RIGIDITY OF LIPSCHITZ MAP USING HARMONIC MAP HEAT FLOW

MAN-CHUN LEE[1] AND LUEN-FAI TAM[2]

Abstract. Motivated by the Lipschitz rigidity problem in scalar curvature geometry, we prove that if a closed smooth spin manifold admits a distance non-increasing continuous map of non-zero degree to a sphere, then either the scalar curvature is strictly less than the sphere somewhere or the map is a distance isometry. Moreover, the property also holds for continuous metrics with scalar curvature lower bound in some weak sense. This extends a result in the recent work of Cecchini-Hanke-Schick [6] and answers a question of Gromov [17]. The method is based on studying the harmonic map heat flow coupled with the Ricci flow from rough initial data to reduce the case to smooth metrics and smooth maps so that results by Llarull [23] can be applied. As a corollary, we obtain some comparison results on metrics on domains in the standard sphere in terms of scalar curvature and the mean curvature of the boundary.

1. introduction

In recent years, there have been many exciting developments in understanding the scalar curvature of a Riemannian manifold. For a comprehensive overview, we refer readers to Gromov's lecture notes [17] on scalar curvature. In [23], Llarull proved the following striking result which confirms one of Gromov's conjectures.

Theorem 1.1. *[Llarull] Let M^n be a compact spin manifold with a smooth metric g and $n \geq 2$. If $f : (M, g) \to (\mathbb{S}^n, g_{sphere})$ is a smooth distance non-increasing map of non-zero degree into the unit sphere in $\mathbb{R}^{n+1}$ with standard metric and if the scalar curvature $\mathcal{R}(g)$ of g is greater than or equal to $n(n-1)$, then f must be an isometry.*

In fact, Llarull proved a more general result when $n \geq 3$. First recall that f is said to be $(1, \wedge^k)$-contracting map from M to N if for any $x \in M$ the norm of the map $\bigwedge^k f : \bigwedge^k(T_x(M)) \to \bigwedge^k(T_{f(x)}(N)$ is not greater then 1. Hence a distance non-increasing map is a $(1, \wedge^1)$-contracting map, which implies that

Date: June 17, 2024.

2020 *Mathematics Subject Classification.* Primary 51F30, 53C24.

[1]Research partially supported by Hong Kong RGC grant (Early Career Scheme) of Hong Kong No. 24304222 and a direct grant of CUHK.

[2]Research partially supported by Hong Kong RGC General Research Fund #CUHK 14300420.

it is a $(1, \wedge^k)$-contracting map for all $k \geq 1$. Llarull proved that for $n \geq 3$, a $(1, \wedge^2)$-contracting map from a spin manifold M^n with $\mathcal{R} \geq n(n-1)$ to $\mathbb{S}^n$ with non-zero degree must be an isometry. If we only assume the map is $(1, \wedge^k)$-contracting for $k \geq 3$, then the result is not true, as shown by an example in [23].

Llarull's theorem was later generalized by Goette-Semmelmann [14] to maps into manifolds with non-zero Euler characteristic and non-negative curvature operator, replacing the sphere. Their method is based on the study of twisted Dirac operators. Gromov then posed the question of whether the rigidity still holds if f is only a distance-decreasing continuous map, see [17, Section 4.5, question (b)]. Since distance can be defined even if we only assume that the metric g is C^0 [6], one may ask whether Theorem 1.1 remains true when g is C^0 and f is Lipschitz and distance non-increasing. In this case, we need to define a notion on the lower bound of scalar curvature for C^0 metrics.

In this direction, Cecchini-Hanke-Schick [6] prove that Llarull's theorem is still true in the sense that f is a metric isometry, under the following assumptions: (i) g is $W^{1,p}$ for some $p > n$; (ii) the scalar curvature is bounded below by $n(n-1)$ in the distribution sense as introduced by Lee-LeFloch [21]; (iii) f is a Lipschitz map that is distance non-increasing or more generally is $(1, \Lambda^2)$-contracting; (iv) n is even.

We note that assumption (i) implies that g is C^0 by Sobolev embedding theorem. Furthermore, the Lipschitz continuity of f ensures that df can be defined almost everywhere, allowing us to discuss the $(1, \Lambda^2)$-contracting property. As in [23], Cecchini-Hanke-Schick employ Dirac operator under these weaker regularity conditions. They conjecture that the result should hold even if the dimension of M is odd.

Motivated by Gromov's question and the results of Cecchini-Hanke-Schick, in this work we address the problem for general dimension n and metric g with C^0 regularity. With C^0 metric structure, the Lipschitz constant of a continuous map can be defined:

Definition 1.1. *Let M and N be two smooth manifolds. Suppose g and h are continuous metrics on M and N respectively. A continuous map $f : (M, d_g) \to (N, d_h)$ is said to be Λ-Lipschitz if*

$$\mathrm{Lip}_{g,h}(f) = \sup \left\{ \frac{d_h\left(f(x), f(y)\right)}{d_g(x, y)} : x, y \in M, x \neq y \right\} \leq \Lambda.$$

We say that f is Lipschitz continuous if f is Λ-Lipschitz for some $\Lambda > 0$. Here the Riemannian distance $d_{\hat{g}}$ of a continuous metric $\hat{g}$ is defined by minimizing the length of smooth regular curve between points with respect to $\hat{g}$, see [6, Reminder 1.2] *and* [4, (2.2)].

Remark 1.1. When both g, h and f are smooth, then the map f being Λ-Lipschitz implies that

$$f^* h \leq \Lambda^2 \cdot g$$

on M by letting $x \to y$. The converse holds trivially by integrating along geodesics. In the non-smooth case, it still holds almost everywhere, for instance see [6, Proposition 2.1], but we will not rely on this fact in this work.

The definition of Lipschitz continuity of a map with respect to a continuous metric g can be defined naturally while the lower bound of scalar curvature is very subtle. We adopt the following definition:

Definition 1.2. *Let g_0 be a C^0 metric on a compact manifold M. We say that the scalar curvature $\mathcal{R}(g_0)$ of g_0 satisfies $\mathcal{R}(g_0) \geq \sigma_0$ if there exists a sequence of smooth metrics g_i on M such that the scalar curvature $\mathcal{R}(g_i)$ of g_i satisfies $\mathcal{R}(g_i) \geq \sigma_0$ on M for all i and $g_i \to g_0$ in $C^0(M)$ as $i \to +\infty$.*

The definition can be rephrased as follows: Let G_{σ_0} be the subset of C^2 metrics with scalar curvature bounded from below by σ_0. A metric $g \in C^0(M)$ is defined to have scalar curvature bounded below by σ_0 if $g \in \overline{G}_{\sigma_0}$, where the closure is taken in terms of C^0 norm. This definition is natural because Gromov [16] proved that if $g \in \overline{G}_{\sigma_0}$ and if g is C^2, then the scalar curvature of g is bounded below by σ_0 in the usual sense. See also the work of Bamler [2], which is based on the Ricci flow.

Suppose g is a C^0 metric. If g satisfies one of the following, then its scalar curvature is bounded below by σ in the sense of Definition 1.2:

(a) g has scalar curvature bounded below by σ in the sense of Burkhardt-Guim [3]. This can be proved using regularizing Ricci flow.
(b) $g \in W^{1,p}, p > n$ with $\mathcal{R} \geq \sigma$ in the sense of distribution as in [21]. This follows from the work of Jiang-Sheng-Zhang [18].
(c) g is smooth away from singularity Σ of co-dimension at least three and has $\mathcal{R}(g) \geq \sigma$ outside Σ. This is a result by authors [22].
(d) There exist smooth metrics g_i such that $g_i \to g$ in C^0 norm so that some integral form of lower bound of $\mathcal{R}(g_i)$ is satisfied. See the work of Huang and the first named author [12].

