

SOLITON RESOLUTION FOR EQUIVARIANT SELF-DUAL CHERN–SIMONS–SCHRÖDINGER EQUATION IN WEIGHTED SOBOLEV CLASS

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ABSTRACT. We consider the self-dual Chern–Simons–Schrödinger equation (CSS) under equivariant symmetry, which is a L^2 -critical equation. It is known that (CSS) admits solitons and finite-time blow-up solutions. In this paper, we show *soliton resolution* for any solutions with equivariant data in the weighted Sobolev space $H^{1,1}$: every maximal solution decomposes into at most one modulated soliton and a radiation. A striking fact is that the nonscattering part must be a single modulated soliton. To our knowledge, this is the first result on soliton resolution in a class of nonlinear Schrödinger equations which are not known to be completely integrable. The key ingredient is the defocusing nature of the equation in the exterior of a soliton profile. This is a consequence of two distinctive features of (CSS): self-duality and non-local nonlinearity.

CONTENTS

1. Introduction	1
2. Preliminaries	9
2.1. Notation	9
2.2. Linearization of (CSS)	11
2.3. Adapted function spaces	12
2.4. Duality estimates	13
3. Proof of Theorem 1.1 when $m < 0$	16
4. Proof of Theorem 1.1 when $m \geq 0$	17
4.1. Decomposition of small energy solutions	18
4.2. Upper bounds for $\lambda(t)$	24
4.3. Existence and regularity of asymptotic profile	27
References	29

1. INTRODUCTION

We study the long time dynamics of the self-dual Chern–Simons–Schrödinger equation (CSS) under equivariant symmetry. Our main result (Theorem 1.1)

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is soliton resolution of solutions in the weighted Sobolev space $H^{1,1}$. Moreover, in the case of finite-time blow-up, our proof works for all finite energy solutions.

The *self-dual Chern–Simons–Schrödinger* equation within m -equivariance is

$$(CSS) \quad i(\partial_t + iA_t[u])u + \partial_r^2 u + \frac{1}{r}\partial_r u - \left(\frac{m + A_\theta[u]}{r}\right)^2 u + |u|^2 u = 0,$$

where $m \in \mathbb{Z}$ (called *equivariance index*), and the connection components $A_t[u]$ and $A_\theta[u]$ are given by

$$(1.1) \quad A_t[u] = -\int_r^\infty (m + A_\theta[u])|u|^2 \frac{dr'}{r'}, \quad A_\theta[u] = -\frac{1}{2} \int_0^r |u|^2 r' dr'.$$

The Chern–Simons–Schrödinger equation was introduced by Jackiw–Pi [11] as a nonrelativistic planar quantum electromagnetic model that exhibits *self-duality* (to be discussed more below). It is a gauge-covariant cubic nonlinear Schrödinger equation on \mathbb{R}^2 . We refer to [5, 10–13] for more physical backgrounds. The model (CSS) is derived after fixing the Coulomb gauge condition and imposing the equivariant symmetry on the scalar field ϕ :

$$\phi(t, x) = u(t, r)e^{im\theta},$$

where (r, θ) are the polar coordinates on \mathbb{R}^2 . For more details on this reduction, we refer to the introduction of [15–17].

(CSS) enjoys various symmetries and conservation laws. Among the most basic symmetries are the time translation and the phase rotation symmetries. Associated to these are the conservation laws for the *energy* and the *mass* (the physical interpretation of the quantity $M[u]$ is the *total charge*, but in this paper we shall call it mass following the widespread convention for NLS):

$$(1.2) \quad E[u] := \int \frac{1}{2} |\partial_r u|^2 + \frac{1}{2} \left(\frac{m + A_\theta[u]}{r}\right)^2 |u|^2 - \frac{1}{4} |u|^4,$$

$$(1.3) \quad M[u] := \int |u|^2,$$

where we denoted $\int f(r) = 2\pi \int f(r)rdr$. With this energy functional, (CSS) admits a *Hamiltonian* structure

$$\partial_t u = -i\nabla E[u],$$

where ∇ (acting on a functional) is the Fréchet derivative with respect to the real inner product $\int \operatorname{Re}(\bar{u}v)$. Of particular importance in this work are the L^2 -*scaling symmetry* and the *pseudoconformal symmetry*; if $u(t, r)$ is a solution to (CSS), then the functions u_λ and $\mathcal{C}u$ also solve (CSS):

$$(1.4) \quad u_\lambda(t, r) := \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{r}{\lambda}\right), \quad \forall \lambda > 0,$$

$$(1.5) \quad [\mathcal{C}u](t, r) := \frac{1}{|t|} u\left(-\frac{1}{t}, \frac{r}{|t|}\right) e^{\frac{ir^2}{4t}}, \quad \forall t \neq 0.$$

Associated to (1.4) and (1.5) are the *virial identities*:

$$(1.6) \quad \partial_t \int r^2 |u|^2 = 4 \int \operatorname{Im}(\bar{u} \cdot r \partial_r u),$$

$$(1.7) \quad \partial_t \int \operatorname{Im}(\bar{u} \cdot r \partial_r u) = 4E[u].$$

In this aspect, (CSS) shares many similarities with the cubic NLS

$$(NLS) \quad i\partial_t \psi + \Delta \psi + |\psi|^2 \psi = 0 \quad \text{on } \mathbb{R}^{1+2}.$$

A notable feature of (CSS) in comparison to NLS is the *self-duality*. Indeed, the energy functional can be written in the self-dual form

$$(1.8) \quad E[u] = \int \frac{1}{2} |\mathbf{D}_u u|^2,$$

where \mathbf{D}_u is the (covariant) *Cauchy–Riemann operator* defined by

$$(1.9) \quad \mathbf{D}_u f := \partial_r f - \frac{m + A_\theta[u]}{r} f.$$

We call the operator $u \mapsto \mathbf{D}_u u$ the *Bogomol’nyi operator*. Due to (1.8) and the Hamiltonian structure, any static solutions to (CSS) are given by solutions to the *Bogomol’nyi equation*:

$$(1.10) \quad \mathbf{D}_Q Q = 0.$$

For $m \geq 0$, there is an *explicit* m -equivariant static solution (Jackiw–Pi vortex) to the Bogomol’nyi equation which is unique up to the symmetries of the equation [10]:

$$(1.11) \quad Q(r) = \sqrt{8}(m+1) \frac{r^m}{1 + r^{2m+2}}, \quad m \geq 0.$$

Note that we suppressed the m -dependences in \mathbf{D}_u and Q for the simplicity of notation. Moreover, applying the pseudoconformal transform (1.5) to Q , we obtain an explicit finite-time blow-up solution:

$$S(t, r) := \frac{1}{|t|} Q\left(\frac{r}{|t|}\right) e^{-i \frac{r^2}{4|t|}}, \quad t < 0.$$

We note that $S(t)$ has finite energy if and only if $m \geq 1$.

Let us briefly discuss some known results on the covariant Chern–Simons–Schrödinger equation without symmetry. The local well-posedness has been studied by many authors: [1, 9, 21, 23]. However, the best known result by Liu–Smith–Tataru [23] still misses the critical L^2 -space. There are also results on the long-term dynamics [1, 2, 25].

If one restricts to the equivariant self-dual Chern–Simons–Schrödinger equation, i.e., (CSS), then much more is known. First of all, as there is no derivative nonlinearity, (CSS) is well-posed in L^2 [22, Section 2]. The global-in-time large data dynamics are partially known. Here, the ground state Q provides a natural threshold for the nonscattering dynamics. Indeed, Liu–Smith [22] proved the following *subthreshold theorem*: for $m \geq 0$, any m -equivariant L^2 -solutions u with $M[u] < M[Q]$ scatter both forwards and backwards in time. At the threshold mass $M[u] = M[Q]$ (necessarily $m \geq 0$), there are two typical examples of non-scattering solutions: Q and

$S(t)$. These are indeed the only examples in the energy space due to the classification result of Li-Liu [20] ($S(t)$ for the radial case $m = 0$ is an exception because it does not have finite energy due to the slow spatial decay of Q). Above the threshold, [15–18] provide a variety of finite-time blow-up solutions (and global-in-time nonscattering solutions) with quantitative descriptions of the dynamics near the blow-up time.

It is widely believed that, for arbitrary large data, the maximal solutions asymptotically decompose into the sum of decoupled solitons and a radiation. This is referred to as the *soliton resolution conjecture*. This has been known for a wide range of completely integrable equations, but the focus of the present paper is on soliton resolution for (possibly) *non-integrable* models without exploiting complete integrability techniques. Recently, the remarkable works [4, 6–8, 14] established soliton resolution for the radial critical nonlinear wave equation (in various dimensions) and energy-critical equivariant wave maps. However, to our knowledge, there is no earlier result for non-integrable Schrödinger type equations.

The main result of this paper is *the proof of soliton resolution for the equivariant self-dual Chern–Simons–Schrödinger equation in a suitable weighted Sobolev class*. Our proof is based on a remarkable consequence of the non-local nonlinearity and the self-duality of (CSS), namely, the defocusing nature of the equation in the exterior of a soliton profile. This property also results in a strong rigidity of the dynamics of (CSS): the non-existence of multi-soliton configurations separated by scales. See the remarks following Theorem 1.1.

We are now ready to state the result. Let us denote the modulated soliton by

$$Q_{\lambda,\gamma}(r) := \frac{e^{i\gamma}}{\lambda} Q\left(\frac{r}{\lambda}\right), \quad \lambda \in (0, \infty), \quad \gamma \in \mathbb{R}/2\pi\mathbb{Z}.$$

We also denote by $H_m^{1,1}$ and H_m^1 the (weighted) Sobolev spaces $H^{1,1}$ and H^1 restricted to m -equivariant functions, equipped with the inherited norms. We denote by $\Delta^{(m)} = \partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}$ the Laplacian acting on m -equivariant functions.

Theorem 1.1 (Soliton resolution for equivariant $H^{1,1}$ -data). *Let $m \in \mathbb{Z}$. When $m \geq 0$, we have soliton resolution for $H_m^{1,1}$ -solutions:*

- (Finite-time blow-up solutions) *If u is a H_m^1 -solution to (CSS) that blows up forwards in time at $T < +\infty$, then $u(t)$ admits the decomposition*

$$(1.12) \quad u(t, \cdot) - Q_{\lambda(t), \gamma(t)} \rightarrow z^* \text{ in } L^2 \text{ as } t \rightarrow T^-,$$

for some continuous $\lambda(t) \in (0, \infty)$ and $\gamma(t) \in \mathbb{R}/2\pi\mathbb{Z}$, and $z^ \in L^2$ with the following properties:*

- (Further regularity of z^*) *We have $\partial_r z^*, \frac{1}{r}z^* \in L^2$. Moreover, if u is a $H_m^{1,1}$ finite-time blow-up solution, then we also have $rz^* \in L^2$.*
- (Bound on the blow-up speed) *As $t \rightarrow T$, we have*

$$(1.13) \quad \lambda(t) \lesssim_{M[u]} \sqrt{E[u]}(T - t).$$

When $m = 0$, we further have the improved bound as $t \rightarrow T$

$$(1.14) \quad \lambda(t) \lesssim_{M[u]} \frac{\sqrt{E[u]}(T-t)}{|\log(T-t)|^{\frac{1}{2}}}.$$

- (Global solutions) If u is a $H_m^{1,1}$ -solution to (CSS) that exists globally forwards in time, then either $u(t)$ scatters forwards in time, or $u(t)$ admits the decomposition

$$(1.15) \quad u(t, \cdot) - Q_{\lambda(t), \gamma(t)} - e^{it\Delta^{(-m-2)}} u^* \rightarrow 0 \text{ in } L^2 \text{ as } t \rightarrow +\infty,$$

for some continuous $\lambda(t) \in (0, \infty)$ and $\gamma(t) \in \mathbb{R}/2\pi\mathbb{Z}$, and $u^* \in L^2$ with the following properties:

- (Further regularity of u^*) We have $\partial_r u^*, \frac{1}{r} u^*, ru^* \in L^2$.
- (Bound on the scale) As $t \rightarrow +\infty$, we have

$$(1.16) \quad \lambda(t) \lesssim_{M[u]} \sqrt{E[\mathcal{C}u]},$$

where $\mathcal{C}u$ is the pseudoconformal transform (1.5) of u . When $m = 0$, we further have as $t \rightarrow +\infty$

$$(1.17) \quad \lambda(t) \lesssim_{M[u]} \frac{\sqrt{E[\mathcal{C}u]}}{|\log t|^{\frac{1}{2}}}.$$

On the other hand, when $m < 0$, any $H_m^{1,1}$ -solution to (CSS) scatters forwards in time. Due to the time-reversal symmetry, all the above statements also hold for backward-in-time evolutions.

We note that one can further choose smooth modulation parameters in Theorem 1.1 because the theorem is invariant under replacing $\lambda(t)$ by any function $\tilde{\lambda}(t)$ with $\tilde{\lambda}(t)/\lambda(t) \rightarrow 1$ (and similarly for $\gamma(t)$).

Remark 1.2 (The dynamics for $m \geq 0$ and $m < 0$). The dynamics of (CSS) for $m \geq 0$ and $m < 0$ are completely different. In fact, we will show that (CSS) for $m < 0$ is *defocusing* in the sense that

$$(1.18) \quad E[u] \sim_{M[u]} \|u(t)\|_{\dot{H}_m^1}^2$$

and hence there are no nontrivial Jackiw–Pi vortices for $m < 0$. See Lemma 3.1 for the proof.

Remark 1.3 (Nonexistence of multi-solitons). It is remarkable that at most one soliton can appear in the resolution. This is a distinctive feature of (CSS). Indeed, as a consequence of the self-duality and non-locality, we observe a *defocusing* nature, i.e., the strict positivity of the energy, of (CSS) at the exterior of a soliton profile. Hence two solitons at different scales cannot exist simultaneously. We will obtain this defocusing nature by combining our two observations: (i) (CSS) at the exterior of soliton resembles (CSS) for $m < 0$ (observed in [15]) and (ii) the defocusing nature (1.18) when $m < 0$. See Lemma 3.1 for the proof.

Even without equivariant symmetry, by essentially the same mechanism, we expect that there is no bubble tree (i.e., a multi-soliton separated *only* by scales) for the self-dual Chern–Simons–Schrödinger equation. However, multi-solitons separated by spatial distances may exist.

