FREE BOUNDARY MINIMAL SURFACES IN THE EUCLIDEAN THREE-BALL CLOSE TO THE BOUNDARY

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ABSTRACT. We construct free boundary minimal surfaces (FBMS) embedded in the unit ball in the Euclidean three-space which are compact, lie arbitrarily close to the boundary unit sphere, are of genus zero, and their boundary has an arbitrarily large number of connected boundary components. The construction is by PDE gluing methods and the surfaces are desingularizations of unions of many catenoidal annuli and two flat discs. The union of the boundaries of the catenoidal annuli and discs is the union of a large finite number of parallel circles contained in the unit sphere, with each parallel circle contained in the boundary of exactly two of the catenoidal annuli and discs.

1. INTRODUCTION

Brief discussion of the results.

In recent years there has been much interest on free boundary minimal surfaces (FBMS) in the unit ball $\mathbb{B}^3 \subset \mathbb{R}^3$, as for example in [1–7, 14, 18, 20–23, 25–27]. In this article we construct by PDE gluing methods compact embedded FBMS in the unit ball $\mathbb{B}^3$ which lie arbitrarily close to the boundary $\mathbb{S}^2 = \partial\mathbb{B}^3$. The construction uses desingularization ideas from [8], FBMS construction ideas from [14, 18], and some auxiliary ideas from [15, 16]. Note in particular that although the desingularization approach of [8] has been used in various constructions before, for example in [13, 14, 24], this is the first time since [8] it is used without imposing simplifying extra symmetries.

The first step of our construction is to determine for each $k \in \mathbb{N} \setminus \{1, 2\}$ a *family of configurations* $\mathcal{W}[\underline{\sigma}:k]$ parametrized by $\underline{\sigma}$ (see 2.24). This step resembles the construction of RLD's for $O(2) \times \mathbb{Z}_2$-symmetric backgrounds [15, 16], but the minimality ODE is nonlinear resulting in a different proof and construction. Each $\mathcal{W}[\underline{\sigma}:k]$ we construct is $O(2) \times \mathbb{Z}_2$-symmetric and consists of two flat discs (at the top and bottom of the configuration) and $k-1$ catenoidal annuli. Here $O(2)$ acts on $\mathbb{R}^3$ by rotations and reflections fixing the y-axis pointwise, and the generator of $\mathbb{Z}_2$ by reflection with respect to the xz-plane (Definition 1.4). The boundaries of the discs and the catenoidal annuli are contained in the union of k parallel circles on $\mathbb{S}^2$, with each of the circles contained in the boundary of exactly two of the discs and annuli. Moreover $\mathcal{W}[0:k]$ is balanced in the sense that the angles at a common

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boundary circle the catenoidal annuli (or disc) make with $\mathbb{S}^2$ are equal. Small *unbalancing* (in the sense of slightly violating the equality) is prescribed continuously by $\underline{\sigma}$ in order to determine $\mathcal{W}[\underline{\sigma}:k]$. Finally for k large enough the configurations $\mathcal{W}[\underline{\sigma}:k]$ are arbitrarily close to $\mathbb{S}^2$, the catenoidal annuli arbitrarily narrow, and the discs arbitrarily small (Proposition 2.30).

Moving forward (leading to the construction of our FBMS), we impose the symmetries of a finite subgroup of $O(2) \times \mathbb{Z}_2$, namely $D_{2m} \times \mathbb{Z}_2$, where D_{2m} is a dihedral group of order $2m$ (Definition 1.4). Imposing such a dihedral subgroup of $O(2)$ symmetries is common for constructions where the background is rotationally symmetric but the surfaces constructed are not, for example [8], [15], or [16, Parts II and III]. The $\mathbb{Z}_2$ symmetry is imposed to simplify the presentation and can be removed as for example in [19]. Another simplifying assumption we adopt in this article is using exactly m "necks" in the desingularizations of each circle (unlike in [8, 17] for example where arbitrary multiples of m depending on the circle are used). As a result we produce FBMS's of genus zero with exactly km connected boundary components.

In the second step we construct the initial surfaces. Their construction is based on appropriately modifying the singly periodic Scherk surfaces so that they can "desingularize" the boundary circles of the configurations. As in [8], the initial surfaces consist of three regions smoothly joined together. The first region consists of the modified *cores* of *half* singly periodic Scherk surfaces. Here the Scherk surfaces are subdivided in two halves by a symmetry plane through their axis (the yz-plane in 3.1). The "waists" of the Scherk surfaces on this plane are modified in the construction to form the boundary of the initial surfaces. Particular care is taken to ensure as in [18] that the initial surfaces are orthogonal to $\mathbb{S}^2$ at the boundary.

The second region consists of graphical regions over catenoidal annuli (or discs) imitating the graphical property of the wings of the Scherk surfaces over their asymptotic half-planes. "Bending" the asymptotic half-planes to catenoids (or planes) instead of cones is an important feature of the construction and as in [8] ensures good estimates for the mean curvature of the initial surfaces. There are *two* (instead of *four* as in [8]) such graphical annular regions (*wings*) for each desingularizing half Scherk surface. The third region consists of perturbations of the catenoidal annuli and discs of the configuration being desingularized.

In the third step we estimate the mean curvature H of the initial surfaces. In Proposition 4.16 H is effectively decomposed into two parts: a part in the *extended substitute kernel* $\mathcal{K}$, which is desirable and amounts more or less to the linearized change of the mean curvature under the action of the unbalancing parameters, and an error part which has to be corrected by solving the PDE. The error part is the sum of the gluing error, which is of order $\tau = 1/m$, and the unbalancing higher order terms, which are controlled by the squares of the unbalancing parameters (which in Proposition 4.16 are $\alpha^+ - \alpha^-$, ϕ^+, and ϕ^-).

Note that the action of the unbalancing parameters is inspired by the *geometric principle* (see [9, 10] for a general discussion) and relocates the various

regions of the initial surfaces relative to each other as in [8, 12]. There are *three* unbalancing parameters for each desingularizing surface (equivalently per neck modulo symmetries), unlike [8] where there are *six*, or [12] where there are *seven* (see [9, page 519] or [10, section 5.3]). One of them corresponds to the translation consistent with the construction and unbalances the wings creating substitute kernel (see 4.2), and the other two reposition the two wings relative to the core creating extended substitute kernel (see 4.6) used to ensure decay along the wings.

In the fourth step we solve the linearized PDE globally on the initial surfaces (Proposition 6.15). By solving the linearized PDE modulo $\mathcal{K}$ we can make the inhomogeneous term orthogonal to the corresponding eigenfunctions (Lemma 4.3(iii)) so we can find a solution and moreover (using the "extended" part of $\mathcal{K}$) we can ensure exponential decay of the solution along the wings (Lemma 4.10(v)). The proof of the main Proposition 6.15) is in the style of (for example) [11, 14] rather than [8]: We first solve the linearized equation on the "standard models", that is on the half Scherk surfaces (Proposition 6.1 and catenoidal annuli or discs (Lemma 6.8). By comparing the operators on the standard models and the initial surfaces (Lemma 6.9), we can transfer the results on the standard models to the initial surfaces, and combine them to prove the main Proposition 6.15 using an iteration to correct small errors.

In the fifth and final step we first estimate the nonlinear terms of the mean curvature (Proposition 7.1) of graphs over the initial surfaces. Note that following [18], we use a simple auxiliary metric, which is a product metric, to define these graphs; this ensures a simple description of the free boundary condition. We then apply Schauder's fixed point theorem to prove the main theorem of this article which less formally is as follows.

Theorem 1.1 (Theorem 7.4). *Given $k \in \mathbb{N}$ large enough in absolute terms and $m \in \mathbb{N}$ large enough in terms of k, there is a compact embedded two-sided free boundary minimal smooth surface (FBMS) $\mathbf{M}_{k,m}$ of genus zero and km connected boundary components (with m of them in the vicinity of each of the k boundary circles of $\mathcal{W}[0{:}k]$), which is symmetric under the action of $\mathcal{G} = D_{2m} \times \mathbb{Z}_2$ (Definition 1.4) and is the graph of a small function $\tilde{v}$ in an auxiliary ambient metric g_A (Definition 5.17) over a smooth initial surface $M[\xi_0]$ (see 5.12 and 5.15). Moreover $\mathbf{M}_{k,m}$ converges on compact subsets of the interior of $\mathbb{B}^3$ in all norms to the configuration $\mathcal{W}[0{:}k]$ as $m \to \infty$; the Hausdorff distance of $\mathbf{M}_{k,m}$ from $\mathcal{W}[0{:}k]$ tends to 0 as $m \to \infty$; and in turn the Hausdorff distance of $\mathcal{W}[0{:}k]$ from $\mathbb{S}^2 = \partial \mathbb{B}^3$ tends to 0 as $k \to \infty$. Finally $\lim_{k \to \infty} \lim_{m \to \infty} \mathbf{M}_{k,m} = \mathbb{S}^2$ in the varifold sense.*

The construction here can be generalized to produce FBMS close to the boundary in rotationally symmetric convex domains in Euclidean three-space [19]. We remark also that very recently Karpukhin-Stern proved that the FBMS in the unit ball $\mathbb{B}^3$, maximizing the normalized first Steklov eigenvalue among oriented surfaces of genus zero and k_∂ boundary components, converge as $k_\partial \to \infty$ to $\mathbb{S}^2 = \partial \mathbb{B}^3$ in the sense of varifolds [20, Corollary 1.4]. This result had been conjectured by Girouard-Lagacé [7] and is

based on Fraser-Schoen's result that such maximizers exist and moreover can be realized as compact embedded smooth FBMS in the unit ball $\mathbb{B}^3$ [4]. At the moment however little is known about the geometry of these eigenvalue maximizing surfaces, unlike the ones constructed here, which are small perturbations of explicit *initial surfaces*. It seems unlikely that the surfaces constructed here are the maximizers of the normalized first Steklov eigenvalue, but it may be possible to construct some of the maximizers by enhancing our approach.
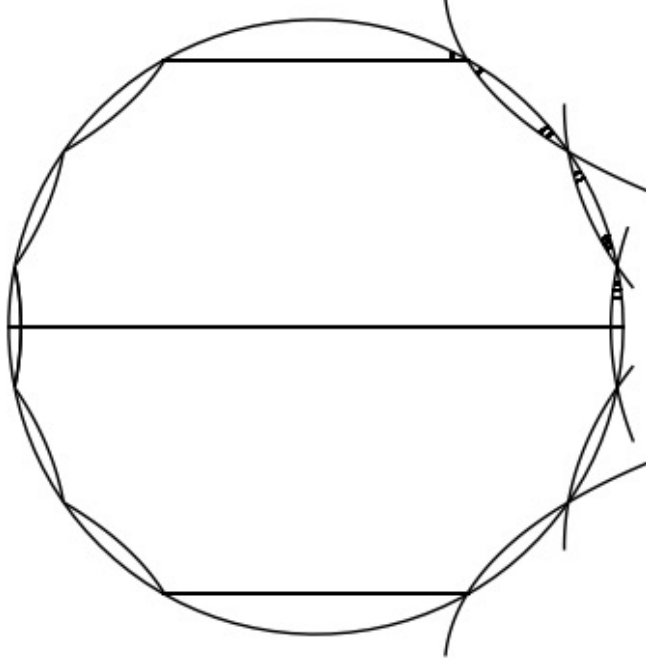

FIGURE 1. Cross section of $\mathcal{W}[0{:}6]$.

General notation and conventions.

Definition 1.2. *We identify the Euclidean space (defined as a Riemannian manifold with no distinguished points) E^3 with $\mathbb{R}^3$ by fixing a standard Cartesian coordinate system $O.xyz$. We define the unit ball $\mathbb{B}^3 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\}$ and its boundary $\mathbb{S}^2 := \partial \mathbb{B}^3$. For convenience we will identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$ by identifying (x, y) with $(x, y, 0)$. Moreover denoting this identification by "$\simeq$", we introduce the notation*

(1.3) $\mathbb{R}^2_+ := \{(x, y, 0) \in \mathbb{R}^3 \ : \ x \ge 0\} \simeq \{(x, y) \in \mathbb{R}^2 \ : \ x \ge 0\}.$

Finally we take the north pole of the sphere to be the point $p_{north} := (0, 1, 0) \in \mathbb{R}^2 \subset \mathbb{R}^3$ and the south pole to be the point $p_{south} := (0, -1, 0) \in \mathbb{R}^2 \subset \mathbb{R}^3$.

Definition 1.4 (Action of symmetry groups). *We let $O(2) \times \mathbb{Z}_2$ act on $\mathbb{R}^3$ (with coordinates as in 1.2) by assuming that $O(2)$ acts the usual way on the xz-plane while fixing the y-axis pointwise, and the generator of $\mathbb{Z}_2$ by*

reflection with respect to the xz-plane. Given $m \in \mathbb{N} \setminus \{1, 2\}$, we let D_{2m} denote the dihedral subgroup of $O(2)$ of order $2m$ whose action on the xz-plane includes reflections with respect to the lines $\{\cos \frac{l\pi}{m} z - \sin \frac{l\pi}{m} x = 0\}$ for $l \in \mathbb{Z}$. We finally define $\mathcal{G} := D_{2m} \times \mathbb{Z}_2$ with action on $\mathbb{R}^3$ as a subgroup of $O(2) \times \mathbb{Z}_2$.

Definition 1.5. *Assuming that Ω is a domain inside a manifold, g is a Riemannian metric on the manifold, $f : \Omega \to (0, \infty)$ are given functions, $k \in \mathbb{N}$, $\beta \in [0, 1)$, $u \in C^{k,\beta}_{loc}(\Omega)$ or more generally u is a $C^{k,\beta}_{loc}$ tensor field (section of a vector bundle) on Ω, and that the injectivity radius in the manifold around each point x in the metric g is at least $1/10$, $\|u : C^{k,\beta}(\Omega, g, f)\|$ is defined by*

$$\|u : C^{k,\beta}(\Omega, g, f)\| := \sup_{x \in \Omega} \frac{\|u : C^{k,\beta}(\Omega \cap B_x, g)\|}{f(x)},$$

where B_x is a geodesic ball centered at x and of radius $1/100$ in the metric g. For simplicity any of β or f may be omitted, when $\beta = 0$ or $f \equiv 1$, respectively.

The function f can be thought of as a "weight" function because $f(x)$ controls the size of u at the vicinity of each point x. Note that from the definition it follows that

(1.6) $$\|\nabla u : C^{k-1,\beta}(\Omega, g, f)\| \leq \|u : C^{k,\beta}(\Omega, g, f)\|,$$

and the multiplicative property

(1.7) $$\|u_1 u_2 : C^{k,\beta}(\Omega, g, f_1 f_2)\|$$
$$\leq C(k) \|u_1 : C^{k,\beta}(\Omega, g, f_1)\| \|u_2 : C^{k,\beta}(\Omega, g, f_2)\|.$$

Cut-off functions will be used extensively, and for this reason the following is adopted.

Definition 1.8. *A smooth function $\Psi : \mathbb{R} \to [0, 1]$ is fixed with the following properties:*
(i). Ψ is nondecreasing.
(ii). $\Psi \equiv 1$ on $[1, \infty)$ and $\Psi \equiv 0$ on $(-\infty, -1]$.
(iii). $\Psi - \frac{1}{2}$ is an odd function.

Given now $a, b \in \mathbb{R}$ with $a \neq b$, the smooth function $\psi[a, b] : \mathbb{R} \to [0, 1]$ is defined by

(1.9) $$\psi[a, b] := \Psi \circ L_{a,b},$$

where $L_{a,b} : \mathbb{R} \to \mathbb{R}$ is the linear function defined by the requirements $L(a) = -3$ and $L(b) = 3$.

Clearly then $\psi[a, b]$ has the following properties:
(i). $\psi[a, b]$ is weakly monotone.
(ii). $\psi[a, b] = 1$ on a neighborhood of b and $\psi[a, b] = 0$ on a neighborhood of a.
(iii). $\psi[a, b] + \psi[b, a] = 1$ on $\mathbb{R}$.

Definition 1.10. *g is defined to be the Euclidean metric in E^3 and g_M is defined to be the restriction of g on the embedded surface M in E^3.*

Definition 1.11. *For an oriented embedded surface M in E^3, A_M is defined to be the second fundamental form of M and $|A|^2_M$ is defined to be the squared norm of the second fundamental form. Also the operator $\mathcal{L}_M$ is defined by*

$$\mathcal{L}_M = \Delta_M + |A|^2_M \,,$$

where Δ_M is the Laplacian on M defined by g_M.

Notation 1.12 (Symmetric functions). Given a manifold M invariant under the action of some group G and a space of functions $\mathcal{X} \subset C^0(\Omega)$, where Ω is a subset of M invariant under the action of G, we use a subscript "G" to denote the subspace $\mathcal{X}_G \subset \mathcal{X}$ consisting of the functions in $\mathcal{X}$ which are invariant under the action of G.

Notation 1.13. For a function s defined on a surface M which has values in $\mathbb{R} \cup \{\infty\}$, if V is subset of M we use the notation

$$V_{s \leq a} := \{p \in V : s(p) \leq a\}, \quad V_{s \geq a} := \{p \in V : s(p) \geq a\}.$$

2. The Initial Configurations

Recall that we use the standard Cartesian coordinate system $O.xyz$ in $\mathbb{R}^3$ as in 1.2. In this section we concentrate on objects in $\mathbb{R}^3$ which are rotationally invariant around the y-axis (recall 1.2) and so are determined uniquely by their intersection with $\mathbb{R}^2_+$.

Definition 2.1 (Polar coordinates). *We define polar coordinates (r, β) on the xy-plane by $(x, y) = r (\sin \beta, \cos \beta)$ and so $\beta = 0$ at the north pole and $\beta = \pi$ at the south pole. Moreover given a function $\mathrm{r} : B \to \mathbb{R}_+$ where $B \subset [0, \pi]$, we define the radial graph of r to be the set*

$$\{ \mathrm{r}(\beta) (\sin \beta, \cos \beta) \, : \, \beta \in B \} \subset \mathbb{R}^2_+.$$

The Catenoidal Annuli.

Lemma 2.2 (Catenoids around the y-axis). *For a catenoid $\mathbb{K} \subset \mathbb{R}^3$ which is rotationally invariant around the y-axis the following hold.*

   (i) There are unique $a = a^{\mathbb{K}} \in \mathbb{R}_+$ and $b = b^{\mathbb{K}} \in \mathbb{R}$ such that $\mathbb{K}' := \mathbb{K} \cap \mathbb{R}^2_+ = \left\{ (x, y) \in \mathbb{R}^2_+ : x = a \cosh(\frac{y-b}{a}) \right\}$.

  (ii) There are exactly two straight lines through the origin which are tangent to $\mathbb{K}'$. We call the points of tangency $p^{\mathbb{K}}_{\top \pm} \in \mathbb{K}'$, chosen so that their y-coordinates $y^{\mathbb{K}}_{\top \pm} \in \mathbb{R}$ satisfy $y^{\mathbb{K}}_{\top -} < y^{\mathbb{K}}_{\top +}$.

 (iii) The arc $\mathbb{K}'_{\top}$ of $\mathbb{K}'$ with endpoints $p^{\mathbb{K}}_{\top \pm}$ is graphical in polar coordinates (that is r is a function of β on $\mathbb{K}'_{\top}$). The inward normal $\nu = \nu^{\mathbb{K}}$ and the position vector X of $\mathbb{K}$ satisfy $\nu \cdot X > 0$ on the interior of $\mathbb{K}'_{\top}$ and $\nu \cdot X < 0$ on $\mathbb{K}' \setminus \mathbb{K}'_{\top}$.

(iv) $v := \frac{d}{d\beta} \log \frac{1}{r}$ is a strictly decreasing function of β on the interior of $\mathbb{K}'_\top$—equivalently $\log r$ is strictly convex on $\mathbb{K}'_\top$—and satisfies the first order equation

$$(2.3) \qquad \frac{dv}{d\beta} = -(\cot \beta)\, v\, (v^2 + 1) - 2\, (v^2 + 1).$$

(v) $\mathbb{K} \cap \mathbb{B}^3$ is connected and when nonempty it is an annulus or a circle, and there exist (uniquely determined) $y^{\mathbb{K}}_-, y^{\mathbb{K}}_+ \in (-1, 1)$ with $y^{\mathbb{K}}_- \leq y^{\mathbb{K}}_+$, such that
  (a) $\mathbb{K} \cap \mathbb{B}^3 = \{(x, y, z) \in \mathbb{K} : y^{\mathbb{K}}_- \leq y \leq y^{\mathbb{K}}_+\}$,
  (b) $\mathbb{K} \cap \mathbb{S}^2 = \{(x, y, z) \in \mathbb{S}^2 : y = y^{\mathbb{K}}_-\ or\ y = y^{\mathbb{K}}_+\}$,
  (c) $\operatorname{sgn} b^{\mathbb{K}} = \operatorname{sgn}(y^{\mathbb{K}}_- + y^{\mathbb{K}}_+)$.

Proof. (i) follows by the equation of catenary $\mathbb{K}'$.

For (ii), consider the function $\frac{y}{x}$ on $\mathbb{K}'$. By definition, the tangency happens exactly at the critical point of the function $\frac{y}{x}$. By (i),

$$\frac{d}{dy} \frac{y}{x} = \frac{d}{dy} \frac{y}{a \cosh(\frac{y-b}{a})} = \frac{1 - \frac{y}{a} \tanh(\frac{y-b}{a})}{a \cosh(\frac{y-b}{a})}.$$

However,

$$\frac{d}{dy} y \tanh\left(\frac{y-b}{a}\right) = \operatorname{sech}^2\left(\frac{y-b}{a}\right) \left(\sinh\left(\frac{y-b}{a}\right) \cosh\left(\frac{y-b}{a}\right) + \frac{y}{a}\right).$$

The function $\sinh(\frac{y-b}{a}) \cosh(\frac{y-b}{a}) + \frac{y}{a}$ is increasing and only has one zero, say y_0. Therefore, the function $1 - \frac{y}{a} \tanh(\frac{y-b}{a})$ has only one critical point y_0, which is a maximum, and it is increasing in $(-\infty, y_0)$ and decreasing in (y_0, ∞). And at $y = 0$, it equals 1; moreover, $\lim_{y \to \pm\infty} 1 - \frac{y}{a} \tanh(\frac{y-b}{a}) = \mp\infty$. Thus it has only two zeroes. Finally, these two points are the only two critical points of $\frac{y}{x}$, which correspond to $p^{\mathbb{K}}_{\top\pm}$.

The fact that the arc $\mathbb{K}'_\top$ of $\mathbb{K}'$ is graphical in polar coordinates follows by (ii) because the graphical property fails only at the tangency points. A straightforward calculation shows that $X = (a \cosh(\frac{y-b}{a}), y)$ and $\nu = (\operatorname{sech}(\frac{y-b}{a}), -\tanh(\frac{y-b}{a}))$, and thus $X \cdot \nu = a - y \tanh(\frac{y-b}{a})$, which has the same sign as the function $\frac{d}{dy} \frac{y}{x}$. The rest part of (iii) then follows by the results in the proof of (ii).

The catenary $\mathbb{K}'$ satisfies the differential equation

$$(2.4) \qquad \frac{d^2 x}{dy^2} = \frac{1 + (\frac{dx}{dy})^2}{x}.$$

From the definitions of v, β and their relationships with x, y, the equation could be rewritten with the new variables. The equation (2.3) then follows.

Assume that $\beta \leq \pi/2$. From equation (2.3), we have $\frac{dv}{d\beta} < 0$ when $v \geq 0$; v is then decreasing. Now suppose that at some $\beta^1 \leq \pi/2$, $\frac{dv}{d\beta}(\beta^1) > 0$ and $v(\beta^1) < 0$. Thus there is $\beta^0 < \beta^1$, such that $\frac{dv}{d\beta}(\beta^0) = 0$, $v(\beta^0) < 0$ and $\frac{dv}{d\beta}(\beta) > 0$ for all $\beta \in (\beta^0, \beta^1)$. Therefore, $v(\beta^0) < v(\beta^1)$. Moreover, $0 =$

$\frac{\mathrm{d}v}{\mathrm{d}\beta}(\beta^0) = -\cot(\beta^0)v(\beta^0)((v(\beta^0))^2 + 1) - 2((v(\beta^0))^2 + 1)$, or $\cot(\beta^0)v(\beta^0) = -2$. However, as $v(\beta^0) < v(\beta^1) < 0$, $\cot\beta^0 > \cot\beta^1 > 0$

$$\frac{\mathrm{d}v}{\mathrm{d}\beta}(\beta^1) = -(\cot(\beta^1)v(\beta^1) + 2)((v(\beta^1))^2 + 1) < 0,$$

which contradicts the assumption. This shows that v is a nonincreasing functions when $\beta \le \pi/2$. As the equation does not have constant solution, v must be strictly decreasing. The case when $\beta > \pi/2$ then simply follows by the symmetry.

