GEOMETRY OF THREE-DIMENSIONAL MANIFOLDS WITH POSITIVE SCALAR CURVATURE

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Abstract. The purpose of this paper is to derive volume and other geometric information for three-dimensional complete manifolds with positive scalar curvature. In the case that the Ricci curvature is nonnegative, it is shown that the volume of the manifold must be of linear growth when the scalar curvature is bounded from below by a positive constant. This answers a question of Gromov in the affirmative for dimension three. Volume growth estimates are also obtained for the case when scalar curvature decays to zero. In fact, results of similar nature are established for the more general case that the Ricci curvature is asymptotically nonnegative.

1. Introduction

One of our purposes in this paper is to provide an affirmative answer in the case of dimension $n = 3$ to the following question posed by Gromov [15].

Question 1.1. *Let (M^n, g) be a complete manifold with nonnegative Ricci curvature. If its scalar curvature $S \geq 1$, is there a positive constant C such that $\mathrm{V}_p(R) \leq C\, R^{n-2}$ for all $R > 0$, where $\mathrm{V}_p(R)$ is the volume of the geodesic ball $B_p(R)$ centered at a point p and of radius R?*

In fact, Yau [35] asked even more ambitiously whether

$$\limsup_{R\to\infty} R^{2-n} \int_{B_p(R)} S < \infty$$

on a complete manifold (M^n, g) with nonnegative Ricci curvature.

In this paper we establish the following result.

Theorem 1.2. *Let (M^3, g) be a complete three-dimensional manifold with nonnegative Ricci curvature.*

- *If the scalar curvature is bounded below by $S \geq 1$ on M, then there exists a universal constant $C > 0$ such that*

$$\mathrm{V}_p\left(R\right) \leq C\, R$$

  for all $R > 0$ and $p \in M$.
- *If the scalar curvature is bounded below by*

$$S\left(x\right) \geq \frac{C_0}{r^{\alpha}\left(x\right) + 1}$$

  for some $\alpha \in [0, 1]$ and $C_0 > 0$, where $r(x)$ is the geodesic distance from x to a fixed point $p \in M$, then

$$\mathrm{V}_p\left(R_i\right) \leq \frac{C}{C_0}\, R_i^{\alpha+1}$$

for a sequence $R_i \to \infty$, where C is a universal constant.

In fact, we consider the more general case that the Ricci curvature is only assumed to be asymptotically nonnegative, namely, the Ricci curvature satisfies

$$\mathrm{Ric}(x) \geq -k\left(r(x)\right), \tag{1.1}$$

for a nonnegative non-increasing function $k(r)$ with

$$\int_0^\infty r\, k(r)\, dr < \infty.$$

Theorem 1.3. *Let (M^3, g) be a complete three-dimensional manifold with finite first Betti number and finitely many ends. Suppose that its Ricci curvature is asymptotically nonnegative. If the scalar curvature S is bounded below by a positive constant, then M is parabolic, i.e., it does not admit any positive Green's function.*

Our next result relates the lower bound of the scalar curvature with the volume of unit balls.

Theorem 1.4. *Let (M^3, g) be a complete three-dimensional manifold with finite first Betti number and finitely many ends. Suppose that its Ricci curvature is asymptotically nonnegative. Then there exists a constant C depending only on the function k in (1.1) such that the scalar curvature S of M satisfies*

$$\liminf_{x\to\infty} S\left(x\right) \leq \frac{C}{\mathrm{V}_p\left(1\right)},$$

where $\mathrm{V}_p\left(1\right)$ denotes the volume of the unit ball $B_p\left(1\right)$.

It should be pointed out that the assumption of the Ricci curvature being asymptotically nonnegative is necessary for Theorem 1.4 to hold. Indeed, consider a complete three-dimensional manifold (M, g) which is isometric to $[2, \infty) \times \mathbb{S}^2$ with the warped product metric $ds_M^2 = dt^2 + \frac{1}{\ln t} ds_{\mathbb{S}^2}^2$ outside a compact set. Then its scalar curvature $S\left(x\right) \to \infty$ as $x \to \infty$. Note that its Ricci curvature satisfies $\mathrm{Ric}(x) \geq -k\left(r\left(x\right)\right)$ for $k\left(r\right) = \frac{C}{r^2 \ln r}$ when r is sufficiently large. We also remark that the upper bound estimate of scalar curvature S in terms of the volume $\mathrm{V}_p(1)$ is sharp by considering the example of cylinder $\mathbb{R} \times \mathbb{S}^2(r)$, where $\mathbb{S}^2(r)$ is the sphere of radius r in Euclidean space $\mathbb{R}^3$.

We now indicate some of the ideas involved in the proofs. The proof of Theorem 1.2 uses a variant of the following monotonicity result in [26].

Theorem 1.5. *Let $\left(M^3, g\right)$ be a complete noncompact three-dimensional manifold with nonnegative scalar curvature. Assume that M has one end and its first Betti number $b_1\left(M\right) = 0$. If M is nonparabolic and the minimal positive Green's function $G\left(x\right) = G\left(p, x\right)$ satisfies $\lim_{x\to\infty} G(x) = 0$, then*

$$\frac{d}{dt}\left(\frac{1}{t}\int_{\{G=t\}} \left|\nabla G\right|^2 - 4\pi t\right) \leq 0$$

for all $t > 0$. Moreover, equality holds for some $T > 0$ if and only if the super level set $\{G > T\}$ is isometric to a ball in the Euclidean space $\mathbb{R}^3$.

This type of result has been applied in [2] to reprove the positive mass theorem in [31]. Recently, using a variant of the above theorem, Chodosh and Li [8] have affirmed a conjecture of Schoen that a stable minimal hypersurface in Euclidean space $\mathbb{R}^4$ must be flat. Historically, more general versions of monotonicity formulas were established by Colding [9] and Colding-Minicozzi [12] for n-dimensional manifolds with nonnegative Ricci curvature and applied to the study of uniqueness of the tangent cones for Ricci flat manifolds with Euclidean volume growth [11]. We refer the readers to [10] for an exposition on monotonicity formulas in geometric analysis, and [1] for their applications to Willmore type inequalities.

The proof of Theorem 1.5 relies on a crucial observation that on a three-dimensional manifold

$$\mathrm{Ric}\,(\nabla u, \nabla u)\,|\nabla u|^{-2} = \frac{1}{2}S - \frac{1}{2}S_t + \frac{1}{|\nabla u|^2}\left(|\nabla\,|\nabla u||^2 - \frac{1}{2}\left|\nabla^2 u\right|^2\right) \tag{1.2}$$

for any harmonic function u, where S_t is the scalar curvature of the level set $l\,(t)$ of u. The proof then proceeds by applying (1.2) to the Green's function G and integrating the following Bochner formula over the level sets $l(t)$ of G.

$$\Delta\,|\nabla G| = \left(|G_{ij}|^2 - |\nabla\,|\nabla G||^2\right)|\nabla G|^{-1} + \mathrm{Ric}\,(\nabla G, \nabla G)\,|\nabla G|^{-1}\,.$$

The term S_t is handled with the help of the Gauss-Bonnet theorem as the assumptions ensure $l(t)$ is compact and connected.

The idea of rewriting the Ricci curvature term as (1.2) has origin in Schoen and Yau [30], where they made the important observation that on a minimal surface N in a three-dimensional manifold M,

$$\mathrm{Ric}\,(\nu, \nu) = \frac{1}{2}S - \frac{1}{2}S_N - \frac{1}{2}\,|A|^2 \tag{1.3}$$

with ν, S_N and A being the unit normal vector, the scalar curvature and the second fundamental form of N, respectively. This observation enabled them to classify compact stable minimal surfaces in a three-dimensional manifold with nonnegative scalar curvature. In fact, it can be used to reprove the well-known classification by Fisher-Colbrie and Schoen [13] for complete stable minimal surfaces as well (see [22]). More generally, an identity of the nature (1.3) was derived for any surface N, not necessarily minimal, by Jezierski and Kijowski in [17]. It was applied to level sets of suitably chosen functions to prove both positive energy and positive mass results in [17, 16]. Recently, this identity was rediscovered by Stern [32] for the level sets of harmonic functions. In [5] it has been subsequently used to reprove the positive mass theorem of Schoen and Yau [31].

The observation we make here is that Theorem 1.5 can be refined and localized to an end of M. Applying it to the so-called barrier functions of ends then yields Theorem 1.2. Recall by Li and Tam [20] that each end E of a complete manifold admits a barrier function, namely, a positive harmonic function u satisfying $u = 0$ on the boundary of E. Such u is bounded if and only if E is nonparabolic. On the other hand, by a result of Nakai [28], u can be chosen to be proper in the case E is parabolic.