Remark 1.4 (Regularity assumptions on data). As seen in the above, we cover all H_m^1 finite-time blow-up solutions. For global solutions, we will reduce the situation to the H_m^1 finite-time blow-up case in the spirit of the pseudoconformal transform, which requires the $H_m^{1,1}$ -assumption. Note that $E[\mathcal{C}u]$ is well-defined for $H_m^{1,1}$ -solutions u . Soliton resolution for any global H_m^1 -solutions (or, more ambitiously L_m^2 -solutions) is an interesting open problem.

Remark 1.5 (Equivariance index on the scattering part). We choose to state the scattering for the radiative part of (1.15) under the $(-m-2)$ -equivariant free Schrödinger flow, because the scattering part $u(t) - Q_{\lambda(t), \gamma(t)}$ approximately solves $(-m-2)$ -equivariant (CSS). This fact is already observed in [15].

However, the equivariance index for the scattering part of (1.15) is irrelevant if one only considers the scattering in L^2 -norm. Indeed, for any $m, k \in \mathbb{Z}$ and radial functions $u^*, v^* \in L^2$ (we equip L^2 with the rdr -measure), we have

$$\|e^{it\Delta^{(m)}}u^* - e^{it\Delta^{(k)}}v^*\|_{L^2} \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

if u^* and v^* satisfy the relation

$$u^* = \mathcal{F}_m^{-1} \mathcal{F}_k v^*,$$

where \mathcal{F}_m is the rescaled version of the Hankel transform of order m :

$$[\mathcal{F}_m f](\rho) = \frac{i^{m-1}}{2} \int_0^\infty f(r) J_m\left(\frac{r\rho}{2}\right) r dr$$

with J_m Bessel function of the first kind of order m . Note that the above can be verified using the pseudoconformal transform and the identity $e^{it\Delta^{(m)}}u^* = [\mathcal{C}e^{i(\cdot)\Delta^{(m)}}\mathcal{F}_m u^*](t)$ for $t > 0$, for any $m \in \mathbb{Z}$. Since $\mathcal{F}_m^{-1}\mathcal{F}_k$ is unitary in L^2 , the L^2 -scattering is independent of equivariance indices. However, the scattering with different equivariance indices might not be equivalent under topologies other than L^2 . For example, the properties $\partial_r u^*, \frac{1}{r}u^*, ru^* \in L^2$ (as stated in Theorem 1.1) may not be preserved under changes of equivariance indices, i.e., under $\mathcal{F}_m^{-1}\mathcal{F}_k$.

Remark 1.6 (Bounds for scaling parameter). When $m \geq 1$, the explicit blow-up solution $S(t)$ and the pseudoconformal blow-up solutions constructed in [15, 16] are finite-energy finite-time blow-up solutions that saturate the bound (1.13). Similarly, the soliton Q itself saturates (1.16).

When $m = 0$, the blow-up solution $S(t)$ and the soliton Q do not satisfy the bounds (1.14) and (1.17), respectively. This is consistent with Theorem 1.1 because Q does not belong to $H_0^{1,1}$ and the explicit blow-up solution $S(t)$ does not have finite energy, and hence Q and $S(t)$ are not covered by our theorem. Note that (1.17) says that any global-in-time nonscattering $H_0^{1,1}$ -solution must blow up in infinite time. On the other hand, the authors [17, 18] construct finite energy finite-time blow-up solutions with the speed $\lambda(t) \sim (T-t)|\log(T-t)|^{-2}$ and $\lambda(t) \sim (T-t)^p|\log(T-t)|^{-1}$ for all $p > 1$, respectively. However, we do not know whether the upper bound (1.14) is sharp or not.

The bounds (1.13)-(1.14) and (1.16)-(1.17) may have to be relaxed for more general class of solutions. Note that such a relaxation is necessary for $m = 0$ by the concrete examples $S(t)$ and Q .

Remark 1.7 (On the phase rotation parameter). The phase rotation parameter does not necessarily stabilize as $t \rightarrow T$ (or $t \rightarrow +\infty$). Indeed, the finite-time blow-up solutions constructed in [18] for the $m = 0$ case exhibit infinite amount of phase rotations. The $m \geq 1$ case is open.

Remark 1.8 (Comparison with (NLS)). For the finite-time blow-up case, there are similar results [24, 26] in (NLS) for solutions having slightly supercritical mass (i.e., $M[u] - M[Q] \ll 1$). Under this assumption, a standard variational argument in the blow-up scenario ensures that solutions eventually undergo the near-soliton dynamics in the L^2 -topology. Note that in Theorem 1.1 we do not have L^2 -proximity to solitons.

For the near-soliton dynamics of (NLS), it is known from [26] that any finite energy finite-time blow-up solutions satisfy either $\lambda(t) \sim ((T-t)/\log|\log(T-t)|)^{\frac{1}{2}}$ or $\lambda(t) \lesssim (T-t)$. The former log-log rate essentially arises from negative energy solutions, which is impossible for the self-dual (CSS). It is expected that such log-log rates are possible for the focusing non-self-dual (CSS) [2].

Strategy of the proof. We use the notation in Section 2.1.

For $m < 0$, we will prove the global coercivity of energy (1.18), which renders (CSS) essentially defocusing for $m < 0$. Thus the scattering for $H_m^{1,1}$ -data follows from a classical argument using the pseudoconformal transform, see e.g., [3].

The interesting case is when $m \geq 0$, where solitons do exist. By the pseudoconformal transform, it suffices to prove the finite-time blow-up case of Theorem 1.1. *Our key input is the nonlinear coercivity of energy (1.20) after extracting out the soliton profile, which holds for solutions with possibly large mass.* As explained in Remark 1.3, this nonlinear coercivity is a consequence of the *self-duality* and *non-locality*, which are distinctive features of (CSS).

1. Variational argument. For a finite energy finite-time blow-up solution $u(t)$, not necessarily close to the modulated soliton $Q_{\lambda,\gamma}$ in L^2 , we work with the renormalized solution $v(t)$ (using the L^2 -scaling) near the blow-up time (say T) of u to have $\|v(t)\|_{\dot{H}_m^1} = \|Q\|_{\dot{H}_m^1}$ and $E[v(t)] \rightarrow 0$.

In view of the *uniqueness* of the zero energy solution (i.e., $E[w] = 0$ if and only if $w = Q_{\lambda,\gamma}$ or $w = 0$) and renormalization, we expect that each $v(t)$ is close to $e^{i\gamma(t)}Q$. We remark that the closeness of v to $e^{i\gamma}Q$ cannot be measured in L^2 , because we do not assume that the mass of v is close to that of Q . In fact, we are able to show that v is close to $e^{i\gamma}Q$ in the \dot{H}^1 -topology (Lemma 4.2). Therefore, we roughly have

$$(1.19) \quad u(t) = [Q + \epsilon(t, \cdot)]_{\lambda(t), \gamma(t)} \quad \text{with} \quad \|\epsilon(t)\|_{\dot{H}^1} \rightarrow 0$$

for some $\lambda(t)$ and $\gamma(t)$. We may fix the decomposition by imposing suitable orthogonality conditions on ϵ . We note that $\|\epsilon(t)\|_{L^2}$ might be large. We also note that (1.19) is a consequence of the uniqueness of zero-energy solutions to (CSS); one cannot expect (1.19) for (NLS) for arbitrary solutions with large mass.

2. *Nonlinear coercivity of energy.* For the proof of Theorem 1.1, the qualitative information $\|\epsilon(t)\|_{\dot{H}^1} \rightarrow 0$ is not sufficient. Our next crucial input is the following *nonlinear coercivity of the energy* (Lemma 4.4):

$$(1.20) \quad E[Q + \epsilon] \gtrsim_{\|\epsilon\|_{L^2}} \|\epsilon\|_{\dot{H}^1}^2$$

for ϵ satisfying the orthogonality conditions and $\|\epsilon\|_{\dot{H}^1} \ll 1$. Here, the point is that *the coercivity holds even for* $\|\epsilon(t)\|_{L^2} \gtrsim 1$. If we were to have L^2 -smallness $\|\epsilon(t)\|_{L^2} \ll 1$, then all the higher order terms of $E[Q + \epsilon]$ are perturbative and (1.20) is merely a consequence of the linear coercivity (around Q). When $\epsilon(t)$ has large L^2 -norm, the higher order terms of $E[Q + \epsilon]$ are no longer perturbative. Instead, we have (using the *self-duality* (1.8))

$$\begin{aligned} E[Q + \epsilon] &= \frac{1}{2} \int |\mathbf{D}_{Q+\epsilon}(Q + \epsilon)|^2 \\ &\approx \frac{1}{2} \int |L_Q(\chi_R \epsilon)|^2 + \left| \left(\partial_r - \frac{m + A_\theta[Q] + A_\theta[\epsilon]}{r} \right) ((1 - \chi_R)\epsilon) \right|^2, \end{aligned}$$

where L_Q is the linearized Bogomol'nyi operator around Q (see (2.2)). The interior term is simply handled by a localized version of the linear coercivity for L_Q . However, the exterior term contains non-perturbative higher order terms like $|\frac{A_\theta[\epsilon]}{r}\epsilon|^2$. At this point, we use the *non-locality* of the problem, particularly the fact that $m + A_\theta[Q] \approx -(m+2)$ is negative. Thus the exterior term can be viewed as the energy of ϵ for the $-(m+2)$ -equivariant (CSS). Using the boundary condition $[(1 - \chi_R)\epsilon](R) = 0$ and the fact that both $m + A_\theta[Q]$ and $A_\theta[\epsilon]$ are negative, we can prove unconditional coercivity (4.12) for the exterior term (which we call nonlinear Hardy's inequality). Note that this argument also shows the nonexistence of nontrivial zero energy solutions to (CSS) for negative equivariance indices. As a result, the nonlinear coercivity of energy (1.20) follows.

3. *Bound on the blow-up rate.* The proof of (1.13) is standard and very similar to the pseudoconformal regime in Raphaël [26]. Indeed, by a standard modulation analysis, we obtain a modulation estimate in the renormalized spacetime variables (2.1):

$$\left| \frac{\lambda_s}{\lambda} \right| \lesssim \|\epsilon\|_{\dot{H}_m^1}.$$

Then, thanks to the nonlinear coercivity (1.20) of energy, we get

$$|\lambda_t| = \frac{1}{\lambda} \left| \frac{\lambda_s}{\lambda} \right| \lesssim \frac{1}{\lambda} \sqrt{E[Q + \epsilon]} = \sqrt{E[u]},$$

whose integration yields the bound (1.13).

However, the proof of (1.14) for $m = 0$ is trickier. We will use the fact that $yQ \notin L^2$ (logarithmic divergence). Indeed, motivated by the generalized nullspace relations (2.7) of the linearized operator $i\mathcal{L}_Q$ (see (2.4)), the time variation of λ can be tracked by looking at the time evolution of the inner product $(\epsilon, y^2 Q)_r$; we roughly have an estimate of the form

$$\frac{\lambda_s}{\lambda} (\Lambda Q, y^2 Q \chi_R)_r - \partial_s (\epsilon, y^2 Q \chi_R)_r \approx -(i\mathcal{L}_Q \epsilon, y^2 Q \chi_R)_r \lesssim \|\epsilon\|_{\dot{H}_m^1} \|yQ \chi_R\|_{L^2}.$$

The point here is that we have different logarithmic divergences of the quantities:

$$\begin{aligned} (\Lambda Q, y^2 Q \chi_R)_r &\sim \log R, \\ \|y Q \chi_R\|_{L^2} &\sim \sqrt{\log R}. \end{aligned}$$

Next, we choose $R = R(t)$ which diverges polynomially so that $\log R \sim |\log(T-t)|$ and absorb $\partial_s(\epsilon, y^2 Q \chi_R)_r$ into a total derivative to roughly have

$$\left| \frac{\lambda_s}{\lambda} \right| \lesssim \frac{1}{|\log(T-t)|^{\frac{1}{2}}} \|\epsilon\|_{\dot{H}^1}.$$

Using (1.20) again, this implies (1.14).

4. Existence of the asymptotic profile. The existence of the asymptotic profile z^* as in (1.12) as well as its regularity can be proved in a very similar manner to Merle–Raphaël [24]. To obtain z^* as the strong L^2 -limit of $\epsilon^\sharp(t)$ as $t \rightarrow T^-$, we again take advantage of the nonlinear coercivity of energy in the form $\|\epsilon^\sharp(t)\|_{H_m^1} \lesssim 1$. This means that $\epsilon^\sharp(t)$ (and hence z^*) is not only controlled on the obvious soliton scale $r \lesssim \lambda$, but also up to scale $r \lesssim 1$.

Organization of the paper. In Section 2, we collect notation and preliminaries for our analysis. In Section 3, we prove Theorem 1.1 for $m < 0$. The heart of this paper is contained in Section 4, where we prove Theorem 1.1 for $m \geq 0$.

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2. PRELIMINARIES

In this section, we collect notation and preliminary facts on linearization, adapted function spaces, and duality estimates for (CSS).

2.1. Notation. For $A \in \mathbb{C}$ and $B \geq 0$, we use the standard asymptotic notation $A \lesssim B$ or $A = O(B)$ if there is a constant $C > 0$ such that $|A| \leq CB$. $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. The dependencies of C are specified by subscripts, e.g., $A \lesssim_E B \Leftrightarrow A = O_E(B) \Leftrightarrow |A| \leq C(E)B$. In this paper, any dependencies on the equivariance index m will be omitted.

We also use the notation $\langle x \rangle$, $\log_+ x$, $\log_- x$ defined by

$$\langle x \rangle := (|x|^2 + 1)^{\frac{1}{2}}, \quad \log_+ x := \max\{\log x, 0\}, \quad \log_- x := \max\{-\log x, 0\}.$$

We let $\chi = \chi(x)$ be a smooth spherically symmetric cutoff function such that $\chi(x) = 1$ for $|x| \leq 1$ and $\chi(x) = 0$ for $|x| \geq 2$. For $A > 0$, we define its rescaled version by $\chi_A(x) := \chi(x/A)$.

We mainly work with equivariant functions on \mathbb{R}^2 , say $\phi : \mathbb{R}^2 \rightarrow \mathbb{C}$, or equivalently their radial part $u : \mathbb{R}_+ \rightarrow \mathbb{C}$ with $\phi(x) = u(r)e^{im\theta}$, where

$\mathbb{R}_+ := (0, \infty)$ and $x_1 + ix_2 = re^{i\theta}$. We denote by $\Delta^{(m)} = \partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}$ the Laplacian acting on m -equivariant functions.