Finally, by the fact that the function $r^2(y) := x^2(y) + y^2$ on $\mathbb{K}'$ is convex, where $x(y) := a\cosh\left(\frac{y-b}{a}\right)$ as in (i), $\mathbb{K}' \cap \mathbb{D}^2$ is connected, where $\mathbb{D}^2$ is the unit disk. Moreover, as $\mathbb{K}'$ and $\mathbb{S}^1 \cap \mathbb{R}^2_+$ are both graphical over the y-axis, there exist uniquely determined $y^{\mathbb{K}}_-, y^{\mathbb{K}}_+ \in (-1, 1)$ with $y^{\mathbb{K}}_- \le y^{\mathbb{K}}_+$ such that

$$\mathbb{K}' \cap \mathbb{D}^2 = \{(x, y) \in \mathbb{K}' : y^{\mathbb{K}}_- \le y \le y^{\mathbb{K}}_+\},$$
$$\mathbb{K}' \cap \mathbb{S}^1 = \{(x, y) \in \mathbb{S}^1 : y = y^{\mathbb{K}}_- \text{ or } y = y^{\mathbb{K}}_+\}.$$

The first two items in (v) then follow from the relationship between $\mathbb{K}'$ and $\mathbb{K}$. Finally, suppose that $\mathbb{K}' \cap \mathbb{S}^1 = \{(x^{\mathbb{K}}_-, y^{\mathbb{K}}_-), (x^{\mathbb{K}}_+, y^{\mathbb{K}}_+)\}$, if $y^{\mathbb{K}}_- + y^{\mathbb{K}}_+ \ge 0$, then $x^{\mathbb{K}}_- \le x^{\mathbb{K}}_+$. By viewing $x(y) := a\cosh\left(\frac{y-b^{\mathbb{K}}}{a^{\mathbb{K}}}\right)$ as a graph over the y-axis, the symmetry axis $y = b^{\mathbb{K}}$ is above the line $y = (y^{\mathbb{K}}_- + y^{\mathbb{K}}_+)/2$. Thus $b^{\mathbb{K}} > 0$ and the last item of (v) follows by symmetry. $\qquad\square$

Definition 2.5. *Given a catenoid $\mathbb{K}$ as in 2.2 with $\mathbb{K} \cap \mathbb{B}^3$ an annulus, we define the arc $\mathbb{K}'_{\mathbb{B}} := \mathbb{K} \cap \mathbb{R}^2_+ \cap \mathbb{B}^3$. We define $\beta^{\mathbb{K}}_{in}, \beta^{\mathbb{K}}_{ex} \in (0, \pi)$ with $\beta^{\mathbb{K}}_{in} < \beta^{\mathbb{K}}_{ex}$ to be the latitudes of the endpoints of $\mathbb{K}'_{\mathbb{B}}$, and hence by 2.2(v) we have $\cos\beta^{\mathbb{K}}_{in} = y^{\mathbb{K}}_+$ and $\cos\beta^{\mathbb{K}}_{ex} = y^{\mathbb{K}}_-$. The endpoints of $\mathbb{K}'_{\mathbb{B}}$ will be denoted by $p^{\mathbb{K}}_{in} := (\sin\beta^{\mathbb{K}}_{in}, \cos\beta^{\mathbb{K}}_{in}) \in \mathbb{S}^1$ and $p^{\mathbb{K}}_{ex} := (\sin\beta^{\mathbb{K}}_{ex}, \cos\beta^{\mathbb{K}}_{ex}) \in \mathbb{S}^1$. Finally we define $\alpha^{\mathbb{K}}_{in}$ ($\alpha^{\mathbb{K}}_{ex}$) to be the angle at $p^{\mathbb{K}}_{in}$ ($p^{\mathbb{K}}_{ex}$) between $\mathbb{K}'_{\mathbb{B}}$ and the arc of $\mathbb{S}^1$ with endpoints $p^{\mathbb{K}}_{in}, p_{south}$ ($p^{\mathbb{K}}_{ex}, p_{north}$).*

Lemma 2.6. *Given a catenoid $\mathbb{K}$ as in 2.5 we have $\alpha^{\mathbb{K}}_{in} \in (0, \pi - \beta^{\mathbb{K}}_{in})$. Conversely given $\beta_{in} \in (0, \pi)$ and $\alpha_{in} \in (0, \pi - \beta_{in})$ there is a unique catenoid which will be denoted by $\mathbb{K}[\beta_{in}, \alpha_{in}]$ and is as in 2.5 and satisfies $\beta^{\mathbb{K}[\beta_{in}, \alpha_{in}]}_{in} = \beta_{in}$ and $\alpha^{\mathbb{K}[\beta_{in}, \alpha_{in}]}_{in} = \alpha_{in}$.*

Proof. Note that the condition $\alpha^{\mathbb{K}}_{in} < \pi - \beta^{\mathbb{K}}_{in}$ amounts by Lemma 2.2 to the fact that $\mathbb{K}'_{\mathbb{B}}$ lies below p_{in}. $\pi - \beta^{\mathbb{K}}_{in}$ in the angle between the line segment $\{(x, y) : y = \cos\beta^{\mathbb{K}}_{in}, 0 < x \le \sin\beta^{\mathbb{K}}_{in}\}$ and the arc of $\mathbb{S}^1$ with endpoints p_{south}. Conversely, given α_{in} and β_{in} with $0 < \alpha_{in} < \pi - \beta_{in}$, consider the ordinary differential equation (2.4) with the initial conditions: $x = \sin\beta_{in}$, $y = \cos\beta_{in}$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = \tan(\alpha_{in} + \beta_{in} - \frac{\pi}{2})$. The graph $(x(y), y)$ of the solution is a catenary $\mathbb{K}'$ going through the point $p_{in} := (\sin\beta_{in}, \cos\beta_{in})$. Moreover, at p_{in}, the inner product $(x, y) \cdot (\frac{\mathrm{d}x}{\mathrm{d}y}, 1) = \sin\beta_{in}(\cot\beta_{in} - \cot(\alpha_{in} + \beta_{in})) > 0$ because of the condition $\alpha_{in} + \beta_{in} < \pi$. This means that the graph is going in the circle $\mathbb{S}^1$ at p_{in} if y is decreasing. A calculation along with the definition of $\beta^{\mathbb{K}}_{in}$ and $\alpha^{\mathbb{K}}_{in}$ in 2.5 shows that $p^{\mathbb{K}}_{in} = p_{in}$, $\beta^{\mathbb{K}}_{in} = \beta_{in}$ and $\alpha^{\mathbb{K}}_{in} = \alpha_{in}$.

On the other hand, each catenary $\mathbb{K}'$ corresponding to the catenoid $\mathbb{K}$ in the lemma must satisfy the equation and initial conditions as above, the uniqueness then follows. $\qquad\square$

Notation 2.7. We will often simplify the notation related to $\mathbb{K} = \mathbb{K}[\beta_{in}, \alpha_{in}]$ (defined as in 2.6) by using "$[\beta_{in}, \alpha_{in}]$" to emphasize the dependence on the parameters, for example we may write $\beta_{ex}[\beta_{in}, \alpha_{in}]$ instead of $\beta_{ex}^{\mathbb{K}[\beta_{in},\alpha_{in}]}$.

Lemma 2.8. *For α_{in}, β_{in} as in 2.6 we have $\mathbb{K}'_{\mathbb{B}}[\beta_{in}, \alpha_{in}] \subset \mathbb{K}'_{\top}[\beta_{in}, \alpha_{in}]$ if and only if $\alpha_{in} < \pi/2$ and $\alpha_{ex}[\beta_{in}, \alpha_{in}] < \pi/2$.*

Proof. By the definition 2.5, the conditions $\alpha_{in} < \pi/2$ and $\alpha_{ex}[\beta_{in}, \alpha_{in}] < \pi/2$ are equivalent with $X \cdot \nu > 0$ at $p_{in}^{\mathbb{K}}$ and $p_{ex}^{\mathbb{K}}$. The result then follows by 2.2(iii). $\qquad\square$

Lemma 2.9. *We have the following for $\mathbb{K} = \mathbb{K}[\beta_{in}, \alpha_{in}]$ as in 2.6, $a = a[\beta_{in}, \alpha_{in}]$, $b = b[\beta_{in}, \alpha_{in}]$ as in 2.2, $\beta_{ex} = \beta_{ex}[\beta_{in}, \alpha_{in}]$, and $\alpha_{ex} = \alpha_{ex}[\beta_{in}, \alpha_{in}]$.*

(i) $a = \sin(\beta_{in}) \sin(\alpha_{in} + \beta_{in}) = \sin(\beta_{ex}) \sin(-\alpha_{ex} + \beta_{ex})$,
$b = \cos(\beta_{in}) - \operatorname{sgn}(\alpha_{in} + \beta_{in} - \pi/2) \cdot$
                $\cdot \sin(\beta_{in}) \sin(\alpha_{in} + \beta_{in}) \operatorname{arcosh}(1/\sin(\alpha_{in} + \beta_{in}))$.
   Moreover when $\beta_{ex} \leq \pi/2$,
   $b = \cos(\beta_{ex}) + \sin(\beta_{ex}) \sin(-\alpha_{ex} + \beta_{ex}) \operatorname{arcosh}(1/\sin(-\alpha_{ex} + \beta_{ex}))$.

(ii) $\operatorname{sgn}(\alpha_{in} - \alpha_{ex}) = \operatorname{sgn} b = \operatorname{sgn}(\pi - \beta_{in} - \beta_{ex})$.

(iii) *If $\alpha_{in} < \pi/2$ and $\alpha_{ex}[\beta_{in}, \alpha_{in}] < \pi/2$, $v := \frac{d}{d\beta} \log \frac{1}{r}$ is strictly decreasing on $\mathbb{K}'_{\mathbb{B}}[\beta_{in}, \alpha_{in}]$ and satisfies (2.3) with boundary conditions $v(\beta_{in}) = \tan(\alpha_{in})$, $v(\beta_{ex}) = -\tan(\alpha_{ex})$.*

Proof. At $p_{in}^{\mathbb{K}} = (\sin \beta_{in}^{\mathbb{K}}, \cos \beta_{in}^{\mathbb{K}})$, the equation of $\mathbb{K}'$ satisfies $\frac{dx}{dy} = \tan(\alpha_{in} + \beta_{in} - \frac{\pi}{2})$; while At $p_{ex}^{\mathbb{K}} = (\sin \beta_{ex}^{\mathbb{K}}, \cos \beta_{ex}^{\mathbb{K}})$, the equation of $\mathbb{K}'$ satisfies $\frac{dx}{dy} = \tan(-\alpha_{ex} + \beta_{ex} - \frac{\pi}{2})$. (i) then follows by the equation and definitions directly.

By the definition of v in 2.2(iv), $\frac{dx}{dy} = \frac{v \sin \beta - \cos \beta}{v \cos \beta + \sin \beta}$. The formulae for $v(\beta_{in})$ and $v(\beta_{ex})$ in (iii) then follow. The rest part of (iii) follows by 2.2(iv) and 2.8.

If $\beta_{in} + \beta_{ex} \leq \pi$, then $\pi/2 > \pi/2 - \beta_{in} \geq \beta_{ex} - \pi/2 > -\pi/2$, thus $y_-^{\mathbb{K}} = \cos \beta_{in} = \sin(\pi/2 - \beta_{in}) \geq \sin(\beta_{ex} - \pi/2) = -\cos \beta_{ex} = -y_+^{\mathbb{K}}$, or $y_-^{\mathbb{K}} + y_+^{\mathbb{K}} \geq 0$. The second equality of (ii) then follows by 2.2(v)*(c)*.

From a symmetry argument, the first equality of (ii) holds as long as $\alpha_{in} > \alpha_{ex}$ if $b > 0$. After a scaling and a translation, this could be reduced to the problem of the comparison of the two angles made by a line $x = py + q$ and the standard catenary $x = \cosh y$ with the assumptions that $p > 0$.

Suppose the line intersects with the catenary at points $P_1(\cosh(y_1), y_1)$ and $P_2(\cosh(y_2), y_2)$, with $y_2 > y_1$ and $\cosh(y_2) > \cosh(y_1)$. Then the unit tangent vectors at P_1, P_2 that direct to the positive x-axis could be written as $\mathbf{v}_1 = \left(\frac{\sinh(y_1)}{\cosh(y_1)}, \frac{1}{\cosh(y_1)} \right)$ and $\mathbf{v}_2 = \left(\frac{\sinh(y_2)}{\cosh(y_2)}, \frac{1}{\cosh(y_2)} \right)$. Therefore,

for $i = 1, 2$,

$$\left| \overrightarrow{P_1 P_2} \right| \cos \left(\langle \widehat{\overrightarrow{P_1 P_2}, \mathbf{v}_i} \rangle \right) = \overrightarrow{P_1 P_2} \cdot \mathbf{v}_i$$

$$= \frac{y_2 - y_1}{\cosh(y_i)} + \frac{(\cosh(y_2) - \cosh(y_1)) \sinh(y_i)}{\cosh(y_i)},$$

where $\langle \widehat{\overrightarrow{P_1 P_2}, \mathbf{v}_i} \rangle$ is the angle made by the vectors $\overrightarrow{P_1 P_2}$ and $\mathbf{v}_i$; and thus

$$\cosh(y_2) \cosh(y_1) \left| \overrightarrow{P_1 P_2} \right| \left(\cos \left(\langle \widehat{\overrightarrow{P_1 P_2}, \mathbf{v}_2} \rangle \right) - \cos \left(\langle \widehat{\overrightarrow{P_1 P_2}, \mathbf{v}_1} \rangle \right) \right)$$

$$= -(y_2 - y_1)(\cosh(y_2) - \cosh(y_1)) +$$

$$(\sinh(y_2) \cosh(y_1) - \sinh(y_1) \cosh(y_2))(\cosh(y_2) - \cosh(y_1))$$

$$= (\cosh(y_2) - \cosh(y_1))(\sinh(y_2 - y_1) - (y_2 - y_1)) > 0,$$

or

$$\langle \widehat{\overrightarrow{P_1 P_2}, \mathbf{v}_1} \rangle > \langle \widehat{\overrightarrow{P_1 P_2}, \mathbf{v}_2} \rangle.$$

This shows the first equality in (ii). $\qquad \square$

Proposition 2.10. *If $\alpha_{in} < \pi/2$ and $\alpha_{ex}[\beta_{in}, \alpha_{in}] < \pi/2$, then $\beta_{ex} = \beta_{ex}[\beta_{in}, \alpha_{in}]$ and $\alpha_{ex} = \alpha_{ex}[\beta_{in}, \alpha_{in}]$ are strictly increasing functions of β_{in} and α_{in}.*

Proof. By the definition of v in 2.2(iv), $\int_{\beta_{in}}^{\beta_{ex}} v(\beta) \mathrm{d}\beta = 0$. By the monotonicity in 2.2(iv), $v(\beta)$ has a unique inverse function $\beta(v)$ and thus

$$(2.11) \qquad \int_0^{v(\beta_{in})} \beta(v) - \beta_{in} \mathrm{d}v = \int_{v(\beta_{ex})}^0 \beta_{ex} - \beta(v) \mathrm{d}v.$$

From the theory of first order ordinary differential equations, if v^1 and v^2 are two solutions of the equation (2.3), the curves of the solutions $(\beta, v^1(\beta))$ and $(\beta, v^2(\beta))$ do not intersect with each other. Therefore, if $\beta_{in}^1 < \beta_{in}^2$, $\alpha_{in}^1 = \alpha_{in}^2$, v^1, v^2 are corresponding solutions, then

$$v^1(\beta) < v^2(\beta),$$

when $\beta \in (\beta_{in}^1, \beta_{ex}^1) \cap (\beta_{in}^2, \beta_{ex}^2)$. Moreover, when $v^1(\beta) = v^2(\beta')$, $\beta' > \beta$. Therefore, from (2.3),

$$\frac{\mathrm{d}v^1}{\mathrm{d}\beta}(\beta) < \frac{\mathrm{d}v^2}{\mathrm{d}\beta}(\beta') < 0,$$

when $v^1(\beta) = v^2(\beta') > 0$; and

$$0 > \frac{\mathrm{d}v^1}{\mathrm{d}\beta}(\beta) > \frac{\mathrm{d}v^2}{\mathrm{d}\beta}(\beta'),$$

when $v^1(\beta) = v^2(\beta') < 0$. Let β^1, β^2 denote the two inverse functions for v^1, v^2, this means $0 > \frac{\mathrm{d}\beta^1}{\mathrm{d}v} > \frac{\mathrm{d}\beta^2}{\mathrm{d}v}$ when $v > 0$ and $0 > \frac{\mathrm{d}\beta^2}{\mathrm{d}v} > \frac{\mathrm{d}\beta^1}{\mathrm{d}v}$ when $v < 0$. As $v^1(\beta_{in}^1) = \tan \alpha_{in}^1 = \tan \alpha_{in}^2 = v^2(\beta_{in}^2)$ from 2.9(iii), for any $v > 0$,

$\beta^1(v) - \beta^1_{in} < \beta^2(v) - \beta^2_{in}$. Moreover, when $v' < v < 0$, $\beta^1(v') - \beta^1(v) > \beta^2(v') - \beta^2(v)$. This means

$$(2.12) \qquad \int_0^{v^1(\beta^1_{in})} \beta^1(v) - \beta^1_{in} dv < \int_0^{v^2(\beta^2_{in})} \beta^2(v) - \beta^2_{in} dv$$

as $v^1(\beta^1_{in}) = v^2(\beta^2_{in})$. Moreover,

$$(2.13) \qquad \int_{v^2(\beta^2_{ex})}^0 \beta^2_{ex} - \beta^2(v) dv < \int_{v^1(\beta^1_{ex})}^0 \beta^1_{ex} - \beta^1(v) dv,$$

if $\beta^2_{ex} \leq \beta^1_{ex}$ or $-v^1(\beta^1_{ex}) \geq -v^2(\beta^2_{ex})$.

But by the equation (2.11) for v^1 along with (2.12) and (2.13),

$$\int_{v^2(\beta^2_{ex})}^0 \beta^2_{ex} - \beta^2(v) dv < \int_{v^1(\beta^1_{ex})}^0 \beta^1_{ex} - \beta^1(v) dv$$
$$= \int_0^{v^1(\beta^1_{in})} \beta^1(v) - \beta^1_{in} dv < \int_0^{v^2(\beta^2_{in})} \beta^2(v) - \beta^2_{in} dv,$$

which contradicts the equation (2.11) for v^2.

Therefore, it must be $\beta^1_{ex} < \beta^2_{ex}$ and $\tan\alpha^1_{ex} = -v^1(\beta^1_{ex}) < \tan\alpha^2_{ex} = -v^2(\beta^2_{ex})$ from 2.9(iii), i.e. β_{ex} and α_{ex} are increasing functions of β_{in}.

By the same argument, β_{ex} and α_{ex} are also strictly increasing functions of α_{in}. $\square$

Assumption 2.14. We assume from now on that $\beta_{in} \in (0, \pi/2]$ and $\alpha_{in} \in (0, \pi/3]$.

Corollary 2.15. *If 2.14 holds then $\alpha_{in} < \pi/2$ and $\alpha_{ex}[\beta_{in}, \alpha_{in}] < \pi/2$, and so 2.9(iii) holds.*

Proof. First it can be seen that the result holds if and only if $\alpha_{ex}[\beta_{in}, \alpha_{in}] < \pi/2$ under the assumptions 2.14. A direct calculation using 2.9(i) shows that the result is correct when $\beta_{in} = \pi/2$, $\alpha_{in} = \pi/3$. Therefore, $\beta_{in} < \pi/2$ and close enough to $\pi/2$, $\alpha_{ex}[\beta_{in}, \pi/3] < \pi/2$. Define β^0_{in} to be

$\beta^0_{in} := \sup\{\beta_{in} \in (0, \pi/2] :$

On $\mathbb{K}'_\mathbb{B}[\beta_{in}, \pi/3]$ r is a single-valued function with $|dr/d\beta| < \infty.\}$

If $\beta^0_{in} > 0$, then $\alpha_{ex}[\beta^0_{in}, \pi/3] \geq \pi/2$. However, for $\pi/2 \geq \beta_{in} > \beta^0_{in}$, on $\mathbb{K}'_\mathbb{B}[\beta_{in}, \pi/3]$ r is a single-valued function. Thus 2.10 implies that

$$\alpha_{ex}[\beta_{in}, \pi/3] \leq \alpha_{ex}[\pi/2, \pi/3] < \pi/2.$$

Therefore, $\alpha_{ex}[\beta^0_{in}, \pi/3] \leq \alpha_{ex}[\pi/2, \pi/3] < \pi/2$ by the continuity, which contradicts the assumption. And thus the result holds for $\mathbb{K}'_\mathbb{B}[\beta_{in}, \pi/3]$ with $\beta_{in} \in (0, \pi/2]$. A similar argument for α_{in} then implies the result for any β_{in}, α_{in} under the assumptions 2.14. $\square$

Existence and Uniqueness of the Initial Configurations.

Definition 2.16 (Catenoidal configurations). *We define a* catenoidal configuration *to be a union of a disc and catenoidal annuli $\mathcal{W} := \bigcup_{i=0}^{k'} \mathcal{A}_i$ such that $\mathcal{W} \cap \mathbb{S}^2 = \bigcup_{j=1}^{k'+1} \mathcal{C}_j$ and the following hold.*

(i) *For each $i = 1, \ldots, k'+1$, $\mathcal{C}_i$ is a parallel circle lying on $\mathbb{S}^2$ at latitude β_i.*

(ii) *$0 < \beta_1 < \cdots < \beta_{k'} < \pi/2$.*

(iii) *$\mathcal{A}_0$ is a flat disk and $\partial \mathcal{A}_0 = \mathcal{C}_1$.*

(iv) *For each $i = 1, \ldots, k'$ we have $\partial \mathcal{A}_i = \mathcal{C}_i \cup \mathcal{C}_{i+1}$ and $\mathcal{A}_i = \mathbb{K}[\beta_i, \alpha_i^+] \cap \mathbb{B}^3$ for a unique $\alpha_i^+ \in (0, \pi - \beta_i)$ (recall 2.6).*

(v) *For each $i = 1, \ldots, k'$, $\mathcal{C}_i = \mathcal{A}_{i-1} \cap \mathcal{A}_i$.*

For each $i = 1, \ldots, k'$ we have then $\beta_{i+1} = \beta_{ex}[\beta_i, \alpha_i^+]$ and we define $\alpha_{i+1}^- := \alpha_{ex}[\beta_i, \alpha_i^+]$ and $\alpha_1^- := \beta_1$. $k' \in \mathbb{N}$ is called the order *of $\mathcal{W}$. Finally we define the* unbalancing parameters *of $\mathcal{W}$, $\underline{\sigma}_{\mathcal{W}} = \{\sigma_{\mathcal{W},i}\}_{i=1}^{k'} \in \mathbb{R}^{k'}$, by requesting $e^{\sigma_{\mathcal{W},i}} = \alpha_i^+/\alpha_i^-$ for $i = 1, \ldots, k'$; if $\underline{\sigma}_{\mathcal{W}}$ vanishes we call $\mathcal{W}$* balanced.

Clearly we can try to construct catenoidal configurations by starting with a disk $\mathcal{A}_0$ and then proceeding inductively to construct $\mathcal{A}_i$ assuming $\underline{\sigma}_{\mathcal{W}}$ given. It is not clear however how this process will end. One possibility is that some $\beta_{k'+1}$ will exceed $\pi/2$ which is what we hope for. Although we will prove later that there is always a finite k' where this happens, we cannot exclude apriori the possibility that the $\mathcal{A}_i$'s become very narrow for large i and the β_i's form an increasing infinite sequence which is bounded above by $\pi/2$. This motivates the following definition.

Definition 2.17 (Complete catenoidal configurations). *We define a* complete catenoidal configuration *to be either a catenoidal configuration as in 2.16 which moreover satisfies $\beta_{k'+1} \geq \pi/2$, or a pair $\mathcal{W} := \{\underline{\mathcal{A}}, \underline{\mathcal{C}}\}$, where $\underline{\mathcal{A}} = \{\mathcal{A}_i\}_{i \in \mathbb{N}_0}$ and $\underline{\mathcal{C}} = \{\mathcal{C}_j\}_{j \in \mathbb{N}}$, and such that for $\forall k' \in \mathbb{N}$ the pair $\left\{ \{\mathcal{A}_i\}_{i=0}^{k'}, \{\mathcal{C}_j\}_{j=1}^{k'+1} \right\}$ is a catenoidal configuration as in 2.16. In the latter case we say that* the order *of $\mathcal{W}$ is ∞.*

Definition 2.18. *We define ℓ^1 the space of $\mathbb{R}$-valued sequences of finite ℓ^1 norm equipped with the ℓ^1 norm which is defined by $\| \{a_i\}_{i \in \mathbb{N}} : \ell^1 \| := \sum_{i=1}^{\infty} |a_i|$. We identify $\mathbb{R}^k$ with a subspace of ℓ^1 by the map which sends $\{a_i\}_{i=1}^k \in \mathbb{R}^k$ to the sequence $\{a_i\}_{i \in \mathbb{N}} \in \ell^1$ with $a_i = 0$ for $i > k$. Finally given $\underline{a} = \{a_i\}_{i \in \mathbb{N}} \in \ell^1$ we define $\underline{a}|_k := \{a_i\}_{i=1}^k \in \mathbb{R}^k$.*

Lemma 2.19 (Existence and uniqueness of $\mathcal{W}[\widehat{\beta}; \underline{\sigma}]$). *There is an absolute constant $\delta'_{\sigma} > 0$ such that $\forall \widehat{\beta} \in (0, 2\pi/7)$ and $\underline{\sigma} = \{\sigma_i\}_{i \in \mathbb{N}} \in \ell^1$ with $\|\underline{\sigma} : \ell^1\| < \delta'_{\sigma}$, there is a complete catenoidal configuration as in 2.17 which we denote by $\mathcal{W}[\widehat{\beta}; \underline{\sigma}]$, and is uniquely determined by the following properties, where we may use "$[\widehat{\beta}; \underline{\sigma}]$" to specify the quantities as in 2.16 associated to $\mathcal{W}[\widehat{\beta}; \underline{\sigma}]$.*

(i) *$\beta_1[\widehat{\beta}; \underline{\sigma}] = \widehat{\beta}$.*

(ii) $\sigma_{\mathcal{W}[\widehat{\beta};\underline{\sigma}],i} = \sigma_i$ for $i \leq k'[\widehat{\beta};\underline{\sigma}]$ if the order $k'[\widehat{\beta};\underline{\sigma}] < \infty$ and $\forall i \in \mathbb{N}$ otherwise.

