infty} \otimes W_{\varepsilon, \infty}^{\mathrm{coh}} \otimes 1
$$

Here, $\omega_{\infty}$ is the fixed generator of the line $\bigwedge^{\operatorname{dim} \mathscr{Y}}(\mathfrak{h} /(\mathfrak{g k} \cap \mathfrak{h}))^{*}$ (independent of $\varepsilon$ ) corresponding to our choice of invariant measure on $H(\mathbf{R})^{0} / \widetilde{K}_{\infty} \cap H(\mathbf{R})^{0}, 1 \in \mathbf{C}_{(j)}$ is a generator as before, and

$$
W_{\varepsilon, \infty}^{\mathrm{coh}, j} \in \mathscr{W}\left(\Pi_{\infty} \widehat{\otimes} \Sigma_{\infty}, \psi_{\infty} \otimes \psi_{\infty}^{-1}\right)
$$

The latter vector is commonly refered to as a cohomological test vector. We emphasize that this vector possibly does depend on $j$.

On the one hand, we obtain for each such $\varepsilon$ a global Whittaker vector

$$
W_{\varepsilon}^{\mathrm{coh}, j}:=W_{\varepsilon, \infty}^{\mathrm{coh}, j} \otimes W_{p} \otimes\left(\otimes_{v \nmid p \infty} W_{v}^{0}\right) \in \mathscr{W}\left(\Pi \widehat{\otimes} \Sigma, \psi \otimes \psi^{-1}\right)
$$

By [55, 84] (cf. also section 8 in [52]), we know

$$
\begin{equation*}
\Omega_{\infty}\left(j, W_{\varepsilon, \infty}^{\mathrm{coh}, j}\right) \neq 0 \tag{95}
\end{equation*}
$$

Set

$$
W^{\mathrm{coh}, j}:=\sum_{\varepsilon} W_{\varepsilon}^{\mathrm{coh}, j} .
$$

Then for every finite order character $\chi$ unramified outside $p \infty$,

$$
\Omega^{S(p)}\left(j,\left(W^{\mathrm{coh}, j}\right)_{\chi}\right)=\Omega_{\infty}\left(j, W_{\chi \infty}^{\mathrm{coh}, j}\right) \neq 0
$$

only depends on $j$ and $\chi_{\infty}$.
On the other hand, we have cohomology classes

$$
\left[\varphi_{\varepsilon, \infty} \otimes W_{p} \otimes\left(\otimes_{v \nmid p \infty} W_{v}^{0}\right)\right] \in H^{\operatorname{dim} \mathscr{Y}}\left(\mathfrak{g}, \mathfrak{g k} ; \mathscr{W}\left(\Pi \widehat{\otimes} \Sigma, \psi \otimes \psi^{-1}\right) \otimes L_{\lambda, \mathbf{C}}\right)_{\varepsilon}
$$

which by inverse Fourier transform give rise to global cohomology classes

$$
\phi_{\varepsilon} \in H_{\mathrm{c}}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, \mathbf{C}}\right)_{\varepsilon}
$$

We may normalize the classes $\phi_{\varepsilon}$ such that they lie p-optimally in

$$
\phi_{\varepsilon} \in H_{\mathrm{c}}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, \mathbf{Q}(\Pi, \Sigma)}\right)_{\varepsilon}
$$

where $p$-optimality is understood with respect to the embedding $i_{p}: \mathbf{Q}(\Pi, \Sigma) \rightarrow E$ (the latter induces a $p$-adic valuation on $\mathbf{Q}(\Pi, \Sigma)$ ).

By the hypothesis (93) on the $U_{p}$-eigenvalue of $W_{p}$, we find a preimage

$$
\phi_{\Pi \widehat{\otimes} \Sigma}:=\sum_{\varepsilon} \phi_{\varepsilon} \in H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}\right) .
$$

The resulting $p$-adic measure will be indepentent of this choice of preimage. Choose any $j$ as above and put

$$
\mu_{\Pi \widehat{\otimes} \Sigma}:=\omega_{F}^{-j}(-)\langle-\rangle_{F}^{-j} \mu_{\mathcal{O}}^{\lambda, j}\left(\phi_{\Pi \widehat{\otimes} \Sigma}\right) .
$$

By Theorem 5.4, $\mu_{\Pi \widehat{\otimes} \Sigma}$ is independent of $j$. Hence, for any $\chi$ and any $j$ as in the statement of the Theorem, if $\beta \geq \alpha$ such that $f_{\chi \vartheta} \mid p^{\beta}$, then