Under this general notion on the lower bound of scalar curvature, we obtain the following:

Theorem 1.2. *Let M^n be a compact Riemannian spin manifold of dimension $n \geq 2$ and g_0 is a C^0 metric on M with $\mathcal{R}(g_0) \geq n(n-1)$ in the sense of Definition 1.2. Suppose there is 1-Lipschitz continuous map $f : (M, d_{g_0}) \to \mathbb{S}^n$ with non-zero degree, then f is a distance isometry.*

In particular, this provides a complete answer to the question posed in [17, Section 4.5, question (b)] for all dimension n. We also confirm the conjecture in [6] that their result holds for the odd dimensional cases, assuming the map is 1-Lipschitz. However, we are unable to prove the result under the weaker condition that the map is $(1, \Lambda^2)$-contracting.

Our proof involves reducing the cases to smooth ones and then applying Llarull's theorem in the smooth case. Since in this case, the metric and the map are both non-smooth, we will employ Ricci-DeTurck flow and the results of Simon [27] to regularize the metric and use the harmonic map heat flow to regularize the Lipschitz map. To utilize Llarull's result, the key is to establish a new monotonicity for harmonic map heat flow coupled with the Ricci flow. Furthermore, when adopting the geometric flow approach, if g_0 has better regularity as considered in [6], one can obtain a slightly stronger conclusion regarding the regularity of g_0, see Theorem 4.2 for details.

Llarull's results have been generalized in another direction: namely, maps between manifolds with boundary. It was previously explored by Lott [24] using boundary value problems for Dirac operators. See also [1, 7] for related works. For the case when $n = 3$ and the boundary is a spherical cap, Hu-Liu-Shi [11] have considered this using μ-bubbles, which allows for a further relaxation of boundary conditions compared to the method of Dirac operators. In this context, by employing a gluing method with Theorem 1.2, we have a relatively simple rigidity result for domains inside sphere which holds for all dimensions:

Corollary 1.1. *Suppose Ω be a domain inside the standard sphere $(\mathbb{S}^n, h)$ with smooth boundary and $n \geq 2$. If g_0 is a smooth metric on Ω such that*

(i) *$g_0 \geq h$ where h is the standard spherical metric;*
(ii) *$\mathcal{R}(g_0) \geq n(n-1)$ on Ω;*
(iii) *$H(g) \geq H(h)$ on $\partial\Omega$, where $H(g), H(h)$ are the mean curvatures with respect to the unit outward normals and with respect to g, h;*
(iv) *$g_0 = h$ on $\partial\Omega$,*

then $g_0 = h$ on Ω. Moreover, if Ω is the hemisphere, then the same conclusion holds without assumption (iv).

Corollary 1.1 in particular gives an affirmative answer to [11, Conjecture 1.3]. When n is even, it was already known to be true by the work of Lott [24]. Hence the only new results in the corollary are for odd dimensions. One may indeed obtain an analogous statement of [6, Theorem B] using the same strategy for all $n \geq 2$. We work on the sphere only to illustrate the application of Theorem 1.2.

The paper is organized as follows. In Section 2, we review some basic definitions of harmonic map heat flow coupled with the Ricci flow and establish a new monotonicity formula. In Section 3, we construct the harmonic map heat flow coupled with a smooth Ricci flow starting from Lipschitz initial data and obtain estimates under scaling invariant curvature control. In Section 4, we provide proofs of the main results.

2. Monotonicity along harmonic heat flow

In this section, we will prove that some quantities will be monotone along the harmonic heat flow coupled with the Ricci flow on compact manifolds. Using these, we can regularize the Lipschitz map and metric while preserving the rigidity structure. Let us first recall the definition of harmonic map heat flow.

Suppose $f : (M, g) \to (N, h)$ is a smooth map, then df is a section of $T^*(M) \otimes f^{-1}(T(N))$ where $f^{-1}(T(N))$ is the pull-back of $T(N)$ by f. Here $T(X), T^*(X)$ are the tangent bundle and cotangent bundle of a manifold X respectively. Let D be the covariant derivative induced by the Riemannian connections of g and h. Then the second fundamental form Ddf is a section of $T^*(M) \otimes T^*(M) \otimes f^{-1}(T(N))$. The trace $\tau(f)$ of Ddf with respect to g is called the *tension field* which is a vector field along f. The energy density $e(f)$ of the map is $|df|^2$ where the inner product is taken with respect to the metric in $T^*(M) \otimes f^{-1}(T(N))$ induced by g and h. Equivalently, we have $e(f) = \mathrm{tr}_g f^*(h)$.

If $g(t)$ is a smooth family of metrics on M, then the harmonic map heat flow is given by

$$\frac{\partial}{\partial t} F = \tau(F) \tag{2.1}$$

where $F : M \times [0, T] \to N$ and $\tau(F)$ at time t is the tension field of $F(\cdot, t)$ with respect to the metric $g(t)$ in the domain and metric h on N. Note that $\frac{\partial}{\partial t} F$ is indeed $F_*(\frac{\partial}{\partial t})$. We want to discuss the behaviours of the eigenvalues of $(F(t))^*(h)$ with respect to $g(t)$.

Theorem 2.1. *Let (M^n, g_0) and (N^n, h) be two compact manifolds such that $f^*h \le \Lambda^2 g_0$ on M for some $\Lambda > 0$. Suppose $g(t)$ is a smooth family of metrics on $M \times [0, T]$ satisfying*

$$\partial_t g_{ij} = 2k_{ij}$$

such that $k + \mathrm{Ric}(g) \ge 0$. Let $F(t) : (M, g(t)) \to (N, h)$ be a family of smooth map satisfying the harmonic map heat flow:

$$\partial_t F = \tau(F), \quad F(0) = f. \tag{2.2}$$

If either one of the followings hold:

(i) *the sectional curvature of h satisfies $K(h) \le \kappa$ for some $\kappa \ge 0$ or;*
(ii) *the sectional curvature of h is non-negative and* $\mathrm{Ric}(h) \le (n-1)\kappa$,

then

$$(\Lambda^{-2} - 2(n-1)\kappa t) F^* h \le g(t),$$

on $M \times [0, T]$.

Proof. It is sufficient to prove the Theorem for those t with $\Lambda^{-2} - 2(n-1)\kappa t > 0$. Let $H = F^*(h)$ so that $H_{ij}(x) = F_i^\alpha F_j^\beta h_{\alpha\beta}(F(x))$ for $x \in M$. Then

$$\partial_t H_{ij} = \left(F_{it}^\alpha F_j^\beta + F_i^\alpha F_{tj}^\beta \right) h_{\alpha\beta}, \tag{2.3}$$

because the derivative of h is zero with respect to the connection on the pull-back bundle $F^{-1}(T(N))$. On the other hand,

$$\begin{aligned}\Delta_t H_{ij} =& g^{pq}(F_i^\alpha F_j^\beta h_{\alpha\beta})_{;pq} \\ =& g^{pq}\left((F_i^\alpha)_{|pq}F_j^\beta + F_i^\alpha (F_j^\beta)_{|pq} + 2(F_i^\alpha)_{|p}(F_j^\beta)_{|q}\right) h_{\alpha\beta}\end{aligned}$$