The proofs of both Theorem 1.3 and Theorem 1.4 are very much in the same spirit of Theorem 1.2. However, the technical details differ. Again, we work with a barrier function f. For Theorem 1.3, the difficulty here is that f may not be proper and the Gauss-Bonnet formula can not be applied directly to the level sets of f. To remedy this, we consider $u = f\,\psi$ instead, where ψ is a smooth cut-off function on M. While this guarantees that the positive level sets of u are compact, the price we pay is that the function u is no longer harmonic. This creates many new technical issues. For example, it becomes unclear how to control the number of connected components of the level sets of u. To get around the issue, we consider only the component $L(t,\infty)$ of the super level set $\{u > t\}$ with the fixed point $p \in L(t,\infty)$. It turns out that for each end E of M, the unbounded component of $E \setminus L(t,\infty)$ has exactly one component of the boundary of $L(t,\infty)$. This fact more or less suffices for our purpose, though there could be other components of the boundary of $L(t,\infty)$. Another issue is that the Bochner formula for u introduces many extra terms. Fortunately, those terms can be controlled by a judicious choice of the cut-off function ψ. The choice is based on a result from [4].

We refer to [27, 33, 36] for some of the progress on the aforementioned questions. In particular, it should be mentioned that Theorem 1.2 was proved by Zhu [36] with a different approach under the additional assumption that all unit balls of M have a fixed amount of volume.

The structure of the paper is as follows. In Section 2, we present a proof of Theorem 1.2. Although more general results are proved later on, we choose to do so as the proof is much less complicated and seems to better illuminate the main ideas. Section 3 is devoted to the proof of Theorem 1.3 and Section 4 to Theorem 1.4.

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2. Proof of Theorem 1.2

We first establish a localized version of Theorem 1.5, see Lemma 2.3, which moreover does not require curvature bounds.

Recall that an end E with respect to a smooth connected bounded domain D is simply an unbounded component of $M \setminus D$. According to [20], each end E of a complete manifold carries a positive harmonic function u, which is either bounded and satisfies

$$u = 1 \text{ on } \partial E, \text{ with } \liminf_{x\to\infty} u(x) = 0, \tag{2.1}$$

or

$$u = 0 \text{ on } \partial E \text{ with } \limsup_{x\to\infty} u(x) = \infty. \tag{2.2}$$

In the former case, E is called nonparabolic and in the the latter case, parabolic.

Definition 2.1. *The positive harmonic function satisfying either (2.1) or (2.2) is called a barrier function of the end E.*

The barrier function in the nonparabolic case has finite energy $\int_M |\nabla u|^2 < \infty$. A result of Nakai [28] further implies that u can be chosen to be proper whenever E is parabolic. Let us also note that in both cases,

$$\int_{\partial E} \frac{\partial u}{\partial \nu} \neq 0.$$

In the following, when (M,g) has finitely many ends and finite first Betti number, the domain D is assumed to be sufficiently large such that all representatives of

$H_1(M)$ lie in D and $M \setminus D$ has the maximal number of ends. For a harmonic function u on an end E, denote by

$$\begin{aligned} L(a,b) &= \{x \in E : a < u(x) < b\} \\ l(t) &= \{x \in E : u(x) = t\}. \end{aligned}$$

The following result is essentially contained in [26]. See also Lemma 2.3 in [21].

Lemma 2.2. *Let (M^n, g) be a complete n-dimensional manifold with finite first Betti number and finite number of ends. For a proper harmonic function u on an end E, its level set $l(t)$ is connected for all t.*

We now derive a refined and localized version of Theorem 1.5. For any regular value t of u, let

$$w(t) = \int_{l(t)} |\nabla u|^2. \tag{2.3}$$

For $\alpha \in \mathbb{R}$, define

$$\begin{aligned} H_\alpha(t) &= t^\alpha \frac{dw}{dt}(t) - (\alpha+3)\, t^{\alpha-1} w(t) + \frac{4\pi}{\alpha+1} t^{\alpha+1} \\ &= t^\alpha \int_{l(t)} \frac{\langle \nabla |\nabla u|, \nabla u\rangle}{|\nabla u|} - (\alpha+3)\, t^{\alpha-1} \int_{l(t)} |\nabla u|^2 \\ &\quad + \frac{4\pi}{\alpha+1} t^{\alpha+1}, \end{aligned} \tag{2.4}$$

where the last term should be replaced by $4\pi \ln t$ when $\alpha = -1$.

Lemma 2.3. *Let (M^3, g) be a complete three-dimensional manifold with finite first Betti number and finite number of ends. Assume that u is a proper harmonic function on an end E of M. Then*

$$H_\alpha(T) \geq H_\alpha(s) - \alpha(\alpha+2) \int_s^T w(t)\, t^{\alpha-2} dt + \frac{1}{2} \int_{L(s,T)} u^\alpha S |\nabla u|$$

for all $s < T$.

Proof. For regular values $s < T$ we have

$$\begin{aligned} & \left(T^\alpha \frac{dw}{dT}(T) - \alpha T^{\alpha-1} w(T)\right) - \left(s^\alpha \frac{dw}{ds}(s) - \alpha s^{\alpha-1} w(s)\right) \\ = & \left(\int_{l(T)} u^\alpha \frac{\langle \nabla |\nabla u|, \nabla u\rangle}{|\nabla u|} - \int_{l(s)} u^\alpha \frac{\langle \nabla |\nabla u|, \nabla u\rangle}{|\nabla u|}\right) \\ & - \left(\int_{l(T)} |\nabla u| \frac{\langle \nabla u^\alpha, \nabla u\rangle}{|\nabla u|} - \int_{l(s)} |\nabla u| \frac{\langle \nabla u^\alpha, \nabla u\rangle}{|\nabla u|}\right) \\ = & \int_{L(s,T)} \left(u^\alpha \Delta |\nabla u| - |\nabla u| \Delta u^\alpha\right) \\ = & \int_{L(s,T)} \frac{u^\alpha}{|\nabla u|} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u)\right) \\ & - \alpha(\alpha-1) \int_{L(s,T)} |\nabla u|^3 u^{\alpha-2}. \end{aligned}$$

Note that the term $\frac{1}{|\nabla u|}\left|\mathrm{Ric}\left(\nabla u,\nabla u\right)\right|\leq\left|\mathrm{Ric}\right|\left|\nabla u\right|$ is integrable even if u has critical points in $L(s,T)$. The same can be concluded for $\frac{1}{|\nabla u|}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}\right)$ via a classical regularization procedure [24] from the above identity by noticing that it is nonnegative due to the Kato inequality.

Using the co-area formula to rewrite the last term

$$\int_{L(s,T)}\left|\nabla u\right|^{3}u^{\alpha-2}=\int_{s}^{T}t^{\alpha-2}w\left(t\right)dt,$$

we conclude

$$\begin{aligned}
&\left(T^{\alpha}\frac{dw}{dT}\left(T\right)-\alpha T^{\alpha-1}w\left(T\right)\right)-\left(s^{\alpha}\frac{dw}{ds}(s)-\alpha s^{\alpha-1}w\left(s\right)\right) \qquad (2.5)\\
=\ &\int_{L(s,T)}\frac{u^{\alpha}}{\left|\nabla u\right|}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\mathrm{Ric}\left(\nabla u,\nabla u\right)\right)\\
&-\alpha\left(\alpha-1\right)\int_{s}^{T}t^{\alpha-2}w\left(t\right)dt.
\end{aligned}$$

On the other hand, by (1.2),

$$\begin{aligned}
&\int_{l(t)}\frac{1}{\left|\nabla u\right|^{2}}\left(\left|u_{ij}\right|^{2}-\left|\nabla\left|\nabla u\right|\right|^{2}+\mathrm{Ric}\left(\nabla u,\nabla u\right)\right) \qquad (2.6)\\
=\ &\frac{1}{2}\int_{l(t)}\left(S-S_{t}+\frac{1}{\left|\nabla u\right|^{2}}\left|u_{ij}\right|^{2}\right)
\end{aligned}$$

on any regular level set $l\left(t\right),$ where S_t is the scalar curvature of $l\left(t\right).$ Since by Lemma 2.2 the level set $l\left(t\right)$ is connected for all $t\geq1$, the Gauss-Bonnet theorem and the Kato inequality imply that

$$\frac{1}{2}\int_{l(t)}\left(S-S_{t}+\frac{1}{\left|\nabla u\right|^{2}}\left|u_{ij}\right|^{2}\right)\geq\frac{1}{2}\int_{l(t)}S-4\pi+\frac{3}{4}\int_{l(t)}\frac{1}{\left|\nabla u\right|^{2}}\left|\nabla\left|\nabla u\right|\right|^{2}. \qquad (2.7)$$

Observe that

$$\left|w'\left(t\right)\right|\leq\int_{l(t)}\left|\nabla\left|\nabla u\right|\right|\leq\left(\int_{l(t)}\frac{1}{\left|\nabla u\right|^{2}}\left|\nabla\left|\nabla u\right|\right|^{2}\right)^{\frac{1}{2}}\left(\int_{l(t)}\left|\nabla u\right|^{2}\right)^{\frac{1}{2}},$$

which implies

$$\int_{l(t)}\frac{1}{\left|\nabla u\right|^{2}}\left|\nabla\left|\nabla u\right|\right|^{2}\geq\frac{\left(w'\right)^{2}\left(t\right)}{w\left(t\right)}.$$

