

# NUMBER OF SOLUTIONS TO A SPECIAL TYPE OF UNIT EQUATIONS IN TWO UNKNOWN

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*Dedicated to Professor Kálmán Győry on the occasion of his 80th birthday  
and to Professor Hirofumi Tsumura on the occasion of his 60th birthday*

ABSTRACT. For any fixed relatively prime positive integers  $a, b$  and  $c$  with  $\min\{a, b, c\} > 1$ , we prove that the equation  $a^x + b^y = c^z$  has at most two solutions in positive integers  $x, y$  and  $z$ , except for one specific case which exactly gives three solutions. Our result is essentially sharp in the sense that there are infinitely many examples allowing the equation to have two solutions in positive integers. From the viewpoint of a well-known generalization of Fermat's equation, it is also regarded as a 3-variable generalization of the celebrated theorem of Bennett [M.A.Bennett, On some exponential equations of S. S. Pillai, *Canad. J. Math.* 53(2001), no.2, 897-922] which asserts that Pillai's type equation  $a^x - b^y = c$  has at most two solutions in positive integers  $x$  and  $y$  for any fixed positive integers  $a, b$  and  $c$  with  $\min\{a, b\} > 1$ .

## 1. INTRODUCTION

The history of the  $S$ -unit equations related to Diophantine equations is very rich (cf. [Gy3, EvGyStTi, EvScSch, EvGy2]). Indeed, many diophantine problems can be reduced to  $S$ -unit equations over number fields. Especially, the simplest one among those is the  $S$ -unit equation in two unknowns over the rationals, which is written as follows:

$$(1.1) \quad \alpha X + \beta Y + \gamma Z = 0,$$

where  $\alpha, \beta, \gamma$  are given non-zero integers, and  $X, Y, Z$  are unknown integers composed of finitely many given primes. The set of the predetermined primes for unknowns is as usual denoted by  $S$ . From the theory on Diophantine approximations we know that there are only finitely many solutions to equation (1.1), and effective upper bounds for their sizes can be obtained by means of Baker's theory of linear forms in logarithms (cf. [Gy, Gy2, BuGy, GyYu]). Since any unknown can be expressed as a product of powers of given primes, equation (1.1) is an exponential Diophantine equation. Based upon this, one of the simplest examples of equation (1.1) is

$$(1.2) \quad a^x - b^y = c,$$

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where  $a, b, c$  are fixed positive integers with  $\min\{a, b\} > 1$ , and  $x, y$  are unknown positive integers. This equation is a special case of Pillai's equation, and Pillai's famous conjecture says that there are only finitely many pairs of distinct powers with their difference fixed. It is also worth noting that the case where  $c = 1$  corresponds to a special one of Catalan's equation (cf. [Mi]).

In a series of papers in the 1930's, Pillai [Pi, Pi2] actively studied equation (1.2) and obtained some finiteness results on the number of solutions (for more detail see [Be, Be2]). Early 1990's, Scott [Sc] extensively investigated equation (1.2) in the case where  $a$  is a prime with a motivation to a classical problem listed in R.K.Guy's book [Gu], and he used strictly elementary methods in quadratic fields to obtain very sharp upper bounds for the number of solutions in several cases. For more details on these topics or some other related ones, see for example [ShTi, Be2, BerHa].

In the direction on the number of solutions to equation (1.2), Bennett [Be] established the following definitive result.

**Theorem** (M.A. Bennett). *For any fixed positive integers  $a, b$  and  $c$  with  $\min\{a, b\} > 1$ , equation (1.2) has at most two solutions in positive integers  $x$  and  $y$ .*

This result is essentially sharp in the sense that there are a number of examples where there are two solutions to equation (1.2) (cf. [Be, (1.2)]). The proof of Bennett uses lower bounds for linear forms in logarithms of two algebraic numbers together with a 'gap principle', based upon an arithmetic of exponential congruences, which gives rise to a large gap among three hypothetical solutions. It should be also remarked that the non-coprimality case, i.e.,  $\gcd(a, b) > 1$  is handled just by a short elementary observation. Several other works to estimate the number of solutions of more general equations of type (1.2) were made available in the literature (cf. [BuLu, ScSt, ScSt2, ScSt3]).

There is another example of equation (1.1), which is not only regarded as a 3-variable generalization of equation (1.2), but also closely related to the so-called generalized Fermat equation, that is, a Diophantine equation arising from the quest to seek for all the relations that the sum of two powers of 'coprime' positive integers is equal to another power (cf. [BeMiSi], [Co, Ch.14]). It is the main subject in this article, given as follows:

$$(1.3) \quad a^x + b^y = c^z,$$

where  $a, b, c$  are fixed coprime positive integers with  $\min\{a, b, c\} > 1$ , and  $x, y, z$  are unknown positive integers. Equation (1.3) is also one of the simplest one among purely exponential Diophantine equations. It seems that the earliest published work on solving equation (1.3) is due to Sierpiński [Si], who solved the equation for  $(a, b, c) = (3, 4, 5)$ . Just after this work, Jeśmanowicz [Je] (who was a student of Sierpiński) considered equation (1.3) for other primitive Pythagorean triples, and he posed the problem of determining the solutions to the equation for any Pythagorean triple. His problem is the most known unsolved problem concerning equation (1.3). In a series of papers in the 1990's, Terai gave some pioneer works on equation

(1.3) for some families of  $(a, b, c)$  including Pythagorean triples, and later he posed several problems including a generalization of the mentioned one of Sierpiński-Jeśmanowicz, now called Terai's conjecture (cf. [Te]). For more details on these topics or some other related ones, see some recent papers [CiMi, Lu2, Miy] and the references therein.

As mentioned before, from the theory of  $S$ -unit equations, we know that equation (1.3) has at most finitely many solutions, moreover, the number of solutions can be bounded by an absolute large number (see the excellent survey of Györy [Gy4]). However, apparently a kind of such estimates is far from the actual number, indeed, many existing works in the literature, and also a much more general treatment to equation (1.1) (cf. [Theorem 1, EvGyStTi]), suggest that equation (1.3) has few solutions in general. In this direction, some important progresses have been made in the recent years by several researchers. In a series of papers, Hu and Le [HuLe, HuLe2, HuLe3] discussed equation (1.3) over the invertible residue classes modulo powers of the base numbers of the equation. Using various elementary methods including continued fractions, they found a large gap among three hypothetical solutions, so-called 'gap principle', and the combination of their gap principle and Baker's method implies that equation (1.3) has at most two solutions whenever the maximal value of  $a, b, c$  is sufficiently large (see Proposition 2.1 below). Here we mention that there are infinitely many examples where there are two solutions to equation (1.3) (see (15.1) in the final section). Since the exponential unknowns  $x, y$  and  $z$  are bounded by an explicit constant depending only on  $a, b$  and  $c$  by Baker's method, just a finite search remains to be done in order to obtain the definitive result on the number of solutions to equation (1.3) corresponding to Bennett's mentioned theorem. However, a kind of brute force computations is never enough to settle that finite search. Related to this study, the work of Scott and Styer [ScSt4] should be referred. They considerably improved the argument over imaginary quadratic fields in [Sc] dealing with the case where  $c$  is a prime, to obtain the same conclusion as that of the mentioned work of Hu and Le, whenever  $c$  is odd (see Proposition 2.2 below). Actually, the main content of this article is to completely handle the remaining finite search mentioned before, as follows:

**Theorem 1** (Main Theorem). *For any fixed relatively prime positive integers  $a, b$  and  $c$  with  $\min\{a, b, c\} > 1$ , equation (1.3) has at most two solutions in positive integers  $x, y$  and  $z$ , except when  $(a, b, c)$  is  $(3, 5, 2)$  or  $(5, 3, 2)$  which exactly gives three solutions.*

This result is essentially sharp as indicated in the mentioned work of Hu and Le, and its exceptional case comes from the identities  $3+5 = 2^3$ ,  $3^3+5 = 2^5$  and  $3+5^3 = 2^7$  (cf. (15.1)).

The proof of our theorem proceeds under the assumption that all  $a, b, c$  are bounded from above explicitly and  $c$  is even, and consists of three main important steps. The first step is to improve the gap principle of Hu and Le. Their gap principle is derived by examining equation (1.3) modulo powers of each of  $a, b$  and  $c$ , and is expressed as some divisibility relation or

some inequality. The main idea for the improving is to consider two congruences ‘simultaneously’ in their treatment using modulus of each power of the base numbers. Roughly saying, this replaces a factor in the inequality from Hu and Le’s gap principle as a common factor of two of exponential unknowns of the equation, where the value of an appearing factor is strictly restricted from the viewpoint of generalized Fermat equations. In this stage, by combining the improved gap principle together with the 2-adic analogue of Baker’s method, the required bounds for  $\max\{a, b, c\}$  for which theorem holds true are substantially reduced. The second step is to find very sharp upper bounds for all the exponential unknowns of at least two of three equations, which in what follows are called the first two equations, derived from equation (1.3) by assuming the existence of three hypothetical solutions. This is done elementarily by comparing the 2-adic valuations of both sides of each of those three equations, a procedure that works only in the case where  $c$  is even, which can be assumed from the mentioned work of Scott and Styer. Working out these two steps together with several elementary number theory methods yields at most finitely many possible values of all parameters appearing in the first two equations. Finally, in each of those cases, we check whether those two equations hold or not. At this point it is worth noting that although the derived general bounds for all letters in those equations are relatively sharp, a direct enumeration of all possible solutions of the system formed by the first two equations is impossible. Therefore, we worked very carefully and found efficient methods for solving that system in reasonable computation time.

The organization of this article is as follows. In the next section, we prepare some useful conditions which are consequences of previous existing results related to equations (1.2) and (1.3). Section 3 is devoted to find a sharp upper bound for  $z$  using a result of Bugeaud [Bu] on the 2-adic estimate of the difference between two powers of algebraic numbers. On the other hand, we find some 2-adic properties of  $z$  in Section 4, where one of those yields an exact information in a certain case. We summarize the contents of these sections together with the second main idea in the forthcoming section, in particular, we deduce relatively small upper estimates for all exponential unknowns of the first two equations. In Section 6, we improve the gap principle of Hu and Le, and give some of its applications in Section 7. In Section 8, we quote several existing works on the generalized Fermat equation, and give their applications to the improved gap principle. Section 9 is devoted to study a certain Diophantine equation related to equation (1.2). In Section 10, we use results established in the previous section together with the preparations in Sections 6, 7 and 8 to settle the case where the values of two  $z$  of the first two equations coincide. Sections 11 and 12 are devoted to exactly find all possible values of letters in the first two equations, and we sieve those completely in Sections 13 and 14, and the proof is completed. In the final section, we make a few remarks concerning an extension of Theorem 1.

All computations in this paper were performed using the computer package MAGMA [BoCaPl]. The total computation time needed for this article did not exceed 17 hours.

## 2. PREVIOUS RESULTS AND THEIR CONSEQUENCES

We begin by quoting the following results on equations (1.2) or (1.3). The first two play important roles in the proof of our theorem.

**Proposition 2.1** ([HuLe3]). *If  $\max\{a, b, c\} > 10^{62}$ , then Theorem 1 is true.*

**Proposition 2.2** ([ScSt4]). *If  $c$  is odd, then Theorem 1 is true.*

The following is a direct consequence of [Sc, Theorem 6].

**Proposition 2.3.** *If  $c = 2$ , then Theorem 1 is true.*

The following is a direct consequence of [ScSt, Theorem 1], and it is a relevant analogue to Bennett's result recalled above.

**Proposition 2.4.** *For any fixed positive integers  $a, b$  and  $c$  with  $\min\{a, b\} > 1$ , the equation*

$$a^x + b^y = c$$

*has at most two solutions in positive integers  $x$  and  $y$ .*

Using Propositions 2.2, 2.3 and 2.4, we show a technical lemma.

**Lemma 2.1.** *Theorem 1 is true in each of the following cases:*

- $a \equiv 1 \pmod{4}$ ,  $b \equiv 1 \pmod{4}$ ;
- $\max\{a, b\} < 11$ ,  $c \equiv 0 \pmod{2}$ .

*Proof.* In the first case, we take equation (1.3) modulo 4 to see that  $c^z \equiv 2 \pmod{4}$ , implying  $z = 1$ . Proposition 2.4 completes the proof.

In the second case, suppose that equation (1.3) has three solutions. By Proposition 2.4, there exists at least one solution with  $z > 1$  among them. Since both  $a, b$  are composed of only primes in  $\{3, 5, 7\}$ , we have  $c = 2$  by [BeBi, Theorem 7.2]. Our lemma follows from Proposition 2.3.  $\square$

In order to prove our theorem, it suffices to consider the case where none of  $a, b, c$  is a perfect power, and  $a, b, c$  are pairwise coprime. Moreover, in view of Propositions 2.1, 2.2 and Lemma 2.1, we may assume any of the conditions in (\*) below.

$$(*) \quad \begin{cases} \text{none of } a, b, c \text{ is a perfect power, } a, b, c \text{ are pairwise coprime;} \\ a \equiv -1 \pmod{4} \text{ or } b \equiv -1 \pmod{4}; \\ \max\{a, b\} \geq 11, \quad \max\{a, b, c\} \leq 10^{62}; \\ 2 \mid c, \quad c > 4. \end{cases}$$

In particular, in the sequel, we always assume that  $a, b, c$  are pairwise coprime, both  $a, b$  are odd and  $c$  is even.

3. GENERAL UPPER BOUND FOR  $z$ 

Here we find a relatively sharp upper bound for  $z$  in equation (1.3). For this we prepare some lemmas.

The following is a slight improvement of a special case of [PedW, Lemma 2.2].

**Lemma 3.1.** *Let  $v$  be a positive number. Assume that  $\frac{t}{\log^2 t} \leq v$  for some positive number  $t$  with  $t > e^2$ , where  $e = \exp(1)$ . Then*

$$t < \left(1 + \frac{\log \log v_0}{\log v_0 - 1}\right)^2 v \log^2(4v),$$

where  $v_0$  is any number with  $e < v_0 < 2v^{1/2}$ .

*Proof.* Firstly, note that the function  $\frac{T}{\log^2 T}$  in  $T$  is increasing for  $T > e^2$ . Thus it suffices to consider the case where  $\frac{t}{\log^2 t} = v$ . Define  $w$  and  $Y$  as  $w = \frac{2t^{1/2}}{\log t}$  and  $(1 + Y)w \log w = t^{1/2}$ . It is easy to see that  $Y > 0$  as  $t > e^2$ . Observe that

$$(1 + Y)w \log w = t^{1/2} = \frac{w}{2} \log t = w \log((1 + Y)w \log w).$$

Thus

$$Y \log w = \log(1 + Y) + \log \log w.$$

Since  $\log(1 + Y) < Y$ , we have  $Y < \frac{\log \log w}{\log w - 1}$ , that is,

$$t^{1/2} < \left(1 + \frac{\log \log w}{\log w - 1}\right) w \log w.$$

Since  $w = 2v^{1/2} = \frac{2t^{1/2}}{\log t} > e$  by assumption, and the function  $\frac{\log \log W}{\log W - 1}$  in  $W$  is decreasing for  $W > e$ , the above displayed inequality leads to the assertion.  $\square$

For a rational prime  $p$  and a non-zero integer  $A$ , let  $\nu_p(A)$  denote as usual the  $p$ -adic valuation of  $A$ , that is, the exponent of  $p$  in the prime factorization of  $A$ .

The next lemma is well-known and gives a precise information on the 2-adic valuations of integers in certain forms.

**Lemma 3.2.** *Let  $s$  be an odd integer with  $s \neq \pm 1$ . For any positive integer  $n$ , the following hold.*

$$\begin{aligned} \nu_2(s^n - 1) &= \nu_2(s^2 - 1) - 1 + \nu_2(n), & \text{if } s \equiv 1 \pmod{4} \text{ or } 2 \mid n; \\ \nu_2(s^n - 1) &= 1, & \text{if } s \equiv 3 \pmod{4} \text{ and } 2 \nmid n; \\ \nu_2(s^n + 1) &= \nu_2(s + 1), & \text{if } 2 \nmid n; \\ \nu_2(s^n + 1) &= 1, & \text{if } 2 \mid n. \end{aligned}$$

Define the function  $\log_*$  as follows:

$$\log_*(x) := \log \max\{x, e\} \quad (x > 0).$$

Note that  $(\log_* x)^2$  is an increasing function.

The following proposition is a special case of [Bu, Theorem 2.13].

**Proposition 3.1.** *Let  $u_1, u_2$  be coprime odd integers with  $u_1, u_2 \neq \pm 1$ . Assume that a positive integer  $g$  satisfies*

$$\nu_2(u_j^g - 1) \geq E \quad (j = 1, 2)$$

*for some number  $E$  with  $E > 2$ . Let  $H_1, H_2$  be real numbers satisfying*

$$H_j \geq \log \max\{|u_j|, 2^E\} \quad (j = 1, 2).$$

*Put*

$$\Lambda = u_1^{b_1} - u_2^{b_2} \quad (\neq 0),$$

*where  $b_1, b_2$  are positive integers. If  $\nu_2(u_2 - 1) \geq 2$ , then*

$$\nu_2(\Lambda) \leq \frac{36.1 g B^2}{(\log 2)^4 E^3} H_1 H_2,$$

*where*

$$B = \max\left\{\log\left(\frac{b_1}{H_2} + \frac{b_2}{H_1}\right) + \log(E \log 2) + 0.4, 6E \log 2\right\}.$$

We give an application of Proposition 3.1 as follows.

**Lemma 3.3.** *Assume that  $\max\{a, b\} \geq 11$ . Put*

$$\alpha = \min\{\nu_2(a^2 - 1), \nu_2(b^2 - 1)\} - 1, \quad \beta = \nu_2(c).$$

*Let  $(x, y, z)$  be a solution of equation (1.3) with  $z > 1$ . Then*

$$z < \max\{c_1, c_2 \log_*^2(c_3 \log c)\} (\log a) \log b,$$

*where*

$$(c_1, c_2, c_3) = \begin{cases} \left(\frac{1803.3m_2}{\beta}, \frac{23.865m_2}{\beta}, \frac{143.75(m_2 + 1)}{\beta}\right), & \text{if } \alpha = 2, \\ \left(\frac{2705m_3}{\alpha\beta}, \frac{156.39m_3(1 + \frac{\log v_\alpha}{v_\alpha - 1})^2}{\alpha^3\beta}, \frac{646.9(m_3 + 1)}{\alpha^2\beta}\right), & \text{if } \alpha \geq 3 \end{cases}$$

*with  $v_\alpha = 3\alpha \log 2 - \log(3\alpha \log 2)$ , and*

$$m_2 = \begin{cases} \frac{\log 8}{\log \min\{a, b\}}, & \text{if } \min\{a, b\} \leq 7, \\ 1, & \text{if } \min\{a, b\} > 9, \end{cases} \quad m_3 = \frac{\alpha \log 2}{\log(2^\alpha - 1)}.$$

**Remark 1.**  $c_1, c_2, c_3$  are explicit constants depending only on  $\alpha, \beta$  for  $\alpha \geq 3$ , and also on  $m_2$  only if  $\alpha = 2$ . Also, these numbers are decreasing on  $\alpha$  ( $\geq 3$ ) and on  $\beta$ .

*Proof of Lemma 3.3.* Since  $c$  is even and  $z > 1$ , it follows from equation (1.3) that  $a^x + b^y \equiv 0 \pmod{4}$ . Therefore, one of the following cases holds.

$$(3.1) \quad \begin{cases} a \equiv 1, b \equiv -1 \pmod{4}, & 2 \nmid y; \\ a \equiv -1, b \equiv 1 \pmod{4}, & 2 \nmid x; \\ a \equiv b \equiv -1 \pmod{4}, & x \not\equiv y \pmod{2}. \end{cases}$$

Put  $\Lambda = a^x + b^y$ . Since  $\Lambda = c^z$ , we have

$$(3.2) \quad z = \frac{1}{\beta} \cdot \nu_2(\Lambda).$$

In order to find an upper bound for  $\nu_2(\Lambda)$ , let us apply Proposition 3.1. For this, set  $u_1, u_2, b_1, b_2$  as follows:

$$(u_1, u_2; b_1, b_2) = \begin{cases} (a, -b; x, y), & \text{if } a \equiv 1, b \equiv -1 \pmod{4}, \\ (-a, b; x, y), & \text{if } a \equiv -1, b \equiv 1 \pmod{4}, \\ (-a, -b; x, y), & \text{if } a \equiv b \equiv -1 \pmod{4}. \end{cases}$$

Then, by (3.1),

$$\begin{aligned} \pm\Lambda &= u_1^{b_1} - u_2^{b_2}, \\ u_1 &= \pm a, \quad u_2 = \pm b, \quad \min\{\nu_2(u_1 - 1), \nu_2(u_2 - 1)\} = \alpha. \end{aligned}$$

For any positive integer  $g$  and  $i \in \{1, 2\}$ , observe from Lemma 3.2 that

$$\nu_2(u_i^g - 1) = \nu_2(u_i^2 - 1) - 1 + \nu_2(g) = \nu_2(u_i - 1) + \nu_2(g) \geq \alpha + \nu_2(g).$$

Thus we may set

$$(g, E) := \begin{cases} (2, 3), & \text{if } \alpha = 2, \\ (1, \alpha), & \text{if } \alpha \geq 3. \end{cases}$$

In what follows, let us separately consider the cases where  $\alpha \geq 3$  and  $\alpha = 2$ . By symmetry of  $a$  and  $b$ , we may assume that  $a > b$  without loss of generality.

First, consider the case where  $\alpha \geq 3$ . Observe that

$$|u_1| = a > b = |u_2| \geq 2^\alpha - 1.$$

Thus we may set  $H_1 := \log a$  and  $H_2 := m_3 \log b$ . Proposition 3.1 together with (3.2) gives

$$(3.3) \quad z \leq \frac{36.1 m_3 \mathcal{B}^2}{\beta (\log 2)^4 \alpha^3} (\log a) \log b,$$

where

$$\mathcal{B} = \max \left\{ \log \left( \frac{x}{m_3 \log b} + \frac{y}{\log a} \right) + \log(\alpha \log 2) + 0.4, 6\alpha \log 2 \right\}.$$

Since  $x < \frac{\log c}{\log a} z$ ,  $y < \frac{\log c}{\log b} z$ , we have

$$\mathcal{B} \leq \log \max\{zL', 2^{6\alpha}\}$$

with

$$L' = \frac{(1 + \frac{1}{m_3}) \alpha (\log 2) e^{0.4 \log c}}{(\log a) \log b}.$$

If  $zL' \leq 2^{6\alpha}$ , then inequality (3.3) yields

$$z \leq \frac{36.1 m_3 (6\alpha \log 2)^2}{\beta (\log 2)^4 \alpha^3} (\log a) \log b < \frac{2705 m_3}{\alpha \beta} (\log a) \log b.$$

Suppose that  $zL' > 2^{6\alpha}$ . Then

$$z \leq \frac{36.1 m_3 \log^2(zL')}{\beta (\log 2)^4 \alpha^3} (\log a) \log b,$$

that is,

$$\frac{zL'}{\log^2(zL')} \leq \frac{36.1 m_3 L'}{(\log 2)^4 \alpha^3 \beta} (\log a) \log b = \frac{p(m_3 + 1) \log c}{\alpha^2 \beta}$$



with  $p = \frac{36.1 \cdot e^{0.4}}{(\log 2)^3}$ . Let us apply Lemma 3.1 to the above inequality with

$$v := \frac{p(m_3 + 1) \log c}{\alpha^2 \beta}, \quad v_0 := \frac{2^{3\alpha}}{3\alpha \log 2}, \quad t := zL'.$$

Indeed, as  $zL' > 2^{6\alpha}$ ,

$$2v^{1/2} \geq 2 \sqrt{\frac{zL'}{\log^2(zL')}} > v_0.$$

Putting  $V = \left(1 + \frac{\log \log v_0}{\log v_0 - 1}\right)^2$ , we see that

$$\begin{aligned} z &< \frac{1}{L'} V v \log^2(4v) \\ &= \frac{V(\log a) \log b}{\left(1 + \frac{1}{m_3}\right) \alpha (\log 2) e^{0.4} \log c} \frac{\frac{36.1 \cdot e^{0.4} (m_3 + 1)}{(\log 2)^3} \log c}{\alpha^2 \beta} \log^2(4v) \\ &= \frac{36.1 V m_3}{(\log 2)^4 \alpha^3 \beta} \log^2(4v) (\log a) \log b \\ &< \frac{156.39 V m_3}{\alpha^3 \beta} \log_*^2 \left( \frac{646.9 (m_3 + 1) \log c}{\alpha^2 \beta} \right) (\log a) \log b. \end{aligned}$$

To sum up, the assertion is shown in this case.

Next, consider the case where  $\alpha = 2$ . We proceed almost similarly to the previous case, and use the same notation as seen in there. Observe that  $2^E = 8$ , and  $|u_1| = a = \max\{a, b\} > 10$  by assumption. Thus we may set  $H_1 := \log a$  and  $H_2 := m_2 \log b$ . Proposition 3.1 gives

$$(3.4) \quad z \leq \frac{36.1 g m_2 \mathcal{B}^2}{\beta (\log 2)^4 (\alpha + 1)^3} (\log a) \log b,$$

where

$$\mathcal{B} = \max \left\{ \log \left( \frac{x}{m_2 \log b} + \frac{y}{\log a} \right) + \log((\alpha + 1) \log 2) + 0.4, 6(\alpha + 1) \log 2 \right\}.$$

Observe that

$$\mathcal{B} \leq \log \max\{zL', 2^{6(\alpha+1)}\}$$

with

$$L' = \frac{\left(1 + \frac{1}{m_2}\right) (\alpha + 1) (\log 2) e^{0.4} \log c}{(\log a) \log b}.$$

If  $zL' \leq 2^{6(\alpha+1)}$ , then inequality (3.4) yields

$$z \leq \frac{36.1 g m_2 (6(\alpha + 1) \log 2)^2}{\beta (\log 2)^4 (\alpha + 1)^3} (\log a) \log b < \frac{1803.3 m_2}{\beta} (\log a) \log b.$$

Suppose that  $zL' > 2^{6(\alpha+1)}$ . Then

$$\frac{zL'}{\log^2(zL')} \leq \frac{36.1 g m_2 \cdot L'}{\beta (\alpha + 1)^3 (\log 2)^4} (\log a) \log b = p(m_2 + 1) \frac{\log c}{(\alpha + 1)^2 \beta}$$

with  $p = \frac{36.1g e^{0.4}}{(\log 2)^3}$ . Let us apply Lemma 3.1 to the above inequality with

$$v := p(m_2 + 1) \frac{\log c}{(\alpha + 1)^2 \beta}, \quad v_0 := 82.073, \quad t := zL'.$$

Indeed, as  $zL' > 2^{6(\alpha+1)} = 2^{18}$ ,

$$2v^{1/2} \geq 2 \sqrt{\frac{zL'}{\log^2(zL')}} > 2 \sqrt{\frac{2^{18}}{\log^2(2^{18})}} > v_0.$$

Putting  $V = \left(1 + \frac{\log \log v_0}{\log v_0 - 1}\right)^2$ , we see that

$$\begin{aligned} z &< \frac{1}{L'} V v \log^2(4v) \\ &= \frac{V(\log a) \log b}{\left(1 + \frac{1}{m_2}\right)(\alpha + 1)(\log 2) e^{0.4} \log c} \frac{p(m_2 + 1) \log c}{(\alpha + 1)^2 \beta} \log^2(4v) \\ &= \frac{p m_2 V}{(\log 2) e^{0.4} (\alpha + 1)^3 \beta} \log_*^2 \left( \frac{4p(m_2 + 1) \log c}{(\alpha + 1)^2 \beta} \right) (\log a) \log b \\ &= \frac{36.1 g m_2 V}{(\log 2)^4 (\alpha + 1)^3 \beta} \log_*^2 \left( \frac{\frac{4 \cdot 36.1 g e^{0.4}}{(\log 2)^3} (m_2 + 1) \log c}{(\alpha + 1)^2 \beta} \right) (\log a) \log b \\ &< \frac{23.865 m_2}{\beta} \log_*^2 \left( \frac{143.75(m_2 + 1) \log c}{\beta} \right) (\log a) \log b. \end{aligned}$$

To sum up, the lemma is proved.  $\square$

#### 4. 2-ADIC INVESTIGATION OF $z$

In this section, we show some results related concerning the 2-adic properties of  $z$  in equation (1.3). For this we prepare some notation as follows:

$$\begin{aligned} \alpha_a &:= \nu_2(a^2 - 1) - 1, \quad \alpha_b := \nu_2(b^2 - 1) - 1, \\ \alpha &:= \min\{\alpha_a, \alpha_b\}, \quad \beta := \nu_2(c). \end{aligned}$$

Note that both  $a, b$  are congruent to  $\pm 1$  modulo  $2^\alpha$  and  $\alpha \geq 2$ .

**Lemma 4.1.** *Let  $(x, y, z)$  be a solution of equation (1.3). Then either  $(\beta, z) = (1, 1)$  or  $\beta z \geq \alpha$ .*

*Proof.* We take equation (1.3) modulo  $2^\alpha$  to see that either

$$c^z \equiv \pm 2 \pmod{2^\alpha} \quad \text{or} \quad c^z \equiv 0 \pmod{2^\alpha}.$$

The first congruence implies that  $2 \parallel c^z$  since  $\alpha \geq 2$ , and the second one means that  $\nu_2(c^z) \geq \alpha$ .  $\square$

**Lemma 4.2.** *Let  $(x, y, z) = (X, Y, Z), (X', Y', Z')$  be two solutions of equation (1.3). Then  $XY' \neq X'Y$ , and*

$$\beta \cdot \min\{Z, Z'\} \leq \alpha + \nu_2(XY' - X'Y).$$

*Proof.* Let  $\mathcal{A} \in \{a, b\}$ . Since  $c$  is even, we use Lemma 6.2 (see below) for  $(A, B, C; \lambda) = (a, b, c; 1)$  to see that

$$(4.1) \quad \mathcal{A}^{\mathcal{E}} \equiv \delta \pmod{2^{\beta \min\{Z, Z'\}}}$$

for some  $\delta \in \{1, -1\}$ , where  $\mathcal{E} = |XY' - X'Y|$  is a positive integer. Lemma 3.2 gives

$$\beta \min\{Z, Z'\} \leq \nu_2(\mathcal{A}^{\mathcal{E}} - \delta) \leq \nu_2(\mathcal{A}^2 - 1) - 1 + \nu_2(\mathcal{E}).$$

This shows the lemma as  $\alpha = \min_{\mathcal{A} \in \{a, b\}} \nu_2(\mathcal{A}^2 - 1) - 1$ .  $\square$

Finally, we show a sufficient condition to give an exact information on  $z$  in a certain case.