where Δ_t is the Laplacian with respect to $g(t)$ and $_{|p}$ denotes the covariant derivative of D. Using $(F_i^\alpha)_{|pq} = (F_p^\alpha)_{|iq}$, the Ricci identity, and the fact that $\tau(F)^\alpha = g^{pq}F^\alpha_{p|q}$ with $g = g(t)$, we have

$$\begin{aligned}\Delta_t H_{ij} =& F_j^\beta\left((\tau(F)^\alpha)_{|i} + R_i^l F_l^\alpha + g^{kl}\widetilde{R}_{\gamma\sigma\delta}{}^{\alpha} F_k^\delta F_l^\gamma F_i^\sigma\right) h_{\alpha\beta} \\ &+ F_i^\alpha\left((\tau(F)^\beta)_{|j} + R_j^l F_l^\beta + g^{kl}\widetilde{R}_{\gamma\sigma\delta}{}^{\beta} F_k^\delta F_l^\gamma F_j^\sigma\right) h_{\alpha\beta} \\ &+ 2g^{pq}(F_i^\alpha)_{|p}(F_j^\beta)_{|q}h_{\alpha\beta} \\ =& F_j^\beta(\tau(F)^\alpha)_{|i}h_{\alpha\beta} + F_i^\alpha(\tau(F)^\beta)_{|j}h_{\alpha\beta} + R_i^l H_{lj} + R_j^l H_{il} \\ &+ 2g^{kl}\widetilde{R}(u_l, u_i, u_k, u_j) + 2g^{pq}(F_i^\alpha)_{|p}(F_j^\beta)_{|q}h_{\alpha\beta}\end{aligned}$$

where $\widetilde{\mathrm{Rm}}$ denotes the curvature of h and $u_i = F_*(\partial_i)$ in local coordinates x^i with $\partial_i = \partial_{x^i}$. Hence

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_t\right) H_{ij} =& \left(F_{it}^\alpha F_j^\beta + F_i^\alpha F_{tj}^\beta\right) h_{\alpha\beta} \\ &- \Big[F_j^\beta(\tau(F)^\alpha)_{|i}h_{\alpha\beta} + F_i^\alpha(\tau(F)^\beta)_{|j}h_{\alpha\beta} + R_i^l H_{lj} + R_j^l H_{il} \\ &+ 2g^{kl}\widetilde{R}(u_l, u_i, u_k, u_j) + 2g^{pq}(F_i^\alpha)_{|p}(F_j^\beta)_{|q}h_{\alpha\beta}\Big] \\ =& -\left[R_i^l H_{lj} + R_j^l H_{il} + 2g^{kl}\widetilde{R}(u_l, u_i, u_k, u_j) + 2g^{pq}(F_i^\alpha)_{|p}(F_j^\beta)_{|q}h_{\alpha\beta}\right]\end{aligned}$$

because $F_{it}^\alpha = F_{ti}^\alpha$ and $\partial_t F^\alpha = \tau(F)^\alpha$ for all i, α.

If we define $A(t) = \lambda(t)g - F^*h = \lambda(t)g - H$ at time t for some function $\lambda(t)$, then whenever $\lambda(t) > 0$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_t\right) A_{ij} \geq& \lambda' g_{ij} + 2\lambda k_{ij} + R_i^l H_{lj} + R_j^l H_{il} + 2g^{kl}\widetilde{R}(u_l, u_i, u_k, u_j) \\ &+ 2g^{pq}(F_i^\alpha)_{|p}(F_j^\beta)_{|q}h_{\alpha\beta} \\ =:& B_{ij}.\end{aligned} \tag{2.4}$$

We want to find $\lambda(t) > 0$ so that B_{ij} satisfies the null-eigenvector assumption at every point (x, t), $t > 0$. Namely, suppose $A_{ij}w^iw^j \geq 0$ for all $w \in T_x(M)$, and v is such that $A_{ij}v^j = 0$ for all i. Then we want to prove that $B_{ij}v^iv^j \geq 0$.

Let v be such a vector. If $v = 0$, obviously, $B_{ij}v^iv^j = 0$. So we may assume that v is a unit vector by scaling. At x, we choose a local coordinates x^i so that $\partial_i = \partial_{x^i}$ are orthonormal and $(F(t))^*h$ is diagonalized with respect to $g(t)$ at x. Since $A(w,w) \geq 0$ for all w and $A(v,v) = 0$, v is an eigenvector of A. We may assume that $v = \partial_1$. $A(\partial_i, \partial_i) \geq 0$ is equivalent to say that $||F_*(\partial_i)||_h^2 \leq \lambda$ for all i. Also $A_{ij}v^j = 0$ implies $H_{ij}v^j = \lambda g_{ij}v^j$. Hence

(2.5)

$$\begin{aligned}
B_{ij}v^iv^j =&\lambda' g_{ij}v^iv^j + 2\lambda k_{ij}v^iv^j + R_i^l H_{lj}v^iv^j + R_j^l H_{il}v^iv^j + 2g^{kl}\widetilde{R}(u_l, u_i, u_k, u_j)v^iv^j \\
&+ 2g^{pq}(F_i^\alpha)_{|p}(F_j^\beta)_{|q}h_{\alpha\beta}v^iv^j \\
\geq&\lambda' + 2\lambda(k_{ij} + R_{ij})v^iv^j - 2\sum_{i=1}^n \widetilde{R}(F_*(\partial_1), F_*(\partial_i), F_*(\partial_i), F_*(\partial_1)) \\
\geq&\lambda' - 2\sum_{i=1}^n \widetilde{R}(F_*(\partial_1), F_*(\partial_i), F_*(\partial_i), F_*(\partial_1)),
\end{aligned}$$

because $k + \mathrm{Ric} \geq 0$. Here F refers to $F(t)$. Since $h(F_*(\partial_i), F_*(\partial_j)) = \beta_i\delta_{ij}$ for some $\beta_i \geq 0$, one can find $\hat{u}_i$ which are orthonormal at $F(x)$ so that $F_*(e_i) = \beta_i^{\frac{1}{2}}\hat{u}_i$. Note that $\beta_i \leq \lambda$.

Hence if the sectional curvature of h is bounded above by κ, then

$$\sum_{i=1}^n \widetilde{R}(F_*(\partial_1), F_*(\partial_i), F_*(\partial_i), F_*(\partial_1)) = \sum_{i=1}^n \widetilde{R}(\beta_1^{\frac{1}{2}}\hat{u}_1, \beta_i^{\frac{1}{2}}\hat{u}_i, \beta_i^{\frac{1}{2}}\hat{u}_i, \beta_1^{\frac{1}{2}}\hat{u}_1) \leq (n-1)\kappa\lambda^2.$$

If the sectional curvature of h is non-negative and $\mathrm{Ric}(h) \leq (n-1)\kappa$, then

$$\sum_{i=1}^n \widetilde{R}(F_*(\partial_1), F_*(\partial_i), F_*(\partial_i), F_*(\partial_1)) \leq \lambda^2 \sum_{i=1}^n \widetilde{R}(\hat{u}_1, \hat{u}_i, \hat{u}_i, \hat{u}_1) \leq (n-1)\kappa\lambda^2.$$

Putting this back to (2.5), we have

$$B_{ij}v^iv^j \geq \lambda' - 2(n-1)\kappa\lambda^2.$$

In either cases, if we let

$$\lambda(t) = \left(\Lambda^{-2} - 2(n-1)\kappa t\right)^{-1}.$$

Then $\lambda(t) > 0$ as long as $\Lambda^{-2} - 2(n-1)\kappa t > 0$ and B satisfies the null eigenvector assumption. Moreover, $\lambda(0) = \Lambda^{-2}$. By assumption we have $A(0) \geq 0$. By the weak maximum principle for tensor [9, Theorem A.21], we conclude that the Theorem holds. □

Remark 2.1.
(i) In Theorem 2.1, if $g(t)$ is the Ricci flow, then $k = -\mathrm{Ric}(g)$ and $k + \mathrm{Ric}(g) = 0$. If $g(t)$ is a fixed metric, then $k = 0$. If g has non-negative Ricci curvature, then we also have $k + \mathrm{Ric}(g) \geq 0$. Hence the theorem can be applied to these cases. We may have corresponding results under the assumption that $k + \mathrm{Ric} \geq -K$, for $K \geq 0$.
(ii) If the sectional curvature of h is non-positive, then we have $(F(t))^*(h) \leq \Lambda^2 g(t)$ for all $t > 0$.