$$
\begin{aligned}
& \int_{C_{F}\left(p^{\infty}\right)} \chi(x) \omega_{F}^{j}(x)\langle x\rangle_{F}^{j} d \mu_{\Pi \widehat{\otimes} \Sigma}(x) \\
= & \int_{C_{F}\left(p^{\infty}\right)} \chi d \mu_{\widehat{O}}^{\lambda, j}\left(\phi_{\Pi \widehat{\otimes} \Sigma}\right) \\
= & \sum_{x \in C\left(p^{\beta}\right)} \chi(x) \mathscr{P}_{\mathcal{O}, x, \beta}^{\lambda, j}\left(\phi_{\Pi \widehat{\otimes} \Sigma}\right) \\
= & \sum_{x \in C\left(p^{\beta}\right)} \chi(x) \int_{\mathscr{Y}\left(g_{\beta} K_{\alpha, \alpha} g_{\beta}^{-1} \cap H\left(\mathbf{A}^{(\infty)}\right)\right)[x]} \eta_{j, \mathcal{O}} i^{*}\left[\lambda^{\vee}\left(t_{p}^{\beta}\right) \cdot t_{g_{\beta}}^{\lambda}\right]\left(U_{p}^{-\beta} \phi_{\Pi \widehat{\otimes} \Sigma}\right) \\
= & \lambda^{\vee}\left(t_{p}^{\beta}\right) \cdot\left[H(\widehat{\mathbf{Z}}): H\left(\mathbf{A}^{(\infty)}\right) \cap g_{\beta} K_{\alpha, \alpha} g_{\beta}^{-1}\right] . \\
& \int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \phi\left(\lambda \vee\left(t_{p}^{-\beta}\right) U_{p}^{-\beta} W^{\text {cob }, j}\right)_{(-1)^{j}} \\
& \left(g \cdot h t_{p^{\beta}}\right) \chi(\operatorname{det}(g))|\operatorname{det}(g)|^{j} d g,
\end{aligned}
$$

By Proposition 5.1,

$$
\begin{aligned}
& {\left[H(\widehat{\mathbf{Z}}): H\left(\mathbf{A}^{(\infty)}\right) \cap g_{\beta} K_{\alpha, \alpha} g_{\beta}^{-1}\right] } \\
= & {\left[H\left(\widehat{\mathbf{Z}}^{(p)}\right): K^{(p)} \cap H\left(\mathbf{A}^{(p \infty)}\right)\right] \cdot\left[H\left(\mathbf{Z}_{p}\right): H\left(\mathbf{Q}_{p}\right) \cap g_{\beta} I_{\alpha, \alpha} g_{\beta}^{-1}\right] } \\
= & {\left[H\left(\widehat{\mathbf{Z}}^{(p)}\right): K^{(p)} \cap H\left(\mathbf{A}^{(p \infty)}\right)\right] \cdot\left[H\left(\mathbf{Z}_{p}\right): \mathfrak{I}_{\beta}^{n}\right] } \\
= & {\left[H\left(\widehat{\mathbf{Z}}^{(p)}\right): K^{(p)} \cap H\left(\mathbf{A}^{(p \infty)}\right)\right] \cdot \prod_{v \mid p} \prod_{\mu=1}^{n}\left(1-q_{v}^{-\mu}\right)^{-1} . } \\
& \mathfrak{N}\left(p \mathcal{O}_{p}\right)^{\beta \frac{(n+2)(n+1) n+(n+1) n(n-1)}{6}} .
\end{aligned}
$$

By the global Birch Lemma (cf. Theorem 2.9),

$$
\begin{aligned}
& \vartheta\left(t_{p}^{-\beta}\right) \int_{H(\mathbf{Q}) \backslash H(\mathbf{A})} \phi_{\left(W^{\mathrm{coh}, j}\right)_{(-1)^{j}}\left(g \cdot h t_{p^{\beta}}\right) \chi(\operatorname{det}(g))|\operatorname{det}(g)|^{j} d g}^{=} \\
& \vartheta\left(t_{p}^{-\beta}\right) \cdot \vartheta\left(t_{p}^{\beta}\right) \cdot \mathfrak{N}\left(p \mathcal{O}_{p}\right)^{-\beta \frac{(n+2)(n+1) n+(n+1) n(n-1)}{6}} \cdot \mathfrak{N}\left(\mathfrak{f}_{\chi \vartheta}\right)^{-\frac{(n+1) n(n-1)}{6}} \cdot\left|t_{f_{\chi \vartheta}}\right|^{-j} . \\
& \Omega_{\infty}\left(j, W_{(-1)^{j} \chi_{\infty}}^{\mathrm{coh}, j}\right) \cdot \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} G\left(\chi \vartheta_{\mu, \nu}\right) \cdot L(j, \Pi \widehat{\otimes}(\Sigma \otimes \chi))
\end{aligned}
$$

Collecting terms concludes the proof of existence of $\mu_{\Pi \widehat{\otimes} \Sigma}$ with

$$
\Omega_{(-1)^{j} \operatorname{sgn} \chi, j}:=\Omega_{\infty}\left(j, W_{(-1)^{j} \chi_{\infty}}^{\mathrm{coh}, j}\right)^{-1}
$$

Using Lemma 10.2 in [30], it is easy to see that the interpolation property in Theorem 6.1 at a single critical $s_{0}=\frac{1}{2}+j$ determines the measure $\mu_{\Pi \widehat{\otimes} \Sigma}$ uniquely, cf. Corollary 6.9 in [51].

