

GEHRING LINK PROBLEM, FOCAL RADIUS AND OVER-TORICAL WIDTH

JIAN GE

ABSTRACT. In this note, we study the Gehring link problem in the round sphere, which motivates our study of the width of a band in positively curved manifolds. Using the same idea, we are able to prove a sphere theorem for hypersurface in the round \mathbf{S}^n provided that its normal injectivity radius is large. A rigidity theorem for Clifford hypersurfaces in \mathbf{S}^n is also proved. The 3-dimensional case of our theorems confirm two conjectures raised by Gromov in [Gro18].

0. INTRODUCTION

Let $(Y = \mathbb{T}^2 \times [0, 1], g)$ be a Riemannian torical band, where \mathbb{T}^2 is the 2-dimensional torus. In [Gro18], Gromov studied the width of Y assuming the scalar curvature of Y is bounded from below. Here the width is defined to be the distance between two boundary components of Y . He also conjectured that the upper bound of the width of those bands which can be isometrically embedded in the round 3-sphere, is $\pi/2$. A related conjecture posed in [Gro18] is the normal injectivity radius of an embedded torus in the round n -sphere.

Our starting point is to find a geometric intuition behind the conjectural upper bound $\pi/2$ of the width. Basically all the proofs in this paper can be summarized as:

“If two sets are linked in the sphere, they cannot be more than $\pi/2$ -apart.”

The following example serves as our motivation. We take $\mathbb{T}_{\text{Cl}}^2 \subset \mathbf{S}^3$, where $\mathbf{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ is the unit 3-sphere and the 2-dimensional Clifford Torus is defined by

$$\mathbb{T}_{\text{Cl}}^2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2| = 1/\sqrt{2}\}.$$

Let $B(\mathbb{T}_{\text{Cl}}^2, r)$ be the r -tubular neighborhood of \mathbb{T}_{Cl}^2 inside \mathbf{S}^3 . It is clear that $\mathbf{S}^3 \setminus B(\mathbb{T}_{\text{Cl}}^2, r)$ has two connected component if $r < \pi/4$, each of which is a solid torus. The central circles of these solid tori form a Hopf link in \mathbf{S}^3 . The distance between this two circles is $\pi/2$. This reminds us the classical Gehring linking problem in \mathbb{R}^3 .

2000 *Mathematics Subject Classification.* Primary: 53C23; Secondary: 51K10.

Key words and phrases. Normal injectivity radius, focal radius, over-toric band, sphere theorem, Gehring link problem, width,

Partially supported by National Key R&D Program of China grant 2020YFA0712800 and the Fundamental Research Funds for the Central Universities.

Theorem 0.1 ([Ort75], [BS83]). *Let A and B be two closed curves smoothly embedded in \mathbb{R}^3 . Suppose they are linked and $d(A, B) \geq 1$, then the lengths of A and B must be greater or equal to 2π .*

We now state our first result.

Theorem 0.2 (Gehring Link Problem in 3-sphere). *Let A, B be two disjoint linked Jordan curves in the unit 3-sphere \mathbf{S}^3 . Then*

$$d(A, B) \leq \pi/2,$$

with equality holds if and only if A and B are the dual great circles in a Hopf fibration.

Our method of the proof can be easily generalized to higher dimension. The classical Gehring Link problem in \mathbb{R}^n is proved in [Gag80].

Theorem 0.3 (Spherical Gehring Link Problem). *Let A^k and B^l be two embedded spheres of dimension k and l in \mathbf{S}^n . Suppose that A and B are linked, then we have*

$$d(A, B) \leq \pi/2,$$

*with the equality holds if and only if A and B are the dual great sub-spheres in \mathbf{S}^n , i.e. $\mathbf{S}^n = A * B$, where $*$ denotes the spherical join.*

[Theorem 0.2](#) explains heuristically why the width of a torical band cannot be too large: the complement of the band in \mathbf{S}^3 are ‘linked’. However, this is only partially correct. In general the complement of a torical band might not retract to a link, therefore ‘linking’ is not well defined. For example, let K be the figure-eight knot embedded in \mathbf{S}^3 . Let $R > 0$ such that the R -tubular neighborhood $B(K, R)$ is an embedded solid torus. Consider the embedded torical band $TB := B(K, R) \setminus B(K, R/2)$, its complement in \mathbf{S}^3 clearly has two components. One of the components is a solid torus, therefore it can be retracted to the knot K . The other component is a once-punctured torus bundle over a circle, which cannot be retracted to a circle. Nevertheless, we can still make use the idea of ‘being linked’ as an obstruction to have large width. We call this obstruction ‘boundary irreducible’ see [Definition 2.1](#). In fact, using this idea we are able to prove the following theorem, which confirms a conjecture of Gromov in [Gro18].