Moreover we have the following.

(a) If the order $k'[\widehat{\beta};\underline{\sigma}] < \infty$, then $\mathcal{W}[\widehat{\beta};\underline{\sigma}]$ depends only on $\widehat{\beta}$ and $\{\sigma_i\}_{i=1}^{k'}$.

(b) The assumption 2.14 holds for all catenoidal annuli in $\mathcal{W}[\widehat{\beta};\underline{\sigma}]$, and moreover $\mathcal{W}[\widehat{\beta};\underline{\sigma}] \cap \mathbb{R}_+^2$ is the radial graph of a piecewise smooth function $\mathrm{r}[\widehat{\beta};\underline{\sigma}] : B \to (0,1]$ (recall 2.1), where $B = [0, \beta_{k'+1}]$ if the order $k'[\widehat{\beta};\underline{\sigma}] < \infty$ and $B = [0, \sup_{i \in \mathbb{N}} \beta_i)$ otherwise.

(c) We have $\alpha_{i+1}^- < \alpha_i^+$ and $\alpha_i^+ < e^{\sigma_1 + \cdots + \sigma_i}\widehat{\beta}$, for $1 \leq i \leq k'[\widehat{\beta};\underline{\sigma}]$ if the order $k'[\widehat{\beta};\underline{\sigma}] < \infty$ and $\forall i \in \mathbb{N}$ otherwise; and we have $\alpha_i^+ = e^{\sigma_i}\alpha_i^- < e^{\sigma_i}\alpha_{i-1}^+$, for $2 \leq i \leq k'[\widehat{\beta};\underline{\sigma}]$ if the order $k'[\widehat{\beta};\underline{\sigma}] < \infty$ and $\forall i \in \mathbb{N} \setminus \{1\}$ otherwise.

Proof. From the definition 2.16 and the assumption $\widehat{\beta} \in (0, 2\pi/7)$, $\alpha_1^+ = e^{\sigma_1}\alpha_1^- = e^{\sigma_1}\beta_1 < \pi - \beta_1$ if $|\sigma_1|$ is smaller than a constant, then the existence and uniqueness of $\mathcal{A}_1$ simply follows by 2.6. Suppose now the catenoidal annulus $\mathcal{A}_n$ exists in the definition 2.17. We have for $i = 2, \ldots, n$

$$\alpha_i^+ = e^{\sigma_i}\alpha_i^- < e^{\sigma_i}\alpha_{i-1}^+,$$

where the equality follows from 2.16 and the inequality from 2.9(ii); and thus by induction

$$\alpha_n^+ < e^{\sigma_1 + \cdots + \sigma_n}\widehat{\beta}.$$

Therefore, if $\|\underline{\sigma} : \ell^1\|$ is small enough, $\alpha_n^+ < \pi - \widehat{\beta} < \pi - \beta_n$ as $\widehat{\beta} < 2\pi/7$. The existence and uniqueness of $\mathcal{A}_{n+1}$ then follows by 2.6. (a) and (c) then follow. Moreover, (c) along with the assumption that $\widehat{\beta} < 2\pi/7$ implies that the assumption 2.14 holds for all $\mathcal{A}_i \subset \mathbb{K}[\beta_i, \alpha_i^+]$ in $\mathcal{W}[\widehat{\beta};\underline{\sigma}]$ if $\|\underline{\sigma} : \ell^1\|$ is small enough. Therefore, by 2.15 and then 2.8, the function $\mathrm{r}[\widehat{\beta};\underline{\sigma}]$ exists as the radial graph of $\mathcal{W}[\widehat{\beta};\underline{\sigma}] \cap \mathbb{R}_+^2$. This proves (b). $\square$

Proposition 2.20. *For $\widehat{\beta}$ and $\underline{\sigma}$ as in 2.19 the following hold for each fixed $\underline{\sigma}$.*

(i) The order $k'[\widehat{\beta};\underline{\sigma}] < \infty$ and it is a decreasing function of $\widehat{\beta}$ with $k'[\widehat{\beta};\underline{\sigma}] \to \infty$ as $\widehat{\beta} \to 0$.

(ii) $\beta_i[\widehat{\beta};\underline{\sigma}]$, $\alpha_i^\pm[\widehat{\beta};\underline{\sigma}]$ (when defined) are continuous strictly increasing functions of $\widehat{\beta}$.

(iii) $\forall i \in \mathbb{N}$ we have $\lim_{\widehat{\beta} \to 0} \beta_i[\widehat{\beta};\underline{\sigma}] = 0$.

Proof. By the theory of ordinary differential equations and an inductive construction of $\mathcal{W}[\widehat{\beta};\underline{\sigma}]$ we conclude that $\beta_i[\widehat{\beta};\underline{\sigma}]$, $\alpha_i^+[\widehat{\beta};\underline{\sigma}]$, and $\alpha_i^-[\widehat{\beta};\underline{\sigma}]$ are continuous functions of β_1 and $\underline{\sigma}$. Moreover, they are strictly increasing functions of $\widehat{\beta}$ by 2.19(b), 2.15 and 2.10 and hence (ii) follows.

(iii) for $i = 1$ is clear by the definition. Now for $i \geq 2$, suppose the result holds for $\beta_1, \ldots, \beta_{i-1}$. Consider the chord tangent to $\mathcal{A}_{i-1}'$ at the position

$r = 1$, $\beta = \beta_{i-1}$. Clearly the chord will also intersect $\mathbb{S}^1$ at $\beta = \beta_{i-1} + 2\alpha_{i-1}^+$, and thus by convexity of the catenoid

$$(2.21) \qquad\qquad \beta_i < \beta_{i-1} + 2\alpha_{i-1}^+.$$

Therefore, the result for β_i follows by the assumption for β_{i-1} and 2.19(c).

Clearly the definition implies that $k'[\widehat{\beta}; \underline{\sigma}] \in \mathbb{N} \cup \{\infty\}$, by (ii) it is a decreasing function of $\widehat{\beta}$, and by (iii) its limit as $\widehat{\beta} \to 0$ is ∞. It remains to prove that it is never ∞. We argue by contradiction and assume that it is for some $\widehat{\beta}$ which can be assumed as small as needed by the monotonicity of k' in $\widehat{\beta}$. $\{\beta_i\}$ is increasing and bounded by $\pi/2$ and so it converges to some $\beta_\infty \leq \pi/2$. By 2.2 $\forall i \in \mathbb{N}$ there is a catenoid $\mathbb{K}_i \supset \mathcal{A}_i$. Let $a_i := a^{\mathbb{K}_i}$ and $b_i := b^{\mathbb{K}_i}$. Clearly they are both positive and uniformly bounded above.

If any subsequence of $\{\alpha_i^+\}$ have a positive lower bound, then $\sum_{i=1}^\infty \alpha_i^+ = \infty$. On the other hand, if $\lim_{i \to \infty} \alpha_i^+ = 0$, then an elementary calculation now using Taylor's expansion for cosh and the uniform bounds for a_i and b_i implies that for some $C > 0$ independent of i we have

$$|\alpha_i^+ - \alpha_{i+1}^-| \leq C \left(\cos \beta_{i+1} - \cos \beta_i\right)^2.$$

By (2.21) and 2.19(c) we have $\cos \beta_i - \cos \beta_{i+1} \leq \beta_{i+1} - \beta_i < 2\alpha_i^+$ and hence

$$\alpha_i^+ - 4C \left(\alpha_i^+\right)^2 \leq \alpha_{i+1}^- = e^{-\sigma_{i+1}} \alpha_{i+1}^+.$$

which implies since $\lim_{i \to \infty} \alpha_i^+ = 0$ that

$$\frac{1}{\alpha_{i+1}^+} \leq \frac{e^{-\sigma_{i+1}}}{\alpha_i^+ (1 - 4C\alpha_i^+)} = \frac{e^{-\sigma_{i+1}}}{\alpha_i^+} + \frac{4e^{-\sigma_{i+1}} C}{1 - 4C\alpha_i^+}$$

$$\leq e^{-\sigma_{i+1}} \left(\frac{1}{\alpha_i^+} + 8C\right) \leq e^{\|\underline{\sigma}:\ell^1\|} \left(\frac{1}{\widehat{\beta}} + 8Ci\right),$$

where the last inequality follows by induction. We conclude that $\sum_{i=1}^\infty \alpha_i^+ = \infty$ by comparing with the harmonic series.

If $\beta_\infty < \pi/2$, we can assume that $\beta_i + \alpha_i^+ < (\pi/2 + \beta_\infty)/2 < \pi/2$. Therefore, by 2.9(i) and mean value theorem,

$$(2.22) \quad a_i - a_{i-1} = 2\sin(\beta_i) \cos\left(\frac{2\beta_i + \alpha_i^+ - \alpha_i^-}{2}\right) \sin\left(\frac{\alpha_i^+ + \alpha_i^-}{2}\right)$$

$$> C\left(\alpha_i^- + \alpha_i^+\right) > 0$$

as $2\beta_i + \alpha_i^+ - \alpha_i^- < \pi$. Thus a_i is increasing, and there is a limit $a_\infty := \lim_{n \to \infty} a_n$. We conclude from 2.22 that $\sum_{i=1}^\infty \alpha_i^+ < \infty$, a contradiction.

Hence $\beta_\infty = \pi/2$. By applying 2.8(i) with $\beta_{in} = \beta_n$, $\alpha_{in} = \alpha_n^+$, or with $\beta_{ex} = \beta_n$, $\alpha_{ex} = \alpha_n^- = e^{-\sigma_n} \alpha_n^+$, and subtracting, we obtain

$$b_{n-1} - b_n = \sin \beta_n \left(f(\beta_n + \alpha_n^+) - f(\beta_n - e^{-\sigma_n} \alpha_n^+)\right)$$

where $f(t) := \mathrm{sgn}(t - \pi/2) \sin t \, \mathrm{arcosh}(1/\sin t)$. Since $\mathrm{arcosh}(1/\sin t)$ $= \log \frac{1 + |\cos t|}{\sin t} = \frac{1}{2} \log \frac{1 + |\cos t|}{1 - |\cos t|}$, we see that we can expand in odd powers of $|\cos t|$. Clearly then f is a smooth function in the vicinity of $\pi/2$ and moreover its derivative at $\pi/2$ is positive. This implies by continuity and

the mean value theorem that $b_{n-1} - b_n \geq C\alpha_n$ for large n, which implies that $\sum_{i=1}^{\infty} \alpha_i^+ < \infty$, a contradiction. This completes the proof. □

Definition 2.23 (Symmetrically extendible catenoidal configurations)**.**
We define a symmetrically extendible catenoidal configuration *to be a complete catenoidal configuration $\mathcal{W}$ as in 2.17 which moreover satisfies either $\beta_{k'+1} = \pi/2$ or $\beta_{k'+1} + \beta_{k'} = \pi$. The* total order k *of $\mathcal{W}$ is defined to be $2k' + 1$ in the former case and $2k'$ in the latter.*

Corollary 2.24 (Existence and uniqueness of $\mathcal{W}[\underline{\sigma}{:}k]$)**.** *Given $k \in \mathbb{N} \setminus \{1, 2\}$ and $\underline{\sigma} = \{\sigma_i\}_{i \in \mathbb{N}} \in \ell^1$ with $\|\underline{\sigma} : \ell^1\| < \delta_\sigma'' \in (0, \delta_\sigma')$, where δ_σ' is as in 2.19, there is a unique $\widehat{\beta} = \widehat{\beta}[\underline{\sigma}{:}k] \in (0, \pi/2)$ such that $\mathcal{W} = \mathcal{W}[\widehat{\beta}; \underline{\sigma}]$ is symmetrically extendible and has total order k (recall 2.23). Moreover $\widehat{\beta}[\underline{\sigma}{:}k]$ is decreasing in k for fixed $\underline{\sigma}$, $\widehat{\beta}[\underline{\sigma}{:}k] \to 0$ when $k \to \infty$, and $\widehat{\beta}[\underline{\sigma}{:}k]$ depends continuously on $\underline{\sigma}|_{[k/2]}$ for fixed k.*

Proof. In this proof, we define an *extended complete catenoidal configuration* to be a catenoidal configuration same as the catenoidal configuration in 2.17 except it satisfies $\beta_{k'+1} > \pi/2$ and $\beta_{k'} \leq \pi/2$. As in 2.19, given $\forall \widehat{\beta} \in (0, 2\pi/7)$ and $\underline{\sigma} = \{\sigma_i\}_{i \in \mathbb{N}} \in \ell^1$ with $\|\underline{\sigma} : \ell^1\| < \delta_\sigma'$, there is a unique extended complete catenoidal configuration satisfying the properties in 2.19, which we denote by $\mathcal{W}[\![\widehat{\beta}; \underline{\sigma}]\!]$. As in 2.19, we will use "$[\![\widehat{\beta}; \underline{\sigma}]\!]$" to specify the quantities associated to $\mathcal{W}[\![\widehat{\beta}; \underline{\sigma}]\!]$.

If $\beta_1 < 2\pi/7$ and $\|\underline{\sigma} : \ell^1\| \leq \delta_\sigma'$, by 2.20(ii), each $\beta_i[\widehat{\beta}; \underline{\sigma}]$ is an increasing function of β_1, which then implies the uniqueness of $\mathcal{W}[\underline{\sigma}{:}k]$. By 2.19(b), the assumption 2.14 holds if $\widehat{\beta}[\underline{\sigma}{:}k]$ exists.

We first show the existence of $\widehat{\beta}[0{:}k]$ with $\widehat{\beta}[0{:}3] \in (\pi/4, 2\pi/7)$. By the definition, $\widehat{\beta}W[0{:}3]$ exists if and only if there exists a catenoidal arc $\mathbb{K}_{\mathbb{B}}'[\beta_{in}, \beta_{in}]$ such that $\beta_{ex}[\beta_{in}, \beta_{in}] = \pi/2$.

By the formulae in 2.9(i), the catenoidal arc $\mathbb{K}_{\mathbb{B}}'[\pi/4, \pi/4]$ satisfies the equation

$$x = \frac{\sqrt{2}}{2} \cosh\left(\sqrt{2}y - 1\right),$$

which intersects $y = 0$ at $(\frac{\sqrt{2}\cosh 1}{2}, 0) \approx (1.0911, 0)$ outside $\mathbb{S}^1$; while the catenoidal arc $\mathbb{K}_{\mathbb{B}}'[2\pi/7, 2\pi/7]$ satisfies the equation

$$x = \sin\frac{2\pi}{7} \sin\frac{4\pi}{7} \cosh\left(\frac{y - \cos\frac{2\pi}{7} - \sin\frac{2\pi}{7}\sin\frac{4\pi}{7}\mathrm{arcosh}\frac{1}{\sin\frac{4\pi}{7}}}{\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}}\right),$$

which intersects $y = 0$ at

$$\left(\sin\frac{2\pi}{7} \sin\frac{4\pi}{7} \cosh\left(\cot\frac{2\pi}{7}\csc\frac{4\pi}{7} + \mathrm{arcosh}\frac{1}{\sin\frac{4\pi}{7}}\right), 0\right) \approx (0.8996, 0)$$

inside $\mathbb{S}^1$. Therefore, the angle $\beta_{ex}[\pi/4, \pi/4] < \pi/2$ while $\beta_{ex}[2\pi/7, 2\pi/7] > \pi/2$. The result then follows by the intermediate value theorem and the monotonicity 2.10. Moreover, the assumption 2.14 holds.

Now when $\underline{\sigma}|_1$ has absolute value smaller than a value δ''_σ small enough, by implicit function theorem and 2.20(ii), there exists a unique $\widehat{\beta}$ near $\widehat{\beta}[0;3]$, such that $\beta_2[\underline{\sigma};\beta_1] = \pi/2$. Thus $\widehat{\beta}[\underline{\sigma}{:}3]$ exits.

Furthermore, suppose that the results hold for $k = 2k'-1$. Then for $\underline{\sigma} \in \ell^1$ with $\|\underline{\sigma} : \ell^1\| < \delta''_\sigma$, $\widehat{\beta}[\underline{\sigma}{:}2k'-1]$ exists. Therefore, in the extended complete catenoidal configuration $\mathcal{W}[\![\widehat{\beta}[\underline{\sigma}{:}2k'-1];\underline{\sigma}]\!]$ with finite order k', $\beta_{k'} = \pi/2$ and $\beta_{k'} + \beta_{k'+1} > \pi$. As in 2.20(ii), $\beta_{k'}[\![\widehat{\beta};\underline{\sigma}]\!]$ and $\beta_{k'+1}[\![\widehat{\beta};\underline{\sigma}]\!]$ are strictly increasing functions of $\widehat{\beta}$. By intermediate theorem and 2.20(iii), there exists a $\widehat{\beta}[\underline{\sigma}{:}2k'] < \widehat{\beta}[\underline{\sigma}{:}2k'-1]$ such that in $\mathcal{W}[\widehat{\beta}[\underline{\sigma}{:}2k'];\underline{\sigma}] = \mathcal{W}[\![\widehat{\beta}[\underline{\sigma}{:}2k'];\underline{\sigma}]\!]$, $\beta_{k'} + \beta_{k'+1} = \pi$. Thus by 2.23 $\widehat{\beta}[\underline{\sigma}{:}2k']$ exists. A similar argument then shows that $\widehat{\beta}[\underline{\sigma}{:}2k'+1]$ exists.

By the implicit function theorem, 2.20(ii) and 2.23, $\widehat{\beta}[\underline{\sigma}{:}k]$ is a continuous function of $\underline{\sigma}$ in a neighborhood of each $\underline{\sigma}$ with $\|\underline{\sigma} : \ell^1\| < \delta''_\sigma$. Finally, by the monotonicity of k' with respect to $\widehat{\beta}$ in 2.20(i) and the uniqueness of $\widehat{\beta}[\underline{\sigma}{:}k]$ in 2.24, $\widehat{\beta}[\underline{\sigma}{:}k] \to 0$ when $k \to \infty$. $\square$

Notation 2.25. We define $\mathcal{W}[\underline{\sigma}{:}k] := \mathcal{W} \cup \underline{\mathsf{R}}\mathcal{W}$, where $\underline{\mathsf{R}}$ is the reflection with respect to the xz-plane and $\mathcal{W} = \mathcal{W}[\widehat{\beta}[\underline{\sigma}{:}k];\underline{\sigma}]$ is as in 2.16. For $1 \leq i \leq k$ we define $\alpha_i^\pm[\underline{\sigma}{:}k]$ and $\beta_i[\underline{\sigma}{:}k]$ to be the values of $\alpha_i^\pm$ and β_i respectively in the configuration $\mathcal{W}$ for $i \leq k' := [k/2]$ and by $\alpha_i^\pm[\underline{\sigma}{:}k] := \alpha_{k-i+1}^\mp[\underline{\sigma}{:}k]$, $\beta_i[\underline{\sigma}{:}k] := \pi - \beta_{k-i+1}[\underline{\sigma}{:}k]$ for $i > k'$; when $\underline{\sigma} = 0$ we take $\alpha_i[0{:}k] := \alpha_i^+[0{:}k] = \alpha_i^-[0{:}k]$. For $0 \leq j \leq k$ and $1 \leq l \leq k$ we define $\mathcal{A}_j[\underline{\sigma}{:}k]$ and $\mathcal{C}_l[\underline{\sigma}{:}k]$ to be the catenoidal annulus (or disk) $\mathcal{A}_j$ and circle $\mathcal{C}_l$ respectively in the configuration $\mathcal{W}$ if $j \leq k'$ or $l \leq k'+1$, and by $\mathcal{A}_j[\underline{\sigma}{:}k] := \underline{\mathsf{R}}\mathcal{A}_{k-j}[\underline{\sigma}{:}k]$ and $\mathcal{C}_l[\underline{\sigma}{:}k] := \underline{\mathsf{R}}\mathcal{C}_{k-l+1}[\underline{\sigma}{:}k]$ otherwise. Finally we define $\mathrm{r}[\underline{\sigma}{:}k] : [0,\pi] \to (0,1]$ to be the extension of $\mathrm{r}[\widehat{\beta}[\underline{\sigma}{:}k];\underline{\sigma}]$ as in 2.19(b) so that its radial graph is $\mathcal{W}[\underline{\sigma}{:}k] \cap \mathbb{R}_+^2$.

Corollary 2.26. *Assuming that k is large enough in terms of a given $\epsilon > 0$ and $\underline{\sigma}$ is as in 2.24, $\mathrm{r} = \mathrm{r}[\underline{\sigma}{:}k]$ defined as in 2.25 satisfies*

$$\left\| \mathrm{r} - 1 : C^0[0,\pi/2] \right\| + \left\| \mathrm{dr}/\mathrm{d}\beta : C^0[0,\pi/2] \right\| \leq \epsilon.$$

Proof. For the catenary $\mathbb{K}'$ corresponding to each catenoidal annulus $\mathcal{A}_i$ in $\mathcal{W}[\underline{\sigma}{:}k]$, define l_i to be the chord tangent to $\mathbb{K}'$ at $(\sin\beta_i, \cos\beta_i)$. The radial distance between $\mathcal{A}_i$ and the unit sphere is then controlled by the radial distance between l_i and the unit circle. From the inequality $\alpha_i^+ < e^{\|\underline{\sigma}:\ell_1\|}\widehat{\beta}$ by 2.19(c),

$$\max(1 - \mathrm{r}(\beta)) = \max_{i=1,\ldots,k}\{1 - \cos\alpha_i^+\} \leq 1 - \cos(e^{\delta''_\sigma}\widehat{\beta}),$$

where $\alpha_i^+ := \alpha_i^+[\underline{\sigma}{:}k]$. Thus $\max(1 - \mathrm{r}(\beta)) \to 0$ when $k \to \infty$ by 2.24.
   Similarly, from 2.9(ii), 2.9(iii),

$$\max\left|\frac{\mathrm{dr}}{\mathrm{d}\beta}(\beta)/\mathrm{r}(\beta)\right| = \max_{i=1,\ldots,k}\{\tan\alpha_i^+\} \leq \tan(e^{\delta''_\sigma}\widehat{\beta}).$$

Thus from the result above, $\max|\mathrm{dr}/\mathrm{d}\beta| \to 0$ when $k \to \infty$ by 2.24. $\square$

Lemma 2.27. *In the balanced configuration $\mathcal{W}[0{:}k]$,*

(i) *For $i = 1, \ldots, k$, $\alpha_i[0{:}k] \leq \beta_1[0{:}k]$.*
(ii) *For $i = 1, \ldots, k-1$, $\beta_{i+1}[0{:}k] - \beta_i[0{:}k] \leq 2\beta_1[0{:}k]$.*

Proof. By the symmetry of the balanced configuration, it can be assumed that $i \leq \frac{k+1}{2}$. (i) then just follows by the balanced condition and (c). (ii) just follows by (2.21) and (i). $\qquad\square$

Lemma 2.28. *If k is large enough in absolute terms, then the kernel of $\mathcal{L}_{\mathcal{A}_i[0{:}k]}$ with Dirichlet boundary condition for each disk or catenoidal annulus $\mathcal{A}_i[0{:}k]$ in the balanced configuration $\mathcal{W}[0{:}k]$ is trivial.*

Proof. For the disks $\mathcal{A}_0[0{:}k]$ and $\mathcal{A}_k[0{:}k]$, the result simply follows by the smallness of β_1. Suppose $\mathcal{A}_i[0{:}k]$, $i = 1, \ldots, k-1$ satisfies the equation $x = a_i \cosh(\frac{y - b_i}{a_i})$, let $\alpha_i := \alpha_i[0{:}k]$, $\beta_i := \beta_i[0{:}k]$. For any rotationally invariant function u, $\mathcal{L}_{\mathcal{A}_i[0{:}k]} u = 0$ is equivalent with

$$u''(y) + \frac{2}{a_i^2} \operatorname{sech}^2\left(\frac{y - b_i}{a_i}\right) u(y) = 0.$$

The solution space is spanned by $u = \tanh \frac{y - b_i}{a_i}$ and $u = 1 - \frac{y - b_i}{a_i} \tanh \frac{y - b_i}{a_i}$. The Dirichlet boundary condition is equivalent with

$$u\left(\cos \beta_i\right) = 0, \quad u\left(\cos \beta_{i+1}\right) = 0.$$

The solution exists if and only if the following condition holds

$$(2.29) \quad 0 = \det \begin{pmatrix} \tanh z_1 & 1 - z_1 \tanh z_1 \\ \tanh z_2 & 1 - z_2 \tanh z_2 \end{pmatrix}$$
$$= \tanh z_1 - \tanh z_2 + (z_1 - z_2) \tanh z_1 \tanh z_2,$$

where $z_1 = \frac{\cos \beta_i - b_i}{a_i}$ and $z_2 = \frac{\cos \beta_{i+1} - b_i}{a_i}$.