Remark 6.2. The dependence of the periods $\Omega_{ \pm, j}$ on $j$ originates from the fact that for each $j$ we obtain potentially a different 'cohomoligcal' vector inside the archimedean component of $\Pi \widehat{\otimes} \Sigma$. The effect of this dependence on the periods $\Omega_{ \pm, j}$ is studied in [54, 24, 27]. It is a result of the author that each $\varphi_{\epsilon, \infty}$ lies in a $\mathbf{Q}\left(\Pi, \Sigma, \sqrt{(-1)^{\frac{(n+1) n}{2}}}\right)$-rational structure,
and so do $W_{\varepsilon, \infty}^{\text {coh }}$ and $W_{\varepsilon}(c f .[53,54])$. We remark that in the case of $\mathrm{GL}(3) \times \mathrm{GL}(2)$ results in $[37,38]$ indicate, that at least in this case it should be possible to make the period relations between $\Omega_{ \pm, j}$ explicit integrally. For case $F=\mathbf{Q}$ and $n=2$ this was carried out by Hara and Namikawa [26], who showed that the period $\Omega_{(-1)^{j} \operatorname{sgn} \chi, j}$, studied here is, as expected, up to powers of $2 \pi i$ and binomial coefficients, independent of $j$ and factors canonically as a product of cohomological periods attached to the automorphic representations $\Pi$ and $\Sigma$ on GL(3) and GL(2) (cf. Theorem 1.1 in loc. cit.).
6.2. Non-abelian $p$-adic $L$-functions. Assume as before that $F$ is totally real, CM or the validity of Conjecture 4.9. Consider a non-Eisenstein component $\mathrm{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)_{\mathfrak{m}} \vee$ for a non-Eisenstein maximal ideal $\mathfrak{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$. We also assume that $p \nmid(n+1) n$ and $S(K)=S(p)$, i.e. $K$ has full level outside $p$ for technical reasons.

Then we know by Hida's Corollary A. 4 in [36] (see also the proof of Prop. 6.4 in loc. cit.), which is applicable in our situation, that for each dominant weight $\lambda$, inner nearly ordinary cohomology

$$
\begin{equation*}
H_{!, \text {ord }}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ;{\underline{L_{\lambda}, E}}\right) \tag{96}
\end{equation*}
$$

is a semi-simple $\mathrm{h}_{!, \text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, E\right)$-module, each simple cuspidal summand occuring with multiplicity $\left|\pi_{0}(H(\mathbf{R}))\right|$, provided that $K_{\alpha^{\prime}, \alpha}$ is neat and $E$ is sufficiently large (we refer to sections 2.3 and 4.3 in [54] and section 4.3 [27] for an explanation of the multiplicity in the case at hand). In particular, by Theorem 4.10,

$$
H_{\mathrm{ord}}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda^{\vee}, \mathcal{O}}^{\circ}\right)_{\mathfrak{m} \vee}=E\left[\pi_{0}(G(\mathbf{R}))\right] \otimes \mathrm{h}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, E\right)_{\mathfrak{m} \vee}
$$

as $\pi_{0}(H(\mathbf{R})) \times \mathrm{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, E\right)_{\mathfrak{m}}{ }^{\vee}$-modules. Consequently, the space corresponding to the trivial signature at infinity

$$
\begin{equation*}
H_{\mathrm{ord}}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{\lambda}_{\lambda^{\vee}, \mathcal{O}}^{\circ}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))}=\mathrm{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, E\right)_{\mathfrak{m} \vee} \tag{97}
\end{equation*}
$$

is free of rank one. We remark that the hypothesis $p \nmid(n+1) n$ implies that $p$ is odd and therefore $\pi_{0}(H(\mathbf{R}))$ acts semi-simply on the cohomology spaces under consideration.

On the one hand, each $E$ valued point $\xi \in \operatorname{Spech}_{\text {ord }}^{q_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)_{\mathfrak{m}}(E)$ corresponds bijectively to an $E$ valued point $\xi^{\vee} \in \operatorname{Spech}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, \mathcal{O}\right)_{\mathfrak{m}} \vee(E)$, and on the other hand, $\xi$ corresponds bijectively to a cuspidal automorphic representation $\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}$ of $G(\mathbf{A})$ defined over $E$ whose contragredient contributes to (96) (cf. Theorem 4.10), by virtue of the embeddings $i_{\infty}$ and $i_{p}$ from the previous section. For notational simplicity, we may fix once and for all an (algebraic) embedding $i_{E}: E \rightarrow \mathbf{C}$.

We call such a $\xi$ a classical point in $\operatorname{Spec}_{\mathrm{h}_{\text {ord }}^{q_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)_{\mathfrak{m}}(E) \text {. If we do not assume } \mathfrak{m}, ~(1)}$ to be non-Eisenstein and consider inner cohomology, the representation $\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}$ is either cuspidal or residual. By [62], regularity of $\lambda$ always implies cuspidality of $\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}$. In the cuspidal case, Theorem 6.1 provides us with an abelian $p$-adic $L$-function for $\xi$. In the residual case, the proof of Theorem 6.1 shows that the modular symbol vanishes identically by Corollary 5.8 in [39].