Theorem 0.4. *The over-torical width of \mathbf{S}^3 denoted by $\text{width}_{\hat{\tau}}(\mathbf{S}^3)$, is $\pi/2$.*

Another estimate can be drawn from the proof of [Theorem 0.3](#) is the estimate of normal injectivity radius of torus in \mathbf{S}^3 .

Definition 0.5. *Let $\Sigma^2 \subset \mathbf{S}^3$ be a smoothly embedded closed surface. The normal injectivity radius of $\Sigma \subset \mathbf{S}^3$ is the largest number $r > 0$ such that*

$$\exp : \{(p, v) \in \nu\Sigma \mid p \in \Sigma, |v| < r\} \rightarrow \mathbf{S}^3$$

is a diffeomorphism onto its image, where $\nu\Sigma$ is the normal bundle of Σ .

The normal injectivity radius of $\Sigma \subset \mathbf{S}^3$ will be denoted by $\text{rad}^{\odot}(\Sigma) = \text{rad}^{\odot}(M \subset \mathbf{S}^3)$. The normal injectivity radius of a smooth submanifold

N^k in M^n can be defined similarly. We remark that in [Gro18] $\text{rad}^\odot(\Sigma)$ is called the *normal focal radius*.

By the standard comparison argument, it is well known that any hypersurface Σ in \mathbf{S}^3 has normal injectivity radius $\leq \pi/2$. In fact if the normal injectivity radius of a closed surface Σ^2 in \mathbf{S}^3 is equal to $\pi/2$, Σ^2 has to be the equatorial 2-sphere, cf. [GW18]. The following question is asked in [Gro18]: Let Σ^n be a smoothly embedded n -torus in \mathbf{S}^{2n-1} , what is the largest possible normal injectivity radius? Gromov conjectured that the Clifford Torus \mathbb{T}_{Cl}^n is the only torus realizing the conjectured upper bound $\arcsin(1/\sqrt{n})$. The upper bound for $n = 2$ actually follows from Corollary C in [GW20]. We give a different proof of this result and study the rigidity case.

Theorem 0.6. *Let Σ be a smoothly embedded 2-torus in \mathbf{S}^3 . Then*

$$\text{rad}^\odot(\Sigma) \leq \frac{\pi}{4},$$

with equality holds if and only if Σ is a Clifford Torus \mathbb{T}_{Cl}^2 .

In fact, we prove the following stronger result

Theorem 0.7. *Let Σ^{n-1} be an orientable hypersurface embedded in \mathbf{S}^n . Suppose Σ is not homeomorphic to \mathbf{S}^{n-1} , then $\text{rad}^\odot(\Sigma) \leq \pi/4$, with equality holds if and only if Σ is isometric to the Clifford hypersurface $\mathbf{S}^k(1/\sqrt{2}) \times \mathbf{S}^l(1/\sqrt{2})$ for some $k, l \in \mathbb{N}$ such that $n = k + l + 1$.*

Next theorem shows that spheres are the only hypersurfaces with large normal injectivity radius in a positively curved manifold.

Theorem 0.8 (A Topological Sphere Theorem). *Let (M^n, g) be a simply connected closed Riemannian manifold with $\text{sec} \geq 1$ and Σ^{n-1} be a smoothly embedded orientable hypersurface in M . Suppose $\text{rad}^\odot(\Sigma) > \frac{\pi}{4}$. Then M is homeomorphic to \mathbf{S}^n and Σ is homeomorphic to \mathbf{S}^{n-1} .*

Remark 0.9. By h -cobordism theorem, $(n - 1)$ -dimensional exotic sphere cannot be embedded in \mathbb{R}^n for $n \neq 5$. Therefore Σ is diffeomorphic to \mathbf{S}^{n-1} for $n \neq 5$ in Theorem 0.8.