When k is even, for the catenoidal annulus $\mathcal{A}_{\frac{k}{2}}[0{:}k]$, the result follows by the smallness of the catenoidal pieces and the fact that $a_{\frac{k}{2}}$ is close to 1 from 2.26. For all other catenoidal annuli, in odd or even case, by the symmetry it can be assumed that $\beta_{i+1} \leq \pi/2$, then by 2.9(i),

$$\frac{\cos \beta_i - \cos \beta_{i+1}}{a_i} = \frac{\cos \beta_i - \cos \beta_{i+1}}{\sin \beta_{i+1} \sin(\beta_{i+1} - \alpha_{i+1})}$$
$$\leq \frac{(\beta_{i+1} - \beta_i) \sin \beta_{i+1}}{\sin \beta_{i+1} \sin(\beta_{i+1} - \alpha_{i+1})} = \frac{\beta_{i+1} - \beta_i}{\sin(\beta_{i+1} - \alpha_{i+1})}.$$

By 2.9(ii) and the balanced condition, α_i is decreasing for $i \leq \frac{k+1}{2}$, and β_i is increasing, therefore,

$$\frac{\cos \beta_i - \cos \beta_{i+1}}{a_i} \leq \frac{\beta_{i+1} - \beta_i}{\sin(\beta_2 - \alpha_2)}.$$

However, if $\beta_2 - \alpha_2 < \beta_1$, then as $\beta_1 = \alpha_1 > \alpha_2$ by 2.9(ii), $\beta_2 - \beta_1 < \beta_2 - \alpha_2 < \beta_1$, and thus $\beta_2 < 2\beta_1$. But these contradict the formula $\sin \beta_1 \sin(2\beta_1) = a_1 = \sin \beta_2 \sin(\beta_2 - \alpha_2)$ by 2.9(i). This means $\beta_2 - \alpha_2 \geq \beta_1$ and thus by 2.27(ii)

$$\frac{\cos \beta_i - \cos \beta_{i+1}}{a_i} \leq \frac{\beta_{i+1} - \beta_i}{\sin \beta_1} \leq 2 \frac{\beta_1}{\sin \beta_1}.$$

By 2.24, $\lim_{k\to\infty} \frac{\beta_1}{\sin\beta_1} = 1$. This means $z_1 - z_2 = \frac{(\cos\beta_i - b_i) - (\cos\beta_{i+1} - b_i)}{a_i} \le 2 + \epsilon$ when k is big enough, where ϵ is a small positive number. However, then 2.29 can not hold except when $z_1 = z_2$. $\qquad\square$

Proposition 2.30. *For $k \in \mathbb{N}$ large enough in absolute terms there are constants $\delta_\sigma = \delta_\sigma(k) \in (0, \delta''_{\underline{\sigma}})$ (recall 2.24), $c = c(k)$, and $\delta'_\theta = \delta'_\theta(k)$, such that the following hold for all $\underline{\sigma}$ with $\|\underline{\sigma} : \ell_1\| \le \delta_\sigma$.*

* (i) *$\alpha_i^+[\underline{\sigma}{:}k] \in [30\delta'_\theta, \frac{\pi}{2} - 30\delta'_\theta]$ and $\left|\alpha_i^-[\underline{\sigma}{:}k] - \alpha_i^+[\underline{\sigma}{:}k]\right| \le \delta'_\theta$ for each $i \in \mathbb{N}$ with $1 \le i \le k$.*
* (ii) *The catenoid $\mathbb{K}_i$ containing $\mathcal{A}_i[\underline{\sigma}{:}k]$ satisfies $a_{\mathbb{K}_i} > \delta'_\theta$ for each $i \in \mathbb{N}$ with $1 \le i \le k - 1$.*
* (iii) *There are no Dirichlet eigenvalues of $\mathcal{L}_{\mathcal{A}_i[\underline{\sigma}{:}k]}$ on $\mathcal{A}_i[\underline{\sigma}{:}k]$ in the interval $[-c, c]$ for each $i \in \mathbb{N}_0$ with $0 \le i \le k$.*

Proof. For the balanced case, (i) and (ii) follow by choosing δ'_θ small enough. (iii) follows by 2.28. The general case then follows by the continuous dependence of solution to $\underline{\sigma}$ in 2.24 and choosing δ_σ small enough. $\qquad\square$

3. The Desingularizing Surfaces

In this section the desingularizing surface $\Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ is defined, which is similar to the surfaces $\Sigma[T, \phi, \tau]$ in [8], to which we refer for some details and proofs.

The Scherk Surfaces $\Sigma(\theta)$.

Definition 3.1. *For $\theta \in (0, \pi/2)$, the Scherk surface $\Sigma(\theta)$ is defined by the equation (recall 1.2)*

$$(3.2) \qquad \sin^2\theta \cosh\frac{x}{\sin\theta} - \cos^2\theta \cosh\frac{y}{\cos\theta} = \cos z.$$

Notation 3.3. $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ are defined to be the coordinate unit vectors of the Cartesian coordinate system $O.xyz$. $\vec{e}[\theta]$, $\vec{e}'[\theta]$ are defined by

$$\vec{e}[\theta] = \cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y, \quad \vec{e}'[\theta] = -\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_y.$$

The identity map is denoted by $\underline{R}_1$, and reflections with respect to yz-plane, z-axis, xz-plane are denoted by $\underline{R}_2$, $\underline{R}_3$, $\underline{R}_4$ respectively. Let R_ϕ denote the rotation around the z-axis by an angle ϕ.

Note that $\underline{R}_i$ is a symmetry of each $\Sigma(\theta)$ mapping the first quadrant to the ith quadrant for $i = 1, 2, 3, 4$.

Assumption 3.4. To ensure uniform bounds in the geometry of the Scherk surfaces $\Sigma(\theta)$ we assume from now on that $\theta \in [10\delta_\theta, \pi/2 - 10\delta_\theta]$, where $\delta_\theta > 0$ is a small number to be determined later.

Proposition 3.5 (Properties of the Scherk surfaces). *$\Sigma(\theta)$ is a singly periodic embedded complete minimal surface which depends smoothly on θ and moreover satisfies the following for θ as in Assumption 3.4.*

* (i) *$\Sigma(\theta)$ is invariant under $\underline{R}_i$ for $i = 1, 2, 3, 4$ and $\underline{R}_{\{z=n\pi\}}$, the reflection with respect to the plane $\{z = n\pi\}$, for $n \in \mathbb{Z}$.*

(ii) *For given $\varepsilon \in (0, 10^{-3})$ there is a number $a = a(\varepsilon, \delta_\theta)$ and smooth functions $f_\theta := f_{\theta,a} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $A_\theta := A_{\theta,a} : \mathbb{R}_+ \times \mathbb{R} \to E^3$, and $F_\theta := F_{\theta,a} : \mathbb{R}_+ \times \mathbb{R} \to E^3$ depending smoothly on θ (and a), such that $W_\theta := W_{\theta,a} := F_\theta(\mathbb{R}_+ \times \mathbb{R}) \subset \Sigma(\theta)$ and*

$$A_\theta(s, z) = (a + s)\vec{e}[\pi/2 - \theta] + z\vec{e}_z + b_\theta \vec{e}'[\pi/2 - \theta],$$
$$F_\theta(s, z) = A(s, z) + f_\theta(s, z)\vec{e}'[\pi/2 - \theta],$$

where $b_\theta = \sin 2\theta \log(\tan \theta)$. Moreover

(iii) $\Sigma(\theta) \setminus \cup_{i=1}^4 \underline{R}_i(W_\theta)$ *is connected and lies within distance $a + 1$ from the z-axis.*

(iv) $W_\theta \subset \{(r\cos\phi, r\sin\phi, z) : r > a, \phi \in [9\delta_\theta, \pi/2 - 9\delta_\theta], z \in \mathbb{R}\}$.

(v) $\left\| f_\theta : C^5(\mathbb{R}_+ \times \mathbb{R}, e^{-s}) \right\| \le \varepsilon$ *and* $\left\| df_\theta/d\theta : C^5(\mathbb{R}_+ \times \mathbb{R}, e^{-s}) \right\| \le \varepsilon$.

(vi) $|b_\theta| + |db_\theta/d\theta| \le \varepsilon a$.

(vii) $\partial\Sigma(\theta)_{x \le 0}$ *(recall 1.13) is a disjoint union of topological circles contained in the yz-plane.*

Proof. (vii) follows by the equation (3.2). All the others are the same as [8, Proposition 2.4]. □

Definition 3.6 (Core, wings, and coordinates)**.** *For $i = 1, 2, 3, 4$ we call $\underline{R}_i(W_\theta)$ the ith wing of $\Sigma(\theta)$. We define coordinates $(s, z) = (s_a, z)$ on $\underline{R}_i(W_\theta)$ by $(s, z) = (\underline{R}_i \circ F_\theta)^{-1}$. The function s is extended to a continuous function on $\Sigma(\theta)$, by requiring s to vanish on $\Sigma(\theta) \setminus \cup_{i=1}^4 \underline{R}_i(W_\theta)$. The part $\Sigma(\theta)_{s \le 0}$ will be called the core of $\Sigma(\theta)$. Finally, the third and second wing of $\Sigma(\theta)$ will be called the $+$ and $-$ wing of half Scherk surface $\Sigma(\theta)_{x \le 0}$ and the part $\Sigma(\theta)_{s \le 0, x \le 0}$ will be called the core of half Scherk surface $\Sigma(\theta)_{x \le 0}$.*

Proposition 3.7 (The Gauss map of the Scherk surfaces)**.** *The Gauss map ν of $\Sigma(\theta)$ has the following properties:*

(i) *ν restricts to a diffeomorphism from $\Sigma(\theta)_{z \in [0, \pi]}$ (recall 1.13) onto*

$$\mathbb{S}^2 \cap \{z \ge 0\} \setminus \{(\pm\cos\theta, \pm\sin\theta, 0)\}.$$

(ii) *Let E_i, $i = 1, 2, 3, 4$ be the arcs into which the equator $\mathbb{S}^2 \cap \{z = 0\}$ is decomposed by removing the points $(\pm\cos\theta, \pm\sin\theta, 0)$, numbered so that $(1, 0, 0) \in E_1$, $(0, 1, 0) \in E_2$, $(-1, 0, 0) \in E_3$, and $(0, -1, 0) \in E_4$, then*

$$\nu(\Sigma(\theta)_{z=0}) = E_1 \cup E_3, \quad \nu(\Sigma(\theta)_{z=\pi}) = E_2 \cup E_4.$$

(iii) *$\Sigma(\theta)$ has no umbilics and $\nu^* g_{\mathbb{S}^2} = \frac{1}{2} |A|^2_{\Sigma(\theta)} g_{\Sigma(\theta)}$.*

Proof. See [8, Proposition 2.6]. □

Definition 3.8. *For $\Sigma = \Sigma(\theta)$ we define $h := \frac{1}{2} |A|^2_\Sigma g_\Sigma = \nu^* g_{\mathbb{S}^2}$ and $\mathcal{L}_h := \Delta_h + 2$.*

The following proposition for the eigenvalue of $\mathcal{L}_h$ is the same as [8, Proposition 2.8]. We are interested in the eigenfunction invariant under the reflection with respect to the yz-plane $\underline{R}_2$.

Proposition 3.9 (Eigenvalues on Scherk surfaces modulo symmetries)**.**
*Let G be the group generated by reflections across the planes $\{z = 0\}$ and
$\{z = \pi\}$. Let $\Sigma := \Sigma(\theta)$. There is an $\epsilon_\lambda = \epsilon_\lambda(\delta_\theta) > 0$, such that the only
bounded eigenfunctions of $\mathcal{L}_h$ on Σ/G whose eigenvalues lie in $[-\epsilon_\lambda, \epsilon_\lambda]$, are
the ones in the kernel of $\mathcal{L}_h$ on Σ/G. This kernel is two dimensional and is
spanned by $\nu \cdot \vec{e}_x$, $\nu \cdot \vec{e}_y$. In particular, the only bounded eigenfunctions of
$\mathcal{L}_h$ on Σ/G whose eigenvalue lies in $[-\epsilon_\lambda, \epsilon_\lambda]$ invariant under $\underline{R}_2$ have the
form $\lambda\nu \cdot \vec{e}_y$, $\lambda \in \mathbb{R}$.*

Proof. See the proof of [8, Proposition 2.8] .   □

The Surfaces $\Sigma[\underline{\alpha}, \beta]$. The definition of $Z_y[\phi]$ below is similar to [8, Def-
inition 3.2] except for the addition of requirement 3.10(i) to ensure that a
neighborhood of the boundary of the half Scherk surface stays invariant.

Definition 3.10. *We fix a smooth family of diffeomorphisms $Z_y[\phi] : E^3 \to
E^3$ parametrized by $\phi \in [-2\delta_\theta, 2\delta_\theta]$ (recall 3.4) satisfying the following.*

   *(i) $Z_y[\phi]$ is the identity map on $L(\delta_\theta) := \{(x, y, z) : |x| \leq \frac{1}{2}\sin 2\delta_\theta\} \cup
       \{(x, y, z) : y \leq 0\}$.*
   *(ii) On $\{(r\cos\theta', r\sin\theta', z) : r > 1, \theta' \in [9\delta_\theta, \pi/2 - 9\delta_\theta]\}$ $Z_y[\phi]$ acts by
       rotation around the z-axis by ϕ.*
   *(iii) $Z_y[\phi]$ is equivariant under $\underline{R}_2$ (reflection with respect to the yz-
       plane).*

The definition of $\Sigma[\underline{\alpha}, \beta]$ below is similar to the one for $\Sigma[T]$ in [8, Defi-
nition 3.4] with $T = \{\vec{e}[\eta_i]\}_{i=1}^4$, where $\eta_1 = \pi - \alpha^- - \beta$, $\eta_2 = \pi + \alpha^- - \beta$,
$\eta_3 = 2\pi - \alpha^+ - \beta$, $\eta_4 = 2\pi + \alpha^+ - \beta$. Therefore in the notation of [8]
$\theta(T) = \pi/2 - \alpha^+$, $\theta_1(T) = -\theta_2(T) = (\alpha^- - \alpha^+)/2$, $\theta_x(T) = 0$, and
$\theta_y(T) = \alpha^- - \alpha^+$. The tetrad is defined so that the yz-plane is mapped
by the appropriate bending map to a cone tangent to a sphere at its parallel
circle of latitude β.

Definition 3.11. *Given $\beta \in [0, \pi]$ and $\underline{\alpha} := (\alpha^-, \alpha^+) \in \mathbb{R}^2$ such that $\alpha^+ \in
[20\delta_\theta, \frac{\pi}{2} - 20\delta_\theta]$ and $\alpha^+ - \alpha^- \in [-2\delta_\theta, 2\delta_\theta]$ (recall 3.4) we define the surface*

$$\Sigma[\underline{\alpha}, \beta] := R_{\pi/2-\beta} \circ Z_y[\alpha^+ - \alpha^-](\Sigma(\alpha^+)).$$

*Moreover, $R_{\underline{\alpha},\beta,i}$ is defined to be the rotation by which $R_{\pi/2-\beta} \circ Z_y[\alpha^+ - \alpha^-]$
acts on the ith wing of $\Sigma(\alpha^+)$. The images of the core and the ith wing of
$\Sigma(\alpha^+)$ under $R_{\pi/2-\beta} \circ Z_y[\alpha^+ - \alpha^-]$ will be called the core and the ith wing
of $\Sigma[\underline{\alpha}, \beta]$. The function s on $\Sigma[\underline{\alpha}, \beta]$ is defined to be the push forward of s
on $\Sigma(\alpha^+)$ by $R_{\pi/2-\beta} \circ Z_y[\alpha^+ - \alpha^-]$. Finally to simplify the notation we take
$\Sigma[\underline{\alpha}] := \Sigma[\underline{\alpha}, 0]$.*

The Desingularizing Surfaces $\Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$.

Definition 3.12 (The bending map $\mathcal{B}_\tau$ and the spheres $\mathcal{S}_{\tau,\beta}$)**.** *For $\tau \in \mathbb{R}$
and $\beta \in (0, \pi)$ we define $\mathcal{B}_\tau : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathcal{S}_{\tau,\beta} \subset \mathbb{R}^3$ by*

$$\mathcal{B}_\tau(x, y, z) := (\tau^{-1} + x)(\cos\tau z, 0, \sin\tau z) + (-\tau^{-1}, y, 0),$$

$$\mathcal{S}_{\tau,\beta} := \left\{(x, y, z) \in \mathbb{R}^3 \left| (x + \tau^{-1})^2 + (y + \frac{\cos\beta}{\tau\sin\beta})^2 + z^2 = (\frac{1}{\tau\sin\beta})^2 \right.\right\},$$

when $\tau \neq 0$, and by taking $\mathcal{B}_0$ to be the identity and $\mathcal{S}_{0,\beta}$ to be the yz-plane rotated by $\mathsf{R}_{\pi/2-\beta}$.

If we bend the Scherk surfaces using $\mathcal{B}_\tau$, we would get wings which decay to cones, which are not minimal surfaces. To avoid this we bend instead the planes into catenoids as in [8]. This leads to the following definitions.

Definition 3.13 (The bent asymptotic half-planes). *Given $\tau \in [0, 1)$,*

$$\theta \in [20\delta_\theta, \pi/2 - 20\delta_\theta], \qquad \phi \in [-\delta_\theta, \delta_\theta],$$

and R a Euclidean motion fixing the z-axis, we define $A[\theta, \phi, R, \tau] : \mathbb{R}_+ \times \mathbb{R} \to E^3$ as follows: if $\tau = 0$ (recall 3.5(ii)), then

$$A[\theta, \phi, R, 0](s, z) := R' \circ R \circ A_\theta(s, z),$$

where R' is the rotation by an angle ϕ around the line $R \circ A_\theta(\partial\mathbb{R}_+ \times \mathbb{R})$; if $\tau \neq 0$, then

$$A[\theta, \phi, R, \tau](s, z) := (r_0 + \tau^{-1}) (\cosh \tau s + \cos \eta \sinh \tau s) (\cos \tau z, 0, \sin \tau z)$$
$$+ (-\tau^{-1}, y_0 + s(1 + r_0\tau) \sin \eta, 0),$$

where (r_0, y_0) is chosen such that $\{x = r_0, y = y_0\} = R \circ A_\theta(\partial\mathbb{R}_+ \times \mathbb{R})$, and η is determined by the requirement that $\vec{e}[\eta] = \mathsf{R}_\phi \circ R(\vec{e}[\theta])$. Notice that by these choices $A[\theta, \phi, R, \tau]$ and $\mathcal{B}_\tau \circ R \circ A_\theta$ agree on $\partial\mathbb{R}_+ \times \mathbb{R}$. The image of $\partial\mathbb{R}_+ \times \mathbb{R}$ is independent of ϕ and will be called the pivot *of $A[\theta, \phi, R, \tau]$.*

Proposition 3.14. *There exists a small positive number $\delta'_\tau = \delta'_\tau(\delta_\theta)$ such that when $\tau \leq \delta'_\tau$ the map $A[\theta, \phi, R, \tau]$ satisfies the following:*

(i) It is a conformal minimal immersion which depends smoothly on the parameters. The image lies on a catenoid or a plane. Moreover the pivot is the circle of radius $\tau^{-1} + r_0$, centered at $(-\tau^{-1}, y_0, 0)$, and parallel to the xz-plane.

(ii) $\vec{e}[\eta]$ is the inward conormal of the image at $A[\theta, \phi, R, \tau](0, 0)$.

Proof. These follow by [8, Lemma 3.9(iv) and Definition 3.10]. $\qquad\square$

Therefore, when $\tau \leq \delta'_\tau$, the Gauss map $\nu[\theta, \phi, R, \tau]$ could be chosen such that it depends smoothly on the parameters and satisfies the orientation choosing $\nu[\theta, 0, R, 0] \equiv R(\vec{e}'[\pi/2 - \theta])$.

Definition 3.15 (The bent wings). *Fix δ_s a small positive constant which will be determined later, for θ, ϕ as before, R a Euclidean motion fixing z-axis, and $\tau \in [0, \delta'_\tau]$, define $F[\theta, \phi, R, \tau] : \mathbb{R}_+ \times \mathbb{R} \to E^3$ by*

$$F[\theta, \phi, R, \tau](s, z) = \psi[1, 0](s) \mathcal{B}_\tau \circ R \circ F_\theta(s, z)$$
$$+ (1 - \psi[1, 0](s))(A_\theta[\theta, \phi, R, \tau](s, z) + \psi_s(s) f_\theta(s, z) \nu[\theta, \phi, R, \tau](s, z)),$$

where $\psi_s(s) := \psi[4\delta_s\tau^{-1}, 3\delta_s\tau^{-1}](s)$.

We consider now the boundary of the image of the half Scherk surface under the bending map $\mathcal{B}_\tau \circ \mathsf{R}_{\pi/2-\beta}$, where we assume $\tau > 0$ and $\beta \in (0, \pi)$. The image of the yz-plane under this map, where this boundary lies, is the cone $\{(x + \tau^{-1})^2 + z^2 = (\cot \beta y - \tau^{-1})^2\}$, and not, as we would like, the

sphere $\mathcal{S}_{\tau,\beta}$ (recall 3.12). Notice that the sphere $\mathcal{S}_{\tau,\beta}$ is tangent to the cone above along the circle which is the image of the z-axis under $\mathcal{B}_\tau \circ \mathsf{R}_{\pi/2-\beta}$. On the other hand, the preimage of the sphere $\mathcal{S}_{\tau,\beta}$ under the same map is the cylinder

$$(3.16) \qquad \mathcal{C}_{\tau,\beta} := \left\{ (x,y,z) \in \mathbb{R}^3 \, \middle| \, (x + \frac{1}{\tau \sin \beta})^2 + y^2 = (\frac{1}{\tau \sin \beta})^2 \right\}.$$

We would like therefore to modify the Scherk surface in the vicinity of the yz-plane so that the boundary of the half Scherk surface lies on $\mathcal{C}_{\tau,\beta}$; this leads to the definition of $\mathcal{D}[\beta,\tau]$ in 3.18.

Lemma 3.17 (Boundary neighborhoods of the half Scherk surfaces).
There exist constants c_d, $\epsilon_d = \epsilon_d(\delta_\theta)$ and $\delta_\tau'' = \delta_\tau''(\delta_\theta)$, such that the following hold when $\tau \in (0, \delta_\tau'']$ and $\beta \in (0,\pi)$.

  (i) $2c_d \epsilon_d \leq \frac{1}{4} \sin 2\delta_\theta$.
  (ii) The intersection of $\Sigma[\underline{\alpha}]$ with the slab
     $R_{\epsilon_d} := \{(x,y,z)|x \in [-c_d \epsilon_d, c_d \epsilon_d], y \in [-c_d, c_d]\}$ *(or with $2R_{\epsilon_d}$) is a periodic union of annuli. Moreover both intersections are contained in the core of $\Sigma[\underline{\alpha}]$.*
  (iii) $\Sigma[\underline{\alpha}] \cap \mathcal{C}_{\tau,\beta} \subset R_{\epsilon_d}$ (recall (3.16)).

Proof. From the equation of the Scherk surface and the definition of $\Sigma[\underline{\alpha}]$ and 3.10(i), $\Sigma[\underline{\alpha}]_{x=0}$ is the union of topological circles satisfying the equation $\sin^2 \theta - \cos z = \cos^2 \theta \cosh \frac{y}{\cos \theta}$ on the yz-plane. Therefore, $|y| < 2$ on $\Sigma[\underline{\alpha}]_{x=0}$. Thus c_d can be chosen to be 2 and then the existence of ϵ_d and δ_τ'' follows. $\square$

Lemma 3.18 (Adjusting the boundary). *There is a smooth family of diffeomorphisms $\mathcal{D}[\beta,\tau] : E^3 \to E^3$ parametrized by $\beta \in [0,\pi]$, $\tau \in [0, \delta_\tau'']$ such that:*

  (i) $\mathcal{D}[\beta,\tau]$ is the identity map on $E^3_{x \leq 0} \setminus 2R_{\epsilon_d}$.
  (ii) When $\beta > 0$, $\tau > 0$, we have on $\overline{R}_{\epsilon_d} \cap E^3_{x \leq 0}$ that $\mathcal{D}[\beta,\tau](x,y,z) =$

$$= \left(\frac{2\epsilon_d c_d + \sqrt{(\frac{1}{\tau \sin \beta})^2 - y^2} - \frac{1}{\tau \sin \beta}}{2\epsilon_d c_d} x + \sqrt{(\frac{1}{\tau \sin \beta})^2 - y^2} - \frac{1}{\tau \sin \beta}, y, z \right).$$

  (iii) $\mathcal{D}[\beta,\tau]$ is the identity map when $\beta = 0$, $\beta = \pi$ or $\tau = 0$.
  (iv) $\left\| \mathcal{D}[\beta,\tau] - \mathrm{id}_{E^3} : C^4(E^3) \right\| \leq C\tau \sin \beta$ for some constant $C := C(\delta_\theta)$.
  (v) The image of $\Sigma[\underline{\alpha}]_{x=0}$ under $\mathcal{D}[\beta,\tau]$ is contained in the cylinder $\mathcal{C}_{\tau,\beta}$.