Back in the non-Eisenstein situation, let

$$
\xi \in \operatorname{Spec}_{\mathrm{ord}}^{q_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)_{\mathfrak{m}}(E)
$$

be a classical point of balanced weight $\lambda$ and Nebentypus $\vartheta$. By Corollary 4.15,
$\mathcal{H}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}^{\vee}}^{\pi_{0}(H(\mathbf{R}))} \otimes_{\Lambda^{\circ}} \Lambda^{\circ} / P_{\lambda^{\vee} \vartheta^{-1}}^{\circ} \otimes_{\mathcal{O}} E \cong H_{\text {ord }}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda^{\vee}, E}^{\circ}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))} \otimes_{E \otimes_{\mathcal{O}} \Lambda^{\circ}, \vartheta^{-1}} E$,
and

$$
\begin{equation*}
H_{\text {ord }}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda^{\vee}, \mathcal{O}}^{\circ}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))} \otimes_{\mathrm{h}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, \mathcal{O}\right)_{\mathfrak{m} \vee, \xi^{\vee}}} E \tag{98}
\end{equation*}
$$

is of rank 1 over $E$, cf. (97). The image of

$$
H_{\mathrm{ord}}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda \vee, \mathcal{O}}^{\circ}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))} \otimes_{\mathrm{h}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, \mathcal{O}\right)_{\mathfrak{m}} \vee, \xi^{\vee}} \mathcal{O}
$$

is a canonical $\mathcal{O}$-lattice $\mathcal{O}_{\xi}$ in (98). Fix an $\mathcal{O}$-basis $b_{\xi} \in \mathcal{O}_{\xi}$. Identifying $L_{p}^{0,0}$ with its image under the canonical projection map

$$
\mathcal{H}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}} \vee \rightarrow \mathcal{H}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))},
$$

Theorems 5.9 and 6.1 imply
Theorem 6.3. Assume $p \nmid(n+1) n, F$ totally real, CM or Conjecture 4.9. Let $\mathfrak{m}$ denote a non-Eisenstein maximal ideal in $\mathfrak{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$. Then the element

$$
L_{p}^{0,0} \in \mathcal{H}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} V}^{\pi_{0}(H(\mathbf{R}))}
$$

has the following interpolation property. For every classical point

$$
\xi \in \operatorname{Spech}_{\mathrm{ord}}^{q_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)_{\mathfrak{m}}(\bar{E})
$$

of balanced weight $\lambda$ and Nebentypus $\vartheta$, such that $s_{0}=\frac{1}{2}$ is critical for $L\left(s, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}\right)$, we have

$$
\begin{aligned}
\Omega_{\xi, p}^{-1} \cdot \xi^{\vee} \circ \iota_{\lambda^{\vee} \vartheta^{-1}}\left(L_{p}^{0,0}\right) & =\int_{C_{F}\left(p^{\infty}\right)} d \mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}} \cdot b_{\xi} \\
& =\mathfrak{N}\left(\mathfrak{f}_{\vartheta}\right)^{\frac{(n+1) n(n-1)}{6}} \cdot \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} G\left(\vartheta_{\mu, \nu}\right) \cdot \frac{L\left(\frac{1}{2}, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}\right)}{\Omega_{\xi}} \cdot b_{\xi},
\end{aligned}
$$

where the second identity is valid whenever $\vartheta$ has fully supported constant conductor.
Here $\Omega_{\xi, p}^{-1} \in \mathcal{O}[\xi]^{\times}$is a p-adic period, $\Omega_{\xi} \in \mathbf{C}^{\times}$is a complex period and both these periods may be normalized in such a way that they are invariant under twists of $\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}$ by finite order Hecke characters $\chi$ unramified outside $p$.

By remark 5.8, $\Omega_{\xi, p}$ invariant under twisting $\xi$ with finite order Hecke characters $\chi$ unramified outside $p$, provided that we choose the test vectors in the construction of the measures for $\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}$ (or equivalently, the complex periods $\Omega_{ \pm, 0}$ ) coherently. That this is indeed possible is a consequence of Theorem 6.1.
Proof of Theorem 6.3. For simplicity of notation we may assume that $\xi$ is defined over $\mathcal{O}$, i.e. $\mathcal{O}[\xi]=\mathcal{O}$. Then by the preceeding discussion and the definition of $L_{p}^{0,0}$, we have

$$
\iota_{\lambda^{\vee} \vartheta^{-1}}^{\circ}\left(L_{p}^{0,0}\right)=\left[\phi \mapsto \int_{C_{F}\left(p^{\infty}\right)} d \mu_{E / \mathcal{O}}^{\lambda, 0}(\phi)\right],
$$

where $\phi \in H_{\mathrm{c}, \text { ord }}^{q_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right)_{\mathrm{m}}^{\pi_{\mathrm{m}}(H(\mathbf{R}))}$. We emphasize that any possible defect in the control theorem is absorbed by the commutativity of diagram (92).