In Section 1 we prove the Gehring Link problem in sphere. The ideas of the proof are developed further in Section 2, where we provide the proofs of Theorem 0.8 and Theorem 0.4. Another short geometric proof of Theorem 0.6 is provided in Section 3.

Acknowledgement: It is my great pleasure to thank Professor Misha Gromov for his comments and interests in our work and Professor Luis Guijarro for comments after reading the first draft of this paper. I am very grateful to the anonymous referee for the detailed suggestions which greatly improved the clarity of the paper. I would like to thank Professor Yuguang Shi for bringing J. Zhu's paper [Zhu21] to my attention where the author estimated the 3-dimension width using a completely different method.

1. PROOF OF THE SPHERICAL GEHRING LINK PROBLEM

Let A^k, B^l be two submanifolds of \mathbf{S}^n , we call A and B are *unlinked* if there exists an embedded topological $(n-1)$ -sphere S^{n-1} in \mathbf{S}^n such that A and B lie in complementary hemispheres; otherwise A and B are called *linked*. Note that we do not require $k+l+1=n$ in the definition of ‘linked’, it is only required if we want to define the ‘linking number’.

The key idea of the proof goes back to Grove-Shiohama’s proof of the Diameter Sphere Theorem, cf [GS77]. Namely we have the following

Proposition 1.1. *Let (M, g) be a closed Riemannian manifold with sectional curvature $\sec(g) \geq 1$. Suppose $\text{diam}(M, g) \geq \pi/2$. Then for any point $x \in M$ and $r \geq \pi/2$, the set $M \setminus B(x, r)$ is either empty or a k -dimensional topological manifold with strictly convex (possibly empty) boundary whose interior is smooth and totally geodesic, where $k \leq n$.*

For any closed set $A \subset M$, it is clear that for $r \geq \pi/2$

$$N := M \setminus B(A, r) = \bigcap_{x \in A} \{M \setminus B(x, r)\}.$$

It follows that N is either empty or a k -dimensional topological manifold with strictly convex (possibly empty) boundary whose interior is smooth and totally geodesic. Moreover if ∂N is nonempty, N is homeomorphic to the standard k -disk.

Proof of Theorem 0.3 . Suppose $d(A, B) > \pi/2$, then $B \subset \mathbf{S}^n \setminus B(A, \pi/2)$. If A is not a great sub-sphere $\mathbf{S}^k \subset \mathbf{S}^n$, then the set $N := \mathbf{S}^n \setminus B(A, \pi/2)$ is a connected n -dimensional convex set with nonempty boundary. It follows that N is homeomorphic to an n -disk, therefore ∂N is homeomorphic to the sphere S^{n-1} . But $B \subset \mathbf{S}^n \setminus B(A, \pi/2)$. Therefore it contradicts to the assumption that A and B are linked in \mathbf{S}^3 . It follows that $d(A, B) \leq \pi/2$.

Suppose $d(A, B) = \pi/2$, then we set

$$A^{anti} := \mathbf{S}^n \setminus B(A, \pi/2).$$

Clearly A^{anti} is nonempty and of dimension $< n$. If A^{anti} has nonempty boundary, A^{anti} is homeomorphic to a k -disk. Therefore we can choose $\varepsilon > 0$ small enough such that $\Sigma := \partial B(A^{anti}, \varepsilon)$ is homeomorphic to an $(n-1)$ -sphere. Since

$$B \subset A^{anti} \subset B(A^{anti}, \varepsilon),$$

it follows that Σ separates B from A , a contradiction. Therefore A^{anti} must have empty boundary. Since A^{anti} is totally geodesic, B is isometric to \mathbf{S}^l for some $l \in \{1, \dots, (n-1)\}$. Apply the same discussion to B , we know A is isometric to \mathbf{S}^k for some $k \in \{1, \dots, (n-1)\}$. Since A and B are linked, $n = k + l + 1$. \square

The proof given above also works for the case when the sphere \mathbf{S}^n is replaced by a closed Riemannian manifold (M, g) with $\sec \geq 1$.

2. THE COMPLEMENT OF A NON-SPHERICAL BAND ARE LINKED

One crucial step in the proof of [Theorem 0.3](#) in the previous section is to produce an embedded sphere S^{n-1} in \mathbf{S}^n which separates A and B , provided that they are more than $\pi/2$ apart. In this case, the $(n-1)$ -sphere separates \mathbf{S}^n into two disks that contain A and B respectively. It follows that this sphere does not bound any disk in $\mathbf{S}^n \setminus \{A, B\}$, therefore it represents a nontrivial element in $\pi_{n-1}(\mathbf{S}^n \setminus \{A, B\})$. This puts a strong restriction on the topology of the complement of A and B . In this section, we focus on the complement instead of the sets A and B .