**Lemma 4.3.** *Let  $(x, y, z) = (X, Y, Z), (X', Y', Z')$  be two solutions of equation (1.3) with  $\beta Z \neq 1$  and  $\beta Z' \neq 1$ . If  $X \not\equiv X' \pmod{2}$ , then  $Z = \alpha/\beta$  or  $Z' = \alpha/\beta$ .*

*Proof.* Without loss of generality, we may assume that  $X$  is odd and  $X'$  is even. If  $Y'$  is even, then  $a^{X'} + b^{Y'}$  is a sum of two squares of odd integers, so that  $a^{X'} + b^{Y'} \equiv 2 \pmod{4}$ , whereby  $\beta Z' = \nu_2(c^{Z'}) = \nu_2(a^{X'} + b^{Y'}) = 1$ . Thus,  $Y'$  is odd by assumption. To sum up,

$$(4.2) \quad 2 \nmid X, \quad 2 \mid X', \quad 2 \nmid Y'.$$

Take  $\delta_a, \delta_b \in \{1, -1\}$  such that  $a \equiv \delta_a \pmod{4}$  and  $b \equiv \delta_b \pmod{4}$ . By the definition of  $\alpha_a$  and  $\alpha_b$ ,

$$2^{\alpha_a} \parallel (a - \delta_a), \quad 2^{\alpha_b} \parallel (b - \delta_b).$$

Recall that  $\min\{\alpha_a, \alpha_b\} = \alpha$ . Then, for any solution  $(x, y, z)$  of equation (1.3),

$$a^x \equiv \delta_a^x + 2^u \pmod{2^{u+1}}, \quad b^y \equiv \delta_b^y + 2^v \pmod{2^{v+1}},$$

where  $u = \alpha_a + \nu_2(x)$  and  $v = \alpha_b + \nu_2(y)$ . Replacing the modulus each of the above congruences by  $2^{\min\{u, v\}+1}$  and adding the resulting relations yields

$$a^x + b^y \equiv \delta + 2^u + 2^v \pmod{2^{\min\{u, v\}+1}}$$

with  $\delta := \delta_a^x + \delta_b^y \in \{-2, 0, 2\}$ . This congruence implies that

$$a^x + b^y \equiv \begin{cases} 2 \pmod{4}, & \text{if } \delta = \pm 2, \\ 2^{\min\{u, v\}} \pmod{2^{\min\{u, v\}+1}}, & \text{if } \delta = 0, u \neq v. \end{cases}$$

Since  $\nu_2(a^x + b^y) = \nu_2(c^z) = \beta z$ , we have

$$\beta z = \begin{cases} 1, & \text{if } \delta = \pm 2, \\ \min\{u, v\}, & \text{if } \delta = 0, u \neq v. \end{cases}$$

We apply the previous argument with  $(x, y, z) = (X, Y, Z), (X', Y', Z')$ . In view of (4.2), we conclude that

$$\begin{cases} \beta Z = \min\{U, V\}, & \text{if } U \neq V, \\ \beta Z' = \min\{U', V'\}, & \text{if } U' \neq V', \end{cases}$$

where

$$U = \alpha_a, \quad V = \alpha_b + \nu_2(Y), \quad U' = \alpha_a + \nu_2(X'), \quad V' = \alpha_b.$$

Observe that  $U - V < U' - V'$ . To sum up, these relations together imply that

$$\begin{cases} \alpha_a - \alpha_b \geq \nu_2(Y), \beta Z' = V' = \alpha_b, & \text{if } U \geq V, \\ \alpha_b - \alpha_a \geq \nu_2(X'), \beta Z = U = \alpha_a, & \text{if } U' \leq V', \\ \beta Z = U = \alpha_a, \beta Z' = V' = \alpha_b, & \text{if } U < V \text{ and } U' > V'. \end{cases}$$

In particular, this shows the lemma.  $\square$

## 5. PRELIMINARIES FOR THEOREM 1

From now on, let  $(a, b, c)$  be any fixed triple of positive integers satisfying (\*). Positive integers  $\alpha, \beta$  are defined as in the previous section. From (\*),

$$(5.1) \quad \max\{a, b\} \geq \max\{11, 2^\alpha + 1\}, \quad \min\{a, b\} \geq 2^\alpha - 1, \quad c \geq 3 \cdot 2^\beta.$$

Also, we suppose that equation (1.3) has three solutions, say  $(x, y, z) = (x_t, y_t, z_t)$  with  $t \in \{1, 2, 3\}$ , that is,

$$a^{x_1} + b^{y_1} = c^{z_1}, \quad a^{x_2} + b^{y_2} = c^{z_2}, \quad a^{x_3} + b^{y_3} = c^{z_3}.$$

Without loss of generality, we may assume that

$$(5.2) \quad z_1 \leq z_2 \leq z_3.$$

For each  $t \in \{1, 2, 3\}$ , we often refer to the equation  $a^{x_t} + b^{y_t} = c^{z_t}$  as ‘ $t$ -th equation’. The pair consisting of the 1st and 2nd equations is referred to as ‘the first two equations’.

It is obvious that

$$(5.3) \quad x_t < \frac{\log c}{\log a} z_t, \quad y_t < \frac{\log c}{\log b} z_t \quad (t = 1, 2, 3).$$

Lemmas 4.1, 4.2 and 3.3 tell us that

$$(5.4) \quad (\beta, z_t) = (1, 1) \quad \text{or} \quad z_t \geq \frac{\alpha}{\beta} \quad (t = 1, 2);$$

$$(5.5) \quad \beta z_t \leq \alpha + \nu_2(x_t y_{t+1} - x_{t+1} y_t) \quad (t = 1, 2);$$

$$(5.6) \quad z_3 < \mathcal{H}_{\alpha, \beta, m_2}(c; a, b),$$

respectively, where

$$\mathcal{H}_{\alpha, \beta, m_2}(u; v, w) := \max\{c_1, c_2 \log_*^2(c_3 \log u)\} \cdot \log v \cdot \log w.$$

From Remark 1, the lower index  $m_2$  of  $\mathcal{H}$  makes sense only when  $\alpha = 2$ . In what follows, we simply write  $\mathcal{H}_{\alpha, \beta, m_2}(u; v, w) = \mathcal{H}_{\alpha, \beta, m_2}(u)$  when  $u = v = w$ , also,  $\mathcal{H}_{\alpha, \beta, m_2}(u; v) := \mathcal{H}_{\alpha, \beta, m_2}(u; v, w) / \log w$ .  $\mathcal{H}$  is decreasing on  $\alpha$  ( $\geq 3$ ). Also, all indices  $\alpha, \beta, m_2$  are often omitted.

Under this setting, we show some results as the first preliminaries in the proof of Theorem 1.

**Lemma 5.1.** *The following inequalities hold.*

$$\begin{aligned} (i) \quad & \beta z_1 - \frac{\log(z_1 z_2)}{\log 2} < \alpha + \frac{1}{\log 2} \log \left( \frac{\log^2 c}{(\log a) \log b} \right) - \frac{\log g}{\log 2}, \\ (ii) \quad & \beta z_2 - \frac{\log z_2}{\log 2} < \alpha + \frac{\log \mathcal{H}_{\alpha, \beta, m_2}(c)}{\log 2} - \frac{\log g}{\log 2}, \end{aligned}$$

where  $g$  is the greatest odd divisor of  $\gcd(x_2, y_2)$ .

*Proof.* Let  $t \in \{1, 2\}$ . From (5.3),

$$|x_t y_{t+1} - x_{t+1} y_t| < \max\{x_t y_{t+1}, x_{t+1} y_t\} < \frac{\log^2 c}{(\log a) \log b} z_t z_{t+1}.$$

In particular, by (5.6),

$$(5.7) \quad |x_2 y_3 - x_3 y_2| < \frac{\log^2 c}{(\log a) \log b} z_2 z_3 < z_2 \cdot \mathcal{H}_{\alpha, \beta, m_2}(c).$$

Also, by the definition of  $g$ ,

$$\nu_2(x_t y_{t+1} - x_{t+1} y_t) = \nu_2\left(\frac{x_t y_{t+1} - x_{t+1} y_t}{g}\right).$$

These relations together with (5.5) yield

$$\begin{aligned} \beta z_t &\leq \alpha + \nu_2(x_t y_{t+1} - x_{t+1} y_t) \leq \alpha + \frac{\log(|x_t y_{t+1} - x_{t+1} y_t|/g)}{\log 2} \\ &< \alpha + \frac{1}{\log 2} \log\left(\frac{\log^2 c}{g(\log a) \log b} z_t z_{t+1}\right). \end{aligned}$$

Thus

$$\beta z_t - \frac{\log z_t}{\log 2} < \alpha + \frac{1}{\log 2} \log\left(\frac{\log^2 c}{g(\log a) \log b} z_{t+1}\right).$$

This inequality for  $t = 1$  immediately yields (i). Assertion (ii) follows from the above inequality for  $t = 2$  together with (5.6).  $\square$

**Definition 5.1.** For given  $\alpha, \beta, m_2, a, b, c$  and  $g$ , define  $\mathcal{U}_2 = \mathcal{U}_2(\alpha, \beta, m_2, c, g)$  as the largest  $z_2$  among  $z_2$  satisfying inequality (ii) of Lemma 5.1, and define  $\mathcal{U}_1 = \mathcal{U}_1(\alpha, \beta, m_2, a, b, c, g)$  as the largest  $z_1$  among  $z_1$  satisfying inequality (i) of Lemma 5.1 with  $z_2$  replaced by  $\mathcal{U}_2$ .

As remarked before,  $m_2$  affects the definitions of  $\mathcal{U}_1, \mathcal{U}_2$  only when  $\alpha = 2$ . Note that both  $\mathcal{U}_1, \mathcal{U}_2$  are decreasing on  $\beta$  and  $g$ , and are increasing on  $c$ , and on  $m_2$  if  $\alpha = 2$ .

In what follows, let  $(\dagger)$  denote the following case:

$$(\dagger) \quad \begin{cases} x_I \not\equiv x_J \pmod{2} \text{ or } y_I \not\equiv y_J \pmod{2} \\ \text{for some set } \{I, J\} \subset \{1, 2, 3\}. \end{cases}$$

**Lemma 5.2.** In case  $(\dagger)$ , either  $(\beta, z_1) = (1, 1)$  or  $z_1 = \alpha/\beta$ .

*Proof.* Suppose that  $(\beta, z_1) \neq (1, 1)$ . Since  $z_1 \leq z_2 \leq z_3$ , we have  $(\beta, z_t) \neq (1, 1)$  for any  $t$ . Then Lemma 4.3 tells that  $z_t = \alpha/\beta$  for some  $t$ . In particular,  $z_1 \leq z_t \leq \alpha/\beta$ . On the other hand,  $z_1 \geq \alpha/\beta$  by (5.4). These inequalities together show that  $z_1 = \alpha/\beta$ .  $\square$

**Lemma 5.3.**  $z_2 \leq 230$ ,  $\max\{x_1, y_1, x_2, y_2\} < 4300$  and  $\max\{a, b, c\} > 1000$ .

*Proof.* On inequality (ii) of Lemma 5.1, put  $(\beta, c, m_2) = (1, 10^{62}, \frac{\log 8}{\log 3})$ , thereby

$$z_2 - \frac{\log z_2}{\log 2} < \alpha + \frac{\log \mathcal{H}_{\alpha, 1, \log 8 / \log 3}(10^{62})}{\log 2}.$$

For each  $\alpha$  with  $2 \leq \alpha \leq \frac{\log(\min\{a,b\}+1)}{\log 2} \leq \frac{\log 10^{62}}{\log 2} (< 206)$ , the above inequality yields that  $\mathcal{U}_2(\alpha, 1, \frac{\log 8}{\log 3}, 10^{62}, 1) \leq 230$ . The second asserted inequality follows from the inequality  $\min\{a, b\}^{\max\{x_1, y_1, x_2, y_2\}} < c^{\mathcal{U}_2}$  with  $\min\{a, b\} \geq 2^\alpha - 1$ . To obtain the third asserted inequality, for each possible  $a, b$  and  $c$  with  $\max\{a, b, c\} \leq 1000$ , and for each pair  $(z_1, z_2)$  satisfying the inequalities in Lemma 5.1 and the restrictions of Lemma 5.2, one can check by a brute force search (within 7 hours) that at least one of 1st and 2nd equations does not hold for any possible  $(x_1, y_1, x_2, y_2)$ .  $\square$

## 6. IMPROVING WORK OF HU AND LE

In this section, we improve the gap principle of Hu and Le. To follow their strategy, we start with several lemmas derived as consequences of [HuLe3].

**Definition 6.1.** *Let  $r$  and  $s$  be coprime integers with  $s > 2$ . Define  $n(r, s)$  to be the least positive integer among positive integer  $n$ 's for which  $r^n$  is congruent to 1 or  $-1$  modulo  $s$ . Moreover, define  $\delta = \delta(r, s) \in \{1, -1\}$  and a positive integer  $f = f(r, s)$  as follows:*

$$\delta \equiv r^{n(r,s)} \pmod{s}, \quad f = \frac{r^{n(r,s)} - \delta}{s}.$$

In the following lemma, the first statement is elementary, and the second one easily follows from the proof of [HuLe3, Lemma 4.4].

**Lemma 6.1.** *Let  $r$  and  $s$  be coprime integers with  $s > 2$ . Then the following hold.*

- (i) *Let  $n'$  be a positive integer satisfying*

$$r^{n'} \equiv \delta' \pmod{s}$$

*for some  $\delta' \in \{1, -1\}$ . Then*

$$n' \equiv 0 \pmod{n(r, s)},$$

$$r^{n'} - \delta' \equiv 0 \pmod{(r^{n(r,s)} - \delta(r, s))}.$$

- (ii) *Let  $t$  be a positive integer whose prime factors divide  $s$ . Assume that  $s \not\equiv 2 \pmod{4}$ . Let  $n'$  be any positive integer satisfying*

$$r^{n'} \equiv \delta' \pmod{st}$$

*for some  $\delta' \in \{1, -1\}$ . Then*

$$n' \equiv 0 \pmod{\frac{t \cdot n(r, s)}{\gcd(t, f(r, s))}}.$$

Let  $A, B$  and  $C$  be any fixed pairwise coprime integers greater than 1. For each  $\lambda \in \{1, -1\}$ , consider the following equation:

$$(6.1) \quad A^X + \lambda B^Y = C^Z,$$

where  $X, Y, Z$  are unknown positive integers. For our purpose, it suffices to observe equation (6.1) under the following conditions (corresponding to

(\*)):

$$(**) \quad \begin{cases} \text{none of } A, B, C \text{ is a perfect power;} \\ 2 \mid C, C > 4, \max\{A, B\} \geq 11, & \text{if } \lambda = 1; \\ 2 \mid A, A > 4, \max\{B, C\} \geq 11, & \text{if } \lambda = -1. \end{cases}$$

The following lemma easily follows from the proof of [HuLe, Lemma 3.3]. It is worth noting that its first assertion is based upon the primitive divisor theorem of Zsigmondy [Zs].

**Lemma 6.2.** *Let  $(X, Y, Z)$  and  $(X', Y', Z')$  be two solutions of equation (6.1). Assume that  $C^{\min\{Z, Z'\}} > 2$ . Then  $XY' \neq X'Y$ . Moreover, for each  $A \in \{A, B\}$ ,*

$$\mathcal{A}^{|XY' - X'Y|} \equiv \pm 1 \pmod{C^{\min\{Z, Z'\}}}.$$

The following lemma easily follows from the proofs of [HuLe2, Lemma 4.6] and [HuLe3, Lemma 4.4].

**Lemma 6.3.** *Let  $(X, Y, Z)$  and  $(X', Y', Z')$  be two solutions of equation (6.1) such that  $Z < Z'$ . Then the following hold.*

(i) *If  $C^Z > 2$ , then*

$$\gcd(C, f(B, C^Z)) \mid X', \quad \gcd(C, f(A, C^Z)) \mid Y'.$$

(ii) *If  $C^Z \not\equiv 2 \pmod{4}$ , then*

$$\gcd(C^{Z'-Z}, f(B, C^Z)) \mid X', \quad \gcd(C^{Z'-Z}, f(A, C^Z)) \mid Y'.$$

Before going to state the improved gap principle, we show an elementary lemma.

**Lemma 6.4.** *Let  $(X, Y, Z)$  be a solution of equation (6.1) with  $\lambda = -1$ . Put  $G = \gcd(X, Y)$ . If  $G > 1$ , then*

$$X < \frac{G}{G-1} \frac{\log C}{\log A} Z, \quad Y < \frac{G}{G-1} \frac{\log C}{\log B} Z.$$

*Proof.* By the definition of  $G$ ,

$$C^Z = (A^{X/G})^G - (B^{Y/G})^G = (A^{X/G} - B^{Y/G})((A^{X/G})^{G-1} + \dots + (B^{Y/G})^{G-1})$$

with  $A^{X/G} - B^{Y/G} > 0$ . In particular,

$$A^{\frac{G-1}{G}X} < C^Z, \quad B^{\frac{G-1}{G}Y} < C^Z.$$

These give the asserted inequalities.  $\square$

For any positive numbers  $P$  and  $Q$ , we define  $t_{P,Q}$  as follows:

$$t_{P,Q} := \frac{\log \min\{P, Q\}}{\log \max\{P, Q\}}.$$

Now we state our improved gap principle.

**Proposition 6.1.** *Suppose that equation (6.1) has three solutions  $(X, Y, Z) = (X_r, Y_r, Z_r)$  with  $r \in \{1, 2, 3\}$  such that  $Z_1 < Z_2 \leq Z_3$ . Put  $G_2 = \gcd(X_2, Y_2)$ , and*

$$\chi := \begin{cases} 2, & \text{if } Z_1 > 1, \text{ and } \lambda = 1 \text{ or } C > \max\{A, B\}, \\ 1, & \text{otherwise.} \end{cases}$$

(i) *Suppose that  $C^{Z_1} \equiv 2 \pmod{4}$  with  $C^{Z_1} > 2$ . Then*

$$C \mid G_2 \cdot (X_2 Y_3 - X_3 Y_2),$$

$$C \leq \frac{\min\{X_2 \log B, Y_2 \log A\}}{\log(\chi C^{Z_1} - 1)} \cdot |X_2 Y_3 - X_3 Y_2|.$$

*Moreover, if either  $\lambda = 1$ , or  $\lambda = -1$  with  $G_2 > 1$ , then*

$$C < \mathcal{K} \cdot t_{A,B} \cdot \frac{Z_2}{Z_1} \cdot |X_2 Y_3 - X_3 Y_2|,$$

*where*

$$\mathcal{K} = \begin{cases} \frac{Z_1 \log C}{\log(\chi C^{Z_1} - 1)}, & \text{if } \lambda = 1, \\ \frac{Z_1 \log C}{\log(\chi C^{Z_1} - 1)} \cdot \frac{G_2}{G_2 - 1}, & \text{if } \lambda = -1, G_2 > 1. \end{cases}$$

(ii) *Suppose that  $C^{Z_1} \not\equiv 2 \pmod{4}$ . Then*

$$C^{Z_2 - Z_1} \mid G_2 \cdot (X_2 Y_3 - X_3 Y_2),$$

$$C^{Z_2 - Z_1} \leq \frac{\min\{X_2 \log B, Y_2 \log A\}}{\log(\chi C^{Z_1} - 1)} \cdot |X_2 Y_3 - X_3 Y_2|.$$

*Moreover, if either  $\lambda = 1$ , or  $\lambda = -1$  with  $G_2 > 1$ , then*

$$C^{Z_2 - Z_1} < \mathcal{K} \cdot t_{A,B} \cdot \frac{Z_2}{Z_1} \cdot |X_2 Y_3 - X_3 Y_2|,$$

*where  $\mathcal{K}$  is the same as in (i).*

*Proof.* We fix the value of  $\mathcal{A} \in \{A, B\}$ . Applying Lemma 6.2 for  $(X, Y, Z) = (X_2, Y_2, Z_2)$  and  $(X', Y', Z') = (X_3, Y_3, Z_3)$  shows that

$$(6.2) \quad \mathcal{A}^{|X_2 Y_3 - X_3 Y_2|} \equiv \varepsilon \pmod{C^{Z_2}}$$

with  $X_2 Y_3 - X_3 Y_2 \neq 0$  and some  $\varepsilon \in \{1, -1\}$ . Since  $Z_2 \geq Z_1$ , Lemma 6.1 (i) for  $(r, s) = (\mathcal{A}, C^{Z_1})$  together with congruence (6.2) tells us that  $|X_2 Y_3 - X_3 Y_2|$  is divisible by  $n(\mathcal{A}, C^{Z_1})$ . Put

$$n_1 := n(\mathcal{A}, C^{Z_1}).$$

Then

$$(6.3) \quad |X_2 Y_3 - X_3 Y_2| = n_1 n_2$$

for some positive integer  $n_2$ . On the other hand,

$$\mathcal{A}^{n_1} = C^{Z_1} f + \delta,$$

where  $f = f(\mathcal{A}, C^{Z_1})$  and  $\delta = \delta(\mathcal{A}, C^{Z_1})$ .

Suppose that  $f = 1$ . Then  $\mathcal{A}^{n_1} = C^{Z_1} \pm 1$ . Observe that

$$\mathcal{A}^{n_1} \geq C^{Z_1} - 1 \begin{cases} = A^{X_1} + B^{Y_1} - 1 > \mathcal{A}, & \text{if } \lambda = 1, \\ \geq C^2 - 1 > C > \mathcal{A}, & \text{if } C > \mathcal{A} \text{ and } Z_1 > 1. \end{cases}$$



Thus, if  $Z_1 > 1$ , and  $\lambda = 1$  or  $C > \mathcal{A}$ , then  $n_1 > 1$  and so the well-known theorem of [Mi] on Catalan's equation tells us that  $\{\mathcal{A}^{n_1}, C^{Z_1}\} = \{8, 9\}$ , which contradicts (\*\*). By these observations,

$$(6.4) \quad n_1 = n(\mathcal{A}, C^{Z_1}) \geq \frac{\log(\chi C^{Z_1} - 1)}{\log \mathcal{A}}.$$

(i) From (6.2) and (6.3),

$$\mathcal{A}^{n_1 n_2} = C^{Z_2} h + \varepsilon,$$

for some positive integer  $h$ . We substitute the mentioned expression of  $\mathcal{A}^{n_1}$  into the above to see that

$$C^{Z_2} h + \varepsilon = (C^{Z_1} f + \delta)^{n_2} = C^{Z_1} \sum_{i=1}^{n_2} \binom{n_2}{i} (C^{Z_1})^{i-1} f^i \delta^{n_2-i} + \delta^{n_2}.$$

It is easy to see that  $\varepsilon = \delta^{n_2}$  as  $\varepsilon \equiv \delta^{n_2} \pmod{C^{Z_1}}$  with  $\delta, \varepsilon \in \{1, -1\}$  and  $C^{Z_1} > 2$  by assumption. Therefore,

$$\begin{aligned} C^{Z_2-Z_1} h &= \sum_{i=1}^{n_2} \binom{n_2}{i} (C^{Z_1})^{i-1} f^i \delta^{n_2-i} \\ &= n_2 f \delta^{n_2-1} + C^{Z_1} \sum_{i=2}^{n_2} \binom{n_2}{i} (C^{Z_1})^{i-2} f^i \delta^{n_2-i}. \end{aligned}$$

Since  $Z_2 > Z_1$  by assumption, we have  $n_2 f \equiv 0 \pmod{C}$ , and so

$$n_2 \gcd(f, C) \equiv 0 \pmod{C}.$$

By (6.3),

$$(6.5) \quad n_1 C \mid \gcd(f, C) \cdot |X_2 Y_3 - X_3 Y_2|.$$

On the other hand, Lemma 6.3 (i) for  $(X, Y, Z) = (X_1, Y_1, Z_1), (X', Y', Z') = (X_2, Y_2, Z_2)$  and  $n = n_1$  tells us that

$$\begin{cases} \gcd(f, C) \mid Y_2, & \text{if } \mathcal{A} = A, \\ \gcd(f, C) \mid X_2, & \text{if } \mathcal{A} = B. \end{cases}$$

This together with (6.5) gives that

$$(6.6) \quad \begin{cases} n(A, C^{Z_1}) \cdot C \mid Y_2 \cdot |X_2 Y_3 - X_3 Y_2|, & \text{if } \mathcal{A} = A, \\ n(B, C^{Z_1}) \cdot C \mid X_2 \cdot |X_2 Y_3 - X_3 Y_2|, & \text{if } \mathcal{A} = B. \end{cases}$$

This enables us to deduce the first asserted divisibility relation.

Moreover, from (6.4), inequalities (6.6) imply

$$(6.7) \quad \begin{cases} \frac{\log(\chi C^{Z_1} - 1)}{\log \mathcal{A}} C \leq Y_2 \cdot |X_2 Y_3 - X_3 Y_2|, \\ \frac{\log(\chi C^{Z_1} - 1)}{\log B} C \leq X_2 \cdot |X_2 Y_3 - X_3 Y_2|. \end{cases}$$

This implies the first asserted upper bound for  $C$ .

Suppose that  $\lambda = 1$ . Since  $X_2 < \frac{\log C}{\log A} Z_2$  and  $Y_2 < \frac{\log C}{\log B} Z_2$ , we use (6.7) to see that

$$\begin{aligned} C &< \frac{\log A}{\log(\chi C^{Z_1} - 1)} \cdot \frac{\log C}{\log B} Z_2 \cdot |X_2 Y_3 - X_3 Y_2|, \\ C &< \frac{\log B}{\log(\chi C^{Z_1} - 1)} \cdot \frac{\log C}{\log A} Z_2 \cdot |X_2 Y_3 - X_3 Y_2|, \end{aligned}$$

thereby

$$C < \frac{Z_1 \log C}{\log(\chi C^{Z_1} - 1)} \cdot t_{A,B} \cdot \frac{Z_2}{Z_1} \cdot |X_2 Y_3 - X_3 Y_2|.$$

Suppose that  $\lambda = -1$  and  $G_2 > 1$ . Applying Lemma 6.4 for  $(X, Y, Z) = (X_2, Y_2, Z_2)$  gives

$$X_2 < \frac{G_2}{G_2 - 1} \frac{\log C}{\log A} Z_2, \quad Y_2 < \frac{G_2}{G_2 - 1} \frac{\log C}{\log B} Z_2.$$

These inequalities together with (6.7) gives the remaining assertion.

(ii) Apply Lemma 6.1 (ii) for

$$(r, s, t, n') = (\mathcal{A}, C^{Z_1}, C^{Z_2 - Z_1}, |X_2 Y_3 - X_3 Y_2|),$$

together with congruence (6.2). Then

$$n_1 C^{Z_2 - Z_1} \mid \gcd(C^{Z_2 - Z_1}, f) \cdot |X_2 Y_3 - X_3 Y_2|.$$

Using this divisibility relation together with Lemma 6.3 (ii) for  $(X, Y, Z) = (X_1, Y_1, Z_1)$ ,  $(X', Y', Z') = (X_2, Y_2, Z_2)$  and  $n = n_1$ , we can show the assertions almost similarly to case (i).  $\square$

In the remaining parts of this section, we apply Proposition 6.1 to special cases concerning equation (6.1). For this we prepare two lemmas from the works of Hu and Le.

The following lemma directly follows from the proofs of [HuLe3, Lemmas 3.2 and 3.4].

**Lemma 6.5.** *Let  $(X, Y, Z)$  be a solution of equation (6.1) for  $\lambda = 1$ . Then the following hold.*

(i) *If  $A^{2X} < C^Z$ , then*

$$0 < \frac{\log C}{\log B} - \frac{Y}{Z} < \frac{2}{Z C^{Z/2} \log B}.$$

(ii) *Let  $(X', Y', Z')$  be another solution of equation (6.1) for  $\lambda = 1$ . If  $X > X'$  and  $Z \leq Z'$ , then*

$$0 < \frac{\log C}{\log B} - \frac{Y'}{Z'} < \frac{2}{Z' A^{X - X'} C^{Z' - Z} \log B}.$$

The following lemma directly follows from the argument in [HuLe3, Section 5].

