SINGULAR LIMITS OF KÄHLER-RICCI FLOW ON FANO G-MANIFOLDS

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Abstract. Let M be a Fano compactification of a semisimple complex Lie group G and ω_0 a $K \times K$-invariant metric in $2\pi c_1(M)$, where K is a maximal compact subgroup of G. Then we prove that the solution of Kähler-Ricci flow with ω_0 as an initial metric on M, is of type II, if M admits no Kähler-Einstein metrics. As an application, we found two Fano compactifications of $\mathrm{SO}_4(\mathbb{C})$ and one Fano compactification of $\mathrm{Sp}_4(\mathbb{C})$, on which the Kähler-Ricci flow will develop singularities of type II. To the authors' knowledge, these are the first examples of Ricci flow with singularities of type II on Fano manifolds in the literature.

1. Introduction

Ricci flow was introduced by Hamilton in early 1980's and preserves the Kählerian structure [17]. The Kähler-Ricci flow is simply the Ricci flow restricted to Kähler metrics. If M is a Fano manifold, that is, a compact Kähler manifold with positive first Chern class $c_1(M)$, we usually consider the following normalized Kähler-Ricci flow,

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) + \omega(t),\ \omega(0) = \omega_0, \tag{1.1}$$

where ω_0 and $\omega(t)$ denote the Kähler forms of a given Kähler metric g_0 and the solutions of Ricci flow with initial metric g_0, respectively (For simplicity, we will denote a Kähler metric by its Kähler form thereafter). It is proved in [7] that (1.1) has a global solution $\omega(t)$ for all $t \geq 0$ whenever ω_0 represents $2\pi c_1(M)$. A long-standing problem concerns the limiting behavior of $\omega(t)$ as $t \to \infty$. If M admits a Kähler-Einstein metric ω_{KE} with Kähler class $2\pi c_1(M)$, then $\omega(t)$ converges to ω_{KE} (cf. [35, 36]), but in general, $\omega(t)$ may not have a limit on M. A conjecture, referred as the Hamilton-Tian conjecture, was stated in [30] that any sequence of $(M, \omega(t))$ contains a subsequence converging to a length space $(M_\infty, \omega_\infty)$ in the Gromov-Hausdorff topology and $(M_\infty, \omega_\infty)$ is a smooth Kähler-Ricci soliton outside a closed subset S, called the singular set, of codimension at least 4. Moreover, this subsequence of $(M, \omega(t))$ converges locally to the regular part of $(M_\infty, \omega_\infty)$ in the Cheeger-Gromov topology. Recall that a Kähler-Ricci soliton on a complex manifold M is a pair (X, ω), where X is a holomorphic vector field on M and ω is a Kähler metric on M, such that

$$\mathrm{Ric}(\omega) - \omega = L_X(\omega), \tag{1.2}$$

where L_X is the Lie derivative along X. If $X = 0$, the Kähler-Ricci soliton becomes a Kähler-Einstein metric. The uniqueness theorem in [33, 34] states that a Kähler-Ricci soliton on a compact complex manifold, if it exists, must be unique modulo $\mathrm{Aut}(M)$ (In the case of Kähler-Einstein metrics, this uniqueness theorem is due to Bando-Mabuchi [6]). Furthermore, X lies in the center of the Lie algebra of a reductive part of $\mathrm{Aut}(M)$.

The Gromov-Hausdorff convergence part in the Hamilton-Tian conjecture follows from Perelman's non-collapsing result and Zhang's upper volume estimate [26, 42, 43]. More recently, there were very significant progresses on this conjecture, first by Tian and Zhang in dimension less than

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4 [37], then by Chen-Wang [9] and Bamler [5] in higher dimensions. In fact, Bamler proved a generalized version of the conjecture.

A natural problem is how regular the limit space $(M_\infty, \omega_\infty)$ is. Assuming that the Hamilton-Tian conjecture is affirmed, Tian and Zhang proved in [37] that M_∞ is a Q-Fano variety whose singular set coincides with S. They proved this by establishing a parabolic version of the partial C^0-estimate. *Is this the best regularity we can have*? In fact, there was a folklore speculation that $(M_\infty, \omega_\infty)$ is actually a smooth Ricci soliton, equivalently, $S = \emptyset$. We recall that a solution $\omega(t)$ of (1.1) is called type I if the curvature of $\omega(t)$ is uniformly bounded, otherwise, we call $\omega(\cdot, t)$ a solution of type II. By using Perelman's entropy [26], we see that in the case of type I solutions, the limit $(M_\infty, \omega_\infty)$ has to be a Kähler-Ricci soliton. Then the above folklore speculation simply means that (1.1) has no type II solutions. The second-named author believed that this speculation can not be true and raised the problem of finding a Fano manifold whose Kähler-Ricci flow develops type II singularity at ∞.

In this paper, we will show that the above folklore speculation does not hold. We will prove

Theorem 1.1. *Let G be a complex semisimple Lie group and M be a Fano G-manifold which admits no Kähler-Einstein metrics. Then any solution of Kähler-Ricci flow (1.1) on M with a $K \times K$-invariant initial metric $\omega_0 \in 2\pi c_1(M)$ is of type II, where K is a maximal compact subgroup of G.*

Here by a G-manifold, we mean a *(bi-equivariant) compactification of* G which admits a holomorphic $G \times G$-action and has an open and dense orbit isomorphic to G as a $G \times G$-homogeneous space. There are examples of G-manifolds which admit neither Kähler-Einstein metrics nor Kähler-Ricci solitons, more precisely, we will show

Theorem 1.2. *There are two* $\mathrm{SO}_4(\mathbb{C})$*-manifolds and one* $\mathrm{Sp}_4(\mathbb{C})$*-manifold on which the Kähler-Ricci flow (1.1) develops singularities of type II.*

Since $\mathrm{SO}_4(\mathbb{C})$ and $\mathrm{Sp}_4(\mathbb{C})$ are both semisimple, Theorem 1.2 is deduced directly from Theorem 1.1. Theorem 1.2 provides the first example of Fano manifolds on which the Kähler-Ricci flow develops singularity of type II and solved the problem raised by the second named author. We would also like to thank the referee for telling us that Székelyhidi and Delcroix have a related speculation on the limit behavior of Kähler-Ricci flow on those Fano manifolds (cf. [11, Page 79]).

We note that the Kähler metrics of flow (1.1) preserves $K \times K$-invariance if the initial metric ω_0 is $K \times K$-invariant. Then by the contradiction argument, the proof of Theorem 1.1 reduces to studying the geometric deformation of G-manifolds with $K \times K$-invariant metrics ω_i in the smooth topology (cf. Proposition 4.1 and Proposition 5.3). Although we shall assume the $K \times K$-invariant condition on those ω_i in proofs of both of Proposition 4.1 and Proposition 5.3, the $K \times K$-invariant condition for ω_0 in Theorem 1.1 can be removed by using a recent result for the uniqueness problem of limits of Kähler-Ricci flow in [39, 18].(cf. Theorem 6.2).

There is a way to remove the semi-simplicity condition on G in Theorem 1.1 by examining all possible Fano G-compactifications which admit Kähler-Einstein metrics or Kähler-Ricci solitons with a underlying differential structure, since the Cheeger-Gromov limit is a Kähler-Ricci soliton by Perelman's result. For examples, this can be done for $\mathrm{SO}_4(\mathbb{C})$-manifolds and $\mathrm{Sp}_4(\mathbb{C})$-manifolds in our case based on the computation of associated polytopes in [12] and [27] (also see Section 6). In fact, there are two ways to prove Theorem 1.2 by using only Proposition 4.1. The one is that the volumes of G-manifolds of $\mathrm{SO}_4(\mathbb{C})$ ($\mathrm{Sp}_4(\mathbb{C})$) are different by the volume formula (cf. [11, 21]) since volumes of corresponding ploytopes are different. Thus these Fano manifolds can not be related by jumping complex structures. The other is to check that these Fano manifolds are all K-unstable (cf. Section 6). Then the limit in the flow can not be a Kähler-Einstein manifold, to see details in the end of proof of Theorem 1.1. Hence, the flow must develop singularities of type II.

The proof of Theorem 1.1 contains two main steps: proofs of Proposition 4.1 and Proposition 5.3. In Proposition 4.1, we prove that the Cheeger-Gromov limit M_∞ of ω_i is still a G-manifold.

Our idea is to study the deformation of holomorphic vector fields induced by the group G under ω_i (cf. Proposition 3.4) and to prove that the limit vector fields induce an open G-orbit $\hat{\mathcal{O}}^0_\infty$ (cf. (4.14)). The $K \times K$-invariant condition of ω_i will play a crucial role so that the convergence of toric vector fields can control other holomorphic vector fields (cf. Lemma 4.4). In the proof of Lemma 4.4, we use a technique of partial C^0-estimate from [29, 31] to compare the original metrics ω_i with the induced metrics by the Fubini-Study metric from the Kodaira embeddings (cf. Lemma 4.2). The advantage to use the Kodaira embeddings is: there is a natural G-action on the limit space M_∞ (cf. (4.7)), which generates holomorphic vector fields on $\mathbb{C}P^N$ with a free torus action on $\hat{\mathcal{O}}^0_\infty$ as well as the diffeomorphisms between the complex submanifolds of embedding images can be controlled (cf. (4.1)).

Proposition 5.3 is a corollary of Theorem 5.2, where we prove a uniqueness result about complex structures of G-manifolds when G is semisimple. Theorem 5.2 is an independent result on the uniqueness of complex structures, even in the sense of G-equivariant deformation of G-manifolds [25]. Our proof reduces to proving a uniqueness result of complex structures on a product of toric manifolds (cf. Lemma 5.1).

The organization of this paper is as follows. In Section 2, we review an existence result of Kähler-Einstein metrics on G-manifolds by Delcroix. In Section 3, we study the deformation of holomorphic vector fields induced by the group G and prove Proposition 3.4. Theorem 1.1 is proved in Section 5, while Proposition 4.1 and Theorem 5.2 are proved in Section 4 and Section 5, respectively. At last, in Section 6, we give all Fano compactifications of $\mathrm{SO}_4(\mathbb{C})$ and $\mathrm{Sp}_4(\mathbb{C})$.

Note. The preprint of this paper was first posted in the summer of 2018 [22]. After that, we find the assumption that G is semisimple can be removed in Theorem 1.1. In the appendix of paper, we will give an analytic proof of Proposition 5.3 without this assumption and so we get the improvement of Theorem 1.1 (cf. Theorem 7.3).

2. Preliminaries on G-manifolds

In this paper, we always assume that G is a reductive Lie group which is a complexification of compact Lie group K. Let $T^{\mathbb{C}}$ be an r-dimensional maximal complex torus of G with its Lie algebra $\mathfrak{t}^{\mathbb{C}}$ and $\mathfrak{M}$ the group of characters of $\mathfrak{t}^{\mathbb{C}}$. Denote the root system of $(G, T^{\mathbb{C}})$ in $\mathfrak{M}$ by Φ and choose a set of positive roots by Φ_+. Then each element in Φ can be regarded as the one of $\mathfrak{a}^*$, where $\mathfrak{a}^*$ is the dual of the non-compact part $\mathfrak{a}$ of $\mathfrak{t}^{\mathbb{C}}$.

2.1. **Local holomorphic coordinates.**

In this subsection, we recall local holomorphic coordinates on G used in [11]. By the standard Cartan decomposition, we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{t}^{\mathbb{C}} \oplus \left(\oplus_{\alpha\in\Phi} V_\alpha\right), \tag{2.1}$$

where $V_\alpha = \{X \in \mathfrak{g} |\ \mathrm{ad}_H(X) = \alpha(H)X,\ \forall\ H \in \mathfrak{t}^{\mathbb{C}}\}$ is the eigenspace of complex dimension 1 with respect to the root α. By [19], one can choose $X_\alpha \in V_\alpha$ such that $X_{-\alpha} = -\iota(X_\alpha)$ and $[X_\alpha, X_{-\alpha}] = \alpha^\vee$, where ι is the Cartan involution and $\alpha^\vee$ is the dual of α by the Killing form. Let $E_\alpha = X_\alpha - X_{-\alpha}$ and $E_{-\alpha} = J(X_\alpha + X_{-\alpha})$. Denote by $\mathfrak{k}_\alpha$, $\mathfrak{k}_{-\alpha}$ the real line spanned by E_α, $E_{-\alpha}$, respectively. Then we have the Cartan decomposition of Lie algebra $\mathfrak{k}$ of K,

$$\mathfrak{k} = \mathfrak{t} \oplus \left(\oplus_{\alpha\in\Phi_+} (\mathfrak{k}_\alpha \oplus \mathfrak{k}_{-\alpha})\right),$$

where $\mathfrak{t} = \mathfrak{t}^{\mathbb{C}} \cap \mathfrak{k}$ is the compact part of Lie algebra of $T^{\mathbb{C}}$. Choose a real basis $\{E^0_1, ..., E^0_r\}$ of $\mathfrak{t}$. Then $\{E^0_1, ..., E^0_r\}$ together with $\{E_\alpha, E_{-\alpha}\}_{\alpha\in\Phi_+}$ form a real basis of $\mathfrak{k}$, which is indexed by $\{E_1, ..., E_n\}$. $\{E_1, ..., E_n\}$ can be also regarded as a complex basis of $\mathfrak{g}$.

For any $g \in G$, we define local coordinates $\{z^i_{(g)}\}_{i=1,...,n}$ on a neighborhood of g by

$$(z^i_{(g)}) \to \exp(z^i_{(g)} E_i) g.$$

It is easy to see that $\theta^i|_g = dz^i_{(g)}|_g$, where θ^i is the dual of E_i, which is a right-invariant holomorphic 1-form. Thus $\wedge_{i=1}^n \left(dz^i_{(g)} \wedge dz^{\bar{i}}_{(g)}\right)|_g$ is also a right-invariant (n,n)-form, which defines a Haar measure dV_G.

For a smooth $K \times K$-invariant function Ψ on G, we define a Weyl-invariant convex function ψ on $\mathfrak{a}$ (called the associated function of Ψ [4]) by

$$\Psi(\exp(\cdot)) = \psi(\cdot): \ \mathfrak{a} \to \mathbb{R}. \tag{2.2}$$

The complex Hessian of the $K \times K$-invariant function Ψ in the above local coordinates was computed by Delcroix as follows [11, Theorem 1.2].

Lemma 2.1. *Let Ψ be a $K \times K$ invariant function on G, and ψ the associated function on $\mathfrak{a}$. Let $\Phi_+ = \{\alpha_{(1)}, ..., \alpha_{(\frac{n-r}{2})}\}$. Then for $x \in \mathfrak{a}_+ = \{x' \in \mathfrak{a} |\ \alpha(x') > 0,\ \forall\ \alpha \in \Phi_+\}$, the complex Hessian matrix of Ψ in the above coordinates is diagonal by blocks, and equals to*

$$\mathrm{Hess}_{\mathbb{C}}(\Psi)(\exp(x)) = \begin{pmatrix} \frac{1}{4}\mathrm{Hess}_{\mathbb{R}}(\psi)(x) & 0 & & 0 \\ 0 & M_{\alpha_{(1)}}(x) & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & M_{\alpha_{(\frac{n-r}{2})}}(x) \end{pmatrix}, \tag{2.3}$$

where

$$M_{\alpha_{(i)}}(x) = \frac{1}{2}\langle \alpha_{(i)}, \nabla\psi(x)\rangle \begin{pmatrix} \coth \alpha_{(i)}(x) & \sqrt{-1} \\ -\sqrt{-1} & \coth \alpha_{(i)}(x) \end{pmatrix}.$$

2.2. **Kähler-Einstein metrics on G-manifolds.** Let M be a G-manifold as a compactification of G. We call (M, L) a *polarized compactification* of G if L is a $G \times G$-linearized ample line bundle on M. In this paper, we just consider $L = K_M^{-1}$. Since M contains an r-dimensional toric manifold Z, there is an associated polytope P of Z induced by (M, L), which is a lattice polytope in the lattice $\mathfrak{M}$ [1, 2]. Let P_+ be the positive part of P defined by Φ_+ such that $P_+ = \{y \in P |\ \langle\alpha, y\rangle > 0,\ \forall\ \alpha \in \Phi_+\}$. Here $\langle\cdot,\cdot\rangle$ denotes the Cartan-Killing inner product on $\mathfrak{a}^*$. We call $W_\alpha = \{y \in \mathfrak{a}^* |\ \langle\alpha, y\rangle = 0\}$ the Weyl wall associated to $\alpha \in \Phi_+$.

Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi_+} \alpha$ be as a character in $\mathfrak{a}^*$ and Ξ the relative interior of the cone generated by Φ_+. We set a function on $\mathfrak{a}^*$ by

$$\pi(y) = \prod_{\alpha\in\Phi_+} \langle\alpha, y\rangle^2,\ y \in \mathfrak{a}^*.$$

Clearly, $\pi(y)$ vanishes on W_α for each $\alpha \in \Phi_+$. Denote by $2P_+$ a dilation of P_+ by 2. We define the barycentre of $2P_+$ with respect to the weighted measure $\pi(y)dy$ by

$$\mathrm{bar}(2P_+) = \frac{\int_{2P_+} y\pi(y)\, dy}{\int_{2P_+} \pi(y)\, dy}.$$

In [11], Delcroix proved the following the existence result of Kähler-Einstein metrics on G-manifolds.

Theorem 2.2. *Let M be a Fano G-manifold. Then M admits a Kähler-Einstein metric if and only if*

$$\mathrm{bar}(2P_+) \in 4\rho + \Xi. \tag{2.4}$$

By extending the argument for toric Fano manifolds in [40], Delcroix obtained a prior C^0-estimate for a class of real Monge-Ampère equations on the cone $\mathfrak{a}_+ \subset \mathfrak{a} = \mathbb{R}^r$ to prove Theorem 2.2, where $\mathfrak{a}_+ = \{x \in \mathfrak{a} |\ \alpha(x) > 0,\ \forall\ \alpha \in \Phi_+\}$. Another proof of Theorem 2.2 was latterly given in [21] by verifying the properness of K-energy $\mu(\phi)$ for $K \times K$-invariant Kähler potentials ϕ modulo the center $Z(G)$ of G. In fact, it was proved under (2.4) that there exist two positive constants δ, C_δ such that

$$\mu(\phi) \geq \delta \inf_{\sigma\in Z(G)} I(\phi_\sigma) - C_\delta,$$

where $I(\phi) = \int_M \phi(\omega^n - \omega_\phi^n)$ with a $K\times K$-invariant background Kähler metric $\omega \in 2\pi c_1(M)$, and ϕ_σ is an induced Kähler potential defined by

$$\omega_{\phi_\sigma} = \sigma^*(\omega + \sqrt{-1}\partial\bar\partial\phi) = \omega + \sqrt{-1}\partial\bar\partial\phi_\sigma.$$

It was also showed in [21] that (2.4) is actually a K-stability condition in terms of [30] and [13] by constructing a $\mathbb{C}^*$-action through a Weyl-invariant piece-wise rationally linear function. In particular, M is K-unstable if $\mathrm{bar}(2P_+) \notin \overline{4\rho + \Xi}$. A more general construction of $\mathbb{C}^*$-actions was also discussed in [12].

3. Deformation of holomorphic vector fields

In this section, we give a description on the deformation of holomorphic vector fields generated by G on a G-manifold M. We introduce

Definition 3.1. *Let (M, g_i) be a sequence of Riemannian metrics on a closed manifold M which has a limit (M_∞, g_∞) in sense of Cheeger-Gromov topology. A sequence of tangent vector fields (M, X_i, g_i) is called convergent to $(M_\infty, X_\infty, g_\infty)$ if there are diffeomorphisms $F_i : M_\infty \to M$ such that*

$$F_i^* g_i \xrightarrow{C^\infty} g_\infty,\ (F_i^{-1})_* X_i \xrightarrow{C^\infty} X_\infty,\ \text{on } M_\infty.$$

Remark 3.2. *Let (M, g_i') be another convergent sequence of Riemannian metrics on M with a Cheeger-Gromov limit (M_∞, g_∞'), which satisfies*

$$F_i^* g_i' \xrightarrow{C^\infty} g_\infty',$$

Then the tangent vector field (M, X_i, g_i') is also convergent to $(M_\infty, X_\infty, g_\infty')$ on M_∞.

Let σ_i be a sequence of auto-diffeomorphisms of M and (M, g_i) be a sequence of Riemannian metrics in Definition 3.1. Then

$$(\sigma_i^{-1}\cdot F_i)^*(\sigma_i^* g_i) \xrightarrow{C^\infty} g_\infty,$$

but, the sequence of $((\sigma_i^{-1}\cdot F_i)^{-1})_* X_i$ may converge to a different limit of tangent vector fields on M_∞. Thus Definition 3.1 is not intrinsic in general. However, there are some cases in which the limit of tangent vector fields does not change after holomorphic transformations.

Example 3.3. *Let M be an n-dimensional toric manifold and $X = \sum_\alpha a_\alpha \frac{\partial}{\partial z^\alpha}$ be a torus vector field in the affine coordinates $(z^1, ..., z^n)$. Then any torus action σ on M is given by*

$$z \to z + z_0'$$

for some z_0'. Thus $\sigma_* X = X$. Let σ_i be a sequence of torus actions. We consider a sequence of torus invariant Kähler metrics g_i. Suppose that (M, X, g_i) is convergent to $(M_\infty, X_\infty, g_\infty)$ in sense of Definition 3.1. Then we still get

$$(\sigma_i^{-1}\cdot F_i)^*(\sigma_i^* g_i) \xrightarrow{C^\infty} g_\infty,\ ((\sigma_i^{-1}\cdot F_i)^{-1})_* X = (F_i^{-1})_* X \xrightarrow{C^\infty} X_\infty.$$

Now we assume that X is a right-invariant holomorphic vector field on a G-manifold M as an element of Lie algebra $\mathfrak{g}$ of G with $\mathrm{im}(X) \in \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K. We choose a $K\times K$-invariant metric $\omega \in 2\pi c_1(M)$ as in Section 2. Then by the Hodge theorem, there is a real-valued smooth function f (usually called a potential of X) on M such that

$$L_X\omega = \sqrt{-1}\partial\bar\partial f.$$

Set

$$M_X^1 = \{x \in M |\ f(x) = \max_M f\} \tag{3.1}$$

and

$$M_X^2 = \{x \in M | \ f(x) = \min_M f\}. \tag{3.2}$$

The following is our main result in this section.

Proposition 3.4. *Let G be a reductive Lie group and M a Fano G-manifold with an open dense G-orbit $\mathcal{O}$. Suppose that (M, ω_i, J) is a sequence of $K \times K$-invariant metrics in $2\pi c_1(M)$ which converges to a smooth limit $(M_\infty, \omega_\infty, J_\infty)$ in the Cheeger-Gromov topology. Let M_X^1, M_X^2 be two sets as in (3.1) and (3.2), respectively. Then the following is true:*

(a) *There is a dense subset $\tilde{M}_X^1 \subset M_X^1$ such that any integral curve $\exp\{t\mathrm{re}(X)\} \cdot y_0$ generated by $\mathrm{re}(X)$ from $y_0 \in \mathcal{O}$ converges to a point in $\tilde{M}_X^1$. Similarly, $\exp\{t\mathrm{re}(-X)\} \cdot y_0$ converges to a point in a dense subset $\tilde{M}_X^2 \subset M_X^2$.*

(b) *(M, X, ω_i) converges to a non-trivial holomorphic vector field X_∞ on M_∞ such that*

$$M_{\infty,X}^1 \cup M_{\infty,X}^2 \subset M_{X_\infty} = \{x \in M_\infty | \ X_\infty(x) = 0\}, \tag{3.3}$$

where $M_{\infty,X}^1$ and $M_{\infty,X}^2$ are the limits of M_X^1 and M_X^2 in M_∞ in the Gromov-Hausdorff topology, respectively, and both of them are non-empty and disjoint.

Proof. Clearly, $M_X \subset (M \setminus \mathcal{O})$ and it can be decomposed into a union of disjointed subsets

$$M_X = \{x \in M | \ X(x) = 0\} = M_X^1 \cup M_X^2 \cup M_X^3, \tag{3.4}$$

where

$$M_X^3 = \{x \in M | \ f(x) = c \text{ is a critical value of } f, \ c \neq \max_M f \text{ or } \min_M f\}.$$

We need to show that any $\exp\{t\mathrm{re(X)}\}$-orbit in $\mathcal{O}$ converges to a point in M_X^1.

For any c above, define a level set of f in M_X^3 by

$$M_c = \{x \in M_X^3 | \ f(x) = c\}. \tag{3.5}$$

Then there is a finite set of such c such that $M_c \neq \emptyset$ since each $M_c \subset M_X$ is an analytic subvariety of M. Suppose that $M_{c_0} \neq \emptyset$ for some c_0 and $x \in M_{c_0}$ and assume that there is a base point $y_0 \in \mathcal{O}$ such that

$$\lim_{t\to\infty} \exp\{t\mathrm{re}(X)\} \cdot y_0 = x.$$

Since f is increasing along the integral curve $\exp\{t\mathrm{re}(X)\} \cdot y_0$ by the relation

$$\frac{df}{dt}(\exp\{t\mathrm{re(X)}\} \cdot y_0) = |\nabla f|^2(\exp\{t\mathrm{re}(X)\} \cdot y_0) > 0, \tag{3.6}$$

the limit

$$c_0 = \lim_{t\to\infty} f(\exp\{t\mathrm{re}(X)\} \cdot y_0) \tag{3.7}$$

is well defined. On the other hand, there is an integral curve from another base point $y' \in \mathcal{O}$ such that

$$\lim_{t\to\infty} \exp\{t\mathrm{re}(X)\} \cdot y' = x' \in M_X^1$$

and

$$f(x') = \max_M f = \lim_{t\to\infty} f(\exp\{t\mathrm{re}(X)\} \cdot y').$$

Thus we can define a class of subsets in $\mathcal{O}$ associated to numbers $c \in [c_0, \max_M f]$ by

$$\mathcal{O}_c = \{y \in \mathcal{O} | \ \lim_{t\to\infty} f(\exp\{t\mathrm{re}(X)\} \cdot y) = c\}. \tag{3.8}$$

Note that the number c in (3.8) must be a critical value of the function f. Hence, there are finitely many such numbers c as in (3.5).

Claim 1: Each $\mathcal{O}_c$ is an open set.

Let $y_0 = g_0 \in \mathcal{O}_c$ and $x_0 \in M \setminus \mathcal{O}$ be the limit of $\exp\{t\mathrm{re}(X)\} \cdot y_0$ as $t \to \infty$. Then there are two neighborhood U_{x_0} and V_{x_0} with $\overline{V_{x_0}} \subset U_{x_0}$ of x_0 such that $|f - c| \le \epsilon$ on U_{x_0} and $\exp\{t\mathrm{re}(X)\} \cdot y_0 \subset V_{x_0}$ for any $t \ge t_0$, where ϵ is a small number and t_0 is a large number. Since

$$\exp\{t\mathrm{re}(X)\} \cdot y = [\exp\{t\mathrm{re}(X)\} \cdot g_0] \cdot (g_0^{-1} \cdot g),\ \forall\ y = g \in \mathcal{O},$$

and $g_0^{-1} \cdot g$ is a smooth map on M, we see that

$$\exp\{t\mathrm{re}(X)\} \cdot y \in U_{x_0},\ \forall\ t \ge t_0, \tag{3.9}$$

as long as $\mathrm{dist}(g_0^{-1} \cdot g, \mathrm{Id}) << 1$. It follows that

$$c - \epsilon \le \lim_{t\to\infty} f(\exp\{t\mathrm{re}(X)\} \cdot y) \le c + \epsilon.$$

Thus $\lim_{t\to\infty} f(\exp\{t\mathrm{re}(X)\} \cdot y)$ must be c since there is no other critical value of f near c. **Claim 1** is proved.

By **Claim 1**, there are finitely many disjoint open subsets $\mathcal{O}_{c_j}$ such that

$$\mathcal{O} = \cup_j \mathcal{O}_{c_j}.$$

On the other hand, from (3.9), one can show that each $\Omega \cap \mathcal{O}_{c_j}$ is a closed set for any closed set $\Omega \subset \mathcal{O}$. It follows that there is only one $\mathcal{O}_{c_j}$ through $\Omega \cap \mathcal{O}$, and as a consequence, it must be $\mathcal{O}_{\max_M f}$ if $\mathcal{O}_{c_j} \cap \Omega \ne \emptyset$. Thus $c_0 = \max_M f$ and M_c in (3.5) with c defined by (3.7) must be empty. In another word, any curve $\exp\{t\mathrm{re}(X)\} \cdot y_0$ with $y_0 \in \mathcal{O}$ converges to a point in M_X^1.

Next, we show that the set

$$\tilde{M}_X^1 = \{x \in M_X^1 |\ x = \lim_{t\to\infty} \exp\{t\mathrm{re}(X)\} \cdot y,\ \text{for some } y \in \mathcal{O}\} \tag{3.10}$$

is dense in M_X^1. On contrary, if

$$\mathcal{E}_1 = \overline{\tilde{M}_X^1} \ne M_X^1, \tag{3.11}$$

then there is another closed subset $\mathcal{E}_2$, which is disjointed with $\mathcal{E}_1$, such that

$$\mathcal{E}_1 \cup \mathcal{E}_2 \subset M_X^1.$$

Note that

$$M_X^1 = \cap_{\epsilon>0} \mathcal{M}^\epsilon,$$

where

$$\mathcal{M}^\epsilon = \{x |\max_M f - \epsilon < f(x) \le \max_M f\}.$$

Thus for sufficiently small ϵ, there are two disjoint open sets $\mathcal{U}_1^\epsilon, \mathcal{U}_2^\epsilon$ such that

$$\mathcal{U}_1^\epsilon \cup \mathcal{U}_2^\epsilon \subset \mathcal{M}^\epsilon \text{ and } \mathcal{E}_i \subset \mathcal{U}_i^\epsilon\ (i = 1, 2)$$

with the property

$$f \ge \max_M f - \epsilon,\ \forall\ x \in \mathcal{U}_i^\epsilon \text{ and } f \equiv \max_M f - \epsilon,\ \forall\ x \in \partial \mathcal{U}_i^\epsilon,\ i = 1, 2.$$

On the other hand, by the monotonicity (3.6) and the definition (3.10), any integral curve generated by $\mathrm{re}(X)$ starting in each $\mathcal{U}_i$ can not leave it. Since $\mathcal{U}_2 \cap \mathcal{O} \ne \emptyset$, there is a point $y_0 \in \mathcal{U}_2$ such that

$$z_0 = \lim_{t\to\infty} \exp\{t\mathrm{re}(X)\} \cdot y_0 \in \mathcal{E}_2 \subset M_X^1,$$

which contradicts to the fact $z_0 \in \mathcal{E}_1$ by (3.10) and (3.11)! Hence, $\tilde{M}_X^1$ is a dense set of M_X^1.

Similarly, we can show that

$$\tilde{M}_X^2 = \{x \in M_X^2 |\ x = \lim_{t\to\infty} \exp(t\mathrm{re}(-X)) \cdot y,\ \text{for some } y \in \mathcal{O}\}$$

is a dense set of M_X^2. Part (a) of the proposition is proved.