Together with the elementary inequality

$$\frac{\left(w'\right)^{2}}{w}\left(t\right)\geq\frac{4}{t}w'\left(t\right)-\frac{4}{t^{2}}w\left(t\right),$$

one concludes from (2.7) that

$$\frac{1}{2}\int_{l(t)}\left(S-S_{t}+\frac{1}{\left|\nabla u\right|^{2}}\left|u_{ij}\right|^{2}\right)\geq\frac{1}{2}\int_{l(t)}S-4\pi+\frac{3}{t}w'\left(t\right)-\frac{3}{t^{2}}w\left(t\right).$$

Hence, by (2.6) we get

$$\begin{aligned}&\int_{l(t)} \frac{1}{|\nabla u|^2}\left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right)\\ \geq\ & \frac{1}{2}\int_{l(t)} S - 4\pi + \frac{3}{t}w'(t) - \frac{3}{t^2}w(t)\end{aligned}$$

for any regular level set $l(t)$. We now use it and the co-area formula to conclude

$$\begin{aligned}&\int_{L(s,T)} \frac{u^\alpha}{|\nabla u|}\left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right)\\ \geq\ & \frac{1}{2}\int_{L(s,T)} u^\alpha S|\nabla u| + \int_s^T \left(-4\pi + \frac{3}{t}w'(t) - \frac{3}{t^2}w(t)\right)t^\alpha dt.\end{aligned}$$

Integrating by parts and noting that w is Lipschitz, we obtain

$$\begin{aligned}&\int_{L(s,T)} \frac{u^\alpha}{|\nabla u|}\left(|u_{ij}|^2 - |\nabla|\nabla u||^2 + \operatorname{Ric}(\nabla u, \nabla u)\right)\\ \geq\ & \frac{1}{2}\int_{L(s,T)} u^\alpha S|\nabla u| - \frac{4\pi}{\alpha+1}\left(T^{\alpha+1} - s^{\alpha+1}\right)\\ +\ & 3T^{\alpha-1}w(T) - 3s^{\alpha-1}w(s) - 3\alpha\int_s^T t^{\alpha-2}w(t)\,dt.\end{aligned}$$

Plugging this into (2.5) yields

$$\begin{aligned}&\left(T^\alpha \frac{dw}{dT}(T) - \alpha T^{\alpha-1}w(T)\right) - \left(s^\alpha\frac{dw}{ds}(s) - \alpha s^{\alpha-1}w(s)\right)\\ \geq\ & \frac{1}{2}\int_{L(s,T)} u^\alpha S|\nabla u| - \frac{4\pi}{\alpha+1}\left(T^{\alpha+1} - s^{\alpha+1}\right) + 3T^{\alpha-1}w(T) - 3s^{\alpha-1}w(s)\\ & -\alpha(\alpha+2)\int_s^T t^{\alpha-2}w(t)\,dt\end{aligned}$$

for any $s < T$. This proves the result. □

We are now ready to prove Theorem 1.2. We may assume that the number of ends is one as otherwise M must be a cylinder by the splitting theorem [6]. Note also that for a complete manifold M with non-negative Ricci curvature, its first Betti number is always bounded by its dimension [3, 25].

Hence, without loss of generality, we may assume that all representatives of $H_1(M)$ lie in some domain D, and $D \subset B_p(r_0)$ for some r_0 large enough. We let u denote the barrier function on $E = M \setminus D$, satisfying either (2.1) or (2.2).

If M is nonparabolic, then the barrier function must be proper as the minimal positive Green's function goes to 0 at infinity [23]. Consequently, Lemma 2.3 is applicable to the barrier function u of $M \setminus D$.

Lemma 2.4. *Let (M^3, g) be a complete three-dimensional manifold with non-negative Ricci curvature. If its scalar curvature is bounded below by $S(x) \geq \frac{c}{r(x)+1}$ on M, then M does not admit any positive Green's function, i.e., M is parabolic.*

Proof. Suppose otherwise, that M is nonparabolic. Apply Lemma 2.3 to the barrier function u of $M \setminus D$ with $\alpha = 0$. Then

$$H_0\left(T\right) \geq H_0\left(t\right) + \frac{1}{2}\int_{L(t,T)} \left|\nabla u\right| S$$

for all $0 < t < T \leq 1$, where

$$H_0\left(t\right) = w'\left(t\right) - 3\frac{w\left(t\right)}{t} + 4\pi t.$$

Note that $w\left(t\right) \leq Ct^2$ by the gradient estimate in [7], hence

$$\liminf_{t\to 0} H_0(t) = 0.$$

Therefore, there exists a constant $C > 0$ so that

$$\int_{M\setminus D} \left|\nabla u\right| S = \int_{L(0,1)} \left|\nabla u\right| S \leq C.$$

On the other hand, for any $R > r_0$,

$$\begin{aligned} \int_{B_p(R)\setminus B_p(r_0)} \left|\nabla u\right| S &= \int_{r_0}^{R} \left(\int_{\partial B_p(r)} \left|\nabla u\right| S\right) dr \\ &\geq \int_{r_0}^{R} \frac{c}{r+1} \left(\int_{\partial B_p(r)} \left|\nabla u\right|\right) dr \\ &\geq \frac{1}{C}\ln R, \end{aligned}$$

for some $C > 0$, where we have used the fact that

$$\begin{aligned} \int_{\partial B_p(r)} \left|\nabla u\right| &\geq \left|\int_{\partial B_p(r)} \frac{\partial u}{\partial r}\right| \\ &= \left|\int_{\partial D} \frac{\partial u}{\partial \nu}\right| > 0. \end{aligned}$$

This contradiction completes the proof. □

The following corollary immediately follows from [23].

Corollary 2.5. *Let (M^3, g) be a complete three-dimensional manifold with non-negative Ricci curvature. If its scalar curvature is bounded below by $S(x) \geq \frac{c}{r(x)+1}$ on M, then $\int_1^\infty \frac{t}{\mathrm{V}_p(t)}\, dt = \infty$.*

For the proof of Theorem 1.2 we also need the following lemma, which is true in any dimension.

Lemma 2.6. *Let (M^n, g) be a complete n-dimensional manifold with non-negative Ricci curvature and u a harmonic function on $M \setminus D$ such that*

$$u = 0 \text{ on } \partial D, \quad \lim_{x\to\infty} u(x) = \infty, \text{ and } \int_{\partial D} \frac{\partial u}{\partial \nu} = 1,$$

where ν denotes the outer normal to D. If

$$\mathrm{V}_p(R) \geq \beta\, R^{\alpha+1} \quad \text{for all} \quad R \geq R_0,$$

where $\alpha \geq 0$, $\beta > 0$ *and* $R_0 > 0$ *are all constants, then*

$$\limsup_{x\to\infty} \left(|\nabla u|\ r^{\alpha}\right)(x) \leq \frac{C}{\beta},$$

for some constant C *depending only on dimension.*

Proof. Let $r_0 > 0$ be large enough so that $D \subset B_p(r_0)$. For $R \geq r_0$, let

$$F(R) = \max_{x\in\partial B_p(R)} |\nabla u|(x).$$

We assert that there exists $C > 0$ such that for any $r_0 \leq R < T$,

$$|\nabla u|(x) - F(R) - \frac{C}{T} u(x) \leq 0 \tag{2.8}$$

on $B_p(T) \setminus B_p(R)$. Indeed, the function on the left hand side is subharmonic as $|\nabla u|$ is subharmonic and u is harmonic. Also, it is nonpositive on the boundary of $B_p(T) \setminus B_p(R)$ by the gradient estimate of Cheng and Yau [7], which says that $|\nabla u|(x) \leq \frac{C}{r(x)} u(x)$. So, (2.8) immediately follows from the maximum principle.

By first letting $T \to \infty$ and then maximizing over all $x \in M \setminus B_p(R)$, one concludes from (2.8) that

$$F(R) = \max_{x\in M\setminus B_p(R)} |\nabla u|(x) \tag{2.9}$$

for any $R > r_0$. In particular, $F(R)$ is non-increasing in R. Let

$$\Gamma = \limsup_{R\to\infty} \left(R^{\alpha}\, F(R)\right). \tag{2.10}$$

It suffices to show that

$$\Gamma \leq \frac{C}{\beta},$$

for C depending only on dimension.