**Lemma 6.6.** *Let  $(X, Y, Z)$  and  $(X', Y', Z')$  be two solutions of equation (6.1) for  $\lambda = 1$ . Assume that both  $Y/Z$  and  $Y'/Z'$  are convergents in the simple continued fraction expansion to  $\frac{\log C}{\log B}$ . If  $Y/Z < Y'/Z' < \frac{\log C}{\log B}$ , then*

$$Z' > \frac{1}{Z \left( \frac{\log C}{\log B} - \frac{Y}{Z} \right)}.$$

**Proposition 6.2.** *Suppose that equation (6.1) for  $\lambda = 1$  has three solutions  $(X, Y, Z) = (X_r, Y_r, Z_r)$  with  $r \in \{1, 2, 3\}$  such that  $Z_1 = Z_2 < Z_3$ . Then one of the following inequalities holds.*

$$C^{Z_2/2}/Z_2 < \max_{t \in \{1, 2\}} \{|X_3 Z_2 - X_t Z_3|, |Y_3 Z_2 - Y_t Z_3|\};$$

$$C^{Z_2/2} < \frac{2}{\log \min\{A, B\}} Z_3.$$

*Proof.* Since  $Z_1 = Z_2$ ,

$$(6.8) \quad A^{X_1} + B^{Y_1} = A^{X_2} + B^{Y_2} = C^{Z_2}.$$

Note that  $X_1 \neq X_2, Y_1 \neq Y_2$ , and that  $X_1 < X_2$  if and only if  $Y_2 < Y_1$ . By symmetry of indices 1 and 2 in the assertion, we may assume that

$$(6.9) \quad X_1 < X_2, \quad Y_2 < Y_1.$$

Also, from the equations in (6.8), observe that

$$B^{Y_2} \mid (A^{X_2-X_1} - 1), \quad A^{X_1} \mid (B^{Y_1-Y_2} - 1), \quad A^{X_2-X_1} \cdot B^{Y_1-Y_2} \equiv 1 \pmod{C^{Z_2}}.$$

These imply that

$$(6.10) \quad A^{X_2-X_1} > B^{Y_2}, \quad B^{Y_1-Y_2} > A^{X_1},$$

$$(6.11) \quad \max\{A^{X_2-X_1}, B^{Y_1-Y_2}\} > C^{Z_2/2}.$$

By symmetry of  $A$  and  $B$  in the assertion, we may assume that  $A^{X_2-X_1} > B^{Y_1-Y_2}$ . From (6.11),

$$(6.12) \quad A^{X_2-X_1} > C^{Z_2/2}.$$

Suppose that  $X_2 \leq X_3$ . From (6.9) observe that the equation  $C^{X'} - B^{Y'} = A^{Z'}$  has three solutions  $(X', Y', Z') = (Z_r, Y_r, X_r)$  with  $r \in \{1, 2, 3\}$  satisfying  $X_1 < X_2 \leq X_3$ . Since  $A^{X_1}$  is odd, Proposition 6.1 (ii) yields that

$$\gcd(Y_2, Z_2) \mid Z_2 Y_3 - Z_3 Y_2 \geq A^{X_2-X_1}.$$

It follows from (6.12) that

$$(6.13) \quad \gcd(Y_2, Z_2) \mid Z_2 Y_3 - Z_3 Y_2 > C^{Z_2/2}.$$

Suppose that  $X_3 < X_2$ . From (6.8) and (6.10) observe that

$$C^{Z_1} = C^{Z_2} > A^{X_2} = A^{X_1} A^{X_2-X_1} > A^{X_1} B^{Y_1-Y_2} > A^{2X_1}.$$

Lemma 6.5 (i) for  $(X, Y, Z) = (X_1, Y_1, Z_1)$  tells us that

$$(0 <) \quad \frac{\log C}{\log B} - \frac{Y_1}{Z_1} < \frac{2}{Z_1 C^{Z_1/2} \log B}.$$

If the RHS above is greater than  $1/(2Z_1^2)$ , then  $\frac{C^{Z_1/2}}{Z_1} < \frac{4}{\log B}$ . It is easy to see that this leads to a contradiction to (\*\*). Thus,

$$(6.14) \quad \frac{\log C}{\log B} - \frac{Y_1}{Z_1} < \frac{2}{Z_1 C^{Z_1/2} \log B} \leq \frac{1}{2Z_1^2}.$$

On the other hand, since  $X_3 < X_2$  and  $Z_2 < Z_3$ , Lemma 6.5 (ii) for  $(X, Y, Z) = (X_2, Y_2, Z_2), (X', Y', Z') = (X_3, Y_3, Z_3)$  gives

$$(0 <) \quad \frac{\log C}{\log B} - \frac{Y_3}{Z_3} < \frac{2}{Z_3 A^{X_2-X_3} C^{Z_3-Z_2} \log B}.$$

Suppose that the RHS above is greater than  $1/(2Z_3^2)$ . Then

$$4Z_3 > (\log B) A^{X_2-X_3} C^{Z_3-Z_2} \geq (A \log B) C^{Z_3-Z_2} \geq (A \log B) \cdot C.$$

Put  $\alpha := \min\{\nu_2(A^2-1), \nu_2(B^2-1)\} - 1$ . Since  $\max\{A, B\} \geq \max\{11, 2^\alpha + 1\}$  by (\*\*), and  $\min\{A, B\} \geq 2^\alpha - 1$ , we have  $A \log B \geq c(\alpha) \geq 3 \log 11$ , where  $c(\alpha) := (2^\alpha - 1) \log \max\{11, 2^\alpha + 1\}$ . Then  $Z_3 > \frac{1}{4}(A \log B)C > 10$ , so that

$$Z_2 > Z_3 - \frac{\log(4Z_3)}{\log C} > \frac{4}{5} Z_3.$$

This gives rise to a sharp lower bound for  $Z_2$ , that is,  $Z_2 \geq \lceil \frac{1}{5} c(\alpha)C \rceil$  ( $\geq 9$ ). However, this is incompatible, for any  $\alpha \geq 2$  and  $C \geq 6$ , with

$$\frac{2^{\nu_2(C)Z_2}}{Z_2^2} < \frac{2^\alpha \log^2 C}{\log(2^\alpha - 1) \cdot \log \max\{11, 2^\alpha + 1\}},$$

which is shown in the same way as Lemma 5.1 (i). Therefore,

$$(6.15) \quad \frac{\log C}{\log B} - \frac{Y_3}{Z_3} < \frac{2}{Z_3 A^{X_2-X_3} C^{Z_3-Z_2} \log B} \leq \frac{1}{2Z_3^2}.$$

To sum up, by a well-known criterion of Legendre on the continued fraction, we may conclude, from inequalities (6.14) and (6.15), that both  $\frac{Y_1}{Z_1}, \frac{Y_3}{Z_3}$  are convergents in the simple continued fraction expansion to  $\frac{\log C}{\log B}$ . The fact that  $\frac{Y_1}{Z_1} \neq \frac{Y_3}{Z_3}$  follows from Lemma 6.2 for the two solutions  $(X_1, Y_1, Z_1)$  and  $(X_3, Y_3, Z_3)$ . Moreover, Lemma 6.6 for  $(X, Y, Z) = (X_1, Y_1, Z_1)$  and  $(X', Y', Z') = (X_3, Y_3, Z_3)$ , together with (6.14) and (6.15) tells us that

$$(6.16) \quad Z_3 > \frac{\log B}{2} C^{Z_1/2} = \frac{\log B}{2} C^{Z_2/2},$$

or  $Z_1 > \frac{\log B}{2} A^{X_2-X_3} C^{Z_3-Z_2}$ . It is easily observed that the latter inequality does not hold similarly to the observation used to show (6.15). The assertion follows from (6.13) and (6.16), having in view symmetries of indices 1 and 2, and of  $A$  and  $B$ .  $\square$

The second assertion of the next proposition is not directly used below (cf. proof of Proposition 12.3 (iii)).

**Proposition 6.3.** *Assume that  $A < C$ . Suppose that equation (6.1) for  $\lambda = -1$  has three solutions  $(X, Y, Z) = (X_r, Y_r, Z_r)$  with  $r \in \{1, 2, 3\}$  such that  $Z_1 = Z_2 < Z_3$ . Then the following hold.*

(i) If  $X_3 > \max\{X_1, X_2\}$ , then one of the following inequalities holds.

$$C^{Z_2}/Z_2 < \max_{t \in \{1,2\}} |X_t Z_3 - X_3 Z_2|;$$

$$C^{Z_2/2}/Z_2 < \max_{t \in \{1,2\}} |Y_t Z_3 - Y_3 Z_2|;$$

$$C^{Z_2/2} < \frac{2}{\log C} X_3.$$

(ii) If  $X_3 \leq \max\{X_1, X_2\}$ , then one of the following inequalities holds.

$$C^{Z_2} < Z_3 \cdot \max_{t \in \{1,2\}} |X_t Z_3 - X_3 Z_2|;$$

$$C^{Z_2}/Z_2^2 < |X_1 - X_2| \cdot Z_3 \cdot \max_{t \in \{1,2\}} |Y_t Z_3 - Y_3 Z_2|.$$

*Proof.* Since  $Z_1 = Z_2$ ,

$$(6.17) \quad A^{X_1} - B^{Y_1} = A^{X_2} - B^{Y_2} = C^{Z_2}.$$

Note that  $X_1 \neq X_2$ ,  $Y_1 \neq Y_2$ , and that  $X_1 < X_2$  if and only if  $Y_1 < Y_2$ . Also,  $X_1 > 1$  as  $A < C$  in equations (6.17). By symmetry of indices 1 and 2 in the assertion, we may assume that  $X_1 < X_2$  and  $Y_1 < Y_2$ . Thus,

$$(6.18) \quad 1 < X_1 < X_2.$$

In particular,  $A^{X_1} \equiv 0 \pmod{4}$ . Also, from (6.17),

$$B^{Y_1} \mid (A^{X_2-X_1} - 1), \quad A^{X_1} \mid (B^{Y_2-Y_1} - 1).$$

Thus

$$(6.19) \quad A^{X_2-X_1} > B^{Y_1}, \quad B^{Y_2-Y_1} > A^{X_1}.$$

Let us consider several cases separately.

First, suppose that

$$Y_2 \leq Y_3.$$

Observe that the equation  $A^{X'} - C^{Y'} = B^{Z'}$  has three solutions  $(X', Y', Z') = (X_r, Z_r, Y_r)$  with  $r \in \{1, 2, 3\}$  satisfying  $Y_1 < Y_2 \leq Y_3$ . Since  $B^{Y_1}$  is odd, Proposition 6.1 (ii) yields

$$\gcd(X_2, Z_2) \mid |X_2 Z_3 - X_3 Z_2| \geq B^{Y_2-Y_1}.$$

As  $B^{Y_2-Y_1} > A^{X_1} > C^{Z_2}$  by (6.17) and (6.19), we have

$$(6.20) \quad \gcd(X_2, Z_2) \mid |X_2 Z_3 - X_3 Z_2| > C^{Z_2}.$$

Second, suppose that

$$Y_3 < Y_2, \quad X_2 < X_3.$$

Then the equation  $C^{X'} + B^{Y'} = A^{Z'}$  has three solutions  $(X', Y', Z') = (Z_r, Y_r, X_r)$  with  $r \in \{1, 2, 3\}$  satisfying  $X_1 < X_2 < X_3$ . Proposition 6.1 (ii) yields

$$\gcd(Z_2, Y_2) \mid |Z_2 Y_3 - Y_3 Z_2| \geq A^{X_2-X_1}.$$

If  $B^{2Y_1} \geq A^{X_1}$ , then, since  $A^{X_2-X_1} > B^{Y_1}$  by (6.19),

$$(6.21) \quad \gcd(Y_2, Z_2) \mid |Y_2 Z_3 - Y_3 Z_2| > B^{Y_1} \geq A^{X_1/2} > C^{Z_2/2}.$$

Suppose that  $B^{2Y_1} < A^{X_1}$ . Lemma 6.5 (i) for the solution  $(Y_1, Z_1, X_1)$  gives

$$(0 <) \quad \frac{\log A}{\log C} - \frac{Z_1}{X_1} < \frac{2}{X_1 A^{X_1/2} \log C}.$$

Similarly to (6.14), we can show that the right hand side above is at most  $\frac{1}{2X_1^2}$ . On the other hand, since  $Y_3 < Y_2$  and  $X_2 < X_3$ , Lemma 6.5 (ii) for the two solutions  $(Y_2, Z_2, X_2), (Y_3, Z_3, X_3)$  gives

$$(0 <) \quad \frac{\log A}{\log C} - \frac{Z_3}{X_3} < \frac{2}{X_3 B^{Y_2-Y_3} A^{X_3-X_2} \log C}.$$

Suppose that the right hand side above is greater than  $\frac{1}{2X_3^2}$ . Then

$$4X_3 > (\log C) B^{Y_2-Y_3} A^{X_3-X_2} \geq (B \log C) \log A.$$

By using the same argument as in the proof of Proposition 6.2 (where it was shown that  $\frac{Y_3}{Z_3}$  is a convergent of  $\frac{\log C}{\log B}$ ), the above inequalities give  $X_2 > \frac{1}{5}c(\alpha)A$ , where  $\alpha = \min\{\nu_2(B^2 - 1), \nu_2(C^2 - 1)\} - 1$  and  $c(\alpha)$  is defined as in Proposition 6.2. Moreover, by applying Lemma 5.1 we obtain

$$\frac{2^{\nu_2(A)X_1}}{X_1 X_2} < \frac{2^\alpha \log^2 A}{\log(2^\alpha - 1) \cdot \log \max\{11, 2^\alpha + 1\}}, \quad \frac{2^{\nu_2(A)X_2}}{X_2} < 2^\alpha \mathcal{H}_{\alpha, \beta, \frac{\log 8}{\log 3}}(A).$$

We check that there is no quadruple  $(\alpha, A, X_1, X_2)$  satisfying both of these inequalities. Thus, similarly to the arguments in the proof of Proposition 6.2, we can conclude that  $\frac{Z_1}{X_1}, \frac{Z_3}{X_3}$  are distinct convergents to  $\frac{\log A}{\log C}$ . Then Lemma 6.6 shows that

$$(6.22) \quad X_3 > \frac{\log C}{2} A^{X_1/2} > \frac{\log C}{2} C^{Z_1/2},$$

or  $X_1 > \frac{\log C}{2} B^{Y_2-Y_3} A^{X_3-X_2}$ . It is shown that the latter inequality does not hold as observed in showing (6.15).

To sum up, assertion (i) follows from the combination of inequalities (6.20), (6.21), (6.22), in view of the symmetry of indices 1 and 2.

Finally, suppose that

$$Y_3 < Y_2, \quad X_3 \leq X_2.$$

If  $Y_1 \leq Y_3$  and  $X_3 \leq X_1$ , then  $A^{X_1} = B^{Y_1} + C^{Z_1} < B^{Y_3} + C^{Z_3} = A^{X_3} \leq A^{X_1}$ , which is absurd. Thus,  $Y_3 < Y_1$  or  $X_1 < X_3$ .

Consider the case where  $Y_3 < Y_1 (< Y_2)$  and  $X_3 \leq X_1$ . Observe that the equation  $A^{X'} - C^{Y'} = B^{Z'}$  has three solutions  $(X', Y', Z') = (X_r, Z_r, Y_r)$  with  $r \in \{3, 1, 2\}$  satisfying  $Y_3 < Y_1 < Y_2$ . Then Proposition 6.1 (ii) yields

$$\gcd(X_1, Z_1) | X_1 Z_2 - X_2 Z_1 | \geq B^{Y_1-Y_3}.$$

On the other hand,

$$A^{X_3} - B^{Y_3} = C^{Z_3} = C^{Z_3-Z_1} (A^{X_1} - B^{Y_1}).$$

Since  $X_3 \leq X_1$ , we reduce this relation modulo  $A^{X_3}$  to see that

$$C^{Z_3-Z_1} B^{Y_1-Y_3} \equiv 1 \pmod{A^{X_3}}.$$

This gives that  $C^{Z_3-Z_1} B^{Y_1-Y_3} > A^{X_3} > C^{Z_3}$ , so that

$$B^{Y_1-Y_3} > C^{Z_1}.$$

These inequalities together yield

$$(6.23) \quad \gcd(X_1, Z_1) |X_1 Z_2 - X_2 Z_1| > C^{Z_1}.$$

Consider the case where  $Y_3 < Y_1 (< Y_2)$  and  $X_1 < X_3 \leq X_2$ . Since the equation  $C^{X'} + B^{Y'} = A^{Z'}$  has three solutions  $(X', Y', Z') = (Z_r, Y_r, X_r)$  with  $r \in \{1, 3, 2\}$  satisfying  $X_1 < X_3 \leq X_2$ , Proposition 6.1 (ii) yields

$$\gcd(Z_3, Y_3) |Z_3 Y_2 - Z_2 Y_3| \geq A^{X_3 - X_1}.$$

On the other hand, reducing the relation  $B^{Y_3} + C^{Z_3} = A^{X_3} = A^{X_3 - X_1} (B^{Y_1} + C^{Z_1})$  modulo  $C^{Z_1}$  yields  $A^{X_3 - X_1} B^{Y_1 - Y_3} \equiv 1 \pmod{C^{Z_1}}$ , in particular,

$$A^{X_3 - X_1} B^{Y_1 - Y_3} > C^{Z_1}.$$

Since  $Z_2 \gcd(X_1, Z_1) |X_1 - X_2| \geq B^{Y_1 - Y_3}$  as seen in the previous case, these inequalities together yield

$$(6.24) \quad \gcd(Y_3, Z_3) |Y_2 Z_3 - Y_3 Z_2| \cdot Z_2 \gcd(X_1, Z_1) |X_1 - X_2| > C^{Z_1}.$$

Consider the case where  $Y_1 \leq Y_3 < Y_2$  and  $X_1 < X_3 \leq X_2$ . Reducing equation  $B^{Y_3} + C^{Z_3} = A^{X_3 - X_1} (B^{Y_1} + C^{Z_1})$  modulo  $C^{Z_1}$  yields  $A^{X_3 - X_1} \equiv B^{Y_3 - Y_1} \pmod{C^{Z_1}}$ , in particular,

$$\max\{A^{X_3 - X_1}, B^{Y_3 - Y_1}\} > C^{Z_1}.$$

Suppose that  $A^{X_3 - X_1} > C^{Z_1}$ . Since the equation  $C^{X'} + B^{Y'} = A^{Z'}$  has three solutions  $(X', Y', Z') = (Z_r, Y_r, X_r)$  with  $r \in \{1, 3, 2\}$  satisfying  $X_1 < X_3 \leq X_2$ , Proposition 6.1 (ii) yields

$$(6.25) \quad \gcd(Z_3, Y_3) |Z_3 Y_2 - Z_2 Y_3| \geq A^{X_3 - X_1} > C^{Z_1}.$$

Suppose that  $B^{Y_3 - Y_1} > C^{Z_1}$ . Since  $Y_1 < Y_3$ , and the equation  $A^{X'} - C^{Y'} = B^{Z'}$  has three solutions  $(X', Y', Z') = (X_r, Z_r, Y_r)$  with  $r \in \{1, 3, 2\}$  satisfying  $Y_1 < Y_3 < Y_2$ , Proposition 6.1 (ii) yields

$$(6.26) \quad \gcd(X_3, Z_3) |X_3 Z_2 - X_2 Z_3| \geq B^{Y_3 - Y_1} > C^{Z_1}.$$

To sum up, assertion (ii) follows from the combination of inequalities (6.20), (6.23), (6.24), (6.25) and (6.26), and the symmetry of indices 1 and 2.  $\square$

## 7. APPLICATIONS

Here we give three applications of Proposition 6.1 to equation (1.3) having three solutions. For this we prepare some notation.

Based upon (5.2), let  $(i, j, k)$  and  $(l, m, n)$  be permutations of  $\{1, 2, 3\}$  such that

$$x_i \leq x_j \leq x_k, \quad y_l \leq y_m \leq y_n.$$

To ensure the uniqueness of these, we assume that  $i < j$  if  $x_i = x_j$ , and  $j < k$  if  $x_j = x_k$ , and that  $l < m$  if  $y_l = y_m$ , and  $m < n$  if  $y_m = y_n$ . Also, define non-negative integers  $d_z, d_x, d_y$  and positive integers  $g_2, g_x, g_y$  as follows:

$$\begin{aligned} d_z &:= z_2 - z_1, & d_x &:= x_j - x_i, & d_y &:= y_m - y_l, \\ g_2 &:= \gcd(x_2, y_2), & g_x &:= \gcd(y_j, z_j), & g_y &:= \gcd(x_m, z_m). \end{aligned}$$

**Lemma 7.1.** *Suppose that  $d_z > 0$  with  $c^{z_1} \equiv 2 \pmod{4}$ . Then*

$$c < \min \left\{ 2^{\alpha+1-z_2} \frac{(g_2')^2}{g_2}, \frac{(g_2')^2}{g_2}, \frac{\log c}{\log(c-1)} t_{a,b} z_2 \right\} \cdot z_2 \mathcal{H}_{\alpha,1,m_2}(c)$$

with  $g_2' = \gcd(c, g_2)$ .

*Proof.* Note that  $(\beta, z_1) = (1, 1)$ . We apply Proposition 6.1 (i) for  $(A, B, C; \lambda) = (a, b, c; 1)$  and  $(X_r, Y_r, Z_r) = (x_r, y_r, z_r)$  with  $r \in \{1, 2, 3\}$ . Then

$$(7.1) \quad c \mid (g_2')^2 \cdot \frac{x_2 y_3 - x_3 y_2}{g_2},$$

$$(7.2) \quad c < \frac{\log c}{\log(c-1)} \cdot t_{a,b} z_2 \cdot |x_2 y_3 - x_3 y_2|.$$

By (5.7), it is easy to see that the assertion for the second part in min follows from (7.1), and the third one follows from (7.2). It remains to consider the first part.

Since  $2 \parallel c$ , and  $g_2$  is odd (cf. Lemma 8.2 (i)), we use divisibility relation (7.1) to see that  $(cg_2/2)/(g_2')^2$  is an odd positive divisor of  $x_2 y_3 - x_3 y_2$ . Thus

$$\begin{aligned} \nu_2(x_2 y_3 - x_3 y_2) &= \nu_2 \left( \frac{x_2 y_3 - x_3 y_2}{(cg_2/2)/(g_2')^2} \right) \\ &\leq \frac{1}{\log 2} \log \left( \frac{|x_2 y_3 - x_3 y_2|}{(cg_2/2)/(g_2')^2} \right) \\ &= 1 - \frac{\log c}{\log 2} + \frac{1}{\log 2} \log \left( \frac{(g_2')^2}{g_2} \cdot |x_2 y_3 - x_3 y_2| \right). \end{aligned}$$

Since  $z_2 \leq \alpha + \nu_2(x_2 y_3 - x_3 y_2)$  by (5.5) for  $t = 2$ , it follows that

$$\frac{\log c}{\log 2} \leq -z_2 + \alpha + 1 + \frac{1}{\log 2} \log \left( \frac{(g_2')^2}{g_2} \cdot |x_2 y_3 - x_3 y_2| \right).$$

This together with (5.7) implies the remaining assertion.  $\square$

**Lemma 7.2.** *Suppose that  $d_z > 0$  with  $c^{z_1} \equiv 0 \pmod{4}$ . Then*

$$c^{d_z} < \min \left\{ 2^{\alpha-\beta z_1} \frac{(g_2')^2}{g_2}, \frac{z_1 \log c}{\log(\chi c^{z_1} - 1)} t_{a,b} \frac{z_2}{z_1} \right\} \cdot \frac{\log^2 c}{(\log a) \log b} z_2 z_3$$

with  $g_2' = \gcd(c^{d_z}, g_2)$ , where  $\chi = 2$  if  $z_1 > 1$ , and  $\chi = 1$  if  $z_1 = 1$ .

*Proof.* The proof proceeds along similar lines to that of Lemma 7.1. We apply Proposition 6.1 (ii) for  $(A, B, C; \lambda) = (a, b, c; 1)$  and  $(X_r, Y_r, Z_r) = (x_r, y_r, z_r)$  with  $r \in \{1, 2, 3\}$ . Then

$$(7.3) \quad c^{z_2-z_1} \mid (g_2')^2 \cdot \frac{x_2 y_3 - x_3 y_2}{g_2},$$

$$(7.4) \quad c^{z_2-z_1} < \frac{z_1 \log c}{\log(\chi c^{z_1} - 1)} \cdot t_{a,b} \cdot \frac{z_2}{z_1} \cdot |x_2 y_3 - x_3 y_2|.$$

By (5.7), the assertion for the second part in min follows from (7.4). It remains to consider the first part.



Since  $2^\beta \parallel c$ , and  $g_2$  is odd as  $4 \mid c^{z_1}$ , we use (7.3) to see that  $(c/2^\beta)^{z_2-z_1}g_2/(g_2')^2$  is an odd positive divisor of  $x_2y_3 - x_3y_2$ . Thus

$$\begin{aligned} \nu_2(x_2y_3 - x_3y_2) &= \nu_2\left(\frac{x_2y_3 - x_3y_2}{(c/2^\beta)^{z_2-z_1}g_2/(g_2')^2}\right) \\ &\leq \frac{1}{\log 2} \log\left(\frac{|x_2y_3 - x_3y_2|}{(c/2^\beta)^{z_2-z_1}g_2/(g_2')^2}\right) \\ &= \beta(z_2 - z_1) - (z_2 - z_1) \frac{\log c}{\log 2} + \frac{\log\left(\frac{(g_2')^2}{g_2} \cdot |x_2y_3 - x_3y_2|\right)}{\log 2}. \end{aligned}$$

Since  $\beta z_2 \leq \alpha + \nu_2(x_2y_3 - x_3y_2)$  by (5.5) for  $t = 2$ , it follows that

$$(z_2 - z_1) \frac{\log c}{\log 2} \leq \alpha - \beta z_1 + \frac{1}{\log 2} \log\left(\frac{(g_2')^2}{g_2} \cdot |x_2y_3 - x_3y_2|\right).$$

This together with (5.7) implies the remaining assertion.  $\square$

**Lemma 7.3.** *Suppose that  $d_x > 0$ . Then*

$$(i) \quad a^{d_x} < \frac{(g_x')^2}{g_x} \cdot \frac{\log c}{\log b} \cdot z_j z_k \leq \frac{\log c}{\log b} \cdot z_j^2 z_k$$

with  $g_x' = \gcd(a^{d_x}, g_x)$ . Moreover, if  $g_x > 1$ , then

$$\begin{aligned} (ii) \quad & a^{d_x} < \frac{(g_x')^2}{g_x - 1} \cdot \frac{\log a}{\log b} \cdot x_j z_k; \\ (iii) \quad & a^{d_x} < \left(\frac{g_x}{g_x - 1}\right)^2 \cdot \frac{\log^2 a}{\log(a-1) \log b} \cdot t_{b,c} \cdot (x_j + d_x + d_x^2) z_k. \end{aligned}$$

Note that Lemma 7.3 holds with  $(b, y, l, m, n)$  instead of  $(a, x, i, j, k)$  by symmetry of  $a$  and  $b$ .

*Proof of Lemma 7.3.* Apply Proposition 6.1 (ii) for  $(A, B, C; \lambda) = (c, b, a; -1)$  and  $(X_r, Y_r, Z_r) = (z_t, y_t, x_t)$  with  $(r, t) \in \{(1, i), (2, j), (3, k)\}$ . Then

$$(7.5) \quad a^{x_j - x_i} \mid (g_x')^2 \cdot \frac{y_j z_k - y_k z_j}{g_x}.$$

Moreover, if  $g_x > 1$ , then

$$(7.6) \quad a^{x_j - x_i} < \frac{\log a}{\log(a-1)} \cdot \frac{g_x}{g_x - 1} \cdot t_{b,c} \cdot \frac{x_j}{x_i} \cdot |y_j z_k - y_k z_j|.$$

From (5.3) for  $t \in \{j, k\}$ ,

$$|y_j z_k - y_k z_j| < \max\{y_j z_k, y_k z_j\} < \frac{\log c}{\log b} z_j z_k.$$

Since  $a^{x_j - x_i} \leq \frac{(g_x')^2}{g_x} \cdot |y_j z_k - y_k z_j|$  by (7.5), the above inequality yields (i).

Suppose that  $g_x > 1$ . By Lemma 6.4 for  $(A, B, C) = (c, b, a)$  and  $(X, Y, Z) = (z_j, y_j, x_j)$ ,

$$y_j < \frac{g_x}{g_x - 1} \frac{\log a}{\log b} x_j, \quad z_j < \frac{g_x}{g_x - 1} \frac{\log a}{\log c} x_j.$$

Since  $y_k < \frac{\log c}{\log b} z_k$ , we have

$$|y_j z_k - y_k z_j| < \max\{y_j z_k, y_k z_j\} < \frac{g_x}{g_x - 1} \frac{\log a}{\log b} x_j z_k.$$

This together with (7.5) yields (ii). Similarly, (iii) follows from (7.6) and the inequality  $x_j^2/x_i \leq x_j + d_x + d_x^2$ .  $\square$

## 8. RESTRICTIONS ON COMMON DIVISORS AMONG SOLUTIONS

The following is a well-known conjecture as a generalization of Fermat's last theorem, known as the generalized Fermat conjecture.

**Conjecture 1.** *Let  $p, q$  and  $r$  be any positive integers satisfying  $1/p + 1/q + 1/r < 1$ . Then all solutions  $(X, Y, Z)$  with  $XYZ \neq 0$  and  $\gcd(X, Y) = 1$  of the Diophantine equation*

$$(8.1) \quad X^p + Y^q = Z^r$$

*come from the following ten identities:*

$$\begin{aligned} 1^p + 2^3 &= 3^2, \quad 7^2 + 2^5 = 3^4, \quad 13^2 + 7^3 = 2^9, \\ 17^3 + 2^7 &= 71^2, \quad 11^4 + 3^5 = 122^2, \quad 1549034^2 + 33^8 = 15613^3, \\ 96222^3 + 43^8 &= 30042907^2, \quad 2213459^2 + 1414^3 = 65^7, \\ 15312283^2 + 9262^3 &= 113^7, \quad 76271^3 + 17^7 = 21063928^2. \end{aligned}$$

The following is just a collection, needed for our purpose, from the existing results on Conjecture 1 (cf. [BeChDaYa], [BeMiSi], [Co, Ch.14]).

**Lemma 8.1.** *Conjecture 1 is true for any  $(p, q, r)$  in the following table:*

$(p, q, r)$	reference(s)
$(N, N, N), N \geq 3$	[Wi], [TaWi]
$(N, N, 2), N \geq 4$	[DarMe], [Po]
$(N, N, 3), N \geq 3$	[DarMe], [Po]
$(2, 4, N), N \geq 4$	[El], [BeElNg], [Br]
$(2, N, 4), N \geq 4$	[BeSk], [Br]
$(2, N, 6), N \geq 3$	[BeChDaYa], [Br]
$(2, 6, N), N \geq 3$	[BeCh], [Br]
$(3, 3, N), 3 \leq N \leq 10^9$	[Kr], [Br2], [Da], [DaSik2]
$(2, 3, N), N \in \{7, 8, 9, 10, 15\}$	[PoShSt], [Br], [Zu], [Sik], [SikSt]
$(3, 4, 5)$	[SikSt]
$(5, 5, 7), (7, 7, 5)$	[DaSik]
$(5, 5, N), N \geq 2, 5 \nmid Z$	[DaSik]

*The result of [DaSik] in the last line of the above table indicates that equation (8.1) with  $(p, q, r) = (5, 5, N)$  and  $N \geq 2$  has no solutions satisfying  $5 \nmid Z$ .*

As almost direct consequences of Lemma 8.1, together with Lemma 5.3, we can show the following lemmas which are useful to restrict the values of  $g_2, g_x$  and  $g_y$  appearing in the previous section.