(iii) If f is a Λ-Lipschitz map, then $\mathrm{tr}_g(f^*(h)) = e(f) \le n\Lambda^2$. We may instead assume $\mathrm{tr}_g(f^*(h)) \le n\Lambda^2$ and study the behaviour under the flow. The bound of $\mathrm{tr}_{g(t)}(F^*(h))$ is well-known. However, one may get the following sharp bound too. Namely, if the sectional curvature of h is bounded above by κ, then $(\Lambda^{-2} - 2(n-1)\kappa t)\,\mathrm{tr}_{g(t)}(F^*(h)) \le n$. One may wonder if there are similar relations of other symmetric functions of eigenvalues of $F^*(h)$.

In the case of Ricci flow (i.e. $k = -\mathrm{Ric}$), Theorem 2.1 is sharp when comparing with the following well-known scalar curvature estimate.

Theorem 2.2. *Suppose M is a compact manifold and $g(t)$ is a smooth solution to the Ricci flow on $M \times [0,T]$ with initial metric $g(0) = g_0$. If $\mathcal{R}(g_0) \ge \sigma_0$ on M for some $\sigma_0 > 0$, then we have*

$$\mathcal{R}(g(t)) \ge \frac{n\sigma_0}{n - 2\sigma_0 t}. \tag{2.6}$$

In particular, we must have $T < n(2\sigma_0)^{-1}$.

Proof. It is well-known that the scalar curvature evolves by

$$\left(\frac{\partial}{\partial t} - \Delta_t\right)\mathcal{R} = 2|\mathrm{Ric}|^2 \ge \frac{2}{n}\mathcal{R}^2. \tag{2.7}$$

The desired estimate of $\mathcal{R}(g(t))$ follows from the maximum principle. □

3. Harmonic map heat flow from Lipschitz initial data

In Theorem 1.2, we want to study continuous metric g_0 in the domain and Lipschitz continuous map f from the domain manifold into the a compact manifold with smooth metric. We want to use the results in the previous section. We want to construct Ricci flow with initial data g_0 and harmonic map heat flow with initial map f. We will do this by approximation. In this section, we will first construct harmonic heat flow coupled with Ricci flow assuming that g_0 is smooth and f is Lipschitz continuous. To be precise, let $g(t)$ be a smooth solution of Ricci flow on $M \times [0,T]$:

$$\begin{cases} \partial_t g(t) = -2\mathrm{Ric}(g(t)); \\ g(0) = g_0. \end{cases} \tag{3.1}$$

Let (N,h) be another compact manifold with smooth metric, and let $f : (M,g_0) \to (N,h)$ be a Lipschitz continuous map.

Theorem 3.1. *Let (M,g_0) and (N,h) be two smooth compact Riemannian manifolds. Suppose the sectional curvature of h is bounded from above by 1 and $g(t)$ is a solution to (3.1) on $M \times [0,T]$ such that*

$$|\mathrm{Rm}(g(t))| \le at^{-1}$$

for some $a > 0$ on $(0, T]$. If $f : (M, g_0) \to (N, h)$ is a Λ-Lipschitz map, then there exist $T_0(n, \Lambda), C_0(n, a, \Lambda, h) > 0$ and $F(t) \in C^\infty(M, N),\ t \in (0, \min\{T, T_0\}]$ satisfying the harmonic map heat flow

$$\partial_t F = \tau(F)$$

where $\tau(F)$ is the tensor field of the map $F(\cdot, t)$ with respect to the metrics $g(t), h$ such that

$$\begin{cases} (\Lambda^{-2} - 2(n-1)t)\,(F(t))^* h \le g(t); \\ \sup_{x\in M} d_h\,(F(x,t), f(x)) \le C_0\sqrt{t} \end{cases} \tag{3.2}$$

Moreover, for any integer $\ell \ge 0$, there exists $C(n, \ell, a, h, \Lambda) > 0$ such that for all $t \in (0, \min\{T, T_0\}]$,

$$|D^\ell dF|^2 \le \frac{C(n, \ell, a, h, \Lambda)}{t^\ell}.$$

We should remark that the existence time and the estimates of F do not depend on the curvature of the initial metric g_0, as long as $|\mathrm{Rm}(g(t))| \le at^{-1}$ holds. This is important for the application.

The theorem will follow from the results for smooth f by approximation. First we have the following:

Lemma 3.1. *Suppose (M, g) and (N, h) are smooth compact Riemannian manifolds and $f : (M, g) \to (N, h)$ is a continuous map such that f is Lipschitz continuous. Then for any $\varepsilon > 0$, there exists a smooth map $f_\varepsilon : (M, g) \to (N, h)$ such that*

$$\mathrm{Lip}_{g,h}(f_\varepsilon) \le (1+\varepsilon)\mathrm{Lip}_{g,h}(f) \quad \textit{and} \quad \sup_{x\in M} d_h\,(f_\varepsilon(x), f(x)) < \varepsilon.$$

Proof. This follows from the method in a well-known result by Greene-Wu [15], see also [20, Theorem 1.3]. For the sake of completeness, we give a proof as follows. First we isometrically embed N to $\mathbb{R}^K$ for some large $K > 0$. In this setting, the map f can be expressed as a vector valued function $\mathbf{u} : M \to N \subset \mathbb{R}^K$. It is easy to see that $\mathbf{u}$ is Lipschitz with Lipschitz constant $L =: \mathrm{Lip}_{g,h}(f)$. Suppose for any $\eta > 0$, we can find a smooth function $\mathbf{v}$ such that $\sup_{x\in M} |\mathbf{u}(x) - \mathbf{v}(x)| \le \eta$ and $|D\mathbf{u}(w)| \le (L+\eta)|w|_g$ for all $w \in T(M)$, then the result follows. In fact, if this is true, then $\mathbf{v}(x)$ will be in the η neighbourhood of N in $\mathbb{R}^K$. Then $\pi \circ \mathbf{v}$ will be the required map if η is small enough, where π is the nearest point projection from $\mathbb{R}^K$ to N.