The integral symbol \int means

$$\int = \int_{\mathbb{R}^2} dx = 2\pi \int r dr.$$

For complex-valued functions f and g , we define their *real* inner product by

$$(f, g)_r := \int \operatorname{Re}(\bar{f}g).$$

For a real-valued functional F and a function u , we denote by $\nabla F[u]$ the *functional derivative* of F at u under this real inner product.

We denote by Λ the L^2 -scaling generator:

$$\Lambda := r\partial_r + 1.$$

Given a scaling parameter $\lambda \in \mathbb{R}_+$, phase rotation parameter $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$, and a function f , we write

$$f_{\lambda, \gamma}(r) := \frac{e^{i\gamma}}{\lambda} f\left(\frac{r}{\lambda}\right).$$

When λ and γ are clear from the context, we will also denote the above by f^\sharp as in [15], i.e.,

$$f^\sharp := f_{\lambda, \gamma}.$$

Similarly, we define its inverse by \flat :

$$g^\flat(y) := \lambda e^{-i\gamma} g(\lambda y).$$

When a time-dependent scaling parameter $\lambda(t)$ is given, we define rescaled spacetime variables s, y by the relations

$$(2.1) \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)} \quad \text{and} \quad y = \frac{r}{\lambda(t)}.$$

The raising operation \sharp converts a function $f = f(y)$ to a function of r : $f^\sharp = f^\sharp(r)$. Similarly, the lowering operation \flat converts a function $g = g(r)$ to a function of y : $g^\flat = g^\flat(y)$. In the modulation analysis in this paper, the dynamical parameters such as λ, γ, b, η are functions of either the variable t or s under $\frac{ds}{dt} = \frac{1}{\lambda^2}$.

For $k \in \mathbb{N}$, we define

$$\begin{aligned} |f|_k &:= \max\{|f|, |r\partial_r f|, \dots, |(r\partial_r)^k f|\}, \\ |f|_{-k} &:= \max\{|\partial_r^k f|, |\frac{1}{r}\partial_r^{k-1} f|, \dots, |\frac{1}{r^k} f|\}. \end{aligned}$$

We note that $|f|_k \sim r^k |f|_{-k}$. The following Leibniz rules hold:

$$|fg|_k \lesssim |f|_k |g|_k, \quad |fg|_{-k} \lesssim |f|_k |g|_{-k}.$$

The relevant function spaces will be discussed in Section 2.3.

2.2. Linearization of (CSS). We quickly record the linearization of (CSS) around Q . For more detailed exposition, see the corresponding sections of [15–17].

We first linearize the Bogomol'nyi operator $w \mapsto \mathbf{D}_w w$. We can write

$$(2.2) \quad \mathbf{D}_{w+\epsilon}(w + \epsilon) = \mathbf{D}_w w + L_w \epsilon + (\text{h.o.t.}),$$

where

$$L_w \epsilon := \mathbf{D}_w \epsilon - \frac{2}{y} A_\theta[w, \epsilon] w,$$

and $A_\theta[\psi_1, \psi_2]$ is defined through the polarization

$$A_\theta[\psi_1, \psi_2] := -\frac{1}{2} \int_0^r \text{Re}(\overline{\psi_1} \psi_2) r' dr'.$$

The L^2 -adjoint L_w^* of L_w takes the form

$$L_w^* v = \mathbf{D}_w^* v + w \int_y^\infty \text{Re}(\overline{w} v) dy'.$$

We remark that the operator L_w and its adjoint L_w^* are *only* \mathbb{R} -linear. From $\mathbf{D}_Q Q = 0$ and (1.8), we have the following expansion for the energy:

$$(2.3) \quad E[Q + \epsilon] = \frac{1}{2} \|L_Q \epsilon\|_{L^2}^2 + (\text{h.o.t.}).$$

Next, we linearize (CSS), which we write in the Hamiltonian form $\partial_t u + i \nabla E[u] = 0$. We decompose

$$(2.4) \quad \nabla E[w + \epsilon] = \nabla E[w] + \mathcal{L}_w \epsilon + R_w(\epsilon),$$

where $\mathcal{L}_w \epsilon$ collects the linear terms in ϵ and $R_w(\epsilon)$ collects the remainders. Note that \mathcal{L}_w is the Hessian of E , i.e.,

$$\nabla^2 E[w] = \mathcal{L}_w.$$

Being the Hessian of the energy, \mathcal{L}_w is formally symmetric with respect to the real inner product:

$$(\mathcal{L}_w f, g)_r = (f, \mathcal{L}_w g)_r.$$

If one recalls (1.8), we have $\nabla E[u] = L_u^* \mathbf{D}_u u$. Thus (2.4) and (2.2) yield

$$\begin{aligned} \mathcal{L}_w \epsilon &= L_w^* L_w \epsilon + \left(\frac{1}{y} \int_0^y \text{Re}(\overline{w} \epsilon) y' dy' \right) \mathbf{D}_w w \\ &\quad + w \int_y^\infty \text{Re}(\overline{\epsilon} \mathbf{D}_w w) dy' + \epsilon \int_y^\infty \text{Re}(\overline{w} \mathbf{D}_w w) dy'. \end{aligned}$$

Again, we remark that the operator \mathcal{L}_w is only \mathbb{R} -linear. In particular, from $\mathbf{D}_Q Q = 0$, we observe the self-dual factorization of $i \mathcal{L}_Q$:

$$(2.5) \quad i \mathcal{L}_Q = i L_Q^* L_Q.$$

This identity was first observed in [19]. Thus, the linearization of (CSS) at Q is

$$(2.6) \quad \partial_t \epsilon + i \mathcal{L}_Q \epsilon = 0, \quad \text{or} \quad \partial_t \epsilon + i L_Q^* L_Q \epsilon = 0.$$

Finally, we briefly recall the formal generalized kernel relations of the linearized operator $i \mathcal{L}_Q$. We have

$$N_g(i \mathcal{L}_Q) = \text{span}_{\mathbb{R}} \{ \Lambda Q, i Q, \frac{i}{4} r^2 Q, \rho \}$$

with the relations (see [15, Proposition 3.4])

$$(2.7) \quad \begin{cases} i\mathcal{L}_Q(i\frac{r^2}{4}Q) = \Lambda Q, & i\mathcal{L}_Q\rho = iQ, \\ i\mathcal{L}_Q(\Lambda Q) = 0, & i\mathcal{L}_Q(iQ) = 0, \end{cases}$$

where the existence of ρ is given in [15, Lemma 3.6]. In fact, we have:

$$(2.8) \quad \begin{cases} L_Q(i\frac{r^2}{4}Q) = i\frac{r}{2}Q, & L_Q\rho = \frac{1}{2(m+1)}rQ, \\ L_Q^*(i\frac{r}{2}Q) = -i\Lambda Q, & L_Q^*(\frac{1}{2(m+1)}rQ) = Q, \\ L_Q(\Lambda Q) = 0, & L_Q(iQ) = 0. \end{cases}$$

2.3. Adapted function spaces. In this subsection, we quickly recall the equivariant Sobolev spaces and the adapted function space $\dot{\mathcal{H}}_m^1$. For more details, see [15, 17].

For $s \geq 0$, we denote by H_m^s and \dot{H}_m^s the restriction of the usual Sobolev spaces $H^s(\mathbb{R}^2)$ and $\dot{H}^s(\mathbb{R}^2)$ on m -equivariant functions. For high equivariance indices, we have the generalized Hardy's inequality [15, Lemma A.7]: whenever $0 \leq k \leq |m|$, we have

$$(2.9) \quad \| |f|_{-k} \|_{L^2} \sim \|f\|_{\dot{H}_m^k}.$$

Specializing this to $k = 1$, we have the *Hardy-Sobolev inequality* [15, Lemma A.6]: whenever $|m| \geq 1$, we have

$$(2.10) \quad \|r^{-1}f\|_{L^2} + \|f\|_{L^\infty} \lesssim \|f\|_{\dot{H}_m^1}.$$

Note in general that $H_0^1 \hookrightarrow L^\infty$ is *false*. Finally, we define the weighted Sobolev space $H_m^{1,1}$ equipped with the norm

$$\|f\|_{H_m^{1,1}}^2 := \|f\|_{H_m^1}^2 + \|rf\|_{L^2}^2.$$

Next, we define the adapted function space $\dot{\mathcal{H}}_m^1$. This space is motivated by the linear coercivity of energy, namely the coercivity estimates for the linearized Bogomol'nyi operator L_Q at the \dot{H}^1 -level. The available Hardy-type controls on f from $\|L_Q f\|_{L^2}$ are different for the cases $m = 0$ and $m \geq 1$. When $m \geq 1$, we have a coercivity of L_Q in terms of the usual \dot{H}_m^1 -norm. Thus we let

$$\dot{\mathcal{H}}_m^1 := \dot{H}_m^1 \quad \text{when } m \geq 1.$$

When $m = 0$, the adapted function space $\dot{\mathcal{H}}_0^1$ is defined by the norm

$$\|f\|_{\dot{\mathcal{H}}_0^1} := \|\partial_r f\|_{L^2} + \|\langle \log_- r \rangle^{-1} r^{-1} f\|_{L^2}.$$

We remark that the logarithmic loss near the origin $r = 0$ is introduced due to the failure of Hardy's inequality when $m = 0$. Let us note $\dot{\mathcal{H}}_0^1 \hookrightarrow \dot{H}_0^1$ and $\dot{\mathcal{H}}_0^1 \cap L^2 = H_0^1$. One also has the following weighted L^∞ -estimate

$$(2.11) \quad \|\langle \log_- r \rangle^{-\frac{1}{2}} f\|_{L^\infty} \lesssim \|f\|_{\dot{\mathcal{H}}_0^1},$$

which follows from integrating the inequality

$$\left| \partial_r \left(\frac{|f|^2}{\langle \log_- r \rangle} \right) \right| \leq \left(|\partial_r f| + \frac{|f|}{r \langle \log_- r \rangle} \right) \cdot \frac{|f|}{\langle \log_- r \rangle}$$

and applying the fundamental theorem of calculus.

We now state the coercivity estimates of L_Q at the \dot{H}^1 -level. To obtain the coercivity of L_Q , it is necessary to preclude the kernel elements ΛQ and iQ of L_Q . We do this by imposing suitable orthogonality conditions. We fix profiles $\mathcal{Z}_1, \mathcal{Z}_2 \in C_{c,m}^\infty$ satisfying *the transversality condition*

$$(2.12) \quad \det \begin{pmatrix} (\Lambda Q, \mathcal{Z}_1)_r & (iQ, \mathcal{Z}_1)_r \\ (\Lambda Q, \mathcal{Z}_2)_r & (iQ, \mathcal{Z}_2)_r \end{pmatrix} \neq 0.$$

Lemma 2.1 (Coercivity of L_Q ; [15, 17]). *Let $m \geq 0$. Let $\mathcal{Z}_1, \mathcal{Z}_2 \in C_{c,m}^\infty$ satisfy (2.12). Then,*

$$(2.13) \quad \|L_Q f\|_{L^2} \sim \|f\|_{\dot{\mathcal{H}}_m^1}, \quad \forall f \in \dot{\mathcal{H}}_m^1 \text{ with } (f, \mathcal{Z}_1)_r = (f, \mathcal{Z}_2)_r = 0.$$

2.4. Duality estimates. In this subsection, we collect estimates for the nonlinearity of (CSS). Some of the following multilinear estimates already appeared in [15]. Here we slightly generalize them for our needs.

We first introduce several more pieces of notation, in order to estimate systematically the errors from the nonlinearity of (CSS). Denote by $\mathcal{N}(u)$ the nonlinearity of (CSS):

$$\mathcal{N}(u) := (-|u|^2 + \frac{2m}{r^2} A_\theta[u] + \frac{1}{r^2} A_\theta^2[u] + A_t[u])u.$$

The nonlinearity $\mathcal{N}(u)$ decomposes into the sum of the cubic and quintic nonlinearities:

$$\mathcal{N} = \mathcal{N}_{3,0} + m(\mathcal{N}_{3,1} + \mathcal{N}_{3,2}) + \mathcal{N}_{5,1} + \mathcal{N}_{5,2},$$

where we abbreviate $\mathcal{N}_*(u) := \mathcal{N}_*(u, \dots, u)$ (where $*$ is a place-holder) and denote the cubic nonlinearities by

$$\begin{aligned} \mathcal{N}_{3,0}(\psi_1, \psi_2, \psi_3) &:= -\operatorname{Re}(\overline{\psi_1} \psi_2) \psi_3, \\ \mathcal{N}_{3,1}(\psi_1, \psi_2, \psi_3) &:= \frac{2}{r^2} A_\theta[\psi_1, \psi_2] \psi_3, \\ \mathcal{N}_{3,2}(\psi_1, \psi_2, \psi_3) &:= -(\int_r^\infty \operatorname{Re}(\overline{\psi_1} \psi_2) \frac{dr'}{r'}) \psi_3, \end{aligned}$$

and the quintic nonlinearities by

$$\begin{aligned} \mathcal{N}_{5,1}(\psi_1, \dots, \psi_5) &:= \frac{1}{r^2} A_\theta[\psi_1, \psi_2] A_\theta[\psi_3, \psi_4] \psi_5, \\ \mathcal{N}_{5,2}(\psi_1, \dots, \psi_5) &:= -(\int_r^\infty A_\theta[\psi_1, \psi_2] \operatorname{Re}(\overline{\psi_3} \psi_4) \frac{dr'}{r'}) \psi_5. \end{aligned}$$

We remark that $\mathcal{N}_{3,1}$ and $\mathcal{N}_{3,2}$ do not appear in the case $m = 0$.