To prove Part (b) in the proposition, we write ω_i as $\omega_i = \omega + \sqrt{-1}\partial\bar\partial\varphi_i$ for some φ_i. Then

$$L_X(\omega_i) = \sqrt{-1}\partial\bar\partial f_i,$$

where $f_i = f + X(\varphi_i)$. It follows that

$$X = g^{k\bar{l}}(\omega_i)(f + X(\varphi_i))_{\bar{l}}\frac{\partial}{\partial z_k}. \tag{3.12}$$

Thus

$$M_X = \{x \in M|\ \nabla f(x) = 0\} = \{x \in M|\ \nabla f_i(x) = 0\}.$$

Hence, $\max_M f_i = \max_M f$ and $\min_M f_i = \min_M f$ by (3.12). In particular,

$$\mathrm{osc}_M f_i = \mathrm{osc}_M f. \tag{3.13}$$

Let h_i be the Ricci potential of ω_i which is normalized by

$$\int_M e^{h_i}\omega_i^n = (2\pi c_1(M))^n. \tag{3.14}$$

Note that ω_i is $K \times K$-invariant. Then by adding a constant f_i satisfies the following equation (cf. [16, 34]),

$$\Delta_i f_i + f_i + \langle \partial f_i, \partial h_i\rangle = \Delta_i f_i + f_i + \frac{1}{2}\langle \nabla f_i, \nabla h_i\rangle = 0, \tag{3.15}$$

where $\triangle_i$ is the Laplace operator associated to ω_i. Since h_i satisfies equation $\Delta_i h_i = R_i - n$, where R_i is the scalar curvature of ω_i, h_i is C^k-uniformly bounded for any k. Thus f_i is C^k-uniformly bounded associated to the metric ω_i for any k, and so f_i converges subsequently to a smooth function f_∞ on M_∞. We need to show that

$$\mathrm{osc}_{M_\infty} f_\infty = \mathrm{osc}_M f. \tag{3.16}$$

Case 1): There is a uniform constant δ_0 such that

$$\mathrm{dist}_{\omega_i}(M_X^1, M_X^2) \geq \delta_0,\ \forall\ i.$$

Then there are Gromov-Hausdorff limits $M^1_{\infty,X}$ and $M^2_{\infty,X}$ of M^1_X and M^2_X in M_∞, respectively, such that $M^1_{\infty,X} \cap M^2_{\infty,X} = \emptyset$. Thus there are neighborhoods B_1 and B_2 of $M^1_{\infty,X}$ and $M^2_{\infty,X}$ in M_∞, respectively, such that $B_1 \cap B_2 = \emptyset$. By the convergence of f_i, we have

$$\max_{M_\infty} f_\infty = \sup_{B_1} f_\infty = \max_M f = f_\infty(x),\ \forall\ x \in M^1_{\infty,X}, \tag{3.17}$$

and

$$\min_{M_\infty} f_\infty = \inf_{B_2} f_\infty = \min_M f = f_\infty(x),\ \forall\ x \in M^2_{\infty,X}. \tag{3.18}$$

Hence, we get (3.16).

Case 2):

$$\mathrm{dist}_{\omega_i}(M_X^1, M_X^2) \to 0,\ \text{as } i \to +\infty.$$

This means that

$$M^1_{\infty,X} \cap M^2_{\infty,X} \neq \emptyset.$$

Thus there is a point $p \in M^1_{\infty,X} \cap M^2_{\infty,X}$ and two sequences of points $\{p_i\}$ and $\{q_i\}$ in M^1_X and M^2_X, respectively, such that

$$p_i \to p \text{ and } q_i \to p.$$

As a consequence, there is an open set U_p around p in M_∞ and two sequences of open sets U_{p_i} and U_{q_i} around p_i and q_i in M, respectively, such that

$$U_{p_i} \to U_p \text{ and } U_{q_i} \to U_p$$

in the topology of Gromov-Hausdorff. By the convergence of f_i, we get

$$f_\infty(p) \equiv \min_M f \text{ and } f_\infty(p) \equiv \max_M f.$$

This is impossible! Hence, Case 2) is impossible and we prove (3.16) by Case 1).

By (3.16), we see that $\nabla f_\infty \neq 0$. Moreover, by (3.15), f_∞ satisfies

$$\Delta_\infty f_\infty + f_\infty + \langle \partial f_\infty, \partial h_\infty\rangle = 0, \tag{3.19}$$

where Δ_∞ is the Laplace operator associated to ω_∞. Thus $g^{k\bar{l}}(\omega_\infty)(f_\infty)_{\bar{l}}\frac{\partial}{\partial z_k}$ defines a holomorphic vector field X_∞ on M_∞. On the other hand, by the convergence of ω_i, there are diffeomorphisms $F_i : M_\infty \to M$ such that

$$F_i^*\omega_i \xrightarrow{C^\infty} \omega_\infty, \text{ on } M_\infty.$$

Since ∇f_i is C^k-uniformly bounded associated to ω_i, we also get

$$(F_i^{-1})_*(g^{k\bar{l}}(\omega_i)(f_i)_{\bar{l}}\frac{\partial}{\partial z_k}) \xrightarrow{C^\infty} g^{k\bar{l}}(\omega_\infty)(f_\infty)_{\bar{l}}\frac{\partial}{\partial z_k}, \text{ on } M_\infty, \tag{3.20}$$

which is the limit of (M, X, ω_i) in the sense of Definition 3.1.

By Case 1), it is clear that $M^1_{\infty,X} \cap M^2_{\infty,X} = \emptyset$. Furthermore,

$$\nabla f_\infty \equiv 0, \ \forall x \in M_{\infty,X} = M^1_{\infty,X} \cup M^2_{\infty,X}.$$

As a consequence,

$$M_{\infty,X} \subset M_{X_\infty}.$$

Part (b) of Proposition 3.4 is proved.

□

4. Deformation of G-structures

In this section, we prove the following proposition on preservation of G-structures on limits in the Cheeger-Gromov topology.

Proposition 4.1. *Let $(M_\infty, \omega_\infty, J_\infty)$ be a smooth limit of $K \times K$-invariant metrics (M, ω_i, J) in $2\pi c_1(M)$ on a Fano G-manifold M in the Cheeger-Gromov topology as in Proposition 3.4. Then $(M_\infty, \omega_\infty)$ is also a Fano G-manifold.*

We use the Kodaira embedding to prove the proposition. This means that there exists an integer m such that M can be embedded into $\mathbb{C}P^N$ by a unitary orthogonal basis $\{s^i_A, \ i = 1, ..., N+1\}$ of holomorphic sections of K_M^{-m} with L^2-norm induced by ω_i. Note that M_∞ is diffeomorphic to M. Thus by the convergence of ω_i, we see that there is a uniform integer N independent of i such that the following properties are satisfied:

1) There is a holomorphic embedding Φ_i from M to $\mathbb{C}P^N$ for each (M, ω_i).

2) There is a holomorphic embedding Φ_∞ from M_∞ to $\mathbb{C}P^N$ for the Kähler manifold $(M_\infty, \omega_\infty)$.

3) The image $\Phi_i(M) = \hat{M}_i$ converges to the image $\Phi_\infty(M_\infty) = \hat{M}_\infty$ according to the topology of complex submanifolds.

The above properties 1)-3) come from the partial C^0-estimate as in [14, 32, 31] (or simply a variant of Tian's almost isometry theorem [28]) for a sequence of Kähler-Einstein metrics or conical Kähler-Einstein metrics. We note that the curvature of ω_i in Proposition 4.1 is uniformly bounded and the sequence $\{\omega_i\}$ can be regarded as a special case in their papers. As a consequence, the norm of sections s^i_A with respect to ω_i as functions on M_∞ is uniformly C^∞-bounded and so the basis $\{s^i_A\}$ is C^∞-convergent to a basis of $\{s^\infty_A\}$ on $H^0(M_\infty, K_{M_\infty}^{-m})$. Thus by 3), we can choose a covering $\{U_\alpha\}$ of $\hat{M}_\infty$ with local holomorphic coordinates and diffeomorphisms $F_i : \hat{M}_\infty \to \hat{M}_i$ such that for each $\hat{M}_i$ there is a covering $\{U^i_\alpha \subset F_i(U_\alpha)\}$ with local holomorphic coordinates and uniform norms of transformation functions. Moreover,

$$F_i^*\hat{\omega}_i \xrightarrow{C^\infty} \hat{\omega}_\infty, \tag{4.1}$$

where $\hat{\omega}_i = \frac{1}{m}\omega_{FS}|_{\hat{M}_i}$ and $\hat{\omega}_\infty = \frac{1}{m}\omega_{FS}|_{\hat{M}_\infty}$.

In the following, we compare ω_i with the induced metric $\hat{\omega}_i$. Write

$$(\Phi_i^{-1})^*\omega_i = \hat{\omega}_i + \sqrt{-1}\partial\bar{\partial}\varphi_i$$

for some Kähler potential φ_i in $\hat{M}_i$. Then by using the regularity theory of complex Monge-Ampère equation, we prove

Lemma 4.2. *There is a uniform constant $A > 0$ such that*

$$A^{-1}(\Phi_i^{-1})^*\omega_i \le \hat\omega_i \le A(\Phi_i^{-1})^*\omega_i, \text{ on } \hat M_i. \tag{4.2}$$

Moreover, for any integer $k > 0$ it holds

$$\|\varphi_i\|_{C^k(\hat M_i)} \le A_k, \tag{4.3}$$

where A_k is a uniform constant independent of i.

Proof. Let h_i and $\hat h_i$ be the Ricci potentials of ω_i and $\tilde\omega_i$, respectively. Then by the convergence of ω_i and $\hat\omega_i$, both of h_i and $\tilde h_i$ are uniformly bounded. Moreover, φ_i (maybe different by a constant) satisfies the following complex Monge-Ampère equation,

$$(\hat\omega_i + \sqrt{-1}\partial\bar\partial\varphi_i)^n = e^{-\varphi_i + \hat h_i - h_i}\hat\omega_i^n, \text{ in } \hat M_i. \tag{4.4}$$

By the partial C^0-estimate and gradient estimate of $\{s_\alpha^i\}$ (cf. [31]), we know that

$$|\varphi_i| \le C \text{ and } \hat\omega_i \le A(\Phi_i^{-1})^*\omega_i.$$

Thus by (4.4) we also get

$$A^{-1}(\Phi_i^{-1})^*\omega_i \le \hat\omega_i$$

possibly by choosing a bigger A. Hence, (4.2) is true.

Note that

$$\Delta_{\omega_i} h_i = R_i - n, \text{ in } M,$$

where R_i is the scalar curvature of ω_i, which is uniformly bounded. By (4.2), we have

$$|\Delta_{\hat\omega_i} h_i| \le C, \text{ in } \hat M_i.$$

It follows that

$$\|h_i\|_{C^{1,\alpha}(\hat M_i)} \le C_1.$$

Hence, the regularity theory of (4.4) (cf. [41]) implies that

$$\|\varphi_i\|_{C^{3,\alpha}(\hat M_i)} \le C_3.$$

Repeating the above argument, we will get (4.3).

□

By (4.1) together with Lemma 4.2, we get

$$F_i^*((\Phi_i^{-1})^*\omega_i) \xrightarrow{C^\infty} (\Phi_\infty^{-1})^*\omega_\infty. \tag{4.5}$$

Let $\{E_1, ..., E_n\}$ be a basis of the Lie algebra $\mathfrak{g}$. Then the left (right) action of G induces a space $\mathrm{span}\{e_1, ..., e_n\}$ of holomorphic vector fields with $\mathrm{im}(e_a) \in \mathfrak{k}$ on M, and so the holomorphism Φ_i induces a space $\mathrm{span}\{\hat e_1^i, ..., \hat e_n^i\}$ of holomorphic vector fields on $\hat M_i$. Since by the Part (b) of Proposition 3.4, for each a, (e_a, ω_i) converges to a holomorphic vector field $(e_a^\infty, \omega_\infty)$ on M_∞, $(\hat e_a^i, (\Phi_i^{-1})^*\omega_i)$ converges to a holomorphic vector field $(\hat e_a^\infty, (\Phi_\infty^{-1})^*\omega_\infty)$ on $\hat M_\infty$. In fact, by (4.5),

$$(F_i^{-1})_*\hat e_a \xrightarrow{C^\infty} \hat e_a^\infty = (\Phi_\infty)_* e_a^\infty, \ \forall\, a = 1, ..., n. \tag{4.6}$$

Hence, by Remark 3.2, it follows that $(\hat e_a^i, \hat\omega_i)$ converges to a holomorphic vector field $(\hat e_\alpha^\infty, \hat\omega_\infty)$ on $\hat M_\infty$. As a consequence, for each $\hat e_a^i$, the holomorphic coefficients of $\hat e_a^i$ are uniformly bounded under local holomorphic coordinates on U_α^i.

For any $g \in G$, the map Φ_i induces a left (right) action on

$$H^0(M, K_M^{-m}) = \mathrm{span}\{s_A^i, \ i = 1, ..., N+1\},$$

thus G can be regarded as a subgroup of $\mathrm{PGL}(N+1, \mathbb{C})$ induced by the map Φ_i. Those subgroups induced by different ϕ_i are conjugate to each other by automorphisms induced by $\sigma_{ij}^{-1} \cdot G \cdot \sigma_{ij}$, where $\sigma_{ij} \in \mathrm{PGL}(N+1, \mathbb{C})$ induced by the Kodaira embeddings ϕ_i and ϕ_j. Without confusion, we still denote by G each of such subgroups which may vary on ϕ_i. Furthermore, any one-parameter subgroup σ_t generated by $\mathrm{Im}(e_a^i)$ induces a family of isomorphisms on $H^0(M, K_M^{-m})$. Taking the derivative on t, we get a lifting holomorphic vector field of $\hat{e}_a^i$ on $\mathbb{C}P^N$.

For any $\hat{x}_\infty \in \hat{M}_\infty$ and any sequence $\{\hat{x}_i\}$ such that

$$\hat{x}_i \in \hat{M}_i \text{ and } \hat{x}_i \to \hat{x}_\infty,$$

we define

$$g(\hat{x}_\infty) = \lim_{i\to\infty} g(\hat{x}_i) \in \hat{M}_\infty, \ \forall\, g \in G. \tag{4.7}$$

Using the convergence of holomorphic vector fields $(\hat{e}_a^i, \hat{\omega}_i)$ on M, one can easily show that the limit $g(\hat{x}_\infty)$ is independent of the choice of $\{\hat{x}_i\}$. Moreover, we have

$$g'(g(\hat{x}_\infty)) = (g' \cdot g)(\hat{x}_\infty), \ \forall\, g, g' \in G. \tag{4.8}$$

Thus, by (4.7) we define a left G-action on $\hat{M}_\infty$, which induces the one on M_∞ by the relation $\Phi_\infty \cdot g = g \cdot \Phi_\infty$ through the holomorphism Φ_∞. Similarly, we can define a right G-action on M_∞ and so get a $G \times G$-action on M_∞.

The following lemma shows that G acts on $\hat{M}_\infty$ effectively. Namely, $\{\hat{e}_1^\infty, ..., \hat{e}_n^\infty\}$ becomes a basis of Lie algebra of G acting on $\hat{M}_\infty$.