Now application of $\xi^{\vee}$ sends $\iota_{\lambda^{\vee} \vartheta-1}^{\circ}\left(L_{p}^{0,0}\right)$ into the rank one $\mathcal{O}$-lattice $\mathcal{O} \cdot b_{\xi}$ in (98). By Poincaré-Pontryagin duality this operation corresponds to the precomposition of the map $\iota_{\lambda^{\vee}, \vartheta^{-1}}^{\circ}\left(L_{p}^{0,0}\right)(-)$ with the dual map

$$
\xi: \quad E / \mathcal{O} \rightarrow H_{\mathrm{c}, \mathrm{ord}}^{q_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, E / \mathcal{O}}\right)_{\mathfrak{m}}^{\pi_{0}(H(\mathbf{R}))} .
$$

However, appealing to Poincaré duality over $\mathcal{O}$ instead, we see that $\xi^{\vee} \circ \iota_{\lambda^{\vee} \vartheta^{-1}}\left(L_{p}^{0,0}\right)$ corresponds to the map

$$
H_{\mathrm{c}, \text { ord }}^{q_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}\right)_{\mathrm{m}}^{\pi_{0}(H(\mathbf{R}))} \rightarrow \mathcal{O}_{(0)}, \quad \phi \mapsto \int_{C_{F}\left(p^{\infty}\right)} d \mu_{\mathcal{O}}^{\lambda, 0}(\phi),
$$

evaluated at the element
$\xi \in \operatorname{Hom}_{\mathcal{O}}\left(H_{\text {ord }}^{q_{0}+l_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda^{\vee}, \mathcal{O}}^{\circ}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))}, \mathcal{O}\right)=H_{\mathrm{c}, \text { ord }}^{q_{0}}\left(\mathscr{X}\left(K_{\alpha^{\prime}, \alpha}\right) ; \underline{L}_{\lambda, \mathcal{O}}\right)_{\mathfrak{m}}^{\pi_{0}(H(\mathbf{R}))} /($ torsion $)$.

We conclude the proof with the observation that

$$
\int_{C_{F}\left(p^{\infty}\right)} d \mu_{\mathcal{O}}^{\lambda, 0}(\phi)=\widetilde{\Omega}_{\xi, p} \cdot \int_{C_{F}\left(p^{\infty}\right)} d \mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}} \cdot \eta_{0, \mathcal{O}}\left(v_{0}\right)
$$

with $\widetilde{\Omega}_{\xi, p} \in \mathcal{O}^{\times}$, since $\mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}}$ is only defined up to $p$-integral units, since the same applies to the image of the cohomology class $\phi_{\Pi_{\xi}} \widehat{\otimes} \Sigma_{\xi}$ after inverting $p$.

To obtain an integral version of Theorem 6.3 we need to formulate
Conjecture 6.4. For any non-Eisenstein maximal ideal $\mathfrak{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$,

$$
\mathcal{H}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))}
$$

is a free module of rank one over $\mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}$.
Remark 6.5. In the totally real case with $n=1$, Conjecture 6.4 is known under mild hypotheses on the residual representation $\bar{\rho}_{\mathfrak{m}}$ and $p$, cf. Theorem 4.6 in [14]. This result relies on an ' $\mathcal{R}=\mathbb{T}$ ' Theorem, which allows allows Dimitrov to interpret his non-abelian $p$-adic $L$-function as an element of a universal deformation ring.
Remark 6.6. Hansen-Thorne's Theorem 4.9 in [25] conditionally implies Conjecture 6.4 up to $\mathcal{O}$-torsion in the case $F=\mathbf{Q}$.

Following section 5 in [14] and assuming Conjecture 6.4, we define the universal $p$-adic $L$-function

$$
L_{p, \mathfrak{m}}^{\mathrm{univ}} \in \mathbf{h}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}
$$

as the image of $L_{p}^{0,0}$ under the isomorphism

$$
\mathcal{H}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}^{\pi_{0}(H(\mathbf{R}))} \rightarrow \mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}
$$

Let $\xi^{\vee} \in \operatorname{Spec} \mathrm{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda, \mathcal{O}\right)_{\mathfrak{m}} \vee(\mathcal{O})$ denote a classical point of weight $\lambda^{\vee}$ and Nebentypus $\vartheta^{-1}$. Assume that $\eta_{0}$ is admissible for $\lambda$ (in particular, $\lambda$ is balanced).

By Theorem 5.9, the canonical isogeny from Corollary 4.16,

$$
\begin{aligned}
& \mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}^{\vee}} / P_{\lambda^{\vee} \vartheta-1} \mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}} \\
& \quad \rightarrow \mathrm{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, \mathcal{O}\right)_{\mathfrak{m} \vee} / P_{\vartheta^{-1}} \mathrm{~h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, \mathcal{O}\right)_{\mathfrak{m} \vee},
\end{aligned}
$$

composed with $\xi^{-1}$, maps $L_{p, \mathfrak{m}}^{\text {univ }}$ onto

$$
\xi^{-1} \circ \lambda^{\vee} \vartheta^{-1}\left(L_{p, \mathfrak{m}}^{\mathrm{univ}}\right)=\Omega_{\xi, p} \cdot \int_{C_{F}\left(p^{\infty}\right)} d \mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}}
$$