**Lemma 8.2.** *The following hold.*

- (i) If  $2 \mid g_2$ , then  $(\beta, z_1, z_2) = (1, 1, 1)$ .
- (ii) If  $3 \mid g_2$ , then  $z_2 \leq 2$ .
- (iii) If  $5 \mid g_2$ , then  $5 \nmid c$  or  $z_2 = 1$ .
- (iv) Suppose that  $2 \mid z_2$ . Then  $g_2 \in \{1, 3\}$ . Moreover,  $g_2 = 1$  if  $z_2 > 2$ .
- (v) If  $3 \mid z_2$ , then  $g_2 = 1$ .

**Lemma 8.3.** *The following hold.*

- (i) If  $3 \mid g_x$  and  $x_j \leq 10^9$ , then  $x_j \leq 2$ .
- (ii) If  $4 \mid g_x$  or  $6 \mid g_x$ , then  $x_j = 1$ .
- (iii) If  $5 \mid g_x$ , then  $5 \nmid a$  or  $x_j = 1$ .
- (iv) Suppose that  $2 \mid x_j$ . Then  $g_x \in \{1, 3\}$ . Moreover, if  $4 \mid x_j$ , then  $g_x = 1$ .
- (v) If  $3 \mid x_j$ , then  $g_x \leq 2$ .

Note that Lemma 8.3 holds with  $(b, y, l, m, n)$  instead of  $(a, x, i, j, k)$  by symmetry of  $a$  and  $b$ .

## 9. DIOPHANTINE EQUATION $A^x + B^y = A^X + B^Y$

Let  $A$  and  $B$  be coprime integers greater than 1. Here we study the following purely exponential equation:

$$(9.1) \quad A^x + B^y = A^X + B^Y,$$

where  $x, y, X, Y$  are unknown positive integers with  $x \neq X$  and  $y \neq Y$ . It is easy to see that  $x < X$  if and only if  $y > Y$ .

Below, we give two results on equation (9.1).

**Lemma 9.1.** *Let  $(x, y, X, Y)$  be a solution of equation (9.1) with  $x < X$  and  $y > Y$ . Then the following hold.*

- (i)  $B^Y \mid (A^{X-x} - 1)$ ,  $A^x \mid (B^{y-Y} - 1)$ .
- (ii)  $x/X + Y/y < 1$ .
- (iii) If  $A > B > 2$ , then  $y > X$  and  $y \geq 4$ .

*Proof.* (i) The assertions readily follow from the equation

$$(9.2) \quad A^x(A^{X-x} - 1) = B^Y(B^{y-Y} - 1)$$

with  $\gcd(A, B) = 1$ .

(ii) From (i),  $A^x < B^{y-Y}$  and  $B^Y < A^{X-x}$ , that is,  $\frac{x}{y-Y} < \frac{\log B}{\log A} < \frac{X-x}{Y}$ , so that  $\frac{xY}{Xy} < (1 - \frac{x}{X})(1 - \frac{Y}{y})$ . This yields the assertion.

(iii) We follow an argument in [Lu, p. 213] to see that if  $y \leq X$  then

$$A^{X-1} + B^{X-1} \leq A^X - B^X \leq A^X - B^y = A^x - B^Y < A^x \leq A^{X-1}.$$

This contradiction shows that  $y > X$ .

Suppose that  $y \leq 3$ . Then  $(X, x, y) = (2, 1, 3)$  as  $y > X > x$ . Thus,  $Y = 1$  by (ii), so that  $A + B^3 = A^2 + B$ . This yields an integral point  $(\mathcal{X}, \mathcal{Y}) = (B, A)$  of the elliptic curve  $\mathcal{Y}^2 - \mathcal{Y} = \mathcal{X}^3 - \mathcal{X}$ . However, none of those points gives a proper pair  $(A, B)$  with  $A > B > 2$ . Thus,  $y \geq 4$ .  $\square$

**Lemma 9.2.** *Let  $(x, y, X, Y)$  be a solution of equation (9.1) with  $x < X$  and  $y > Y$ . Suppose that  $X - x = 1$ . Then  $A > B$ , and the following hold.*

- (i)  $A \equiv -xB^{2Y} - B^Y + 1 \pmod{B^{3Y}}$ . In particular,  $A \geq B^{3Y} - xB^{2Y} - B^Y + 1$ .
- (ii) Assume that  $B > 2$ . Then one of the following cases holds.
- (ii-1)  $y > (3Y - 1)X$ , and

$$A \geq \begin{cases} \frac{1}{2}B^{3Y} + \frac{1}{2}B^{2Y} - B^Y + 1, & \text{if } B \text{ is odd,} \\ \frac{1}{2}B^{3Y} - B^Y + 1, & \text{if } B \text{ is even.} \end{cases}$$

- (ii-2) It holds that

$$A = rB^{2Y} - B^Y + 1, \quad x \geq B^Y - r,$$

where  $r$  is some positive integer satisfying  $r \equiv -x \pmod{B^Y}$  with  $r \leq \lfloor B^Y/2 \rfloor$ .

In particular, case (ii-1) holds if  $x < B^Y/2$ .

*Proof.* First, under the assumption that  $(X - x) \mid x$ , we show the following congruence:

$$(9.3) \quad A^{X-x} \equiv -B^Y + 1 \pmod{B^{2Y}}.$$

By Lemma 9.1 (i),  $A^{X-x} = 1 + KB^Y$  with some  $K \in \mathbb{N}$ . Substituting this into (9.2) yields

$$A^x \cdot K = B^{y-Y} - 1.$$

Suppose that  $(X - x) \mid x$ . Then  $A^x \equiv 1 \pmod{B^Y}$ . We reduce the last displayed equality modulo  $B^{\min\{y-Y, Y\}}$  to see that

$$K \equiv -1 \pmod{B^{\min\{y-Y, Y\}}}.$$

Thus, for obtaining (9.3), it suffices to show that  $y \geq 2Y$ . Since  $X \leq 2x$ , this inequality follows from Lemma 9.1 (ii). In what follows, suppose that  $X - x = 1$ .

(i) By congruence (9.3),  $A = LB^{2Y} - B^Y + 1$  with some  $L \in \mathbb{N}$ . Substituting this into (9.2) yields

$$(LB^{2Y} - B^Y + 1)^x (LB^Y - 1) = B^{y-Y} - 1.$$

Observe that

$$\begin{aligned} (LB^{2Y} - B^Y + 1)^x (LB^Y - 1) &\equiv (-xB^Y + 1)(LB^Y - 1) \pmod{B^{2Y}} \\ &\equiv (x + L)B^Y - 1 \pmod{B^{2Y}}. \end{aligned}$$

We reduce the previous equality modulo  $B^{2Y}$  to see that  $(x + L)B^Y \equiv B^{y-Y} \pmod{B^{2Y}}$ , so that

$$x + L \equiv B^{y-2Y} \pmod{B^Y}.$$

It suffices to show that  $y \geq 3Y$ . This follows from the inequalities  $A \geq B^{2Y} - B^Y + 1 > B^{2Y-1}$ , and  $A \leq A^x < B^{y-Y}$ .

(ii) Set  $r$  be the integer satisfying  $r \equiv x \pmod{B^Y}$  with  $|r| \leq \lfloor B^Y/2 \rfloor$ . By (i),  $A = (TB^Y - r)B^{2Y} - B^Y + 1$  with some  $T \in \mathbb{Z}$ . It is clear that  $TB^Y - r \geq 1$ . If  $T < 0$ , then  $-r \geq 1 - TB^Y \geq 1 + B^Y > \lfloor B^Y/2 \rfloor$ , which is absurd. If  $T = 0$ , then  $A = -rB^{2Y} - B^Y + 1$  with  $r < 0$ , and  $x \geq B^Y - r (> B^Y/2)$  since  $x \equiv r \pmod{B^Y}$ , so case (ii-2) holds.

Finally, suppose that  $T > 0$ . Then  $A \geq B^{3Y} - rB^{2Y} - B^Y + 1$ . This together with  $r \leq \lfloor B^Y/2 \rfloor$  easily gives the asserted lower bounds for  $A$  in

case (ii-1). Also, from those observe that  $A \geq (1/k) \cdot B^{3Y}$ , where  $k = 2$  if  $2 \nmid B$ , and  $k = 64/31$  if  $2 \mid B$  with  $B > 2$ . On the other hand,

$$B^y = A^X \cdot \frac{1 - \frac{1}{A}}{1 - \frac{1}{B^{y-Y}}} > A^X \cdot (1 - \frac{1}{A}).$$

Since  $B/k \geq 4/3$  as  $B > 2$ , these inequalities together show that

$$y > \frac{\log A}{\log B} X + \frac{\log(1 - \frac{1}{A})}{\log B} > (3Y - \frac{\log k}{\log B}) X - \frac{\log(4/3)}{\log B} \geq (3Y - 1)X.$$

Thus, case (ii-1) holds.  $\square$

In the forthcoming two sections, we apply results in Sections 6, 7, 8 and 9 to find all possible values of  $a, b, c$  (together with those of  $\alpha, \beta$ ) and  $x_1, y_1, z_1, x_2, y_2, z_2$ . More precisely, in the next section, we sieve all those remaining cases with  $z_1 = z_2$  by using the system formed of the first two equations, that is,

$$(9.4) \quad \begin{cases} a^{x_1} + b^{y_1} = c^{z_1}, \\ a^{x_2} + b^{y_2} = c^{z_2}. \end{cases}$$

For these purposes, we prepare several notation and give a few remarks as follows. In each of any forthcoming situations, let  $\mathcal{M}_a, \mathcal{M}_b, \mathcal{M}_c$  denote any uniform upper bound for  $a, b, c$ , respectively. Then any computer program used for sieving depends on the sizes of these numbers, and it proceeds faster for smaller values of them. Thus, throughout those programs, we always replace  $\mathcal{M}_a, \mathcal{M}_b, \mathcal{M}_c$  by any smaller ones whenever those are found. The details on the iterations coming from these are omitted in most cases in the text. The situation is similar for the lower bounds for  $a, b$  and  $c$ . In what follows, in each of the situations, let  $a_0, b_0, c_0$  denote any uniform lower bound for  $a, b, c$ , respectively. These numbers may be chosen appropriately according to each case together with (\*), (5.1) and Lemma 5.3. For example, in any case with  $a > b$ , we may choose those numbers as follows:

$$\begin{aligned} a_0 &= \max\{1000, 2^\alpha + 1, 3 \cdot 2^\beta + 1\}, \quad c_0 = 3 \cdot 2^\beta, & \text{if } a > \max\{b, c\}, \\ a_0 &= \max\{11, 2^\alpha + 1\}, \quad c_0 = \max\{1000, 3 \cdot 2^\beta, 2^\alpha + 2\}, & \text{if } c > a > b, \end{aligned}$$

with  $b_0 = 2^\alpha - 1$ .

To treat system (9.4), it is very efficient to rely upon the existing results on ternary Diophantine equations, which are summarized in Lemma 8.1. Indeed, those results restrict the divisibility properties of the exponential unknowns (as already seen in Lemmas 8.2 and 8.3), and reduce considerably the computation time for showing results. However, we often omit the details on those applications for simplicity of the presentation.

## 10. CASE WHERE $z_1 = z_2$

Here we examine the case of  $z_1 = z_2$ , where system (9.4) is

$$(10.1) \quad a^{x_1} + b^{y_1} = a^{x_2} + b^{y_2} = c^{z_1}.$$

Without loss of generality, we may assume that  $x_1 < x_2$  and  $y_1 > y_2$ . Applying Lemma 9.1 for  $(A, B) = (a, b)$  and  $(x, y, X, Y) = (x_1, y_1, x_2, y_2)$  gives

$$(10.2) \quad \begin{cases} a^{x_1} \mid (b^{D_y} - 1), & b^{y_2} \mid (a^{D_x} - 1); \\ \max\{x_2, y_1\} \geq 4 & x_1/x_2 + y_2/y_1 < 1; \\ y_1 > x_2, & x_1 < D_y, \quad \text{if } a > b; \\ x_2 > y_1, & y_2 < D_x, \quad \text{if } b > a, \end{cases}$$

where  $D_x := x_2 - x_1$ ,  $D_y := y_1 - y_2$ . Also, recall from Lemma 5.2 that either  $(\beta, z_1) = (1, 1)$  or  $z_1 = \alpha/\beta$  if one of  $d_x, d_y, D_x, D_y$  is odd. In case  $(\dagger)$  with  $c < \max\{a, b\}$ , we have  $z_1 = \alpha/\beta$  as  $z_1 > 1$ . We often use these conditions implicitly below.

Let us begin with the following lemma.

**Lemma 10.1.** *If  $d_z = 0$ , then*

$$c^{z_1/2} < \frac{\log c}{\log \min\{a, b\}} z_1^2 z_3, \quad \min\{a, b\} < c^{z_1/4}.$$

*Proof.* Suppose that  $z_1 = z_2$ . The second inequality immediately follows from system (10.1) with  $\max\{x_2, y_1\} \geq 4$  by (10.2). Apply Proposition 6.2 with  $(A, B, C) = (a, b, c)$  and  $(X_r, Y_r, Z_r) = (x_r, y_r, z_r)$  for  $r = 1, 2, 3$ . Then  $c^{z_2/2} < \frac{2}{\log \min\{a, b\}} z_3$ , or

$$\begin{aligned} \frac{c^{z_2/2}}{z_2} &< \max_{t \in \{1, 2\}} \{ |x_3 z_2 - x_t z_3|, |y_3 z_2 - y_t z_3| \} \\ &< \max_{t \in \{1, 2\}} \{ x_3 z_2, x_t z_3, y_3 z_2, y_t z_3 \} \\ &< \max_{t \in \{1, 2\}} \left\{ \frac{\log c}{\log a} z_3 z_2, \frac{\log c}{\log a} z_t z_3, \frac{\log c}{\log b} z_3 z_2, \frac{\log c}{\log b} z_t z_3 \right\} = \frac{\log c}{\log \min\{a, b\}} z_2 z_3. \end{aligned}$$

These together show the first asserted inequality as  $z_2 = z_1$ .  $\square$

Using this lemma, we first deal with the case where  $c > \max\{a, b\}$ .

**Proposition 10.1.** *If  $c > \max\{a, b\}$ , then  $d_z > 0$ .*

*Proof.* It suffices to consider the case where  $a > b$ . Suppose on the contrary that  $c > a > b$  and  $z_1 = z_2$ . Since  $\frac{\log c}{\log b} z_3 < \mathcal{H}(c; a, c) < \mathcal{H}(c)$  by (5.6), Lemma 10.1 yields

$$(10.3) \quad c^{z_1/2} < z_1^2 \cdot \mathcal{H}_{\alpha, \beta, \frac{\log 8}{\log 3}}(c).$$

We use this inequality to find all possible values of the letters in system (10.1), and we sieve them as follows.

By (5.1),

$$(10.4) \quad \beta \leq \left\lfloor \frac{\log(\mathcal{M}_c/3)}{\log 2} \right\rfloor, \quad 2 \leq \alpha \leq \left\lfloor \frac{\log \mathcal{M}_{\min\{a, b\}}}{\log 2} \right\rfloor.$$

Also, from (5.4),

$$(10.5) \quad \beta = z_1 = 1 \quad \text{or} \quad \left\lceil \frac{\alpha}{\beta} \right\rceil \leq z_1 \leq \mathcal{U}_1(\alpha, \beta, \frac{\log 8}{\log 3}, a_0, b_0, \mathcal{M}_c, 1).$$

First, let us find a smaller upper bound for  $c$ . Since  $c^{1/2} \leq c^{z_1/2}/z_1^2$ , inequality (10.3) yields

$$c^{1/2} < \max\{\mathcal{H}_{2,1,\frac{\log 8}{\log 3}}(c), \mathcal{H}_{3,1}(c)\}.$$

This implies that  $c < 1.4 \cdot 10^{13}$ . Thus we set  $\mathcal{M}_c$  and  $\mathcal{M}_{\min a, b}$  as this upper bound.

Next, for each of the values of  $\beta, \alpha$  and  $z_1$  satisfying (10.4) and (10.5), we use inequality (10.3) to find an upper bound for  $c$ , say  $c_u$ . At the same time, an upper bound for  $b (= \min\{a, b\})$  is found, say  $b_u$ , by the second inequality of Lemma 10.1, that is,  $b_u := \lfloor c_u^{z_1/4} \rfloor$ . Let LIST be the list composed of all possible tuples  $(\beta, \alpha, z_1, b_u, c_u)$ .

Third, for each tuple in LIST, we find all possible values of  $b, y_1, y_2, a, x_2$  and  $x_1$  in turn by using the following relations:

$$b \leq b_u, \quad 4 \leq y_1 \leq \left\lfloor \frac{\log c_u}{\log b} z_1 \right\rfloor, \quad y_2 < y_1 - 1, \quad a \leq a_u, \quad a \mid (b^{D_y} - 1),$$

$$1 < x_2 \leq \min\left\{\left\lfloor \frac{\log c_u}{\log a} z_1 \right\rfloor, y_1 - 1\right\}, \quad x_1 < \min\{x_2, D_y\}.$$

Finally, for each of the found tuples  $(a, b, x_1, y_1, x_2, y_2)$ , we check whether the system (10.1) holds or not, as well as the divisibility in (10.2). As it turns result, the only remaining tuple is  $(a, b, x_1, y_1, x_2, y_2) = (13, 3, 1, 7, 3, 1)$  with  $c = 2200$ , where a short modular arithmetic argument shows that there is no other triple  $(x, y, z)$  satisfying  $13^x + 3^y = 2200^z$ . The proof is completed.  $\square$

**Remark 2.** The information on  $\beta$  and  $z_1$  can be also used to check whether system (10.1) holds or not.

In what follows, we keep the notation in the proof of Proposition 10.1 and set  $m_2 = \frac{\log 8}{\log 3}$  uniformly. Also, inequalities (10.4) and (10.5) are implicitly used.

For dealing with the case where  $c < \max\{a, b\}$ , we show two lemmas.

**Lemma 10.2.** *If  $d_z = 0$ , then  $d_x > 0$  and  $d_y > 0$ .*

*Proof.* By symmetry of  $a$  and  $b$ , it suffices to show that  $d_x > 0$ . Suppose on the contrary that  $z_1 = z_2$  and  $x_i = x_j (< x_k)$ . Then  $\{i, j\} \ni 3$ . Let  $(I, J)$  be the permutation of  $\{i, j\}$  such that  $J = 3$ . Note that  $\{I, k\} = \{1, 2\}$ ,  $z_I = z_k = z_1$  and  $x_I = x_J = x_3$ .

Since  $z_I = z_k$  and  $x_I = x_j < x_k$ , it follows that  $y_k < y_I$ . Also, observe that  $c^{z_I} - b^{y_I} = a^{x_I} = a^{x_J} = c^{z_J} - b^{y_J}$  and  $z_I = z_1 \leq z_3 = z_J$ . Thus,  $y_I \leq y_J$ . To sum up,  $y_k < y_I \leq y_J$ , so that  $d_y = y_I - y_k > 0$  with  $g_y = \gcd(x_I, z_1)$ . Now Lemma 7.3(i) with the base  $b$  gives

$$b^{d_y} < g_y \cdot \frac{\log c}{\log a} \cdot z_I z_J = g_y \cdot z_1 \cdot \frac{\log c}{\log a} z_3.$$

By (5.6),

$$(10.6) \quad b^{d_y} < g_y \cdot z_1 \cdot \mathcal{H}(c; b, c).$$

Similarly to the use of (10.3), we use inequality (10.6) to find all possible values of the letters in system (10.1), and we sieve them. We distinguish

two cases according to whether  $a > \max\{b, c\}$  or  $b > \max\{a, c\}$ . Note that the conditions in (10.2) correspond to

$$a^{x_I} \mid (b^{d_y} - 1), \quad b^{y_k} \mid (a^{D_x} - 1),$$

$$\max\{x_k, y_I\} \geq 4, \quad \begin{cases} y_I > x_k, & \text{if } a > b, \\ x_k > y_I, & \text{if } b > a, \end{cases} \quad x_I/x_k + y_k/y_I < 1$$

with  $D_x = x_k - x_I$ .

*Case where  $a > \max\{b, c\}$ .*

Since  $a^{x_I} < b^{d_y}$  and  $g_y \leq x_I$ , we see from (10.6) that  $a \leq a^{x_I}/x_I < b^{d_y}/g_y < z_1 \mathcal{H}(c; b, c) < z_1 \mathcal{H}(a)$ , so that

$$(10.7) \quad a < z_1 \mathcal{H}(a).$$

Also, since  $c < a < b^{d_y/x_I}$ ,

$$(10.8) \quad b^{d_y} < g_y \cdot z_1 \cdot d_y/x_I \cdot \mathcal{H}(b^{d_y/x_I}; b, b).$$

First, we use inequality (10.7) with  $\beta = 1$  to find that  $a < 2.3 \cdot 10^7$ . Thus we set  $\mathcal{M}_a := 2.3 \cdot 10^7$ . Note that

$$x_I \leq z_1 - 2, \quad x_I < d_y < \left\lfloor \frac{\log \mathcal{M}_c}{\log b_0} z_1 \right\rfloor.$$

Next, for each of possible tuples  $(\beta, \alpha, z_1, x_I, d_y)$ , we use inequality (10.8) to find an upper bound for  $b$ , say  $b_u$ , thereby an upper bound for  $a$  is also obtained, say  $a_u$ , from the divisibility relation  $a^{x_I} \mid (b^{d_y} - 1)$ . At the same time, we find an upper bound for  $c$  and another upper bound for  $b$ , say  $c_u, b_u'$ , respectively, by using the following inequalities from Lemma 10.1:

$$c^{z_1/2} < z_1^2 \mathcal{H}(c; a_u, c), \quad b < c_u^{z_1/4}.$$

Third, for each of the found tuples  $(\beta, \alpha, z_1, x_I, d_y, a_u, \min\{b_u, b_u'\}, c_u)$ , we find all possible values of  $b, a, y_I, y_k$  and  $x_k$  in turn by using the following relations:

$$b \leq \min\{b_u, b_u'\}, \quad a \leq a_u, \quad a^{x_I} \mid (b^{d_y} - 1), \quad d_y < y_I \leq \left\lfloor \frac{\log c_u}{\log b} z_1 \right\rfloor,$$

$$y_k = y_I - d_y, \quad x_I < x_k \leq \min\left\{ \left\lfloor \frac{\log c_u}{\log a} z_1 \right\rfloor, y_I - 1 \right\}.$$

Finally, we verify that system (10.1) does not hold for any found tuple  $(a, b, x_1, y_1, x_2, y_2)$ .

*Case where  $b > \max\{a, c\}$ .*

In case  $(\dagger)$ , we have  $z_1 = \alpha/\beta$ , and  $y_k < y_I < z_I = z_1 \leq \alpha < \frac{2^\alpha - 1}{2} \leq \frac{a}{2}$ . Thus, if  $d_y = 1$ , by Lemma 9.2, we can use the following relations:

$$b \equiv -y_k a^{2x_I} - a^{x_I} + 1 \pmod{a^{3x_I}},$$

$$b \geq \frac{1}{2} a^{3x_I} + \frac{1}{2} a^{2x_I} - a^{x_I} + 1, \quad x_k > (3x_I - 1)y_I.$$



Since  $c < b$ , inequality (10.6) yields

$$(10.9) \quad b^{d_y} < \gcd(x_I, z_1) \cdot z_1 \cdot \mathcal{H}(b).$$

First, we use the inequality  $b < z_1^2 \cdot \mathcal{H}(b)$  with  $\beta = 1$  to see that  $b < 3.1 \cdot 10^6$ , and set  $\mathcal{M}_b := 3.1 \cdot 10^6$ . Note that

$$x_I \leq x_k - 2 \leq \left\lfloor \frac{\log \mathcal{M}_c}{\log a_0} z_1 \right\rfloor - 2, \quad d_y \leq z_1 - 2.$$

Next, for each possible tuple  $(\beta, \alpha, z_1, x_I, d_y)$ , we use inequality (10.9) to find an upper bound for  $b$ , say  $b_u$ . Similarly to the previous case, we use the inequalities  $a < b_u^{d_y/x_I}$ ,  $c^{z_1/2} < z_1^2 \mathcal{H}(c; b_u, c)$  and  $a < c^{z_1/4}$ , to find upper bounds for  $a$  and  $c$ , say  $a_u, c_u$ , respectively. Finally, we verify that system (10.1) does not hold for any possible tuples  $(a, b, x_1, y_1, x_2, y_2)$  coming from all possible tuples  $(\beta, \alpha, z_1, x_I, d_y, a_u, b_u, c_u)$  found similarly to the case where  $a > \max\{b, c\}$ .

To sum up, the lemma is proved by Proposition 10.1.  $\square$

**Lemma 10.3.** *Suppose that  $d_z = 0$ . Then*

$$\begin{cases} k = 3, & \text{if } a > \max\{b, c\}; \\ n = 3, & \text{if } b > \max\{a, c\}. \end{cases}$$

*Proof.* By symmetry of  $a$  and  $b$ , it suffices to consider the case where  $a > b$ . Suppose on the contrary that  $a > \max\{b, c\}$  and  $k \neq 3$ . Then  $\{i, j\} \ni 3$ . Let  $(I, J)$  be the permutation of  $\{i, j\}$  such that  $J = 3$ . Note that  $\{I, k\} = \{1, 2\}$  and  $z_I = z_k = z_1$ . Further, we know that  $d_x > 0$  by Lemma 10.2.

We claim that

$$(10.10) \quad z_3 < \mathcal{U}_3 := \max \left\{ (1 + \varepsilon)z_1 + \frac{(1 + \varepsilon) \log a}{\log c} \cdot d_x + 1, 2531 \log b \right\}$$

with  $\varepsilon = 250$ . First, suppose that  $a^{(1+\varepsilon)x_J} > b^{y_J}$ . Then  $c^{z_J} = a^{x_J} + b^{y_J} < 2a^{(1+\varepsilon)x_J}$ , so that

$$z_J < 1 + \frac{(1 + \varepsilon) \log a}{\log c} x_J.$$

Since  $x_J = x_I - (x_I - x_J)$  and  $x_I < \frac{\log c}{\log a} z_I$ , it follows that

$$z_3 < 1 + (1 + \varepsilon)z_1 + \frac{(1 + \varepsilon) \log a}{\log c} |x_I - x_J|.$$

Next, suppose that  $a^{(1+\varepsilon)x_J} < b^{y_J}$ . From  $J$ -th equation,

$$\frac{c^{z_J}}{b^{y_J}} = \frac{a^{x_J}}{b^{y_J}} + 1 < \frac{1}{b^{\frac{\varepsilon}{1+\varepsilon} y_J}} + 1.$$

Put  $\lambda := z_J \log c - y_J \log b (> 0)$ . We find that

$$\log \lambda < -\frac{\varepsilon}{1 + \varepsilon} y_J \log b.$$

On the other hand, from [La, Corollary 2;  $(m, C_2) = (10, 25.2)$ ],

$$\log \lambda > -25.2 \log b \log c \left( \max \left\{ \log \left( \frac{z_J}{\log b} + \frac{y_J}{\log c} \right) + 0.38, 10 \right\} \right)^2.$$

These inequalities together yield

$$\frac{y_J}{\log c} < 25.2(1 + 1/\varepsilon) \left( \max \left\{ \log \left( \frac{2y_J}{\log c} + 1 \right) + 0.38, 10 \right\} \right)^2.$$

This implies that  $y_J/\log c < 2520(1 + 1/\varepsilon)$ . Inequality (10.10) for this case follows from the fact that  $\lambda$  is small.

Secondly, in several cases according to the value of  $g_x$ , we apply Lemma 7.3 together with inequality (10.10) to find all possible values of the letters in (10.1) and sieve them. We proceed basically along similar lines to the proof of the previous lemma.

*Case where  $g_x = 1$ .*

Lemma 7.3 (i) gives

$$a^{d_x} < \frac{\log c}{\log b} \cdot z_j z_k < z_1 \mathcal{U}_3', \quad a^{d_x} < z_1 \mathcal{H}(a).$$

where  $\mathcal{U}_3' = \max \left\{ \frac{(1+\varepsilon)z_1 \log c}{\log b_0} + \frac{(1+\varepsilon) \log a}{\log b_0} \cdot d_x + \log c, 2523 \log c \right\}$ . The second inequality above gives a smaller bound for  $a$ , that is, we can set  $\mathcal{M}_a := 2.7 \cdot 10^{11}$  and also that  $d_x \leq 2$ . For each possible tuple  $(\beta, \alpha, d_x, z_1)$ , we use the first inequality above to find an upper bound for  $a$ , say  $a_u$ . Also, upper bounds for  $c$  and  $b$ , say  $c_u, b_u$ , respectively, are found by using the following inequalities from Lemma 10.1:

$$c^{z_1/2} < z_1^2 \mathcal{U}_3', \quad b < c_u^{z_1/4}.$$

Finally, we check that system (10.1) does not hold for any tuple  $(a, b, x_1, y_1, x_2, y_2)$  coming from all possible tuples  $(\beta, \alpha, z_1, a_u, b_u, c_u)$ .

*Case where  $g_x \in \{2, 5\}$ .*

Since  $\gcd(g'_x, a) = 1$  by Lemma 8.3 (iii), and  $x_j \leq x_k < \frac{\log c}{\log a} z_k$ , Lemma 7.3 (ii) gives

$$a^{d_x} < \frac{\log a}{\log b} \cdot x_j z_k < \frac{\log c}{\log b} \cdot z_k^2 < \frac{\log a}{\log b_0} \cdot z_1^2.$$

It turns out that there is no quadruple  $(\beta, \alpha, d_x, z_1)$  satisfying the above inequalities.

*Case where  $g_x = 3$ .*

Since  $2 \leq x_j < \min\{z_j, z_k\} \leq z_2$ , it follows from Lemmas 5.3 and 8.3 (i) that  $x_j = 2$ . Thus,  $x_i = 1, d_x = 1$ , so that Lemma 7.3 (ii) gives

$$a < \frac{3^2}{3-1} \cdot \frac{\log a}{\log b} \cdot 2 \cdot z_k = \frac{9 \log a}{\log b_0} \cdot z_1.$$

It turns out that there is no triple  $(\beta, \alpha, z_1)$  satisfying the above inequalities.