Lemma 4.3. *$\hat{e}_1^\infty, ..., \hat{e}_n^\infty$ are all linearly independent on $\hat{M}_\infty$.*

Proof. It suffices to prove that $e_1^\infty, ..., e_n^\infty$ are all linearly independent M_∞. In fact, if

$$\sum_\alpha a_\alpha e_\alpha^\infty \equiv 0, \text{ for some } a_\alpha \neq 0, \tag{4.9}$$

then, by the Part (b) of Proposition 3.4, the vector field $\sum_\alpha a_\alpha e_\alpha^i$ converges to a nontrivial holomorphic vector field, which should be $\sum_\alpha a_\alpha e_\alpha^\infty$ on M_∞. This is a contradiction with (4.9). □

Let $\mathcal{O}$ be an open dense G-orbit in M. Since M has finitely many $G \times G$-orbits [1, 2], there are basis points $x_\delta \in M \backslash \mathcal{O}$, $\delta = 1, ..., k$, such that

$$M = \mathcal{O} \cup_\delta (G \times G)x_\delta. \tag{4.10}$$

Note that the closure of each $G\times G$-orbit $(G\times G)x_\delta$ is a smooth algebraic variety whose dimension is less than n. Then up to a subsequence, the closure of $\Phi_i((G \times G)x_\delta)$ converges to an algebraic limit in $\mathbb{C}P^N$. As a consequence, $\Phi_i(M\backslash\mathcal{O})$ has an algebraic limit $D\hat{M}_\infty$ in $\hat{M}_\infty \subset \mathbb{C}P^N$.

For any i and $g \in G \times G$, we have

$$g \circ \Phi_i = \Phi_i \circ g. \tag{4.11}$$

Then by (4.7) and (4.10), for any $\hat{x}_\infty \in \hat{M}_\infty$ there is a sequence of $g_i \in G \times G$ such that

$$\hat{x}_\infty = \lim_i g_i \cdot \Phi_i(x_0), \text{ or } \hat{x}_\infty = \lim_i g_i \cdot \Phi_i(x_\delta), \text{ for some } \delta \in \{1, ..., k\}, \tag{4.12}$$

where $x_0 \in \mathcal{O}$. We define an open set in $\hat{M}_\infty$ by $\hat{\mathcal{O}}_\infty = \hat{M}_\infty \setminus D\hat{M}_\infty$. Thus for any $\hat{x}_{0,\infty} \in \hat{\mathcal{O}}_\infty$, there exists a $\delta_0 > 0$ such that

$$\mathrm{dist}(\hat{x}_{0,\infty}, D\hat{M}_\infty) \geq 2\delta_0 > 0.$$

It follows that there are $\hat{x}_{0,i} \in \hat{M}_i$ such that $\mathrm{dist}(\hat{x}_{0,i}, \Phi_i(M\backslash\mathcal{O})) \geq \delta_0 > 0$.

Since G induces an action on $H^0(M, K_M^{-m})$ for each i through an orthonormal basis corresponding to ϕ_i, by taking limit as in (4.7), G induces an action on $H^0(M_\infty, K_{M_\infty}^{-m}) = \mathrm{span}\{s_1^\infty, ..., s_{N+1}^\infty\}$. It follows that each holomorphic vector field $\hat{e}_a^\infty$, where $a = 1, \cdots, n$, can be lifted to a vector field on $\mathbb{C}P^N$ as $\hat{e}_a^i$ does. Namely, $\{\hat{e}_1^\infty, .., \hat{e}_n^\infty\}$ can be induced by a basis of the Lie algebra of G which acts on $\hat{M}_\infty$. In particular, a maximal r-dimensional torus subgroup $T^{\mathbb{C}}$ of G acting on $\hat{M}_\infty$, which generated by a basis $\{X_1, ..., X_r\}$ of $\mathfrak{a}$, induces an r-dimensional torus subgroup $\tilde{T}^{\mathbb{C}}$ of $\mathrm{PSL}(N + 1, \mathbb{C})$ generated by an r-dimensional torus vector fields on $\mathbb{C}P^N$.

Let $\tilde{W}_1, ..., \tilde{W}_{N+1}$ be the $(N+1)$ hyperplanes in $\mathbb{C}P^N$ where $\tilde{T}^{\mathbb{C}}$ does not act freely. Then for any induced holomorphic vector field $\tilde{X}$ of X in $\mathfrak{t}^{\mathbb{C}}$ on $\hat{M}_\infty$, it holds

$$\{\hat{x} \in \hat{M}_\infty |\ \tilde{X}(\hat{x}) = 0\} \subset \cup_\alpha \tilde{W}_\alpha. \tag{4.13}$$

Set

$$\hat{\mathcal{O}}^0_\infty = \hat{\mathcal{O}}_\infty \backslash (\cup_\alpha \tilde{W}_\alpha). \tag{4.14}$$

To prove that (M_∞, J_∞) is a G-manifold, it suffices to show that $\hat{\mathcal{O}}^0_\infty$ is isomorphic to G. Without loss of generality, we may assume that $\hat{x}_{0,\infty} \in \hat{\mathcal{O}}^0_\infty$ above. Thus it reduces to proving that G acts on $\hat{x}_{0,\infty}$ freely and $G \cdot \hat{x}_{0,\infty} = \hat{\mathcal{O}}^0_\infty$.

The following key lemma shows that any holomorphic vector field induced by G is non-degenerate on $\hat{\mathcal{O}}^0_\infty$.

Lemma 4.4. *For any induced holomorphic vector field $\tilde{X}$ of $X \in \mathfrak{g}$ on $\hat{M}_\infty$, it holds*

$$\tilde{X}(\hat{x}_\infty) \neq 0,\ \forall\ \hat{x}_\infty \in \hat{\mathcal{O}}^0_\infty. \tag{4.15}$$

Proof. Let $\{E_1, ..., E_n\}$ be a basis of $\mathfrak{g}$ such that $X_1 = JE_1, ..., X_r = JE_r \in \mathfrak{a}$, and $E_{a'} = E_\alpha$ and $E_{a'+\frac{n-r}{2}} = E_{-\alpha}$, $a' = r+1, ..., \frac{n+r}{2}$, which satisfy $V_\alpha \oplus V_{-\alpha} = \mathrm{span}\{E_\alpha, E_{-\alpha}\}$ as in (2.1), where V_α are eigenvectors associated to the positive roots $\alpha \in \Phi_+$. Let $\{e_1, ..., e_n\}$ be holomorphic vector fields with $\mathrm{im}(e_a) \in \mathfrak{k}$ on M induced by $\{E_1, ..., E_n\}$. Then the induced holomorphic vector fields $\{\hat{e}^i_1, ..., \hat{e}^i_n\}$ on $\hat{M}_i$ by Φ_i converge to a basis $\{\hat{e}^\infty_1, ..., \hat{e}^\infty_n\}$ of holomorphic vector fields on $\hat{M}_\infty$ as in (4.6).

Let

$$\mathcal{O}^0_\infty = \Phi^{-1}_\infty(\hat{\mathcal{O}}^0_\infty) \subset M_\infty.$$

Note that

$$e^\infty_a = (\Phi^{-1}_\infty)_* \hat{e}^\infty_a,\ a = 1, ..., n.$$

Then (4.15) is equivalent to

$$\bar{X}(x_\infty) \neq 0,\ \forall\ x_\infty \in \mathcal{O}^0_\infty, \tag{4.16}$$

where $\bar{X} \in \mathrm{span}\{e^\infty_1, ..., e^\infty_n\}$ is any nontrivial vector field. By (4.13), we have already known that (4.16) holds for any non-trivial $\bar{X} \in \mathrm{span}\{e^\infty_1, ..., e^\infty_r\}$. More precisely, we have

$$\begin{aligned}\omega_\infty(\bar{X}, \bar{X})(x_\infty) &= (\Phi^{-1}_\infty)^* \omega_\infty(\tilde{X}, \tilde{X})(\hat{x}_\infty)\\ &\geq c\hat{\omega}_\infty(\tilde{X}, \tilde{X})(\hat{x}_\infty) \neq 0,\end{aligned} \tag{4.17}$$

where $\tilde{X} = (\Phi_\infty)_* \bar{X} \in \mathrm{span}\{\hat{e}^\infty_1, ..., \hat{e}^\infty_r\}$. In the following, we want to show that (4.17) implies (4.16).

By (4.6) and (4.17), there are a constant $c' > 0$ and a small ball $B_\delta(\hat{x}_\infty) \subset \mathbb{C}P^N$ near $\hat{x}_\infty$ such that

$$(\hat{\omega}_i(\hat{e}^i_a, \hat{e}^i_b))_{r\times r} \geq \frac{1}{2}(\hat{\omega}_\infty(\hat{e}^\infty_a, \hat{e}^\infty_b))_{r\times r} \geq c'\mathrm{Id},\ \text{ in } B_\delta(\hat{x}_\infty) \cap \hat{M}_i.$$

Write

$$\omega_i = \sqrt{-1}\partial\bar{\partial}\psi^i = \sqrt{-1}\partial\bar{\partial}\Psi^i$$

for some convex function ψ^i on $\mathfrak{a}$ as in (2.2). Then by Lemma 4.2, we get

$$\begin{aligned}(\psi^i_{ab}) &= (\omega_i(e_a, e_b))\\ &= ((\Phi^{-1}_i)^* \omega_i(\hat{e}^i_a, \hat{e}^i_b))\\ &\geq A^{-1}(\hat{\omega}_i(\hat{e}^i_a, \hat{e}^i_b)) \geq \delta_0 \mathrm{Id},\ \text{ in } \Phi^{-1}_i(B_\delta(\hat{x}_\infty) \cap \hat{M}_i),\end{aligned} \tag{4.18}$$

where $\delta_0 > 0$ is a uniform constant. Note that $|e_a|_{\omega_i}$ is uniformly bounded as in the proof of (3.20). Hence, we derive

$$\delta_0 \mathrm{Id} \leq (\psi^i_{ab}) \leq \frac{1}{\delta_0}\mathrm{Id},\ \text{ in } \Phi^{-1}_i(B_\delta(\hat{x}_\infty) \cap \hat{M}_i),$$

as long as δ_0 is small enough. Choose $\hat{x}_i \in \hat{M}_i \to \hat{x}_\infty = \Phi_\infty(x_\infty) \in \hat{M}_\infty$ and let $x_i = \Phi_i^{-1}(\hat{x}_i) \in M$. Then $x_i \to x_\infty \in M_\infty$ in the Gromov-Hausdroff topology. Note that $B_\delta(\hat{x}_\infty) \cap \hat{M}_i$ contains a uniform small geodesic ball centered at $\hat{x}_i$ associated to the metric $\hat{\omega}_i$. Therefore, again by Lemma 4.2, it is easy to see that there is a sequence of open sets $U_{x_i} \subset \mathcal{O}$, each of which contains an ϵ-geodesic ball $(B_\epsilon(x_i), \omega_i)$ centered at x_i associated to ω_i, where the radius ϵ is a uniform small constant, such that

$$\delta_0 \mathrm{Id} \leq (\psi^i_{ab}) \leq \frac{1}{\delta_0}\mathrm{Id}, \text{ in } U_{x_i}. \tag{4.19}$$

Claim 1: There is a uniform small constant ϵ_0 such that $\Delta_{2\epsilon_0} \subset (B_\epsilon(x_i), \omega_i) \subset U_{x_i}$, where $\Delta_{2\epsilon_0} = \{z = (z^1, ..., z^n) |\ |z^l - x_i^l| < 2\epsilon_0\}$ is an $2\epsilon_0$-square of dimension n centered at x_i in the local coordinates $\{z^l_{(g)}\}_{l=1,...,n}$ on $\mathcal{O}$ introduced in Section 2.1.

In fact, as in the proof of (3.20), we see that each of

$$|e_a|^2_{\omega_i} = \omega_i(\frac{\partial}{\partial z^a}, \overline{\frac{\partial}{\partial z^a}})\ (a = 1, ..., n) \tag{4.20}$$

is uniformly bounded. In particular, ω_i as a metric tensor is uniformly bounded under the local coordinates $\{z^l_{(g)}\}_{l=1,...,n}$ in $(B_\epsilon(x_i), \omega_i)$. Then **Claim 1** follows immediately.

Claim 2: There is a uniform constant $c_0 > 0$ such that

$$c_0 \leq \langle \alpha, \nabla\psi^i(x)\rangle \coth\alpha(x) \leq \frac{1}{c_0},\ \forall\ x \in \Delta_{\epsilon_0}. \tag{4.21}$$

By (2.3) and (4.20), we have

$$|e_\alpha|^2_{\omega_i} = \langle \alpha, \nabla\psi^i(x)\rangle \coth\alpha(x). \tag{4.22}$$

Then the upper bound of (4.21) is true. Thus it suffices to get the lower bound.

Case 1: $\alpha(x) << 1$. Then there exists $x' \in W_\alpha$ on a Weyl wall W_α such that $x = x' + t\alpha^*$ for some small $t \geq 0$, where $\alpha^* = (\alpha^1, ..., \alpha^r) \in \mathfrak{a}$ is the dual point of α associated to the Killing inner product. By the fact $\langle \alpha, \nabla\psi^i(x')\rangle = 0$, it follows that

$$\langle \alpha, \nabla\psi^i(x)\rangle = \alpha^a\alpha^b\psi^i_{ab}((x' + t'\alpha^*))t,$$

where $t' \leq t$. Since $\psi^i_{ab}((x' + t'\alpha))$ satisfies (4.19), by **Claim 1**, we get

$$\langle \alpha, \nabla\psi^i(x)\rangle \coth\alpha(x) = \alpha^a\alpha^b\psi^i_{ab}((x' + t'\alpha^*))t\coth(|\alpha|^2 t) \geq c_0.$$

Case 2: there exists a $\delta_0' > 0$ such that $\alpha(x) \geq \delta_0'$ for any $x \in \Delta_{2\epsilon_0}$ (ϵ_0 may be replaced by a smaller number if necessary). We need to prove

$$\langle \alpha, \nabla\psi^i(x)\rangle \geq c_0',\ \forall\ x \in \Delta_{\epsilon_0}, \tag{4.23}$$

where $c_0' > 0$ is a small uniform constant. On contrary, there is a sequence of $y_i \in \Delta_{\epsilon_0}$ such that

$$\langle \alpha, \nabla\psi^i(y_i)\rangle \to 0.$$

Then we choose $z_i = y_i - t_0\alpha \in \Delta_{2\epsilon_0}$ such that

$$\alpha(z_i) \geq \frac{\delta_0'}{2}.$$

Thus $z_i \in \mathfrak{a}_+$. On the other hand, we have

$$\langle \alpha, \nabla\psi^i(z_i)\rangle = \langle \alpha, \nabla\psi^i(y_i)\rangle - \alpha^a\alpha^b\psi^i_{ab}(y_i - t'\alpha^*)t_0,$$

where $t' \leq t_0$. By **Claim 1** and (4.19), we get

$$\langle \alpha, \nabla\psi^i(z_i)\rangle < 0,\ i >> 1.$$

This is impossible since $z_i \in \mathfrak{a}_+$! Thus (4.23) is true and we also get the lower bound of (4.21) since $\coth\alpha(x) > 1$. **Claim 2** is proved.

Now we can complete the proof of Lemma 4.4. By Lemma 2.1, we have

$$\langle e_{a'}, e_{b'}\rangle_{\omega_i} \equiv 0,\ a' \neq b',\ a', b' = r+1, ..., \frac{n+r}{2},$$
$$\langle e_{a'}, e_{a}\rangle_{\omega_i} \equiv 0,\ a = 1, ..., r,\ a' = r+1, ..., n. \tag{4.24}$$

Thus it remains to check that for each α it holds

$$\bar{X}(x_\infty) \neq 0,\ \forall\ \bar{X} \in \mathrm{span}\{e_\alpha^\infty, e_{-\alpha}^\infty\}. \tag{4.25}$$

By (4.22) and **Claim 2**, we see that

$$e_\alpha^\infty(x_\infty) \neq 0 \text{ and } e_{-\alpha}^\infty(x_\infty) \neq 0. \tag{4.26}$$

Hence, we need to show that e_α^∞ and $e_{-\alpha}^\infty$ are independent.