Let's recall several definitions in [\[Gro18\]](#). A *band* is a manifold Y with two distinguished disjoint non-empty subsets ∂Y_- and ∂Y_+ in the boundary ∂Y :

$$\partial Y = \partial Y_- \cup \partial Y_+.$$

A band Y is called *proper* if ∂Y_{\pm} are unions of connected components of ∂Y . A proper band Y is called *over-torical* if there exists a map

$$f : Y \rightarrow \underline{Y} := \mathbb{T}^{n-1} \times [0, 1],$$

with nonzero degree respecting the boundaries: $\partial Y_{\pm} \rightarrow \partial \underline{Y}_{\pm}$. We introduce the following definition:

Definition 2.1. *Let Y be an n -dimensional proper band. We call Y boundary reducible if there exists an embedded $(n-1)$ -sphere $S^{n-1} \subset Y$, such that S separates ∂Y_- from ∂Y_+ . Otherwise Y is called boundary irreducible.*

For a proper over-torical band, since $\pi_2(\underline{Y}) = 0$, we have the following observation:

Lemma 2.2. *Let Y be a 3-dimensional proper over-torical band, then Y is boundary irreducible.*

Remark 2.3. Note that if Y is boundary irreducible, it is still possible to find an embedded $(n-1)$ -sphere in Y that does not bound an n -ball. For example in 3-dimension, let Y be the connected sum of $\mathbb{T}^2 \times [0, 1]$ with a lens space. Then Y is boundary irreducible but reducible in the classical sense.

Proposition 2.4. *Let (M^n, g) be a closed orientable n -dimensional manifold with $\text{sec} \geq 1$. Let $Y = \Sigma^{n-1} \times [0, 1]$ be a band isometrically embedded in M , where Σ is a closed orientable $(n-1)$ -dimensional manifold. We will identify Y with its image in M . Let Y_- and Y_+ be the boundaries $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ of Y . Suppose $R := d(Y_-, Y_+) > \pi/2$, then Σ is homeomorphic to \mathbf{S}^{n-1} .*

Proof. By the Diameter Sphere Theorem (cf. [\[GS77\]](#)), we know M is homeomorphic to the n -sphere. For ε small enough such that $R - \varepsilon > \pi/2$, we set

$$N := M \setminus B(Y_-, R - \varepsilon).$$

Clearly N is homeomorphic to the standard n -disk and ∂N is homeomorphic to the $(n-1)$ -dimensional sphere. Let B_0 and B_1 be the two connected

components of the closure of the set $M \setminus Y$, where we assume $\partial B_0 = Y_-$ and $\partial B_1 = Y_+$. The manifold M can be decomposed as follows

$$M = B_0 \cup \Sigma \times [0, 1] \cup B_1. \quad (2.1)$$

Set

$$X_t = \Sigma \times [t, 1] \cup B_1.$$

Since N is compact, there exists $0 < t_0 < s_0 < 1$ such that

$$X_{t_0} \supset N \supset X_{s_0}. \quad (2.2)$$

To calculate the homotopy groups of X_{t_0} , we notice that any map $\phi : S^k \rightarrow X_{t_0}$ can be homotopic to a map $\phi' : S^k \rightarrow X_{s_0} \subset N$. Since N is contractible, it follows that X_{t_0} and B_1 are contractible. Similarly we can prove B_0 is contractible. Therefore Σ^{n-1} is an embedded homology sphere.