*Case where  $g_x \notin \{1, 2, 3, 5\}$ .*

Note that  $g_x \geq 7$  as  $4 \nmid g_x$  and  $6 \nmid g_x$  by Lemma 8.3 (ii). Lemma 7.3 (iii) gives

$$\begin{aligned} a^{d_x} &< \frac{(7/6)^2 \log^2 a}{\log(a-1) \log b} \cdot t_{b,c} \cdot (x_j + d_x + d_x^2) z_k \\ &\leq \frac{(49/36) \log a}{\log(a-1)} \cdot \frac{\log a}{\log \max\{b_0, c_0\}} \cdot (z_1 - 1 + d_x + d_x^2) z_1. \end{aligned}$$

It turns out that there is no quadruple  $(\beta, \alpha, d_x, z_1)$  satisfying the above inequalities.  $\square$

**Proposition 10.2.** *If  $c < \max\{a, b\}$ , then  $d_z > 0$ .*

*Proof.* It suffices to consider the case where  $a > b$ . Suppose on the contrary that  $a > \max\{b, c\}$  and  $z_1 = z_2$ . Then  $k = 3$  and  $d_x = x_2 - x_1 > 0$  by the combination of Lemmas 10.2 and 10.3. Since the argument is almost similar to that of Lemma 10.3, we omit the details in the following. The main difference is that  $\mathcal{U}_3$  is replaced by  $\mathcal{H}(c; a, b)$  from (5.6).

*Case where  $g_x \in \{1, 2, 5\}$ .*

Lemma 7.3 (i) together with Lemma 8.3 (iii) gives

$$a^{d_x} < \frac{\log c}{\log b} \cdot z_1 z_3 < z_1 \mathcal{H}(c; a, c).$$

Note that  $a^{d_x} < z_1 \mathcal{H}(a)$  and this implies small upper bounds for  $a$  and  $d_x$ . The remaining part is handled almost similarly to the proof of the previous lemma. However, remark that we can efficiently use the additional condition that  $D_x = d_x$  and  $\gcd(y_2, z_1) \in \{1, 2, 5\}$  in checking system (10.1).

*Case where  $g_x = 3$ .*

This case is also handled almost similarly to the previous lemma, in particular,  $d_x = 1$ , and it is easy to verify that Lemma 9.2 can be used, so that the additional condition that  $a \equiv -x_1 b^{2y_2} - b^{y_2} + 1 \pmod{b^{3y_2}}$  and  $y_1 > (3y_2 - 1)x_2$  can be efficiently used.

*Case where  $g_x \notin \{1, 2, 3, 5\}$ .*

First, Lemma 7.3 (i) gives

$$a^{d_x} < \frac{\log c}{\log b} \cdot z_1^2 z_3 < z_1^2 \mathcal{H}(a).$$

This implies small upper bounds for  $a$  and  $d_x$ . Next, together with  $g_x \geq 7$ , Lemma 7.3 (iii) implies

$$a^{d_x} < \frac{49}{36} \cdot \left( \frac{\log c}{\log a} z_1 + d_x + d_x^2 \right) \mathcal{H}(c; a, a).$$

The remaining part is handled similarly to the previous lemma, where the additional condition  $\gcd(y_2, z_1) \geq 7$  is efficiently used.  $\square$

In view of Propositions 10.1 and 10.2, the conclusion of this section is:

**Proposition 10.3.**  $z_1 \neq z_2$ .

The total computation time for this proposition was about 5 minutes.

By Proposition 10.3, it remains to consider the case where  $z_1 < z_2$ , where the initial upper bound for  $\max\{a, b, c\}$  in (\*) can be replaced by  $5 \cdot 10^{27}$  by [HuLe2, Lemma 4.7].

#### 11. CASE WHERE $z_1 < z_2$ WITH $c > \max\{a, b\}$ : FINDING BOUNDS

The aim of this section is to provide a list of all possible values or upper bounds of some letters in system (9.4) satisfying  $z_1 < z_2$  with  $c > \max\{a, b\}$ . We distinguish two cases according to whether  $c^{z_1}$  is divisible by 4 or not.

Let us begin with the following lemma to give an upper bound for  $g_2$  in terms of  $z_2$ .

**Lemma 11.1.**  $g_2 < \frac{\log c}{\log \max\{a, b\}} z_2$ .

*Proof.* From 2nd equation,  $\min\{x_2, y_2\} < \frac{\log c}{\log \max\{a, b\}} z_2$ . On the other hand,  $g_2 \mid \min\{x_2, y_2\}$  by the definition of  $g_2$ . These relations together readily yield the assertion.  $\square$

In what follows, for any numbers  $P_1, P_2, \dots, P_k$  and  $Q_1, Q_2, \dots, Q_k$ , the notation  $[P_1, P_2, \dots, P_k] \leq [Q_1, Q_2, \dots, Q_k]$  means that  $P_i \leq Q_i$  for any  $i$ .

**Proposition 11.1.** *Suppose that*

$$d_z > 0, \quad c^{z_1} \equiv 2 \pmod{4}.$$

*Then  $\beta = 1, z_1 = 1, c > \max\{a, b\}$ , and the following hold.*

(i) *Suppose that  $g_2 = 1$ . Then*

$$\alpha \leq 17, \quad z_2 \leq 18, \quad c < 1.1 \cdot 10^6.$$

*More exactly, one of the following cases holds.*

- $z_2 \leq 14, c < 7.9 \cdot 10^5, (x_1, y_1) = (1, 1), \min\{a, b\} \leq 7$ ;
- $[\alpha, z_2] \leq [17, 18], c < 1.1 \cdot 10^6, (x_1, y_1) = (1, 1), \min\{a, b\} > 7$ ;
- $z_2 \leq 14, c < 7.9 \cdot 10^5, (x_1, y_1) \neq (1, 1), \min\{a, b\} \leq 7$ ;
- $[\alpha, z_2] \leq [9, 14], c < 1.1 \cdot 10^6, (x_1, y_1) \neq (1, 1), \min\{a, b\} > 7$ .

(ii) *Suppose that  $g_2 > 1$ . Then*

$$\alpha \leq 22, \quad z_2 \leq 23, \quad c < 1.9 \cdot 10^7.$$

*More exactly, one of the following cases holds.*

- $(\alpha, z_2) = (2, 2), c < 5.2 \cdot 10^6, g_2 = 3, (x_1, y_1) \neq (1, 1), \min\{a, b\} > 7$ ;
- $z_2 = 2, c < 8.7 \cdot 10^5, g_2 = 3, (x_1, y_1) \neq (1, 1), \min\{a, b\} \leq 7$ ;
- $z_2 = 11, c < 1600, g_2 = 5, \min\{a, b\} \leq 7$ ;
- $[\alpha, z_2] \leq [13, 13], c < 1.6 \cdot 10^5, g_2 = 5, (x_1, y_1) = (1, 1), \min\{a, b\} > 7$ ;
- $[\alpha, z_2] \leq [5, 11], c < 4500, g_2 = 5, (x_1, y_1) \neq (1, 1), \min\{a, b\} > 7$ ;
- $z_2 \leq 13, c < 1.5 \cdot 10^5, g_2 = 7, \min\{a, b\} \leq 7$ ;
- $[\alpha, z_2] \leq [19, 19], c < 5 \cdot 10^6, g_2 = 7, (x_1, y_1) = (1, 1), \min\{a, b\} > 7$ ;

- $[\alpha, z_2] \leq [11, 17]$ ,  $c < 5 \cdot 10^6$ ,  $g_2 = 7$ ,  $(x_1, y_1) \neq (1, 1)$ ,  $\min\{a, b\} > 7$ ;
- $z_2 = 5$ ,  $c < 1.5 \cdot 10^6$ ,  $g_2 \geq 11$ ,  $\min\{a, b\} \leq 7$ ;
- $[\alpha, z_2] \leq [22, 23]$ ,  $c < 1.9 \cdot 10^7$ ,  $g_2 \geq 11$ ,  $(x_1, y_1) = (1, 1)$ ,  $\min\{a, b\} > 7$ ;
- $[\alpha, z_2] \leq [11, 19]$ ,  $c < 1.2 \cdot 10^7$ ,  $g_2 \geq 11$ ,  $(x_1, y_1) \neq (1, 1)$ ,  $\min\{a, b\} > 7$ .

*Proof.* Here we just indicate how we find a list of all possible pairs  $(\alpha, z_2)$  with the corresponding upper bound for  $c$ . It suffices to consider the case where  $a > b$ .

From  $c = a^{x_1} + b^{y_1}$  by 1st equation, observe that  $c \geq \max\{1000, 2^{\alpha+1} + 2^\alpha - 2\}$ , and

$$\begin{cases} a = \max\{a, b\} \geq c/2 + 2, & \text{if } (x_1, y_1) = (1, 1); \\ c \geq (2^\alpha - 1)^2 + (2^\alpha + 1) = 2^{2\alpha} - 2^\alpha + 2, & \text{if } (x_1, y_1) \neq (1, 1). \end{cases}$$

This affects the choice of the values of  $a_0$  and  $c_0$ . By Lemma 7.1,

$$(11.1) \quad c < \min \left\{ 2^{\alpha+1-z_2} \frac{(g_2')^2}{g_2}, \frac{(g_2')^2}{g_2}, \frac{T \log c}{\log(c-1)} z_2 \right\} \cdot z_2 \mathcal{H}_{\alpha,1,m_2}(c)$$

where  $g_2' = \gcd(c, g_2)$ , and

$$T = \begin{cases} 1, & \text{if } b > 7, \\ \frac{\log b}{\log a_0}, & \text{if } b \in \{3, 5, 7\}. \end{cases}$$

Note that  $g_2$  is odd by Lemma 8.2 (i).

Similarly to Section 10, firstly setting  $\mathcal{M}_c = 5 \cdot 10^{27}$ , we use inequality (11.1) to find an upper bound for  $c$  for each possible pair  $(\alpha, z_2)$  satisfying (10.4) and  $\alpha \leq z_2 \leq \mathcal{U}_2(\alpha, 1, m_2, \mathcal{M}_c, g_2)$  by (5.4), where each of the procedures is implemented in two versions according to whether  $b > 7$  or not, and to  $(x_1, y_1) = (1, 1)$  or not. Moreover, we proceed in several cases according to the value of  $g_2$  as explained below, where we only indicate the specialization or relaxation of the inequality (11.1).

*Case where  $g_2 = 1$ .*

Inequality (11.1) becomes

$$c < \min \left\{ 2^{\alpha+1-z_2}, 1, \frac{T z_2 \log c}{\log(c-1)} \right\} \cdot z_2 \mathcal{H}_{\alpha,1,m_2}(c).$$

*Case where  $g_2 > 1$ .*

Lemma 8.2 (iv, v) tells us that  $\gcd(z_2, 6) = 1$  if  $g_2 > 3$ . Moreover,  $z_2 \not\equiv 0 \pmod{g_2}$  by Lemma 8.1. We proceed in several subcases.

(i) *Case where  $g_2 \equiv 0 \pmod{3}$ .*

Lemma 8.2 (ii, iv) tells us that  $g_2 = 3, z_2 = 2$ , and so  $\alpha = 2, (x_1, y_1) \neq (1, 1)$ . Inequality (11.1) is

$$c < 2 \min \left\{ 3, \frac{2T \log c}{\log(c-1)} \right\} \cdot \mathcal{H}_{2,1,m_2}(c).$$

(ii) Case where  $g_2 = 5$ .

By Lemma 8.2 (iii),  $g_2' = 1$ , and also  $\gcd(z_2, 7) = 1$  by Lemma 8.1. Inequality (11.1) is

$$\min\left\{\frac{2^{\alpha+1-z_2}}{5}, \frac{Tz_2 \log c}{\log(c-1)}\right\} \cdot z_2 \mathcal{H}_{\alpha,1,m_2}(c).$$

(iii) Case where  $g_2 = 7$ .

Inequality (11.1) is used with  $g_2, g_2'$  replaced by 7. Note that  $\gcd(z_2, 5) = 1$ .

(iv) Case where  $g_2 > 7$ .

Note that  $g_2 \geq 11$ . From Lemma 11.1, inequality (11.1) yields

$$c < \min\left\{\frac{2^{\alpha+1-z_2} \log c}{\log a_0}, \frac{\log c}{\log a_0}, \frac{T \log c}{\log(c-1)}\right\} \cdot z_2^2 \mathcal{H}_{\alpha,1,m_2}(c).$$

By these observations, we find the finite list of all possible pairs  $(\alpha, z_2)$  with the corresponding upper bound for  $c$ , and those satisfy the stated conditions.  $\square$

The following lemma is a supplement to Proposition 11.1 and helps to reduce the computation time to sieve the given cases with  $(x_1, y_1) = (1, 1)$  (see Section 13.1.2).

**Lemma 11.2.** *Under the hypothesis of Proposition 11.1, assume that  $\min\{a, b\} > 7$  and  $(x_1, y_1) = (1, 1)$ . Then  $\min\{x_2, y_2\} \leq 7$ . Moreover, if  $\min\{x_2, y_2\} \geq 4$ , then the following holds.*

$$\min\{a, b\} \leq \begin{cases} 11, & \text{if } \min\{x_2, y_2\} = 7, \\ 19, & \text{if } \min\{x_2, y_2\} = 6, \\ 45, & \text{if } \min\{x_2, y_2\} = 5, \\ 177, & \text{if } \min\{x_2, y_2\} = 4. \end{cases}$$

*Proof.* Since  $x_2 \leq x_3$  or  $y_2 \leq y_3$ , one of the following cases holds.

$$\begin{aligned} (i, j, k) &= (1, 2, 3), & d_x &= x_2 - x_1 \geq \min\{x_2, y_2\} - 1; \\ (l, m, n) &= (1, 2, 3), & d_y &= y_2 - y_1 \geq \min\{x_2, y_2\} - 1. \end{aligned}$$

Let us consider only the former case as the latter one is similarly handled. From the assumption that  $\min\{x_2, y_2\} \geq 4$ , we have  $d_x \geq 3$ .

Similarly to Proposition 10.2, we apply Lemma 7.3 (iii) together with Lemmas 5.3 and 8.3 to see that

$$a^{d_x} < \frac{49}{36} \cdot \frac{\log a}{\log(a-1)} \cdot \left( \frac{\log \mathcal{M}_c}{\log a} z_2 + d_x + d_x^2 \right) \mathcal{H}(\mathcal{M}_c; a, a).$$

Note that  $m_2 = 1$  since  $\min\{a, b\} > 7$ , and we can set  $\mathcal{M}_c := 5 \cdot 10^6$  by Proposition 11.1. Finally, for each  $d_x \geq 3$ , similarly to Proposition 11.1, we use the above inequality to find an upper bound for  $a$  for each possible pair  $(\alpha, z_2)$ . The result implies the assertion.  $\square$

**Proposition 11.2.** *Suppose that*

$$d_z > 0, \quad c^{z_1} \equiv 0 \pmod{4}, \quad c > \max\{a, b\}.$$

*Then the following hold.*

(i) *Suppose that  $g_2 = 1$ . Then*

$$[\beta, \alpha, z_2] \leq [10, 18, 19], \quad d_z = 1, \quad c < 1.5 \cdot 10^6.$$

*More exactly, one of the following cases holds.*

- $[\beta, z_2] \leq [7, 14], \quad c < 1.5 \cdot 10^6, \quad \min\{a, b\} \leq 7.$
- $[\beta, \alpha, z_2] \leq [10, 18, 19], \quad c < 1.1 \cdot 10^6, \quad \min\{a, b\} > 7.$

(ii) *Suppose that  $g_2 > 1$ . Then*

$$[\beta, \alpha, z_2, d_z] \leq [10, 19, 23, 4], \quad c < 3.4 \cdot 10^6.$$

*More exactly, one of the following cases holds.*

- $\beta \leq 7, \quad c < 5.5 \cdot 10^5, \quad \min\{a, b\} \leq 7, \quad (z_2, z_1, g_2) = (2, 1, 3);$
- $[\beta, \alpha] \leq [9, 8], \quad c < 6.5 \cdot 10^5, \quad \min\{a, b\} > 7, \quad (z_2, z_1, g_2) = (2, 1, 3);$
- $[\beta, z_2] = [1, 11], \quad c < 1600, \quad \min\{a, b\} = 3, \quad g_2 = 5;$
- $[\beta, \alpha, z_2] \leq [1, 16, 17], \quad c < 7.6 \cdot 10^4, \quad \min\{a, b\} > 7, \quad g_2 = 5;$
- $[\beta, z_2] \leq [1, 13], \quad c < 1.5 \cdot 10^5, \quad \min\{a, b\} \leq 7, \quad g_2 = 7;$
- $[\beta, \alpha, z_2] \leq [1, 18, 19], \quad c < 7.5 \cdot 10^5, \quad \min\{a, b\} > 7, \quad g_2 = 7;$
- $[\beta, z_2] \leq [2, 13], \quad c < 2.9 \cdot 10^6, \quad \min\{a, b\} \leq 7, \quad g_2 \geq 11.$
- $[\beta, \alpha, z_2] \leq [2, 19, 23], \quad c < 3.4 \cdot 10^6, \quad \min\{a, b\} > 7, \quad g_2 \geq 11.$

*Proof.* Here we just indicate how we find a list of all possible triples  $(\beta, \alpha, z_2)$  with the corresponding upper bound for  $c$ . It suffices to consider the case where  $a > b$ . We proceed along similar lines to that of Proposition 11.1.

By Lemma 7.2,

$$(11.2) \quad c^{d_z} < \min \left\{ 2^{\alpha - \beta z_1} \frac{(g_2')^2}{g_2}, \frac{T \log c}{\log(c-1)} \frac{z_2}{z_1} \right\} \cdot z_2 \mathcal{H}(c),$$

where  $g_2' = \gcd(c^{z_2 - z_1}, g_2)$  and  $T$  is the same as in the proof of Proposition 11.1. Note that  $g_2$  is odd by Lemma 8.2(i). We use inequality (11.2) to find an upper bound for  $c$  for each  $d_z$  and for each triple  $(\beta, \alpha, z_2)$  satisfying (10.4) and  $\lceil \alpha/\beta \rceil \leq z_1 \leq \mathcal{U}_1(\alpha, \beta, m_2, a_0, b_0, \mathcal{M}_c, g_2)$  with  $z_2 = z_1 + d_z$  and  $\mathcal{M}_c = 5 \cdot 10^{27}$ , where each of the procedures is implemented in two versions according to whether  $b > 7$  or not. Moreover, we proceed in several cases according to the value of  $g_2$  as indicated below.

*Case where  $g_2 = 1$ .*

Inequality (11.2) becomes

$$c^{d_z} < \min \left\{ 2^{\alpha - \beta z_1}, \frac{T \log c}{\log(c-1)} \frac{z_2}{z_1} \right\} \cdot z_2 \mathcal{H}(c).$$

*Case where  $g_2 > 1$ .*

From Lemma 11.1, inequality (11.2) yields

$$(11.3) \quad c^{d_z} < \min \left\{ \frac{\log c}{\log a_0} 2^{\alpha - \beta z_1}, \frac{T \log c}{\log(c-1)} \frac{1}{z_1} \right\} \cdot z_2^2 \mathcal{H}(c).$$

Note that  $\gcd(z_2, 6) = 1$  if  $g_2 > 3$  and that  $z_2 \not\equiv 0 \pmod{g_2}$ . We proceed in several subcases.

(i) *Case where  $g_2 \equiv 0 \pmod{3}$ .*

Since  $g_2 = 3$  and  $z_2 = 2$ , it follows that  $z_1 = 1, d_z = 1$ , so that inequality (11.2) gives

$$c < \min \left\{ 6 \cdot 2^{\alpha-\beta}, \frac{4T \log c}{\log(c-1)} \right\} \cdot \mathcal{H}(c).$$

(ii) *Case where  $d_z \geq 2$ .*

Note that  $g_2 \geq 5$  by a previous case. It turns out that there is no case satisfying inequality (11.3).

(iii) *Case where  $g_2 = 5, d_z = 1$ .*

Since  $g_2' = 1$ , inequality (11.2) is

$$c < \min \left\{ \frac{2^{\alpha-\beta z_1}}{5}, \frac{T \log c}{\log(c-1)} \frac{z_2}{z_1} \right\} \cdot z_2 \mathcal{H}(c)$$

with  $\gcd(z_2, 7) = 1$ .

(iv) *Case where  $g_2 = 7, d_z = 1$ .*

Inequality (11.2) is used with both  $g_2$  and  $g_2'$  replaced as 7 and with  $\gcd(z_2, 5) = 1$ .

(v) *Case where  $g_2 > 7, d_z = 1$ .*

Inequality (11.3) is used with  $g_2 \geq 11$ .

By these observations, we find a finite list of all possible tuples  $(d_z, \beta, \alpha, z_2)$  with the corresponding upper bound for  $c$ , and those satisfy the stated conditions.  $\square$

## 12. CASE WHERE $z_1 < z_2$ WITH $c < \max\{a, b\}$ : FINDING BOUNDS

The aim of this section is to provide a list of all possible values or upper bounds of some letters in system (9.4) satisfying  $z_1 < z_2$  with  $c < \max\{a, b\}$ . It suffices for us to do this when  $a > b$ . We proceed basically in two cases according to whether  $d_x = 0$  or not.

We begin with a technical lemma that gives relatively small upper bounds for  $d_z, d_x$  and  $d_y$ . This helps to reduce the computation time for establishing the forthcoming propositions.

**Lemma 12.1.** *Suppose that  $d_z > 0$  and  $a > \max\{b, c\}$ . Then  $d_z \leq 7, d_x \leq 2$  and  $d_y \leq 10$ .*

*Proof.* Observe that  $z_1 > 1$  and  $g_2 < z_2$  as  $a > c$ , in particular,  $c^{z_1} \equiv 0 \pmod{4}$ . First, Lemma 7.2 yields  $c^{d_z} < \min\{2^{\alpha-\beta z_1}, 1/z_1\} \cdot z_2^2 \mathcal{H}(c)$ . Similarly to the previous sections, we use this inequality, for all possible triples  $(\beta, \alpha, z_2)$ , to restrict the values of  $c$  and  $d_z$ . The result gives the asserted bound for  $d_z$  and also  $c < 5.2 \cdot 10^6$ . Second, we apply Lemma 7.3 similarly as in the proofs of Lemma 10.3 and Proposition 10.2. It reveals that  $a^{d_x} < 9/4 \cdot (z_2 - 1 + d_x + d_x^2) \mathcal{H}(\mathcal{M}_c; a, a)$  with  $\mathcal{M}_c = 5.2 \cdot 10^6$ . For



all possible triples  $(\beta, \alpha, z_2)$ , we use this inequality to restrict the values of  $a$  and  $d_x$ , which gives the asserted bound for  $d_x$ . Finally, using Lemma 7.3 with the base  $b$ , we find that

$$\begin{aligned} b^{d_y} &< \frac{(9/4) \log^2 b \log c}{\log(b-1) \log^2 a} \left( \frac{\log c}{\log b} z_2 + d_y + d_y^2 \right) z_3 \\ &< \frac{9 \log b}{4 \log(b-1)} (z_2 + d_y + d_y^2) \mathcal{H}(\mathcal{M}_c; b, \mathcal{M}_c). \end{aligned}$$

It is easy to see that if  $d_y > 10$ , then this inequality holds for no possible triple  $(\beta, \alpha, z_2)$ .  $\square$

Lemma 12.1 is implicitly used in the sequel.

**Proposition 12.1.** *Suppose that*

$$d_z > 0, \quad a > \max\{b, c\}, \quad d_x > 0.$$

*Then  $k = 3$ , and the following hold.*

(i) *Suppose that  $g_x = 1$ . Then*

$$[\beta, \alpha, z_1, z_2] \leq [6, 18, 18, 21], \quad a < 1.9 \cdot 10^6.$$

*More exactly, one of the following cases holds.*

- $[\beta, z_1, z_2] \leq [1, 14, 17]$ ,  $a < 1.9 \cdot 10^6$ ,  $c < 2.1 \cdot 10^5$ ,  $b \leq 7$ ;
- $[\beta, \alpha, z_1, z_2] \leq [6, 18, 18, 21]$ ,  $a < 1.1 \cdot 10^6$ ,  $b > 7$ .

(ii) *Suppose that  $g_x > 1$ . Then*

$$a < 3.4 \cdot 10^6, \quad z_2 \leq 22.$$

*More exactly, one of the following cases holds.*

- $[\beta, \alpha, z_1, z_2] \leq [1, 2, 12, 13]$ ,  $a < 3000$ ,  $d_x = 2$ .
- $[\beta, \alpha, z_1, z_2] \leq [4, 17, 17, 21]$ ,  $a < 7.3 \cdot 10^5$ ,  $(g_x, d_x) = (2, 1)$ .
- $[\beta, z_1, z_2] \leq [1, 3, 8]$ ,  $a < 4.2 \cdot 10^6$ ,  $c < 2.1 \cdot 10^5$ ,  $(g_x, d_x) = (3, 1)$ ,  $b \leq 7$ ;
- $[\beta, \alpha, z_1, z_2] \leq [7, 17, 17, 21]$ ,  $a < 3.8 \cdot 10^6$ ,  $c < 1.7 \cdot 10^6$ ,  $(g_x, d_x) = (3, 1)$ ,  $b > 7$ ;
- $[\beta, \alpha, z_1, z_2] = [1, 15, 15, 17]$ ,  $a < 4.9 \cdot 10^4$ ,  $c < 400$ ,  $(g_x, d_x) = (5, 1)$ ,  $b > 7$ ;
- $[\beta, \alpha, z_1, z_2] \leq [3, 19, 19, 22]$ ,  $a < 3.4 \cdot 10^6$ ,  $c < 6.7 \cdot 10^5$ ,  $g_x \geq 7$ ,  $d_x = 1$ ,

*where  $z_1 = \alpha/\beta$  if  $d_x = 1$ .*

*Proof.* Here we just indicate how we find a list of all possible tuples  $(\beta, \alpha, z_1, z_2)$  with the corresponding upper bounds for  $a$  and  $c$ . We proceed basically similarly to the proofs of Propositions 11.1 and 11.2.

First, we use Lemma 7.2. Since  $a > c$ ,  $z_1 > 1$  and  $g_2 < z_2$ , it follows that

$$(12.1) \quad c^{d_z} < \min \left\{ 2^{\alpha - \beta z_1} (z_2 - 1), \frac{T z_2}{z_1} \right\} \cdot z_2 \mathcal{H}(c),$$

where  $T = 1$  if  $b > 7$  and  $T = \frac{\log b}{\log(c+1)}$  if  $b \leq 7$ . For each  $d_z$  and for each possible tuple  $(\beta, \alpha, z_1)$  satisfying (10.4) and  $\lceil \alpha/\beta \rceil \leq z_1 \leq \mathcal{U}_1(\alpha, \beta, m_2, a_0, b_0, \mathcal{M}_c, 1)$  with  $\mathcal{M}_c = 5 \cdot 10^{27}$ , we use inequality (12.1) to find an upper bound for  $c$ , say

$c_u$ , where each of the procedures is implemented in two versions according to whether  $b > 7$  or not.

Next, for each of the found tuples  $(d_z, \beta, \alpha, z_2, c_u)$ , we use an inequality from Lemma 7.3 to find an upper bound for  $a$ , where the used inequality depends on the size of  $c$ . For this we proceed in several cases according to the value of  $g_x$ . Below, we just indicate the used inequality from Lemma 7.3 with additional remarks.

*Case where  $g_x = 1, k = 3$ .*

By Lemma 7.3 (i) with (5.6),

$$a^{d_x} < \frac{\log c}{\log b} \cdot z_j z_k \leq \frac{\log c}{\log b} \cdot z_2 z_3 \leq z_2 \mathcal{H}(c_u; a, c_u).$$

*Case where  $g_x = 1, k \neq 3$ .*

By Lemma 7.3 (i), we have

$$a^{d_x} < \frac{\log c_u}{\log b_0} \cdot z_2^2.$$

*Case where  $g_x > 1, k \neq 3$ .*

Since  $x_j \leq x_k < z_k \leq z_2$ , Lemma 7.3 yields

$$a^{d_x} < \frac{\frac{9}{4} \log^2 a}{\log(a-1) \log \max\{b_0, c_0\}} \cdot (z_2 - 1 + d_x + d_x^2) z_2.$$

*Case where  $g_x > 1, k = 3, d_x \geq 2$ .*

Since  $x_j < \frac{\log c}{\log a} z_2$ , Lemma 7.3 implies

$$a^{d_x} < \frac{9}{4} \left( \frac{\log c_u}{\log a} z_2 + d_x + d_x^2 \right) \mathcal{H}(c_u; a, a).$$

*Case where  $g_x > 1, k = 3, d_x = 1$ .*

By Lemma 5.2,  $z_1 = \alpha/\beta$ . Note that  $j \in \{1, 2\}$  and  $j$ -th equation is  $a^{x_j} + b^{y_j} = c^{z_j}$  with both  $y_j, z_j$  divisible by  $g_x$ . Then  $z_j \geq \max\{g_x, x_j + 1\}$ . We proceed in several subcases.

(i) *Case where  $g_x = 2$ .*

Note that  $2 \nmid x_j$ , so  $x_j \geq 3$ . Thus,  $z_j \geq 4$  and  $2 \mid z_j$ . From Lemma 8.1,  $3 \nmid z_j$ . Since  $g_x' = 1$ , Lemma 7.3 (ii) yields

$$a < \frac{\log a}{\log b} \cdot x_j z_3 < \frac{\log c}{\log b} \cdot z_2 z_3 < z_2 \mathcal{H}(c_u; a, c_u).$$

(ii) *Case where  $g_x \equiv 0 \pmod{3}$ .*

Note that  $(x_j, x_i) = (2, 1), g_x = 3$  and  $z_j$  is not divisible by any of 6, 7, 8, 9, 10 and 15. Lemma 7.3 yields that  $a < 9 \min\{1, T\} \mathcal{H}(c_u; a, a)$ , where  $T = 1$  if  $b > 7$ , and  $T = \frac{\log a}{\log(a-1)} \frac{\log b}{\log c_0}$  if  $b \leq 7$ .

(iii) Case where  $g_x = 5$ .

Since  $\gcd(x_j, 2 \cdot 3 \cdot 5 \cdot 7) = 1$ , we have  $x_j \geq 11$ , so that  $z_j \geq 12$ . Since  $g_x' = 1$ , Lemma 7.3 (ii) yields that  $a < \frac{1}{4}z_2\mathcal{H}(c_u; a, c_u)$ .

(iv) Case where  $g_x \notin \{1, 2, 3, 5\}$ .