Case 1: $\alpha(x_i) \to \infty$. Then

$$\alpha(x) \to \infty,\ \forall\ x \in \Delta_{2\epsilon_0}.$$

Moveover, by (4.23), there exists a $c_0' > 0$ such that

$$\langle \alpha, \nabla\psi^i(x)\rangle \geq c_0',\ \forall\ x \in \Delta_{\epsilon_0}.$$

Thus the matrix block $M_\alpha^i(x)$ of ω_i on Δ_{ϵ_0} in (2.3) uniformly converges to

$$h(x)\begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix},$$

where $h(x)$ is a smooth positive function Δ_{ϵ_0}. Note that the above matrix is degenerate and both of e_α^∞ and $e_{-\alpha}^\infty$ do not vanish in $(B_\epsilon(x_\infty), \omega_\infty)$ by (4.26). Hence, e_α^∞ and $e_{-\alpha}^\infty$ must be linearly dependent at $x_\infty \in (B_\epsilon(x_\infty), \omega_\infty)$. In fact, we have

$$|e_\alpha^i - \sqrt{-1}e_{-\alpha}^i|(x_i) \to 0,\ \text{as } i \to \infty,$$

and we get

$$e_\alpha^\infty(x_\infty) = \sqrt{-1}e_{-\alpha}^\infty(x_\infty).$$

Let

$$U = \{z = (z^1, ..., z^n)|\ |z^a - x_i^a| \leq \epsilon_0,\ a = 1, ..., r\} \subset M$$

be a subset in M and

$$\pi : U \to U' = \{z' = (z^1, ..., z^r)|\ |z^a - x_i^a| < \epsilon_0,\ a = 1, ..., r\} \subset \mathfrak{a}$$

be the projection. Then for any curve γ starting from x_i such that $\pi(\gamma) \cap \partial U' \neq \emptyset$, it is easy to see by (4.19) and (4.24) that there is a uniform small constant ϵ' such that

$$\mathrm{length}(\gamma) \geq \epsilon'.$$

On the other hand, for any minimal geodesic ray γ starting from x_i such that $\pi(\gamma) \cap \partial U' = \emptyset$ its length is bigger than $r_{\mathrm{inj}}(x_i)$ which has a uniform lower bound by the convergence of ω_i. Thus there is a uniform constant ϵ' such that the ϵ'-geodesic ball $(B_{\epsilon'}(x_i), \omega_i)$ is contained in U. Since the convergence of $M_\alpha^i(x)$ is independent of the coordinate variables of $z^{a'}$ ($a' = r+1, ..., n$), we can actually prove that e_α^∞ and $e_{-\alpha}^\infty$ are globally linearly dependent in $(B_{\epsilon'}(x_\infty), \omega_\infty)$. But this is impossible by Lemma 4.3. In the other words, Case 1 will not happen. Therefore, we need to consider the following case.

Case 2: $\alpha(x_i) \leq A$ for some uniform constant A. Then

$$\tilde{e}_\alpha^i = \frac{\sqrt{2}e_\alpha}{\sqrt{\langle \alpha, \nabla\psi^i(x_i)\rangle \coth \alpha(x_i)}},$$
$$\tilde{e}_{-\alpha}^i = \frac{\sqrt{2}(e_{-\alpha} - \frac{\sqrt{-1}}{\coth \alpha(x_i)}e_\alpha)}{\sqrt{\langle \alpha, \nabla\psi^i(x_i)\rangle \left(\coth \alpha(x_i) - \coth^{-1} \alpha(x_i)\right)}} \tag{4.27}$$

form a unitary orthogonal basis on $\mathrm{span}\{\tilde e_\alpha, \tilde e_{-\alpha}\}(x_i) \subset (TM|_{x_i}, \omega_i)$. As in the proof of (4.21), there is a uniform constant $a_0 > 0$ such that

$$a_0 \le \langle \alpha, \nabla \psi^i(x_i)\rangle \left(\coth \alpha(x_i) - \coth^{-1} \alpha(x_i)\right) \le a_0^{-1}.$$

Thus the potentials $\tilde f^i_\alpha$ ($\tilde f^i_{-\alpha}$) of $\tilde e^i_\alpha$ associated to ω_i are uniformly bounded by (4.27). Hence, as in the proof of Part (b) in Proposition (3.4), $\{\tilde e^i_\alpha, \tilde e^i_{-\alpha}\}$ converges to a subbasis $\{\tilde e^\infty_\alpha, \tilde e^\infty_{-\alpha}\}$, which is orthogonal and unitary at x_∞ with respect to ω_∞. Moreover, again by (4.27), we see that there are constants $a \ne 0, d \ne 0, c$ such that

$$\tilde e^\infty_\alpha = a e^\infty_\alpha,\ \tilde e^\infty_{-\alpha} = d(e^\infty_{-\alpha} - c e^\infty_\alpha). \tag{4.28}$$

Therefore, e^∞_α and $e^\infty_{-\alpha}$ must be independent. The proof of lemma is finished.

□

Let Γ be the set of stabilizers of $\hat x_{0,\infty}$

$$\Gamma = \{g \in G|\ g \cdot \hat x_{0,\infty} = \hat x_{0,\infty}\}. \tag{4.29}$$

Then it is a closed subgroup of G. Moreover, by Lemma 4.4, we have

Corollary 4.5. *Suppose that $\hat x_{0,\infty} \in \hat{\mathcal O}^0_\infty$. Then there is a small neighborhood U_{Id} of* Id *in G such that*

$$\Gamma \cap U_{\mathrm{Id}} = \{\mathrm{Id}\}.$$

Proof. Let $\{E_1, ..., E_n\}$ be a basis of $\mathfrak g$ chosen as in Lemma 4.4. Then there exists a small $\epsilon > 0$ such that

$$(\exp\{z_1 E_1\}......\exp\{z_n E_n\})\hat x_{0,\infty} \subset \hat{\mathcal O}^0_\infty,\ \forall\ |z_l| \le \epsilon.$$

Since the set $\{\exp\{z_1E_1\}......\exp\{z_nE_n\}||z_i| \le \epsilon\} \subset G$ covers an open set of Id, there exists $U_{\mathrm{Id}} \subset G$ such that

$$U_{\mathrm{Id}} \subset \{\exp\{z_1E_1\}......\exp\{z_nE_n\}||z_i| \le \epsilon\}.$$

Note that for any $g \in U_{\mathrm{Id}}$, there is a $|t_0| \le \epsilon$ and $X \in \mathfrak g$ such that $g = \exp\{t_0 X\}$. By (4.15),

$$X(\hat x) \ne 0,\ \forall\ \hat x = \exp\{sX\} \cdot \hat x_{0,\infty},\ |s| \le |t_0|.$$

It follows that

$$g \cdot \hat x_{0,\infty} \ne \hat x_{0,\infty},\ \forall\ g \ne \mathrm{Id} \in U_{\mathrm{Id}}.$$

Thus $g \in \Gamma \cap U_{\mathrm{Id}}$ if and only if $g = \mathrm{Id}$. The corollary is proved.

□

By Corollary 4.5, Γ is a discrete set. Next we show that

$$\#\Gamma = N_0 < \infty. \tag{4.30}$$

We use the contradiction argument to prove (4.30) and suppose that $\#\Gamma = \infty$. Then by Corollary 4.5, there is an infinite sequence of $\{g_l \in \Gamma|\ l \in \mathbb Z\}$ such that $\mathrm{dist}(g_l, \mathrm{Id}) \to \infty$ as $l \to \infty$. On the other hand, by the KAK decomposition of G [20], we see that there are $k_l, k_l' \in K$ and $a_l \in T$ such that $g_l = k_l' \cdot a_l \cdot k_l$ and $\mathrm{dist}(a_l, \mathrm{Id}) \to \infty$ as $l \to \infty$. It follows that there exists a $\delta_1 > 0$ such that

$$\mathrm{dist}(k_l \cdot \hat x_{0,\infty}, D\hat M_\infty) \ge \delta_1,\ \forall\ l. \tag{4.31}$$

In fact, if (4.31) is not true, there is a subsequence $\{k_{\alpha_l}\}$ which converges to $k_0 \in K$ and $k_0 \cdot \hat x_{0,\infty} \in D\hat M_\infty$. Note that any $g \in G$ fixes the set $D\hat M_\infty$. Then

$$\hat x_{0,\infty} = k_0^{-1}(k_0 \cdot \hat x_{0,\infty}) \in D\hat M_\infty,$$

which contradicts to the fact that $\hat x_{0,\infty} \in \hat{\mathcal O}_\infty$.

By (4.31), there is a compact set $\bar V \subset \hat{\mathcal O}_\infty$ such that $k_l \cdot \hat x_{0,\infty} \subset \bar V$ for all k_l. Furthermore, we have

Claim 2: For any small $\delta > 0$, there is a large number c_δ such that

$$\mathrm{dist}(a \cdot \hat y, D\hat M_\infty) \le \delta,\ \forall\ \hat y \in \bar V, \tag{4.32}$$

as long as $\mathrm{dist}(a, \mathrm{Id}) \geq c_\delta$, where $a \in T$.

By (4.32), we see that there is a subsequence of integers α_l such that

$$a_{\alpha_l}(k_{\alpha_l} \cdot \hat{x}_{0,\infty}) \to \hat{z} \in D\hat{M}_\infty, \text{ as } \alpha_l \to \infty.$$

It follows that

$$\mathrm{dist}(k'_{\alpha_l}[a_{\alpha_l}(k_{\alpha_l} \cdot \hat{x}_{0,\infty})], D\hat{M}_\infty) \to 0, \text{ as } \alpha_l \to \infty.$$

But this is impossible since $g_l \cdot \hat{x}_{0,\infty} = \hat{x}_{0.\infty} \in \hat{\mathcal{O}}_\infty$. Hence, (4.30) is true.

To prove **Claim 2**, we consider any element $X \in \mathfrak{g}$ with $\mathrm{im}(X) \in \mathfrak{k}$ and its potential function f_X associated to the Fubini-Study metric $\frac{1}{m}\omega_{FS}$ as in Proposition 3.4 for the torus manifold $\mathbb{C}P^N$. Let M_X^1 be a subset in $\mathbb{C}P^N$ defined by

$$M_X^1 = \{x \in \mathbb{C}P^N |\ f_X(x) = \max_{\mathbb{C}P^N} f_X\}.$$

Then

$$W_X = \{x \in \mathbb{C}P^N |\ x = \lim_{t\to\infty} \exp(t\mathrm{re}(X)) \cdot y, \text{ for some } y \in \hat{\mathcal{O}}_\infty^0\} \subset M_X^1. \tag{4.33}$$

Moreover, if $X \in \mathfrak{a}$ is a torus vector field, it can be shown that M_X^1 is a subplane in $\mathbb{C}P^N$.

Lemma 4.6. *$W_X \cap \hat{\mathcal{O}}_\infty = \emptyset$ for any torus vector field $X \in \mathfrak{a}$.*

Proof. On the contrary, we suppose that there is a point $\hat{x} \in W_X \cap \hat{\mathcal{O}}_\infty$. Then there is a point $\hat{y}_\infty \in \hat{\mathcal{O}}_\infty^0$ such that

$$\hat{x} = \lim_{t\to\infty} \exp\{t\mathrm{re}(X)\} \cdot \hat{y}_\infty.$$

Let X^i be a sequence of holomorphic vector fields on $\hat{M}^i$ which converges to X with respect to $K \times K$-invariant metric $(\Phi_i^{-1})^*\omega_i$. Take a sequence of $\hat{y}_i \in \hat{\mathcal{O}}_i$ such that $\hat{y}_i \to \hat{y}_\infty$. Then by Proposition 3.4, there is a point $\hat{x}_i \in \hat{M}_{X^i}^1$ for each $\hat{y}_i$ such that

$$\hat{x}_i = \lim_{t\to\infty} \exp\{t\mathrm{re}(X_i)\} \cdot \hat{y}_i,$$

where

$$\hat{M}_{X^i}^1 = \{x \in \hat{M}_i |\ f_{X^i}(x) = \max_{\hat{M}_i} f_{X^i}\} \subset (\hat{M}_i \setminus \hat{\mathcal{O}}_i) = D\hat{M}_i,$$

and f_{X^i} is a potential of X^i with respect to the metric $(\Phi_i^{-1})^*\omega_i$ on $\hat{M}_i$, which converges to a potential f_X^∞ with respect to the metric $(\Phi_\infty^{-1})^*\omega_\infty$ on $\hat{M}_\infty$. Thus there is a limit $\hat{x}_\infty \in D\hat{M}_\infty \cap \hat{M}_{\infty,X}^1$ of $\hat{x}_i$ in Gromov-Hausdorff topology, where $(\hat{M}_{\infty,X}^1, (\Phi_\infty^{-1})^*\omega_\infty)$ is the Gromov-Hausdorff topology limit of $(\hat{M}_{X^i}^1, (\Phi_i^{-1})^*\omega_i)$ as in Proposition 3.4. Note that

$$\max_{\mathbb{C}P^N} f_X = \max_{\hat{M}_\infty} f_X^\infty$$

since $(\Phi_\infty^{-1})^*\omega_\infty$ and $\frac{1}{m}\omega_{FS}|_{\hat{M}_\infty}$ are both invariant under the S^1-group generated by $\mathrm{im}(X)$ [44]. Hence, we get

$$\hat{M}_{\infty,X}^1 = \hat{M}_\infty \cap M_X^1.$$

Moreover, by Proposition 3.4 and the above relation, we have

$$\max_{\hat{M}_i} f_{X^i} = \max_{\mathbb{C}P^N} f_X = A_0.$$

Choose a small neighborhood T_δ around the set $\hat{M}_{\infty,X}^1$ in $\mathbb{C}P^N$ such that

$$1)\ f_X^\infty(x) \geq A_0 - \delta,\ \forall x \in \hat{M}_\infty \cap T_\delta;$$

$$2)\ f_X^\infty(x) = A_0 - \delta,\ \forall x \in \hat{M}_\infty \cap \partial T_\delta;$$

$$3)\ f_{X^i}(x) \geq A_0 - 2\delta,\ \forall x \in \hat{M}_i \cap T_\delta,\ \forall i \geq i_0;$$

$$4)\ \hat{x} \notin T_\delta. \tag{4.34}$$

3) can be guaranteed since f_{X^i} converges to f_X^∞ smoothly and $\hat{M}^1_{X^i}$ converges to $\hat{M}^1_{\infty,X}$ in Gromov-Hausdorff topology, and 4) comes from the assumption that $\hat{x} \in W_X \cap \hat{\mathcal{O}}_\infty$ so that $\mathrm{dist}(\hat{x}, D\hat{M}_i) \geq \delta_0 >> \delta$ as $i >> 1$.