Proof. The results are clear when $\beta = 0$ or $\tau = 0$. In the other cases, $\mathcal{D}[\beta,\tau]$ can be constructed by connecting the map in (ii) with identity map in (i) by a cut-off function. (v) then follows by the lemma above. (iv) follows by an expansion of the formula in (ii). $\square$

We define now the maps which we will use to define the desingularizing surfaces which unlike in [8] use only half a Scherk surface. We also set the ϕ_i angles symmetrically so that when $\tau = 0$, the image of a Scherk surface under the map $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ is symmetric with respect to the image of the yz-plane.

Definition 3.19 (The maps $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$). *Given* $\underline{\alpha} := (\alpha^-, \alpha^+) \in \mathbb{R}^2$, β, $\underline{\phi} = (\phi^-, \phi^+) \in \mathbb{R}^2$, *and* τ *satisfying*

$$\alpha^+ \in [30\delta_\theta, \frac{\pi}{2} - 30\delta_\theta], \quad \alpha^+ - \alpha^- \in [-\delta_\theta, \delta_\theta], \quad \beta \in [\delta_\theta, \pi - \delta_\theta],$$

$$|\underline{\phi}| := \max\{|\phi^-|, |\phi^+|\} \le \delta_\theta, \quad \tau \in [0, \delta_\tau''],$$

we define $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau] = \mathcal{Z}_a[\underline{\alpha}, \beta, \underline{\phi}, \tau] : \Sigma(\alpha^+) \to E^3$ *by (recall 3.6 and for the rotation* $\mathsf{R}_{\underline{\alpha}, \beta, i}$ *3.11)*

$$\mathcal{Z}_a[\underline{\alpha}, \beta, \underline{\phi}, \tau] := \mathcal{B}_\tau \circ \mathsf{R}_{\pi/2 - \beta} \circ \mathcal{D}[\beta, \tau] \circ Z_y[\alpha^+ - \alpha^-] \qquad \text{on the core of} \quad \Sigma(\alpha^+),$$

and by requesting that for $i = 1, 2, 3, 4$ *we have on* $\mathbb{R}_+ \times \mathbb{R}$ *that*

$$\mathcal{Z}_a[\underline{\alpha}, \beta, \underline{\phi}, \tau] \circ \underline{\mathsf{R}}_i \circ F_{\alpha^+} = F_i[\underline{\alpha}, \beta, \underline{\phi}, \tau] := F[\alpha^+, \phi_i, \mathsf{R}_{\underline{\alpha}, \beta, i} \circ \underline{\mathsf{R}}_i, \tau],$$

$$A_i[\underline{\alpha}, \beta, \underline{\phi}, \tau] := A[\alpha^+, \phi_i, \mathsf{R}_{\underline{\alpha}, \beta, i} \circ \underline{\mathsf{R}}_i, \tau],$$

$$\text{where} \qquad (\phi_1, \phi_2, \phi_3, \phi_4) := (\phi^-, -\phi^-, \phi^+, -\phi^+).$$

When $\tau = 0$ *and* $\beta = \pi/2$, $\mathcal{Z}[\underline{\alpha}, \underline{\phi}] = \mathcal{Z}_a[\underline{\alpha}, \underline{\phi}]$ *and* $\Sigma[\underline{\alpha}, \underline{\phi}] = \Sigma_a[\underline{\alpha}, \underline{\phi}]$ *are defined by*

$$\mathcal{Z}_a[\underline{\alpha}, \underline{\phi}] := \mathcal{Z}_a[\underline{\alpha}, \pi/2, \underline{\phi}, 0], \quad \Sigma_a[\underline{\alpha}, \underline{\phi}] := \mathcal{Z}_a[\underline{\alpha}, \underline{\phi}](\Sigma(\alpha^+)).$$

In particular, when $\underline{\phi} = 0$, $\Sigma[\underline{\alpha}, \underline{\phi}] = \Sigma[\underline{\alpha}]$.

Finally the ith *pivot of* $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *is defined to be* $A_i[\underline{\alpha}, \beta, \underline{\phi}, \tau](\partial \mathbb{R}_+ \times \mathbb{R})$.

Proposition 3.20 (Properties of $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$). *The following are satisfied for each* $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *as in 3.19.*

(i) *It is a smooth immersion depending smoothly on its parameters.*

(ii) *For each* $n \in \mathbb{Z}$, *it is equivariant under the reflection of the domain with respect to the plane* $\{z = n\pi\}$, *and of the range with respect to the plane which is parallel to the* y-axis, *contains* $(-\tau^{-1}, 0, 0)$, *and forms an angle* $n\tau^{-1}\pi$ *with the positive* x-axis *when* $\tau \ne 0$ *or is parallel to the* x-axis *when* $\tau = 0$.

(iii) *The boundary components of* $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau](\Sigma(\alpha^+)_{x \le 0})$ *are contained in* $\mathsf{S}_{\tau, \beta}$ *(recall 3.12).*

(iv) *If* $\tau^{-1} \in \mathbb{N}$, *then* $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau](\Sigma(\alpha^+)_{x \le 0})$ *(recall 1.13) is an embedded surface with* τ^{-1} *boundary components.*

(v) *If* $\tau^{-1} \in \mathbb{N}$, $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau](\Sigma(\alpha^+)_{x \le 0})$ *contains* $2\tau^{-1}$ *fundamental regions in (ii), and the circles* $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau](\Sigma(\alpha^+)_{x \le 0, s = \frac{5\delta_s}{\tau}})$ *have catenoidal neighborhoods.*

Proof. (iii) follows by the fact that $\mathcal{B}_\tau \circ \mathsf{R}_{\pi/2 - \beta}(\mathcal{C}_{\tau, \beta}) = \mathsf{S}_{\tau, \beta}$ when $\tau \ne 0$ and 3.18(v). All others follow by the definitions. $\square$

Definition 3.21 (The desingularizing surfaces). *Given* $[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *as above, and* $\tau^{-1} \in \mathbb{N}$, *the smooth embedded surface with boundary* $\Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *is defined by (recall 3.19 and 1.13)*

$$\Sigma := \Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau] := \mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau](\Sigma(\alpha^+)_{x \le 0, s \le \frac{5\delta_s}{\tau}}).$$

The function s *on* Σ *is defined to be the push forward by* $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *of* s *on* $\Sigma_{x \le 0}$. *The* $\pm$ *wing is defined to be the image under* $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *of*

the third or the second wing of $\Sigma(\alpha^+)_{x \leq 0, s \leq \frac{5\delta_s}{\tau}}$. Moreover, the $\pm$ boundary circle $\partial_\pm[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ of Σ is defined to be the image of $\Sigma(\alpha^+)_{x \leq 0, s=\frac{5\delta_s}{\tau}}$ on the $\pm$ wing, and the $\pm$ pivot circle $\mathcal{C}_\pm[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ is defined to be the third or the second pivot of $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$. The $\pm$ catenoidal region $\Omega_\pm[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ is defined to be the image of $A_3[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ and $A_2[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ respectively. Finally, the Neumann boundary of Σ is defined by

$$\partial_n \Sigma := \mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau](\Sigma(\alpha^+)_{x=0}).$$

4. The Mean Curvature of the Desingularizing Surfaces

In this section, the conditions

$$(4.1) \quad \alpha^+ \in [30\delta_\theta, \frac{\pi}{2} - 30\delta_\theta], \qquad \alpha^+ - \alpha^- \in [-\delta_\theta, \delta_\theta],$$
$$\beta \in [2\delta_\theta, \pi - 2\delta_\theta], \qquad |\underline{\phi}| \leq \delta_\theta, \qquad \tau \in [0, \delta_\tau''],$$

for the parameters $[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ are always assumed. The constant C could depend on δ_θ and other quantities which will be denoted.

The substitute kernel and the functions w.
The mean curvature introduced by the bending which changes $\Sigma(\alpha^+)$ to $\Sigma[\underline{\alpha}, \beta]$ is close to a scalar multiple of w we define next. Unlike in [8], there is only one such function in this case because we have only two wings. The w's for the desingularizing surfaces span the substitute kernel.

Definition 4.2 (The function w). *Let H_ϕ be the mean curvature of the surface $Z_y[\phi](\Sigma(\alpha^+))$ and then $w : \Sigma(\alpha^+) \to \mathbb{R}$ is defined by*

$$w := \frac{d}{d\phi}\bigg|_{\phi=0} H_\phi \circ Z_y[\phi].$$

The push forward of w on $\Sigma[\underline{\alpha}]$ from $\Sigma(\alpha^+)$ by $Z_y[\alpha^+ - \alpha^-]$ will also be denoted by w.

Lemma 4.3 (Properties of w). *The function w defined on $\Sigma := \Sigma(\alpha^+)$ satisfies the following:*

    (i) w is supported on $\Sigma_{s \leq 0}$ and $\|w : C^{0,\alpha}(\Sigma)\| \leq C$.
    (ii) w is identical to 0 on $L(\delta_\theta) \cap \Sigma(\alpha^+)$ (recall 3.10(i)).
    (iii) Let $\mathcal{P} : \mathbb{R} \to V$, where V is the span of $\nu \cdot \vec{e}_y$ on $\Sigma(\alpha^+)$, be the linear map which assigns to η the orthogonal projection in $L^2(\Sigma, |A|_\Sigma^2 g_\Sigma/2)$ of $\eta w/|A|_\Sigma^2$ into V. $\mathcal{P}$ is then invertible and $\mathcal{P}^{-1} \leq C$.

Proof. (i) and (ii) follow by the definitions of w and $Z_y[\phi]$. (iii) follows by using the balancing formula to check that the integral

$$\frac{d}{d\phi} \int H_\phi \circ Z_y[\phi]\nu \cdot \vec{e}_y,$$

is bounded below by a positive absolute constant similarly to the proof of [8, Lemma 4.19(iii)]. $\qquad\square$

The next corollary is simpler than the corresponding [8, Lemma 7.4] as we are on the Scherk surface now instead of the desingularizing surface.

Corollary 4.4. *Let $\Sigma := \Sigma(\alpha^+)$, there is a positive constant C such that given $E \in L^2(\Sigma/G, h)$ there is θ_E such that $(E - \theta_E w)/|A|_\Sigma^2$ is $L^2(\Sigma/G, h)$-orthogonal to the kernel of $\mathcal{L}_h$ invariant under $\underline{R}_2$, moreover,*

$$|\theta_E| \leq C \left\| E/|A|_\Sigma^2 : L^2(\Sigma/G, h) \right\|.$$

Proof. The result follows by 4.3(iii) and 3.9. $\qquad\square$

The extended substitute kernel and the functions $\overline{u}_\pm$ and $\overline{w}_\pm$.

We discuss now the functions $\overline{w}_\pm$ on the desingularizing surfaces and the functions $\overline{u}_\pm$ on $\Sigma[\underline{\alpha}]$. The functions $\overline{w}_\pm$ on the desingularizing surfaces together with the substitute kernel span the extended substitute kernel $\mathcal{K}$ on the initial surfaces (see 6.11). Unlike in [8], we need just two such functions for each desingularizing surface, and we also need to define the related functions $\overline{u}_\pm$ only on the models $\Sigma[\underline{\alpha}]$. The choice of $\underline{\alpha}'$ and $\underline{\phi}'$ in the following lemma is made such that for $i = 1, 2, 3, 4$ the asymptotic planes of the ith wings of $\Sigma[\underline{\alpha}', \underline{\phi}']$ and $\Sigma[\underline{\alpha}]$ are parallel to each other.

Lemma 4.5. *There is $\delta_\phi = \delta_\phi(\delta_\theta) \in (0, \delta_\theta)$ such that for a given $[\underline{\alpha}, \underline{\phi}'] \in \mathbb{R}^4$, where $|\underline{\phi}'| \leq \delta_\phi$, there is an $\underline{\alpha}'$ which depends smoothly on $\underline{\alpha}, \underline{\phi}'$ satisfying condition 4.1, and is characterized by the following properties:*

(i) $\underline{\alpha}' = \underline{\alpha}$ when $\underline{\phi} = 0$.

(ii) For $i, j = \pm$,

$$\frac{\partial \alpha'^i}{\partial \phi'^j} = \delta_{ij}.$$

(iii) There is a smooth function $f_{\underline{\phi}'} := f_{\underline{\phi}', a}$ on $\Sigma[\underline{\alpha}]$ invariant under $\underline{R}_2$ which depends smoothly on $\underline{\alpha}, \underline{\phi}'$, and its graph on $\Sigma[\underline{\alpha}]$ is contained in $\Sigma[\underline{\alpha}', \underline{\phi}']$ (recall 3.19).

Proof. By 3.13 and 3.5(v), for given $\underline{\alpha}$, $\underline{\alpha}'$ and $\underline{\phi}'$ as in the lemma with $|\underline{\alpha} - \underline{\alpha}'| \leq 2\delta_\phi$, there is a function f on $\Sigma[\underline{\alpha}]$ which depends smoothly on $\underline{\alpha}$, $\underline{\alpha}'$ and $\underline{\phi}'$, and whose graph over $\Sigma[\underline{\alpha}]$ is contained in $\Sigma[\underline{\alpha}', \underline{\phi}']$. The results then follow by choosing $\alpha'^\pm = \alpha^\pm + \phi'^\pm$. $\qquad\square$

We define now the functions $\overline{u}$ and $\overline{w}$ similarly to [8]. The negative signs in the definitions are chosen to improve the expressions in 4.10(v) and 4.16.

Definition 4.6 (The functions $\overline{u}_\pm$ and $\overline{w}_\pm$). *The function $\overline{u}_\pm = \overline{u}_\pm[\underline{\alpha}] = \overline{u}_{\pm,a}[\underline{\alpha}] : \Sigma[\underline{\alpha}] \to \mathbb{R}$ is defined by*

$$\overline{u}_\pm := -\left.\frac{\partial}{\partial \phi'^\pm}\right|_{\underline{\phi}'=0} f_{\underline{\phi}'}.$$

We denote the pull back of $\overline{u}_\pm$ to $\Sigma(\alpha^+)$ by $Z_y[\alpha^+ - \alpha^-]$ also by $\overline{u}_\pm = \overline{u}_\pm[\underline{\alpha}]$ and we define the function $\overline{w}_\pm = \overline{w}_\pm[\underline{\alpha}] = \overline{w}_{\pm,a}[\underline{\alpha}] : \Sigma(\alpha^+) \to \mathbb{R}$ by

$$\overline{w}_\pm := -\mathcal{L}_\Sigma \overline{u}_\pm.$$

Finally we denote the push forwards of $w, \overline{u}_\pm, \overline{w}_\pm$ on $\Sigma[\underline{\alpha}, \underline{\phi}]$ and $\Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ by $\mathcal{Z}[\underline{\alpha}, \underline{\phi}]$ and $\mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ also by $w, \overline{u}_\pm, \overline{w}_\pm$.

Before we discuss further the functions $\overline{u}_\pm$ and $\overline{w}_\pm$ in 4.10 we have two useful lemmas.

Lemma 4.7. *For $\alpha, \gamma \in (0,1)$, let the cylinder $\Omega := [0, \infty) \times \mathbb{R}/G'$, where G' is the group generated by $(s, z) \to (s, z+2\pi)$. Then there is an $\varepsilon_0 = \varepsilon_0(\alpha, \gamma)$, such that for any $\varepsilon < \varepsilon_0$, and correspondingly $a = a(\varepsilon)$, $s = s_a$, $F := F_{\theta,a} \circ R$, R is the reparametrization $(s, z) \to (s + s_0, z)$, where s_0 is any positive number, there is a linear map*

$$\mathcal{R}_\Omega : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s}) \to C^{2,\alpha}(\Omega, g_0, e^{-\gamma s}),$$

where g_0 is the flat metric on Ω, and a constant $C = C(\alpha, \gamma)$, such that for $v = \mathcal{R}_\Omega(E)$ the following are true, where the constants C depends only on α and γ

  (i) $\underline{\mathcal{L}}v = E$, where the operator $\underline{\mathcal{L}}$ on Ω defined by:

$$\underline{\mathcal{L}} := \Delta_{F^* g_\Sigma} + |A|^2_\Sigma \circ F.$$

  (ii) v is a constant on $\partial\Omega$.
  (iii) $\left\| v : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s}) \right\| \leq C \left\| E : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s}) \right\|.$

Proof. By 3.5(v), the quantity
(4.8)
$$N(\underline{\mathcal{L}}) := \left\| F^* g_\Sigma - g_0 : C^2(\Omega, g_0, e^{-s}) \right\| + \left\| |A|^2_\Sigma \circ F : C^2(\Omega, g_0, e^{-s}) \right\| \leq C\varepsilon.$$

The lemma then follows by using [8, Proposition A.4].   □

Lemma 4.9. *For $\alpha, \gamma \in (0,1)$, let the cylinder Ω as before. Then there is an $\varepsilon_0 = \varepsilon_0(\alpha, \gamma)$, such that for any $\varepsilon < \varepsilon_0$, and correspondingly $a = a(\varepsilon)$, $s = s_a$, $F := \underline{R}_i \circ F_{\theta,a} \circ R$, R is the reparametrization $(s, z) \to (s + s_0, z)$, where s_0 is any positive number, and for any $f \in C^{0,\alpha}(\partial\Omega, F^* h)$, there is a unique function $v \in C^{2,\alpha}(\Omega, F^* h)$ with $\left\| v : C^{2,\alpha}(\Omega, F^* h) \right\| < \infty$, such that*

  (i) $\underline{\mathcal{L}}v = 0$, where the operator $\underline{\mathcal{L}}$ on Ω is as 4.7.
  (ii) $v = f$ on $\partial\Omega$.
Moreover, the limit $v_0 := \lim_{s \to \infty} v$ exists. And

$$\left\| v - v_0 : C^{2,\alpha}(\Omega, F^* g_\Sigma, e^{-\gamma s}) \right\| \leq C \left\| f : C^{0,\alpha}(\Omega, F^* h) \right\|.$$

Proof. By 3.8, $\underline{\mathcal{L}}v = 0$ is equivalent to $\mathcal{L}_{F^* h}v = 0$. Then by 3.7, the problem could be transformed into the Dirichlet problem on a small disk on $\mathbb{S}^2$, and the result then follows by the standard theory as Laplacian has no small eigenvalue for the disk when the radius is small enough and (4.8). The estimate follows by the standard elliptic estimate, 3.8 and 3.5(v). □

Notice that our definition of "wings", and thus the construction of desingularizing surfaces, depend on the choice of the parameter a (or ε) in 3.5. The choice of ε will not be specified until the study of linearized equations, therefore, we will mention the dependence on ε explicitly until its choice is specified.

Lemma 4.10 (Properties of $\overline{u}_\pm$ and $\overline{w}_\pm$). *There exists ε_0, such that for $\varepsilon < \varepsilon_0$ and corresponding $a(\varepsilon)$, the functions $\overline{u}_\pm$ and $\overline{w}_\pm$ defined on $\Sigma := \Sigma(\alpha^+)$ satisfy the following:*

(i) *They depend continuously on $\underline{\alpha}$ and a.*

(ii) *They are invariant under $\underline{R}_2$. In particular, on $\Sigma_{x=0}$ (recall 1.13),*
$$\partial_x \overline{u}_\pm = 0.$$

(iii) *$\overline{w}_\pm$ is supported on $\Sigma_{s\leq 1}$ (recall 1.13).*

(iv) *$\left\| \overline{u}_\pm : C^{2,\alpha}(\Sigma) \right\| \leq Ca$ and $\left\| \overline{w}_\pm : C^{0,\alpha}(\Sigma) \right\| \leq Ca$.*

(v) *$\left\| \overline{u}_\pm : C^{2,\alpha}(\Sigma, h) \right\| < \infty$ and for $i, j = 1, 2, 3, 4$, $a_{ij} := \lim_{s\to\infty} \overline{u}_i$ exist on the jth wing, where $(\overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4) := (\overline{u}_-, \overline{u}_-, \overline{u}_+, \overline{u}_+)$. Moreover*

$$|a_{ij} - \delta_{ij}a| \leq C\varepsilon a.$$

Proof. (i) and (ii) follow by the definitions and 4.5(iii). The minimality of $\Sigma[\underline{\alpha}, \underline{\phi}]_{s\geq 1}$ from the definition implies that $\overline{w}_\pm = 0$ on $\Sigma_{s\geq 1}$. This implies (iii).

From 4.5, 3.5(ii) and 3.13, the distance between the asymptotic planes $A_i[\underline{\alpha}', \pi/2, \underline{\phi}', 0]$ and $A_i[\underline{\alpha}, \pi/2, 0, 0]$ is ($-$ for $i = 1, 2$, $+$ for $i = 3, 4$)

$$\left| b_{\alpha'+} \cos \phi'^\pm - a \sin \phi'^\pm - b_{\alpha+} \right|.$$

Therefore, from 3.5(v), 3.5(vi), for $s = s_a$ defined on $\Sigma[\underline{\alpha}]$,

(4.11) $$\left\| f_{\underline{\phi}'} + a \sin \phi'^\pm : C^4(\Sigma[\underline{\alpha}]_{s\geq 1, \mp y > 0}) \right\| \leq C\varepsilon a.$$

This implies (iv) when $s \geq 1$ along with 4.6.

Now let ε_0 be a fixed small number and $a_0 := a(\varepsilon_0)$, s_{a_0} correspondingly (recall 3.5 and 3.13). Then the result

$$\left\| \overline{u}_{\pm, a_0} : C^{2,\alpha}(\Sigma) \right\| \leq C = C(\varepsilon_0)$$

follows by the construction in 4.6 and (4.11) when $s_{a_0} \geq 1$ and compactness when $s_{a_0} \leq 1$. In this proof, we use Π to denote the map from $\Sigma[\underline{\alpha}]$ to $\Sigma_{a_0}[\underline{\alpha}', \underline{\phi}']$ by the graph function $f_{\underline{\phi}', a_0}$ defined in 4.5.

Now for general $a \geq a_0$, from the definitions in 3.5(ii) and 4.5, we have $f_{\underline{\phi}', a} = f_{\underline{\phi}', a_0}$ when $s_{a_0} \circ \Pi \leq 0$, where the function s_{a_0} is defined on $\Sigma_{a_0}[\underline{\alpha}', \underline{\phi}']$. On the other hand, the distance between the two asymptotic planes $A_{i, a_0}[\underline{\alpha}', \pi/2, \underline{\phi}', 0]$ and $A_{i, a}[\underline{\alpha}, \pi/2, \underline{\phi}', 0]$ is $(a - a_0) \left| \sin \phi^{\pm'} \right|$ when $s_{a_0} \circ \Pi \geq 0 \geq (a - a_0) \cos \phi^{\pm'}$; and when $0 \leq s_{a_0} \circ \Pi \leq (a - a_0) \cos \phi^{\pm'}$, the asymptotic plane $A_{i, a}[\underline{\alpha}', \pi/2, \underline{\phi}', 0]$ is a graph of $A_{i, a_0}[\underline{\alpha}, \pi/2, 0, 0]$ with graph function $s_{a_0} \sin \phi^{\pm'}$. So by 3.5(v) and 3.5(vi) again

$$\left\| f_{\underline{\phi}', a} - f_{\underline{\phi}', a_0} : C^4(\Sigma[\underline{\alpha}]_{0 \leq s_{a_0} \circ \Pi, \mp y > 0}) \right\|$$
$$\leq \left\| s_{a_0} \circ \Pi \sin \phi'^\pm : C^4(\Sigma[\underline{\alpha}]_{0 \leq s_{a_0}, \mp y > 0}) \right\| + C(\varepsilon_0)a.$$

The function s_{a_0} defined on Σ is close to the pull back of the function $s_{a_0} \circ \Pi$ on $\Sigma[\underline{\alpha}]$ for s_{a_0} defined on $\Sigma_{a_0}[\underline{\alpha}', \underline{\phi}']$ when $\underline{\alpha}', \underline{\phi}'$ is small. The region $s_a \leq 1$ on Σ is the same as the region $s_{a_0} \leq a - a_0 + 1$ for functions s_{a_0} and

s_a defined on Σ. Therefore,

$$\left\| \frac{\overline{u}_{\pm,a}}{a} : C^{2,\alpha}(\Sigma_{s_a \leq 1}) \right\| \leq \left\| \frac{\overline{u}_{\pm,a}}{a} : C^{2,\alpha}(\Sigma[\underline{\alpha}]_{s_{a_0} \circ \Pi \leq a-a_0+2}) \right\|$$

$$\leq \frac{a_0}{a} \left\| \frac{\overline{u}_{\pm,a_0}}{a_0} : C^{2,\alpha}(\Sigma_{s_{a_0} \leq a-a_0+2}) \right\| + \frac{a-a_0+2}{a} + C(\varepsilon_0) \leq C(\epsilon_0)$$

This then implies (iv) when $s \leq 1$ along with 4.6.