In view of the Hamiltonian structure of (CSS), the nonlinearity of (CSS) arises as a part of the functional derivative of the energy, i.e.,

$$\nabla E[u] = -\Delta^{(m)} u + \mathcal{N}(u).$$

In order to relate \mathcal{N}_* with each component of the energy, we decompose

$$E[u] = \frac{1}{2} \int (|\partial_r u|^2 + \frac{|m|^2}{r^2} |u|^2) + \mathcal{M}_{4,0}[u] + m \mathcal{M}_{4,1}[u] + \mathcal{M}_6[u],$$

where we abbreviate $\mathcal{M}_*(u) := \mathcal{M}_*(u, \dots, u)$ and denote the multilinear forms by

$$\begin{aligned}\mathcal{M}_{4,0}(\psi_1, \dots, \psi_4) &:= -\frac{1}{4}\text{Re}(\overline{\psi_1}\psi_2)\text{Re}(\overline{\psi_3}\psi_4), \\ \mathcal{M}_{4,1}(\psi_1, \dots, \psi_4) &:= \int \frac{1}{r^2} A_\theta[\psi_1, \psi_2]\text{Re}(\overline{\psi_3}\psi_4), \\ \mathcal{M}_6(\psi_1, \dots, \psi_6) &:= \frac{1}{2} \int \frac{1}{r^2} A_\theta[\psi_1, \psi_2] A_\theta[\psi_3, \psi_4]\text{Re}(\overline{\psi_5}\psi_6).\end{aligned}$$

It is then easy to verify that

$$(2.14) \quad \begin{cases} (\mathcal{N}_{3,0}(\psi_1, \psi_2, \psi_3), \psi_4)_r = 4\mathcal{M}_{4,0}(\psi_1, \psi_2, \psi_3, \psi_4), \\ (m\mathcal{N}_{3,1}(\psi_1, \psi_2, \psi_3), \psi_4)_r = 2m\mathcal{M}_{4,1}(\psi_1, \psi_2, \psi_3, \psi_4), \\ (m\mathcal{N}_{3,2}(\psi_1, \psi_2, \psi_3), \psi_4)_r = 2m\mathcal{M}_{4,1}(\psi_3, \psi_4, \psi_1, \psi_2), \\ (\mathcal{N}_{5,1}(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5), \psi_6)_r = 2\mathcal{M}_6(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6), \\ (\mathcal{N}_{5,2}(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5), \psi_6)_r = 4\mathcal{M}_6(\psi_1, \psi_2, \psi_5, \psi_6, \psi_3, \psi_4). \end{cases}$$

We remark that $\mathcal{M}_{4,1}$ does not appear in the case $m = 0$.

We turn to study the boundedness properties of \mathcal{M}_* and \mathcal{N}_* . Note that the above relations, in view of duality, tell us that estimates for the multilinear forms \mathcal{M}_* might transfer to those of \mathcal{N}_* . We start with the mapping properties of the integral operators:

Lemma 2.2 (Mapping properties for integral operators). *Let $1 \leq p, q \leq \infty$ and $s \in [0, 2]$ be such that $(p, q, s) = (1, \infty, 0)$ or $\frac{1}{q} + 1 = \frac{1}{p} + \frac{s}{2}$ with $p > 1$. Then, we have*

$$(2.15) \quad \left\| \frac{1}{r^s} \int_0^r f(r') r' dr' \right\|_{L^q} \lesssim_p \|f\|_{L^p}.$$

Proof. Note by the definition of $\|f\|_{L^1}$ that the estimate is immediate when $(p, q, s) = (1, \infty, 0)$. Henceforth, we assume $p > 1$. When $q = p$ and $s = 2$, then the proof follows from a change of variables and Minkowski's inequality:

$$\begin{aligned}\left\| \frac{1}{r^2} \int_0^r f(r') r' dr' \right\|_{L^p} &= \left\| \int_0^1 f(ru) u du \right\|_{L^p(rdr)} \\ &\leq \int_0^1 \|f(ru)\|_{L^p(rdr)} u du = \int_0^1 u^{1-\frac{2}{p}} \|f\|_{L^p} du = \frac{p}{2(p-1)} \|f\|_{L^p}.\end{aligned}$$

When $q = \infty$ and $s = 2 - \frac{2}{p}$, then by Hölder we have

$$\left\| \frac{1}{r^s} \int_0^r f(r') r' dr' \right\|_{L^\infty} \leq \|f\|_{L^p} \cdot \sup_{r \in (0, \infty)} \frac{1}{r^s} \|\mathbf{1}_{r' \leq r}\|_{L^{p'}(r' dr')} \lesssim_p \|f\|_{L^p},$$

where $\frac{1}{p'} = 1 - \frac{1}{p}$. For $q \in (p, \infty)$, the estimate follows from the interpolation:

$$\left\| \frac{1}{r^s} \int_0^r f(r') r' dr' \right\|_{L^q} \leq \left\| \frac{1}{r^2} \int_0^r f(r') r' dr' \right\|_{L^p}^\theta \left\| \frac{1}{r^{2-(2/p)}} \int_0^r f(r') r' dr' \right\|_{L^\infty}^{1-\theta} \lesssim_p \|f\|_{L^p},$$

where $\theta = \frac{p}{q} \in (0, 1)$. This completes the proof of (2.15). \square

We then record the Hölder- and weighted L^1 -type estimates for the multilinear forms \mathcal{M}_* .

Lemma 2.3 (Duality estimates (Hölder-type)). *The following estimates hold.*

- (For $\mathcal{M}_{4,*}$) Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$|\mathcal{M}_{4,0}(\psi_1, \psi_2, \psi_3, \psi_4)| \lesssim \|\psi_1 \psi_2\|_{L^p} \|\psi_3 \psi_4\|_{L^q},$$

$$|\mathcal{M}_{4,1}(\psi_1, \psi_2, \psi_3, \psi_4)| \lesssim_p \|\psi_1 \psi_2\|_{L^p} \|\psi_3 \psi_4\|_{L^q}, \quad \text{if } (p, q) \neq (1, \infty).$$
- (For \mathcal{M}_6) Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$ and $(p, q, r) \neq (1, 1, \infty)$. Then, we have

$$|\mathcal{M}_6(\psi_1, \dots, \psi_6)| \lesssim_{p,q} \|\psi_1 \psi_2\|_{L^p} \|\psi_3 \psi_4\|_{L^q} \|\psi_5 \psi_6\|_{L^r}.$$

Proof. For $\mathcal{M}_{4,0}$, this is just Hölder's inequality. For $\mathcal{M}_{4,1}$, we assume $p > 1$ and apply (2.15) with $s = 2$ to have

$$|\mathcal{M}_{4,1}(\psi_1, \psi_2, \psi_3, \psi_4)| \lesssim_p \|\frac{1}{r^2} A_\theta[\psi_1, \psi_2]\|_{L^p} \|\psi_3 \psi_4\|_{L^q} \lesssim \|\psi_1 \psi_2\|_{L^p} \|\psi_3 \psi_4\|_{L^q}.$$

For \mathcal{M}_6 , assume $(p, q, r) \neq (1, 1, \infty)$. By symmetry, we may assume $q > 1$. We then use (2.15) to have

$$\begin{aligned} |\mathcal{M}_6(\psi_1, \dots, \psi_6)| &\lesssim \|\frac{1}{r^{2-(2/p)}} A_\theta[\psi_1, \psi_2]\|_{L^\infty} \|\frac{1}{r^{2/p}} A_\theta[\psi_3, \psi_4]\|_{L^{r'}} \|\psi_5 \psi_6\|_{L^r} \\ &\lesssim_{p,q} \|\psi_1 \psi_2\|_{L^p} \|\psi_3 \psi_4\|_{L^q} \|\psi_5 \psi_6\|_{L^r}, \end{aligned}$$

where we denoted $\frac{1}{r'} := 1 - \frac{1}{r}$. This completes the proof. \square

Lemma 2.4 (Duality estimates (weighted L^1 -type)). *The following estimates hold.*

- (For $\mathcal{M}_{4,1}$) Let $w_{12}, w_{34} : (0, \infty) \rightarrow \mathbb{R}_+$ be decreasing functions such that $w_{12}(r)w_{34}(r) = \frac{1}{r^2}$. Then, we have

$$|\mathcal{M}_{4,1}(\psi_1, \psi_2, \psi_3, \psi_4)| \lesssim \|w_{12}\psi_1\psi_2\|_{L^1} \|w_{34}\psi_3\psi_4\|_{L^1}.$$

- (For \mathcal{M}_6) Let $w_{12}, w_{34}, w_{56} : (0, \infty) \rightarrow \mathbb{R}_+$ be decreasing functions such that $w_{12}(r)w_{34}(r)w_{56}(r) = \frac{1}{r^2}$. Then, we have

$$|\mathcal{M}_6(\psi_1, \dots, \psi_6)| \lesssim \|w_{12}\psi_1\psi_2\|_{L^1} \|w_{34}\psi_3\psi_4\|_{L^1} \|w_{56}\psi_5\psi_6\|_{L^1}.$$

Proof. Let us only prove the lemma for \mathcal{M}_6 . We start from writing

$$\mathcal{M}_6 = \frac{1}{2} \int w_{12} w_{34} w_{56} A_\theta[\psi_1, \psi_2] A_\theta[\psi_3, \psi_4] \text{Re}(\overline{\psi_5 \psi_6}).$$

Since w_{12} and w_{34} are decreasing, we have

$$|w_{12}(r) \int_0^r \text{Re}(\overline{\psi_1 \psi_2}) r' dr'| \leq \int_0^r |w_{12}\psi_1\psi_2| r' dr' \leq \|w_{12}\psi_1\psi_2\|_{L^1}$$

and a similar estimate for $\psi_3\psi_4$. Thus

$$\begin{aligned} |\mathcal{M}_6(\psi_1, \dots, \psi_6)| &\leq \int_0^\infty \|w_{12}\psi_1\psi_2\|_{L^1} \|w_{34}\psi_3\psi_4\|_{L^1} |w_{56}\psi_5\psi_6| r dr \\ &\leq \|w_{12}\psi_1\psi_2\|_{L^1} \|w_{34}\psi_3\psi_4\|_{L^1} \|w_{56}\psi_5\psi_6\|_{L^1}. \end{aligned}$$

This completes the proof. \square

The following two corollaries follow from the duality relations (2.14) and the above two lemmas.

Corollary 2.5 (Nonlinear estimates (Hölder-type)). *For $p \in [1, \infty]$, denote by p' the Hölder conjugate exponent. The following estimates hold.*

- (For $\mathcal{N}_{3,k}$) For any $1 \leq p_1, \dots, p_4 \leq \infty$ with $\sum_{j=1}^4 \frac{1}{p_j} = 1$ and $\#\{j : p_j = \infty\} \leq 1$, we have

$$\|\mathcal{N}_{3,k}(\psi_1, \psi_2, \psi_3)\|_{L^{p'_4}} \lesssim_{p_1, p_2, p_3} \|\psi_1\|_{L^{p_1}} \|\psi_2\|_{L^{p_2}} \|\psi_3\|_{L^{p_3}}.$$

- (For $\mathcal{N}_{5,k}$) For any $1 \leq p_1, \dots, p_6 \leq \infty$ with $\sum_{j=1}^6 \frac{1}{p_j} = 2$ and $\#\{j : p_j = \infty\} \leq 1$, we have

$$\|\mathcal{N}_{5,k}(\psi_1, \dots, \psi_5)\|_{L^{p'_6}} \lesssim_{p_1, \dots, p_5} \prod_{j=1}^5 \|\psi_j\|_{L^{p_j}}.$$

Corollary 2.6 (Nonlinear estimates (weighted L^2 -type)). *The following estimates hold.*

- (For $\mathcal{N}_{3,1}$ and $\mathcal{N}_{3,2}$) Let $w_1, \dots, w_3 : (0, \infty) \rightarrow \mathbb{R}_+$ be decreasing functions such that $\prod_{j=1}^3 w_j(r) = \frac{1}{r^2}$. Then, for any $k \in \{1, 2\}$, we have

$$\|\mathcal{N}_{3,k}(\psi_1, \psi_2, \psi_3)\|_{L^2} \lesssim \prod_{j=1}^3 \|w_j \psi_j\|_{L^2}.$$

- (For $\mathcal{N}_{5,1}$ and $\mathcal{N}_{5,2}$) Let $w_1, \dots, w_5 : (0, \infty) \rightarrow \mathbb{R}_+$ be decreasing functions such that $\prod_{j=1}^5 w_j(r) = \frac{1}{r^2}$. Then, for any $k \in \{1, 2\}$, we have

$$\|\mathcal{N}_{5,k}(\psi_1, \psi_2, \psi_3)\|_{L^2} \lesssim \prod_{j=1}^5 \|w_j \psi_j\|_{L^2}.$$

3. PROOF OF THEOREM 1.1 WHEN $m < 0$

In this short section, we prove Theorem 1.1 when $m < 0$. In this case, the only scenario for the long-term dynamics is the scattering. We first show that (CSS) is *defocusing* in the sense that the energy is globally coercive:

Lemma 3.1 (Nonlinear coercivity for $m < 0$). *Let $m < 0$. For any $u \in \dot{H}_m^1$, we have*

$$E[u] \sim_{M[u]} \|u\|_{\dot{H}_m^1}^2.$$

In particular, there is no nontrivial finite energy solution to the Bogomol'nyi equation (1.10) for $m < 0$.

Proof. As the inequality $E[u] \lesssim_{M[u]} \|u\|_{\dot{H}_m^1}^2$ is obvious, we focus on the proof of the reverse inequality $E[u] \gtrsim_{M[u]} \|u\|_{\dot{H}_m^1}^2$. By density, we may assume that u is an m -equivariant Schwartz function. In particular, $u(0) = 0$. We note that

$$\begin{aligned} \int_0^\infty |\mathbf{D}_u u|^2 r dr &= \int_0^\infty \left| \partial_r u - \frac{m + A_\theta[u]}{r} u \right|^2 r dr \\ &= \int_0^\infty \left\{ |\partial_r u|^2 + \frac{(m + A_\theta[u])^2}{r^2} |u|^2 - 2\operatorname{Re}(\overline{\partial_r u} \cdot \frac{A_\theta[u]}{r} u) \right\} r dr \\ &\quad - \int_0^\infty 2\operatorname{Re}(\overline{\partial_r u} \cdot m u) dr. \end{aligned}$$

The last term *vanishes*, thanks to integration by parts. Thus, we have proved

$$(3.1) \quad \int_0^\infty |\mathbf{D}_u u|^2 r dr = \int_0^\infty \left\{ |\partial_r u|^2 + \frac{(m + A_\theta[u])^2}{r^2} |u|^2 - 2\operatorname{Re}(\overline{\partial_r u} \cdot \frac{A_\theta[u]}{r} u) \right\} r dr.$$

The last term of RHS(3.1) will be absorbed into the sum of the first and second terms. Indeed, since $m < 0$ and $0 \leq -A_\theta[u](r) \leq \frac{1}{4\pi} M[u]$, there exists $c = c(M[u]) > 0$ such that

$$|A_\theta[u]| \leq (1 - c)|m + A_\theta[u]|.$$

Thus the last term of RHS(3.1) can be estimated by

$$\begin{aligned} |\int -2\operatorname{Re}(\overline{\partial_r u} \cdot \frac{A_\theta[u]}{r} u)| &\leq (1-c) \int |2\overline{\partial_r u} \cdot \frac{m+A_\theta[u]}{r} u| \\ &\leq (1-c) \int \{|\partial_r u|^2 + \frac{(m+A_\theta[u])^2}{r^2} |u|^2\} r dr. \end{aligned}$$

Substituting this into (3.1), we have

$$E[u] = \frac{1}{2} \int |\mathbf{D}_u u|^2 \geq c \int \{|\partial_r u|^2 + \frac{(m+A_\theta[u])^2}{r^2} |u|^2\} \geq c \|u\|_{\dot{H}_m^1}^2,$$

completing the proof. \square

As is standard, the coercivity of energy directly implies the scattering for all $H_m^{1,1}$ -solutions via the pseudoconformal transform.