Note that f_{X^i} is monotone along the integral curve $\exp\{t\mathrm{re}(X^i)\} \cdot \hat{y}_i$. Then there is a uniform constant $T_N > 0$ such that

$$\exp\{t\mathrm{re}(X^i)\} \cdot \hat{y}_i \subset T_\delta,\ \forall\ t \geq T_N,\ i \geq i_0.$$

Thus we can choose a sequence of $z_i = \exp\{t_i\mathrm{re}(X^i)\} \cdot \hat{y}_i \in \hat{M}_i \cap (T_\delta \setminus T_{2\delta})$ which converges to a point $z_\infty \in \hat{M}_\infty \cap \overline{T_\delta}$, where $T_{2\delta} \subset T_\delta$ is another small neighborhood around $\hat{M}^1_{\infty,X}$ in $\mathbb{C}P^N$ such that

$$\mathrm{dist}(\hat{M}^1_{X^i}, \hat{M}_i \cap \partial T_{2\delta}) \geq \delta'$$

for some sufficiently small δ'. It follows that t_i converges subsequently to some $T_0 < \infty$ as $i \to \infty$. On the other hand, by the convergence of X^i, we see that

$$\lim_i \exp\{t\mathrm{re}(X^i)\} \cdot \hat{y}_i = \exp\{t\mathrm{re}(X)\} \cdot \hat{y}_\infty,\ \forall t \leq 2T_0.$$

Hence we derive

$$z_\infty = \exp\{T_0\mathrm{re}(X)\} \cdot \hat{y}_\infty.$$

By the monotonicity of f_X along the integral curve $\exp\{t\mathrm{re}(X)\} \cdot \hat{y}_\infty$, we conclude that

$$\exp\{t\mathrm{re}(X)\} \cdot \hat{y}_\infty \in \hat{M}_\infty \cap T_\delta,\ \forall t > T_0$$

and consequently, $\hat{x} \in \hat{M}_\infty \cap T_\delta$. Therefore, we get a contradiction with 4) in (4.34). The lemma is proved.

□

*Proof of **Claim 2**.* Suppose that **Claim 2** is not true. Then there exist a δ_0, a sequence of $a_l \in T$ and a sequence of $\hat{y}_l \in \bar{V}$ such that

$$\mathrm{dist}(a_l \cdot \hat{y}_l, D\hat{M}_\infty) \geq \delta_0, \tag{4.35}$$

where $\mathrm{dist}(a_l, \mathrm{Id}) \to \infty$ as $l \to \infty$. Write each a_l as $a_l = \exp\{\sum b_l^i X_i\}$ for some real numbers $b_l^1, ..., b_l^r$, where $\{X_1, ..., X_r\}$ is a basis of $\mathfrak{a}$. Then $\sum_i |b_l^i| \to \infty$ as $l \to \infty$. Without loss of generality, we may assume that

$$\sum b_l^i X_i = b_l^1(X_1 + Y_l),$$

where $b_l^1 \to \infty$ and $|Y_l| \to 0$ as $l \to \infty$. Then by Lemma 4.6, for any fixed $\hat{y} \in \bar{V}$ it holds

$$\mathrm{dist}(\exp\{b_l^1(X_1 + Y_l)\} \cdot \hat{y}, D\hat{M}_\infty \cap (\cup_\alpha \tilde{W}_\alpha)) \to 0,\ \text{as } b_l^1 \to \infty,$$

where $\tilde{W}_1, ..., \tilde{W}_{N+1}$ are the $(N+1)$ hyperplanes in $\mathbb{C}P^N$ as in (4.13). Since $\bar{V}$ is a compact set away from $\cup_\alpha \tilde{W}_\alpha$, as in the proof of **Claim 1** in Section 3, the above convergence is uniform. It follows that

$$\mathrm{dist}(a_l \cdot \hat{y}_l, D\hat{M}_\infty) \to 0,\ \text{as } a_l \to \infty,$$

which contradicts to (4.35). **Claim 2** is proved.

□

By (4.30), we can finish the proof of Proposition 4.1.

Completion of proof of Proposition 4.1. For any $\hat{x} = h \cdot \hat{x}_{0,\infty}$, we have

$$(hgh^{-1}) \cdot \hat{x} = (hgh^{-1})(h \cdot \hat{x}_{0,\infty}) = \hat{x},\ \forall\ g \in \Gamma.$$

It follows that $h\Gamma h^{-1}$ is the set of stabilizers of $\hat{x}_\infty$. By (4.30), $G \cdot \hat{x}_{0,\infty}$ is a finite quotient space. Since the above argument works for any $\hat{x}_\infty \in \hat{\mathcal{O}}^0_\infty$, in particular, both Corollary 4.5 and (4.30) hold. Thus, each orbit $G \cdot \hat{x}_\infty$ is isomorphic to $G/\Gamma_{\hat{x}_\infty}$, where $\Gamma_{\hat{x}_\infty}$ is a finite subgroup of $\mathrm{PU}(N+1, \mathbb{C})$. Moreover $G \cdot \hat{x}_\infty \cap G \cdot \hat{x}'_\infty = \emptyset$ for any $\hat{x}_\infty, \hat{x}'_\infty \in \hat{\mathcal{O}}^0_\infty$. Otherwise $G \cdot \hat{x}_\infty = G \cdot \hat{x}'_\infty$. This means that any two different orbits are disjoint. Note that

$$\hat{\mathcal{O}}^0_\infty = \cup_{\hat{x}_\infty \in \hat{\mathcal{O}}^0_\infty} G \cdot \hat{x}_\infty.$$

It is easy to see that for any bounded set U in $\hat{\mathcal{O}}^0_\infty$ there are finitely many different orbits passing through U. Since $\mathcal{O}_\infty \setminus \hat{\mathcal{O}}^0_\infty$ consists of finitely many subvarieties of codimension at least 1 in $\mathcal{O}_\infty$, $\hat{\mathcal{O}}^0_\infty$ is connected. As a consequence, there is only one orbit $G\cdot \hat{x}_{0,\infty}$ through U. Otherwise U will be disconnected. Therefore, we prove that $\hat{\mathcal{O}}^0_\infty = G\cdot \hat{x}_{0,\infty}$.

It remains to show that $\Gamma=\{\mathrm{Id}\}$ in (4.30). For any compact set $K^\epsilon_\infty\subset \hat{\mathcal{O}}^0_\infty$ with

$$\mathrm{vol}_{\hat{\omega}_\infty}(\hat{M}_\infty\backslash K^\epsilon_\infty)<\epsilon, \tag{4.36}$$

where $\hat{\omega}_\infty=\frac{1}{m}\omega_{FS}|_{\hat{M}_\infty}$, we choose a family of disjointed geodesic balls $\hat{B}_{r_l}$ in $\hat{\mathcal{O}}^0_\infty$ such that the following holds:

1) $\sum_l \mathrm{vol}_{\hat{\omega}_\infty}(\hat{B}_{r_l})\geq \mathrm{vol}_{\hat{\omega}_\infty}(K^\epsilon_\infty)-\epsilon$.

2) For each $\hat{B}_{r_l}$, there are disjointed geodesic open sets $B^\alpha_{r_l}\subset\mathcal{O}$, $\alpha=1,...,N_0$, such that $\pi^{-1}(\hat{B}_{r_l})=\cup_\alpha B^\alpha_{r_l}$, where $\pi:\ \mathcal{O}\to\hat{\mathcal{O}}^0_\infty$ is the projection by $\lim_i\Phi_i(\Gamma\cdot x)=\Phi_\infty(\Gamma\cdot x_\infty)=\hat{x}_\infty$.

3) $(\Phi_i(B^\alpha_{r_l}),\hat{\omega_i})$ is isometric to $(\Phi_i(B^\beta_{r_l}),\hat{\omega_i})$ for any α,β.

By Lemma 4.4 (also see (4.18)), we see that $(\Phi_i(B^\alpha_{r_l}),\hat{\omega}_i)$ converges to $(\hat{B}_{r_l},\hat{\omega}_\infty)$ uniformly as open submanifolds when $i\to\infty$. In particular, it holds that for each α,

$$\lim_i\sum_l \mathrm{vol}_{\hat{\omega}_i}(\Phi_i(B^\alpha_{r_l}))=\sum_l\mathrm{vol}_{\hat{\omega}_\infty}(\hat{B}_{r_l}).$$

Note that $B^\alpha_{r_l}$ are disjointed for each l,α. Thus

$$\begin{aligned}\mathrm{vol}_{\hat{\omega}_i}(\Phi_i(M))&\geq\sum_{l,\alpha}\mathrm{vol}_{\hat{\omega}_i}(\Phi_i(B^\alpha_{r_i}))\\&\geq N_0\sum_l\mathrm{vol}_{\hat{\omega}_\infty}(\hat{B}_{r_l})-N_0\epsilon\end{aligned}$$

as long as i is large enough. By (4.36), it follows that

$$\mathrm{vol}_{\hat{\omega}_i}(\Phi_i(M))\geq N_0\mathrm{vol}_{\hat{\omega}_\infty}(\hat{M}_\infty)-(2N_0+2)\epsilon.$$

But this is impossible if $N_0\geq 2$ since

$$\mathrm{vol}_{\hat{\omega}_i}(\Phi_i(M))=\mathrm{vol}_{\hat{\omega}_\infty}(\hat{M}_\infty)=c_1(M)^n. \tag{4.37}$$

Thus $\Gamma=\{\mathrm{Id}\}$, and so G acts on $\hat{x}_{0,\infty}$ freely. Hence, we prove that (M_∞,J_∞) is a G-manifold. □

5. Uniqueness of complex structures on semisimple G-compactifications

In this section, we first prove a uniqueness result about complex structures on G-manifolds when G is semisimple. Then we complete the proof of Theorem 1.1.

We begin with following elemental lemma.

Lemma 5.1. *Let (Z,J) be an r-dimensional toric manifold with an r-dimensional torus T^r-action. Let $T^r=T^m\times T^{r-m}$ and $J'=\mathrm{diag}(-J|_{T^m},J|_{T^{r-m}})$ be an integral almost complex structure on the open T^r-orbit $\mathcal{O}$ of Z. Suppose that J' can be extended to a smooth complex structure on Z. Then Z must be a product of m-dimensional toric manifold and $(r-m)$-dimensional toric manifold. Furthermore, (Z,J) and (Z,J') are bi-holomorphic.*

Proof. On the open T^r-orbit $\mathcal{O}$, we choose log-affine coordinates $w_1,...,w_r$. Let Σ be the fan of Z and σ_a an r-dimensional cone in it. Then on the corresponding chart $U_a\subset Z$, we have local coordinates $z^1,...,z^r\in\mathbb{C}$ such that on $U_a\cap\mathcal{O}$,

$$\begin{cases} z^i=\exp(\sum_j w^j\alpha^i_j+\sum_\beta w^\beta a^i_\beta)\\ z^\alpha=\exp(\sum_j w^j a^\alpha_j+\sum_\beta w^\beta a^\alpha_\beta)\end{cases},1\leq i,j\leq m<\alpha,\beta\leq r, \tag{5.1}$$

where

$$A=\begin{pmatrix} (a_i^j)_{m\times m} & (a_\alpha^j)_{m\times(r-m)} \\ (a_i^\beta)_{(r-m)\times m} & (a_\alpha^\beta)_{(r-m)\times(r-m)} \end{pmatrix}\in GL_r(\mathbb{Z}). \tag{5.2}$$

On the open orbit $\mathcal{O}$, we have

$$\begin{aligned} J' =&\sqrt{-1}\Bigg[-\sum_i(dw^i\otimes\frac{\partial}{\partial w^i}-d\bar{w}^i\otimes\frac{\partial}{\partial\bar{w}^i}) \\ &+\sum_\alpha(dw^\alpha\otimes\frac{\partial}{\partial w^\alpha}-d\bar{w}^\alpha\otimes\frac{\partial}{\partial\bar{w}^\alpha})\Bigg]. \end{aligned} \tag{5.3}$$

By (5.2), it follows that

$$\begin{aligned} J'|_{U_a\cap\mathcal{O}} =&\sqrt{-1}\left[-\sum_i(dz^i\otimes\frac{\partial}{\partial z^i}-d\bar{z}^i\otimes\frac{\partial}{\partial\bar{z}^i})+\sum_\alpha(dz^\alpha\otimes\frac{\partial}{\partial z^\alpha}-d\bar{z}^\alpha\otimes\frac{\partial}{\partial\bar{z}^\alpha})\right] \\ &-4\mathrm{Im}\Bigg[\sum_{k,j}(A^{-1})_k^\alpha a_\alpha^j\frac{z^j}{z^k}dz^k\otimes\frac{\partial}{\partial z^j}+\sum_{i,\gamma}(A^{-1})_i^\alpha a_\alpha^\gamma\frac{z^\gamma}{z^i}dz^i\otimes\frac{\partial}{\partial z^\gamma} \\ &-\sum_{\beta,i}(A^{-1})_\beta^j a_j^i\frac{z^i}{z^\beta}dz^i\otimes\frac{\partial}{\partial z^\beta}+\sum_{\beta,\gamma}(A^{-1})_\beta^j a_j^\gamma\frac{z^\gamma}{z^\beta}dz^\beta\otimes\frac{\partial}{\partial z^\gamma}\Bigg], \end{aligned} \tag{5.4}$$

where $A^{-1}=((A^{-1})_q^p)$ is the inverse matrix of A with elements $(A^{-1})_q^p$. Note that J' can be smoothly extended on whole U_a. By taking any variable z^l of $\{z^1,...,z^r\}$ to 0, it is easy to see that

$$\begin{cases} (A^{-1})_j^\alpha a_\alpha^k=0,\ j\neq k \\ (A^{-1})_j^\alpha a_\alpha^\gamma=0 \end{cases} \text{and}\ \begin{cases} (A^{-1})_\alpha^j a_j^k=0 \\ (A^{-1})_\alpha^j a_j^\beta=0,\ \alpha\neq\beta \end{cases}. \tag{5.5}$$

Thus, by the fact $J'^2=-1$, we get

$$\begin{aligned} J'|_{U_a} =&\sqrt{-1}\Bigg[\sum_i\epsilon_i(dz^i\otimes\frac{\partial}{\partial z^i}-d\bar{z}^i\otimes\frac{\partial}{\partial\bar{z}^i}) \\ &+\sum_\alpha\epsilon_\alpha(dz^\alpha\otimes\frac{\partial}{\partial z^\alpha}-d\bar{z}^\alpha\otimes\frac{\partial}{\partial\bar{z}^\alpha})\Bigg], \end{aligned} \tag{5.6}$$

where each of ϵ_i and ϵ_α is 1 or -1.

By (5.3) and (5.6), there must be m numbers of -1 and $(r-m)$ numbers of 1 in $\{\epsilon_1,...,\epsilon_r\}$. Without of loss of generality, we may assume that

$$\epsilon_i=-1,\epsilon_\alpha=1.$$

Then by (5.4), we get

$$\begin{cases} (A^{-1})_j^\alpha a_\alpha^k=0, \\ (A^{-1})_j^\alpha a_\alpha^\gamma=0 \end{cases} \text{and}\ \begin{cases} (A^{-1})_\alpha^j a_j^k=0 \\ (A^{-1})_\alpha^j a_j^\beta=0, \end{cases}\ \forall\ i,j,\alpha,\beta. \tag{5.7}$$

On the other hand, the matrices

$$\begin{pmatrix} (a_\alpha^j)_{m\times(r-m)} \\ (a_\alpha^\beta)_{(r-m)\times(r-m)} \end{pmatrix} \text{ and } \begin{pmatrix} (a_i^j)_{m\times m} \\ (a_i^\beta)_{(r-m)\times m} \end{pmatrix}$$

are both of full ranks. Thus by (5.7), we have

$$(A^{-1})_j^\alpha=0,(A^{-1})_\alpha^j=0,$$

i.e.,

$$a_j^\alpha=0,a_\alpha^j=0.$$

As a consequence, by (5.1), it follows that

$$
\begin{cases} z^i = \exp(\sum_j w^j a^i_j) \\ z^\alpha = \exp(\sum_\beta w^\beta a^\alpha_\beta) \end{cases}, \text{ on } U_a \cap \mathcal{O}. \tag{5.8}
$$

Hence, the first equation in (5.8) defines a toric manifold Z_1 with T^m-action, while the second equation in (5.8) defines a toric manifold Z_2 with T^{r-m}-action. This proves that $Z = Z_1 \times Z_2$.