For $x \in \partial B_p(R)$ with $R \geq 2r_0$, by the Bishop-Gromov volume comparison theorem,

$$\frac{\mathrm{V}_p(R)}{\mathrm{V}_x\left(\frac{R}{2}\right)} \leq \frac{\mathrm{V}_x(2R)}{\mathrm{V}_x\left(\frac{R}{2}\right)} \leq C.$$

Therefore, as $\mathrm{V}_p(R) \geq \beta\, R^{\alpha+1}$ for $R \geq R_0$, it follows that

$$\mathrm{V}_x\left(\frac{R}{2}\right) \geq \frac{\beta}{C}\, R^{\alpha+1}.$$

We now follow an argument in [29]. Let

$$a_x = \min_{B_x(\frac{R}{2})} u \text{ and } b_x = \max_{B_x(\frac{R}{2})} u.$$

Since u is harmonic and $\int_{\partial D} \frac{\partial u}{\partial \nu} = 1$, we have $\int_{l(t)} |\nabla u| = 1$. The co-area formula implies that

$$\int_{L(a,b)} |\nabla u|^2 = \int_{a_x}^{b_x} \int_{l(t)} |\nabla u| = b_x - a_x \leq F\left(\frac{R}{2}\right) R,$$

where the last inequality follows from (2.9). Applying the mean value inequality to the subharmonic function $|\nabla u|^2$ (see Theorem 7.2 in [18]) we arrive at

$$\begin{aligned} |\nabla u|^2(x) &\leq \frac{C}{\mathrm{V}_x\left(\frac{R}{2}\right)} \int_{B_x\left(\frac{R}{2}\right)} |\nabla u|^2 \\ &\leq \frac{C}{\beta R^{\alpha+1}} \int_{L(a_x,b_x)} |\nabla u|^2 \\ &\leq \frac{C}{\beta R^{\alpha}} F\left(\frac{R}{2}\right). \end{aligned}$$

As $x \in \partial B_p(R)$ is arbitrary, this implies that $F^2(R) \leq \frac{C}{\beta R^\alpha} F\left(\frac{R}{2}\right)$. Rewrite it into

$$W^2(R) \leq \frac{C}{\beta} W\left(\frac{R}{2}\right), \tag{2.11}$$

where $W(R) = R^\alpha F(R)$. By a simple induction, we conclude $W(R) \leq C(R_0)$. Hence, Γ is finite. From (2.11), after letting $R \to \infty$, one obtains $\Gamma \leq \frac{C}{\beta}$ as desired. By (2.10), this proves the lemma. □

For the reader's convenience, we restate Theorem 1.2 from the Introduction.

Theorem 2.7. *Let (M^3, g) be a complete three-dimensional manifold with non-negative Ricci curvature.*

- *If the scalar curvature is bounded below by $S \geq 1$ on M, then there exists a universal constant $C > 0$ such that*

$$\mathrm{V}_p(R) \leq C\, R \tag{2.12}$$

  for all $R > 0$ and $p \in M$.
- *If the scalar curvature*

$$S(x) \geq \frac{C_0}{r^\alpha(x) + 1}$$

  for some $\alpha \in [0,1]$, where $r(x)$ is the geodesic distance from x to a fixed point $p \in M$, then

$$\mathrm{V}_p(R_i) \leq \frac{C}{C_0} R_i^{\alpha+1} \tag{2.13}$$

  for a sequence $R_i \to \infty$, where $C > 0$ is a universal constant.

Proof. By Lemma 2.4, M must be parabolic, hence by (2.2) it admits a positive harmonic function u on $M \setminus D$ such that $u = 0$ on ∂D and $\lim_{x\to\infty} u(x) = \infty$. Normalizing u we have

$$u = 0 \text{ on } \partial D, \quad \lim_{x\to\infty} u(x) = \infty, \text{ and } \int_{\partial D} \frac{\partial u}{\partial \nu} = 1.$$

According to [34], $\mathrm{V}_p(R) \geq \frac{1}{C}\mathrm{V}_p(1)\, R$, for all $R \geq 1$, where $C > 0$ is a universal constant. Applying Lemma 2.6 for $\alpha = 0$ and $\beta = \frac{\mathrm{V}_p(1)}{C}$, it follows that

$$|\nabla u| \leq \frac{C}{\mathrm{V}_p(1)} \text{ on } M \setminus B_p(r_1) \tag{2.14}$$

for some $r_1 > 0$ sufficiently large. The monotonicity formula from Lemma 2.3 implies that

$$H_0(t) \geq H_0(1) + \frac{1}{2} \int_{L(1,t)} |\nabla u| \, S \tag{2.15}$$

for all $t \geq 1$, where

$$H_0(t) = w'(t) - 3\frac{w(t)}{t} + 4\pi t$$

and $w(t) = \int_{l(t)} |\nabla u|^2$.

Note that $w(t)$ is bounded, as $\int_{l(t)} |\nabla u| = 1$ and $|\nabla u|$ is bounded by (2.14). In particular, there exists some constant $C(r_1) > 0$, depending on $\sup_{B_p(r_1)\setminus D} |\nabla u|$, such that for each $t > 1$ we have $w'(\xi) \leq C(r_1)$, for some $\xi \in (t, t+1)$.

In conclusion, (2.15) implies that for all $t > 1$,

$$\int_{L(1,t)} |\nabla u| \, S \leq \int_{L(1,\xi)} |\nabla u| \, S \leq 8\pi \, t + C(r_1). \tag{2.16}$$

To prove the first part (2.12), by scaling, it suffices to show there exists a universal constant $C > 0$ so that

$$\mathrm{V}_p(1) \leq \frac{C}{A} \quad \text{if} \quad S \geq A > 0. \tag{2.17}$$

According to (2.14),

$$u(x) \leq \frac{C}{\mathrm{V}_p(1)} \, r(x) + C(r_1)$$

for all $x \in M \setminus B_p(r_1)$. Hence, there exists a universal constant $C > 0$ so that

$$B_p(R) \setminus B_p(r_1) \subset L\left(1, \frac{C}{\mathrm{V}_p(1)} \, R + C(r_1)\right)$$

for all $R > r_1$, by additionally assuming that r_1 is large enough such that $u(x) \geq 1$ outside $B_p(r_1)$. From (2.16) and $S \geq A$, we conclude

$$A \int_{B_p(R)\setminus B_p(r_1)} |\nabla u| \leq \frac{C}{\mathrm{V}_p(1)} \, R + C(r_1)$$

for all $R > r_1$. However,

$$\int_{B_p(R)\setminus B_p(r_1)} |\nabla u| \geq \int_{r_1}^{R} \left(\int_{\partial B_p(r)} \frac{\partial u}{\partial r} \right) dr = R - r_1.$$

Hence, $\mathrm{V}_p(1) \leq \frac{C}{A}$ after letting $R \to \infty$. This proves (2.17).

Now we turn to the second part (2.13). Let us assume that for some $\varepsilon > 0$ we have

$$\mathrm{V}_p(R) \geq \frac{1}{\varepsilon} \, R^{\alpha+1}, \tag{2.18}$$

for all $R \geq R_0$. Therefore, by Lemma 2.6,

$$|\nabla u| \leq C\varepsilon r^{-\alpha} \quad \text{on } M \setminus B_p(R_1), \tag{2.19}$$

where C is a universal constant and $R_1 \geq R_0$ a fixed constant. For $0 \leq \alpha < 1$, integrating (2.19) along minimal geodesics we obtain

$$u \leq \frac{C\,\varepsilon}{1-\alpha} r^{1-\alpha} + C(R_1) \quad \text{on } M \setminus B_p(R_1).$$

This implies that

$$B_p(R) \setminus B_p(R_1) \subset L\left(1, \frac{C\varepsilon}{1-\alpha} R^{1-\alpha} + C(R_1)\right)$$

by assuming additionally that R_1 is large enough so such that $u(x) \geq 1$ outside $B_p(R_1)$. By (2.16),

$$\begin{aligned} \int_{B_p(R)\setminus B_p(R_1)} S|\nabla u| &\leq \int_{L\left(1, \frac{C\varepsilon}{1-\alpha}R^{1-\alpha}+C(R_1)\right)} S\,|\nabla u| \\ &\leq \frac{C\varepsilon}{1-\alpha} R^{1-\alpha} + C(R_1), \end{aligned} \tag{2.20}$$

where C is a universal constant. However, by the assumption that $S \geq C_0\, r^{-\alpha}$ on $M \setminus B_p(R_1)$, we obtain from the co-area formula that

$$\begin{aligned} \int_{B_p(R)\setminus B_p(R_1)} S|\nabla u| &\geq \int_{R_1}^{R} \left(C_0\, r^{-\alpha}\right)\left(\int_{\partial B_p(r)} \frac{\partial u}{\partial r}\right) dr \\ &= \frac{C_0}{1-\alpha}\left(R^{1-\alpha} - R_1^{1-\alpha}\right). \end{aligned} \tag{2.21}$$

In conclusion, from (2.20) and (2.21),

$$\frac{C\,\varepsilon}{1-\alpha} R^{1-\alpha} + C(R_1) \geq \frac{C_0}{1-\alpha}\left(R^{1-\alpha} - R_1^{1-\alpha}\right)$$

for all $R > R_1$. Making $R \to \infty$ implies that $\varepsilon \geq \frac{C_0}{C}$. Hence, (2.18) implies that

$$\mathrm{V}_p\left(R_i\right) \leq \frac{C}{C_0} R_i^{\alpha+1},$$

for a universal constant $C > 0$. A similar argument also works for $\alpha = 1$. This proves the result. □

3. Proof of Theorem 1.3

In this section, we focus on the proof of Theorem 1.3. First, we recall a result of Bianchi and Setti (Theorem 2.1 in [4]) as the following lemma. While (3.2) is not explicitly stated by them, it can be easily verified from the construction. Everywhere, $r(x)$ denotes the distance function to a fixed point $p \in M$.

Lemma 3.1. *Let (M^n, g) be an n-dimensional complete Riemannian manifold with Ricci curvature bounded below by $\mathrm{Ric} \geq -\frac{C}{r^2+1}$. Then there exists a smooth proper function η such that*

$$\begin{aligned} \frac{1}{C}\ln(r+2) &\leq \eta \leq C\ln(r+2) \\ |\nabla \eta| &\leq \frac{C}{r+1} \\ |\Delta \eta| &\leq \frac{C}{(r+1)^2} \end{aligned} \tag{3.1}$$

and

$$|\nabla(\Delta\eta)| \leq \frac{C}{(r+1)^3} + \frac{C}{r+1}\left|\nabla^2 \eta\right| \tag{3.2}$$

for some constant $C > 0$.