To show Σ is homeomorphic to the $(n-1)$ -sphere, the argument can be taken with minor changes from Mazur's proof of Corollary 4 in [Maz61]. We include it only for completeness. Reparametrizing the interval $[0, 1]$ in the decomposition (2.1) if necessary, we can assume $t_0 > 1/2$ in (2.2). Let $\xi : [0, 1] \rightarrow [0, 1]$ be the piecewise linear function that maps $[0, 1/2]$ to $[0, 3/4]$ and maps $[1/2, 1]$ to $[3/4, 1]$. Abusing the notation, let $\xi : M \rightarrow M$ be the homeomorphism defined by

$$\begin{aligned} \xi_{B_i} &= id_{B_i} \text{ is the identity map for } i = 0, 1. \\ \xi(x, t) &= (x, \xi(t)), \text{ for } (x, t) \in \Sigma \times [0, 1]. \end{aligned}$$

Clearly $\xi^k(\partial N)$ pushes ∂N towards B_1 for $k > 1$, and therefore

$$B_1 = \bigcap_{k \in \mathbb{N}} \xi^k(N).$$

Since $\xi^k(\partial N)$ is homeomorphic to S^{n-1} and bound a closed n -disk, it follows that B_1 is cellular cf. [Bro60]. Same arguments can be applied to B_0 . Therefore M is homeomorphic to the suspension $\mathbf{S}^0 * \Sigma^{n-1}$, by pinching B_0 and B_1 to points. This is only possible when Σ^{n-1} is homeomorphic to the sphere. □

Clearly Proposition 2.4 and the Diameter Sphere Theorem (cf. [GS77]) imply Theorem 0.8 immediately. Now we move to the study of the non-spherical band.

Proof of Theorem 0.7. We only have to consider the case when $\text{rad}^\odot(\Sigma) = \pi/4$. Let's consider the set

$$N := \mathbf{S}^n \setminus B(\Sigma, \pi/4).$$

Since Σ is orientable, the set N has two connected components, denote them by $\{A, B\}$. Since any geodesic connecting A and B must intersect with Σ , it follows that

$$d(A, B) \geq d(A, \Sigma) + d(B, \Sigma) = \pi/2.$$

By the assumption Σ is not homeomorphic to \mathbf{S}^{n-1} , it follows from Proposition 2.4 that $d(A, B) = \pi/2$. Therefore A and B are both totally geodesic

submanifolds in \mathbf{S}^n with empty boundary. Namely A and B are great spheres in \mathbf{S}^n :

$$A = \mathbf{S}^k \text{ and } B = \mathbf{S}^l$$

for some $k, l \in \{1, \dots, n-1\}$ and $k+l+1 \leq n$. On the other hand

$$A = \mathbf{S}^n \setminus (B(B, \pi/2)),$$

and vice versa, then $k+l+1 = n$. i.e. A and B are the dual great spheres in \mathbf{S}^n . It follows that Σ is the Clifford hypersurface $\mathbf{S}^k(1/\sqrt{2}) \times \mathbf{S}^l(1/\sqrt{2})$. \square

Recall that the over-torical width of \mathbf{S}^3 , denoted by $\text{width}_{\hat{\tau}}(\mathbf{S}^3)$ is defined as the supremum of numbers d such that there exists a proper over-torical band Y of width d and an isometric immersion

$$\phi : Y \rightarrow \mathbf{S}^3.$$

Since the r -neighborhood of Clifford Torus in \mathbf{S}^3 provides a torus band of width arbitrarily close to $\pi/2$, $\text{width}_{\hat{\tau}}(\mathbf{S}^3) \geq \pi/2$. To prove [Theorem 0.4](#), it suffices to show $\text{width}_{\hat{\tau}}(\mathbf{S}^3) \leq \pi/2$. Let Y be as above, we have

Proposition 2.5. *The width of Y is less than or equal to $\pi/2$.*

Proof. Suppose the width $d := d(Y_-, Y_+) > \pi/2$, where we equipped Y with the pull back of the round metric on the sphere. Consider the distance function $\rho : Y \rightarrow \mathbb{R}$, defined by

$$\rho(x) = d(x, Y_-),$$

where the distance d is defined by the shortest curve connecting x to Y_- . Since we make no assumption on the convexity of Y_{\pm} , such a curve might intersect the boundary Y_+ . However, if the width $d = d(Y_-, Y_+) > \pi/2$, we know for any point

$$x \in \Sigma := \rho^{-1}\left(\frac{\pi/2 + d}{2}\right),$$

any geodesic connecting x to Y_- that realizes $\rho(x)$ does not intersect Y_+ . Since Y has constant curvature 1, the standard comparison argument shows that Σ is locally strictly convex in Y . Therefore under the isometric immersion $\phi : Y \rightarrow \mathbf{S}^3$, its image $\bar{\Sigma} := \phi(\Sigma) \subset \mathbf{S}^3$ is also locally strictly convex in \mathbf{S}^3 . By the classical theorem of Hadamard (cf. [[Had97](#), [Hop89](#)]), all such surfaces must be embedded, hence $\bar{\Sigma}$ is an embedded \mathbf{S}^2 . Since ϕ is an immersion, it follows that ϕ is a covering map. Therefore Σ is a disjoint union of embedded separating 2-spheres in Y (It might be disjoint only if Y_- or Y_+ has more than one connected components). By connecting these spheres via cylinders we get an embedded separating 2-sphere in Y . This contradict to [Lemma 2.2](#). Therefore $d \leq \pi/2$. \square