By (5.8) the map

$$
\Phi(z^i, z^\alpha) = (\bar{z}^i, z^\alpha) \tag{5.9}
$$

is well-defined on Z, which satisfies $\Phi^* J' = J$. Thus (Z, J) and (Z, J') are bi-holomorphic.

□

Theorem 5.2. *Let G be a semisimple reductive Lie group. Let (M, K_M^{-1}, J) and $(\tilde{M}, K_{\tilde{M}}^{-1}, \tilde{J})$ be two Fano compactifications of G. Suppose that $\tilde{M}$ is diffeomorphic to M. Then $(\tilde{M}, \tilde{J})$ is bi-holomorphic to (M, J).*

Proof. Let $F : \tilde{M} \to M$ be a diffeomorphism. Then it suffices to show that there is an automorphism Ψ on M such that

$$
(F^{-1})^* \tilde{J} = \Psi^* J, \text{ on } M. \tag{5.10}
$$

Consider the $G \times G$-action on M, which is induced by the one on $\tilde{M}$. Namely, for any $x \in \tilde{M}$, it holds

$$
g \cdot F(x) = F(g \cdot x), \ \forall\, g \in G \times G. \tag{5.11}
$$

Thus $J' = (F^{-1})^* \tilde{J}$ is also a $G \times G$-invariant integral almost complex structure on M and it induces another complex structure on G.

Choose a base point $x_0 \in \mathcal{O}$. Since G is a $2n$-dimensional real Lie group (denoted by $G_\mathbb{R}$) with an adjoint representation $\mathrm{ad}_{\mathfrak{g}_\mathbb{R}}(\cdot)$ of $\mathfrak{g}_\mathbb{R}$ on itself, we have

$$
J'(x_0) \in \mathrm{End}'(\mathfrak{g}_\mathbb{R}) = \{\sigma |\ \sigma \in \mathrm{End}(\mathfrak{g}_\mathbb{R}) \text{ and } \sigma(\mathrm{ad}_X Y) = \mathrm{ad}_X(\sigma(Y)),\ \forall\, X, Y \in \mathfrak{g}_\mathbb{R}\}.
$$

On the other hand, the semisimple complex Lie algebra $\mathfrak{g}$ of (G, J) can be decomposed into irreducible ideals $\mathfrak{s}_i$ of $\mathfrak{g}$, $i = 1, ..., l$, such that

$$
\mathfrak{g} = \oplus_i \mathfrak{s}_i,
$$

with $[\mathfrak{s}_i, \mathfrak{s}_j] = \delta_{ij} \mathfrak{s}_i$. Then it is easy to see that

$$
\mathrm{End}'(\mathfrak{g}_\mathbb{R}) = \oplus_i \mathrm{End}'_i(\mathfrak{s}_{i\mathbb{R}}),
$$

where

$$
\mathrm{End}'_i(\mathfrak{s}_{i\mathbb{R}}) = \{\sigma |\ \sigma \in \mathrm{End}(\mathfrak{s}_{i\mathbb{R}}) \text{ and } \sigma(\mathrm{ad}_X Y) = \mathrm{ad}_X(\sigma(Y)),\ \forall\, X, Y \in \mathfrak{s}_{i\mathbb{R}}\}.
$$

Note that each $\mathfrak{s}_i$ is a complex irreducible representation by $\mathrm{ad}_{\mathfrak{s}_i}(\cdot)$ on $\mathfrak{s}_i$. Thus

$$
\dim_\mathbb{C} \mathrm{End}'_i(\mathfrak{s}_i) = 1.
$$

As a consequence, $\dim_\mathbb{R} \mathrm{End}'(\mathfrak{s}_{i\mathbb{R}}) = 2$, which can be spanned by Id and $J|_{\mathfrak{s}_{i\mathbb{R}}}$. Hence, $J'|_{\mathfrak{s}_{i\mathbb{R}}} = \lambda_i \mathrm{Id} + \mu_i J|_{\mathfrak{s}_{i\mathbb{R}}}$ for some $\lambda_i, \mu_i \in \mathbb{R}$, and

$$
J'|^2_{\mathfrak{s}_{i\mathbb{R}}} = (\lambda_i^2 - \mu_i^2)\mathrm{Id} + 2\lambda_i \mu_i J|_{\mathfrak{s}_{i\mathbb{R}}}.
$$

By the fact $J'^2 = -\mathrm{Id}$, it follows that that $\lambda_i = 0, \mu_i = \pm 1$. Therefore, we prove that

$$
J'(x_0) = \oplus \mu_i J|_{\mathfrak{s}_{i\mathbb{R}}}(x_0), \tag{5.12}
$$

where $\mu_i = 1$, or -1.

By [1, 2], for the G-manifold (M, J) there is an r-dimensional toric complex submanifold $(Z, J|_Z)$ through x_0 associated to a maximal torus $T^\mathbb{C}$ of G. Similarly, there is another r-dimensional toric complex submanifold $(Z', J'|_{Z'})$ through x_0 associated to a maximal torus of the induced G-action by (5.11). Moveover, by (5.12), we have

$$J'|_{TZ'}(x_0) = \oplus \mu_i J|_{\mathfrak{t}_{i\mathbb{R}}},$$

where $\mathfrak{t}_{i\mathbb{R}} = \mathfrak{s}_{i\mathbb{R}} \cap \mathfrak{t}_{\mathbb{R}}$ which is non-empty for each i. Thus there is a decomposition of $\mathfrak{t}_{\mathbb{R}}$ such that

$$\begin{aligned} J'|_{TZ'}(x_0) &= (\oplus_{i=1}^{r_1}(-J)|_{\mathfrak{t}_{i\mathbb{R}}}) \oplus (\oplus_{i=1}^{r_2} J|_{\mathfrak{t}_{i\mathbb{R}}}) \\ &= (-J)|_{\mathfrak{t}_{\mathbb{R}}^m} \oplus J|_{\mathfrak{t}_{\mathbb{R}}^{r-m}}, \end{aligned} \tag{5.13}$$

where $\mathfrak{t}_{\mathbb{R}}^m$ and $\mathfrak{t}_{\mathbb{R}}^{r-m}$ are two Lie subalgebras of $\mathfrak{t}_{\mathbb{R}}$ with dimensions m and $(r-m)$, respectively. Note that $Z|_{T^{\mathbb{C}}x_0} = Z'|_{T^{\mathbb{C}}x_0}$. Hence, by the completeness in the same ambient space M, we get

$$Z = \overline{Z|_{T^{\mathbb{C}}z_0}} = \overline{Z'|_{T^{\mathbb{C}}z_0}} = Z' \subset M. \tag{5.14}$$

By Lemma 5.1, we prove that $(Z, J|_Z)$ and$(Z, J'|_Z)$ are bi-holomorphic.

Let P and P' be two associated polytopes of $(Z, K_M^{-1}|_Z, J|_Z)$ and $(Z, K_M^{-1}|_Z, J'|_Z)$ as in Subsection 2.2, respectively. Since both of m-multiple bundles of K_M^{-1} and $K_{\tilde{M}}^{-1}$ can be regarded as a restricted line bundle of $K_{\mathbb{C}P^N}^{-1}$ by the Kodaira embedding as in Section 4 for M_i and M_∞,

$$(Z, K_M^{-m}|_Z, J|_Z) = (Z, K_{\mathbb{C}P^N}^{-1}|_Z),\ (Z, K_M^{-m}|_Z, J'|_Z) = (F^{-1}(Z), K_{\mathbb{C}P^N}^{-1}|_{F^{-1}(Z)}).$$

It follows that the curvatures of $(Z, K_M^{-1}|_Z, J|_Z)$ and $(Z, K_M^{-1}|_Z, J'|_Z)$ can be induced by the Fubuni-Study of $\mathbb{C}P^N$ and so their cohomology classes on Z are same. Thus P and P' as the moment images of curvature forms of the above line bundles are isomorphic. Hence, by the equivariant classification theory [3, Section 2], two polarized compactifications (M, K_M^{-1}, J) and (M, K_M^{-1}, J') are different from a $G \times G$-equivariant morphism. Therefore, (M, J') must be bi-holomorphic to (M, J). Namely there is an automorphism Ψ on M such that (5.10) holds. The theorem is proved. □

Theorem 5.2 can be also proved without using the equivariant classification theory [3, Section 2]. In fact, we can give a direct construction of automorphism Ψ by Lemma 5.1 and the Cartan involution in the following.

Let S_i be the subgroup of G with Lie algebra $\mathfrak{s}_i$. Then each S_i is semisimple and

$$G = \prod_{i=1}^{l} S_i / \mathrm{diag}(\cap_i S_i), \tag{5.15}$$

where $\cap_i S_i$ is a finite group (cf. [47, Section 3.2]). Fix a maximal compact subgroup K_i in each S_i and let Θ_i be the Cartan involution on S_i which acts trivially on K_i and inverses the complex structure of S_i. We may choose K_i so that $\cap_i K_i$ contains the finite group $\cap_i S_i$. Note that there are $s_i \in S_i$ for any $g \in G$ such that $g = s_1 \cdot ... \cdot s_l$ by (5.15). Thus we can define an automorphism on G by

$$\Theta(g) = \Theta_1^{\epsilon_1}(s_1) \cdot ... \cdot \Theta_l^{\epsilon_l}(s_l), \tag{5.16}$$

where $\epsilon_i = 1$ if $\mu_i = -1$, and $\epsilon_i = 0$ if $\mu_i = 1$. Since each s_i is uniquely determined up to multiplying an element of $\cap_i S_i$, which is fixed under Θ_i, Θ is well-defined. Moreover, we have

$$\Theta^* J = J'. \tag{5.17}$$

It suffices to show that Θ can be extended to a diffeomorphism on M. By Lemma 5.1, we know that $(Z, J|_Z)$ and $(Z, J'|_Z)$ are bi-holomorphic. Note that the restriction of Θ on $T^{\mathbb{C}}$ is just Φ by(5.9). Thus Θ can be extended to a diffemorphism on Z by

$$\Theta(z) = \Phi(z),\ \forall z \in Z. \tag{5.18}$$

Moreover, $\Theta|_Z$ commutes with the W-action.

On the other hand, by a generalized KAK-decomposition of G-compactification (cf. [45, Section 3.4] or [46, Section 9]), for any $x \in M$, there are $k_1, k_2 \in K$ and $z \in Z$ so that $x = (k_1, k_2)z$.

Moreover, z is uniquely determined up to a W-action. Since Θ commutes with the W-action and $\Theta|_K$ is trivial, the following

$$\Theta(x) = (k_1, k_2)\Theta(z)$$

is well-defined by (5.18). Thus we can extend Θ to a diffeomorphism Ψ on M so that (5.10) holds by (5.17). Hence, we also prove Theorem 5.2.

As a corollary of Theorem 5.2, we immediately get

Proposition 5.3. *The limit (M_∞, J_∞) in Proposition 4.1 is bi-holomorphic to (M, J) whenever G is semisimple.*

By Proposition 4.1 and Proposition 5.3, we are able to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. By the definition, if the solution of Kähler-Ricci flow (1.1) has only type I singularities, then the curvature of $\omega(t)$ is uniformly bounded. Thus there is a subsequence $\{\omega(t_i)\}$ which converges to a limit of Kähler-Ricci soliton $(M_\infty, \omega_\infty, J_\infty)$ in Cheeger-Gromov topology. Note that the center of Lie algebra of the reductive part of $\mathrm{Aut}(M_\infty)$ is trivial by Proposition 4.1 since G is semisimple. Thus $(M_\infty, \omega_\infty, J_\infty)$ must be a Kähler-Einstein metric. On the other hand, by Proposition 5.3, (M_∞, J_∞) is biholomorphic to (M, J), which admits no Kähler-Einstein metric by the assumption in the theorem. Hence, we get a contradiction. As a consequence, the curvature of $\omega(t)$ must blow-up as $t \to \infty$. Namely, the solution of flow is of type II.

There is another way to prove Theorem 1.1 without using Proposition 5.3 if in addition we know that M is K-unstable. In fact, the limit $(M_\infty, \omega_\infty, J_\infty)$ is a Kähler-Einstein metric if the solution of flow is of type I. Then by a result in [8] (also see [36, Lemma 7.1]), the K-energy is bounded below on the space of Kähler potentials in $2\pi c_1(M)$. This implies that (M, J) is K-semistable [15, 24]. Thus we get a contradiction. Hence, the curvature of $\omega(t)$ must blow up as $t \to \infty$. □

6. Examples of G-manifolds with rank 2

In this section, we describe Fano compactifications of $\mathrm{SO}_4(\mathbb{C})$ and $\mathrm{Sp}_4(\mathbb{C})$.

6.1. **Fano $\mathrm{SO}_4(\mathbb{C})$-manifolds of dimension 6.** In [12], Delcroix computed three polytopes P_+ associated to Fano compactifications of $\mathrm{SO}_4(\mathbb{C})$. In fact, by checking the Delzant condition of polytope P and the Fano condition of compactified manifold, these three manifolds M are only Fano compactifications of $\mathrm{SO}_4(\mathbb{C})$. In the following, we write down the detailed data associated to P_+, in particular, the values of $\mathrm{bar}(P_+)$.

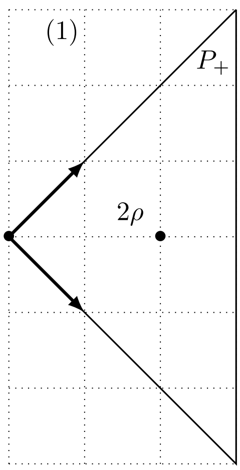

Figure 1.

Choose a coordinate on $\mathfrak{a}^*$ such that the basis are the generator of $\mathfrak{M}$. Then the positive roots are

$$\alpha_1 = (1, -1),\ \alpha_2 = (1, 1),$$

and

$$2\rho = (2, 0).$$

Thus

$$\mathfrak{a}^*_+ = \{x > y > -x\},$$
$$2\rho + \Xi = \{-2 + x > y > 2 - x\},$$

and

$$\pi(x, y) = (x - y)^2(x + y)^2.$$

(A)-**Case (1).** There is one smooth Fano compactification of $SO_4(\mathbb{C})$, which admits a Kähler-Einstein metric. The polytope P_+ is given by (See Figure 1),

$$P_+ = \{y > -x, x > y, 2 - x > 0, 2 + y > 0\}. \tag{6.1}$$

A direct computation shows that $\mathrm{vol}(P_+) = \frac{648}{5}$ and

$$\mathrm{bar}(P_+) = \left(\frac{18}{7}, 0\right).$$

Then

$$\mathrm{bar}(P_+) \in 2\rho + \Xi$$

which implies (2.4). Thus by Theorem 2.2, the $SO_4(\mathbb{C})$-manifold associated to P_+ in (6.1) admits a Kähler-Einstein metric.

(B) There are two smooth Fano compactifications of $SO_4(\mathbb{C})$ with no Kähler-Einstein metrics. Both of P_+ (see Figure 2) do not satisfy (2.4). Moreover, the Futaki invariant vanishes since the center of automorphisms group are finite. Hence there are also no Kähler-Ricci solitons on the compactifications.

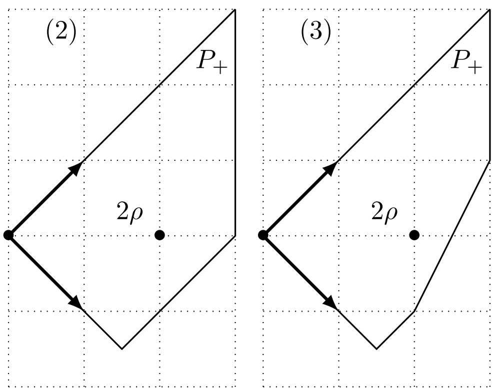

FIGURE 2.