Note that  $g_x \geq 7$ , and Lemma 7.3 (iii) implies that

$$a < \frac{49}{36} \left( \min \left\{ \frac{\log c_u}{\log a} z_2, z_2 - 1 \right\} + 2 \right) \mathcal{H}(c_u; a, a).$$

By these observations, we find a list of finitely many possible tuples  $(d_z, \beta, \alpha, z_2, c_u)$  with the corresponding upper bound for  $a$ , and those satisfy the stated conditions.  $\square$

**Proposition 12.2.** *Suppose that*

$$d_z > 0, \quad a > \max\{b, c\}, \quad d_x = 0, \quad k = 3.$$

*Then*

$$[\beta, \alpha, z_1, z_2] \leq [4, 16, 16, 18], \quad c < 1.1 \cdot 10^6, \quad g_2 = 1.$$

*Proof.* Since the method is similar to that employed to prove Proposition 12.1, we just indicate the inequality used to find an upper bound for  $c$  or  $a$  for each possible tuple  $(\beta, \alpha, z_1, z_2)$ . We distinguish two cases according to whether  $g_2 = 1$  or not. Note that since  $c^{z_1} + b^{y_2} = c^{z_2} + b^{y_1}$  we can apply the restrictions from Lemmas 9.1 and 9.2.

*Case where  $g_2 = 1$ .*

By Lemma 7.2,

$$c^{d_z} < \min \left\{ 2^{\alpha - \beta z_1}, \frac{T z_2}{z_1} \right\} \cdot z_2 \mathcal{H}(c).$$

where  $T$  is the same as in inequality (12.1). This gives an upper bound for  $c$ , and the found tuples satisfy the stated conditions.

*Case where  $g_2 > 1$ .*

By Lemma 8.2 (i, ii), we have  $\gcd(g_2, 6) = 1$ , so that  $g_2 \geq 5$ . Thus,  $x_j = x_2 \geq g_2 \geq 5$ . First, for each possible tuple  $(\beta, \alpha, z_1, z_2)$ , we can use inequality (12.1) to find an upper bound for  $c$ , say  $c_u$ . Next, for each of the found tuples  $(\beta, \alpha, z_1, z_2, c_u)$ , we apply Proposition 6.3 (i) for  $(A, B, C) = (c, b, a)$  and  $(X_r, Y_r, Z_r) = (z_t, y_t, x_t)$  with  $(r, t) \in \{(1, i), (2, j), (3, k)\}$ . Then one of the following inequalities is found.

$$(12.2) \quad a^{x_j} < x_j \cdot \max_{t \in \{1, 2\}} \{|z_t x_3 - z_3 x_j|\},$$

$$(12.3) \quad a^{x_j/2} < x_j \cdot \max_{t \in \{1, 2\}} \{|y_t x_3 - y_3 x_j|\},$$

$$(12.4) \quad a^{x_j/2} < \frac{2}{\log a} z_3.$$

In cases (12.2) and (12.3), respectively, we see that

$$\begin{aligned}
a^{x_j} &< x_j \cdot \max_{t \in \{1,2\}} \{z_t x_3, z_3 x_j\} \\
&\leq x_j \cdot \max_{t \in \{1,2\}} \left\{ z_t \cdot \frac{\log c}{\log a} z_3, z_3 x_j \right\} \\
&\leq x_j \cdot z_3 \cdot \max_{t \in \{1,2\}} \left\{ \frac{\log c}{\log a} z_t, x_j \right\} = x_j \cdot z_3 \cdot \frac{\log c}{\log a} z_2; \\
a^{x_j/2} &< x_j \cdot \max_{t \in \{1,2\}} \{y_t x_3, y_3 x_j\} \\
&< x_j \cdot \max_{t \in \{1,2\}} \left\{ \frac{\log c}{\log b} z_t \cdot \frac{\log c}{\log a} z_3, \frac{\log c}{\log b} z_3 \cdot x_j \right\} \\
&= x_j \cdot \frac{\log c}{\log b} z_3 \cdot \max_{t \in \{1,2\}} \left\{ \frac{\log c}{\log a} z_t, x_j \right\} = x_j \cdot \frac{\log c}{\log b} z_3 \cdot \frac{\log c}{\log a} z_2.
\end{aligned}$$

These together with (12.4) and (5.6) imply one of the following inequalities:

$$a^{x_j}/x_j < z_2 \mathcal{H}(c_u; S, c_u); \quad a^{x_j/2}/x_j < z_2 \mathcal{H}(c_u); \quad a^{x_j/2} < 2\mathcal{H}(c_u; S),$$

where  $S = a - 2$  if  $b > 7$  and  $S = b$  if  $b \leq 7$ . Each of these inequalities together with  $x_j \geq 5$  gives an upper bound for  $a$ , and the found tuples satisfy the stated conditions.  $\square$

**Proposition 12.3.** *Suppose that*

$$d_z > 0, \quad a > \max\{b, c\}, \quad d_x = 0, \quad k \neq 3.$$

*Then  $y_3 \geq \max\{y_1, y_2\}$ , and*

$$[\beta, \alpha, z_1, z_2] \leq [7, 19, 21, 23], \quad a < 1.1 \cdot 10^{10}.$$

*More exactly, one of the following cases holds.*

- (i)  $y_1 \leq y_2$ ,  $d_z \geq 2$ ,  $b^{d_y} \equiv c^{d_z} \pmod{a^{\min\{x_1, x_2\}}}$ ,  $b^{d_y} < c^{d_z}$ , and one of the following cases holds.
  - $[\beta, \alpha, z_1, z_2] \leq [5, 17, 17, 19]$ ,  $c^{d_z} < 1.6 \cdot 10^5$ ,  $g_2 = 1$ ;
  - $[\beta, \alpha, z_1, z_2] \leq [2, 17, 21, 23]$ ,  $c^{d_z} < 2.5 \cdot 10^5$ ,  $g_2 \geq 5$ .
- (ii)  $y_1 \leq y_2$ ,  $d_y \geq 2$ ,  $b^{d_y} \equiv c^{d_z} \pmod{a^{\min\{x_1, x_2\}}}$ ,  $b^{d_y} > c^{d_z}$ , and one of the following cases holds.
  - $[\beta, \alpha, z_1, z_2] \leq [6, 9, 21, 22]$ ,  $b^{d_y} < 4.9 \cdot 10^5$ ,  $g_y \in \{1, 2, 5\}$ ;
  - $[\beta, \alpha, z_1, z_2] \leq [1, 9, 21, 22]$ ,  $b^{d_y} < 6.5 \cdot 10^5$ ,  $g_y \geq 7$ .
- (iii)  $y_1 > y_2$ ,  $x_1 < x_2$ ,  $a^{x_1} \mid (b^{d_y} c^{d_z} - 1)$ ,  $g_y \in \{1, 2\}$ , and one of the following cases holds.
  - $x_1 = 1$ ,  $[\beta, \alpha, z_1, z_2] \leq [7, 19, 20, 21]$ ,  $a < 1.1 \cdot 10^{10}$ ,  $b < 6.5 \cdot 10^5$ ,  $c < 6.5 \cdot 10^5$ ,  $g_2 = 1$ ;
  - $x_1 \geq 2$ ,  $[\beta, \alpha, x_1, z_1, z_2] \leq [4, 15, 3, 20, 21]$ ,  $a < 6.6 \cdot 10^4$ ,  $b < 5.5 \cdot 10^4$ ,  $c < 6.4 \cdot 10^4$ ,  $g_2 = 1$ ;
  - $x_1 = 1$ ,  $[\beta, \alpha, z_1, z_2] \leq [4, 18, 18, 19]$ ,  $a < 1.1 \cdot 10^{10}$ ,  $b < 6.2 \cdot 10^5$ ,  $c < 6.2 \cdot 10^5$ ,  $g_2 \geq 5$ ;
  - $x_1 \geq 2$ ,  $[\beta, \alpha, x_1, z_1, z_2] \leq [3, 15, 3, 18, 19]$ ,  $a < 9.6 \cdot 10^4$ ,  $b < 5.9 \cdot 10^4$ ,  $c < 9.6 \cdot 10^4$ ,  $g_2 \geq 5$ .

*Proof.* First, we rewrite the three equations as

$$a^x + b^{y_I} = c^{z_I}, \quad a^{x_J} + b^{y_J} = c^{z_J}, \quad a^x + b^{y_3} = c^{z_3}$$

with  $\{I, J\} = \{1, 2\}$  and  $x = x_3$ .

Note that  $c^{z_3} < 2 \max\{a^x, b^{y_3}\}$ . If  $c^{z_3} < 2a^x$ , then  $c^{z_3} < 2a^x < 2c^{z_I}$ , so  $z_3 \leq z_I$ . This implies that  $z_I = z_3$ , which is absurd as  $(y_I, z_I) \neq (y_3, z_3)$ . Thus  $c^{z_3} < 2b^{y_3}$ , so that  $2b^{y_3} > c^{z_2} > b^{\max\{y_1, y_2\}}$ , which shows the first assertion. Therefore,  $d_y = |y_2 - y_1|$  with  $n = 3$ .

Next, we show the following:

$$(12.5) \quad b^{y_2 - y_1} \equiv c^{d_z} \pmod{a^{\min\{x_1, x_2\}}},$$

$$(12.6) \quad z_3 < \mathcal{U}_3 := \max\{250z_2, 2531 \log b\}.$$

Congruence (12.5) follows from reducing 1st and 2nd equations modulo  $a^{\min\{x_1, x_2\}}$ . Let  $\varepsilon = 248$ . If  $a^{(1+\varepsilon)x} \geq b^{y_3}$ , then  $c^{z_3} = a^x + b^{y_3} < 2a^{(1+\varepsilon)x} < 2c^{(1+\varepsilon)z_I}$ , so that  $z_3 \leq \frac{\log 2}{\log c} + (1+\varepsilon)z_I < 250z_I$ . If  $a^{(1+\varepsilon)x} < b^{y_3}$ , then inequality  $z_3 < 2531 \log b$  is deduced almost similarly to the proof of inequality (10.10). To sum up, (12.6) holds.

Third, we combine Lemma 7.2 with inequality (12.6). Since  $\frac{(g_2')^2}{g_2} \leq g_2 < z_2$ , we have

$$(12.7) \quad c^{d_z} < \min\left\{2^{\alpha - \beta z_1} Z, \frac{T z_2}{z_1}\right\} \cdot \frac{(\log^2 c) z_2 \mathcal{U}_3'}{\log \max\{a_0, c + 1\}},$$

where  $Z = 1$  if  $g_2 = 1$ , and  $Z = z_2 - 1$  if  $g_2 > 1$ , and  $T$  is the same as in (12.1), and  $\mathcal{U}_3' = \max\{250z_2/\log b_0, 2531\}$ . Note that  $g_2 \geq 5$  if  $g_2 > 1$ .

Fourth, we apply Lemma 7.3 with the base  $b$  together with (12.6) to see that

$$(12.8) \quad \begin{aligned} b^{d_y} &< \frac{\log c_u}{\log C} \cdot z_m \mathcal{U}_3, \quad \text{if } g_y \in \{1, 2, 5\}, \\ b^{d_y} &< \frac{(49/36) \log b}{\log(b-1)} \cdot (z_m + d_y + d_y^2) \frac{\log c_u}{\log C} \mathcal{U}_3, \quad \text{if } g_y \notin \{1, 2, 5\}, \end{aligned}$$

where  $c_u$  is any upper bound for  $c$ , and  $C = \max\{a_0, b + 2, c_u + 1\}$ . Note that  $g_y \geq 7$  if  $g_y \notin \{1, 2, 5\}$ .

In the remaining cases, we proceed in three cases separately. In each of those cases, similarly to previous propositions, we just indicate how we find a list of all possible tuples composed of  $\beta, \alpha, x_1, z_2, d_z, d_y$  and the corresponding upper bounds for some of  $a, b, c$ .

*Case where  $y_2 \geq y_1$  and  $b^{d_y} < c^{d_z}$ .*

Note that  $m = 2$ . By congruence (12.5), we have  $a^{\min\{x_1, x_2\}} < c^{d_z}$ , in particular,  $d_z > 1$  as  $a > c$ . Taking these restrictions into consideration, we use inequality (12.7) to find an upper bound for  $c$  for each possible tuple  $(\beta, \alpha, d_z, z_2)$ , where these procedures are implemented in versions according to whether  $b > 7$  or not, and whether  $g_2 = 1$  or not. The found tuples satisfy the conditions stated in (i).

*Case where  $y_2 \geq y_1$  and  $b^{d_y} > c^{d_z}$ .*

Similarly to the previous case, we have  $a^{\min\{x_1, x_2\}} < b^{d_y}$  and  $d_y > 1$ . Note that  $z_2 \geq 2g_y$  as  $g_y = \gcd(x_2, z_2)$  with  $x_2 < z_2$ . Taking these restrictions into consideration, similarly to the previous case, we find a list of all possible tuples  $(\beta, \alpha, d_z, z_2, c_u)$  with  $c_u$  the corresponding upper bound for  $c$  derived from inequality (12.7). Finally, for each of the found tuples and for each  $d_y$ , we use inequality (12.8) to find an upper bound for  $b$  in four cases according to whether  $g_y = 1, 2, 5$  or  $g_y \geq 7$ . The found tuples satisfy the conditions stated in (ii).

*Case where  $y_1 > y_2$ .*

Note that  $m = 1$  with  $g_y = \gcd(x_1, z_1)$ , and  $z_1 \geq 2g_y$ . We proceed almost similarly to the previous case. Now we have  $a^{x_1} < b^{d_y} c^{d_z}$  from congruence (12.5), in particular,  $d_y + d_z > x_1$  as  $a > \max\{b, c\}$ . Taking these into consideration, for each  $x_1$  we have a list of all possible tuples  $(\alpha, \beta, d_y, d_z, z_2, a_u, b_u, c_u)$  where  $b_u$  is the corresponding upper bound for  $b$  and  $a_u := \lfloor (b_u \cdot c_u)^{1/x_1} \rfloor$  is the corresponding upper bound for  $a$ . The found tuples satisfy the conditions stated in (iii).  $\square$

Under the assumption that equation (1.3) has three solutions  $(x_t, y_t, z_t)$  with  $t \in \{1, 2, 3\}$  satisfying  $z_1 < z_2 \leq z_3$ , the propositions established in the previous two sections provide us middle-sized bounds on the base numbers  $a, b, c$  and on the exponential unknowns  $x_t, y_t, z_t$  for  $t \in \{1, 2\}$ . Although those bounds are relatively sharp, a direct enumeration of the possible solutions of system (9.4) is still impossible. In order to find efficient methods for reducing the obtained bounds, we need to be more sophisticated than in the case where  $z_1 = z_2$ . In the next two sections, we investigate system (9.4) with  $z_1 < z_2$  and explicitly present our reduction algorithms for the cases  $c > \max\{a, b\}$  and  $c < \max\{a, b\}$ , respectively. Note that it suffices to consider the case where  $a > b$ .

### 13. CASE WHERE $z_1 < z_2$ AND $c > \max\{a, b\}$ : SIEVING

The aim of this section is to show that there is no solution of system (9.4) fulfilling the statements of Propositions 11.1 and 11.2, respectively. It suffices to consider the case where  $c > a > b$ , and we put

$$a_0 = \max\{11, 2^\alpha + 1\}, \quad b_0 = 2^\alpha - 1, \quad c_0 = \max\{1000, 3 \cdot 2^\beta, 2^\alpha + 2\}.$$

Recall that these numbers are uniform lower bounds for  $a, b$  and  $c$ , respectively.

We proceed in two cases according to whether  $c^{z_1} \equiv 2 \pmod{4}$  or  $c^{z_1} \equiv 0 \pmod{4}$ .

**13.1. Case where  $c^{z_1} \equiv 2 \pmod{4}$ .** System (9.4) is

$$(13.1) \quad \begin{cases} a^{x_1} + b^{y_1} = c, \\ a^{x_2} + b^{y_2} = c^{z_2} \end{cases}$$

with  $\beta = 1$ .

First, we give several restrictions on the solutions of system (13.1).

**Lemma 13.1.** *Let  $(x_1, y_1, x_2, y_2, z_2)$  be a solution of system (13.1). Then the following hold.*

- (i)  $x_2$  or  $y_2$  is odd.
- (ii) If both  $x_1$  and  $x_2$  are odd, then  $y_1$  or  $y_2$  is odd.
- (iii) If both  $x_1$  and  $x_2$  are even, then  $y_1$  is even.
- (iv) One of  $x_1, x_2, y_1$  and  $y_2$  is even.
- (v)  $x_2 > x_1$  or  $y_2 > y_1$ .
- (vi)  $\max\{x_2, y_2\} \geq z_2$ .
- (vii)  $x_1 y_2 \neq x_2 y_1$ ,  $y_1 z_2 \neq y_2$ ,  $x_1 z_2 \neq x_2$ .
- (viii)  $\min\{x_1, x_2\} < |y_1 z_2 - y_2|$ .
- (ix) Assume that  $b < 11$ . Then  $(2 \nmid x_2 \text{ or } 3 \nmid z_2)$  and  $(3 \nmid x_2 \text{ or } 2 \nmid z_2)$ .

*Proof.* (i) This is a direct consequence of Lemma 8.2 (i).

(ii) Suppose that both  $x_1, x_2$  are odd and both  $y_1, y_2$  are even. Then  $a^{x_i} \equiv a \pmod{4}$  and  $b^{y_i} \equiv 1 \pmod{4}$  for  $i \in \{1, 2\}$ . Since  $c \equiv 2 \pmod{4}$  and  $z_2 > 1$ , 1st equation leads to  $a \equiv c - 1 \equiv 1 \pmod{4}$ , while 2nd one leads to  $a \equiv c^{z_2} - 1 \equiv -1 \pmod{4}$ . These are incompatible.

(iii, iv) These are shown similarly to (ii).

(v, vi) These easily follow from the inequality  $a^{x_1} + b^{y_1} < a^{x_2} + b^{y_2} = c^{z_2}$  with  $c > \max\{a, b\}$ .

(vii) This is a direct consequence of applying Lemma 6.2 to the equations in (13.1).

(viii) We take the equations in (13.1) modulo  $a^{\min\{x_1, x_2\}}$  to see that  $b^{|y_1 z_2 - y_2|} \equiv 1 \pmod{a^{\min\{x_1, x_2\}}}$ . Since  $a > b$ , and  $y_1 z_2 - y_2 \neq 0$  by (vii), the found congruence leads to the assertion.

(ix) If  $2 \mid x_2$  and  $3 \mid z_2$ , then 2nd equation is of the form  $A^2 + b^{y_2} = C^3$  with  $b \in \{3, 5, 7\}$ . For  $S = \{b\}$ , we compute the  $S$ -integral points  $(A, C)$  on this elliptic curve. None of the found points leads to a solution of the system. The remaining case is similarly handled.  $\square$

**Lemma 13.2.** *Let  $(x_1, y_1, x_2, y_2, z_2)$  be a solution of system (13.1). Then*

$$(13.2) \quad \begin{cases} a^{|x_1 z_2 - x_2|} \equiv 1 \pmod{2b^{\min\{y_1, y_2\}}}, \\ b^{|y_1 z_2 - y_2|} \equiv 1 \pmod{2a^{\min\{x_1, x_2\}}}, \end{cases} \\ a \geq a_1, \quad b \leq b_1,$$

where

$$a_1 := \max\{a_0, b + 2\}, \quad b_1 := \lfloor c^{\min\{1/x_1, 1/y_1, z_2/x_2, z_2/y_2\}} \rfloor.$$

Moreover, the following hold.

- (i) Suppose that  $a^{x_2} > b^{y_2}$  and  $a^{x_1} > b^{y_1}$ . Then

$$0 < x_2 - x_1 z_2 \leq t_1, \quad a \leq a_2,$$

where  $t_1 = \lfloor \frac{\log 2}{\log a_0} z_2 \rfloor$ , and

$$a_2 := \min\left\{\left\lfloor 2^{z_2/(x_2 - x_1 z_2)} \right\rfloor, \left\lfloor c^{z_2/x_2} \right\rfloor\right\}.$$

(ii) Suppose that  $a^{x_2} > b^{y_2}$  and  $a^{x_1} < b^{y_1}$ . Then the following hold.

$$(a) \quad \begin{cases} y_1 > x_1, & x_1 y_2 < x_2 y_1, & x_2 - x_1 z_2 \geq 1, \\ y_2 - y_1 z_2 \leq t_2, & x_2 - y_1 z_2 \leq t_3, \end{cases}$$

$$\text{where } t_2 = \lfloor \frac{\log 2}{\log b_0} (z_2 - 1) \rfloor \text{ and } t_3 = \lfloor \frac{\log 2}{\log a_0} z_2 \rfloor.$$

$$(b) \quad a \geq a_4 := \max \left\{ a_1, \lfloor b^{y_1 z_2 / x_2} / 2^{1/x_2} \rfloor + 1 \right\},$$

$$(c) \quad a \leq a_5 := \min \left\{ \lfloor 2^{z_2 / x_2} b^{y_1 z_2 / x_2} \rfloor, \lfloor c^{z_2 / x_2} \rfloor, \lfloor b^{y_1 / x_1} \rfloor \right\}.$$

(iii) Suppose that  $a^{x_2} < b^{y_2}$  and  $a^{x_1} > b^{y_1}$ . Then the following hold.

$$(a) \quad \begin{cases} y_2 > x_2, & x_1 y_2 > x_2 y_1, & y_2 - y_1 z_2 \geq 1, & y_2 \geq x_1 z_2, \\ x_2 - x_1 z_2 \leq t_4, \end{cases}$$

$$\text{where } t_4 = \lfloor \frac{\log 2}{\log a_0} (z_2 - 1) \rfloor.$$

$$(b) \quad a \geq a_6 := \max \left\{ a_1, \lfloor b^{y_2 / (x_1 z_2)} / 2^{1/x_1} \rfloor + 1 \right\},$$

$$(c) \quad a \leq a_7 := \min \left\{ \lfloor 2^{1/(x_1 z_2)} b^{y_2 / (x_1 z_2)} \rfloor, \lfloor c^{1/x_1} \rfloor, \lfloor b^{y_2 / x_2} \rfloor \right\}.$$

(iv) Suppose that  $a^{x_2} < b^{y_2}$  and  $a^{x_1} < b^{y_1}$ . Then the following hold.

$$(a) \quad \begin{cases} y_2 > x_2, & y_1 > x_1, & y_2 - x_1 z_2 \geq 1, \\ x_2 - y_1 z_2 \leq t_5, & 2 \leq y_2 - y_1 z_2 \leq t_6, \end{cases}$$

$$\text{where } t_5 = \lfloor \frac{\log 2}{\log a_0} (z_2 - 1) \rfloor \text{ and } t_6 = \lfloor \frac{\log 2}{\log b_0} z_2 \rfloor.$$

$$(b) \quad a \leq a_8 := \min \left\{ \lfloor c^{z_2 / x_2} \rfloor, \lfloor b^{y_1 / x_1} \rfloor, \lfloor b^{y_2 / x_2} \rfloor \right\}.$$

*Proof.* From (13.1),

$$(13.3) \quad a^{x_2} + b^{y_2} = (a^{x_1} + b^{y_1})^{z_2}.$$

The congruences in (13.2) follow by reducing equation (13.3) modulo  $b^{\min\{y_1, y_2\}}$  and  $a^{\min\{x_1, x_2\}}$ , respectively. The next asserted upper bound for  $b$  follows easily from system (13.1).

(i) From (13.3) with  $a^{x_1} > b^{y_1}$  and  $a^{x_2} > b^{y_2}$ , observe that

$$\begin{aligned} a^{x_2} &< a^{x_2} + b^{y_2} = (a^{x_1} + b^{y_1})^{z_2} < (2a^{x_1})^{z_2} = 2^{z_2} a^{x_1 z_2}, \\ a^{x_1 z_2} &< (a^{x_1} + b^{y_1})^{z_2} = a^{x_2} + b^{y_2} < 2a^{x_2}. \end{aligned}$$

These inequalities together imply

$$(13.4) \quad \frac{1}{2} < a^{x_2 - x_1 z_2} < 2^{z_2}.$$

The left-hand inequality shows that  $x_2 - x_1 z_2 \geq 0$ , so that  $x_2 - x_1 z_2 > 0$  by Lemma 13.1 (vii), while the right-hand one implies that  $x_2 - x_1 z_2 < \frac{\log 2}{\log a} z_2 < t_1 + 1$ .

On the other hand,  $a < 2^{\frac{z_2}{x_2 - x_1 z_2}}$  by the right-hand inequality of (13.4). Also, by (13.1),  $a^{x_2} < c^{z_2}$ , leading to  $a \leq a_2$ .

(ii) Since  $a^{x_1} < b^{y_1}$  and  $b^{y_1} < (a^{x_2/y_2})^{y_1}$  with  $a > b$ , we have  $y_1 > x_1$  and  $x_1 y_2 < y_1 x_2$ . The remaining three inequalities in (a) can be proven in



exactly the same way as the corresponding results in (i). It remains to show (b) and (c).

Observe from (13.3) that

$$b^{y_1 z_2} < a^{x_2} + b^{y_2} < 2a^{x_2}, \quad a^{x_2} < (a^{x_1} + b^{y_1})^{z_2} < 2^{z_2} b^{y_1 z_2}.$$

The former inequality yields  $a > b^{y_1 z_2 / x_2} / 2^{1/x_2}$ , and so (b) holds, while the latter inequality implies that  $a < (2^{z_2} b^{y_1 z_2})^{1/x_2}$ , leading to (c).

(iii)-(iv) These are shown similarly to (i) and (ii).  $\square$

In what follows, we proceed in two cases according to whether  $(x_1, y_1) = (1, 1)$  or not.

13.1.1. *Case where  $c^{z_1} \equiv 2 \pmod{4}$  with  $(x_1, y_1) \neq (1, 1)$ .* As already mentioned at the end of Section 9, it is very efficient to rely upon the existing results on ternary Diophantine equations which are summarized in Lemma 8.1. In our algorithms we use Lemma 8.1 without any further explicit reference and combine it with Lemmas 13.1 and 13.2.

Proposition 11.1 gives us a list of all possible values of  $\alpha$  and  $z_2$  together with the corresponding upper bound for  $c$ , say  $c_u = c_u(\alpha, z_2)$ . We note that this list is fairly short. Namely, the number of elements in this list is at most 122 in each case under consideration.

We divide our algorithm in four parts according to (i)-(iv) of Lemma 13.2. The basic strategy is similar in each of these parts, where the cases  $b > 7$  and  $b \in \{3, 5, 7\}$  are distinguished. First we give the details of our reduction method for case (i) under the assumption that  $b > 7$ .

(i) Case where  $a^{x_2} > b^{y_2}$  and  $a^{x_1} > b^{y_1}$  with  $b > 7$ .

*Step I. Initialization.* We have an explicitly determined list of all possible triples  $(\alpha, z_2, c_u)$  satisfying system (13.1). We put these data into the list named *clist*.

*Step II.* We generate a list named *list1* containing elements of the form  $[x_1, y_1, x_2, y_2, \alpha, z_2, c_u]$ , where the last three elements are the same as the elements of *clist*, while the first four elements are the possible solutions  $(x_1, y_1, x_2, y_2)$  restricted by Lemmas 13.1 and 13.2(i). The construction of *list1* is given by the following program.

```

for each element of clist do
  for  $x_1 := 1$  to  $\lfloor (\log c_u) / \log a_0 \rfloor$  do
    for  $x_2 := 1$  to  $\lfloor z_2 (\log c_u) / \log a_0 \rfloor$  do
      for  $y_1 := 1$  to  $\lfloor (\log c_u) / \log b_0 \rfloor$  do
        for  $y_2 := 1$  to  $\lfloor z_2 (\log c_u) / \log b_0 \rfloor$  do
          sieve using Lemmas 13.1 and 13.2(i)
        end
      end
    end
  end
end

```

In the last line of the above program, we take into account the restrictions from Lemma 13.1 together with the fact that  $0 < x_2 - x_1 z_2 \leq t_1$ , where  $t_1 = \lfloor \frac{\log 2}{\log a_0} z_2 \rfloor$  by Lemma 13.2(i).

*Step III.* Using the elements of *list1* and the bounds  $a_1, a_2$  and  $b_1$ , we check for each of possible values of  $a$  and  $b$  whether congruences (13.2) hold or not. It turned out that in each case at least one of them does not hold. The details are given as follows. First, from  $(*)$ , we are in one of the cases:

$$[a, b] \equiv [-1, 1], [-1, -1], [1, -1] \pmod{2^\alpha},$$

where  $[a, b] \equiv [u, v] \pmod{2^\alpha}$  means  $a \equiv u \pmod{2^\alpha}$  and  $b \equiv v \pmod{2^\alpha}$ . Define the list  $sig = [[-1, 1], [-1, -1], [1, -1]]$  as a possible list of signatures. We proceed as follows.

```

begin
  for each element of list1 do
    for each element  $s$  of sig do
       $d_a := s[1]$  and  $d_b := s[2]$  and  $T_b := \lceil (b_0 - d_b)/2^\alpha \rceil$ 
      for  $b := T_b \cdot 2^\alpha + d_b$  to  $b_1$  by  $2^\alpha$  do
         $T_a := \lceil (a_1 - d_a)/2^\alpha \rceil$ 
        for  $a := T_a \cdot 2^\alpha + d_a$  to  $a_2$  by  $2^\alpha$  do
          if  $(a^{x_2} > b^{y_2})$  and  $(a^{x_1} > b^{y_1})$  then
            sieve using congruences (13.2), equation (13.3)
            and the relation  $\nu_2(a^{x_2} + b^{y_2}) = z_2$ .
          end
        end
      end
    end
  end
end

```

We implemented the above algorithms and it turned out there is no solution of system (13.1). Case (i) with  $b \leq 7$  can be handled similarly, the only difference being that in *Step III* the range for  $b$  is replaced by  $b \in \{3, 5, 7\}$ .

By using the same strategy as above we can handle the cases according to (ii)-(iv) of Lemma 13.2, as well.

**13.1.2. Case where  $c^{z_1} \equiv 2 \pmod{4}$  with  $(x_1, y_1) = (1, 1)$ .** Note that only cases (i) and (iii) of Lemma 13.2 can occur.

(i) Case where  $a^{x_2} > b^{y_2}$ .

We can proceed exactly in the same way as in case (i) of subsection 13.1.1.