where $\Omega_{\xi, p} \in \mathcal{O}^{\times}$is again a $p$-adic period. We obtain
Theorem 6.7. Assume $p \nmid(n+1) n, F$ totally real, CM or Conjecture 4.9. Let $\mathfrak{m}$ denote a non-Eisenstein maximal ideal in $\mathfrak{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$ for which Conjecture 6.4 holds. Then there exists an element

$$
L_{p, \mathfrak{m}}^{\mathrm{univ}} \in \mathbf{h}_{\mathrm{ord}}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m} \vee}
$$

with the following interpolation property. For every classical point

$$
\xi^{\vee} \in \operatorname{Spech}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, \mathcal{O}\right)_{\mathfrak{m}^{\vee}}(\overline{\mathcal{O}})
$$

of balanced weight $\lambda^{\vee}$ and Nebentypus $\vartheta^{-1}$, such that $s_{0}=\frac{1}{2}$ is critical for $L\left(s, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}\right)$, we have

$$
\begin{aligned}
\Omega_{\xi, p}^{-1} \cdot \xi^{\vee} \circ \lambda^{\vee} \vartheta^{-1}\left(L_{p, \mathfrak{m}}^{\mathrm{univ}}\right) & =\int_{C_{F}\left(p^{\infty}\right)} d \mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}} \\
& =\mathfrak{N}\left(\mathfrak{f}_{\vartheta}\right)^{\frac{(n+1) n(n-1)}{6}} \cdot \prod_{\mu=1}^{n} \prod_{\nu=1}^{\mu} G\left(\vartheta_{\mu, \nu}\right) \cdot \frac{L\left(\frac{1}{2}, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}\right)}{\Omega_{\xi}}
\end{aligned}
$$

where the second identity is valid whenever $\vartheta$ has fully supported constant conductor.
Here $\Omega_{\xi, p}^{-1} \in \mathcal{O}[\xi]^{\times}$is a p-adic period, $\Omega_{\xi} \in \mathbf{C}^{\times}$is a complex period and both these periods may be normalized in such a way that they are invariant under twists of $\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}$ by finite order Hecke characters $\chi$ unramified outside $p$.
Remark 6.8. In general, twisting by finite order characters also twists the residual representation by a character of the finite torsion subgroup $\Delta$ of $C_{F}\left(p^{\infty}\right)$. However, there are only finitely many such twists and hence only a finite set of corresponding non-Eisenstein maximal ideals $\mathfrak{m}_{1}, \ldots, \mathfrak{m}_{\delta},(1 \leq \delta \leq|\Delta[1 / p]|)$, corresponding to finite order twists with characters unramified outside $p \infty$. Let

$$
\mathfrak{n}:=\bigcap_{i=1}^{\delta} \mathfrak{m}_{i}
$$

Our construction generalizes to Hecke algebras localized at $\mathfrak{n}$, i.e. the product of their localizations at each $\mathfrak{m}_{i}$, provided that Conjecture 6.4 holds for every $\mathfrak{m}_{i}$. By the Chinese Remainder Theorem, we obtain an isomorphism

$$
\mathcal{H}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{n}^{-1}} \cong \mathcal{O}\left[\pi_{0}(H(\mathbf{R}))\right] \otimes_{\mathcal{O}} \mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{n}^{-1}},
$$

which allows us to define a $p$-adic $L$-function

$$
L_{p, \mathfrak{n}}^{\text {univ }} \in \mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{n}^{-1}},
$$

with the corresponding interpolation property for classical points in

$$
\operatorname{Spech}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\alpha^{\prime}, \alpha} ; \lambda^{\vee}, \mathcal{O}\right)_{\mathfrak{n}^{-1}}(\overline{\mathcal{O}}) .
$$

Remark 6.9. The interest in considering $L_{p, n}^{\text {univ }}$ is that it allows us to recover the full abelian $p$-adic $L$-functions from Theorem 6.1 as follows.

The universal nearly ordinary Hecke algebra $\mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{n}^{-1}}$ carries a canonical $\mathcal{O}\left[\left[C_{F}\left(p^{\infty}\right)\right]\right]$-algebra structure and any $\mathcal{O}\left[\left[C_{F}\left(p^{\infty}\right)\right]\right]$-algebra homomorphism

$$
\Xi: \quad \mathbf{h}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{n}^{-1}} \rightarrow \mathcal{O}\left[\left[C_{F}\left(p^{\infty}\right)\right]\right]
$$

for which the composition with the trivial character of $C_{F}\left(p^{\infty}\right)$ yields a classical point $\xi^{\vee}$ of balanced weight, sends $L_{p, n}^{\text {univ }}$ to $\mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}}$. Likewise, we can recover $\mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}}$ from $L_{p}^{0,0}$ localized at $\mathfrak{n}$.
6.3. Applications to non-vanishing of central $L$-values. As an application of Theorem 6.1 we prove