To find $\mathbf{v}$, we follow the idea in [15]. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function with support contained in $[-1, 1]$, so that

$$\int_{\mathbb{R}^n} \rho(|x|)dx = 1.$$

Define

$$\mathbf{u}_\varepsilon(p) = \varepsilon^{-n} \int_{v\in T_p(M)} \mathbf{u}(\exp_p(v)) \cdot \rho\left(\frac{|v|}{\varepsilon}\right) dV_p. \tag{3.3}$$

dV_p is the volume form with respect to $g|_{T_p(M)}$. For ε small enough, $\mathbf{u}_\varepsilon$ is smooth and $\mathbf{u}_\varepsilon \to \mathbf{u}$ uniformly as $\varepsilon \to 0$. Let $p \in M$ and let $w \in T_p(M)$ with $||w|| = 1$. Let $\gamma(t)$ be the unique minimal geodesic with $\gamma(0) = p$, $\gamma'(0) = w$, $0 \le t \le a$. We may fix a so that $B_x(2a)$ is convex for all $x \in M$. Then

$$D\mathbf{u}_\varepsilon(w) = \lim_{t\to 0} \frac{1}{t} \left(\mathbf{u}_\varepsilon(\gamma(t)) - \mathbf{u}_\varepsilon(\gamma(0))\right).$$

Now,

$$\begin{aligned}
&\mathbf{u}_\varepsilon(\gamma(t)) - \mathbf{u}_\varepsilon(\gamma(0)) \\
=&\varepsilon^{-n} \int_{v\in T_{\gamma(t)}(M)} \mathbf{u}(\exp_{\gamma(t)}(v)) \cdot \rho\left(\frac{|v|}{\varepsilon}\right) dV_{\gamma(t)} - \varepsilon^{-n} \int_{v\in T_p(M)} \mathbf{u}(\exp_p(v)) \cdot \rho\left(\frac{|v|}{\varepsilon}\right) dV_p \\
=&\varepsilon^{-n} \int_{v\in T_p(M), |v|\le\varepsilon} \left(\mathbf{u}(\exp_{\gamma(t)}(P_t(v))) - \mathbf{u}(\exp_p(v))\right) \rho\left(\frac{|v|}{\varepsilon}\right) dV_p
\end{aligned}$$

where P_t is the parallel translation along γ from $\gamma(0)$ to $\gamma(t)$. Hence by Minkowski integral inequality [10, Theorem 202], we have

$$\begin{aligned}
&|\mathbf{u}_\varepsilon(\gamma(t)) - \mathbf{u}_\varepsilon(\gamma(0)| \\
\le&\varepsilon^{-n} \int_{v\in T_p(M), |v|\le\varepsilon} |\left(\mathbf{u}(\exp_{\gamma(t)}(P_t(v)) - \mathbf{u}(\exp_p(v))\right)| \rho\left(\frac{|v|}{\varepsilon}\right) dV_p \\
\le& \sup_{v\in T_p(M), |v|\le\varepsilon} |\left(\mathbf{u}(\exp_{\gamma(t)}(P_t(v)) - \mathbf{u}(\exp_p(v))\right)| \\
\le&L \sup_{v\in T_p(M), |v|\le\varepsilon} |d_g\left(\mathbf{u}(\exp_{\gamma(t)}(P_t(v)), \mathbf{u}(\exp_p(v))\right)|.
\end{aligned}$$

For each t, let $\gamma(t,s)$ be the geodesic from $\gamma(t)$ with tangent vector $P_t(v)$, $0 \le s \le 1$. Then $|\partial_t\gamma|_{s=0} = 1$. Hence for any $\eta > 0$, there is $\varepsilon > 0$ such that if $|v| \le \varepsilon$, then $|\partial_t\gamma| \le 1+\varepsilon$ for all $0 \le s \le 1$. By compactness and the fact that exp is smooth, and solutions of ODE depends smoothly on initial data, one can see that $\varepsilon > 0$ can be chosen so that for all $p, w \in T_p(M)$ with $|w| = 1$ and $v \in T(M)$ with $|v| \le \varepsilon$, we have $|\partial_t\gamma| \le 1+\varepsilon$. This implies that the length of the curve $\gamma(1,t)$, $0 \le t \le t_0$ is less than or equal to $(1 + C\varepsilon^2)t_0$. So

$$d_g\left(\mathbf{u}(\exp_{\gamma(t)}(P_t(v)), \mathbf{u}(\exp_p(v))\right) \le (1+\varepsilon)t.$$

From this we conclude that $|D\mathbf{u}_\varepsilon(w)| \le L(1+\varepsilon)$ and the result follows. □

Proof of Theorem 3.1. Let $L = \mathrm{Lip}_{g_0,h}(f)$ so that $L \le \Lambda$. By Lemma 3.1, there exist sequence of smooth maps f_i from M to N so that

$$\mathrm{Lip}_{g_0,h}(f_i) \le (1+\varepsilon_i)L \quad \text{and} \quad \sup_{x\in M} d_h\left(f_i(x), f(x)\right) < \varepsilon_i$$

with $\varepsilon_i \to 0$ as $i \to \infty$. In particular, the energy density of the maps f_i are uniformly bounded by $(n+1)L^2$, say if i is sufficiently large. By [13, Theorem 1.1], for each i there is a solution F_i to the harmonic heat flow coupled with

$g(t)$ on $M\times[0,\min\{T,T_0\}]$ for some $T_0(n,\Lambda)$ so that for any integer $\ell\ge 0$, there exists $C(n,\ell,a,h,\Lambda)>0$ such that for all $t\in(0,\min\{T,T_0\}]$,

$$|D^\ell dF_i|^2\le \frac{C(n,\ell,a,h,\Lambda)}{t^\ell}. \tag{3.4}$$

In [13], only the energy density and the tension field have been estimated. But the higher order estimates of $|D^\ell dF_i|^2$ can be done similarly using Bernstein-Shi trick as in [8]. The only difference is that in [8], $|\mathrm{Rm}(g(t)|$ is uniformly bounded and in our case, $|\mathrm{Rm}(g(t))|\le at^{-1}$. In fact, by the estimates of Shi [25], $|\nabla^k\mathrm{Rm}(g(t))|\le Ct^{-(1+\frac{k}{2})}$ for any integer $k\ge 0$, where C depends only on a,n,k. Denote $|D^\ell dF_i|^2$ by P_ℓ, where P_0 is just the energy density. So (3.4) is true for $\ell=0$. Suppose the estimates are true up to $\ell-1$. We will use C_i to denote any constants depending only on n,T,a,Λ,h,ℓ. By a direct computation [8, Lemma 2.10 and p.141], we have

$$\left(\frac{\partial}{\partial t}-\Delta_t\right)P_\ell\le -2P_{\ell+1}+C_1\left(t^{-1}P_\ell+t^{-1-\frac{\ell}{2}}P_\ell^{\frac12}\right).$$

Hence whenever $P_\ell>0$, we have

$$\left(\frac{\partial}{\partial t}-\Delta_t\right)(t^{\frac{1+\ell}{2}}P_\ell^{\frac12})\le C_3(t^{\frac{\ell-1}{2}}P_\ell^{\frac12}+t^{-\frac12})$$

On the other hand,

$$\left(\frac{\partial}{\partial t}-\Delta_t\right)P_{\ell-1}\le -2P_\ell+C_4t^{-\ell}$$

and so

$$\left(\frac{\partial}{\partial t}-\Delta_t\right)(t^{\ell-\frac12}P_{\ell-1})\le -2t^{\ell-\frac12}P_\ell+C_5t^{-\frac12}.$$

Let $G=t^{\frac{1+\ell}{2}}P^{\frac12}+t^{\ell-\frac12}P_{\ell-1}-\alpha t^{\frac12}$, for $\alpha>0$, we have:

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta_t\right)G\le& C_3(t^{\frac{\ell-1}{2}}P_\ell^{\frac12}+t^{-\frac12})-2t^{\ell-\frac12}P_\ell+C_5t^{-\frac12}-\frac12\alpha t^{-\frac12}\\ \le& C_6t^{-\frac12}-\frac12\alpha t^{-\frac12}.\end{aligned}$$

Let α be such that $\frac12\alpha=C_1+1$. We have

$$\left(\frac{\partial}{\partial t}-\Delta_t\right)G<0$$

at the points where $P_\ell>0$. Since $G=0$ at $t=0$, we conclude that $G\le 0$. Hence

$$t^{\frac{\ell}{2}}P_\ell^{\frac12}\le C_7.$$

By induction, one can conclude that the estimates (3.4) are true.