(iii) Case where  $a^{x_2} < b^{y_2}$ .

We basically follow the method described in (iii) of the case  $(x_1, y_1) \neq (1, 1)$  with one important modification when  $b > 7$ . Namely, in order to increase the efficiency of our algorithm, we make use of Lemma 11.2, which says that  $\min\{x_2, y_2\} \leq 7$  and provides us the sharp upper bounds for  $b$ , that is, 177, 45, 19 and 11, according to the cases  $\min\{x_2, y_2\} = 4, 5, 6$  and 7, respectively. We built in these information in our program and it turned out there is no solution to the system.

The total computation time in Subsection 13.1 did not exceed 3 hours.

**13.2. Case where  $c^{z_1} \equiv 0 \pmod{4}$ .** Note that  $\beta > 1$  or  $z_1 > 1$ . Also, Proposition 11.2 provides us a list of possible tuples  $(\beta, \alpha, z_1, z_2, c_u)$ , where  $c_u$  is the corresponding upper bound for  $c$ . We call this list by *clist*. From

(9.4),

$$(13.5) \quad (a^{x_1} + b^{y_1})^{z_2} = (a^{x_2} + b^{y_2})^{z_1},$$

$$(13.6) \quad (c^{z_1} - b^{y_1})^{x_2} = (c^{z_2} - b^{y_2})^{x_1}.$$

First, we deal with a special case.

**Lemma 13.3.** *Under the hypothesis of Proposition 11.2, the system (9.4) has no solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$  satisfying  $(z_1, z_2) \in \{(1, 2), (2, 3)\}$ .*

*Proof.* We proceed in two cases according to whether  $(z_1, z_2) = (1, 2)$  or  $(2, 3)$ . By Proposition 11.2, we may assume that  $c \leq c_u$ , where  $c_u = 6.5 \cdot 10^5$  if  $(z_1, z_2) = (1, 2)$ , and  $c_u = 1.5 \cdot 10^6$  if  $(z_1, z_2) = (2, 3)$ .

*I. Case where  $(z_1, z_2) = (1, 2)$ .*

System (9.4) is

$$(13.7) \quad a^{x_1} + b^{y_1} = c, \quad a^{x_2} + b^{y_2} = c^2.$$

Note that  $2x_1 \neq x_2$  by Lemma 6.2. We further consider several subcases.

*I/(i). Case where  $x_1 \geq 2$  or  $x_2 \geq 4$ .*

From system (13.7), observe that

$$a < \min\{c^{1/x_1}, c^{2/x_2}\} \leq c^{1/2} \leq c_u^{1/2}.$$

Then  $a$  is small. It is not hard to enumerate all possible tuples  $(a, b, x_1, y_1, x_2, y_2)$ , and to verify that none of those satisfies equation (13.7).

*I/(ii). Case where  $(x_1, x_2) = (1, 3)$ .*

It is not hard to see that  $y_1 > 1$ . From (13.7),  $a < c_u^{2/3}$  and  $b < c_u^{1/2}$ , thereby both  $a$  and  $b$  are small enough to deal with this case similarly to case I/(i).

*I/(iii). Case where  $(x_1, x_2) = (1, 1)$ .*

Actually, this case can be handled by the methods described in Section 10. However, we tackle this with an important idea to find a good restriction on solutions, which will play an important role in other difficult cases.

From (13.7), we have

$$(13.8) \quad c + b^{y_2} = c^2 + b^{y_1}, \quad c - b^{y_2/2} = \frac{a}{c + b^{y_2/2}}.$$

We apply Lemma 9.2 with  $(A, B) = (c, b)$  and  $(x, y, X, Y) = (1, y_2, 2, y_1)$  to see that

$$y_2 > 6y_1 - 2 \geq 4, \quad c \equiv -b^{2y_1} - b^{y_1} + 1 \pmod{b^{3y_1}}.$$

In particular,  $b$  is small as  $b < c^{z_2/y_2} \leq c_u^{2/5}$ . On the other hand, since  $a < c$ , the second equation in (13.8) leads to  $0 < c - b^{y_2/2} < 1$ . Therefore,

$$c = \lfloor b^{y_2/2} \rfloor + 1.$$

These restrictions on the values of  $b, c, y_1$  and  $y_2$  are so strong that we can verify by brute force that the first equation in (13.8) does not hold in any possible cases.

II. Case where  $(z_1, z_2) = (2, 3)$ .

System (9.4) is

$$(13.9) \quad a^{x_1} + b^{y_1} = c^2, \quad a^{x_2} + b^{y_2} = c^3.$$

We proceed similarly to case I.

II/(i). Case where  $x_1 \geq 4$  or  $x_2 \geq 5$ .

Since  $a < c_u^{3/5}$  by (13.9),  $a$  is small enough to deal with this case similarly to I/(i).

II/(ii). Case where  $x_2 = 4$  and  $y_2 \geq 7$ .

Since  $a < c_u^{3/4}$  and  $b < c_u^{3/7}$  by (13.9),  $a$  and  $b$  are small enough to deal with this case similarly to I/(ii).

II/(iii). Case where  $x_2 = 4$  and  $y_2 \leq 2$ .

Note that  $y_2 = 1$ . Since  $a > b$  and  $y_2 = 1$ , it follows from 2nd equation that  $c^{3/2} - a^2 = \frac{b}{c^{3/2} + a^2} < \frac{1}{a}$ , which yields that  $\lceil a^{4/3} \rceil \leq c \leq \lfloor (a^2 + 1/a)^{2/3} \rfloor$ . Using these inequalities, we apply the algorithm described in Lemma 13.5 (see below) to deal with this case.

II/(iv). Case where  $(x_2 = 4 \text{ and } y_2 \in \{3, 4, 5, 6\}) \text{ or } x_2 = 3$ .

This case is handled by applying Lemma 8.1 to 2nd equation.

II/(v). Case where  $x_1 = 3$  and  $x_2 \leq 2$ .

Since  $a$  is relatively small as  $a < c_u^{2/3}$ , and  $x_2$  is very small, this case can be handled similarly to case II/(i).

II/(vi). Case where  $(x_1, x_2) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ .

The case where  $x_1 = 1$  or  $x_2 = 1$  can be dealt with by the same algorithm as in case II/(iii). Finally, assume that  $x_1 = x_2 = 2$ . Since  $c^2 + b^{y_2} = c^3 + b^{y_1}$  from system (13.9), this case is dealt with by methods similar to those described in Section 10.  $\square$

By Proposition 11.2 together with Lemma 13.3, we may assume in system (9.4) that

$$z_2 - z_1 = 1, \quad z_2 \geq 4.$$

The next lemma is an analogue to Lemma 13.1 from the case where  $c^{z_1} \equiv 2 \pmod{4}$ , and it can be proved almost similarly.

**Lemma 13.4.** *Let  $(x_1, y_1, z_1, x_2, y_2, z_2)$  be a solution of system (9.4). Assume that  $c^{z_1} \equiv 0 \pmod{4}$ . Then the following hold.*

- (i) *If  $a \equiv 1 \pmod{4}$  and  $b \equiv -1 \pmod{4}$  then both  $y_1, y_2$  are odd.*
- (ii) *If  $a \equiv -1 \pmod{4}$  and  $b \equiv 1 \pmod{4}$  then both  $x_1, x_2$  are odd.*
- (iii) *If  $a \equiv b \equiv -1 \pmod{4}$ , then  $x_1 \not\equiv y_1 \pmod{2}$  and  $x_2 \not\equiv y_2 \pmod{2}$ .*
- (iv) *One of  $x_1$  and  $y_1$  is odd, and one of  $x_2$  and  $y_2$  is odd.*
- (v)  *$x_1 < x_2$  or  $y_1 < y_2$ .*
- (vi)  *$(x_1 \geq z_1 \text{ or } y_1 \geq z_1) \text{ and } (x_2 \geq z_2 \text{ or } y_2 \geq z_2)$ .*
- (vii)  *$x_1 y_2 \neq x_2 y_1$ ,  $x_1 z_2 \neq x_2 z_1$ ,  $y_1 z_2 \neq y_2 z_1$ .*

- (viii)  $\min\{x_1, x_2\} < |y_1 z_2 - y_2 z_1|$ .
- (ix)  $(x_1 \neq z_1 \text{ or } y_1 \geq z_1) \text{ and } (y_1 \neq z_1 \text{ or } x_1 \geq z_1) \text{ and } (x_2 \neq z_2 \text{ or } y_2 \geq z_2) \text{ and } (y_2 \neq z_2 \text{ or } x_2 \geq z_2)$ .
- (x) If  $b < 11$ , then  $(2 \nmid x_1 \text{ or } 3 \nmid z_1) \text{ and } (3 \nmid x_1 \text{ or } 2 \nmid z_1) \text{ and } (2 \nmid x_2 \text{ or } 3 \nmid z_2) \text{ and } (3 \nmid x_2 \text{ or } 2 \nmid z_2)$ .

Finally, using the established lemmas, we further show three lemmas, where the latter two of them together show the contrary to the condition from Lemma 13.4 saying that  $x_1 < x_2$  or  $y_1 < y_2$  in (9.4).

**Lemma 13.5.** *Under the hypothesis of Proposition 11.2, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$ , then*

$$\min\{a^{x_1}, b^{y_1}\} \geq c, \quad \min\{a^{x_2}, b^{y_2}\} \geq c^2.$$

*Proof.* First, we illustrate the method to show that  $a^{x_1} \geq c$ . Suppose on the contrary that  $a^{x_1} < c$ . If  $y_1 \leq z_1$ , then  $c > a^{x_1} = c^{z_1} - b^{y_1} \geq c^{z_1} - b^{z_1} \geq c^{z_1-1}$ , so that  $z_1 < 2$ , which is absurd as  $z_1 \geq 3$ . Thus  $y_1 > z_1$ . On the other hand, from 1st equation, observe that

$$c^{z_1/2} - b^{y_1/2} = \frac{a^{x_1}}{c^{z_1/2} + b^{y_1/2}} < \frac{c}{c^{z_1/2}} < 1.$$

Thus

$$\lceil b^{y_1/z_1} \rceil =: c_L \leq c \leq c_U := \lfloor (1 + b^{y_1/2})^{2/z_1} \rfloor.$$

Since  $y_1 > z_1$ , it is very often observed that  $c_L > c_U$  for given  $b, y_1$  and  $z_1$ . For each of the elements  $(\beta, \alpha, z_1, c_u)$  in *clist* and for each possible tuple  $(b, c, x_1, y_1, x_2, y_2)$  satisfying

$$z_1 < y_1 \leq \left\lfloor \frac{\log c_u}{\log b_0} z_1 \right\rfloor, \quad b_0 \leq b \leq \lfloor c_u^{z_1/y_1} \rfloor, \quad \max\{c_0, c_L\} \leq c \leq \min\{c_U, c_u\},$$

$$y_2 \leq \left\lfloor \frac{\log c}{\log b} z_2 \right\rfloor, \quad x_1 \leq \left\lfloor \frac{\log c}{\log a_0} z_1 \right\rfloor, \quad x_2 \leq \left\lfloor \frac{\log c}{\log a_0} z_2 \right\rfloor$$

with  $z_2 = z_1 + 1$ , we check equation (13.6) does not hold. Thus the inequality  $a^{x_1} \geq c$  holds. The remaining inequalities can be shown exactly in the same way by changing the roles of  $a, b$  and indices 1, 2, respectively.  $\square$

**Lemma 13.6.** *Under the hypothesis of Proposition 11.2, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$ , then  $x_1 \geq x_2$ .*

*Proof.* Suppose that  $x_1 < x_2$ . Recall that we may assume that  $z_2 = z_1 + 1$ .

First, consider the case where  $y_1 < y_2$ . From (9.4),  $a^{x_1}(c - a^{x_2-x_1}) = -b^{y_1}(c - b^{y_2-y_1})$ . This implies that

$$a^{x_1} \mid (c - b^{y_2-y_1}), \quad b^{y_1} \mid (c - a^{x_2-x_1})$$

with  $(c - b^{y_2-y_1})(c - a^{x_2-x_1}) < 0$ . These together yield that  $a^{x_1} \leq c - b^{y_2-y_1}$  or  $b^{y_1} \leq c - a^{x_2-x_1}$ , thereby  $a^{x_1} < c$  or  $b^{y_1} < c$ . However, this contradicts Lemma 13.5.

Second, consider the case where  $y_1 \geq y_2$ . From (9.4),  $a^{x_1}(a^{x_2-x_1} - c) = b^{y_2}(cb^{y_1-y_2} - 1)$  with  $a^{x_2-x_1} - c > 0$  and  $cb^{y_1-y_2} - 1 > 0$ . Since  $a^{x_1} \mid (cb^{y_1-y_2} - 1)$  and  $b^{y_2} \mid (a^{x_2-x_1} - c)$ , we have

$$a^{x_1} < cb^{y_1-y_2}, \quad b^{y_2} < a^{x_2-x_1}.$$

These together with equation (13.5) yield

$$\begin{aligned} a^{x_2 z_1} &< (b^{y_1} + c b^{y_1 - y_2})^{z_2} = b^{y_1 z_2} (1 + c/b^{y_2})^{z_2}, \\ b^{y_1 z_2} &< (a^{x_2} + a^{x_2 - x_1})^{z_1} = a^{x_2 z_1} (1 + 1/a^{x_1})^{z_1}. \end{aligned}$$

Thus

$$\frac{1}{(1 + 1/a^{x_1})^{1/x_2}} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} < a < (1 + c/b^{y_2})^{z_2/(x_2 z_1)} \cdot b^{\frac{y_1 z_2}{x_2 z_1}}.$$

Since  $b < a < c$ , and  $c^2 < b^{y_2}$  by Lemma 13.5, it follows that

$$(13.10) \quad \frac{1}{(1 + 1/a_1^{x_1})^{1/x_2}} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} < a < (1 + 1/c_{LL})^{z_2/(x_2 z_1)} \cdot b^{\frac{y_1 z_2}{x_2 z_1}},$$

where  $a_1 = \max\{a_0, b + 2\}$  and  $c_{LL} = \max\{c_0, b^{y_2/2}\}$ .

We are now in the position to give the details of our reduction algorithm.

*Step I.* In the sequel, we call a pair of integers  $[u, v]$  with  $u, v \in \{1, -1\}$  the *signature* of  $[a, b]$  denoted by  $s = s([a, b])$  if  $[a, b] \equiv [u, v] \pmod{2^\alpha}$ . From (\*), we know that  $s \in \{-1, -1\}, [1, -1], [-1, 1]\}$ . On the other hand, Lemma 13.4 (i,ii,iii) shows, for instance, that if a tuple  $[x_1, y_1, x_2, y_2]$  is a solution of (9.4) with  $[x_1, y_1, x_2, y_2] \equiv [0, 1, 1, 0] \pmod{2}$ , then  $s([a, b]) = [-1, -1]$ . On distinguishing between the 16 possible cases of  $[x_1, y_1, x_2, y_2]$  according to the parities of  $x_1, y_1, x_2$  and  $y_2$ , between the 3 cases of possible signatures of  $[a, b]$  and using Lemma 13.4, we can assign to each tuple  $[x_1, y_1, x_2, y_2]$  the corresponding signatures of  $[a, b]$ . This way we can rule out 36 cases of the total of  $16 \times 3 = 48$  cases, and we obtain a list of possible parities and signatures denoted by *parsig*. The elements of *parsig* are of the form  $[px_1, py_1, px_2, py_2, [s_a, s_b]]$ , where, for  $i = 1, 2$ , we write  $px_i, py_i = 1$  or  $2$  according to whether  $x_i, y_i$  are odd or even, respectively. Moreover,  $[s_a, s_b]$  denotes the corresponding signatures of  $[a, b]$ . *parsig* is explicitly given as follows:

$$\begin{aligned} \text{parsig} = & \left[ [2, 1, 2, 1, [-1, -1]], [2, 1, 1, 2, [-1, -1]], [1, 2, 1, 2, [-1, -1]], \right. \\ & [1, 2, 2, 1, [-1, -1]], [2, 1, 2, 1, [1, -1]], [2, 1, 1, 1, [1, -1]], \\ & [1, 1, 2, 1, [1, -1]], [1, 1, 1, 1, [1, -1]], [1, 2, 1, 2, [-1, 1]], \\ & \left. [1, 2, 1, 1, [-1, 1]], [1, 1, 1, 2, [-1, 1]], [1, 1, 1, 1, [-1, 1]] \right]. \end{aligned}$$

Now, for each element in *clist* and each element in *parsig*, we use Lemma 8.1 together with Lemmas 13.4 and 13.5 (see also Remark 3 below) to sieve considerably the possible solutions  $[x_1, y_1, z_1, x_2, y_2, z_2]$  of system (9.4). This way we obtain a list named *list1* having elements of the form

$$[\alpha, \beta, x_1, y_1, z_1, x_2, y_2, z_2, c_u, b_{\max}, [s_a, s_b]],$$

where  $[s_a, s_b]$  denotes the signature of  $[a, b]$  and  $b_{\max}$  is defined as

$$b_{\max} := \min\{c_u, \lfloor c_u^{z_1/x_1} \rfloor, \lfloor c_u^{z_1/y_1} \rfloor, \lfloor c_u^{z_2/x_2} \rfloor, \lfloor c_u^{z_2/y_2} \rfloor\}.$$

The above algorithm for generating *list1* is given by the following program.

```
begin
  for each element of clist do
```

```

 $z_1 := z_2 - 1$ 
for each element of parsig do
for  $x_1 := px_1$  to  $\lfloor z_1(\log c_u)/\log a_0 \rfloor$  by 2 do
for  $y_1 := py_1$  to  $\lfloor z_1(\log c_u)/\log b_0 \rfloor$  by 2 do
for  $x_2 := px_2$  to  $\lfloor z_2(\log c_u)/\log a_0 \rfloor$  by 2 do
for  $y_2 := py_2$  to  $\lfloor z_2(\log c_u)/\log b_0 \rfloor$  by 2 do
if  $x_1 y_2 - x_2 y_1 \bmod 2^{\beta z_1 - \alpha} = 0$  then
sieve using Lemmas 13.4 and 13.5
put the result  $[\alpha, \beta, x_1, y_1, z_1, x_2, y_2, z_2, c_u, b_{max}, [s_a, s_b]]$  into list1
end

```

*Step II.* In order to create *list2* composed of all possible tuples  $[a, b, x_1, y_1, z_1, x_2, y_2, z_2]$ , by using inequalities (13.10), we proceed as follows.

```

begin
for each element of list1 do
 $T_b := \lceil (b_0 - s_b)/2^\alpha \rceil$ 
for  $b := T_b \cdot 2^\alpha + s_b$  to  $b_{max}$  by  $2^\alpha$  do
 $a_{min} := \max \left\{ a_1, \left\lceil (1 + 1/a_1^{x_1})^{-1/x_2} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} \right\rceil \right\}$ 
 $a_{max} := \min \left\{ c_u, \lfloor c_u^{z_1/x_1} \rfloor, \lfloor c_u^{z_2/x_2} \rfloor, \left\lfloor (1 + 1/c_{LL})^{z_2/(x_2 z_1)} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} \right\rfloor \right\}$ 
 $T_a := \lceil (a_{min} - s_a)/2^\alpha \rceil$ 
for  $a := T_a \cdot 2^\alpha + s_a$  to  $a_{max}$  by  $2^\alpha$  do
test whether equation (13.5) holds or not
put the result  $[a, b, x_1, y_1, z_1, x_2, y_2, z_2]$  into list2
end

```

It turned out that *list2* is empty.

Finally, we mention that the restriction from Lemma 13.4 (x) was very efficient for the case where  $b \leq 7$ .  $\square$

**Remark 3.** Throughout our program implemented in the proof of Lemma 13.6, we may assume by Lemma 13.5 that  $\min\{x_1, x_2\} = x_1 \geq 2$  and  $\min\{y_1, y_2\} = y_2 \geq 3$ . On the one hand, for generating *list1* in *Step I*, we combined that information with Lemma 13.4. This way we excluded a lot of candidates from our *list1* since the number of tuples satisfying  $x_1 = 1$  or  $y_2 \leq 2$  is large. On the other hand, the second advantage is that  $a_{min}$  in *Step II* becomes larger as  $x_1$  increases.

**Lemma 13.7.** *Under the hypothesis of Proposition 11.2, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$ , then  $y_1 \geq y_2$ .*

*Proof.* We may assume that  $x_1 \geq x_2$  by Lemma 13.6. Suppose on the contrary that  $y_1 < y_2$ . Starting with these two inequalities, we can proceed as in the proof of Lemma 13.6. Thus we just indicate the key points on the implemented algorithms. First, we generate the list *list1* exactly in the same way as in *Step I* of Lemma 13.6. Second, we closely follow the method of *Step II* of Lemma 13.6, where the only difference arises from the fact that  $\min\{y_1, y_2\} = y_1$  and  $\min\{x_1, x_2\} = x_2$ . Namely, system (9.4) with  $z_2 - z_1 = 1$  implies that  $b^{y_1}(b^{y_2-y_1} - c) = a^{x_2}(ca^{x_1-x_2} - 1)$ , whence

$b^{y_1} < ca^{x_1-x_2}$  and  $a^{x_2} < b^{y_2-y_1}$ . These together with equation (13.5) yield

$$\frac{1}{(1 + 1/\max\{c_0, a_1^{x_2/2}\})^{1/x_1}} \cdot b^{\frac{y_2 z_1}{x_1 z_2}} < a < (1 + 1/\max\{b^{y_1}, c_0\})^{z_1/(x_1 z_2)} \cdot b^{\frac{y_2 z_1}{x_1 z_2}}.$$

We can proceed exactly in the same way as in Lemma 13.6 by using the corresponding parameters  $a_{min}$  and  $a_{max}$  indicated by the above inequalities.  $\square$

The total computation time in Subsection 13.2 did not exceed 1 hour.

#### 14. CASE WHERE $z_1 < z_2$ AND $c < \max\{a, b\}$ : SIEVING

The aim of this section is to show that there is no solution of system (9.4) fulfilling the statements of Propositions 12.1, 12.2 and 12.3, respectively. However, the case under the hypothesis of Proposition 12.2 can be handled similarly to Section 10, since the system is reduced to the equation  $c^{z_1} + b^{y_2} = c^{z_2} + b^{y_1}$  and both  $b, c$  are relatively small.

It suffices to consider the case where  $a > \max\{b, c\}$ , and we put

$$a_0 = \max\{1001, 2^\alpha + 1, 3 \cdot 2^\beta + 1\}, \quad b_0 = 2^\alpha - 1, \quad c_0 = 3 \cdot 2^\beta.$$

These numbers are lower bounds for  $a, b$  and  $c$ , respectively. We can use both equations (13.5) and (13.6). Moreover, since  $z_1 \geq 2$  as  $a > c$ , we have  $c^{z_1} \equiv 0 \pmod{4}$ , so that Lemma 13.4 can be used. The restrictions from Lemmas 8.1 and 13.4 will be used several times in our reduction procedure without any further explicit reference.

We proceed in two cases according to Propositions 12.1 or 12.3.

##### 14.1. On the system under the hypothesis of Proposition 12.1.

Proposition 12.1 provides us all possible tuples  $(\alpha, \beta, z_1, z_2, a_u, c_u)$ , where  $a_u$  and  $c_u$  are the corresponding upper bounds for  $a$  and  $c$ , respectively. We put these data in the list named *aclist*. In the sequel, we proceed in two cases according to whether  $d_z = 1$  or not.

14.1.1. *Case where  $d_z > 1$ .* Note that  $2 \leq d_z \leq 6$  and each  $c_u$  is very small. The order of magnitude of  $c_u$  is between 7 and 826, where smaller values occur for larger  $d_z$ 's.

We begin with the following lemma.

**Lemma 14.1.** *Under the hypothesis of Proposition 12.1, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$  with  $d_z > 1$ , then  $z_1 \geq 2d_z$ .*

*Proof.* Suppose on the contrary that  $z_1 < 2d_z$ . Since  $z_2 - z_1 = d_z \in \{2, 3, 4, 5, 6\}$ , the possible pairs  $[z_1, z_2]$  are given as follows:

$$\begin{aligned} [z_1, z_2] \in & [[2, 4], [3, 5], [2, 5], [3, 6], [4, 7], [5, 8], [2, 6], [3, 7], [4, 8], [5, 9], \\ & [6, 10], [7, 11], [2, 7], [3, 8], [4, 9], [5, 10], [6, 11], [7, 12], [8, 13], \\ & [9, 14], [2, 8], [3, 9], [4, 10], [5, 11], [6, 12]]. \end{aligned}$$

We set *dzlist* as the list composed of these pairs. If  $b > c$ , then  $\max\{x_i, y_i\} < z_i$  for  $i \in \{1, 2\}$ . Now, for each element of *aclist* and for  $x_i, y_i$  in that ranges,



we use the restrictions from Lemmas 8.1 and 13.4 to generate a list named *list1* of the form  $[\alpha, \beta, x_1, y_1, z_1, x_2, y_2, z_2, a_u, c_u]$ , as follows:

```

for each element of aclist do
  for  $x_1 := 1$  to  $z_1 - 1$  do
    for  $y_1 := 1$  to  $z_1 - 1$  do
      for  $x_2 := 1$  to  $z_2 - 1$  do
        for  $y_2 := 1$  to  $z_2 - 1$  do
          if  $x_1 y_2 - x_2 y_1 \bmod 2^{\beta z_1 - \alpha} = 0$ 
            sieve using Lemmas 8.1 and 13.4 and put the result into
              list1
        end
      end
    end
  end
end

```

Note that once *list1* is generated, we have not only a list of possible solutions  $x_i, y_i, z_i$  of (9.4) but also upper bounds for  $a, b, c$ , as well (i.e.,  $b < a \leq a_u$  and  $c \leq c_u$ ). Using these bounds we basically check for each possible case whether equation (13.6) holds or not. We proceed as follows.

```

for each element of list1 do
   $T_c := \lceil (c_0/2^\beta - 1)/2 \rceil$ ;
  for  $c := (2T_c + 1) \cdot 2^\beta$  to  $c_u$  by  $2^{\beta+1}$  do
    for  $s$  in  $[-1, 1]$  do
      for  $b := \lceil (\max\{b_0, c + 1\} - s)/2^\alpha \rceil \cdot 2^\alpha + s$  to  $a_u$  by  $2^\alpha$  do
        if  $(c^{z_1} - b^{y_1} > 0)$  and  $(c^{z_2} - b^{y_2} > 0)$  then
          if equation (13.6) holds
            then
               $a' := (c^{z_1} - b^{y_1})^{1/x_1}$ 
              if  $a'$  is an integer and  $\gcd(a', b, c) = 1$  then put the result
                 $[a', b, c, x_1, y_1, z_1, x_2, y_2, z_2]$  into list2
            end
          end
        end
      end
    end
  end
end

```

It turned out that *list2* is empty. If  $b < c$ , then we can use the inequality  $b < c$  to proceed exactly in the same way as above with appropriate upper bounds for  $y_1$  and  $y_2$  on creating *list1*.  $\square$

The next lemma is an analogue to Lemma 13.5, where the condition  $d_z > 1$  is very important to run the described algorithm in a reasonable amount of time.

**Lemma 14.2.** *Under the hypothesis of Proposition 12.1, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$  with  $d_z > 1$ , then*

$$\min\{a^{x_1}, b^{y_1}\} \geq c^{d_z}, \quad \min\{a^{x_2}, b^{y_2}\} \geq c^{d_z+1}.$$

*Proof.* The proof is similar to that of Lemma 13.5. We only show that  $a^{x_1} \geq c^{d_z}$ , since the treatment of the remaining inequalities is similar. By Lemma 14.1, we may assume that  $z_1 \geq 2d_z$ . Suppose on the contrary that  $a^{x_1} < c^{d_z}$ . Since  $a^{x_1} < c^{d_z} \leq c^{z_1/2}$ , from 1st equation, we have  $c^{z_1/2} - b^{y_1/2} = \frac{a^{x_1}}{c^{z_1/2} + b^{y_1/2}} < 1$ , thereby

$$(14.1) \quad \left\lceil (c^{z_1/2} - 1)^{2/y_1} \right\rceil =: b_L \leq b \leq b_U := \lfloor c^{z_1/y_1} \rfloor.$$

Recall that the bounds  $c \leq c_u$  in *clist* are (very) sharp. Moreover, on combining this information with inequalities (14.1), it very often holds that  $b_L > b_U$  for given  $c, y_1$  and  $z_1$ . We construct a list named *list1* consisting of elements of the form  $[b, c, y_1, z_1, z_2]$ .

```

for each element of aclist do
   $T_c := \lceil (c_0/2^\beta - 1)/2 \rceil$ ;
  for  $c := (2T_c + 1) \cdot 2^\beta$  to  $c_u$  by  $2^{\beta+1}$  do
    for  $y_1 := 1$  to  $\lfloor z_1(\log c)/\log b_0 \rfloor$  do
       $b_{min} := \max\{b_0, b_L\}$  and  $b_{max} := \min\{b_U, a_u - 2\}$ 
      for  $s$  in  $[-1, 1]$  do
         $T_b := \lceil (b_{min} - s)/2^\alpha \rceil$ 
        for  $b := T_b \cdot 2^\alpha + s$  to  $b_{max}$  by  $2^\alpha$  do
          put  $[b, c, y_1, z_1, z_2]$  into list1
        end
      end
    end
  end
end

```

Since  $z_1 \geq 2d_z$  and  $a^{x_1} < c^{d_z}$ , we observe that  $x_1 < d_z$ . Finally, using *list1* and the above range for  $x_1$ , we basically check whether equation (13.6) holds or not.

```

for each element of list1 do
  for  $x_1 := 1$  to  $d_z - 1$  do
    for  $x_2 := 1$  to  $z_2 - 1$  do
      for  $y_2 := 1$  to  $\lfloor z_2(\log c)/\log b \rfloor$  do
        if equation (13.6) holds then
           $a' := (c^{z_1} - b^{y_1})^{1/x_1}$ 
          if  $(a' > \max\{b, c\})$  and  $\gcd(a', b, c) = 1$  and  $(a'^{x_1} < c^{d_z})$  then
            print  $[a', b, c, x_1, y_1, z_1, x_2, y_2, z_2]$ .
          end
        end
      end
    end
  end
end

```

It turned out that there is no output. □

In the following two lemmas together, we show the contrary to the condition from Lemma 13.4 (iv), saying that  $x_1 < x_2$  or  $y_1 < y_2$  in (9.4).