We henceforth denote with

$$D(t) = \{x \in M : \eta(x) < t\}. \tag{3.3}$$

For given $R > 0$, let $\psi : (0, \infty) \to \mathbb{R}$ be a smooth function such that $\psi = 1$ on $(0, \ln R)$ and $\psi = 0$ on $(2\ln R, \infty)$, and

$$|\psi'| \le \frac{C}{\ln R}, \quad |\psi''| \le \frac{C}{\ln^2 R} \text{ and } |\psi'''| \le \frac{C}{\ln^3 R}.$$

Composing it with η, we obtain a cut-off function $\psi(x) = \psi(\eta(x))$. Obviously,

$$\psi = 1 \text{ on } D(\ln R) \quad \text{and } \psi = 0 \text{ on } M \setminus D(2\ln R).$$

In the following we assume that (M, g) is nonparabolic and let f be a barrier function on $M \setminus D$, see Definition 2.1. We extend f to M by setting $f = 1$ on D. The next result is well-known [7].

Lemma 3.2. *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with Ricci curvature bounded below by $\mathrm{Ric} \ge -\frac{C}{r^2+1}$. Then*

$$|\nabla \ln f| \le \frac{C}{r} \quad \text{on } M \setminus D.$$

We now consider the function u given by

$$u = f\psi. \tag{3.4}$$

In view of Lemma 3.1, it is straightforward to verify the following lemma.

Lemma 3.3. *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with Ricci curvature bounded below by $\mathrm{Ric} \ge -\frac{C}{r^2+1}$. Then u is harmonic on $D(\ln R) \setminus D$ and on $M \setminus D$,*

$$\begin{aligned} |\nabla u| &\le u\,|\nabla \ln f| + \frac{C}{r}\frac{1}{\ln R} \\ |\Delta u| &\le \frac{C}{r^2 \ln R} + \frac{C}{\ln R}|\nabla f|^2 \\ |\nabla(\Delta u)| &\le \frac{Cr}{\ln R}\left(\left|\nabla^2 \eta\right|^2 + \left|\nabla^2 f\right|^2\right) + \frac{C}{r^3 \ln R} + \frac{C}{\ln R}|\nabla f|^2. \end{aligned}$$

As mentioned before, in the case that M has finite first Betti number and finitely many, say m, ends, D is chosen to be large enough so that all representatives of the first homology group $H_1(M)$ are included in D. Moreover, $M \setminus D$ has exactly m unbounded connected components. In this section, let $L(t, \infty)$ denote the connected component of the super-level set $\{u > t\}$ that contains D and

$$l(t) = \partial L(t, \infty).$$

Lemma 3.4. *Let (M^n, g) be an n-dimensional complete Riemannian manifold with m ends and finite first Betti number $b_1(M)$. Then for all $0 < t < 1$,*

$$l(t) \cap \overline{M_t} = \partial M_t \text{ has } m \text{ connected components,}$$

where M_t is the union of all unbounded connected components of $M \setminus \overline{L(t, \infty)}$.

Proof. Recall that all representatives of the first homology $H_1(M)$ lie in D and $M \setminus D$ has exactly m unbounded components. Therefore, as $D \subset L(t, \infty)$, it follows that $L(t, \infty)$ contains all representatives of $H_1(M)$ and M_t has m connected components for any $0 < t < 1$. Note also $\partial M_t \subset l(t)$.

For fixed $0<\delta<1$ such that $t+\delta<1$, define U to be the union of $L(t,\infty)$ with all bounded components of $M\setminus\overline{L(t+\delta,\infty)}$ and V the union of all unbounded components of $M\setminus\overline{L(t+\delta,\infty)}$. Since $M=U\cup V$, we have the following Mayer-Vietoris sequence

$$H_1(U)\oplus H_1(V)\overset{j_*}{\to}H_1(M)\overset{\partial}{\to}H_0(U\cap V)\overset{i_*}{\to}H_0(U)\oplus H_0(V)\overset{j'_*}{\to}H_0(M).$$

The map j_* is onto because all representatives of $H_1(M)$ lie inside U. The map j'_* is also onto. Note also that V has m components and U is connected. The latter is true because each component of $M\setminus\overline{L(t+\delta,\infty)}$ intersects with $L(t,\infty)$ as $\overline{L(t+\delta,\infty)}\subset L(t,\infty)$. We therefore obtain the short exact sequence

$$0\to H_0(U\cap V)\to\mathbb{Z}\oplus..\oplus\mathbb{Z}\to\mathbb{Z}\to 0$$

with $m+1$ summands. In conclusion,

$$H_0(U\cap V)=\mathbb{Z}\oplus..\oplus\mathbb{Z}$$

with m summands. Since $\delta>0$ can be arbitrarily small, this proves that

$$l(t)\cap\overline{M_t}\text{ has }m\text{ components}$$

for $0<t<1$. □

Recall that an n-dimensional manifold M has asymptotically nonnegative Ricci curvature if its Ricci curvature is bounded by

$$\mathrm{Ric}(x)\geq -k(r(x))$$

for a continuous nonincreasing function $k:[0,\infty)\to[0,\infty)$ with

$$\int_0^\infty rk(r)\,dr<\infty.$$

It is well known [14] that (M^n,g) has at most Euclidean area growth, that is, the area $A_p(t)$ of geodesic sphere $\partial B_p(t)$ satisfies

$$\mathrm{A}_p(t)\leq C\,t^{n-1} \tag{3.5}$$

for all $t>0$. It is also clear that $\mathrm{Ric}\geq-\frac{C}{r^2+1}$. In passing, we mention that Li and Tam [20] have shown that M^n has finitely many ends under the stronger assumption that $\int_0^\infty r^{n-1}k(r)\,dr<\infty$.

We also note that according to Lemma 3.1,

$$D(2\ln R)\subset B_p\left(R^C\right) \tag{3.6}$$

for some $C>0$, where $D(t)$ was defined in (3.3).

We are now ready to prove Theorem 1.3 which is restated below.

Theorem 3.5. *Let $\left(M^3,g\right)$ be a three-dimensional complete noncompact Riemannian manifold with asymptotically nonnegative Ricci curvature. Suppose that its scalar curvature is bounded below by a positive constant and that M has finitely many ends and finite first Betti number $b_1(M)$. Then (M,g) is parabolic.*

Proof. Assume by contradiction that (M,g) is nonparabolic. Let f be a barrier function on $M\setminus D$, see (2.1), where the smooth connected compact domain D contains all representatives of $H_1(M)$ and $M\setminus D$ has m ends.

For given small $0 < \varepsilon < 1$, consider R large so that

$$\ln R > \frac{1}{\varepsilon^2}. \tag{3.7}$$

In the following, the constant $C > 0$ is independent of R and ε, but its value may change from line to line.

Define the function

$$u = f\psi$$

as in (3.4). By the Bochner formula and the inequality $\langle \nabla \Delta u, \nabla u \rangle \geq - |\nabla (\Delta u)| \, |\nabla u|$ it follows that

$$\Delta |\nabla u| \geq \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 \right) |\nabla u|^{-1} + \mathrm{Ric}\,(\nabla u, \nabla u) \, |\nabla u|^{-1} - |\nabla (\Delta u)|$$

holds on $M \setminus D$ whenever $|\nabla u| \neq 0$.

Let $\phi(x)$ be the smooth cut-off function given by

$$\phi(x) = \begin{cases} \phi(u(x)) & \text{on } L(\varepsilon, \infty) \\ 0 & \text{on } M \setminus L(\varepsilon, \infty) \end{cases}$$

Here, the function $\phi(t)$ is smooth, $\phi(t) = 1$ for $2\varepsilon \leq t \leq 1$ and $\phi(t) = 0$ for $t < \varepsilon$. Moreover,

$$|\phi'|(t) \leq \frac{C}{\varepsilon} \text{ and } |\phi''|(t) \leq \frac{C}{\varepsilon^2} \text{ for } \varepsilon < t < 2\varepsilon. \tag{3.8}$$

Clearly, the function $\phi(x)$ satisfies $\phi = 1$ on $L(2\varepsilon, \infty)$ and $\phi = 0$ on $M \setminus L(\varepsilon, \infty)$. It follows that

$$\begin{aligned} &\int_{M \setminus D} (\Delta |\nabla u|) \, \phi^2 \\ \geq &\int_{M \setminus D} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}\,(\nabla u, \nabla u) \right) |\nabla u|^{-1} \phi^2 \\ &- \int_{M \setminus D} |\nabla (\Delta u)| \, \phi^2. \end{aligned} \tag{3.9}$$