Theorem 6.10. Let $F$ be a number field, $\Pi \widehat{\otimes} \Sigma$ be a cuspidal automorphic representation of $G(\mathbf{A})$ satisfying the hypotheses of Theorem 6.1. Assume that $L(s, \Pi \widehat{\otimes} \Sigma)$ admits at least two critical values and that $w_{\lambda}$ is even. Then the p-adic measure $\mu_{\Pi \widehat{\otimes} \Sigma, \lambda, w_{\lambda} / 2} \in$ $\mathcal{O}\left[\left[C_{F}\left(p^{\infty}\right)\right]\right]$ is non-zero. Furthermore, for each $\chi \in \mathscr{X}_{F}^{0}$ the measure

$$
\left.\chi \cdot \mu_{\Pi \widehat{\otimes} \Sigma, \lambda, w_{\lambda} / 2}\right|_{C_{Q}^{\text {cyc }}\left(p^{\infty}\right)} \in \mathcal{O}[\chi]\left[\left[C_{F}^{\text {cyc }}\left(p^{\infty}\right)\right]\right]
$$

is non-zero.
Proof. We have an identity

$$
\begin{equation*}
\left.\langle\cdot\rangle_{F} \cdot \omega_{F} \cdot \chi \cdot \mu_{\Pi \widehat{\otimes} \Sigma, \lambda, w_{\lambda} / 2}\right|_{C_{F}^{\text {čc }}\left(p^{\infty}\right)}=\left.\chi \cdot \mu_{\Pi \widehat{\otimes} \Sigma, \lambda, 1+w_{\lambda} / 2}\right|_{C_{F}^{\text {cyc }}\left(p^{\infty}\right)} \tag{99}
\end{equation*}
$$

of measures on $C_{F}^{\text {cyc }}\left(p^{\infty}\right)$. We know that the measure

$$
\left.\chi \cdot \mu_{\Pi \widehat{\otimes} \Sigma, \lambda, 1+w_{\lambda} / 2}\right|_{C_{F}^{\text {cyc }}\left(p^{\infty}\right)}
$$

is non-zero by Theorem 6.1, because the complex $L$-function in the interpolation formula is non-zero whenever the real part of $s$ is larger than $\frac{w_{\lambda}}{2}+1$, cf. [46]. Whence $\left.\chi \cdot \mu_{\Pi \widehat{\otimes} \Sigma, \lambda_{\lambda}, w_{\lambda} / 2}\right|_{C_{F}^{\text {cyc }}\left(p^{\infty}\right)}$ is non-zero.

Corollary 6.11. For every finite order Hecke character $\chi$ of $F$ we have

$$
L\left(\frac{1+w_{\lambda}}{2}, \Pi \widehat{\otimes} \Sigma \otimes \chi \chi^{\prime}\right) \neq 0
$$

for all but finitely many characters $\chi^{\prime}: C_{F}^{\text {cyc }}\left(p^{\infty}\right) \rightarrow \mathbf{C}^{\times}$.
 statement about the central $L$-value. By [79] it is known that near-central critical values are always non-zero.

Proof. Remark that if $\Pi \widehat{\otimes} \Sigma$ satisfies the hypotheses of Theorem 6.1 , then so does $\Pi \widehat{\otimes} \Sigma \otimes \chi$. Assume without loss of generality that $\chi$ takes values in $E$. By the above theorem,

$$
\begin{aligned}
0 \neq\left.\mu_{\Pi \widehat{\otimes} \Sigma \otimes \chi, \lambda, w_{\lambda} / 2}\right|_{F} ^{\mathrm{cyc}}\left(p^{\infty}\right) & \in \mathcal{O}\left[\left[C_{F}^{\mathrm{cyc}}\left(p^{\infty}\right)\right]\right] \\
& \cong \mathcal{O}[[X]] .
\end{aligned}
$$

By the Weierstrass Peparation Theorem, a non-zero power series in one variable over $\mathcal{O}$ admits only finitely many zeroes and the claim follows.
6.4. Non-vanishing via non-abelian deformations. As an application of Theorems 6.1 and 5.9 we prove (independently of Conjecture 6.4 ),

Theorem 6.13. Let $p \nmid(n+1) n, F$ totally real, $C M$ or assume the validity of Conjecture 4.9 over $F$. Let $\mathfrak{m}$ denote a non-Eisenstein maximal ideal in $\mathfrak{m}$ in $\mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)$. Assume the existence of a classical point $\xi \in \operatorname{Spec} \mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}^{-1}}$ of balanced weight such that $L\left(s, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}\right)$ admits at least two critical values.

Then the image $L_{p, \mathfrak{m}}$ of $L_{p}^{0,0}$ in $\mathcal{H}_{\text {ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}^{-1}}^{\pi_{0}(H(\mathbf{R}))}$ is non-zero. Moreover, its projection to

$$
\begin{equation*}
\mathcal{H}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}^{-1}}^{\pi_{0}(H(\mathbf{R}))} \otimes_{\mathbf{h}_{\mathrm{ord}}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}^{-1}, \xi^{\prime}}} \overline{\mathcal{O}} \tag{100}
\end{equation*}
$$

is non-zero for some classical point $\xi^{\prime} \in \operatorname{Spec} \mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}}$ of balanced weight.
Proof. By Theorem 5.5, we may assume without loss of generality that $\eta_{0}$ is admissible for the weight $\lambda$ of $\xi$.