Since $|\tau(F_i)| \le |DdF_i| \le Ct^{-\frac{1}{2}}$, we conclude that

$$d_h(F_i(x,t), f(x)) \le C_8 t^{\frac{1}{2}}.$$

Since N is compact, we can find a subsequence of F_i which converges uniformly in C^∞ to a map F in compact subsets of $M \times (0, \min\{T, T_0\}]$. Hence F satisfies the harmonic heat flow with estimates of $|D^\ell dF|$ in the theorem. Moreover,

$$d_h(F(x,t), f(x)) \le C_8 t^{\frac{1}{2}}.$$

Finally, apply Theorem 2.1 to F_i and let $i \to \infty$, we have

$$(\Lambda^{-2} - 2(n-1)t)(F(t))^* h \le g(t).$$

This completes the proof of the theorem.

□

4. Proof of Theorem 1.2

In this section, we will construct a solution to the harmonic map heat flow coupled with the Ricci flow where both f and g_0 are allowed to be non-smooth. Due to the weak parabolicity of the Ricci flow, there will be some technical issue when discussing the time zero regularity of the Ricci flow if the initial data is only C^0. To avoid this, we will work on the Ricci-DeTurck G-flow instead, where G is a smooth background metric. Recall that a smooth family of metrics $g(t)$ on $M \times (0,T]$ is said to be a solution to the Ricci-DeTurck G-flow if it satisfies

$$\begin{cases} \partial_t g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i; \\ W^k = g^{pq}\left(\Gamma^k_{pq} - \widetilde{\Gamma}^k_{pq}\right) \end{cases} \tag{4.1}$$

where Γ and $\widetilde{\Gamma}$ denote the connections of $g(t)$ and G respectively. If the initial metric g_0 is smooth, it is well-known that the Ricci flow is equivalent to the Ricci-DeTurck G-flow in the following sense. Let Φ_t be the diffeomorphism given by

$$\begin{cases} \partial_t \Phi_t(x) = -W\left(\Phi_t(x), t\right); \\ \quad \Phi_0(x) = x. \end{cases} \tag{4.2}$$

Then the pull-back of the Ricci-DeTurck flow $\widehat{g}(t) = \Phi_t^* g(t)$ is a Ricci flow solution with $\widehat{g}(0) = g(0) = g_0$. We also need the following smoothing result of M. Simon [27] which allows us to regularize a C^0 metric on a compact manifold using the Ricci-DeTurck flow, see also the work of Koch-Lamm [19] on $\mathbb{R}^n$.

Theorem 4.1. *[M. Simon] There exists a sufficiently small $\varepsilon_n > 0$ depending only on n such that if M^n is a compact manifold with a smooth metric h on M and g_0 is a continuous metric on M so that $(1-\varepsilon_n)h \le g_0 \le (1+\varepsilon_n)h$ on M, then there exists $T(n,h) > 0$ and a solution to the h-flow on $M \times (0,T]$ such that the following holds on $M \times (0,T]$:*

(i) $(1-2\varepsilon_n)h \le g(t) \le (1+2\varepsilon_n)h$;

(ii) *for all* $k \in \mathbb{N}$, *there exists* $C(n,k,h) > 0$ *such that*

$$\sup_M |\tilde{\nabla}^k g(t)| \le \frac{C(n,k,h)}{t^{k/2}};$$

(iii) $\lim_{t\to 0} ||g(t) - g_0||_{L^\infty(M),h} = 0$.

Proof. This follows from [27, Theorem 5.2] since M is compact. □

Now we are ready to prove our main Theorem.

Proof of Theorem 1.2. Let g_0 be a C^0 metric on M with scalar curvature at least $n(n-1)$ in the sense of Definition 1.2 and let (N,h) be the standard unit n-sphere. Then there exist a sequence of smooth metrics $g_{i,0}$ with $\mathcal{R}(g_{i,0}) \ge n(n-1)$ so that $g_{i,0} \to g_0$ in C^0-norm. We may assume that

$$\left(1+\frac{1}{i}\right)^{-1} g_{i,0} \le g_0 \le \left(1+\frac{1}{i}\right) g_{i,0}. \tag{4.3}$$

Fix i_0 large enough depending only on n and let $G =: g_{i_0}$ so that we can apply the results Theorem 4.1 by M. Simon to obtain a constant $T > 0$ depending only on n and $G =: g_{i_0}$ such that the Ricci-DeTurck G-flow (4.1) with initial data $g_{i,0}$ has a solution defined on $M \times [0,T]$ which is smooth up to $t = 0$. Denote this solution by $g_i(t)$. Moreover, for any $\ell \in \mathbb{N}$, there is $C_\ell > 0$ depending only on $n, k_0, ..., k_{\ell+1}$ so that

$$\begin{cases} \frac{1}{2}(1+\frac{1}{i_0})^{-1} g_i \le G \le 2(1+\frac{1}{i_0}) g_i; \\ \sup_M |\widetilde{\nabla}^\ell g_i(t)| \le \dfrac{C_\ell}{t^{\ell/2}}, \end{cases} \tag{4.4}$$

where $\widetilde{\nabla}$ is the covariant derivative with respect to G and k_m is the bound of the norm of the m-th derivative of the curvature tensor of G. Passing to some subsequence, we may assume that $g_i(t)$ will converge to a solution $g(t)$ to the Ricci-DeTurck G-flow on $M \times (0,T]$. Moreover, by Theorem 4.1 again, $g(t)$ converges to g_0 in C^0 norm as $t \to 0$. Since $\mathcal{R}(g_{i,0}) \ge n(n-1)$, Theorem 2.2 implies

$$\mathcal{R}\left(\lambda^{-1}(t) g(t)\right) \ge n(n-1). \tag{4.5}$$

where $\lambda(t) = 1 - 2(n-1)t$ as long as $\lambda(t) > 0$. We may assume that this is true for $0 < t \le T$ by choosing a smaller $T > 0$.

Next we want to construct "harmonic" map heat flow coupled with $g(t)$. For each i, let $W^{(i)}$ be the vector field given by (4.1) with $\Gamma = \Gamma^{(i)}$ to be the connection of $g_i(t)$ and let $\Phi_t^{(i)}$ be as in (4.2) with respect to $W^{(i)}$. Let $\widehat{g}_i(t) = (\Phi_t^{(i)})^*(g_i(t))$. Then $\widehat{g}_i(t)$ is a smooth solution of the Ricci flow in $[0,T]$ such that $\widehat{g}_i(0) = g_{i,0}$. By (4.3) and the assumption that f is a 1-Lipschitz map from (M, g_0) to (N,h), we conclude that f is a $(1+\frac{1}{i})$-Lipschitz maps from $(M, g_{i,0})$ to (N,h). By (4.4) and the fact that $\widehat{g}_i(t)$ is isometric to $g_i(t)$, we conclude that $|\mathrm{Rm}(\widehat{g}_i(t)| \le a t^{-1}$ on $M \times (0,T]$ for some $a > 0$, independent of

i. By Theorem 3.1, we can solve the harmonic heat flow $\widehat{F}^{(i)}(t)$ coupled with the Ricci flow $\widehat{g}_i(t)$ in $M \times [0,T]$ (with a possibly smaller T but independent of i) with the following properties:

(i) $\widehat{F}^{(i)}(t)$ is a $L^{(i)}(t)$-Lipschitz map from $(M, \widehat{g}_i(t))$ to (N,h), with

$$L^{(i)}(t) = \frac{(1+\frac{1}{i})^2}{1-2(n-1)(1+\frac{1}{i})^2 t}.$$

(ii) For each $\ell \geq 0$, $|D^\ell d\widehat{F}^{(i)}(t)|^2 \leq C_\ell t^{-\ell}$.