Proof of Theorem 1.1 when $m < 0$. Let $m < 0$. Suppose that there is a non-scattering maximal $H_m^{1,1}$ -solution u to (CSS). By the time-reversal and time-translational symmetry, we may assume that u is defined on $[1, T_+)$ and is non-scattering forwards in time, where $T_+ \in (1, +\infty]$ is the forward maximal time of existence.

If $T_+ < +\infty$, then u is a finite-time blow-up solution with finite energy. The standard blow-up criterion (as a consequence of the H_m^1 -subcritical local well-posedness) says that $\|u(t)\|_{\dot{H}_m^1} \rightarrow \infty$ as $t \rightarrow T_+$. This is inconsistent with the nonlinear coercivity (Lemma 3.1) and the conservation of energy.

If $T_+ = +\infty$ but u does not scatter, then the standard equivariant L^2 -Cauchy theory [22] says that u has infinite $L_{t,x}^4$ -norm

$$\|u\|_{L_{t,x}^4([1,+\infty) \times \mathbb{R}^2)} = +\infty.$$

Let $v := \mathcal{C}u$ be the pseudoconformal transform of u (see (1.5)). Note that v is defined on the time interval $[-1, 0)$ (having well-defined extension past the time $t = -1$). Moreover, since the pseudoconformal transform preserves the space $H_m^{1,1}$ as well as the $L_{t,x}^4$ -norm of the solution, we see that v is a $H_m^{1,1}$ -solution with

$$\|v\|_{L_{t,x}^4([-1,0) \times \mathbb{R}^2)} = +\infty,$$

meaning that $t = 0$ is the forward maximal time of existence. In particular, v blows up at $t = 0$. This is impossible due to the previous paragraph. This completes the proof. \square

4. PROOF OF THEOREM 1.1 WHEN $m \geq 0$

In this section, we prove Theorem 1.1 when $m \geq 0$. As before, we first reduce the proof of Theorem 1.1 to the case of finite-time blow-up solutions.

Proof of Theorem 1.1 for global solutions assuming the finite-time blow-up case.

Assume that u is a $H_m^{1,1}$ -solution on the time interval $[1, +\infty)$. If u scatters forwards in time, then there is nothing to prove. Suppose that u does not scatter forwards in time. Similarly as in the proof for the $m < 0$ case (see the previous section), the pseudoconformal transformed solution $v := \mathcal{C}u$ becomes a $H_m^{1,1}$ finite-time blow-up solution that blows up at $t = 0$.

According to Theorem 1.1 for the finite-time blow-up case, v admits the decomposition

$$v(t) - Q_{\lambda(t), \gamma(t)} \rightarrow z^* \text{ in } L^2 \text{ as } t \rightarrow 0^-,$$

with $\lambda(t), \gamma(t), z^*$ satisfying the properties stated in Theorem 1.1. Since $\partial_r z^*, \frac{1}{r} z^*, r z^* \in L^2$, we can view z^* as a radial part of a $H_{-m-2}^{1,1}$ function. We rewrite the above decomposition as

$$v(t) - Q_{\lambda(t), \gamma(t)} - z_{\text{lin}}(t) \rightarrow 0 \text{ in } L^2 \text{ as } t \rightarrow 0^-,$$

where $z_{\text{lin}}(t) := e^{it\Delta^{(-m-2)}} z^*$.

Inverting the pseudoconformal transform, we have

$$(4.1) \quad u(t) - e^{i\frac{r^2}{4t}} Q_{\hat{\lambda}(t), \hat{\gamma}(t)} - [\mathcal{C}z_{\text{lin}}](t) \rightarrow 0 \text{ in } L^2 \text{ as } t \rightarrow +\infty,$$

where $\hat{\lambda}(t) := t\lambda(-1/t)$ and $\hat{\gamma}(t) := \gamma(-1/t)$. In particular, (1.13)-(1.14) for λ implies (1.16)-(1.17) for $\hat{\lambda}$. Combining the facts $Q \in L^2$ and $\hat{\lambda}(t) \lesssim 1$ with the DCT, we can replace the pseudoconformal factor $e^{i\frac{r^2}{4t}}$ in the display (4.1) by 1. Finally, since z_{lin} is a $H_{-m-2}^{1,1}$ -solution to the $(-m-2)$ -equivariant free Schrödinger equation, so is $\mathcal{C}z_{\text{lin}}$. In other words, $\mathcal{C}z_{\text{lin}} = e^{it\Delta^{(-m-2)}} u^*$ for some $u^* \in H_{-m-2}^{1,1}$ and further regularities $\partial_r u^*, \frac{1}{r} u^*, r u^* \in L^2$ follow. \square

The rest of this section is devoted to the proof of Theorem 1.1 for the finite-time blow-up case.

4.1. Decomposition of small energy solutions. Let u be a finite-time blow-up solution with finite energy E . By the standard Cauchy theory of (CSS), we have $\|u(t)\|_{\dot{H}_m^1} \rightarrow \infty$ as $t \rightarrow T$ with T the blow-up time of u , whereas $E[u(t)] = E$ by conservation of energy. Renormalizing $u(t)$, i.e., introducing $v(t, r) := \hat{\lambda}(t)u(t, \hat{\lambda}(t)r)$ with $\hat{\lambda}(t) = \|Q\|_{\dot{H}_m^1} / \|u(t)\|_{\dot{H}_m^1}$, we have $\|v(t)\|_{\dot{H}_m^1} = \|Q\|_{\dot{H}_m^1}$ and $E[v(t)] \rightarrow 0$.

Since we know that $E[w] = 0$ if and only if $w = 0$ or w is a modulated soliton, it is natural to expect that $v(t)$ is in some sense near Q (modulo phase rotation). This is done in Lemma 4.2 below in a qualitative way. In practice, we further need to quantify this proximity to Q . So we will fix the decomposition of u into

$$u = [Q + \epsilon]_{\lambda, \gamma}$$

by imposing suitable orthogonality conditions on ϵ (Lemma 4.3), and then quantify the smallness of ϵ in terms of the energy E (Lemma 4.4). This motivates the following proposition.

Proposition 4.1 (Decomposition). *Let $m \geq 0$; let $\mathcal{Z}_1, \mathcal{Z}_2 \in C_{c,m}^\infty$ be the profiles as in (2.12). For any $M > 1$, there exist $0 < \alpha^* \ll \eta \ll 1$ such that the following properties hold for all $u \in H_m^1$ with $\|u\|_{L^2} \leq M$ satisfying the small energy condition $\sqrt{E[u]} \leq \alpha^* \|u\|_{\dot{H}_m^1}$:*

- (1) (Decomposition) *there exists unique $(\lambda, \gamma) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ such that $\epsilon \in H_m^1$ defined by the relation*

$$u = [Q + \epsilon]_{\lambda, \gamma}$$

satisfies the orthogonality conditions

$$(4.2) \quad (\epsilon, \mathcal{Z}_1)_r = (\epsilon, \mathcal{Z}_2)_r = 0$$

and smallness

$$(4.3) \quad \|\epsilon\|_{\dot{\mathcal{H}}_m^1} < \eta.$$

(2) (Estimate for λ) We have

$$(4.4) \quad \left| \frac{\|u\|_{\dot{H}_m^1}}{\|Q\|_{\dot{H}_m^1}} \lambda - 1 \right| \lesssim \|\epsilon\|_{\dot{H}_m^1}.$$

(3) (Improved smallness of ϵ) We have

$$(4.5) \quad \|\epsilon\|_{\dot{\mathcal{H}}_m^1} \sim_M \lambda \sqrt{E[u]}.$$

The rest of this subsection is devoted to the proof of Proposition 4.1. The proof is separated into three lemmas.

Firstly, we show that the smallness of the ratio $\sqrt{E[u]}/\|u\|_{\dot{H}_m^1}$ implies that u is close to a modulated soliton in the $\dot{\mathcal{H}}_m^1$ -topology:

Lemma 4.2 (Orbital stability for small energy solutions). *For any $M > 1$ and $\delta > 0$, there exists $\alpha^* > 0$ such that the following holds. Let $u \in H_m^1$ be a nonzero profile satisfying $\|u\|_{L^2} \leq M$ and the small energy condition $\sqrt{E[u]} \leq \alpha^* \|u\|_{\dot{H}_m^1}$. Then, there exists $\hat{\gamma} \in \mathbb{R}/2\pi\mathbb{Z}$ such that*

$$\|e^{-i\hat{\gamma}} \hat{\lambda} u(\hat{\lambda} \cdot) - Q\|_{\dot{\mathcal{H}}_m^1} < \delta,$$

where $\hat{\lambda} := \|Q\|_{\dot{H}_m^1} / \|u\|_{\dot{H}_m^1}$.

Proof. In view of scaling symmetry, we may assume $\hat{\lambda} = 1$.

Suppose not. Then, there exist $\eta > 0$ and a sequence $\{w_n\}_{n \in \mathbb{N}}$ in H_m^1 such that

$$E[w_n] \rightarrow 0, \quad \|w_n\|_{L^2} \leq M, \quad \|w_n\|_{\dot{H}_m^1} = \|Q\|_{\dot{H}_m^1},$$

and

$$(4.6) \quad \|w_n - e^{i\gamma} Q\|_{\dot{\mathcal{H}}_m^1} \geq \eta \text{ for any } n \in \mathbb{N} \text{ and } \gamma \in \mathbb{R}/2\pi\mathbb{Z}.$$

Passing to a subsequence, and using compact embeddings, we may assume that

$$\begin{aligned} w_n &\rightharpoonup w_\infty \text{ in } H_m^1, \\ w_n &\rightarrow w_\infty \text{ in } L^p \text{ for any } p \in (2, \infty), \end{aligned}$$

for some $w_\infty \in H_m^1$.

We show that w_∞ cannot be zero. Indeed, using [20, Lemma 3.1]

$$\|\mathbf{D}_x w_n\|_{L^2}^2 := \|\partial_r w_n\|_{L^2}^2 + \left\| \frac{m + A_\theta[w_n]}{r} w_n \right\|_{L^2}^2 \sim_M \|w_n\|_{\dot{H}_m^1}^2,$$

the definition (1.2) of energy, and $E[w_n] \rightarrow 0$, we have

$$\begin{aligned} \|w_\infty\|_{L^4}^4 &= \lim_{n \rightarrow \infty} \|w_n\|_{L^4}^4 = \lim_{n \rightarrow \infty} (-4E[w_n] + 2\|\mathbf{D}_x w_n\|_{L^2}^2) \\ &= \lim_{n \rightarrow \infty} 2\|\mathbf{D}_x w_n\|_{L^2}^2 \gtrsim_M \liminf_{n \rightarrow \infty} \|w_n\|_{\dot{H}_m^1}^2 = \|Q\|_{\dot{H}_m^1}^2. \end{aligned}$$

Thus $w_\infty \neq 0$.

We now show that $w_\infty = Q_{\lambda,\gamma}$ for some $\lambda \in (0, \infty)$ and $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$. Indeed, on one hand, $E[w_n] \rightarrow 0$ implies that $\mathbf{D}_{w_n} w_n \rightarrow 0$ in L^2 . On the other hand, [20, Lemma 3.2] says that $\mathbf{D}_{w_n} w_n \rightharpoonup \mathbf{D}_{w_\infty} w_\infty$. Therefore, we have $\mathbf{D}_{w_\infty} w_\infty = 0$, which combined with $w_\infty \neq 0$ and the uniqueness of zero energy solutions implies that $w_\infty = Q_{\lambda,\gamma}$ for some $\lambda \in (0, \infty)$ and $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$.

Next, we show that $w_n \rightarrow Q_{\lambda,\gamma}$ in $\dot{\mathcal{H}}_m^1$. Let us write $w_n = [Q + \tilde{w}_n]_{\lambda,\gamma}$. Note that $\tilde{w}_n \rightharpoonup 0$ in H_m^1 and $\tilde{w}_n \rightarrow 0$ in L^p for any $p \in (2, \infty)$. Expanding the expression $E[Q + \tilde{w}_n] = \lambda^{-2} E[w_n] \rightarrow 0$ using (2.3) and applying the duality estimate (Lemma 2.3), we see that

$$\frac{1}{2} \|L_Q \tilde{w}_n\|_{L^2}^2 = E[Q + \tilde{w}_n] + O_M(\|\tilde{w}_n\|_{L^4}^3) \rightarrow 0.$$

Combining this with the subcoercivity estimate (see [16, Lemma A.5] for $m \geq 1$ and [17, Lemma A.3] for $m = 0$) and $\tilde{w}_n \rightarrow 0$ in L^p for some $p \in (2, \infty)$, we conclude that $\tilde{w}_n \rightarrow 0$ in $\dot{\mathcal{H}}_m^1$. This shows $w_n \rightarrow Q_{\lambda,\gamma}$ in $\dot{\mathcal{H}}_m^1$.

We are now ready to derive a contradiction. Note that $\lambda = 1$ because

$$\|Q\|_{\dot{\mathcal{H}}_m^1} = \lim_{n \rightarrow \infty} \|w_n\|_{\dot{\mathcal{H}}_m^1} = \|Q_{\lambda,\gamma}\|_{\dot{\mathcal{H}}_m^1} = \lambda^{-1} \|Q\|_{\dot{\mathcal{H}}_m^1}.$$

Thus $w_n \rightarrow e^{i\gamma} Q$ in $\dot{\mathcal{H}}_m^1$, contradicting (4.6). This completes the proof. \square

Having established that u is close to a modulated soliton, we fix the decomposition $u = [Q + \epsilon]_{\lambda,\gamma}$ by imposing the orthogonality conditions (4.2). In fact, we prove the following:

Lemma 4.3 (Decomposition near Q). *For $\delta > 0$, let us denote by \mathcal{T}_δ the set of $u \in \dot{\mathcal{H}}_m^1$ satisfying*

$$(4.7) \quad \inf_{\lambda' \in \mathbb{R}_+, \gamma' \in \mathbb{R}/2\pi\mathbb{Z}} \|u_{(\lambda')^{-1}, -\gamma'} - Q\|_{\dot{\mathcal{H}}_m^1} < \delta.$$

For any sufficiently small $\eta > 0$, there exists $\delta > 0$ such that the following hold for all $u \in \mathcal{T}_\delta$:

- (1) *There exists unique $(\lambda, \gamma) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ such that u admits the decomposition*

$$u = [Q + \epsilon]_{\lambda,\gamma}$$

satisfying the orthogonality conditions (4.2) and smallness $\|\epsilon\|_{\dot{\mathcal{H}}_m^1} < \eta$ (which is (4.3)).