3. YET ANOTHER PROOF OF THE [THEOREM 0.6](#)

In this section, we give another proof of [Theorem 0.6](#) based on Weyl's Tube formula and the solution of Willmore Conjecture [[MN14](#)]. Let $\Sigma^2 \subset \mathbf{S}^3$ be an embedded 2-torus with normal injectivity radius $r \in [\pi/4, \pi/2]$. Let

κ_1, κ_2 be the principal curvatures of Σ in \mathbf{S}^3 , then the Gauss-Bonnet formula gives:

$$\int_{\Sigma} (1 + \kappa_1 \kappa_2) d\mu = 0, \quad (3.1)$$

since the Euler number of Σ vanishes. It follows from the definition of normal injectivity radius that

$$\text{vol}(B(\Sigma, r)) \leq \text{vol}(\mathbf{S}^3) = 2\pi^2. \quad (3.2)$$

By the Weyl's Tube formula [Lemma 4.1](#), which will be proved in next section, and the fact that $\sin(2r) \geq \cot(r)$ if $r \in [\pi/4, \pi/2]$, we have

$$\text{vol}(B(\Sigma, r)) = \sin(2r) \text{area}(\Sigma) \geq \cot(r) \text{area}(\Sigma). \quad (3.3)$$

Combining [\(3.2\)](#) and [\(3.3\)](#), we have

$$\cot(r) \cdot \text{area}(\Sigma) \leq 2\pi^2. \quad (3.4)$$

On the other hand by the solution of Willmore Conjecture we have

$$2\pi^2 \leq \int_{\Sigma} 1 + \left(\frac{\kappa_1 + \kappa_2}{2} \right)^2 d\mu, \quad (3.5)$$

By the Gauss-Bonnet formula [\(3.1\)](#), we have the following

$$\begin{aligned} 2\pi^2 &\leq \int_{\Sigma} 1 + \left(\frac{\kappa_1 + \kappa_2}{2} \right)^2 d\mu \\ &= \int_{\Sigma} -\kappa_1 \kappa_2 + \frac{\kappa_1^2 + 2\kappa_1 \kappa_2 + \kappa_2^2}{4} d\mu \\ &= \int_{\Sigma} \frac{\kappa_1^2 - 2\kappa_1 \kappa_2 + \kappa_2^2}{4} d\mu \\ &\leq \int_{\Sigma} \frac{\kappa_1^2 + \kappa_2^2}{2} d\mu \end{aligned} \quad (3.6)$$

Since the normal injectivity radius of M equals to r , it follows that

$$|\kappa_i| \leq \cot(r)$$

for $i = 1, 2$. In fact this can be seen by touching each point $p \in \Sigma$ by a geodesic sphere with radius r in \mathbf{S}^3 from both sides of Σ , or we can use the theorem 3.1 of [\[GW20\]](#). Plugging them back to [\(3.6\)](#) yields

$$2\pi^2 \leq \cot^2 r \cdot \text{area}(\Sigma). \quad (3.7)$$

Using [\(3.4\)](#), we have

$$2\pi^2 \leq \cot^2 r \cdot \text{area}(\Sigma) \leq \cot r \cdot 2\pi^2,$$

i.e. $\cot(r) \geq 1$, which implies $r = \pi/4$ since we assume $r \geq \pi/4$. The rigidity part of the theorem follows from the rigidity part of the Willmore Conjecture.

4. VOLUME OF THE TUBE

In this section, we calculate the volume of the r -neighborhood of $\Sigma^2 \subset \mathbf{S}^3$. It is a special case of Weyl's Tube Formula. In fact it implies the normal injectivity radius $\leq \pi/4$ by monotonicity of $\sin(2r)$.