We import the notation of Remark 6.8, i.e. let $\mathfrak{m}_{i}, 1 \leq i \leq \delta$, denote the non-Eisenstein maximal ideals corresponding to finite order twists of the residual representation $\bar{\rho}_{\mathfrak{m}}$ by characters unramified outside $p \infty$ and of prime-to- $p$ order.

From the proof of Theorem 6.10 we deduce that the measure $\mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}}$ is non-zero on every translate of $C_{F}^{\text {cyc }}\left(p^{\infty}\right)$ in $C_{F}\left(p^{\infty}\right)$.

By Theorem 5.9 and Remark 5.8, the non-vanishing of $\mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}}$ on every translate of the cyclotomic line implies that the image $L_{p, \mathfrak{m}_{i}}^{0,0}$ of $L_{p}^{0,0}$ in $\mathcal{H}_{\mathrm{c}, \text { ord }}^{q_{0}+l_{0}}\left(K_{\infty, \infty} ; \underline{L}_{\lambda, \mathcal{O}}^{\circ}\right)_{\mathfrak{m}_{i}^{\vee}}^{\pi_{0}(H(\mathbf{R}))}$ is non-zero for every $1 \leq i \leq \delta$ and also non-zero in (100) for some cyclotomic twist $\xi^{\prime}$ of $\xi$ by Theorem 6.3.

Corollary 6.14. Under the assumptions of Theorem 6.13, for every dominant weight $\lambda$ and every $\eta_{j}$ admissible for $\lambda$, the restriction of $\mu_{\alpha}^{\lambda, j}$ to the localization

$$
\mu_{\alpha, \mathfrak{m}}^{\lambda, j}: \quad H_{\mathrm{c}, \text { ord }}^{\operatorname{dim} \mathscr{Y}}\left(\mathscr{X}\left(K_{\alpha, \alpha}\right) ; \underline{L}_{\lambda, p^{-\alpha} \mathcal{O} / \mathcal{O}}\right)_{\mathfrak{m}}^{\pi_{0}(H(\mathbf{R}))} \rightarrow\left(p^{-\alpha} \mathcal{O} / \mathcal{O}\right)_{(j)}\left[\left[C\left(p^{\infty}\right)\right]\right]
$$

is non-zero for every sufficiently large $\alpha$.
Remark 6.15. Corollary 6.14 implies the existence of non-zero $p$-adic $L$-functions for torsion classes in the absence of classical points.

Corollary 6.16. Under the assumptions of Theorem 6.13, let $\lambda$ denote a balanced cohomological weight for which $\eta_{0}$ is the (unique) admissible character. Let furthermore

$$
\mathcal{X} \subseteq \operatorname{Spec} \mathbf{h}_{\text {ord }}\left(K_{\infty, \infty} ; \mathcal{O}\right)_{\mathfrak{m}}(\bar{E})
$$

denote an irreducible component containing the classical point $\xi$ from Theorem 6.13 with the property that its subset $\mathcal{X}_{\mathrm{cl}}^{\lambda} \subseteq \mathcal{X}$ of classical points of balanced weight $\lambda$ is Zariski-dense in $\mathcal{X}$.

Then there exists $\xi \in \mathcal{X}_{\mathrm{cl}}^{\lambda}$ such that $\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}$ is cuspidal and a Hecke character $\chi$ of finite order and unramified outside $p \infty$, satisfying

$$
L\left(\frac{1}{2}, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi} \otimes \chi\right) \neq 0
$$

Moreover,

$$
L\left(\frac{1}{2}, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi} \otimes \chi \chi^{\prime}\right) \neq 0
$$

for all but finitely many Hecke characters $\chi^{\prime} \in \mathscr{X}_{F}^{\text {cyc }}$.
Proof. Assume to the contrary that

$$
L\left(\frac{1}{2}, \Pi_{\xi} \widehat{\otimes} \Sigma_{\xi} \otimes \chi\right)=0
$$

for all $\xi \in \mathcal{X}_{\mathrm{cl}}^{\lambda}$ and all $\chi$ unramified outside $p \infty$. By Theorem 6.1 this implies

$$
\mu_{\Pi_{\xi} \widehat{\otimes} \Sigma_{\xi}}=0
$$

for all $\xi \in \mathcal{X}_{\mathrm{cl}}^{\lambda}$. Therefore, the specialization of $L_{p}^{0,0}$ vanishes at all classical $\xi$ of weight $\lambda$ in $\mathcal{X}$ (cf. Theorem 6.3).

By the Zariski-density of $\mathcal{X}_{\mathrm{cl}}^{\lambda}$, this implies that the evaluation of $L_{p}^{0,0}$ vanishes at all points in $\mathcal{X}$. This contradicts Theorem 6.13. The generic non-vanishing statement on the cyclotomic line follows as in the proof of Theorem 6.10.

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