- (2) *Moreover, the estimate (4.4) for λ holds.*

Proof. The proof will follow from a standard application of the implicit function theorem and L^2 -scaling/phase rotation symmetries.

Equip \mathbb{R}_+ with the metric $d_{\mathbb{R}_+}(\lambda_1, \lambda_2) := |\log(\frac{\lambda_1}{\lambda_2})|$; equip $\mathbb{R}/2\pi\mathbb{Z}$ with the metric inherited by the standard metric on \mathbb{R} . We then equip the parameter space $\mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ with the product metric, which we denote by dist . Next, for $u \in \dot{\mathcal{H}}_m^1$, $\lambda \in \mathbb{R}_+$, and $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$, we define $\epsilon = \epsilon(\lambda, \gamma, u)$ via the relation $u = [Q + \epsilon]_{\lambda,\gamma}$.

Step 1. Application of the implicit function theorem.

A standard application of the implicit function theorem yields the following: there exist $0 < \delta_1, \delta_2 \ll 1$ such that, if $\|u - Q\|_{\dot{\mathcal{H}}_m^1} < \delta_2$, then there

exists unique (λ, γ) in the class $\text{dist}((\lambda, \gamma), (1, 0)) < \delta_1$ such that $(\epsilon, \mathcal{Z}_1)_r = (\epsilon, \mathcal{Z}_2)_r = 0$. Moreover, the Lipschitz bound $\text{dist}((\lambda, \gamma), (1, 0)) \lesssim \|u - Q\|_{\dot{\mathcal{H}}_m^1}$ holds. Note that $\|\epsilon\|_{\dot{\mathcal{H}}_m^1} \lesssim \|u - Q\|_{\dot{\mathcal{H}}_m^1}$ also holds in view of the formula of ϵ .

On the other hand, towards the proof of the global uniqueness of (λ, γ) in $\mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$, let us prove the following: if $\eta \ll \delta_1$ and $[Q + \epsilon]_{\lambda, \gamma} = [Q + \epsilon']_{\lambda', \gamma'}$ for some $\|\epsilon\|_{\dot{\mathcal{H}}_m^1}, \|\epsilon'\|_{\dot{\mathcal{H}}_m^1} < \eta$, then $\text{dist}((\lambda, \gamma), (\lambda', \gamma')) < \delta_1$. Indeed, by the scaling/rotation symmetries and changing the roles of $\lambda, \gamma, \epsilon$ and $\lambda', \gamma', \epsilon'$ if necessary, we may assume $\lambda \geq 1$, $\lambda' = 1$, and $\gamma' = 0$. Then, the identity $Q - Q_{\lambda, \gamma} = \epsilon_{\lambda, \gamma} - \epsilon'$ gives $\|Q - Q_{\lambda, \gamma}\|_{\dot{\mathcal{H}}_m^1} \leq \|\epsilon\|_{\dot{\mathcal{H}}_m^1} + \|\epsilon_{\lambda, \gamma}\|_{\dot{\mathcal{H}}_m^1} \lesssim \eta \ll \delta_1$, which implies $\text{dist}((\lambda, \gamma), (1, 0)) < \delta_1$ as desired.

Step 2. Completion of the proof.

Let $\eta \ll \delta_1$ and $\delta \ll \min\{\eta, \delta_2\}$. By the L^2 -scaling and phase rotation invariances, we may assume

$$\|u - Q\|_{\dot{\mathcal{H}}_m^1} < \delta.$$

(1) By the first result of Step 1, there exists (λ, γ) satisfying the orthogonality conditions (4.2) and the Lipschitz bound $\text{dist}((\lambda, \gamma), (1, 0)) + \|\epsilon\|_{\dot{\mathcal{H}}_m^1} \lesssim \delta \ll \eta$. For the proof of uniqueness, if there exists $(\lambda', \gamma', \epsilon')$ satisfying $(\epsilon', \mathcal{Z}_1)_r = (\epsilon', \mathcal{Z}_2)_r = 0$ and $\|\epsilon'\|_{\dot{\mathcal{H}}_m^1} < \eta$, then the second result of Step 1 says that $\text{dist}((\lambda, \gamma), (\lambda', \gamma')) < \delta_1$. By the local uniqueness result of Step 1, one must have $\lambda = \lambda'$ and $\gamma = \gamma'$.

(2) It now remains to show the estimate (4.4) for λ . Let $\hat{\lambda} := \|Q\|_{\dot{H}_m^1} / \|u\|_{\dot{H}_m^1}$. From the identity

$$\|Q\|_{\dot{H}_m^1} = \hat{\lambda} \|u\|_{\dot{H}_m^1} = \frac{\hat{\lambda}}{\lambda} \|Q + \epsilon\|_{\dot{H}_m^1}$$

and smallness (4.3), we first have $\frac{\hat{\lambda}}{\lambda} \sim 1$. Together with this, the previous display yields the control (4.4). This completes the proof. \square

By Lemmas 4.2 and 4.3, we have proved all the statements of Proposition 4.1 except the improved smallness (4.5) of ϵ . We only know that $\|\epsilon\|_{\dot{\mathcal{H}}_m^1} = o_{\alpha^* \rightarrow 0}(1)$ so far. In our next lemma, we show that this qualitative smallness can be improved to the following quantitative smallness:

Lemma 4.4 (Nonlinear coercivity of energy). *For any $M > 0$, there exists $\eta > 0$ such that the nonlinear coercivity*

$$(4.8) \quad E[Q + \epsilon] \sim_M \|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2$$

holds for any $\epsilon \in H_m^1$ with $\|\epsilon\|_{L^2} \leq M$ satisfying the orthogonality conditions (4.2) and smallness $\|\epsilon\|_{\dot{\mathcal{H}}_m^1} \leq \eta$.

Remark 4.5. The proof of Lemma 4.4 not only relies on the linear coercivity of L_Q (Lemma 2.1), but also on a Hardy inequality (4.12) for the operator $\mathbf{D}_Q - \frac{A_\theta[\epsilon]}{r}$. We will call (4.12) the *nonlinear Hardy inequality*, because the multiplication by $\frac{A_\theta[\epsilon]}{r}$ cannot be treated perturbatively when $\|\epsilon\|_{L^2}$ is allowed to be large. In fact, the proof of (4.12) is reminiscent of the proof of the global coercivity of energy in the negative equivariance case.

Proof. In the proof, we will need an additional parameter R satisfying the parameter dependence $0 < \eta \ll R^{-1} \ll M^{-1}$. Thus we may freely replace the error terms such as $O_M(1) \cdot o_{R \rightarrow \infty}(1)$ and $O_{M,R}(1) \cdot o_{\eta \rightarrow 0}(1)$ by $o_{R \rightarrow \infty}(1)$ and $o_{\eta \rightarrow 0}(1)$, respectively. Moreover, we may bound $o_{\eta \rightarrow 0}(1)$ by $o_{R \rightarrow \infty}(1)$.

We start from writing the energy functional using the self-dual form (1.8):

$$2E[Q + \epsilon] = \|\mathbf{D}_{Q+\epsilon}(Q + \epsilon)\|_{L^2}^2 = \|L_Q \epsilon - \frac{2A_\theta[Q, \epsilon]}{r} \epsilon - \frac{A_\theta[\epsilon]}{r} \epsilon\|_{L^2}^2.$$

Note that the middle term is a perturbative error

$$\begin{aligned} \|\frac{2}{r} A_\theta[Q, \epsilon] \epsilon\|_{L^2} &\lesssim (\int_0^\infty Q(r) |\epsilon| dr) \cdot \|\frac{1}{\langle r \rangle} \epsilon\|_{L^2} \\ &\lesssim \|\epsilon\|_{\dot{\mathcal{H}}_m^1}^{\frac{3}{2}} \|\epsilon\|_{L^2}^{\frac{1}{2}} \lesssim O_M(1) \cdot o_{\eta \rightarrow 0}(1) \cdot \|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2 \lesssim o_{\eta \rightarrow 0}(\|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2), \end{aligned}$$

whereas the remaining terms are not perturbative: $\|L_Q \epsilon\| \sim \|\epsilon\|_{\dot{\mathcal{H}}_m^1}$ due to (2.13) and (using (2.15))

$$\begin{aligned} \|\frac{A_\theta[\epsilon]}{r} \epsilon\|_{L^2} &\lesssim \|\frac{1}{r} \int_0^r |\epsilon|^2 r' dr'\|_{L^\infty} \|\epsilon\|_{L^2} \\ &\lesssim \|\epsilon\|_{L^2}^2 \|\epsilon\|_{L^2} \lesssim \|\epsilon\|_{L^2}^2 \|\epsilon\|_{\dot{\mathcal{H}}_m^1} \lesssim_M \|\epsilon\|_{\dot{\mathcal{H}}_m^1}. \end{aligned}$$

The above estimates in particular yield the boundedness $E[Q + \epsilon] \lesssim_M \|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2$.

For the proof of the reverse inequality, we note

$$(4.9) \quad 2E[Q + \epsilon] = \|L_Q \epsilon - \frac{A_\theta[\epsilon]}{r} \epsilon\|_{L^2}^2 + o_{\eta \rightarrow 0}(\|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2).$$

as a consequence of the above estimates.

We then separately consider the coercivity of $\|L_Q \epsilon - \frac{A_\theta[\epsilon]}{r} \epsilon\|_{L^2}$ in the regions $r \lesssim R$ and $r \gtrsim R$, for $R > 1$ sufficiently large. In the region $r \lesssim R$, we notice that the nonlinear contribution is small:

$$\begin{aligned} \|\frac{A_\theta[\epsilon]}{r} \chi_R \epsilon\|_{L^2} &\lesssim \|\epsilon\|_{L^2}^2 \|\chi_R \epsilon\|_{L^2} \\ &\lesssim \|\epsilon\|_{L^2} \|\epsilon\|_{\dot{\mathcal{H}}_m^1} \cdot R \|\epsilon\|_{\dot{\mathcal{H}}_m^1} \lesssim MR \|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2 \lesssim o_{\eta \rightarrow 0}(\|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2). \end{aligned}$$

On the other hand, the nonlocal term of L_Q is small in the region $r \gtrsim R$, thanks to the spatial decay of Q :

$$\|\frac{Q}{r} \int_0^r \text{Re}[Q(1 - \chi_R) \epsilon] r' dr'\|_{L^2} = o_{R \rightarrow \infty}(\|\epsilon\|_{\dot{\mathcal{H}}_m^1}).$$

Thus we have shown that

$$\|L_Q \epsilon - \frac{A_\theta[\epsilon]}{r} \epsilon\|_{L^2} = \|L_Q(\chi_R \epsilon) + (\mathbf{D}_Q - \frac{A_\theta[\epsilon]}{r})(1 - \chi_R) \epsilon\|_{L^2} + o_{R \rightarrow \infty}(\|\epsilon\|_{\dot{\mathcal{H}}_m^1}).$$

We further observe the following almost orthogonality:

$$\begin{aligned} &\left| \left(L_Q(\chi_R \epsilon), (\mathbf{D}_Q - \frac{A_\theta[\epsilon]}{r})(1 - \chi_R) \epsilon \right)_r \right| \\ &\lesssim_M \|(\mathbf{1}_{(0, 2R]} |\epsilon|_{-1} + \frac{Q}{r} \int_0^r |Q \epsilon| r' dr') \cdot \mathbf{1}_{[R, \infty)} |\epsilon|_{-1}\|_{L^1} \\ &\lesssim_M \|\mathbf{1}_{[R, 2R]} |\epsilon|_{-1}\|_{L^2}^2 + o_{R \rightarrow \infty}(\|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2). \end{aligned}$$

As a result, we have arrived at

$$\begin{aligned} (4.10) \quad \|L_Q \epsilon - \frac{A_\theta[\epsilon]}{r} \epsilon\|_{L^2}^2 &= \|L_Q(\chi_R \epsilon)\|_{L^2}^2 + \|(\mathbf{D}_Q - \frac{A_\theta[\epsilon]}{r})(1 - \chi_R) \epsilon\|_{L^2}^2 \\ &\quad + o_{R \rightarrow \infty}(\|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2) + O_M(\|\mathbf{1}_{[R, 2R]} |\epsilon|_{-1}\|_{L^2}^2). \end{aligned}$$

Since R is sufficiently large, we may assume that \mathcal{Z}_1 and \mathcal{Z}_2 are supported in the region $r \leq R$. Thus $\chi_R \epsilon$ satisfies the same orthogonality conditions as ϵ and hence the first term of RHS(4.10) is coercive by Lemma 2.1:

$$(4.11) \quad \|L_Q(\chi_R \epsilon)\|_{L^2}^2 \sim \|\chi_R \epsilon\|_{\dot{\mathcal{H}}_m^1}^2.$$

To deal with the second term of RHS(4.10), we claim the following *nonlinear Hardy inequality*: under the parameter dependence $R^{-1} \ll M^{-1}$ and $\|\epsilon\|_{L^2} \leq M$, we have, for $f \in \dot{\mathcal{H}}_m^1$ with $f(R) = 0$,

$$(4.12) \quad \|\mathbf{1}_{[R,\infty)}(\mathbf{D}_Q - \frac{A_\theta[\epsilon]}{r})f\|_{L^2}^2 \sim_M \|\mathbf{1}_{[R,\infty)}|f|_{-1}\|_{L^2}^2.$$

Let us assume (4.12) and finish the proof. Substituting (4.11) and (4.12) into (4.10), we have

$$\|L_Q \epsilon - \frac{A_\theta[\epsilon]}{r} \epsilon\|_{L^2}^2 \geq c(M) \|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2 - o_{R \rightarrow \infty}(\|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2) - O_M(\|\mathbf{1}_{[R,2R]}|\epsilon|_{-1}\|_{L^2}^2)$$

for some constant $c(M) > 0$ depending on M . Performing an averaging argument in R for the last term (that is, one replaces R by R' , takes the integral $\frac{1}{\log R} \int_R^{R^2} \cdot \frac{dR'}{R'}$, uses Fubini, and then exploits the smallness $\frac{1}{\log R} = o_{R \rightarrow \infty}(1)$) and applying the parameter dependence $R^{-1} \ll M^{-1}$ yield

$$\|L_Q \epsilon - \frac{A_\theta[\epsilon]}{r} \epsilon\|_{L^2}^2 \gtrsim_M \|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2.$$

Substituting this into (4.9) gives the nonlinear coercivity

$$E[Q + \epsilon] \gtrsim_M \|\epsilon\|_{\dot{\mathcal{H}}_m^1}^2.$$

This completes the proof of (4.8), assuming the nonlinear Hardy inequality (4.12).