To estimate the last term, we use Lemma 3.3 to obtain

$$\begin{aligned} \int_{M \setminus D} |\nabla (\Delta u)| \, \phi^2 \quad \leq \quad & \frac{C}{\ln R} \int_{M \setminus D} \left| \nabla^2 \eta \right|^2 r \phi^2 + \frac{C}{\ln R} \int_{M \setminus D} \left| \nabla^2 f \right|^2 r \phi^2 \\ & + \frac{C}{\ln R} \int_{D(2 \ln R) \setminus D} \frac{1}{r^3} + \frac{C}{\ln R} \int_{M \setminus D} |\nabla f|^2 . \end{aligned} \tag{3.10}$$

By [20], $\int_M |\nabla f|^2 < \infty$. By (3.5) and (3.6),

$$\frac{C}{\ln R} \int_{D(2 \ln R) \setminus D} \frac{1}{r^3} \leq C. \tag{3.11}$$

Now we deal with the first term in (3.10). For convenience, we extend ϕ everywhere on M by setting $\phi(x) = 1$ for $x \in D$. Integrating by parts gives

$$\begin{aligned} \int_M \left|\nabla^2\eta\right|^2 (r+1)\,\phi^2 &= -\int_M \langle \nabla(\Delta\eta), \nabla\eta\rangle (r+1)\,\phi^2 \\ &\quad -\int_M \mathrm{Ric}(\nabla\eta,\nabla\eta)(r+1)\,\phi^2 \\ &\quad -\int_M \eta_{ij}\eta_i r_j \phi^2 \\ &\quad -2\int_M \eta_{ij}\eta_i\phi_j (r+1)\phi. \end{aligned} \tag{3.12}$$

By Lemma 3.1 it follows that

$$\begin{aligned} -\int_M \langle \nabla(\Delta\eta), \nabla\eta\rangle (r+1)\,\phi^2 &\le C\int_M \frac{1}{r+1}\left|\nabla^2\eta\right|\phi^2 + C\int_{D(2\ln R)} \frac{1}{(r+1)^3} \\ &\le \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\int_{D(2\ln R)} \frac{1}{(r+1)^3} \\ &\le \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\ln R, \end{aligned}$$

where in the last line we have used (3.11). According to the Ricci lower bound, we have

$$\begin{aligned} -\int_M \mathrm{Ric}(\nabla\eta,\nabla\eta)(r+1)\,\phi^2 &\le C\int_M \frac{1}{r+1}\left|\nabla\eta\right|^2 \\ &\le C\int_{D(2\ln R)} \frac{1}{(r+1)^3} \\ &\le C\ln R. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} -\int_M \eta_{ij}\eta_i r_j\phi^2 &\le \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + \int_{D(2\ln R)} \frac{1}{(r+1)^3} \\ &\le \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\ln R. \end{aligned}$$

Finally, we estimate the last term in (3.12) by

$$\begin{aligned} -2\int_M \eta_{ij}\eta_i\phi_j\phi(r+1) &\le \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + 4\int_M |\nabla\eta|^2|\nabla\phi|^2(r+1) \\ &\le \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\int_M \frac{1}{r+1}|\nabla\phi|^2. \end{aligned}$$

Lemma 3.3 and Lemma 3.2 imply that on $L(\varepsilon, 2\varepsilon)$,

$$|\nabla u| \le \frac{C\varepsilon}{r}. \tag{3.13}$$

Using the definition of ϕ we see that $|\nabla\phi| \le \frac{C}{r}$. Therefore,

$$\int_M \frac{1}{r+1}|\nabla\phi|^2 \le \int_{D(2\ln R)} \frac{1}{(r+1)^3} \le C\ln R.$$

We have proved that

$$-2\int_M \eta_{ij}\eta_i\phi_j\phi(r+1) \leq \frac{1}{4}\int_M (r+1)\left|\nabla^2\eta\right|^2\phi^2 + C\ln R.$$

Plugging these estimates into (3.12) implies that

$$\frac{C}{\ln R}\int_M \left|\nabla^2\eta\right|^2 (r+1)\,\phi^2 \leq C. \tag{3.14}$$

Since f is harmonic on $M\setminus D$, we similarly have

$$\begin{aligned}
\int_{M\setminus D}\left|\nabla^2 f\right|^2 r\phi^2 &= -\int_{M\setminus D}\mathrm{Ric}\,(\nabla f,\nabla f)\,r\phi^2 - \int_{M\setminus D} f_{ij}f_i r_j\phi^2 \\
&\quad -2\int_{M\setminus D} f_{ij}f_i\phi_j r\phi - \int_{\partial D} f_{ij}f_i\nu_j r\phi^2 \\
&\leq \frac{1}{2}\int_{M\setminus D}\left|\nabla^2 f\right|^2 r\phi^2 + C.
\end{aligned}$$

This proves that

$$\frac{C}{\ln R}\int_{M\setminus D}\left|\nabla^2 f\right|^2 (r+1)\,\phi^2 \leq C. \tag{3.15}$$

In conclusion, plugging (3.11), (3.14), and (3.15) into (3.10) implies that

$$\int_{M\setminus D}\left|\nabla\left(\Delta u\right)\right|\phi^2 \leq C.$$

Consequently, (3.9) becomes

$$\begin{aligned}
&\int_{M\setminus D}\left(\Delta\left|\nabla u\right|\right)\phi^2 \\
&\geq \int_{M\setminus D}\Big(\left|u_{ij}\right|^2 - \left|\nabla\left|\nabla u\right|\right|^2 + \mathrm{Ric}\,(\nabla u,\nabla u)\Big)\left|\nabla u\right|^{-1}\phi^2 - C.
\end{aligned} \tag{3.16}$$

By the co-area formula, we have

$$\begin{aligned}
&\int_{M\setminus D}\Big(\left|u_{ij}\right|^2 - \left|\nabla\left|\nabla u\right|\right|^2 + \mathrm{Ric}\,(\nabla u,\nabla u)\Big)\left|\nabla u\right|^{-1}\phi^2 \\
= &\int_\varepsilon^1 \phi^2(t)\int_{\{u=t\}}\Big(\left|u_{ij}\right|^2 - \left|\nabla\left|\nabla u\right|\right|^2 + \mathrm{Ric}\,(\nabla u,\nabla u)\Big)\left|\nabla u\right|^{-2}dt \\
= &\int_\varepsilon^1 \phi^2(t)\int_{l(t)}\Big(\left|u_{ij}\right|^2 - \left|\nabla\left|\nabla u\right|\right|^2 + \mathrm{Ric}\,(\nabla u,\nabla u)\Big)\left|\nabla u\right|^{-2}dt \\
&+\int_\varepsilon^1 \phi^2(t)\int_{\widetilde{l}(t)}\Big(\left|u_{ij}\right|^2 - \left|\nabla\left|\nabla u\right|\right|^2 + \mathrm{Ric}\,(\nabla u,\nabla u)\Big)\left|\nabla u\right|^{-2}dt,
\end{aligned}$$

where

$$\widetilde{l}\,(t) = \partial\widetilde{L}\,(t,\infty) \tag{3.17}$$

and

$$\widetilde{L}\,(t,\infty) = \left(\{u>t\}\setminus L\,(t,\infty)\right)\cap L\,(\varepsilon,\infty)\,. \tag{3.18}$$

By the Kato inequality and noting that $\mathrm{Ric}(\nabla u, \nabla u) \geq -\frac{C}{(r+1)^2} |\nabla u|^2$ we have

$$
\begin{aligned}
& \int_\varepsilon^1 \phi^2(t) \int_{\widetilde{l}(t)} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) \right) |\nabla u|^{-2} dt \\
\geq & -C \int_\varepsilon^1 \phi^2(t) \int_{\{u=t\}} \frac{1}{(r+1)^2} dt \\
\geq & -C \int_{L(\varepsilon,1)} \frac{1}{(r+1)^2} |\nabla u|,
\end{aligned}
$$

where the last line follows from the co-area formula. However,

$$
\begin{aligned}
\int_{M \setminus D} \frac{1}{r^2} |\nabla u| & \leq \int_{M \setminus D} \frac{1}{r^2} |\nabla f| + \frac{1}{\ln R} \int_{D(2 \ln R)} \frac{1}{(r+1)^3} \\
& \leq C.
\end{aligned}
\tag{3.19}
$$

In conclusion,

$$
\int_\varepsilon^1 \phi^2(t) \int_{\widetilde{l}(t)} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) \right) |\nabla u|^{-2} dt \geq -C.
$$

Thus, we have that

$$
\begin{aligned}
& \int_{M \setminus D} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) \right) |\nabla u|^{-1} \phi^2 \\
\geq & \int_\varepsilon^1 \phi^2(t) \int_{l(t)} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) \right) |\nabla u|^{-2} dt - C.
\end{aligned}
$$

Rewrite it into

$$
\begin{aligned}
& \int_{M \setminus D} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) \right) |\nabla u|^{-1} \phi^2 \\
\geq & \int_\varepsilon^1 \phi^2(t) \int_{l(t) \cap \overline{M_t}} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) \right) |\nabla u|^{-2} dt \\
& + \int_\varepsilon^1 \phi^2(t) \int_{l(t) \cap \left(M \setminus \overline{M_t} \right)} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) \right) |\nabla u|^{-2} dt - C.
\end{aligned}
\tag{3.20}
$$