**Lemma 14.3.** *Under the hypothesis of Proposition 12.1, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$  with  $d_z > 1$ , then  $x_1 \geq x_2$ .*

*Proof.* We closely follow the method described in the proof of Lemma 13.6. Suppose on the contrary that  $x_1 < x_2$ .

If  $y_1 < y_2$ , then  $a^{x_1}(c^{d_z} - a^{x_2-x_1}) = -b^{y_1}(c^{d_z} - b^{y_2-y_1})$ . This equation implies that  $\max\{a^{x_1}, b^{y_1}\} < c^{d_z}$ , which contradicts Lemma 14.2. Thus  $y_1 \geq y_2$ . Then  $a^{x_1}(a^{x_2-x_1} - c^{d_z}) = b^{y_2}(c^{d_z}b^{y_1-y_2} - 1)$ , and this implies that

$$a^{x_1} \mid (c^{d_z}b^{y_1-y_2} - 1), \quad b^{y_2} \mid (a^{x_2-x_1} - c^{d_z})$$

with  $c^{d_z}b^{y_1-y_2} - 1 > 0$  and  $a^{x_2-x_1} - c^{d_z} > 0$ . Thus

$$a^{x_1} < c^{d_z}b^{y_1-y_2}, \quad b^{y_2} < a^{x_2-x_1}.$$

These inequalities together with equation (13.5) yield

$$\begin{aligned} a^{x_2 z_1} &< (b^{y_1} + c^{d_z} b^{y_1 - y_2})^{z_2} = b^{y_1 z_2} (1 + c^{d_z} / b^{y_2})^{z_2}, \\ b^{y_1 z_2} &< (a^{x_2} + a^{x_2 - x_1})^{z_1} = a^{x_2 z_1} (1 + 1/a^{x_1})^{z_1}. \end{aligned}$$

Since  $c^{d_z+1} \leq b^{y_2}$  by Lemma 14.2, it follows that

$$(14.2) \quad \frac{1}{(1 + 1/a_1^{x_1})^{1/x_2}} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} < a < (1 + 1/c_L)^{z_2/x_2 z_1} \cdot b^{\frac{y_1 z_2}{x_2 z_1}}$$

with  $a_1 = \max\{a_0, b + 2\}$  and  $c_L = \max\{c_0, b^{y_2/(d_z+1)}\}$ .

We are now in the position to give the details of our reduction algorithm in this case. We proceed exactly in the same way as in Lemma 13.6 with some appropriate modifications.

*Step I.* We follow *Step I* in the proof of Lemma 13.6 with the following modifications. By using *aclist*, the list *parsig* defined in Lemma 13.6 and the inequalities  $x_i < z_i$  for  $i \in \{1, 2\}$ , we generate a list named *list1* containing elements of the form  $[\alpha, \beta, x_1, y_1, z_1, x_2, y_2, z_2, a_u, b_{max}, [s_a, s_b]]$ , where  $[s_a, s_b]$  denotes the signature of  $[a, b]$ , and  $b_{max}$  is defined by

$$b_{max} := \min\{a_u, \lfloor c_u^{z_1/x_1} \rfloor, \lfloor c_u^{z_1/y_1} \rfloor, \lfloor c_u^{z_2/x_2} \rfloor, \lfloor c_u^{z_2/y_2} \rfloor\}.$$

*Step II.* We follow *Step II* in the proof of Lemma 13.6 with a single modification in the bound  $a_{max}$  for  $a$  according to (14.2), namely:

$$a_{max} := \min\left\{a_u, \lfloor c_u^{z_1/x_1} \rfloor, \lfloor c_u^{z_2/x_2} \rfloor, \left\lfloor (1 + 1/c_L)^{z_2/(x_2 z_1)} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} \right\rfloor\right\}.$$

Using the program occurring in *Step II* in the proof of Lemma 13.6 with the above bounds we check that equation (13.5) holds in no case.  $\square$

**Lemma 14.4.** *Under the hypothesis of Proposition 12.1, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$  with  $d_z > 1$ , then  $y_1 \geq y_2$ .*

*Proof.* Since the method of the proof and the resulting algorithm are similar to the ones used to prove Lemma 14.3, along with the proof of Lemma 13.7, we only indicate the key points of the algorithm. By Lemma 14.3 we may assume that  $x_1 \geq x_2$ , and suppose on the contrary that  $y_1 < y_2$ . In *Step I*, we generate *list1* exactly in the same way as in the proof of Lemma 14.3. In *Step II*, we closely follow the method of Lemma 14.3, where the only difference comes from the fact that  $\min\{y_1, y_2\} = y_1$  and  $\min\{x_1, x_2\} = x_2$ . Namely, in this case,

$$\begin{aligned} &\frac{1}{(1 + 1/\max\{c_0^{d_z}, a_1^{d_z/(d_z+1) \cdot x_2}\})^{1/x_1}} \cdot b^{\frac{y_2 z_1}{x_1 z_2}} \\ &< a < \left(1 + \frac{1}{\max\{b^{y_1}, c^{d_z}\}}\right)^{z_1/(x_1 z_2)} \cdot b^{\frac{y_2 z_1}{x_1 z_2}}. \end{aligned}$$

Using these inequalities, we can proceed exactly in the same way as in the proof of Lemma 14.3 with the corresponding changes of  $a_{min}$  and  $a_{max}$ .  $\square$

14.1.2. *Case where  $d_z = 1$ .* We begin with the following lemma, which is an analogue of Lemma 13.3 in the case where  $a > \max\{b, c\}$  and  $(z_1, z_2) = (2, 3)$ .

**Lemma 14.5.** *Under the hypothesis of Proposition 12.1, system (9.4) has no solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$  satisfying  $(z_1, z_2) = (2, 3)$ .*

*Proof.* Suppose the contrary. Since  $a > c$  and  $a^{x_i} < c^{z_i}$  for  $i \in \{1, 2\}$ , we have  $x_1 = 1$  and  $x_2 \leq 2$ . Note that the case where  $x_2 = 1$  is reduced to the equation  $c^2 + b^{y_2} = c^3 + b^{y_1}$  with both  $b, c$  suitably small, so we only consider the case where  $x_2 = 2$ . Then system (9.4) is

$$(14.3) \quad a + b^{y_1} = c^2, \quad a^2 + b^{y_2} = c^3.$$

Note that  $a_u \leq a_U := 4.5 \cdot 10^6$  in any element in *aclist*.

Suppose that  $y_1 \leq y_2$ . We reduce the equations modulo  $b^{y_1}$  to see that  $b^{y_1} \mid (a - c)$ , so that  $b^{y_1} < a$ . This together with the equations implies that  $c^2 < 2a < 2c^{3/2}$ , yielding a contradiction. Thus we may assume that  $y_1 > y_2$ . From (14.3),  $a(a - c) = b^{y_2}(cb^{y_1-y_2} - 1)$ , leading to  $b^{y_2} < a$ . Since  $a^2 > c^3/2$  by the second equation, we have

$$(14.4) \quad c < 2^{1/3}a_U^{2/3}.$$

If  $y_1 \geq 4$ , then the first equation together with (14.4) implies that  $b < 2^{1/6}a_U^{2/6}$ . By this estimate of  $b$  and (14.4), both  $b$  and  $c$  are small enough to check by a brute force search that equation (13.6) does not hold in any case. Finally, suppose that  $y_1 \leq 3$ . Since  $y_1 > y_2$ , we use Lemma 13.4 (vii) to see that  $(y_1, y_2) = (3, 1)$ . In this case,  $b \mid (c - 1)$ , in particular,  $b < 2^{1/3}a_U^{2/3}$  by (14.4). Then we can deal with this case similarly to the previous case.  $\square$

By Lemma 14.5, in what follows we may assume that  $z_2 \geq 4$ .

The next lemma can be regarded as a common analogue of Lemma 13.5 and of Lemma 14.2 in the case  $d_z = 1$ .

**Lemma 14.6.** *Under the hypothesis of Proposition 12.1, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$  with  $d_z = 1$ , then  $\min\{a^{x_2}, b^{y_2}\} \geq c^2$ .*

*Proof.* We only show that  $a^{x_2} \geq c^2$  since the treatment of the inequality  $b^{y_2} \geq c^2$  is similar. Suppose on the contrary that  $a^{x_2} < c^2$ . Similarly to the proof of Lemma 13.5, we can use 2nd equation together with  $z_2 \geq 4$  to see that

$$y_2 > z_2, \quad \lceil b^{y_2/z_2} \rceil \leq c \leq \lfloor (b^{y_2/2} + 1)^{2/z_2} \rfloor.$$

The details of the algorithm to create the list named *list1* including all possible tuples  $[b, c, z_1, y_2, z_2]$  are given below.

```

for each element of aclist do
  for  $y_2 := 1$  to  $\lfloor z_2(\log a_u)/\log b_0 \rfloor$  do
    for each  $s$  in  $[-1, 1]$  do
      for  $b := \lceil (b_0 - s)/2^\alpha \rceil \cdot 2^\alpha + s$  to  $\lfloor a_u^{z_2/y_2} \rfloor$  by  $2^\alpha$  do
         $c_{\min} := \max\{c_0, \lceil b^{y_2/z_2} \rceil\}$  and  $c_{\max} := \min\{c_u, \lfloor (b^{y_2/2} + 1)^{2/z_2} \rfloor\}$ 
         $T_c := \lceil (c_0/2^\beta - 1)/2 \rceil$ ;
        for  $c := (2T_c + 1) \cdot 2^\beta$  to  $c_{\max}$  by  $2^{\beta+1}$  do

```

put the result  $[b, c, z_1, y_2, z_2]$  into *list1*  
end

Finally, for each element of *list1* and each possible tuple  $(x_1, y_1, x_2)$  we check equation (13.6) does not hold.  $\square$

We finish this subsection by the following lemma giving the contrary to an assertion in Lemma 13.4.

**Lemma 14.7.** *Under the hypothesis of Proposition 12.1, if system (9.4) has a solution  $(x_1, y_1, z_1, x_2, y_2, z_2)$  with  $d_z = 1$ , then  $x_1 \geq x_2$  and  $y_1 \geq y_2$ .*

*Proof.* This can be proved on the lines of the proofs of Lemmas 14.3 and 14.4.  $\square$

The total computation time for Subsection 14.1 did not exceed 4 hours, where the most time consuming part was Lemma 14.6.

**14.2. On the system under the hypothesis of Proposition 12.3.** Proposition 12.3 provides us with some upper bounds and possible solutions of system (9.4) which are classified in three cases denoted by (i)-(iii). We present our reduction algorithms in each of these cases.

(i) We are in the case where

$$d_z \in \{2, 3, 4, 5\}, \quad b^{d_y} < c^{d_z} \leq c_U := 2.5 \cdot 10^5$$

with  $d_y = y_2 - y_1$ . Further, we have a list of all possible tuples  $[\alpha, \beta, z_1, z_2, c_u]$ , where  $c_u$  is the corresponding (sharp) upper bound for  $c$ . By applying the same method as in *Step I* in the proof of Lemma 13.6, we generate a list named *list1* having elements of the form

$$[\alpha, \beta, x_1, y_1, z_1, x_2, y_2, z_2, b_{max}, c_{max}, s_a, s_b],$$

where  $[s_a, s_b]$  is the signature of the pair  $[a, b]$ ,  $c_{max} = c_u$  and  $b_{max}$  is an upper bound for  $b$  defined as

$$b_{max} := \min \left\{ \lfloor c_u^{1/(y_2-y_1)} \rfloor, \lfloor c_u^{z_1/y_1} \rfloor, \lfloor c_u^{z_1/x_1} \rfloor, \lfloor c_u^{z_2/y_2} \rfloor, \lfloor c_u^{z_2/x_2} \rfloor \right\}.$$

Finally, for each element of *list1*, we loop through the values of  $c := c_0$  to  $c_{max}$  by  $2^{\beta+1}$  and the values of  $b := \lceil (b_0 - s_b)/2^\alpha \rceil \cdot 2^\alpha + s_b$  to  $b_{max}$  by  $2^\alpha$ , to verify that equation (13.6) does not hold in any case.

(ii) We are in the case where

$$\max\{a^{\min\{x_1, x_2\}}, c^{d_z}\} < b^{d_y} \leq b_U := 5.4 \cdot 10^5$$

with  $d_y = y_2 - y_1$ . Further, we have a list of all possible tuples  $[\alpha, \beta, z_1, z_2, d_y, b_u]$ , where  $b_u$  is the corresponding (sharp) upper bound for  $b$ . We proceed in two cases according to whether  $d_z = 1$  or not.

If  $d_z > 1$ , then  $c^2 \leq c^{d_z} < b_U$ , so that  $c \leq \lfloor b_U^{1/2} \rfloor (< 10^3)$ . This together with  $b \leq b_u$  shows that both  $b, c$  are so small that we can apply the same algorithm as in (i).

In the case where  $d_z = 1$ , a short modular arithmetic computation (cf. proof of Lemma 13.6) leads to  $x_1 \geq x_2$ . We further proceed in two cases according to whether  $x_2 = 1$  or not.

If  $x_2 \geq 2$ , then  $a^2 \leq a^{x_2} = a^{\min\{x_1, x_2\}} < b_U$ , so that  $a \leq \lfloor b_U^{1/x_2} \rfloor \leq \lfloor b_U^{1/2} \rfloor$ . Thus both  $a, b$  are so small that we can apply the same method as in the case  $d_z \geq 2$  and verify that equation (13.5) (with  $d_z = 1$ ) does not hold in any possible cases.

Finally, in the case where  $d_z = 1$  and  $x_2 = 1$ , we proceed as follows. The case where  $a > c^2$  is dealt with by a previous method since both  $b, c$  are small enough as  $c \leq a^{1/2} < b_U^{1/2}$ . Thus suppose that  $a < c^2$ . Since  $z_2 \geq 4$  as  $k \neq 3$ , it follows from 2nd equation that  $c^{z_2/2} - b^{y_2/2} = \frac{a}{c^{z_2/2} + b^{y_2/2}} < 1$ , which sharply restricts the value of  $c$  in terms of  $b, y_2$  and  $z_2$ . Using this fact and  $b$  is small, we can apply the algorithm described in the proof of Lemma 13.5.

(iii) We are in the case where

$$(14.5) \quad x_1 < x_2, \quad a^{x_1} < b^{d_y} c^{d_z}$$

with  $d_y = y_1 - y_2$ . Further, we have a list named *abclist* containing all possible tuples  $[\alpha, \beta, x_1, z_2, a_u, b_u, c_u, d_z, d_y]$ , where  $a_u, b_u, c_u$  are the corresponding upper bounds for  $a, b, c$ , respectively. A quick check on *abclist* shows that if  $d_z \geq 2$  or  $d_y \geq 2$  then at least one of the bounds  $b_u$  and  $c_u$  is small (about  $10^3$  or less) and the other is middle sized (about  $5 \cdot 10^5$  or less). Then this case can be handled similarly to (i) and (ii) with the parameters  $c_{max}$  and  $b_{max}$  given as  $c_{max} = c_u$  and

$$b_{max} = \min\{b_u, \lfloor c_u^{z_1/y_1} \rfloor, \lfloor c_u^{z_1/x_1} \rfloor, \lfloor c_u^{z_2/y_2} \rfloor, \lfloor c_u^{z_2/x_2} \rfloor\}.$$

Moreover, if  $d_z = d_y = 1$ , then both of these bounds are middle sized, while, unfortunately, the bound  $a_u$  becomes large ( $\approx 10^{10}$ ). Thus, we have to find another reduction procedure which avoids the use of  $a_u$ .

Finally, we consider the case where  $(d_z, d_y) = (1, 1)$ . We follow the method applied in *Step I* of Lemma 13.6 to generate the corresponding list named *list1* having elements satisfying  $(d_z, d_y) = (1, 1)$  of the form

$$[\alpha, \beta, x_1, y_1, z_1, x_2, y_2, z_2, b_{max}, c_{max}, s_a, s_b].$$

The pair  $[s_a, s_b]$  is the signature of  $[a, b]$  while the bounds  $c_{max}$  and  $b_{max}$  are defined by  $c_{max} := c_u$  and  $b_{max} := \min\{b_u, \lfloor c_u^{z_1/y_1} \rfloor\}$ . A quick look shows that *list1* does not include any element satisfying  $(y_1, z_1) = (2, 2)$ . Thus, if  $y_1 \leq z_1$ , then  $z_1 > 2$ , which together with the inequality  $bc > a^{x_1}$  by (14.5) shows that  $bc > a^{x_1} = c^{z_1} - b^{y_1} \geq c^{z_1} - b^{z_1} > c^{z_1-1} + b^{z_1-1} \geq c^2 + b^2$ , a contradiction. Therefore, it remains to consider the case where  $y_1 > z_1$ . We note that since  $y_1 > z_1$  and the orders of magnitude of  $b_u$  and  $c_u$  are the same, the quantity  $\lfloor c_u^{z_1/y_1} \rfloor$  is smaller than  $b_u$ , resulting in a sharper upper bound  $b_{max}$  for  $b$ . This observation is crucial in order to have a reasonable running time. The remaining task can be dealt with similarly to cases (i) and (ii).

The total computation time of Section 14 did not exceed 5 hours.

The conclusion of Sections 13 and 14 together is:

**Proposition 14.1.**  $z_1 = z_2$ .

In view of Propositions 10.3 and 14.1, the proof of Theorem 1 is finally completed.

## 15. CONCLUDING REMARKS

Theorem 1 says that there is only one example which allows equation (1.3) to have three solutions in positive integers. On the other hand, a simple search in a suitably finite region using computer (cf. [ScSt4, Section 3]) finds a number of examples where there are two solutions to (1.3) in positive integers  $x, y$  and  $z$ , corresponding to the following set of equations:

$$\begin{aligned}
 (15.1) \quad & 5 + 3 = 2^3, \quad 5 + 3^3 = 2^5, \quad 5^3 + 3 = 2^7; \\
 & 13 + 3 = 2^4, \quad 13 + 3^5 = 2^8; \\
 & 5 + 2^2 = 3^2, \quad 5^2 + 2 = 3^3; \\
 & 7 + 2 = 3^2, \quad 7^2 + 2^5 = 3^4; \\
 & 3 + 2^3 = 11, \quad 3^2 + 2 = 11; \\
 & 10 + 3 = 13, \quad 10 + 3^7 = 13^3; \\
 & 3 + 2^5 = 35, \quad 3^3 + 2^3 = 35; \\
 & 89 + 2 = 91, \quad 89 + 2^{13} = 91^2; \\
 & 5 + 2^7 = 133, \quad 5^3 + 2^3 = 133; \\
 & 3 + 2^8 = 259, \quad 3^5 + 2^4 = 259; \\
 & 13 + 3^7 = 2200, \quad 13^3 + 3 = 2200; \\
 & 91 + 2^{13} = 8283, \quad 91^2 + 2 = 8283; \\
 & (2^k - 1) + 2 = 2^k + 1, \quad (2^k - 1)^2 + 2^{k+2} = (2^k + 1)^2,
 \end{aligned}$$

where  $k$  is any integer with  $k \geq 2$ .

While Theorem 1 is essentially sharp, as indicated by (15.1), it is natural, in light of a lot of existing works on the solutions of equation (1.3), to believe that something rather stronger is true. A formulation in this direction is posed by Scott and Styer [ScSt4], which is regarded as a 3-variable generalization of [Be, Conjecture 1.3], as follows:

**Conjecture 2.** *For any fixed relatively prime positive integers  $a, b$  and  $c$  with  $\min\{a, b, c\} > 1$ , equation (1.3) has at most one solution in positive integers  $x, y$  and  $z$ , except for those triples  $(a, b, c)$  corresponding to (15.1).*

There are many results in the literature which support this conjecture. However, Conjecture 2 seems completely out of reach. It is worth noting that it solves several open problems on the solutions of equation (1.3) for some infinite families of  $(a, b, c)$ , including not only the conjecture of Sierpiński and Jeśmanowicz on primitive Pythagorean triples, but also its generalization posed by Terai as mentioned in the first section. Finally, we mention that Conjecture 2 seems not to follow directly from some of well-known conjectures closely related to ternary Diophantine equations including generalized Fermat conjecture and any effective version of  $abc$  conjecture.

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#### REFERENCES

- [Be] M.A. Bennett, *On some exponential equations of S. S. Pillai*, *Canad. J. Math.* **53** (2001), 897–922.
- [Be2] M.A. Bennett, *Pillai's conjecture revisited*, *J. Number Theory* **98** (2003), 228–235.
- [BeBi] M.A. Bennett and N. Billerey, *Sums of two S-units via Frey-Hellegouarch curves*, *Math. Comp.* **86** (2017), 1375–1401.
- [BeCh] M.A. Bennett and I. Chen, *Multi-Frey  $\mathbb{Q}$ -curves and the Diophantine equation  $a^2 + b^6 = c^n$* , *Algebra Number Theory* **6** (2012), 707–730.
- [BeChDaYa] M.A. Bennett, I. Chen, S. Dahmen and S. Yazdani, *Generalized Fermat equations: a miscellany*, *Int. J. Number Theory* **11** (2015), 1–28.
- [BeElNg] M.A. Bennett, J.S. Ellenberg and N.C. Ng, *The Diophantine equation  $A^4 + 2^6 B^2 = C^n$* , *Int. J. Number Theory* **6** (2010), 311–338.
- [BeMiSi] M.A. Bennett, P. Mihăilescu and S. Siksek, *The generalized Fermat equation*, in Springer volume *Open Problems in Mathematics*, 2016, 173–205.
- [BeSk] M.A. Bennett and C. Skinner, *Ternary Diophantine equations via Galois representations and modular forms*, *Canad. J. Math.* **56** (2004), 23–54.
- [BerHa] Cs. Bertók and L. Hajdu, *A Hasse-type principle for exponential diophantine equations and its applications*, *Math. Comp.* **85** (2016), 849–860.
- [BoCaPl] W. Bosma, J. Cannon and C. Playoust, *The Magma Algebra System I: The User Language*, *J. Symb. Comp.* **24** (1997), 235–265. (See also <http://www.maths.usyd.edu.au/>)
- [Br] N. Bruin, *The Diophantine equations  $x^2 \pm y^4 = \pm z^6$  and  $x^2 + y^8 = z^3$* , *Compos. Math.* **118** (199), 305–321.
- [Br2] N. Bruin, *On powers as sums of two cubes*, ANTS IV, Leiden 2000, 169–184, *Lecture Notes in Comput. Sci.* **1838** (Springer, 2000).
- [Bu] Y. Bugeaud, *Linear forms in logarithms and applications*, IRMA Lectures in Mathematics and Theoretical Physics 28, European Mathematical Society, 2017.
- [BuGy] Y. Bugeaud and K. Györy, *Bounds for the solutions of unit equations*, *Acta Arith.* **74** (1996), 67–80.
- [BuLu] Y. Bugeaud and F. Luca, *On Pillai's Diophantine equation*, *New York J. Math.* **12** (2006), 193–217.
- [CiMi] M. Cipu and M. Mignotte, *On a conjecture on exponential Diophantine equations*, *Acta Arith.* **140** (2009), 251–269.
- [Co] H. Cohen, *Number Theory. Vol. II: Analytic and Modern Tools*, *Grad. Texts in Math.* 240, Springer, 2007.



- [Da] S. Dahmen, *Classical and modular methods applied to Diophantine equations*, PhD thesis, University of Utrecht, 2008.
- [DaSik] S. Dahmen and S. Siksek, *Perfect powers expressible as sums of two fifth or seventh powers*, Acta Arith. **164** (2014), 65–100.
- [DaSik2] S. Dahmen and S. Siksek, *Perfect powers expressible as sums of two cubes*, J. Algebra **322** (2009), 638–656.
- [DarMe] H. Darmon and L. Merel, *Winding quotients and some variants of Fermat’s last Theorem*, J. Reine. Angew. Math. **490** (1997), 81–100.
- [El] J.S. Ellenberg, *Galois representations attached to  $\mathbb{Q}$ -curves and the generalized Fermat equation  $A^4 + B^2 = C^p$* , Amer. J. Math. **126** (2004), no. 4, 763–787.
- [EvGy] J.H. Evertse and K. Györy, *On unit equations and decomposable form equations*, J. Reine. Angew. Math. **358** (1985), 6–19.
- [EvGyStTi] J.H. Evertse, K. Györy, C.L. Stewart and R. Tijdeman, *S-unit equations in two unknowns*, Invent. math. **92** (1988), 461–477.
- [EvScSch] J.H. Evertse, H.P. Schlickewei and W.M. Schmidt, *Linear equations in variables which lie in a multiplicative group*, Ann. Math. **155** (2002), 1–30.
- [EvGy2] J.H. Evertse and K. Györy, *Unit Equations in Diophantine Number Theory*, Cambridge University Press, 2015.
- [Gu] R.K. Guy, *Unsolved Problems in Number Theory*, Springer, 2004.
- [Gy] K. Györy, *On the number of solutions of linear equations in units of an algebraic number field*, Comment. Math. Helv. **54** (1979), 583–600.
- [Gy2] K. Györy, *On the solutions of linear diophantine equations in algebraic integers of bounded norm*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **22–23** (1979–1980), 225–233.
- [Gy3] K. Györy, *Résultats effectifs sur la représentation des entiers par des formes décomposables*, Kingston, Canada, 1980.
- [Gy4] K. Györy, *Some recent applications of S-unit equations* Astérisque, **209** (1992), 17–38.
- [GyYu] K. Györy and K. Yu, *Bounds for the solutions of S-unit equations and decomposable form equations*, Acta Arith. **123** (2006), 9–41.
- [HuLe] Y. Hu and M. Le, *A note on ternary purely exponential diophantine equations*, Acta Arith. **171** (2015), 173–182.
- [HuLe2] Y. Hu and M. Le, *An upper bound for the number of solutions of ternary purely exponential diophantine equations*, J. Number Theory **183** (2018), 62–73.
- [HuLe3] Y. Hu and M. Le, *An upper bound for the number of solutions of ternary purely exponential diophantine equations II*, Publ. Math. Debrecen **95** (2019), 335–354.
- [Je] L. Jeśmanowicz, *Several remarks on Pythagorean numbers*, Wiadom. Mat. **1** (1955/56), 196–202 (in Polish).
- [Kr] A. Kraus, *Sur l’équation  $a^3 + b^3 = c^p$* , Experiment. Math. **7** (1998), 1–13.
- [La] M. Laurent, *Linear forms in two logarithms and interpolation determinants II*, Acta Arith. **133** (2008), 325–348.
- [Lu] F. Luca, *On the diophantine equation  $p^{x_1} - p^{x_2} = q^{y_1} - q^{y_2}$* , Indag. Math. (N.S.) **14** (2003), no. 2, 207–222.
- [Lu2] F. Luca, *On the system of Diophantine equations  $a^2 + b^2 = (m^2 + 1)^r$  and  $a^x + b^y = (m^2 + 1)^z$* , Acta Arith. **153** (2012), 373–392.
- [Mi] P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan’s conjecture*, J. Reine. Angew. Math. **572** (2004), 167–195.
- [Miy] T. Miyazaki, *Contributions to some conjectures on a ternary exponential Diophantine equation*, Acta Arith. **186** (2018), 1–36.
- [PedW] A. Pethő and B.M.M. de Weger, *Products of prime powers in binary recurrence sequences. I. The hyperbolic case, with an application to the generalized Ramanujan-Nagell equation*, Math. Comp. **47** (1986), no. 176, 713–727.
- [Pi] S.S. Pillai, *On the inequality  $0 < a^x - b^y \leq n$* , J. Indian Math. Soc. **19** (1931), 1–11.
- [Pi2] S.S. Pillai, *On  $a^x - b^y = c$* , J. Indian Math. Soc. (N.S.) **2** (1936), 119–122, and 215.

- [Po] B. Poonen, *Some Diophantine equations of the form  $x^n + y^n = z^m$* , Acta Arith. **86** (1998), 193–205.
- [PoShSt] B. Poonen, E.F. Schaefer and M. Stoll, *Twists of  $X(7)$  and primitive solutions to  $x^2 + y^3 = z^7$* , Duke Math. J. **137** (2007), 103–158.
- [Sc] R. Scott, *On the equations  $p^x - b^y = c$  and  $a^x + b^y = c^z$* , J. Number Theory **44** (1993), 153–165.
- [ScSt] R. Scott and R. Styer, *On the generalized Pillai equation  $\pm a^x \pm b^y = c$* , J. Number Theory **118** (2006), 236–265.
- [ScSt2] R. Scott and R. Styer, *The number of solutions to the generalized Pillai equation  $\pm ra^x \pm sb^y = c$* , J. Theor. Nombres Bordeaux **25** (2013), 179–210.
- [ScSt3] R. Scott and R. Styer, *Bennett’s Pillai theorem with fractional bases and negative exponents allowed*, J. Theor. Nombres Bordeaux **27** (2015), 289–307.
- [ScSt4] R. Scott and R. Styer, *Number of solutions to  $a^x + b^y = c^z$* , Publ. Math. Debrecen **88** (2016), 131–138.
- [ShTi] T.N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge University Press, 1986.
- [Si] W. Sierpiński, *On the equation  $3^x + 4^y = 5^z$* , Wiadom. Mat. **1** (1955/56), 194–195 (in Polish).
- [Sik] S. Siksek, *Explicit Chabauty over number fields*, Algebra Number Theory **7** (2013), 765–793.
- [SikSt] S. Siksek and M. Stoll, *Partial descent on hyperelliptic curves and the generalized Fermat equation  $x^3 + y^4 + z^5 = 0$* , Bull. Lond. Math. Soc. **44** (2012), no. 1, 151–166.
- [TaWi] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Annals of Math. **141** (1995), 553–572.
- [Te] N. Terai, *Applications of a lower bound for linear forms in two logarithms to exponential Diophantine equations*, Acta Arith. **90** (1999), 17–35.
- [Wi] A. Wiles, *Modular elliptic curves and Fermat’s Last Theorem*, Annals of Math. **141** (1995), 443–551.
- [Zu] D. Zureick-Brown, *Primitive integral solutions to  $x^2 + y^3 = z^{10}$* , Int. Math. Res. Not. IMRN **2** (2012), 423–436.
- [Zs] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. **3** (1892), 265–284.

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