By (iv), $\overline{u}_\pm \in H^2(\Sigma, h)$ and thus $\overline{u}_\pm \in C^{0,\alpha}(\Sigma, h)$. Moreover, by (iii) and (iv), $\overline{w}_\pm \in C^{0,\alpha}(\Sigma, h)$. Therefore, by standard theory and the definition of $\overline{w}_\pm$, $\overline{u}_\pm \in C^{2,\alpha}(\Sigma, h)$. By 4.9 and (iii), the limits in (v) exist. Finally, (4.11) implies the last in equality in (v) along with the definition 4.6. □

Notation 4.12. From 4.10(v), we define

$$a_{--} := a_{22}, \quad a_{++} := a_{33},$$

$$a_{-+} := a_{23}, \quad a_{+-} := a_{32}.$$

The Decomposition of the Mean Curvature.

Lemma 4.13 (Estimates on the geometry of catenoidal annuli). *Given* $\Omega_\pm := \Omega_\pm[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *as in 3.21 we have the following.*

(i) $\left\| g_{\Omega_\pm} - g_\pm^0 : C^l(\Omega_\pm, 1+s) \right\| \leq C(l)\tau$ *and also*
$\left\| g_{\Omega_\pm} - g_\pm^0 : C^l(\Omega_\pm) \right\| \leq C(l)\delta_s$, *where* $g_\pm^0$ *is the push forward of the flat metric on* $\mathbb{R}_+ \times \mathbb{R}$ *by* $A_3[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *and* $A_2[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *respectively.*
(ii) $\left\| A_{\Omega_\pm} : C^l(\Omega_\pm) \right\| \leq C(l)\tau.$

Proof. These follow by the definition of $\Omega_\pm$ and 3.13. □

Corollary 4.14. *Given* $\tau^{-1} \in \mathbb{N}$, $\Sigma' := \Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$, $\Sigma := \Sigma(\alpha^+)$, $\mathcal{Z} := \mathcal{Z}[\underline{\alpha}, \beta, \underline{\phi}, \tau]$. *Suppose* τ, $|\underline{\phi}|$ *small enough, then*

(i) $\left\| A_{\Sigma'} : C^3(\Sigma') \right\| \leq C(\varepsilon).$
(ii) $\left\| \mathcal{Z}^* g_{\Sigma'} - g_\Sigma : C^3(\Sigma) \right\| \leq C(\varepsilon)(\tau + |\underline{\phi}| + |\alpha^+ - \alpha^-|) + C(\delta_s + \varepsilon).$
(iii) $\left\| \mathcal{Z}^* A_{\Sigma'} - A_\Sigma : C^3(\Sigma) \right\| \leq C(\varepsilon)(\tau + |\underline{\phi}| + |\alpha^+ - \alpha^-|) + C\varepsilon.$

Proof. On $\Sigma_{s\leq1}$, these follow by the smooth dependence on the parameters on a compact set in 3.20. On $\Sigma_{s\geq1}$, (ii) follows by 4.13(i) and 3.5(v), and (i), (iii) follow by 4.13(ii) and 3.5(v). □

Lemma 4.15 (The mean curvature on the wings). *Given* $\gamma \in (0,1)$, $\tau^{-1} \in \mathbb{N}$, $\Sigma := \Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$, *if* τ *small enough with respect to* γ, *then the mean curvature* H *on* $\overline{\Sigma} = \Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *satisfies*

$$\left\| H_\Sigma : C^2(\Sigma_{s\geq1}, e^{-\gamma s}) \right\| \leq C\tau.$$

Proof. This follows by 4.13, 3.5(v), and [8, Lemma B.1] similarly to the proof of [8, Lemma 4.5]. □

Proposition 4.16 (The mean curvature on the desingularizing surfaces). *There are* $\delta_\phi = \delta_\phi(\delta_\theta) \in (0, \delta_\theta)$ *and* $\delta_\tau = \delta_\tau(\delta_\theta) \in (0, \delta_\tau'')$, *such that for given* $\gamma \in (0,1)$, $[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ *satisfying* (4.1) *and*

$$|\underline{\phi}| \leq \delta_\phi, \quad \tau \in (0, \delta_\tau], \quad \tau^{-1} \in \mathbb{N},$$

the mean curvature H on $\Sigma = \Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$ satisfies

$$\left\| H - (\alpha^+ - \alpha^- - \phi^+ + \phi^-)w - \phi^+ \overline{w}_+ - \phi^- \overline{w}_- : C^{0,\alpha}(\Sigma, e^{-\gamma s}) \right\|$$
$$\leq C(\varepsilon)(\tau + \left| \alpha^+ - \alpha^- \right|^2 + \left| \underline{\phi} \right|^2).$$

Proof. On $\Sigma_{s \geq 1}$, this follows by 4.15 along with 4.3(i) and 4.10(iii).

On $\Sigma_{s \leq 1}$, this follows by comparing the four surfaces $\Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau]$, $\Sigma[\underline{\alpha}, \beta, \underline{\phi}, 0]$, $\Sigma[\underline{\alpha} - \underline{\phi}]$ and $\Sigma(\alpha^+ - \phi^+)$, the smooth dependence on the parameters in 3.20, the definitions of w and $\overline{w}^\pm$ in 4.2, 4.6 and 4.5(iii), along with [8, Lemma B.1]. $\qquad \square$

5. The Initial Surfaces

The construction of the pre-initial surfaces.
In this section, the parameter k in the initial configurations is fixed. So are $\delta'_\theta = \delta'_\theta(k)$ and $\delta_\sigma = \delta_\sigma(k)$ as in 2.30 and all the constants depending on it. We then choose δ_θ in 3.4 to be the same as δ'_θ. Moreover, k is assumed to be big enough. We will construct the pre-initial surfaces as the initial surfaces in [8]. To simplify the notation, in this article (unlike in [8]) we use the same τ for all desingularizing surfaces.

Definition 5.1. *In the rest of this article we take $\delta := \delta_\theta/5$ and $\tau := 1/m$, where τ is assumed as small (equivalently m big enough) as needed and such that*

$$(5.2) \qquad \tau \leq \min\{\delta/3\underline{c}, \delta_\sigma/3\underline{c}, \delta_\tau\},$$

where δ_τ is as 4.16, δ_σ is as 2.30, and the constant $\underline{c} > 0$ will be determined later. Finally constants denoted by C will depend on k and any other quantities we mention.

Definition 5.3 (Parameters of the initial surfaces). *We define the following vector spaces and we will use the maximum norms on them except for $\mathcal{V}_\sigma$ in (i).*

    (i) $\mathcal{V}_\sigma := \{\underline{\sigma} \in \mathbb{R}^k | \sigma_{k-i+1} = -\sigma_i, i = 1, \ldots, [\frac{k}{2}]\}$ *with* $|\underline{\sigma}| := \|\underline{\sigma} : \ell^1\|$.
    (ii) $\mathcal{V}_\theta := \{\underline{\theta} \in \mathbb{R}^k | \theta_{k-i+1} = -\theta_i, i = 1, \ldots, [\frac{k}{2}]\}$.
    (iii) $\mathcal{V}_\phi := \{\underline{\varphi} = \{\underline{\phi}_i\}_{i=1}^k \in \mathbb{R}^{2k} | \underline{\phi}_i = (\phi_i^-, \phi_i^+) \in \mathbb{R}^2, \phi_i^- = \phi_{k-i+1}^+, \phi_i^+ = \phi_{k-i+1}^-, i = 1, \ldots, [\frac{k}{2}]\}$.
    (iv) $\mathcal{V} := \mathcal{V}_\sigma \times \mathcal{V}_\phi$ *and* $\mathcal{V}' := \mathcal{V}_\theta \times \mathcal{V}_\phi$.

Finally we define $\Xi_\mathcal{V} := \{\xi \in \mathcal{V} : |\xi| \leq \underline{c}\tau\}$.

Given $\xi = (\underline{\sigma}, \underline{\varphi}) \in \Xi_\mathcal{V}$, the initial surface $M = M[\xi] = M[\underline{\sigma}, \underline{\varphi}]$ is constructed by the following. For the initial configuration $\mathcal{W}[\underline{\sigma}{:}k]$, there is a unique composition of a homothety followed by a translation which maps the circle $\mathcal{B}_\tau(z\text{-axis})$ onto $\mathcal{C}_i[\underline{\sigma}{:}k]$ (recall 2.25). This map will be called $\mathcal{H}_i$ suppressing the parameters $[\underline{\sigma}{:}k]$ on which it depends. Moreover, each $\mathcal{C}_i[\underline{\sigma}{:}k]$ has latitude $\beta_i[\underline{\sigma}{:}k]$, and there are two angles $\alpha_i^\pm[\underline{\sigma}{:}k]$. The angles $\underline{\alpha}_i(\xi)$, $\beta_i(\xi)$ are defined by the relations:
(5.4)
$\underline{\alpha}_i(\xi) = (\alpha_i^-(\xi), \alpha_i^+(\xi)) := (\alpha_i^-[\underline{\sigma}{:}k] - \phi_i^-, \alpha_i^+[\underline{\sigma}{:}k] - \phi_i^+), \quad \beta_i(\xi) := \beta_i[\underline{\sigma}{:}k].$

Then the surface $\mathring{\mathcal{S}}_i[\xi]$ is defined by

$$(5.5) \qquad \mathring{\mathcal{S}}_i[\xi] := \mathcal{H}_i(\Sigma[\underline{\alpha}_i(\xi), \beta_i(\xi), \underline{\tilde{\phi}}_i(\xi), \tau]),$$

where $\underline{\tilde{\phi}}_i(\xi) = \{\tilde{\phi}_i^-(\xi), \tilde{\phi}_i^+(\xi)\}$ will be determined later.

From 3.21, the $\pm$ pivot of $\Sigma[\underline{\alpha}_i(\xi), \beta_i(\xi), \underline{\tilde{\phi}}_i(\xi), \tau]$ does not depend on $\underline{\tilde{\phi}}_i(\xi)$, hence $\mathcal{C}'_{\pm,i}[\xi]$ could be defined by

$$\mathcal{C}'_{\pm,i}[\xi] := \mathcal{H}_i(\mathcal{C}_\pm[\underline{\alpha}_i(\xi), \beta_i(\xi), \underline{\tilde{\phi}}_i(\xi), \tau]).$$

When $i = 1, \ldots, k-1$, $\mathcal{A}_i[\underline{\sigma}{:}k]$ is an annulus and

$$\partial \mathcal{A}_i[\underline{\sigma}{:}k] = \mathcal{C}_i[\underline{\sigma}{:}k] \cup \mathcal{C}_{i+1}[\underline{\sigma}{:}k].$$

For each $i = 1, \ldots, k-1$, there exists a unique catenoidal annulus $\mathcal{A}'_i[\xi]$, which is a small perturbation of $\mathcal{A}_i[\underline{\sigma}{:}k]$, such that

$$\partial \mathcal{A}'_i[\xi] = \mathcal{C}'_{+,i}[\xi] \cup \mathcal{C}'_{-,i+1}[\xi].$$

There is then a unique value for each $\tilde{\phi}_1^+$, $\tilde{\phi}_k^-$ and $\tilde{\phi}_i^\pm$, $i = 2, \ldots, k-1$ such that

$$\mathcal{H}_{i+1}(\partial_-[\underline{\alpha}_{i+1}(\xi), \beta_{i+1}(\xi), \underline{\tilde{\phi}}_{i+1}(\xi), \tau]) \subset \mathcal{A}'_i[\xi],$$
$$\mathcal{H}_i(\partial_+[\underline{\alpha}_i(\xi), \beta_i(\xi), \underline{\tilde{\phi}}_i(\xi), \tau]) \subset \mathcal{A}'_i[\xi]$$

This means the two surfaces $\mathring{\mathcal{S}}_i[\xi]$ and $\mathcal{A}'_i[\xi]$ (or $\mathcal{A}'_{i-1}[\xi]$) match together except a neighborhood of $\mathcal{C}'_{+,i}[\xi]$ (or $\mathcal{C}'_{-,i}[\xi]$) in $\mathcal{A}'_i[\xi]$ (or $\mathcal{A}'_{i-1}[\xi]$). And $\mathcal{A}''_i[\xi]$ is defined by

$$\mathcal{A}''_i[\xi] := \mathcal{A}'_i[\xi] \setminus \left(\mathcal{H}_i \circ A_+[\underline{\alpha}_i(\xi), \beta_i(\xi), \underline{\tilde{\phi}}_i(\xi), \tau]([0, 4\delta_s/\tau] \times \mathbb{R}) \right.$$
$$\left. \cup \ \mathcal{H}_{i+1} \circ A_-[\underline{\alpha}_{i+1}(\xi), \beta_{i+1}(\xi), \underline{\tilde{\phi}}_{i+1}(\xi), \tau]([0, 4\delta_s/\tau] \times \mathbb{R}) \right).$$

In the case $i = 0$ or k, $\mathcal{A}_i[\underline{\sigma}{:}k]$ is a planar disk, and $\partial \mathcal{A}_0[\underline{\sigma}{:}k] = \mathcal{C}_1[\underline{\sigma}{:}k]$, $\partial \mathcal{A}_k[\underline{\sigma}{:}k] = \mathcal{C}_k[\underline{\sigma}{:}k]$. For $i = 1, k$, there are unique maps $\mathcal{H}[c_i^+, c_i] : E^3 \to E^3$, $\mathcal{H}[c_i^+, c_i](x, y, z) := c_i^+(x, y + c_i, z)$ such that

$$\mathcal{H}[c_1^+, c_1](\mathcal{C}_1[\underline{\sigma}{:}k]) = \mathcal{C}'_{-,1}[\xi], \quad \mathcal{H}[c_k^+, c_k](\mathcal{C}_k[\underline{\sigma}{:}k]) = \mathcal{C}'_{+,k}[\xi],$$

and we define

$$\mathcal{A}'_0[\xi] := \mathcal{H}[c_1^+, c_1](\mathcal{A}_0[\underline{\sigma}{:}k]), \quad \mathcal{A}'_k[\xi] := \mathcal{H}[c_k^+, c_k](\mathcal{A}_k[\underline{\sigma}{:}k]).$$

As before, $\tilde{\phi}_1^-$ and $\tilde{\phi}_k^+$ are defined such that

$$\mathcal{H}_1(\partial_-[\underline{\alpha}_1(\xi), \beta_1(\xi), \underline{\tilde{\phi}}_1(\xi), \tau]) \subset \mathcal{A}'_0[\xi],$$
$$\mathcal{H}_k(\partial_+[\underline{\alpha}_k(\xi), \beta_k(\xi), \underline{\tilde{\phi}}_k(\xi), \tau]) \subset \mathcal{A}'_k[\xi],$$

and $\mathcal{A}''_0[\xi]$, $\mathcal{A}''_k[\xi]$ are defined by

$$\mathcal{A}''_0[\xi] := \mathcal{A}'_0[\xi] \setminus (\mathcal{H}_1 \circ A_-[\underline{\alpha}_1(\xi), \beta_1(\xi), \underline{\tilde{\phi}}_1(\xi), \tau]([0, 4\delta_s/\tau] \times \mathbb{R}))$$
$$\mathcal{A}''_k[\xi] := \mathcal{A}'_k[\xi] \setminus (\mathcal{H}_k \circ A_+[\underline{\alpha}_k(\xi), \beta_k(\xi), \underline{\tilde{\phi}}_k(\xi), \tau]([0, 4\delta_s/\tau] \times \mathbb{R})).$$

Now all angles $\tilde{\phi}_i^\pm$ have been defined.

Definition 5.6 (The pre-initial surfaces). *We define surfaces $\mathring{M}[\xi]$, $M'[\xi]$ and $\mathring{S}[\xi]$ by*

$$\mathring{M} = \mathring{M}[\xi] = \mathring{M}[\underline{\sigma}, \varphi] := \cup_{i=1}^{k} \mathring{S}_i[\xi] \cup_{j=0}^{k} \mathcal{A}''_j[\xi],$$

$$M' = M'[\xi] = M'[\underline{\sigma}, \varphi] := \cup_{j=0}^{k} \mathcal{A}'_j[\xi],$$

$$\mathring{S} = \mathring{S}[\xi] = \mathring{S}[\underline{\sigma}, \varphi] := \cup_{i=1}^{k} \mathring{S}_i[\xi].$$

The push forward of the function s by $\mathcal{H}_i$ to each $\mathring{S}_i$ could be extended to a function on the whole $\mathring{M}$ by taking $s = \max_S s$ on the rest of $\mathring{M}$. The function will also be denoted by s. Similarly, s can be defined on M'.

$\underline{\tilde{\sigma}} = \underline{\tilde{\sigma}}(\xi) \in \mathcal{V}_\sigma$ and $\underline{\tilde{\varphi}} = \underline{\tilde{\varphi}}(\xi) \in \mathcal{V}_\phi$ are defined as the following:

- *(i) $\underline{\tilde{\sigma}} := \{\tilde{\sigma}_i\}_{i=1}^{k}$ where $\tilde{\sigma}_i := \log \frac{\alpha_i^+(\xi)}{\alpha_i^-(\xi)}$.*
- *(ii) $\underline{\tilde{\varphi}} := \{\underline{\tilde{\phi}}_i\}_{i=1}^{k}$, where $\underline{\tilde{\phi}}_i(\xi)$ has already been defined.*

Finally let $\underline{a} := 8 |\log \tau| = 8 \log m$. Therefore for $\mathring{M}[\xi]$ as above, each component of $\mathring{M}_{s \geq \underline{a}}$ ((recall 1.13)) contains exactly one $\mathcal{A}''_j[\xi]$. This component will be called $\mathcal{N}_j[\xi]$.

Proposition 5.7 (Properties of the pre-initial surfaces). *$\mathring{M} = \mathring{M}[\xi]$ with $\xi \in \Xi_\mathcal{V}$ is well defined and has the following properties:*

- *(i) $\mathring{M}$ is a smooth embedded surface which depends smoothly on ξ.*
- *(ii) $\mathring{M}$ is contained in the unit ball $\mathbb{B}^3$ and $\partial \mathring{M}$ is contained in the unit sphere $\mathbb{S}^2$, which is a disjoint union of km topological circles.*
- *(iii) $\mathring{M}$ respects the symmetries in $\mathcal{G}$.*
- *(iv) $|\varphi + \underline{\tilde{\varphi}}| \leq C(\varepsilon)\tau$.*
- *(v) $\mathring{M}_{s \geq 1}$ (recall 1.13) is a graph over $M'_{s \geq 1}$ by a function f_M such that*
$$\left\| f_M : C^5(M'_{s \geq \underline{a}}) \right\| \leq \tau^3.$$

Proof. The facts that $\mathring{M}$ is well defined and satisfies (i) and (ii) follow by the construction and 3.20(iii). (iii) follows by 3.20(ii) and the symmetry of the initial configuration $\mathcal{W}[\underline{\sigma}:k]$. (iv) follows by the fact that the conormals at the boundary of each $\mathcal{A}'_i$ differ the corresponding ones for $\mathcal{A}_i$ by $C(\varepsilon)\tau$ since the corresponding boundaries have been moved by $C(\varepsilon)\tau$. (v) follows by the construction and rescaling 3.5(v). □

Corollary 5.8 (Estimates on the parameters). *If τ is small enough, the parameters of each*

$$\Sigma[\underline{\alpha}, \beta, \underline{\phi}, \tau] = \mathcal{H}_i^{-1}(\mathring{S}_i[\xi])$$

(recall 3.21) satisfy the following:

- *(i) $50\delta \leq \alpha^+ \leq \frac{\pi}{2} - 50\delta$.*
- *(ii) $|\alpha^+ - \alpha^-| \leq 3\underline{c}\tau \leq \delta$.*
- *(iii) $|\underline{\phi}| \leq (\underline{c} + C(\varepsilon))\tau \leq \delta_\phi$, where δ_ϕ is as 4.16.*

Proof. From the definitions $\underline{\alpha} = \underline{\alpha}_i(\xi)$, $\underline{\phi} = \underline{\tilde{\phi}}_i(\xi)$. (i) follows by (5.4), 2.30(i), 5.1, 5.3 and (5.2). From (5.4),

$$\alpha_i^+(\xi) - \alpha_i^-(\xi) = \alpha_i^+[\underline{\sigma}:k] - \alpha_i^-[\underline{\sigma}:k] + (\phi_i^- - \phi_i^+) = (e^{\sigma_i} - 1)\alpha_i^+[\underline{\sigma}:k] + (\phi_i^- - \phi_i^+).$$

Moreover, from 2.26, $\left|\alpha_i^{\pm}[\underline{\sigma}{:}k]\right| < 1$ when k big enough. (ii) then follows by 5.3 and (5.2). (iii) follows by 5.3, 5.7(iv) and by assuming τ small enough. $\square$

The boundary of the pre-initial surfaces.

Unlike in [8] there are no complete ends extending to infinity. On the other hand, we need to ensure that the initial surfaces intersect $\mathbb{S}^2$ orthogonally. To achieve this we study and modify the vicinity of the boundary of the pre-initial surfaces as in [18].

Notation 5.9. $\mathring{X} := \mathring{X}[\xi] : \mathring{M} = \mathring{M}[\xi] \to E^3$ is defined to be the embedding map for $\mathring{M}$ and $\mathring{\nu} := \mathring{\nu}[\xi] : \mathring{M} = \mathring{M}[\xi] \to E^3$ is defined to be the unit normal to $\mathring{M}$ which points outward on each annulus $\mathcal{A}_i''$. $\mathring{\Theta} : \partial\mathring{M} \to \mathbb{R}$ is defined to be the inner product

$$\mathring{\Theta} := \langle \mathring{X}, \mathring{\nu} \rangle.$$

Definition 5.10. *On a neighborhood in $\mathring{\mathcal{S}}_i \subset \mathring{M}$ of each component of $\partial\mathring{M}$ small enough, such that within it the map of the nearest point projection onto $\partial\mathring{M}$ is well-defined and smooth, the smooth coordinates $(\mathring{\rho}, \mathring{t})$ are defined as the following: for any point P in this region, let $\mathring{\rho}$ be the distance of P from $\partial\mathring{M}$, and let $\mathring{t}$ be the distance in $\partial\mathring{M}$ of the nearest point projection of P onto $\partial\mathring{M}$ from an arbitrary fixed reference point on $\partial\mathring{M}$. In particular, along $\partial\mathring{M}$, $\partial_{\mathring{\rho}}$ is the inward unit conormal for $\partial\mathring{M}$, and $\{\mathring{X}_*\partial_{\mathring{\rho}}, \mathring{X}_*\partial_{\mathring{t}}, \mathring{\nu}\}$ is an orthonormal basis at each point of $\mathring{M}$.*

From the definitions, $\langle \mathring{X}, \mathring{X}_*\partial_{\mathring{\rho}} \rangle\big|_{\partial\mathring{M}} = -\sqrt{1 - \mathring{\Theta}^2}$.

Lemma 5.11. *For each $\mathring{\mathcal{S}}_i$, $i = 1,\ldots,k$, $\Sigma = \Sigma[\underline{\alpha}_i(\xi), \beta_i(\xi), \underline{\tilde{\phi}}_i(\xi), \tau] = \mathcal{H}_i^{-1}(\mathring{\mathcal{S}}_i)$, the following results hold:*
(i) There exists $\epsilon' = \epsilon'(k)$, such that $\mathring{\rho}$ is well defined and smooth in the range $[0, \epsilon'\tau]$. Moreover, ϵ' could be chosen to be small enough, such that $\mathring{M}_{\mathring{\rho} \leq \epsilon'\tau} \subset \mathring{M}_{s=0}$ (recall 1.13).
(ii) $\left\| \mathring{\Theta} \circ \mathcal{H}_i : C^3(\partial_n\Sigma) \right\| \leq C\tau$ (recall 3.21).

Proof. (i) follows by the construction of $\mathring{M}$ and a scaling. (ii) follows by the construction in 3.10(iii), 3.18(iv) and the smooth dependence of parameters in 3.20. $\square$

The Construction of the Initial Surfaces.

Definition 5.12 (The initial surfaces). *$\mathring{u} \in C^\infty(\mathring{M})$ is defined by*

$$\mathring{u}(\mathring{\rho}, \mathring{t}) := -\mathring{\rho}\frac{\mathring{\Theta}(\mathring{t})}{\sqrt{1 - \mathring{\Theta}^2(\mathring{t})}}\psi\left[\frac{\epsilon'\tau}{2}, \frac{\epsilon'\tau}{4}\right](\mathring{\rho})$$

on $\mathring{\mathcal{S}}_i$ and 0 on the rest of $\mathring{M}$.
Then the corresponding deformation $\mathring{X}_{\mathring{u}}[\xi] : \mathring{M}[\xi] \to E^3$ is defined by

$$\mathring{X}_{\mathring{u}}[\xi](P) := \mathring{X}(P) + \mathring{u}(P)\mathring{\nu}(P),$$

and the resulting initial surface $M = M[\xi]$ is defined to be the image of $\mathring{M}$ under $\mathring{X}_{\mathring{u}}$:

$$M[\xi] := \mathring{X}_{\mathring{u}}[\xi](\mathring{M}[\xi]).$$

Similarly, $\mathcal{S}_i[\xi]$, $S[\xi]$ are defined by

$$\mathcal{S}_i[\xi] := \mathring{X}_{\mathring{u}}[\xi](\mathring{\mathcal{S}}_i[\xi]), \quad S[\xi] := \mathring{X}_{\mathring{u}}[\xi](\mathring{S}[\xi]) = \cup_{i=1}^{k} \mathcal{S}_i[\xi].$$

Notation 5.13. The embedding map for M will be denoted by $X := X[\xi] : M = M[\xi] \to E^3$. The unit normal to M which points outward on each catenoidal annulus $\mathcal{A}_i''$ will be denoted by $\nu := \nu[\xi] : M = M[\xi] \to E^3$. Moreover, the composition $\mathring{X}_{\mathring{u}} \circ \mathcal{H}_i \circ \mathcal{Z}[\underline{\alpha}_i(\xi), \beta_i(\xi), \underline{\tilde{\phi}}_i(\xi), \tau] : \Sigma(\alpha_i^+(\xi)) \to M$ will be denoted by $\mathcal{Y}_i[\xi]$.