Since the Ricci curvature is bounded below by $\mathrm{Ric}(\nabla u, \nabla u) |\nabla u|^{-2} \geq -\frac{C}{(r+1)^2}$, together with the Kato inequality $|u_{ij}|^2 \geq |\nabla |\nabla u||^2$, it follows that

$$
\begin{aligned}
& \int_\varepsilon^1 \phi^2(t) \int_{l(t) \cap \left(M \setminus \overline{M_t} \right)} \left(|u_{ij}|^2 - |\nabla |\nabla u||^2 + \mathrm{Ric}(\nabla u, \nabla u) \right) |\nabla u|^{-2} dt \\
\geq & -C \int_\varepsilon^1 \int_{\{u=t\}} \frac{1}{r^2} dt \\
\geq & -C \int_{M \setminus D} \frac{1}{r^2} |\nabla u| \\
\geq & -C,
\end{aligned}
$$

where the last line follows from (3.19). In conclusion, (3.20) becomes

$$\begin{aligned}(3.21)\qquad & \int_{M\setminus D}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2+\mathrm{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-1}\phi^2\\ \geq\ & \int_{\varepsilon}^{1}\phi^2\left(t\right)\int_{l(t)\cap\overline{M_t}}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2+\mathrm{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-2}dt-C.\end{aligned}$$

By (1.2) and the Kato inequality,

$$(3.22)\qquad \left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2+\mathrm{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-2}\geq\frac{1}{2}S-\frac{1}{2}S_t.$$

Note that by the Gauss-Bonnet theorem, in view of Lemma 3.4, one has

$$\int_{l(t)\cap\overline{M_t}}S_t\leq 8\pi m\leq C,$$

for all regular values $t\in(0,1)$. Therefore, we conclude from (3.22) that

$$\int_{l(t)\cap\overline{M_t}}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2+\mathrm{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-2}\geq\frac{1}{2}\int_{l(t)\cap\overline{M_t}}S-C.$$

Together with (3.21), this implies that

$$\begin{aligned}(3.23)\qquad & \int_{M\setminus D}\left(\left|u_{ij}\right|^2-\left|\nabla\left|\nabla u\right|\right|^2+\mathrm{Ric}\left(\nabla u,\nabla u\right)\right)\left|\nabla u\right|^{-1}\phi^2\\ \geq\ & \frac{1}{2}\int_{\varepsilon}^{1}\phi^2\left(t\right)\int_{l(t)\cap\overline{M_t}}Sdt-C.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{M\setminus D}\left(\Delta\left|\nabla u\right|\right)\phi^2 &= \int_{M\setminus D}\left|\nabla u\right|\Delta\phi^2-\int_{\partial D}\left|\nabla u\right|_{\nu}\\ &\leq \frac{C}{\varepsilon}\int_{L(\varepsilon,2\varepsilon)}\left|\nabla u\right|\,\left|\Delta u\right|+\frac{C}{\varepsilon^2}\int_{L(\varepsilon,2\varepsilon)}\left|\nabla u\right|^3+C.\end{aligned}$$

First, by (3.13) and Lemma 3.3 we have

$$\frac{C}{\varepsilon}\int_{L(\varepsilon,2\varepsilon)}\left|\nabla u\right|\,\left|\Delta u\right|\leq\frac{C}{\ln R}\int_{D(2\ln R)\setminus D}\left(\frac{1}{r^3}+\left|\nabla f\right|^2\right)\leq C,$$

where the last inequality follows by (3.11). Similarly, we get

$$\frac{C}{\varepsilon^2}\int_{L(\varepsilon,2\varepsilon)}\left|\nabla u\right|^3\leq C\int_{D(2\ln R)\setminus D}\frac{1}{r^2}\left|\nabla u\right|\leq C,$$

where the last inequality is by (3.19). In conclusion,

$$\int_{M\setminus D}\left(\Delta\left|\nabla u\right|\right)\phi^2\leq C.$$

Hence, (3.16) and (3.23) imply that

$$\int_{2\varepsilon}^{1}\left(\int_{l(t)\cap\overline{M_t}}S\right)dt\leq C.$$

As S is bounded from below by a positive constant and $l(t)\cap \overline{M_t}=\partial M_t$, it follows that

$$\int_{2\varepsilon}^{1}\mathrm{A}\left(\partial M_t\right)dt\leq C \tag{3.24}$$

for a constant $C>0$ independent of $\varepsilon>0$ and $R>0$.

Recall that the barrier function defined by (2.1) satisfies $\int_{\partial D}\left(-\frac{\partial f}{\partial \nu}\right)=C>0$. Since f is harmonic on the bounded domain $M\setminus(M_t\cup D)$, it follows that

$$C=-\int_{\partial D}\frac{\partial f}{\partial\nu}=-\int_{\partial M_t}\frac{\partial f}{\partial\nu}\leq\int_{\partial M_t}\left|\nabla f\right|. \tag{3.25}$$

According to Lemma 3.2 we have

$$\left|\nabla f\right|\leq\frac{C}{r}f\ \text{ on } M\setminus D. \tag{3.26}$$

If $x\in\partial M_t\cap D(\ln R)$, then $f(x)=u(x)=t$ and we evidently get $\left|\nabla f\right|(x)\leq Ct$. If $x\in\partial M_t\setminus D(\ln R)$, then (3.26) implies $\left|\nabla f\right|(x)\leq\frac{C}{\ln R}\leq C\varepsilon^2$ in view of (3.7). In conclusion,

$$\left|\nabla f\right|\leq Ct\ \text{ on } \partial M_t \text{ for all } t\in(2\varepsilon,1).$$

Plugging this into (3.25) one obtains that

$$\mathrm{A}\left(\partial M_t\right)\geq\frac{C}{t}\ \text{ for all } t\in(2\varepsilon,1).$$

However, this contradicts (3.24) if ε is sufficiently small. In conclusion, (M,g) must be parabolic. ☐

4. Proof of Theorem 1.4

We now turn to the proof of Theorem 1.4. Since M is assumed to have finite first Betti number and finitely many ends, we may assume that $D\subset M$ is large enough so that all representatives $H_1(M)$ lie in D and $M\setminus D$ has the maximal number of ends. The monotonicity formula in Lemma 2.3 gives the following integral estimate.

Lemma 4.1. *Let $\left(M^3,g\right)$ be a parabolic complete three-dimensional manifold with non-negative scalar curvature, finite first Betti number, and finite number of ends. Assume the barrier u defined on an end E of M by (2.2) has bounded gradient, $\sup_E\left|\nabla u\right|<\infty$. Then there exists a constant $\Upsilon>0$ such that*

$$\int_{L(1,t)}S|\nabla u|\leq 8\pi t+\Upsilon,$$

for all $t>1$.

Proof. Applying Lemma 2.3 with $\alpha=0$ to u we get

$$\frac{1}{2}\int_{L(1,t)}S|\nabla u|\leq\frac{dw}{dt}-3\frac{w(t)}{t}+4\pi t+\Upsilon_0 \tag{4.1}$$

for all $t>1$, where

$$\Upsilon_0=-\left(\frac{dw}{dt}\Big|_{t=1}-3w(1)+4\pi\right).$$

Since u is harmonic, we know that $\int_{l(t)} |\nabla u|$ is a constant. Together with the assumption that u has bounded gradient, we obtain for all $t > 1$

$$w(t) = \int_{l(t)} |\nabla u|^2 \leq \Upsilon_1,$$

where Υ_1 is a constant. So, for every $t > 1$ there exists $\xi \in (t, t+1)$ such that $\frac{dw}{dt}(\xi) \leq \Upsilon_1$. Then (4.1) implies

$$\int_{L(1,t)} S|\nabla u| \leq \int_{L(1,\xi)} S|\nabla u| \leq 8\pi t + \Upsilon,$$

where $\Upsilon = 2\Upsilon_0 + 2\Upsilon_1 + 8\pi$. This proves the result. $\square$

In the following, we again normalize the barrier function u on the end E such that

$$\int_{l(t)} |\nabla u| = 1 \tag{4.2}$$

for all $t \geq 1$. Assume that $D \subset B_p(r_0)$, for some $r_0 > 0$ large enough.

Lemma 4.2. *Let (M^3, g) be a parabolic complete three-dimensional Riemannian manifold. Assume that the Ricci curvature of M is bounded below by $\mathrm{Ric} \geq -k(r(x))$, for a continuous nonincreasing function $k(r)$ satisfying $\int_0^\infty rk(r)\,dr < \infty$. Normalize the barrier u of an end E as in (4.2). Then*

$$M(R) = \sup_{E \cap (B_p(2R) \setminus B_p(R))} |\nabla u|$$

satisfies

$$M^2(R) \leq CR \left(\inf_{x \in E \cap (B_p(2R) \setminus B_p(R))} \mathrm{V}_x\left(\frac{R}{2}\right) \right)^{-1} \left(M\left(\frac{R}{2}\right) + M(R) + M(2R) \right)$$

for a constant C depending only on k and for all $R > 2r_0$.