(iii) $\widehat{F}^{(i)}(0) = f$.

Let $F^{(i)}(x,t) = \widehat{F}^{(i)}\left(\left(\Phi_t^{(i)}\right)^{-1}(x), t\right)$. Then using (4.2), $F^{(i)}$ satisfies:

$$\begin{aligned} \partial_t F^{(i)}(x,t) &= \partial_t \left(\hat{F}^{(i)}((\Phi_t^{(i)})^{-1}(x)\right) \\ &= \tau(\widehat{F}^{(i)}) + d\widehat{F}^{(i)}\left(\partial_t(\Phi_t^{(i)})^{-1}(x)\right) \\ &= \tau(\widehat{F}^{(i)}) + d\hat{F}_i(W^{(i)})|\left(\Phi_{i,t}^{-1}(x)\right) \\ &= \tau(F^{(i)}) + dF^{(i)}(W^{(i)}). \end{aligned} \tag{4.6}$$

Moreover, since $\Phi_t^{(i)}$ is an isometry between $\widehat{g}_i(t)$ and $g_i(t)$, we know that

(i) $F^{(i)}(t)$ is a $L^{(i)}(t)$-Lipschitz map from $(M, g_i(t))$ to (N,h).

(ii) For each $\ell \geq 0$, $|D^\ell dF^{(i)}(t)|^2 \leq C_\ell t^{-\ell}$ where D is the covariant derivative with respect to $g_i(t)$ and h on the pull-back bundle $(F^{(i)})^{-1}(T(N))$.

(iii) $F^{(i)}(0) = f$ because $\Phi_0^{(i)}(x) = x$.

By (4.6), $|\tau(F^{(i)})|_h$ is bounded by $Ct^{-\frac{1}{2}}$, $|dF^{(i)}|_{g_i(t),h}$ is bounded uniformly and $|W_i|_{g_i(t)} \leq Ct^{-1/2}$, for some C independent of i, x, t using (4.4). By computing the length of the curve from $F_i(x,t)$ to $F_i(x,0) = f(x)$ in N, we conclude that for all $x \in M$,

$$d_h\left(F_i(x,t), f(x)\right) \leq C(n,h,G)\sqrt{t}. \tag{4.7}$$

On the other hand, by (ii) above, (4.4) and passing to a subsequence, F_i will converges to a smooth map $F(x,t)$ from $M \times (0, T_1)$ to (N,h) for some $T_1 > 0$. Let $\breve{g}(t) = \lambda(t)^{-1} g(t)$. Then $F(t)$ is a smooth 1-Lipschitz map from $(M, \breve{g}(t))$ to (N,h) for each $t \in (0,T_1)$. By (4.5), the scalar curvature of $\breve{g}(t)$ is at least $n(n-1)$. On the other hand, by (4.7), we conclude that $\sup_M d_h(F(x,t), f(x)) \to 0$ as $t \to 0$. Hence $F(x,t)$ will be homotopy to $f(x)$ if t is small enough. This implies $F(x,t)$ has non-zero degree. Now the result of Llarull [23] applies to show that $F(t)$ is a metric isometry and $\breve{g}(t)$ is the standard sphere via $F(t)$ so that for all $x, y \in M$ and $t \in (0,T]$,

$$d_h\left(F_t(x), F_t(y)\right) = d_{\breve{g}(t)}(x,y).$$

By letting $t \to 0$ using the fact that $g(t)$ converges uniformly to g_0 as $t \to 0$, we conclude that f is a distance isometry. This completes the proof. □

With the Ricci flow smoothing, we can slightly improve the result in [6] when g_0 is of slightly better regularity with scalar curvature bounded below by $n(n-1)$ in the sense of distribution defined by Lee-LeFloch [21]. Namely:

Definition 4.1 ([21]). *Let M be a smooth manifold with background metric h. A $W^{1,2}_{loc}\cap L^\infty$ Riemannian metric g is said to have $\mathcal{R}(g)\geq \sigma$ for $\sigma\in\mathbb{R}$ in the sense of distribution if $\langle \mathcal{R}(g),\varphi\rangle \geq \sigma\int_M \varphi\, \mathrm{dvol}_h$ for all non-negative test function $\varphi\in C^\infty_{loc}(M)$ where*

$$\langle \mathcal{R}(g),\varphi\rangle=\int_M\left\{-\left\langle V,\widetilde\nabla(\varphi\cdot\sqrt{\frac{\det g}{\det h}})\right\rangle_h+F\varphi\cdot\sqrt{\frac{\det g}{\det h}}\right\}\, d\mathrm{vol}_h \tag{4.8}$$

with

$$\begin{cases}\Psi^k_{ij}=\frac{1}{2}g^{kl}\left(\widetilde\nabla_i g_{kl}+\widetilde\nabla_j g_{il}-\widetilde\nabla_l g_{ij}\right);\\ V^k=g^{ij}\Psi^k_{ij}-g^{ik}\Psi^j_{ji};\\ F=\mathrm{tr}_g\,\widetilde{\mathrm{Ric}}-\Psi^k_{ij}\widetilde{i}lde\nabla_k g^{ij}+\Psi^i_{jl}\widetilde\nabla_k g^{ik}+g^{ij}\left(\Psi^k_{kl}\Psi^l_{ij}-\Psi^k_{jl}\Psi^l_{ik}\right)\end{cases} \tag{4.9}$$

Here $\widetilde\nabla$ denotes the connection with respect to the background metric h.

With the slightly stronger assumption, we might conclude a slightly better regularity of g_0.

Theorem 4.2. *Under the assumption in Theorem 1.2, if in addition $g_0\in W^{1,p}(M), p>n$ satisfies $\mathcal{R}(g_0)\geq n(n-1)$ in the sense of distribution, then there exists a $C^{1,\alpha}$ diffeomorphism Ψ of M for some $\alpha>0$ so that Ψ^*g_0 is the standard spherical metric on $\mathbb{S}^n$.*

Proof. The main result follows from refining the proof of Theorem 1.2. The idea is similar to that in [22]. Since $g_0\in W^{1,p}$ for $p>n$, the works in [18, 26] infers that the constructed Ricci-DeTurck G-flow $g(t)$ satisfies a better estimate on $M\times(0,T]$:

$$\int_M|\widetilde\nabla g(t)|^p\leq C. \tag{4.10}$$

Moreover, it is known that each $\breve g(t)=\lambda^{-1}(t)g(t), t\in(0,T]$ is isometric to the standard sphere. We now construct the diffeomorphism to compensate the singularity of (4.10) as $t\to 0$. Using (4.1) and $\mathrm{Ric}(\breve g(t))=n-1$, we deduce that

$$\begin{cases}\partial_t\breve g_{ij}=\breve\nabla_i\breve W_j+\breve\nabla_j\breve W_i;\\ \breve W_j=\lambda(t)^{-1}\breve g_{jk}\breve g^{pq}\left(\breve\Gamma^k_{pq}-\widetilde\Gamma^k_{pq}\right)\end{cases} \tag{4.11}$$