Proof of the nonlinear Hardy inequality (4.12).

As the \lesssim_M -inequality is obvious, we only show the \gtrsim_M -inequality. Let $f \in \dot{\mathcal{H}}_m^1$ be such that $f(R) = 0$. We write

$$\begin{aligned} & \int_R^\infty |(\mathbf{D}_Q - \frac{A_\theta[\epsilon]}{r})f|^2 r dr \\ &= \int_R^\infty \{|\partial_r f|^2 + \frac{(m+A_\theta[Q]+A_\theta[\epsilon])^2}{r^2}|f|^2 - 2\operatorname{Re}(\overline{\partial_r f} \cdot \frac{A_\theta[\epsilon]}{r} f)\} r dr \\ & \quad - \int_R^\infty 2\operatorname{Re}(\overline{\partial_r f} \cdot (m+A_\theta[Q])f) dr \end{aligned}$$

and integrate by parts the last term:

$$- \int_R^\infty 2\operatorname{Re}(\overline{\partial_r f} \cdot (m+A_\theta[Q])f) dr = -\frac{1}{2} \int_R^\infty Q^2 |f|^2 r dr,$$

where we used $f(R) = 0$. Thus we have arrived at the main identity

$$(4.13) \quad \begin{aligned} & \int_R^\infty |(\mathbf{D}_Q - \frac{A_\theta[\epsilon]}{r})\epsilon|^2 r dr \\ &= \int_R^\infty \{|\partial_r f|^2 + \frac{(m+A_\theta[Q]+A_\theta[\epsilon])^2}{r^2}|f|^2 - 2\operatorname{Re}(\overline{\partial_r f} \cdot \frac{A_\theta[\epsilon]}{r} f)\} r dr \\ & \quad - \frac{1}{2} \int_R^\infty Q^2 |f|^2 r dr. \end{aligned}$$

Next, we claim that the first term of RHS(4.13) enjoys a good lower bound:

$$(4.14) \quad \begin{aligned} & \int_R^\infty \{|\partial_r f|^2 + \frac{(m+A_\theta[Q]+A_\theta[\epsilon])^2}{r^2}|f|^2 - 2\operatorname{Re}(\overline{\partial_r f} \cdot \frac{A_\theta[\epsilon]}{r} f)\} r dr \\ & \gtrsim_M \|\mathbf{1}_{[R,\infty)}|f|_{-1}\|_{L^2}^2. \end{aligned}$$

Indeed, since $m + A_\theta[Q]$ and $A_\theta[\epsilon]$ are both negative (on $[R, \infty)$) and $|A_\theta[\epsilon]| \leq \frac{1}{4\pi}M^2$, we notice that there exists $c = c(M) > 0$ satisfying

$$\mathbf{1}_{[R, \infty)}|A_\theta[\epsilon]| \leq (1 - c)\mathbf{1}_{[R, \infty)}|m + A_\theta[Q] + A_\theta[\epsilon]|.$$

Combining this with

$$|2\operatorname{Re}(\overline{\partial_r f} \cdot \frac{A_\theta[\epsilon]}{r} f)| \leq (1 - c)|\partial_r f|^2 + \frac{1}{1-c}|\frac{A_\theta[\epsilon]}{r} f|^2,$$

we obtain the desired lower bound of the integrand

$$\begin{aligned} |\partial_r f|^2 + \frac{(m + A_\theta[Q] + A_\theta[\epsilon])^2}{r^2}|f|^2 - 2\operatorname{Re}(\overline{\partial_r f} \cdot \frac{A_\theta[\epsilon]}{r} f) \\ \geq c(|\partial_r f|^2 + \frac{(m + A_\theta[Q] + A_\theta[\epsilon])^2}{r^2}|f|^2) \gtrsim_M |f|_{-1}^2. \end{aligned}$$

Integrating this on the region $r \geq R$ yields (4.14).

Now the proof of (4.12) is immediate from the identity (4.13), the lower bound (4.14), the estimate

$$\frac{1}{2} \int_R^\infty Q^2 |f|^2 r dr \lesssim \frac{1}{R^2} \|\mathbf{1}_{[R, \infty)}|f|_{-1}\|_{L^2}^2,$$

and the parameter dependence $R^{-1} \ll M^{-1}$. \square

To finish the proof of Proposition 4.1, we recall that it only suffices to show (4.5). This follows from Lemma 4.4:

$$\|\epsilon\|_{\dot{H}_m^1} \sim_M \sqrt{E[Q + \epsilon]} = \lambda \sqrt{E[u]}.$$

This completes the proof of Proposition 4.1.

4.2. Upper bounds for $\lambda(t)$. Here and in the next subsection, we let u be a H_m^1 -solution to (CSS) which blows up forwards in time at $T \in (0, \infty)$. Let M and E be the mass and energy of u , respectively. We recall that, by the standard Cauchy theory of (CSS), $\|u(t)\|_{\dot{H}_m^1} \rightarrow \infty$ as $t \rightarrow T$ and thus $\sqrt{E}/\|u(t)\|_{\dot{H}_m^1} \rightarrow 0$. Therefore, for all t sufficiently close to T , we can decompose $u(t)$ according to Proposition 4.1:

$$u(t, r) = \frac{e^{i\gamma(t)}}{\lambda(t)} [Q + \epsilon(t, \cdot)] \left(\frac{r}{\lambda(t)} \right)$$

with the parameters $\lambda(t)$, $\gamma(t)$, and the remainder $\epsilon(t)$ satisfying the properties as in Proposition 4.1. This subsection is devoted to the proofs of the upper bounds (1.13) and (1.14).

Proof of (1.13). Recall the rescaled spacetime variables (s, y) (see (2.1)). We claim that, as a standard application of the modulation estimate, for all t sufficiently close to T we have

$$(4.15) \quad \left| \frac{\lambda_s}{\lambda} \right| + |\gamma_s| \lesssim \|\epsilon\|_{\dot{H}_m^1} \lesssim_M \lambda \sqrt{E}.$$

Assuming this claim, we have

$$|\lambda_t| = \lambda^{-2} |\lambda_s| \lesssim_M \sqrt{E},$$

from which the bound $\lambda(t) \lesssim_M \sqrt{E}(T - t)$ (which is (1.13)) follows.

Henceforth, we prove the claim (4.15). As the argument is standard, we will be brief. For a (possibly time-dependent) profile ψ , we note the identity

$$(4.16) \quad \begin{aligned} \partial_s(\epsilon, \psi)_r &= \frac{\lambda_s}{\lambda}(\Lambda(Q + \epsilon), \psi)_r - \gamma_s(i(Q + \epsilon), \psi)_r \\ &\quad + (i\mathcal{L}_Q\epsilon + iR_Q(\epsilon), \psi)_r + (\epsilon, \partial_s\psi)_r. \end{aligned}$$

If $\psi \in \{\mathcal{Z}_1, \mathcal{Z}_2\}$, then we use the orthogonality conditions (4.2), anti-symmetry of Λ and i , the self-dual factorization $\mathcal{L}_Q = L_Q^* L_Q$, and $\partial_s\psi = 0$ to obtain

$$\begin{aligned} \frac{\lambda_s}{\lambda}\{(\Lambda Q, \mathcal{Z}_k)_r - (\epsilon, \Lambda \mathcal{Z}_k)_r\} + \gamma_s\{(iQ, \mathcal{Z}_k)_r - (\epsilon, i\mathcal{Z}_k)_r\} \\ = (L_Q\epsilon, L_Q i\mathcal{Z}_k)_r + (R_Q(\epsilon), i\mathcal{Z}_k)_r. \end{aligned}$$

By the transversality assumption (2.12), $\mathcal{Z}_k \in C_{c,m}^\infty$, and $\|\epsilon(t)\|_{\mathcal{H}_m^1} \rightarrow 0$ as $t \rightarrow T$, we get

$$(4.17) \quad \left| \frac{\lambda_s}{\lambda} \right| + |\gamma_s| \lesssim \|L_Q\epsilon\|_{L^2} + \|R_Q(\epsilon)\|_{L^2}.$$

Note that $\|L_Q\epsilon\|_{L^2} \lesssim \|\epsilon\|_{\mathcal{H}_m^1}$ due to (2.13). In the next paragraph, we show that

$$(4.18) \quad \|R_Q(\epsilon)\|_{L^2} \lesssim_M \|\epsilon\|_{\mathcal{H}_m^1}^2.$$

Substituting this into (4.17) completes the proof of the claim (4.15).

Proof of (4.18). The nonlinear term $R_Q(\epsilon)$ is a linear combination of $\mathcal{N}_*(\psi_1, \dots, \psi_*)$ where $\#\{j : \psi_j = \epsilon\} \geq 2$. If $\#\{j : \psi_j = \epsilon\} \geq 3$, then by Corollary 2.5 we have

$$\|\mathcal{N}_*(\psi_1, \dots, \psi_*)\|_{L^2} \lesssim (1 + \|Q\|_{L^2}^2 + \|\epsilon\|_{L^2}^2)\|\epsilon\|_{L^6}^3 \lesssim_M \|\epsilon\|_{L^6}^3 \lesssim_M \|\epsilon\|_{L^2} \|\epsilon\|_{\mathcal{H}_m^1}^2.$$

If $\#\{j : \psi_j = \epsilon\} = 2$, we separately consider the local and nonlocal nonlinearities; for $\mathcal{N}_{3,0}$ we use (2.11) to have

$$\|\mathcal{N}_{3,0}(\psi_1, \psi_2, \psi_3)\|_{L^2} \lesssim \|Q\epsilon^2\|_{L^2} \lesssim \|\langle \log_- y \rangle Q\|_{L^2} \|\langle \log_- y \rangle^{-\frac{1}{2}} \epsilon\|_{L^\infty}^2 \lesssim \|\epsilon\|_{\mathcal{H}_m^1}^2$$

and for the nonlocal nonlinearities we use Corollary 2.6 to have

$$\|\mathcal{N}_*(\psi_1, \dots, \psi_*)\|_{L^2} \lesssim_M \|\langle \log_- y \rangle^2 Q\|_{L^2} \|y^{-1} \langle \log_- y \rangle^{-1} \epsilon\|_{L^2}^2 \lesssim_M \|\epsilon\|_{\mathcal{H}_m^1}^2.$$

This completes the proof of (4.18). \square

When $m = 0$, we can further improve the bound (1.13) to (1.14). The idea is to project the ϵ -equation onto the direction of $y^2 Q$, which is a generalized kernel element that effectively detects the evolution of λ . The logarithmic improvement for the upper bound of λ will follow from the fact that yQ logarithmically fails to lie in L^2 , which holds only in the $m = 0$ case.

Proof of (1.14) for $m = 0$. Let $\psi = y^2 Q \chi_{R(t)}$ with $R(t) := (T - t)^{-\delta}$ with small $\delta > 0$. We note that it suffices to choose any $\delta \in (0, \frac{1}{2})$ in the following analysis. We note the bounds

$$(4.19) \quad \begin{aligned} \left| \frac{R_s}{R} \right| &= \lambda^2 \left| \frac{R_t}{R} \right| \lesssim \frac{\lambda^2}{T-t} \lesssim_M \lambda \sqrt{E}, \\ |\partial_s \psi| &\lesssim \left| \frac{R_s}{R} \right| \mathbf{1}_{y \sim R} \lesssim_M \lambda \sqrt{E} \mathbf{1}_{y \sim R}. \end{aligned}$$

We start by rewriting (4.16) as follows:

$$\begin{aligned} \frac{\lambda_s}{\lambda}(\Lambda Q, \psi)_r - \partial_s(\epsilon, \psi)_r &= \frac{\lambda_s}{\lambda}(\epsilon, \Lambda \psi)_r - \gamma_s(\epsilon, i\psi)_r - (\epsilon, \partial_s \psi)_r \\ &\quad - (i\mathcal{L}_Q \epsilon, \psi)_r - (iR_Q(\epsilon), \psi)_r. \end{aligned}$$

We further rearrange the LHS of the above display as

$$\frac{\lambda_s}{\lambda}(\Lambda Q, \psi)_r - \partial_s(\epsilon, \psi)_r = \frac{1}{\lambda} \partial_s \{ \lambda(\Lambda Q, \psi)_r - \lambda(\epsilon, \psi)_r \} - (\Lambda Q, \partial_s \psi)_r + \frac{\lambda_s}{\lambda}(\epsilon, \psi)_r.$$

As a result, we have obtained

$$\begin{aligned} (4.20) \quad \frac{1}{\lambda} \partial_s \{ \lambda(\Lambda Q, \psi)_r - \lambda(\epsilon, \psi)_r \} &= \frac{\lambda_s}{\lambda}(\epsilon, y \partial_y \psi)_r - \gamma_s(\epsilon, i\psi)_r - (\epsilon, \partial_s \psi)_r \\ &\quad + (\Lambda Q, \partial_s \psi)_r - (i\mathcal{L}_Q \epsilon, \psi)_r - (iR_Q(\epsilon), \psi)_r. \end{aligned}$$

In view of

$$\begin{aligned} (\Lambda Q, \psi)_r &= 16\pi \log R + O(1), \\ |(\epsilon, \psi)_r| &\lesssim \|\epsilon\|_{\dot{H}_0^1} R^2 \lesssim_M \lambda \sqrt{E} (T-t)^{-2\delta} \lesssim_{M,E} (T-t)^{1-2\delta}, \end{aligned}$$

we have

$$(4.21) \quad \text{LHS}(4.20) = \frac{1}{\lambda} \partial_s \left\{ \lambda(16\pi \log R + O(1)) \right\}$$