Case (2). The polytope is

$$P_+ = \{y > -x, x > y, 2 - x > 0, 2 + y > 0, 3 - x + y > 0\}.$$

Then $\mathrm{vol}(P_+) = \frac{1701}{20}$ and the barycenter is

$$\mathrm{bar}(P_+) = \left(\frac{489}{196}, \frac{15}{28}\right).$$

Thus

$$\mathrm{bar}(P_+) \notin \overline{2\rho + \Xi}$$

and consequently, there is no Kähler-Einstein metric in **Case (2)**.

Case (3). The polytope is

$$P_+ = \{y > -x, x > y, 2 - x > 0, 2 + y > 0, 3 - x + y > 0, 5 - 2x + y > 0\}.$$

Then $\mathrm{vol}(P_+) = \frac{10751}{180}$ and the barycenter is

$$\mathrm{bar}(P_+) = \left(\frac{102741}{43004}, \frac{16575}{23156}\right).$$

Thus

$$\mathrm{bar}(P_+) \notin \overline{2\rho + \Xi}$$

and consequently, there is no Kähler-Einstein metric in **Case (3)**.

6.2. **Fano $\mathrm{Sp}_4(\mathbb{C})$-manifolds of dimension 10.** In [11] Delcroix computed two polytopes P_+ associated to toroidal Fano compactifications of $\mathrm{Sp}_4(\mathbb{C})$ (see Cases (1) and (3) below). By the tables listed in [27], one can check that there are in total three smooth Fano compactifiactions. We give the data in the following. The positive roots are

$$\alpha_1 = (1, -1), \alpha_2 = (2, 0), \alpha_3 = (1, 1), \alpha_4 = (0, 2).$$

Consequently, $\rho = (2, 1)$,

$$2\rho + \Xi = \{y > 6 - x, x > 4\}$$

and $\pi(x, y) = 16(x - y)^2(x + y)^2x^2y^2$.

(A) There are two smooth Fano compactifications which admit Kähler-Einstein metrics (see Figure 3).

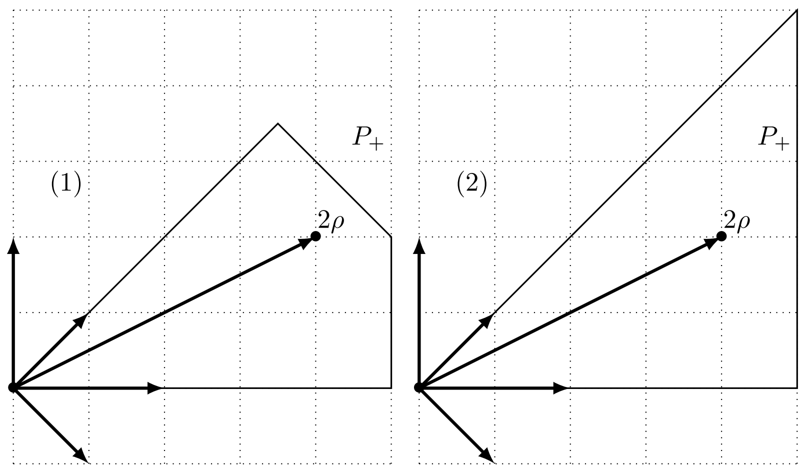

FIGURE 3.

Case (1). The polytope P_+ is given by

$$P_+ = \{y > 0, x > y, 5 - x > 0, 7 > x + y\}.$$

A direct computation shows that $\mathrm{vol}(P_+) = \frac{31702283}{1400}$ and

$$\mathrm{bar}(P_+) = \left(\frac{456413622265}{104829824704}, \frac{186115662215}{104829824704}\right) \in 2\rho + \Xi,$$

which implies (2.4). Thus by Theorem 2.2, the $\mathrm{Sp}_4(\mathbb{C})$-manifold associated to P_+ in **Case (1)** admits a Kähler-Einstein metric.

Case (2). The polytope P_+ is given by

$$P_+ = \{y > 0, x > y, 5 > x\}.$$

A direct computation shows that $\mathrm{vol}(P_+) = \frac{1562500}{21}$ and

$$\mathrm{bar}(P_+) = \left(\frac{50}{11}, \frac{875}{352}\right) \in 2\rho + \Xi.$$

Hence there admits a Kähler-Einstein metric in **Case (2)**.

(B)-**Case (3).** There is one smooth Fano compactification which does not admit Kähler-Einstein metrics (see Figure 4).

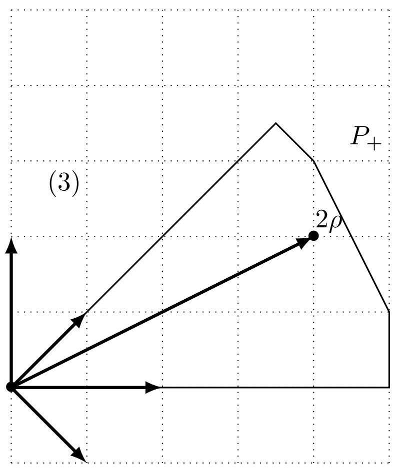

FIGURE 4.

The polytope P_+ is given by

$$P_+ = \{y > 0, x > y, 5 - x > 0, 7 > x + y, 11 > 2x + y\}.$$

A direct computation shows that $\mathrm{vol}(P_+) = \frac{148906001}{4200}$ and

$$\mathrm{bar}(P_+) = \left(\frac{278037566905}{66955221696}, \frac{111498923355}{66955221696}\right) \notin \overline{2\rho + \Xi}.$$

Hence there does not admit a Kähler-Einstein metric in **Case (3)**.

Two $\mathrm{SO}_4(\mathbb{C})$-manifolds in Section 6.1 (B-Cases (2), (3)) and one $\mathrm{Sp}_4(\mathbb{C})$-manifold in Section 6.2 (B-Case (3)) are those examples described as in Theorem 1.2. Moreover, these three Fano manifolds are all K-unstable.

6.3. **Remarks on Theorem 1.1 and Theorem 1.2.** By Theorem 1.1 and the Hamilton-Tian conjecture [30, 37, 5, 9, 38], the Kähler-Ricci flow (1.1) will converge to a Q-Fano variety $\tilde{M}_\infty$ with a singular Kähler-Ricci soliton. One may expect that $\tilde{M}_\infty$ is a $\mathbb{Q}$-Fano compactification of G by extending the argument in the proof of Proposition 4.1. Unfortunately, it is not true in general. In fact, in a sequel of paper [23], we prove

Theorem 6.1. *There is no $\mathbb{Q}$-Fano compactification of $SO_4(\mathbb{C})$ which admits a singular Kähler-Einstein metric with the same volume as in Section 6.1 (B-Cases (2), (3)).*

By the Hamilton-Tian conjecture, the limit M_∞ of (1.1) will preserve the volume (also see [38, Theorem 1.1]). Thus if M_∞ is a $\mathbb{Q}$-Fano compactification of $SO_4(\mathbb{C})$ in case of $G = SO_4(\mathbb{C})$, there will be a contradiction with Theorem 6.1. Theorem 6.1 implies that the limit soliton will has less symmetry than the original one, which is totally different to the situation of smooth convergence as in Proposition 4.1. However, it is still interesting in understanding the $\mathbb{Q}$-Fano structure of M_∞.

Although we shall assume that metrics (M, ω_i, J) are all $K \times K$-invariant in the proofs of both of Proposition 4.1 and Proposition 5.3, the $K \times K$-invariant condition for the initial metric ω_0 in Theorem 1.1 can be removed by using a recent result for the uniqueness of limits of Kähler-Ricci flow with varied initial metrics in [39, 18]. In fact, we have

Theorem 6.2. *Let G be a complex semisimple Lie group and M a Fano G-manifold which admits no Kähler-Einstein metrics. Then any solution of Kähler-Ricci flow (1.1) on M with an initial metric $\omega_0 \in 2\pi c_1(M)$ is of type II.*

Proof. By Theorem 1.1, we claim that the Q-Fano variety limit $\tilde{M}_\infty$ of flow $(M, \omega(t))$ with a $K \times K$-invariant initial metric in the Hamilton-Tian conjecture is a singular variety. In fact, on contrary, by the partial C^0-estimate in [39], the Kodaira images $\tilde{M}_t$ in $\mathbb{C}P^N$ associated to $\omega(t)$ will smoothly converge to $\tilde{M}_\infty$. Then we get the estimates (4.2) and (4.3) for metrics $\omega(t)$ as in Lemma 4.2. In particular, the curvature of $\omega(t)$ is uniformly bounded, which is contradict with Theorem 1.1!

On the other hand, by [39, 18], the singular Q-Fano variety limit $\tilde{M}_\infty$ of (1.1) is independent of the choice of initial metric ω_0. Then by a result [39, Lemma 6.2], the Gromov-Hausdroff limit $(M_\infty, \omega_\infty)$ of any sequence of flow $(M, \omega(t))$ with any initial metric ω_0 could not be a smooth Riemannian manifold since $\tilde{M}_\infty$ is a singular variety. This implies that $(M, \omega(t))$ is of type II.

□

7. Appendix: An analytic proof of Proposition 5.3 by Gang Tian and Xiaohua Zhu

Fix a point $\hat{x}_\infty \in \hat{\mathcal{O}}^0_\infty$ as in Lemma 4.4 and choose a sequence of $x_i \in \mathcal{O} \subset M$ such that $\hat{x}_i = \Phi_i(x_i) \in \hat{M}_i \to \hat{x}_\infty$. Then by the relation (4.19), ψ^i converges to a convex function ψ^∞ on $D_\epsilon \subset \mathfrak{a}$ with the property

$$\nabla\psi^\infty(0) = \lim_i \nabla\psi^i(x_i), \tag{7.1}$$

where D_ϵ is a small ϵ ball centered at the original with coordinates $(y^1, ..., y^r)$ in $\mathfrak{a}$. By the regularity in Lemma 4.2, ψ^∞ is smooth and it satisfies that

$$\nabla^2\psi^\infty(\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}) = \omega_\infty(e^\infty_a, e^\infty_b) = \lim_i \psi^i_{ab}.$$

On the other hand, by the relation (4.7), it is easy to see that the limit metric ω_∞ is also $K \times K$-invariant. Thus by the uniqueness of $K \times K$-invariant functions associated to ω_∞, ψ^∞ (modulo a constant) can be uniquely extended to a Weyl-invariant convex function on $\mathfrak{a}$ such that its associated $K \times K$-invariant function Ψ_∞ on G satisfies

$$\omega_\infty = \sqrt{-1}\partial\bar{\partial}\Psi_\infty, \text{ on } G.$$

Recall the associated polytope $P \subset \mathfrak{a}^*$ of r-dimensional toric complex submanifold Z in M in Section 2.2. Then

$$\mathrm{Im}(\nabla\psi^i) = 2P,$$

which is independent of i. We shall prove

Lemma 7.1.

$$\mathrm{Im}(\nabla\psi^\infty) = 2P.$$

Proof. 1) $\mathrm{Im}(\nabla\psi^\infty) \subset 2P$. This is clear by (7.1). In fact, for any $\hat{x}_\infty \in \hat{\mathcal{O}}^0_\infty$ there is a sequence of $x_i \in M \to x_\infty = \Phi^{-1}_\infty(\hat{x}_\infty)$ such that

$$\nabla\psi^\infty(x_\infty) = \lim_i \nabla\psi^i(x_i) \in 2P. \tag{7.2}$$

2) $2P \subset \mathrm{Im}(\nabla\psi^\infty)$. On contrary, we suppose that there is a point $v \in 2P \setminus \overline{\mathrm{Im}(\nabla\psi^\infty)}$. Then we can choose a square set Δ around v by

$$\Delta = \{v' \in 2P | \ |(v')^a - v^a| \le \delta, a = 1, ..., r\} \subset 2P \setminus \overline{\mathrm{Im}(\nabla\psi^\infty)}.$$

Let

$$U_i = \{x \in \mathcal{O} |\ \nabla\psi^i(x) \in \Delta\}.$$

By Lemma 2.1, we get

$$\begin{aligned}\mathrm{vol}_{\omega_i}(U_i) =& C_0 \int_{B_i} \prod_{\alpha\in\Phi_+} \langle \alpha, \nabla\psi^i\rangle^2 \det(\nabla^2\psi^i)dy \\ =& C_0 \int_{\Delta} \pi(y')dy' \geq \delta_0,\end{aligned} \tag{7.3}$$

where C_0 and δ_0 are constants, and $B_i = \{y \in \mathfrak{a} |\ \nabla\psi^i(y) \in \Delta\}$.

We claim that there are $\epsilon_0 > 0$ and a sequence of $x_i \in U_i$ such that

$$\mathrm{dist}(x_i, M \setminus \mathcal{O}) \geq \epsilon_0. \tag{7.4}$$

In fact, if (7.4) is not true, then there is a subsequence of sets U_i (still denoted by the same sequence of U_i for convenience) such that

$$\mathrm{dist}(x', M \setminus \mathcal{O}) \to 0,\ \forall\ x' \in U_i.$$

Then we can choose a sequence of ϵ_i-tubular neighborhood T_i of $M \setminus \mathcal{O}$ with $\epsilon_i \to 0$ such that

$$U_i \subset T_i.$$

Since (T_i, ω_i) converges to $(DM_\infty, \omega_\infty)$ in the Gromov-Hausdroff topology, where $DM_\infty = \Phi_\infty^{-1}$ $(D\hat{M}_\infty)$, $\mathrm{vol}_{\omega_i}(T_i)$ goes to zero by the volume convergence theorem of Colding [10]. But this is impossible by (7.3).

By (7.4), we see that

$$\mathrm{dist}_{\omega_i}(B_{\frac{\epsilon_0}{2}}(x_i), M \setminus \mathcal{O}) \geq \frac{\epsilon_0}{4}.$$

Let x_∞ be the limit of x_i. Then $\hat{x}_\infty \notin D\hat{M}_\infty$. Since we already know that M_∞ is a G-manifold, $\hat{x}_\infty \in \hat{\mathcal{O}}^0_\infty$. Now we can use (7.2) in the above 1) to conclude that

$$\lim_i \nabla\psi^i(x_i) \in \mathrm{Im}(\nabla\psi^\infty).$$

However, this is impossible since each $\nabla\psi^i(x_i) \notin \overline{\mathrm{Im}(\nabla\psi^\infty)}$.

□

By the equivariant classification theory [3, Section 2], the polarized G-compactification (M, K_M^{-1}, J) is determined by the associated polytope P of $(Z, K_M^{-1}|_Z, J|_Z)$ as in Subsection 2.2. Let $(Z', J_\infty|_{Z'})$ be an r-dimensional toric complex submanifold of M_∞ generated by torus vector fields through a point $x_\infty \in \mathcal{O}^0_\infty \subset M_\infty$. Then by Lemma 7.1, the associated polytope of $(Z', K_{M_\infty}^{-1}|_{Z'}, J_\infty|_{Z'})$ is same as P. Thus we prove

Proposition 7.2. *The limit (M_∞, J_∞) in Proposition 4.1 is bi-holomorphic to (M, J).*

Proposition 7.2 removes the assumption that G is semisimple in Proposition 5.3, so we can improve Theorem 1.1 (also see Theorem 6.2) as follows.

Theorem 7.3. *Let G be a complex reductive Lie group and M a Fano G-manifold which admits no Kähler-Ricci soliton. Then any solution of Kähler-Ricci flow (1.1) on M with any initial metric $\omega_0 \in 2\pi c_1(M)$ is of type II.*

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