Therefore, if we consider the ODE:

$$\begin{cases}\partial_t\Psi_t(x)=\breve W\left(\Psi_t(x),t\right);\\ \Psi_T(x)=x\end{cases} \tag{4.12}$$

for $(x,t)\in M\times(0,T]$, then

$$\partial_t\left[(\Psi_t^{-1})^*\breve g(t)\right]=0.$$

Hence, $\breve g(t)=\Psi_t^*\breve g(T)$ where $\breve g(T)$ is a standard spherical metric. By [5, (5.2)],

$$\frac{\partial^2\Psi_t^m}{\partial x^i\partial x^j}=\breve\Gamma_{ij}^k\frac{\partial\Psi_t^m}{\partial x^k}-\Gamma(\breve g(T))_{kl}^m\frac{\partial\Psi_t^l}{\partial x^i}\frac{\partial\Psi_t^k}{\partial x^j}\tag{4.13}$$

in local coordinate of M. Thanks to the improved estimate (4.10) and metrics equivalence, the right hand side of (4.13) is bounded in $L^p, p>n$ as $t\to 0$. Hence in local coordinates the $W^{2,p}$ norm of Ψ_t^m are uniformly bounded at $t\to 0$. By standard Sobolev embedding, we may pass $\Psi_t\to\Psi_0$ for $t\to 0$ in $C^{1,\alpha}$ for some $\alpha>0$ and a $C^{1,\alpha}$ diffeomorphism Ψ_0 of M. This completes the proof because $\breve g(t)\to g_0$ as $t\to 0$. □

As an application of Theorem 1.2, we will prove Corollary 1.1, which is a rigidity concerning domains inside sphere. To make our statement precise, for a domain Ω in a Riemannian manifold, we shall adopt the convention $H=\mathrm{tr}(\nabla\nu)$ where ν is the outward unit normal along $\partial\Omega$ so that the mean curvature of the boundary of unit ball in $\mathbb{R}^n$ is $n-1$. Now we are ready to prove Corollary 1.1.

Proof of Corollary 1.1. The strategy is similar to that in [6, Theorem B]. We first consider the general case. Consider the metric g on $\mathbb{S}^n$ defined by

$$g=\begin{cases} g_0 & \text{on} \quad \overline{\Omega};\\ h & \text{on} \quad \mathbb{S}^n\setminus\Omega.\end{cases}\tag{4.14}$$

where h is the standard metric on $\mathbb{S}^n$. With this identification, the metric g is Lipschitz on M and [21, Proposition 5.1] applies to show that $\mathcal{R}(g)\geq n(n-1)$ in the sense of distribution. Hence g satisfies the assumptions of Theorem 1.2 by [18, Corollary 1.2] with f being the identity map. Hence, $g=h$ and so $g_0=h$ on Ω.

If Ω is the hemisphere, then we have $H(g)\geq 0$. Now we instead define g by taking the reflected metric on $\mathbb{S}^n\setminus\Omega$. In this case, g is also Lipschitz and satisfies $\mathcal{R}(g)\geq n(n-1)$ in the sense of distribution by [21, Proposition 5.1]. The conclusion follows using Theorem 1.2. □

References

[1] C. Bär; B. Hanke, *Boundary conditions for scalar curvature*, Vol. 2, 325–377, World Sci. Publ., Hackensack, NJ, [2023].

[2] R. Bamler, *A Ricci flow proof of a result by gromov on lower bounds for scalar curvature*, Math. Res. Lett. 23 (2016), no. 2, 325–337.

[3] P. Burkhardt-Guim, *Pointwise lower scalar curvature bounds for C^0 metrics via regularizing Ricci flow*, Geom. Funct. Anal. 29 (2019), no. 6, 1703–1772.

[4] A. Burtscher, *Length structures on manifolds with continuous Riemannian metrics.* New York J. Math., 21:273–296, 2015.

[5] E. Calabi; P. Hartman, *On the smoothness of isometries*, Duke Math. J. 37 (1970), 741–750.

[6] S. Cecchini; B. Hanke; T. Schick, *Lipschitz rigidity for scalar curvature*, arXiv:2206.11796

[7] S. Cecchini; R. Zeidler, *Scalar and mean curvature comparison via the Dirac operator*, Geom. Topol. 28 (2024), no. 3, 1167–1212.

[8] B.-L. Chen; X.-P. Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Differential Geometry 74 (2006), 119–154.

[9] B. Chow; et al, *The Ricci flow: techniques and applications, Part I. Geometric aspects.* Math. Survey and Monographs, 135, Amer. Math. Soc., Prodidence, RI, (2007).

[10] G. H. Hardy; J. E. Littlewood; G. Pólya *Inequalities*, Cambridge Univ. Press, Cambridge (1934).

[11] Y. Hu; P. Liu; Y. Shi, *Rigidity of 3D spherical caps via μ-bubbles*, Pacific J. Math. 323 (2023), no. 1, 89–114.

[12] Y. Huang; M.-C. Lee, *Scalar curvature lower bound under integral convergence*, Math. Z. 303 (2023), no. 1, Paper No. 2, 10 pp.

[13] S. Huang; L.-F. Tam, *Short time existence for harmonic map heat flow with time-dependent metrics*, J. Geom. Anal. 32 (2022), no. 12, Paper No. 287, 32 pp.

[14] S. Goette; U. Semmelmann,*Scalar curvature estimates for compact symmetric spaces*, Diff. Geom. Appl. 16, p. 65-78 (2002)

[15] R. E. Greene; H. Wu, *On the subharmonicity and plurisubharmonicity of geodesically convex functions*, Indiana Univ. Math. J. **22** (1972/73), 641–653.

[16] M. Gromov, *Dirac and Plateau billiards in domains with corners*, Cent. Eur. J. Math. 12 (2014), no. 8, 1109–1156.

[17] M. Gromov, *Four Lectures on Scalar Curvature*, Perspectives in scalar curvature. Vol. 1, 1–514, World Sci. Publ., Hackensack, NJ, [2023].

[18] W. Jiang; W. Sheng; H. Zhang, *Weak scalar curvature lower bounds along Ricci flow*, Sci. China Math. 66 (2023), no. 6, 1141–1160.

[19] H. Koch; T. Lamm, *Geometric flows with rough initial data*. Asian J. Math. 16 (2012), no. 2, 209–235.

[20] K. Kondo; M. Tanaka, *Approximations of Lipschitz maps via immersions and differentiable exotic sphere theorems*. Nonlinear Anal. 155 (2017), 219–249.

[21] D. A. Lee; P. G. LeFloch, *The positive mass theorem for manifolds with distributional curvature*, Comm. Math. Phys. 339 (2015), no. 1, 99–120.

[22] M.-C. Lee; L.-F. Tam, *Continuous metrics and a conjecture of Schoen*, arXiv:2111.05582

[23] M. Llarull, *Sharp estimates and the Dirac operator*. Math. Ann., 310(1):55–71, 1998.

[24] J. Lott, *Index theory for scalar curvature on manifolds with boundary*. Proc. Am. Math. Soc., 149(10):4451–4459, 2021.

[25] W.-X. Shi, *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. 30 (1989), no. 1, 223–301.

[26] Y. Shi; L.-F. Tam, *Scalar curvature and singular metrics*, Pacific J. Math. 293 (2018), no. 2, 427–470.

[27] M. Simon, *Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature*, Comm. Anal. Geom. 10(2002), no. 5, 10331074

(Man-Chun Lee) Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China

E-mail address: mclee@math.cuhk.edu.hk

(Luen-Fai Tam) The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.

E-mail address: lftam@math.cuhk.edu.hk