Proof. We follow an argument in [29]. Pick $x \in E \cap \partial B_p(t)$, where $R \leq t \leq 2R$, such that $M(R) = |\nabla u|(x)$. On $B_x\left(\frac{R}{2}\right)$ we have $\mathrm{Ric} \geq -\frac{C}{R^2}$ and $\Delta |\nabla u| \geq -\frac{C}{R^2} |\nabla u|$. Applying the mean value inequality from [19] we get

$$|\nabla u|^2(x) \leq \frac{C}{\mathrm{V}_x\left(\frac{R}{2}\right)} \int_{B_x\left(\frac{R}{2}\right)} |\nabla u|^2,$$

for a constant C independent of R. This shows that

$$M^2(R) \leq \frac{C}{\mathrm{V}_x\left(\frac{R}{2}\right)} \int_{L(\alpha_x, \beta_x)} |\nabla u|^2, \tag{4.3}$$

where

$$\alpha_x = \min_{B_x\left(\frac{R}{2}\right)} u \text{ and } \beta_x = \max_{B_x\left(\frac{R}{2}\right)} u.$$

By the co-area formula and (4.2) we have

$$\int_{L(\alpha_x, \beta_x)} |\nabla u|^2 = \int_{\alpha_x}^{\beta_x} \int_{l(t)} |\nabla u|\,dt = \beta_x - \alpha_x \leq R \sup_{B_x\left(\frac{R}{2}\right)} |\nabla u|.$$

Since $B_x\left(\frac{R}{2}\right) \subset B_p\left(\frac{3R}{2}\right) \setminus B_p\left(\frac{R}{2}\right)$, it follows that

$$\int_{L(\alpha_x,\beta_x)} |\nabla u|^2 \leq R\left(M\left(\frac{R}{2}\right) + M(R) + M(2R)\right).$$

Together with (4.3) we conclude that

$$M^2(R) \leq \frac{CR}{\mathrm{V}_x\left(\frac{R}{2}\right)}\left(M\left(\frac{R}{2}\right) + M(R) + M(2R)\right)$$

for some $x \in E \cap (B_p(2R) \setminus B_p(R))$ and all $R > 2r_0$. This proves the result. $\square$

We are now ready to prove Theorem 1.4.

Theorem 4.3. *Let $\left(M^3, g\right)$ be a three-dimensional complete noncompact Riemannian manifold with finitely many ends and finite first Betti number $b_1(M) < \infty$. Assume that the Ricci curvature of M is bounded from below by $\mathrm{Ric}(x) \geq -k(r(x))$ for a continuous non-increasing function $k(r)$ satisfying $\int_0^\infty rk(r)\,dr < \infty$. Then there exists a constant $C_0 > 0$, depending only on k, such that*

$$\liminf_{x\to\infty} S(x) \leq \frac{C_0}{\mathrm{V}_p(1)},$$

where $\mathrm{V}_p(1)$ is the volume of the geodesic ball $B_p(1)$.

Proof. Let us assume by contradiction that

$$\liminf_{x\to\infty} S(x) > \frac{\Gamma}{\mathrm{V}_p(1)}, \tag{4.4}$$

where $\Gamma > 0$ is a large enough constant to be specified later. In particular,

$$S \geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p(1)} \quad \text{on } M \setminus B_p(R_0) \tag{4.5}$$

for some large enough R_0. Below, C denotes a constant that depends only on k.

By Theorem 3.5 we know that M is parabolic. By localizing a barrier function to an end as above, we may assume without loss of generality that M has only one end. According to Nakai [28], there exists a proper harmonic function $u : M\setminus D \to (0,\infty)$ with $u = 0$ on ∂D. We normalize u as in (4.2).

Since M has asymptotically nonnegative Ricci curvature, a volume comparison result in [21] implies

$$\mathrm{V}_x\left(\frac{R}{2}\right) \geq \frac{1}{C} R\mathrm{V}_p(1)$$

for any $x \in \partial B_p(R)$. Choosing $R_0 > 0$ large enough so that $D \subset B_p(R_0)$ and applying Lemma 4.2 we get

$$M^2(R) \leq \frac{C_1}{\mathrm{V}_p(1)}\left(M\left(\frac{R}{2}\right) + M(R) + M(2R)\right) \tag{4.6}$$

for all $R > 2R_0$, where C_1 is a constant depending only on the function k.

On the other hand, by Lemma 3.2 we know that $|\nabla \ln u| \leq \frac{C}{r}$ on $M \setminus B_p(2R_0)$. It follows that

$$u \leq r^C \sup_{\partial B_p(2R_0)} u \quad \text{on } M \setminus B_p(2R_0)$$

and furthermore that

$$|\nabla u| \leq r^C \sup_{\partial B_p(2R_0)} u \quad \text{on } M \setminus B_p(2R_0).$$

In particular, there exists C_2 with $C_2 > 2C_1$, where C_1 is the constant in (4.6), and there exists $R_1 > 2R_0$ sufficiently large, such that

$$M(R) \leq \frac{1}{\mathrm{V}_p(1)} R^{\frac{C_2}{2}} \tag{4.7}$$

for all $R \geq R_1$.

We now claim that for $m \geq 1$,

$$M(R) \leq \frac{2^{C_2}}{\mathrm{V}_p(1)} R^{\frac{C_2}{2^m}} \tag{4.8}$$

for any $R \geq 2^m R_1$.

By (4.7) we see that (4.8) holds for $m = 1$. We assume by induction that it holds for m and prove it for $m+1$. In other words, we aim to prove that

$$M(R) \leq \frac{2^{C_2}}{\mathrm{V}_p(1)} R^{\frac{C_2}{2^{m+1}}} \tag{4.9}$$

for all $R \geq 2^{m+1} R_1$. By the induction hypothesis (4.8) we get that

$$\begin{aligned} M\left(\frac{R}{2}\right) &\leq \frac{2^{C_2}}{\mathrm{V}_p(1)} \left(\frac{R}{2}\right)^{\frac{C_2}{2^m}}, \\ M(R) &\leq \frac{2^{C_2}}{\mathrm{V}_p(1)} R^{\frac{C_2}{2^m}}, \\ M(2R) &\leq \frac{2^{C_2}}{\mathrm{V}_p(1)} (2R)^{\frac{C_2}{2^m}}. \end{aligned}$$

Using (4.6) we therefore obtain

$$\begin{aligned} M^2(R) &\leq \frac{2^{C_2} C_1}{(\mathrm{V}_p(1))^2} \left(\left(\frac{1}{2}\right)^{\frac{C_2}{2^m}} + 1 + 2^{\frac{C_2}{2^m}} \right) R^{\frac{C_2}{2^m}} \\ &\leq \frac{2^{2C_2}}{(\mathrm{V}_p(1))^2} R^{\frac{C_2}{2^m}}, \end{aligned}$$

where the second line follows from $C_2 > 2C_1$. In conclusion,

$$M(R) \leq \frac{2^{C_2}}{\mathrm{V}_p(1)} R^{\frac{C_2}{2^{m+1}}}$$

and (4.9) is proved. Hence, (4.8) holds for any $m \geq 1$ and $R \geq 2^m R_1$. By taking $2^m = [\ln R]$, this readily implies that

$$M(R) \leq \frac{C}{\mathrm{V}_p(1)}$$

for all $R \geq R_1^2$. In other words,

$$\sup_{M \setminus B_p(R_2)} |\nabla u| \leq \frac{C}{\mathrm{V}_p(1)} \tag{4.10}$$

for sufficiently large $R_2 > R_1$, where the constant C depends only on the function k. It in turn implies that

$$u(x) \leq \frac{C}{\mathrm{V}_p(1)} r(x) + C(R_2) \quad \text{on } M \setminus B_p(R_2) \tag{4.11}$$

for some constant $C(R_2)$ depending on R_2. By (4.11) we conclude that

$$B_p(R)\setminus B_p(R_2)\subset L\left(1,\frac{C}{\mathrm{V}_p(1)}R+C(R_2)\right).$$

According to Lemma 4.1 we have

$$\begin{aligned}\int_{B_p(R)\setminus B_p(R_2)} S\,|\nabla u| &\leq \int_{L\left(1,\frac{C}{\mathrm{V}_p(1)}R+C(R_2)\right)} S\,|\nabla u| \\ &\leq \frac{C}{\mathrm{V}_p(1)}R+C(R_2)+\Upsilon\end{aligned} \tag{4.12}$$

for all $R>R_2$, where Υ is a constant.

Now by (4.5) it follows that $S\geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p(1)}$ on $B_p(R)\setminus B_p(R_2)$, whereas by (4.2) we have $\int_{\partial B_p(r)}\frac{\partial u}{\partial r}=1$, for all $r>R_2$. The co-area formula then implies

$$\begin{aligned}\int_{B_p(R)\setminus B_p(R_2)} S\,|\nabla u| &\geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p(1)}\int_{B_p(R)\setminus B_p(R_2)}|\nabla u| \\ &\geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p(1)}\int_{R_2}^{R}\left(\int_{\partial B_p(r)}\frac{\partial u}{\partial r}\right)dr \\ &\geq \frac{1}{2}\frac{\Gamma}{\mathrm{V}_p(1)}(R-R_2).\end{aligned}$$

From (4.12) we get

$$\frac{\Gamma-C}{\mathrm{V}_p(1)}R\leq C(R_2)+\Upsilon.$$

Since $R>R_2$ is arbitrary, this forces $\Gamma\leq C$ and the theorem is proved. $\square$

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