Definition 5.14. *The coordinates (ρ, t) are defined on a sufficiently small neighborhood in M of each component of ∂M, so that for each P in this neighborhood $\rho(P)$ is the distance in M from ∂M to P, while $t(P)$ is the distance along ∂M from the same fixed reference point in ∂M as in the definition of $\mathring{t}$ (see 5.10) to the nearest point projection of P onto ∂M. Moreover, the function s on M is defined to be the push forward of s on $\mathring{M}$.*

Proposition 5.15 (Properties of the initial surfaces). *$M = M[\xi]$ with $\xi \in \Xi_{\mathcal{V}}$ is well defined and has the following properties:*

(i) *M is a smooth embedded surface which depends smoothly on ξ. Moreover,*

$$M[\xi] = \cup_{i=1}^{k} \mathcal{S}_i[\xi] \cup_{j=0}^{k} \mathcal{A}_j''[\xi].$$

(ii) *M is contained in the unit ball $\mathbb{B}^3$ and ∂M is contained in the unit sphere $\mathbb{S}^2$, which is a disjoint union of km topological circles.*

(iii) *M intersects $\mathbb{S}^2$ orthogonally along ∂M.*

(iv) *M respects the symmetries in $\mathcal{G}$.*

(v) *$M_{s \geq 1}$ is a graph over $M'_{s \geq 1}$ by a function f_M such that*

$$\left\| f_M : C^5(M'_{s \geq \underline{a}}) \right\| \leq \tau^3.$$

Proof. (i), (ii), (iv), (v) come from the construction of M, 5.7 and 5.11(i). Moreover, from 5.10 and 5.12, $\mathring{u}|_{\partial \mathring{M}} = 0$, $\partial \mathring{u} / \partial \mathring{t}|_{\partial \mathring{M}} = 0$, $\partial \mathring{u} / \partial \mathring{\rho}|_{\partial \mathring{M}} = -\frac{\mathring{\Theta}}{\sqrt{1 - \mathring{\Theta}^2}}$, and thus

$$\nu \circ \mathring{X}_{\mathring{u}} \Big|_{\partial M} = \frac{\mathring{\nu} - \partial \mathring{u} / \partial \mathring{\rho} \mathring{X}_* \partial_{\mathring{\rho}}}{\sqrt{1 + (\partial \mathring{u} / \partial \mathring{\rho})^2}} \Big|_{\partial \mathring{M}}.$$

Therefore,

$$\langle X, \nu \rangle |_{\partial M} = \langle \mathring{X}, \nu \circ \mathring{X}_{\mathring{u}} \rangle \Big|_{\partial \mathring{M}} = 0,$$

as $\langle \mathring{X}, \mathring{\nu} \rangle \Big|_{\partial M} = \mathring{\Theta}$, $\langle \mathring{X}, \mathring{X}_* \partial_{\mathring{\rho}} \rangle \Big|_{\partial \mathring{M}} = -\sqrt{1 - \mathring{\Theta}^2}$. This shows (iii). $\square$

Lemma 5.16. *Let $\beta = \beta_i(\xi)$, $\mathcal{S} = \mathcal{S}_i[\xi]$ and $\mathring{\mathcal{S}} = \mathring{\mathcal{S}}_i[\xi]$, then*

(i) *$\frac{1}{\tau^2} \left\| g \circ \mathring{X}_{\mathring{u}} - \mathring{g} : C^3(\mathring{\mathcal{S}}, (\tau \sin \beta)^{-2} \mathring{g}) \right\| \leq C\tau$, where g and $\mathring{g}$ are the first fundamental forms on $\mathcal{S}$ and $\mathring{\mathcal{S}}$ respectively.*

(ii) $\frac{1}{\tau}\left\|A \circ \mathring{X}_{\mathring{u}} - \mathring{A} : C^3(\mathring{\mathcal{S}}, (\tau \sin \beta)^{-2}\mathring{g})\right\| \leq C\tau$, where A and $\mathring{A}$ are the second fundamental forms on $\mathcal{S}$ and $\mathring{\mathcal{S}}$ respectively.

(iii) $\tau\left\|H \circ \mathring{X}_{\mathring{u}} - \mathring{H} : C^3(\mathring{\mathcal{S}}, (\tau \sin \beta)^{-2}\mathring{g})\right\| \leq C\tau$, where H and $\mathring{H}$ are the mean curvatures on $\mathcal{S}$ and $\mathring{\mathcal{S}}$ respectively.

Proof. These follow by the construction of M and 5.11(ii). $\qquad\square$

Graphs over the Initial Surfaces.

To study the perturbations of the initial surfaces in a convenient manner (including the free boundary condition) we will describe them as graphs in an auxiliary metric we define below following [18].

Definition 5.17. *We define an auxiliary metric $g_A := \Omega^2 g$ on $\mathbb{R}^3$ where $\Omega : \mathbb{R}^3 \to \mathbb{R}_+$ is a fixed smooth function such that $\Omega = r := \sqrt{x^2 + y^2 + z^2}$ when $r > 1/2$, so that the region $\{r \in (\frac{2}{3}, 1]\} \subset \mathbb{B}^3$ with the g_A metric is isometric to the round cylinder $\mathbb{S}^2 \times (-\log 2, 0]$.*

Therefore, the g_A unit normal on M pointing in the same direction as ν is $(\Omega \circ X)^{-1}\nu$.

Definition 5.18. *Given any function $\tilde{u} \in C^2(M)$, the perturbation $X_{\tilde{u}}$ is defined by*

$$X_{\tilde{u}}(P) := \exp^{g_A}_{X(P)} \frac{\tilde{u}(P)\nu(P)}{\Omega \circ X(P)},$$

where $\exp^{g_A}$ is the exponential map for g_A on E^3. For $\tilde{u}$ small enough, $X_{\tilde{u}}$ is an immersion with the Euclidean unit normal $\nu_{\tilde{u}}$, which is taken to have positive inner product with the velocity of g_A geodesic generated by ν. Also $H[\tilde{u}] : M \to \mathbb{R}$ is defined to be the mean curvature of the image of $X_{\tilde{u}}$ relative to g and $\nu_{\tilde{u}}$. $\Theta[\tilde{u}] : \partial M \to \mathbb{R}$ is defined by

$$\Theta[\tilde{u}] := (g \circ X_{\tilde{u}})(X_{\tilde{u}}, \nu_{\tilde{u}}).$$

Moreover, the graph $M_{\tilde{u}} = M_{\tilde{u}}(\xi)$ is defined to be the image of $X_{\tilde{u}}$.

Lemma 5.19. *For each $\tilde{u} \in C^2(M)$ sufficiently small, $\Theta[\tilde{u}] = 0$ if and only if $\partial_\rho \tilde{u} = 0$.*

Proof. See [18, Lemma 5.6]. $\qquad\square$

Notation 5.20. The linearization of $H[\tilde{u}]$ at $\tilde{u} = 0$ will be denoted by $\tilde{\mathcal{L}}$.

Lemma 5.21. *For each $\tilde{u} \in C^2(M)$ we define $u \in C^2(M)$ by (recall 5.17)*

$$u := (\Omega \circ X)^{-1}\tilde{u},$$

and then we have the following.

(i) $\tilde{\mathcal{L}}\tilde{u} = \mathcal{L}_M u.$

(ii) $\partial_\rho \tilde{u}|_{\partial M} = (\partial_\rho + 1)u|_{\partial M}$. *In particular, $\Theta[\tilde{u}] = 0$ if and only if $(\partial_\rho + 1)u|_{\partial M} = 0$.*

Proof. See [18, Lemma 5.19]. $\qquad\square$

6. The Linearized Equation

In this section we study the linearized equation on the initial surfaces. Our approach is closer to the approach in [11, 14] for example, rather than in [8], in that we first solve the equation on the standard models (half Scherk surfaces and catenoidal annuli or discs), and then we transfer the solutions to the initial surfaces, where we combine them to get approximate solutions which we correct by iteration. This shifts some of the difficulty from understanding properties of the linearized equation on the initial surfaces, to comparing the equations and the norms on the initial surfaces versus the standard models. Besides the differences in the approach there are two more differences with [8]: first, we do not have to deal with ends extending to infinity, and second, unlike in [8], we have to deal with a boundary and the free boundary condition.

The linearized equation on $\Sigma(\alpha_i^+(\xi))$.

In this subsection, an $\mathcal{S}_i[\xi]$ is fixed, and so is the corresponding Scherck surface $\Sigma := \Sigma(\alpha_i^+(\xi))$. On Σ, G is defined to be the group of symmetries which is generated by the reflections with respect to xy-plane and the plane $\{z = \pi\}$ as in 3.9. In the rest of this subsection, all the functions on Σ are assumed to be invariant under the action of G. Constants $\gamma \in (\frac{3}{4}, 1)$ and $\alpha \in (0, 1)$ are fixed. Moreover, the Neumann boundary $\partial_n \Sigma(\alpha_i^+(\xi))$ of half-$\Sigma(\alpha_i^+(\xi))$ is defined to be $\Sigma(\alpha_i^+(\xi))_{x=0}$.

Proposition 6.1 (The Jacobi equation on Scherk surfaces). *Given $\Sigma = = \Sigma(\alpha_i^+(\xi))$, there exists a linear map (recall 1.13)*

$$\mathcal{R}_\Sigma = \mathcal{R}_{\Sigma, a}[i, \xi] : C_G^{0,\alpha}(\Sigma(\alpha_i^+(\xi))_{x \leq 0}, e^{-\gamma s}) \times C_G^{1,\alpha}(\partial_n \Sigma(\alpha_i^+(\xi)))$$
$$\to C_G^{2,\alpha}(\Sigma(\alpha_i^+(\xi))_{x \leq 0}, e^{-\gamma s}) \times \mathbb{R} \times \mathbb{R}^2,$$

and a constant C, such that for $(E, f) \in C_G^{0,\alpha}(\Sigma_{x \leq 0}, e^{-\gamma s}) \times C_G^{1,\alpha}(\partial_n \Sigma)$, if $(u_E, \theta_E, \underline{\phi}_E) := \mathcal{R}_\Sigma(E, f)$, where $\underline{\phi}_E := \{\phi_E^-, \phi_E^+\} \in \mathbb{R}^2$, then

(i) $\mathcal{L}_\Sigma u_E = E - \theta_E w - \phi_E^- \overline{w}_- - \phi_E^+ \overline{w}_+$ *on $\Sigma_{x \leq 0}$.*
(ii) $\partial_x u_E = -f$ *on $\partial_n \Sigma$.*
(iii) $|\theta_E| \leq C \|E\|$, *where $\|E\| := \left\| E : C^{0,\alpha}(\Sigma_{x \leq 0}, e^{-\gamma s}) \right\|$.*
(iv) $\left| \underline{\phi}_E \right| \leq \frac{C}{a}(\|E\| + \|f\|)$, *where $\|f\| := \left\| f : C^{1,\alpha}(\partial_n \Sigma) \right\|$.*
(v) $\left\| u_E : C^{2,\alpha}(\Sigma_{x \leq 0}, e^{-\gamma s}) \right\| \leq C(\|E\| + \|f\|)$.

Proof. Fix ε_0 in 4.7, 4.9 and 4.10, and correspondingly a_0, s_{a_0} (recall 3.5, 3.6). First, assume that E has support in $\Sigma_{s_{a_0} \leq 2}$. Then there is the estimate:

$$\left\| E/|A|_\Sigma^2 : L^2(\Sigma/G, h) \right\| \leq C(\varepsilon_0) \|E\|,$$

with h as in 3.8 and 4.4. Now 4.4 could be applied, and there is a θ_E as in 4.4 with (iii) satisfied. Moreover, by 3.9, the inhomogeneous term in the

Neumann boundary value problem on $\Sigma_{x\leq 0}$:

$$(6.2) \qquad \mathcal{L}_h u'_E = (E - \theta_E w)/|A|_\Sigma^2,$$
$$\partial_x u'_E|_{\partial_n \Sigma} = -f,$$

is $L^2(\Sigma/G, h)$-orthogonal to the eigenfunctions with eigenvalues smaller than ϵ_λ, and is bounded above by $\left\| E/|A|_\Sigma^2 : L^2(\Sigma/G, h) \right\| \leq C(\varepsilon_0) \|E\|$ from 4.3(i) and (iii). Then by the implied L^2 estimate and the standard linear theory in h metric, there is a unique solution u'_E such that

$$(6.3) \qquad \left\| u'_E : C^{2,\alpha}(\Sigma_{x\leq 0}, h) \right\| \leq C(\|E\| + \|f\|),$$

where $C = C(\varepsilon_0)$. From the fact that (6.2) is equivalent to

$$\mathcal{L}_\Sigma u'_E = E - \theta_E w,$$

the standard linear theory in g metric and the estimate (6.3), the solution u'_E also satisfies

$$(6.4) \qquad \left\| u'_E : C^{2,\alpha}(\Sigma_{x\leq 0}, g) \right\| \leq C(\|E\| + \|f\|),$$

where $C = C(\varepsilon_0)$.

Now for any $\varepsilon < \varepsilon_0$ and corresponding $a > a_0$ and $s_a \leq s_{a_0}$ as in 3.5 and 3.6, define $s := s_a$. Let $F_- := \underline{\mathsf{R}}_2 \circ F_\theta \circ R$, and $F_+ := \underline{\mathsf{R}}_3 \circ F_\theta \circ R$, where $\theta = \alpha_i^+(\xi)$, F_θ is the map defined in 3.5(ii) and R is the reparametrization $(s, z) \to (s + 2, z)$. For the cylinder $\Omega := [0, \infty) \times \mathbb{R}/G'$, where G' is the group generated by $(s, z) \to (s, z + 2\pi)$ as in 4.9, by the definition, 4.3(i) and the assumption of the support of E, $\underline{\mathcal{L}}(u'_E \circ F_\pm) = 0$ on Ω, where $\underline{\mathcal{L}}$ is as in 4.9. Then by 4.9 and (6.3), for $j = \pm$, a_j can be defined by

$$a_j := \lim_{s\to\infty} u'_E \circ F_j.$$

u_E is then defined by

$$u_E = u'_E + \phi_E^- \overline{u}_- + \phi_E^+ \overline{u}_+,$$

where $\phi_E^\pm$ is determined by the linear equations (recall 4.12):

$$(6.5) \qquad a_j + \sum_{i=\pm} \phi_E^i a_{ij} = 0.$$

Then by 4.10(v) along with (6.3) and the definition, $\phi_E^\pm$ is well defined and satisfies (iv). Moreover, by 4.10(v), the function u_E is in $C^{2,\alpha}(\Sigma_{x\leq 0}, h)$ with $\left\| u_E : C^{2,\alpha}(\Sigma_{x\leq 0}, h) \right\| < \infty$ and by 4.10(iii), 4.10(v) and (6.5), we have on Ω

$$\underline{\mathcal{L}}(u_E \circ F_j) = 0, \quad \lim_{s\to\infty} u_E \circ F_j = 0.$$

Therefore, by (iv), 4.10(iv), (6.4), the uniqueness and the estimate in 4.9, u_E satisfies (v).

Now for general E, let the cylinder Ω as before, $F_- := F_{-,a_0} := \underline{\mathsf{R}}_2 \circ F_{\theta,a_0} \circ R$, and $F_+ := F_{+,a_0} := \underline{\mathsf{R}}_3 \circ F_{\theta,a_0} \circ R$ but R is the reparametrization $(s, z) \to (s + 1, z)$. Then by 4.7, there is a function $v'_{E\pm}$ such that $\underline{\mathcal{L}} v'_{E\pm} = E \circ F_\pm$. Moreover,

$$(6.6) \qquad \left\| v'_{E\pm} : C^{2,\alpha}(\Omega, g_0, e^{-\gamma s}) \right\| \leq C \left\| E \circ F_\pm : C^{0,\alpha}(\Omega, g_0, e^{-\gamma s}) \right\|.$$

We define the function E' by

$$E' := E - \mathcal{L}_\Sigma(\psi[1,2](s_{a_0})F_{\pm *}v'_{E\pm}).$$

Then E' is supported on $\Sigma_{s_{a_0} \leq 2}$, and so the results above could be applied for E' to get $(u_{E'}, \theta_{E'}, \underline{\phi}_{E'}) = \mathcal{R}_\Sigma(E', f)$. Recall 3.6, we can now define $u_E = u_{E'}$ on the core of $\Sigma_{x \leq 0}$ and $u_E = u_{E'} + (\psi[1,2](s_{a_0}))F_{\pm *}v'_{E\pm}$ on the $\pm$-wing of $\Sigma_{x \leq 0}$. And then $\mathcal{R}_\Sigma(E, f)$ is defined by

$$\mathcal{R}_\Sigma(E, f) := (u_E, \theta_{E'}, \underline{\phi}_{E'}).$$

The estimates of $\mathcal{R}_\Sigma$ follow by the definitions, (6.6) and 3.5(v). □

The linearized equation on $\mathcal{N}_j[\xi]$.

In this subsection, an $\mathcal{N} = \mathcal{N}_j[\xi]$ is fixed (recall 5.6).

Definition 6.7 (Norms on catenoidal annuli). *Given $u \in C^{r,\alpha}(\mathcal{N})$ with $r = 0, 2$, we define $\|u : \mathcal{N}\|_r$ by*

$$\|u : \mathcal{N}\|_r := \|u : C^{r,\alpha}(\mathcal{N})\|.$$

Similarly we define $\|u : \mathcal{N}\|'_r$ by

$$\|u : \mathcal{N}\|'_r := \|u : C^{r,\alpha}(\mathcal{N} \setminus S[\xi])\|.$$

Lemma 6.8 (The Jacobi equation on catenoidal annuli). *Given $\alpha \in (0, 1)$ there exist a constant C and a linear map*

$$\mathcal{R}_\mathcal{N} = \mathcal{R}_{\mathcal{N},a}[j, \xi] : C^{0,\alpha}_\mathcal{G}(\mathcal{N}_j[\xi]) \to C^{2,\alpha}_\mathcal{G}(\mathcal{N}_j[\xi]),$$

such that $\forall E \in C^{0,\alpha}_\mathcal{G}(\mathcal{N}_j[\xi])$ and $u_E := \mathcal{R}_\mathcal{N}E$ we have $\mathcal{L}_\mathcal{N}u_E = E$ on $\mathcal{N}$, $u_E = 0$ on $\partial\mathcal{N}$, and

$$\|u_E : \mathcal{N}\|_2 \leq C \|E : \mathcal{N}\|_0.$$

Proof. By 2.30(iii) and the fact that $\mathcal{N}$ is a small perturbation of $\mathcal{A}_j$, the Dirichlet problem for $\mathcal{L}_\mathcal{N}$ has no small eigenvalues. Thus there is a unique solution u_E to $\mathcal{L}_\mathcal{N}u_E = E$, $u_E = 0$ on $\partial\mathcal{N}$, and it satisfies the required estimate by the standard linear theory and the uniform control of the geometry. □

The linearized equation globally on the initial surfaces.

Lemma 6.9. *Let $\beta = \beta_i(\xi)$, $\mathcal{S} := \mathcal{S}_i[\xi]$, $\mathcal{Y} = \mathcal{Y}_i[\xi]$, $\Sigma := \Sigma(\alpha^+_i(\xi))$, and let $\mathcal{Y}_*$ be the push forward by $\mathcal{Y}$ of a function or tensor on $\Sigma(\alpha^+_i(\xi))_{x \leq 0, s \leq 5\delta_s/\tau}$ to a function or tensor on $\mathcal{S}_i[\xi]$, the following results hold:*

(i) $\left\| (\tau \sin\beta)^{-2}g_\mathcal{S}) - \mathcal{Y}_*g_\Sigma : C^3(\mathcal{S}, (\tau \sin\beta)^{-2}g_\mathcal{S}) \right\| \leq$
$$C(\varepsilon)(1 + \underline{c})\tau + C(\delta_s + \varepsilon).$$

(ii) *Suppose $u \in C^{2,\alpha}_G(\Sigma_{x \leq 0, s \leq 5\delta_s/\tau})$, then*

$$\left\| (\tau \sin\beta)^2\mathcal{L}_M \circ \mathcal{Y}_*u - \mathcal{Y}_*\mathcal{L}_\Sigma u : C^{0,\alpha}(\mathcal{S}, (\tau \sin\beta)^{-2}g_\mathcal{S}, e^{-\gamma s}) \right\|$$
$$\leq (C(\varepsilon)(1 + \underline{c})\tau + C(\delta_s + \varepsilon)) \left\| u : C^{2,\alpha}(\mathcal{S}, (\tau \sin\beta)^{-2}g_\mathcal{S}, e^{-\gamma s}) \right\|.$$

(iii) *Suppose* $u \in C_G^{1;\alpha}(\partial_n \Sigma)$ *(recall 1.13), then*

$$\left\| \tau \sin \beta \partial_\rho \circ \mathcal{Y}_{i*} u + \mathcal{Y}_* \partial_x u : C^{0,\alpha}(\partial M \cap \mathcal{S}, (\tau \sin \beta)^{-2} g_{\mathcal{S}}) \right\|$$
$$\leq C\tau \left\| u : C^{1,\alpha}(\partial M \cap \mathcal{S}, (\tau \sin \beta)^{-2} g_{\mathcal{S}}) \right\|.$$

Proof. (i) follows by 4.14(ii), 5.16(i), 5.3 and a scaling. (ii) follows by 4.14(iii), 5.16(ii) along with (i). (iii) follows by the smooth dependence in 3.20, 5.16(i) and the construction. □

Definition 6.10 (Global norms on the initial surfaces $M = M[\xi]$). *Given* $u \in C^{r,\alpha}(M)$ $(r = 0, 2)$, *we define* $\|u\|_r$ *to be the maximum of the following quantities, where* $b_0 = e^{-5\gamma\delta_s/\tau}$ *and* $b_2 = b_0/\tau^{10}$:

(i) *For each* $i = 1, \ldots, k$, *and* $\mathcal{S}_i := \mathcal{S}_i[\xi]$,

$$\tau^{1-r} \left\| u : C^{r,\alpha}(\mathcal{S}_i, (\tau \sin \beta_i)^{-2} g_{\mathcal{S}_i}, \max\{e^{-\gamma s}, b_r\}) \right\|.$$

(ii) *For each* $j = 1, \ldots, k + 1$,

$$b_r^{-1} \left\| u : \mathcal{N}_j[\xi] \right\|_r'.$$

And given $f \in C^{1,\alpha}(\partial M)$, $\|f\|_1$ *is defined to be the maximum of the following quantities: for each* $i = 1, \ldots, k$, *consider the quantity*

$$\left\| f : C^{1,\alpha}(\partial M \cap \mathcal{S}_i, (\tau \sin \beta_i)^{-2} g_{\mathcal{S}_i}) \right\|.$$

Definition 6.11 (W and extended substitute kernel). *We define the extended substitute kernel* $\mathcal{K} = \mathcal{K}[M[\xi]] := W(\mathcal{V}') \subset C_G^\infty(M)$ *(recall 1.12), where* $W : \mathcal{V}' \to C_G^\infty(M)$ *is the linear map (actually a linear isomorphism from* $\mathcal{V}'$ *onto* $\mathcal{K}$*) defined by (recall 5.3)*

$$W(\underline{\theta}', \underline{\varphi}') = \sum_{i=1}^{k} \frac{1}{\tau \sin \beta_i} \mathcal{Y}_{i*}(\theta_i' w + \phi_i'^{-} \overline{w}_- + \phi_i'^{+} \overline{w}_+),$$

where $\underline{\theta}' = \{\theta_i'\}_{i=1}^k \in \mathcal{V}_\theta$, $\underline{\varphi}' = \{\underline{\phi}_i' = \{\phi_i'^{-}, \phi_i'^{+}\}\}_{i=1}^k \in \mathcal{V}_\phi$, *and* $\mathcal{Y}_{i*}$ *is the push forward by* $\mathcal{Y}_i$ *of a function on* $\Sigma(\alpha_i^+(\xi))$ *to a function on* $\mathcal{S}_i[\xi]$ *extended to vanish on the rest of* M.