We turn to estimate each term of RHS(4.20). By (4.15) and (4.19), we have

$$\left| \frac{\lambda_s}{\lambda}(\epsilon, y \partial_y \psi)_r \right| + |\gamma_s(\epsilon, i\psi)_r| + |(\epsilon, \partial_s \psi)_r| \lesssim_{M,E} \lambda \cdot \|\epsilon\|_{\dot{H}_0^1} R^2 \lesssim_{M,E} \lambda (T-t)^{1-2\delta}.$$

Next, by (4.19), we have

$$|(\Lambda Q, \partial_s \psi)_r| \lesssim_M \lambda \sqrt{E}.$$

Next, the linear term can be bounded as

$$\begin{aligned} |(i\mathcal{L}_Q \epsilon, \psi)_r| &= |(L_Q \epsilon, L_Q i\psi)_r| \lesssim \|L_Q \epsilon\|_{L^2} \|L_Q i\psi\|_{L^2} \\ &\lesssim_M \lambda \sqrt{E} \cdot (\log R)^{\frac{1}{2}} \lesssim_M \lambda \sqrt{E} |\log(T-t)|^{\frac{1}{2}}, \end{aligned}$$

Finally, the nonlinear term can be bounded using (4.18):

$$|(iR_Q(\epsilon), \psi)_r| \lesssim R \|R_Q(\epsilon)\|_{L^2} \lesssim_M R \|\epsilon\|_{\dot{H}_m^1}^2 \lesssim_{M,E} \lambda (T-t)^{1-\delta}.$$

Therefore, we have obtained the following bound as $t \rightarrow T$:

$$(4.22) \quad |\text{RHS}(4.20)| \lesssim_M \lambda \sqrt{E} |\log(T-t)|^{\frac{1}{2}}.$$

Combining (4.21) and (4.22), we have as $t \rightarrow T$:

$$\left| \frac{1}{\lambda} \partial_s \left\{ \lambda(16\pi \log R + O(1)) \right\} \right| \lesssim_M \lambda \sqrt{E} |\log(T-t)|^{\frac{1}{2}}.$$

In terms of the t -variable, this reads

$$\left| \partial_t \left\{ \lambda(16\pi \log R + O(1)) \right\} \right| \lesssim_M \sqrt{E} |\log(T-t)|^{\frac{1}{2}}.$$

Integrating the above backwards from the blow-up time with $\log R \sim |\log(T-t)|$ and $\lambda \lesssim_{M,E} T-t$ yields

$$\lambda \cdot |\log(T-t)| \lesssim_M \sqrt{E} (T-t) |\log(T-t)|^{\frac{1}{2}},$$

which completes the proof of (1.14). \square

4.3. Existence and regularity of asymptotic profile. In this subsection, we finish the proof of Theorem 1.1 by showing that (i) $u(t)$ decomposes as (1.12) for some $z^* \in L^2$, (ii) z^* enjoys further regularity $|z^*|_{-1} \in L^2$, and (iii) also $rz^* \in L^2$ if u is a $H_m^{1,1}$ -solution. We closely follow the argument of Merle–Raphaël [24].

We first claim the outer L^2 -convergence:

Lemma 4.6 (Outer L^2 -convergence). *There exists $z^* \in L^2$ such that for any $R > 0$, we have $\mathbf{1}_{r \geq R} \epsilon^\sharp(t) \rightarrow \mathbf{1}_{r \geq R} z^*$ in L^2 as $t \rightarrow T$.*

Proof. In the proof, let φ_R be a smooth radial cutoff function satisfying $\varphi_R(r) = 1$ for $r \geq R$, $\varphi_R(r) = 0$ for $r \leq \frac{R}{2}$, and $|\varphi_R|_2 \lesssim 1$.

We claim that:

$$(4.23) \quad \text{For any } R > 0, \{\varphi_R \epsilon^\sharp(t)\} \text{ is Cauchy in } L^2 \text{ as } t \rightarrow T.$$

Let us finish the proof assuming this claim. For each $R > 0$, the above claim says that $\{\mathbf{1}_{r \geq R} \epsilon^\sharp(t)\}$ is Cauchy in L^2 as $t \rightarrow T$. Thus there exists $z_R^* \in L^2(r \geq R)$ such that $\mathbf{1}_{r \geq R} \epsilon^\sharp(t) \rightarrow z_R^*$. In view of the uniqueness of the limit, we have $z_{R_1}^* = \mathbf{1}_{r \geq R_1} z_{R_2}^*$ whenever $R_1 \geq R_2 > 0$. Therefore, there exists unique profile $z^*(r)$ such that, for any $R > 0$, $\mathbf{1}_{r \geq R} z^* \in L^2$ and $\mathbf{1}_{r \geq R} \epsilon^\sharp(t) \rightarrow \mathbf{1}_{r \geq R} z^*$ in L^2 as $t \rightarrow T$. The fact that $z^* \in L^2$ follows from the uniform boundedness of $\|\epsilon^\sharp(t)\|_{L^2}$ with Fatou's lemma.

We turn to show the claim (4.23). In this paragraph, we will reduce the proof of (4.23) to the proof of (4.24). Fix any $\delta_1 > 0$ and $R > 0$. It suffices to show that: there exists $\delta_2 > 0$ such that $\|\varphi_R \{\epsilon^\sharp(t) - \epsilon^\sharp(s)\}\|_{L^2} < \delta_1$ for all $t, s \in (T - \delta_2, T)$. This will follow from showing that: there exist $t_0 < T$ and $\delta_2 \in (0, T - t_0)$ such that $\|\varphi_R \{\epsilon^\sharp(t + \tau) - \epsilon^\sharp(t)\}\|_{L^2} < \delta_1$ for all $\tau \in (0, \delta_2)$ and $t \in [t_0, T - \tau)$. Now, thanks to $\varphi_R Q_{\lambda(t), \gamma(t)} \rightarrow 0$ in L^2 as $t \rightarrow T$ (due to $\lambda(t) \rightarrow 0$), it suffices to show that:

$$(4.24) \quad \text{There exist } t_0 < T \text{ and } \delta_2 \in (0, T - t_0) \text{ such that} \\ \sup_{\tau \in (0, \delta_2)} \sup_{t \in [t_0, T - \tau)} \|\varphi_R \{u(t + \tau) - u(t)\}\|_{L^2} < \delta_1.$$

Henceforth, we show (4.24). Denote $\tilde{u}^\tau(t) := \varphi_R \{u(t + \tau) - u(t)\}$. Then,

$$(i\partial_t + \Delta_m) \tilde{u}^\tau(t) = [\Delta_m, \varphi_R] u(t + \tau) + \varphi_R \mathcal{N}(u(t + \tau)) \\ - [\Delta_m, \varphi_R] u(t) - \varphi_R \mathcal{N}(u(t)),$$

so a standard L^2 -energy estimate yields

$$\|\tilde{u}^\tau(t)\|_{L^2} \leq \|\tilde{u}^\tau(t_0)\|_{L^2} + 2(T - t_0) \cdot \sup_{s \in [t_0, T)} \|[\Delta_m, \varphi_R] u(s) + \varphi_R \mathcal{N}(u(s))\|_{L^2}.$$

In the next paragraph, we will show that for any t_0 sufficiently close to T

$$(4.25) \quad \sup_{s \in [t_0, T)} \|[\Delta_m, \varphi_R] u(s) + \varphi_R \mathcal{N}(u(s))\|_{L^2} \lesssim_{R, M, E} 1,$$

where $M = M[u]$ and $E = E[u]$. Assuming this, we are led to

$$\sup_{\tau \in (0, \delta_2)} \sup_{t \in [t_0, T - \tau)} \|\tilde{u}^\tau(t)\|_{L^2} \leq \sup_{\tau \in (0, \delta_2)} \|\tilde{u}^\tau(t_0)\|_{L^2} + C(R, M, E) \cdot (T - t_0).$$

Choosing t_0 sufficiently close to T and then choosing $\delta_2 > 0$ small (using continuity of the flow $t \mapsto u(t) \in L^2$ at time $t = t_0$), the claim (4.24) follows.

It remains to show (4.25). We fix a time $s \in [t_0, T)$ and will obtain estimates uniformly in s . For t_0 sufficiently close to T , we have $\lambda^{-1}(s)R > 1$ and the following estimate:

$$\begin{aligned} & \|\mathbf{1}_{r \gtrsim R} \partial_r u\|_{L^2} + \|\mathbf{1}_{r \gtrsim R} u\|_{L^\infty} \\ &= \frac{1}{\lambda} (\|\mathbf{1}_{y \gtrsim \lambda^{-1}R} \partial_y (Q + \epsilon)\|_{L^2} + \|\mathbf{1}_{y \gtrsim \lambda^{-1}R} (Q + \epsilon)\|_{L^\infty}) \\ &\lesssim \frac{1}{\lambda} ((\lambda R^{-1})^{m+2} + \|\epsilon\|_{\dot{H}_m^1}) \lesssim R^{-1} + \lambda^{-1} \|\epsilon\|_{\dot{H}_m^1} \lesssim_{R,M,E} 1. \end{aligned}$$

Using this, the commutator term in (4.25) can be easily treated as

$$\|[\Delta_m, \varphi_R]u\|_{L^2} \lesssim R^{-1} \|\mathbf{1}_{r \sim R} \partial_r u\|_{L^2} + R^{-2} \|\mathbf{1}_{r \sim R} u\|_{L^2} \lesssim_{R,M,E} 1.$$

To estimate the nonlinearity $\mathcal{N}(u)$, we note that

$$\mathcal{N}(u) = -|u|^2 u + \mathcal{N}_*(u),$$

where $\mathcal{N}_*(u)$ consists of the nonlocal nonlinearities $\mathcal{N}_{3,1}, \mathcal{N}_{3,2}, \mathcal{N}_{5,1}, \mathcal{N}_{5,2}$. For the local nonlinearity, we have

$$\||u|^2 \varphi_R u\|_{L^2} \lesssim \|\mathbf{1}_{y \gtrsim R} u\|_{L^\infty}^2 \|u\|_{L^2} \lesssim_R 1.$$

For the nonlocal nonlinearity, by the nonlinear estimate (Corollary 2.6), we have

$$\|\varphi_R \mathcal{N}_*(u)\|_{L^2} \lesssim (1 + \|u\|_{L^2}^2) \|u\|_{L^2}^2 \|\frac{1}{r^2} \varphi_R u\|_{L^2} \lesssim_{R,M} 1.$$

This ends the proof of (4.25). \square

Next, we claim the weak H_m^1 -convergence:

Lemma 4.7 (Weak H_m^1 -convergence). *We have $|z^*|_{-1} \in L^2$ and $\epsilon^\sharp(t) \rightharpoonup z^*$ in H_m^1 . In particular, $\epsilon^\sharp(t) \rightarrow z^*$ in L_{loc}^2 .*

Proof. First, we show that $\frac{1}{r} z^* \in L^2$. For any $R > 0$, we see from $\lambda(t) \rightarrow 0$ that

$$\begin{aligned} \|\mathbf{1}_{r \geq R} \frac{1}{r} z^*\|_{L^2} &= \lim_{t \rightarrow T} \|\mathbf{1}_{r \geq R} \frac{1}{r} \epsilon^\sharp(t)\|_{L^2} \\ &= \lim_{t \rightarrow T} \frac{1}{\lambda} \|\mathbf{1}_{y \geq \lambda^{-1}R} \frac{1}{y} \epsilon(t, y)\|_{L^2} \leq \limsup_{t \rightarrow T} \frac{1}{\lambda} \|\epsilon(t)\|_{\dot{H}_m^1} \lesssim 1, \end{aligned}$$

where the implicit constant is uniform in R . Letting $R \rightarrow 0$, we have $\frac{1}{r} z^* \in L^2$.

Next, we show that $z^* \in H_m^1$ and $\epsilon^\sharp(t) \rightharpoonup z^*$ in H_m^1 . By a further subsequence argument, it suffices to show that (i) $z^* \in H_m^1$ and (ii) for any sequence $t_n \rightarrow T$ there exists a further subsequence $t_{n'} \rightarrow T$ such that $\epsilon^\sharp(t_{n'}) \rightharpoonup z^*$ in H_m^1 . Let $t_n \rightarrow T$ be arbitrary. Since $\{\epsilon^\sharp(t_n)\}$ is H_m^1 -bounded, it has a further subsequence $\{\epsilon^\sharp(t_{n'})\}$ such that $\epsilon^\sharp(t_{n'}) \rightharpoonup z_w^*$ for some $z_w^* \in H_m^1$. It now suffices to show that $z_w^* = z^*$. For this, it suffices show that $\mathbf{1}_{[R^{-1}, R]} z_w^* = \mathbf{1}_{[R^{-1}, R]} z^*$ for any $R > 1$. Fix $R > 1$. On one hand, by the Rellich–Kondrachov theorem, we have $\mathbf{1}_{[R^{-1}, R]} \epsilon^\sharp(t_{n'}) \rightarrow \mathbf{1}_{[R^{-1}, R]} z_w^*$ in L^2 . On the other hand, by the outer L^2 -convergence (Lemma 4.6), we have $\mathbf{1}_{[R^{-1}, R]} \epsilon^\sharp(t_{n'}) \rightarrow \mathbf{1}_{[R^{-1}, R]} z^*$ in L^2 . Thus $\mathbf{1}_{[R^{-1}, R]} z_w^* = \mathbf{1}_{[R^{-1}, R]} z^*$. This completes the proof of the weak H_m^1 -convergence. \square

Lemmas 4.6 and 4.7 show that $z^* \in L^2$, $|z^*|_{-1} \in L^2$, and $\epsilon^\sharp(t) \rightarrow z^*$ in L^2 as $t \rightarrow T$. This proves the decomposition (1.12) and the regularity of z^* in Theorem 1.1 for H_m^1 -solutions. If in addition u is a $H_m^{1,1}$ -solution, then we can appeal to the virial identities (1.6)-(1.7) and observe that $\|ru(t)\|_{L^2}$ is bounded as $t \rightarrow T$. Thus by the outer convergence (Lemma 4.6) and Fatou property we have

$$\begin{aligned} \|rz^*\|_{L^2} &= \lim_{R \rightarrow 0} \|\mathbf{1}_{r \geq R} rz^*\|_{L^2} \leq \lim_{R \rightarrow 0} \liminf_{t \rightarrow T} \|\mathbf{1}_{r \geq R} ru(t)\|_{L^2} \\ &\leq \limsup_{t \rightarrow T} \|ru(t)\|_{L^2} < +\infty. \end{aligned}$$

Hence $rz^* \in L^2$. This ends the proof of Theorem 1.1.

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