Lemma 4.1 (Tube Formula). *Let Σ be an embedded 2-torus in \mathbf{S}^3 . Then for any $r \leq \text{rad}^\odot(\Sigma)$, we have*

$$\text{vol}(B(\Sigma, r)) = \sin(2r) \text{area}(\Sigma).$$

Proof. Let

$$\Sigma(t) := \{x \in \mathbf{S}^3 \mid x = \exp_p(tv), p \in M, v \in \nu_p M\}$$

be the parallel hypersurface with signed distance $t \in [-r, r]$ to M . For any $p \in \Sigma^2$ let $\kappa_1(p), \kappa_2(p)$ be the two principal curvature of Σ at p . Using Fermi coordinate and co-area formula we have

$$\begin{aligned} \text{vol}(B(\Sigma, r)) &= \int_{-r}^r \text{area}(\Sigma(t)) dt \\ &= \int_{-r}^r \cos^2 t \int_{\Sigma} (1 - \kappa_1 \tan t)(1 - \kappa_2 \tan t) d\mu dt \\ &= \int_{\Sigma} \int_{-r}^r (\cos^2 t - (\kappa_1 + \kappa_2) \cos^2 t \cdot \tan t + \kappa_1 \kappa_2 \sin^2 t) dt d\mu \\ &= \int_{\Sigma} \int_{-r}^r (\cos^2 t + \kappa_1 \kappa_2 \sin^2 t) dt d\mu \\ &= \int_{\Sigma} \left(\int_{-r}^r (\cos^2 t - \sin^2 t) dt + \int_{-r}^r ((1 + \kappa_1 \kappa_2) \sin^2 t) dt \right) d\mu \\ &= \int_{\Sigma} \int_{-r}^r (\cos^2 t - \sin^2 t) dt d\mu \\ &= \sin(2r) \text{area}(\Sigma). \end{aligned} \tag{4.1}$$

□

REFERENCES

- [Bro60] M. Brown. A proof of the generalized Schoenflies theorem. *Bull. Amer. Math. Soc.*, 66:74–76, 1960.
- [BS83] E. Bombieri and L. Simon. On the Gehring link problem. In *Seminar on minimal submanifolds*, volume 103 of *Ann. of Math. Stud.*, pages 271–274. Princeton Univ. Press, Princeton, NJ, 1983.
- [Gag80] M. E. Gage. A proof of Gehring's linked spheres conjecture. *Duke Math. J.*, 47(3):615–620, 1980.
- [Gro18] M. Gromov. Metric inequalities with scalar curvature. *Geom. Funct. Anal.*, 28(3):645–726, 2018.
- [GS77] K. Grove and K. Shiohama. A generalized sphere theorem. *Ann. of Math. (2)*, 106(2):201–211, 1977.
- [GW18] L. Guijarro and F. Wilhelm. Focal radius, rigidity, and lower curvature bounds. *Proc. Lond. Math. Soc. (3)*, 116(6):1519–1552, 2018.
- [GW20] L. Guijarro and F. Wilhelm. Restrictions on submanifolds via focal radius bounds. *Math. Res. Lett.*, 27(1):115–139, 2020.

- [Had97] J. Hadmard. Sur certaines propriétés des trajectoires en dynamique. *J. Math. Pures Appl.*, 3:331–387, 1897.
- [Hop89] H. Hopf. *Differential geometry in the large*, volume 1000 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 1989. Notes taken by Peter Lax and John W. Gray, With a preface by S. S. Chern, With a preface by K. Voss.
- [Maz61] B. Mazur. A note on some contractible 4-manifolds. *Ann. of Math. (2)*, 73:221–228, 1961.
- [MN14] F. C. Marques and A. Neves. Min-max theory and the Willmore conjecture. *Ann. of Math. (2)*, 179(2):683–782, 2014.
- [Ort75] M. Ortel. unpublished. 1975.
- [Zhu21] J. Zhu. Width estimate and doubly warped product. *Trans. Amer. Math. Soc.*, 374(2):1497–1511, 2021.

(Ge) SCHOOL OF MATHEMATICAL SCIENCES, LABORATORY OF MATHEMATICS AND COMPLEX SYSTEMS, BEIJING NORMAL UNIVERSITY, BEIJING 100875, P. R. CHINA.
Email address: `jge@bnu.edu.cn`