Definition 6.12. $\psi' \in C_G^\infty(M)$ *is defined by*

$$\psi' := \psi[5\delta_s \tau^{-1}, 5\delta_s \tau^{-1} - 1] \circ s$$

on $\mathcal{S}_i$ *and* 0 *on the rest of* M. *Also* $\psi' \in C_G^\infty(M)$ *is defined by*

$$\psi'' := \psi[\underline{a}, \underline{a} + 1] \circ s.$$

Now the linear map

(6.13) $\mathcal{R}_{M,appr} : C_G^{0,\alpha}(M) \times C_G^{1,\alpha}(\partial M)$
$$\to \quad C_G^{2,\alpha}(M) \times \mathcal{V}' \times C_G^{0,\alpha}(M) \times C_G^{1,\alpha}(\partial M)$$

is constructed by the following. Given $E \in C_G^{0,\alpha}(M)$, $f \in C_G^{1,\alpha}(\partial M)$, let $\Sigma := \Sigma(\alpha_i^+(\xi))$, then

$$\psi' E|_{\mathcal{S}_i} \circ \mathcal{Y}_i \in C_G^{0,\alpha}(\Sigma, e^{-\gamma s}), \quad f \circ \mathcal{Y}_i \in C_G^{1,\alpha}(\partial_n \Sigma).$$

By 6.1, there are $(v_{Si}, \theta_{1i}, \underline{\phi}_{1i}) \in C^{2,\alpha}(\Sigma) \times \mathbb{R} \times \mathbb{R}^2$ such that

$$(v_{Si}, \theta_{1i}, \underline{\phi}_{1i}) = \mathcal{R}_\Sigma(\tau \sin \beta_i \psi' E|_{\mathcal{S}_i} \circ \mathcal{Y}_i, f \circ \mathcal{Y}_i).$$

Define $u_{1S} := \tau \sin \beta_i \mathcal{Y}_{i*} v_{Si}$ on $\mathcal{S}_i$ and 0 on the rest, then $u_{1S} \in C^{2,\alpha}_{\mathcal{G}}(S)$. By 6.8, there is $u_{Nj} \in C^{2,\alpha}_{\mathcal{G}}(\mathcal{N}_j)$, such that

$$u_{Nj} = \mathcal{R}_{\mathcal{N}}((1 - \psi'^2)E - [\mathcal{L}_M, \psi']u_{1S}).$$

Again, let $u_{1N} = u_{Nj}$ on $\mathcal{N}_j$ and 0 on the rest.

Definition 6.14. *The map* $\mathcal{R}_{M,appr} : (E, f) \mapsto (u_1, \underline{\theta}_1, \underline{\varphi}_1, E_1, f_1)$ *is defined by the following:*

(i) $u_1 := \psi' u_{1S} + \psi'' u_{1N}$.
(ii) $\underline{\theta}_1 := \{\theta_{1i}\}_{i=1}^k$.
(iii) $\underline{\varphi}_1 := \{\underline{\phi}_{1i}\}_{i=1}^k$.
(iv) $E_1 := \mathcal{L}_M u_1 - E + W(\underline{\theta}_1, \underline{\varphi}_1)$.
(v) $f_1 := \partial_\rho u_1 + u_1 - f$.

Proposition 6.15 (The Jacobi equation on the initial surfaces)**.** *For some small* δ_s, ε, *when* τ *small enough, a linear map (recall 6.10)*

$$\mathcal{R}_M : C^{0,\alpha}_{\mathcal{G}}(M) \to C^{2,\alpha}_{\mathcal{G}}(M) \times \mathcal{V}'$$

can be defined by

$$\mathcal{R}_M E := (u_E, \underline{\theta}_E, \underline{\varphi}_E) := \sum_{n=1}^\infty (u_n, \underline{\theta}_n, \underline{\varphi}_n)$$

for $E \in C^{0,\alpha}(M)$, *where the sequence* $\{(u_n, \underline{\theta}_n, \underline{\varphi}_n, E_n, f_n)\}_{n \in \mathbb{N}}$ *is defined inductively for* $n \in \mathbb{N}$ *by (recall 6.13)*

$$(u_n, \underline{\theta}_n, \underline{\varphi}_n, E_n, f_n) := -\mathcal{R}_{M,appr}(E_{n-1}, f_{n-1}), \quad E_0 := -E, \quad f_0 = 0.$$

Moreover,

$$\mathcal{L}_M u_E = E - W(\underline{\theta}_E, \underline{\varphi}_E), \quad (\partial_\rho + 1)u_E|_{\partial M} = 0,$$

$$|\underline{\theta}_E| \leq C \|E\|_0, \quad \left|\underline{\varphi}_E\right| \leq C \|E\|_0, \quad \|u_E\|_2 \leq C \|E\|_0.$$

Proof. By the definitions, 4.3(i), 4.10(iii) and the facts $\psi''(1 - \psi'^2) = 1 - \psi'^2$, $\psi'' \nabla \psi' = \nabla \psi'$,

$$\mathcal{L}_M u_1 = -E_0 - W(\underline{\theta}_1, \underline{\varphi}_1) + [\mathcal{L}_M, \psi'']u_{1N}$$
$$+ (\tau \sin \beta_i)^{-1} \psi' \sum_i ((\tau \sin \beta_i)^2 \mathcal{L}_M \circ \mathcal{Y}_{i*} v_{Si} - \mathcal{Y}_{i*} \mathcal{L}_\Sigma v_{Si}),$$

$$\partial_\rho u_1 + u_1 = -f_0 + \sum_i \tau \sin \beta_i \mathcal{Y}_{i*} v_{Si} + \sum_i (\tau \sin \beta_i \partial_\rho \circ \mathcal{Y}_{i*} v_{Si} + \mathcal{Y}_{i*} \partial_x v_{Si}),$$

where $(u_1, \underline{\theta}_1, \underline{\varphi}_1, E_1, f_1) := -\mathcal{R}_{M,appr}(E_0, f_0)$. Therefore, by the definitions,

$$E_1 = [\mathcal{L}_M, \psi'']u_{1N} + (\tau \sin \beta_i)^{-1} \psi' \sum_i ((\tau \sin \beta_i)^2 \mathcal{L}_M \circ \mathcal{Y}_{i*} v_{Si} - \mathcal{Y}_{i*} \mathcal{L}_\Sigma v_{Si}),$$

$$f_1 = \sum_i \tau \sin \beta_i v_{Si} + \sum_i (\tau \sin \beta_i \partial_\rho \circ \mathcal{Y}_{i*} v_{Si} + \mathcal{Y}_{i*} \partial_x v_{Si}).$$

However, since $\operatorname{supp}\nabla\psi' \subset \left[\frac{5\delta_s}{\tau}-1, \frac{5\delta_s}{\tau}\right]$ and $\operatorname{supp}\nabla\psi'' \subset [\underline{a}, \underline{a}+1] = \left[8\log\frac{1}{\tau}, 8\log\frac{1}{\tau}+1\right]$, and by 6.8, the fact that $\gamma \in (3/4, 1)$, and the definitions,

$$
\begin{aligned}
\left\|[\mathcal{L}_M, \psi''] u_{1N}\right\|_0 & \\
\leq \max_i \tau^{-1} & \left\|u_{1N} : C^{2,\alpha}(\mathcal{S}_i \cap M_{s\in[\underline{a},\underline{a}+1]}, (\tau\sin\beta_i)^{-2}g_{\mathcal{S}_i}, e^{-\gamma s})\right\| \\
\leq C\tau^{-9} \max_i & \left\|u_{1N} : C^{2,\alpha}(\mathcal{S}_i \cap M_{s\in[\underline{a},\underline{a}+1]}, (\tau\sin\beta_i)^{-2}g_{\mathcal{S}_i}\right\| \\
\leq C\tau^{-9} & \left\|u_{1N} : C^{2,\alpha}(M, g_M)\right\| \\
\leq C\tau^{-9} & \left\|(1-\psi'^2)E_0 - [\mathcal{L}_M, \psi']u_{1S} : C^{0,\alpha}(M, g_M)\right\| \\
\leq C\tau^{-10} & \left\|E_0 : C^{0,\alpha}(M_{s\geq\frac{5\delta_s}{\tau}-1}, g_M)\right\| \\
& + C\tau^{-12} \left\|u_{1S} : C^{2,\alpha}(M_{s\in\left[\frac{5\delta_s}{\tau}-1, \frac{5\delta_s}{\tau}\right]}, g_M)\right\|.
\end{aligned}
$$

By the definitions,

$$
\begin{aligned}
\left\|E_0 : C^{0,\alpha}(M_{s\geq\frac{5\delta_s}{\tau}-1}, g_M)\right\| & \\
\leq \max_i Ce^{-5\gamma\delta_s/\tau} & \left\|E_0 : C^{0,\alpha}(\mathcal{S}_i, g_{\mathcal{S}_i}, e^{-\gamma s})\right\| + \max_j \left\|E_0 : C^{0,\alpha}(\mathcal{N}_j)\right\|' \\
\leq \max_i Ce^{-5\gamma\delta_s/\tau}\tau^{-1} & \left\|E_0 : C^{0,\alpha}(\mathcal{S}_i, (\tau\sin\beta_i)^{-2}g_{\mathcal{S}_i}, e^{-\gamma s})\right\| \\
& + \max_j \left\|E_0 : C^{0,\alpha}(\mathcal{N}_j)\right\|' \\
& \leq Ce^{-5\gamma\delta_s/\tau}\tau^{-2} \left\|E_0\right\|_0.
\end{aligned}
$$

And by 6.1 and 6.9(i),

$$
\begin{aligned}
\left\|u_{1S} : C^{2,\alpha}(M_{s\in\left[\frac{5\delta_s}{\tau}-1, \frac{5\delta_s}{\tau}\right]}, g_M)\right\| & \leq Ce^{-5\gamma\delta_s/\tau} \left\|u_{1S} : C^{2,\alpha}(S, g_M, e^{-\gamma s})\right\| \\
\leq \max_i Ce^{-5\gamma\delta_s/\tau}\tau^{-3} & \left\|u_{1S} : C^{2,\alpha}(\mathcal{S}_i, (\tau\sin\beta_i)^{-2}g_{\mathcal{S}_i}, e^{-\gamma s})\right\| \\
\leq \max_i Ce^{-5\gamma\delta_s/\tau}\tau^{-2} \left\|v_{Si} : C^{2,\alpha}(\Sigma, g_\Sigma, e^{-\gamma s})\right\| & \leq \max_i Ce^{-5\gamma\delta_s/\tau}\tau^{-2}. \\
\cdot \left(\tau \left\|E_0 \circ \mathcal{Y}_i : C^{0,\alpha}(\Sigma_{x\leq 0}, g_\Sigma, e^{-\gamma s})\right\| + \left\|f_0 \circ \mathcal{Y}_i : C^{1,\alpha}(\partial_n\Sigma, g_\Sigma)\right\|\right) & \\
& \leq Ce^{-5\gamma\delta_s/\tau}\tau^{-2}\left(\left\|E_0\right\|_0 + \left\|f_0\right\|_1\right).
\end{aligned}
$$

Moreover, by 6.9(ii) and the inequality
$$\left\|v_{Si} : C^{2,\alpha}(\Sigma_{x\leq 0}, g_\Sigma, e^{-\gamma s})\right\| \leq C\left(\left\|E_0\right\|_0 + \left\|f_0\right\|_1\right) \text{ above}$$

$$
\begin{aligned}
\left\|\psi'((\tau\sin\beta_i)^2\mathcal{L}_M \circ \mathcal{Y}_{i*}v_{Si} - \mathcal{Y}_{i*}\mathcal{L}_\Sigma v_{Si})\right\|_0 & \\
= \tau \left\|\psi'((\tau\sin\beta_i)^2\mathcal{L}_M \circ \mathcal{Y}_{i*}v_{Si} - \mathcal{Y}_{i*}\mathcal{L}_\Sigma v_{Si}) : C^{0,\alpha}(\mathcal{S}_i, (\tau\sin\beta_i)^{-2}g_{\mathcal{S}_i}, e^{-\gamma s})\right\| \\
\leq (C(\varepsilon)(1+\underline{c})\tau + C(\delta_s+\varepsilon))\tau & \left\|v_{Si} : C^{2,\alpha}(\Sigma_{x\leq 0}, g_\Sigma, e^{-\gamma s})\right\| \\
& \leq (C(\varepsilon)(1+\underline{c})\tau + C(\delta_s+\varepsilon))\tau\left(\left\|E_0\right\|_0 + \left\|f_0\right\|_1\right).
\end{aligned}
$$

Therefore,

$$
\left\|E_1\right\|_0 \leq \left(Ce^{-5\gamma\delta_s/\tau}/\tau^{14} + (C(\varepsilon)(1+\underline{c})\tau + C(\delta_s+\varepsilon))\right)\left(\left\|E_0\right\|_0 + \left\|f_0\right\|_1\right).
$$

By choosing small δ_s, ε and then τ small enough, $\|E_1\|_0 \le \frac{1}{2}(\|E_0\|_0 + \|f_0\|_1)$.
Similarly, by 6.9(iii),

$$\|f_1\|_1 \le C\tau \left\|v_{Si} : C^{1,\alpha}(\partial_n\Sigma), g_\Sigma)\right\| \le C\tau \left(\|E_0\|_0 + \|f_0\|_1\right).$$

By choosing τ small enough, $\|f_1\|_1 \le \frac{1}{2}(\|E\|_0 + \|f_0\|_1)$.
Finally, from the inequalities above and the definitions,

$$\|u_1\|_2 \le C\left(\|E_0\|_0 + \|f_0\|_1\right), \qquad |\underline{\theta}_1| \le C\left(\|E_0\|_0 + \|f_0\|_1\right),$$
$$\left|\underline{\varphi}_1\right| \le C\left(\|E_0\|_0 + \|f_0\|_1\right).$$

By using the estimates and iterating we conclude that the operator $\mathcal{R}_M$ is well defined and satisfies the requirements. $\qquad \square$

From now on the small constants δ_s, ε are fixed as in the proposition.

Corollary 6.16 (The linearized equation for the mean curvature). *There are $u_H \in C^{2,\alpha}_{\mathfrak{G}}(M)$ and $(\underline{\theta}_H, \underline{\varphi}_H) \in \mathcal{V}'$ such that*

$$\mathcal{L}_M u_H = H - W(\underline{\theta}_H, \underline{\varphi}_H), \quad (\partial_\rho + 1)u_H|_{\partial M} = 0,$$

$$|\underline{\theta}_H - \underline{\theta}| \le C\tau, \quad \left|\underline{\varphi}_H + \underline{\varphi}\right| \le C\tau, \quad |\underline{\sigma}_H - \underline{\sigma}| \le C\tau, \quad \|u_H\|_2 \le C\tau,$$

*where H is the mean curvature of M, $\quad M := M[\xi] = M(\underline{\sigma}, \underline{\varphi}), \quad \underline{\theta} :=$
$\{\alpha_i^+[\underline{\sigma}:k] - \alpha_i^-[\underline{\sigma}:k]\}_{i=1}^k, \quad \underline{\sigma}_H := \left\{\frac{\theta_{Hi}}{\alpha_i^+[\underline{\sigma}:k]}\right\}_{i=1}^k \in \mathcal{V}_\sigma.$*

Proof. By 4.16, 5.8 and 5.16(iii),

(6.17) $$\left\|H - W(\underline{\theta}', \tilde{\underline{\varphi}})\right\|_0 \le C\tau,$$

where $\underline{\theta}' = \{\theta_i'\}_{i=1}^k$, with

$$\theta_i' = \alpha_i^+(\xi) - \alpha_i^-(\xi) - \tilde{\phi}_i^+(\xi) + \tilde{\phi}_i^-(\xi)$$
$$= \alpha_i^+[\underline{\sigma}:k] - \alpha_i^-[\underline{\sigma}:k] - \phi_i^+ + \phi_i^- - \tilde{\phi}_i^+(\xi) + \tilde{\phi}_i^-(\xi).$$

Applying the proposition with $E := H - W(\underline{\theta}', \tilde{\underline{\varphi}})$, there are $(u_E, \underline{\theta}_E, \underline{\varphi}_E) := \mathcal{R}_M E$. By the definitions $u_H = u_E$, $\underline{\theta}_H = \underline{\theta}' + \underline{\theta}_E$, $\underline{\varphi}_H = \tilde{\underline{\varphi}} + \underline{\varphi}_E$, all the estimates except the one for $\underline{\sigma}_H$ follow by (6.17), 6.15 and 5.7(iv).

From 5.8(i) and 5.8(ii), $\alpha_i^+[\underline{\sigma}:k] \in [30\delta, \frac{\pi}{2} - 30\delta]$ when τ small enough. Moreover, by the definition, $\sigma_i = \log \frac{\alpha_i^+[\underline{\sigma}:k]}{\alpha_i^-[\underline{\sigma}:k]}$. The result for $\underline{\sigma}_H$ then follows by the estimate of $\underline{\theta}_H$. $\qquad \square$

7. The Main Results

Proposition 7.1 (The nonlinear terms of $H[\tilde{u}]$). *Given $\tilde{u} \in C^{2,\alpha}_{\mathfrak{G}}(M)$ with $\|\tilde{u}\|_2$ small enough (recall 6.10), we have*

$$\|H[\tilde{u}] - H - \mathcal{L}_M u\|_0 \le C \|u\|_2^2,$$

where H is the mean curvature of M, $H[\tilde{u}]$ is the mean curvature of $M_{\tilde{u}}$, $u = (\Omega \circ X)^{-1}\tilde{u}$ as in 5.21.

Proof. Let $\mathcal{S} := \mathcal{S}_i[\xi]$, $\beta := \beta_i(\xi)$, $i = 1 \ldots, k$, by 5.16(i), 5.16(ii) and 4.14(i), $\left\| A_{\mathcal{S}} : C^3(\mathcal{S}, (\tau\beta)^{-2} g_{\mathcal{S}}) \right\| \leq C$. Therefore, by the fact that when $0 \leq s \leq \frac{5\delta_s}{\tau}$,

$$e^{\gamma s} \leq \frac{2}{(e^{-\gamma s} + b_2)^2}$$

if τ small enough, by 5.17, 5.21, and [8, Lemma B.1],

$$(7.2) \quad \tau \left\| H[\tilde{u}] - H - \mathcal{L}_M u : C^{0,\alpha} \left(\mathcal{S}, (\tau\beta)^{-2} g_{\mathcal{S}}, e^{-\gamma s} \right) \right\|$$
$$\leq C\tau^{-2} \left\| u : C^{2,\alpha} \left(\mathcal{S}, (\tau\beta)^{-2} g_{\mathcal{S}}, e^{-\gamma s} + b_2 \right) \right\|^2,$$

when $\tau^{-1} \left\| u : C^{2,\alpha} \left(\mathcal{S}, (\tau\beta)^{-2} g_{\mathcal{S}}, e^{-\gamma s} + b_2 \right) \right\|$ small enough.

On the other hand, let $\mathcal{A} := \mathcal{N}_j[\xi] \setminus S[\xi]$, $j = 1, \ldots, k+1$, by 2.30(ii) and the construction, $\left\| A_{\mathcal{A}} : C^3(\mathcal{A}, g_{\mathcal{A}}) \right\| \leq C$. Therefore, by 5.17, 5.21, and [8, Lemma B.1],

$$(7.3) \quad \left\| H[\tilde{u}] - H - \mathcal{L}_M u : C^{0,\alpha} (\mathcal{A}, g_{\mathcal{A}}) \right\| \leq C \left\| u : C^{2,\alpha} (\mathcal{A}, g_{\mathcal{A}}) \right\|^2,$$

when $\left\| u : C^{2,\alpha} (\mathcal{A}, g_{\mathcal{A}}) \right\|$ small enough. The result then follows by combining (7.2) and (7.3) and using the definitions. $\square$

We combine now the results of the previous sections with the estimates above to prove the main theorem of this article.

Theorem 7.4 (Main Theorem). *Given $k \in \mathbb{N}$ large enough in absolute terms and $m \in \mathbb{N}$ large enough in terms of k, there is a compact embedded two-sided free boundary minimal smooth surface (FBMS) $\mathbf{M}_{k,m}$ of genus zero and km connected components, which is symmetric under the action of $\mathcal{G} = D_{2m} \times \mathbb{Z}_2$ (Definition 1.4) and is the graph of a small function $\tilde{v}$ in an auxiliary ambient metric g_A (Definition 5.17) over a smooth initial surface $M[\xi_0]$ (see 5.12 and 5.15), where $\xi_0 \in \Xi_{\mathcal{V}}$, the constant $\underline{c} > 0$ in the definition of $\Xi_{\mathcal{V}}$ (Definition 5.3) depends only on k, and $\|\tilde{v}\|_2 \leq C(k)/m$ (recall Definition 6.10 for the definition of the norm). Moreover $\mathbf{M}_{k,m}$ converges on compact subsets of the interior of $\mathbb{B}^3$ in all norms to the configuration $\mathcal{W}[0{:}k]$ as $m \to \infty$; the Hausdorff distance of $\mathbf{M}_{k,m}$ from $\mathcal{W}[0{:}k]$ tends to 0 as $m \to \infty$; and in turn the Hausdorff distance of $\mathcal{W}[0{:}k]$ from $\mathbb{S}^2 = \partial\mathbb{B}^3$ tends to 0 as $k \to \infty$. Finally $\lim_{k\to\infty} \lim_{m\to\infty} \mathbf{M}_{k,m} = \mathbb{S}^2$ in the varifold sense.*

Proof. Let $\mathcal{X}$ be the Banach space $C^{2,\alpha'}(M[0])$, where $\alpha' \in (0, \alpha)$ is a fixed constant. Given a function $f \in \mathcal{X}$, the norm $\|f : \mathcal{X}\|$ is defined by

$$\|f : \mathcal{X}\| := \left\| f : C^{2,\alpha'}(M[0]) \right\|.$$

There is a family of smooth diffeomorphisms $D_\xi : M[0] \to M[\xi]$ for $\xi \in \Xi_{\mathcal{V}}$, which depend smoothly on ξ and satisfy the following: there is constant C such that for $f \in C^{2,\alpha}(M[0])$ and $f' \in C^{2,\alpha}(M[\xi])$ (recall 6.10),

$$(7.5) \quad \left\| f \circ D_\xi^{-1} \right\|_2 \leq C \|f\|_2, \quad \left\| f' \circ D_\xi \right\|_2 \leq C \|f'\|_2.$$

Let $\Xi := \{(\xi, u) \in \mathcal{V} \times \mathcal{X} : |\xi| \leq \underline{c}\tau, \|u\|_2 \leq \underline{c}\tau\}$. By the definition, $\Xi \subset \mathcal{V} \times \mathcal{X}$ is compact and convex.

Given $(\xi, u) \in \Xi$. Let $v := u \circ D_\xi^{-1} - u_H$, where $u_H \in C^{2,\alpha}(M[\xi])$ as in 6.16. Then by 6.16 and (7.5),

$$(7.6) \qquad \|v\|_2 \leq C(\underline{c}+1)\tau.$$

By 7.1, the surface $M_{\tilde{v}}$ is well defined and

$$(7.7) \qquad \|H[\tilde{v}] - H - \mathcal{L}_M v\|_0 \leq C(\underline{c}+1)^2\tau^2,$$

where $\tilde{v} := (\Omega \circ X)v$ as in 5.21. Let $E := H[\tilde{v}] - H - \mathcal{L}_M v$, by 6.15, there are $(v_E, \underline{\theta}_E, \underline{\varphi}_E) = \mathcal{R}_M E$. Let $\underline{\sigma}_E := \left\{ \frac{\theta_{Ei}}{\alpha_i^+[\underline{\sigma}:k]} \right\}_{i=1}^k$, by 6.15, 6.16 and (7.7),

$$(7.8) \qquad H[\tilde{v}] = \mathcal{L}_M(v_E + u \circ D_\xi^{-1}) + W(\underline{\theta}_H + \underline{\theta}_E, \underline{\varphi}_H + \underline{\varphi}_E),$$

$$(7.9) \qquad \partial_\rho \tilde{v} = (\partial_\rho + 1)u \circ D_\xi^{-1},$$

$$(7.10) \qquad |\underline{\sigma} - \underline{\sigma}_E - \underline{\sigma}_H| \leq C\tau + C(\underline{c}+1)^2\tau^2,$$

$$(7.11) \qquad \left|\underline{\varphi} + \underline{\varphi}_E + \underline{\varphi}_H\right| \leq C\tau + C(\underline{c}+1)^2\tau^2,$$

$$(7.12) \qquad \|v_E\|_2 \leq C(\underline{c}+1)^2\tau^2.$$

Then define the map $\mathcal{J} : \Xi \to \mathcal{V} \times \mathcal{X}$ by

$$(7.13) \qquad \mathcal{J}(\xi, u) := ((\underline{\sigma} - \underline{\sigma}_E - \underline{\sigma}_H, \underline{\varphi} + \underline{\varphi}_E + \underline{\varphi}_H), -v_E \circ D_\xi).$$

(7.10), (7.11) and (7.12) imply that by choosing $\underline{c}$ large enough and τ small enough, $\mathcal{J}(\Xi) \subset \Xi$. As $\mathcal{J}$ is continuous by the construction, the Schauder fixed point theorem can be applied and there must be a fixed point (ξ_0, u) of $\mathcal{J}$. By (7.8) and (7.13), the corresponding $\mathbf{M}_{k,m} := M_{\tilde{v}}$ is a minimal surface. Moreover, by 5.21, 6.15 and (7.9), it has free boundary in $\mathbb{B}^3$ and by 5.15(iii), there are km boundary components. The estimate of $\tilde{v}$ follows by (7.6) and the uniform boundedness of Ω in the Definition 5.17. The smoothness of $\tilde{v}$ follows by the standard regularity theorem. The embeddedness follows by the embeddedness of $M[\xi_0]$ and the smallness of $\tilde{v}$. The symmetry of $\mathbf{M}_{k,m}$ follows by the construction and 5.15(iv). The convergence of $\mathbf{M}_{k,m}$ follows by 2.26 and the smallness of $\tilde{v}$. $\square$

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