

HECKE ALGEBRAS FOR TAME SUPERCUSPIDAL TYPES

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ABSTRACT. Let F be a non-archimedean local field of residue characteristic $p \neq 2$. Let G be a connected reductive group over F that splits over a tamely ramified extension of F . In [Yu01], Yu constructed types which are called *tame supercuspidal types* and conjectured that Hecke algebras associated with these types are isomorphic to Hecke algebras associated with depth-zero types of some twisted Levi subgroups of G . In this paper, we prove this conjecture. We also prove that the Hecke algebra associated with a *regular supercuspidal type* is isomorphic to the group algebra of a certain abelian group.

1. INTRODUCTION

Let F be a non-archimedean local field of residue characteristic $p \neq 2$ and G be a connected reductive group over F that splits over a tamely ramified extension of F . As explained in [Ber84], the category $\mathcal{R}(G(F))$ of smooth complex representations of $G(F)$ is decomposed into a product $\prod_{[M, \sigma]_G} \mathcal{R}^{[M, \sigma]_G}(G(F))$ of full subcategories $\mathcal{R}^{[M, \sigma]_G}(G(F))$, called *Bernstein blocks*. Bernstein blocks are parametrized by inertial equivalence classes $[M, \sigma]_G$ of cuspidal pairs. Each block $\mathcal{R}^{[M, \sigma]_G}(G(F))$ is equivalent to the category of modules over an algebra if $[M, \sigma]_G$ has an associated type as explained below. Let K be a compact open subgroup of $G(F)$, (ρ, W) be an irreducible representation of K , and \mathfrak{s} be an inertial equivalence class of a cuspidal pair. We say that (K, ρ) is an \mathfrak{s} -type if $\mathcal{R}^{\mathfrak{s}}(G(F))$ is precisely the full subcategory of $\mathcal{R}(G(F))$ consisting of smooth representations which are generated by their ρ -isotypic components. In this case, $\mathcal{R}^{\mathfrak{s}}(G(F))$ is equivalent to the category of modules over the Hecke algebra $\mathcal{H}(G(F), \rho)$ associated with (K, ρ) [BK98, Theorem 4.3]. Therefore, to construct types and determine the structure of Hecke algebras associated with the types are essential to understand the category $\mathcal{R}(G(F))$.

In [MP94] and [MP96], Moy and Prasad defined the notion of *depth* of types and constructed *depth-zero types*. Hecke algebras associated with depth-zero types were calculated in [Mor93]. In [Mor93], Morris gave the generators and relations for Hecke algebras associated with depth-zero types [Mor93, Theorem 7.12].

In [Yu01], Yu constructed types of general depth which are called *tame supercuspidal types*. His construction starts with a tuple $(\vec{G}, y, \vec{r}, {}^\circ\rho_{-1}, \vec{\phi})$, out of which it produces a sequence of types $({}^\circ K^i, {}^\circ\rho_i)$ in $G^i(F)$, where $\vec{G} = (G^0 \subsetneq G^1 \subsetneq \dots \subsetneq G^d = G)$ is a sequence of twisted Levi subgroups of G . Yu conjectured that the Hecke algebras associated with $({}^\circ K^i, {}^\circ\rho_i)$ are all isomorphic [Yu01, Conjecture 0.2]. In particular, Hecke algebras associated with these types are isomorphic to Hecke algebras associated with depth-zero types, which are studied in [Mor93] as explained above.

In [Mis19], Mishra proved [Yu01, conjecture 0.2] under some conditions [Mis19, Theorem 6.4]. Following to this result, Adler and Mishra proved [Yu01, conjecture 0.2] under similar conditions [AM21, Corollary 6.4]. However, these results cover only the cases that Hecke algebras are commutative. In this paper, we prove [Yu01, Conjecture 0.2] without any assumptions. This is the first topic of this paper.

The second topic of this paper is on *regular supercuspidal types*. In [Kal19], Kaletha defined and constructed a large class of supercuspidal representations which he calls *regular*. Kaletha's construction starts with a regular tame elliptic pair (S, θ) , where S is a tame elliptic maximal torus, and θ is a character of $S(F)$ which satisfy some conditions [Kal19, Definition 3.7.5]. As explained in the paragraph following [Kal19, Definition 3.7.3], “most” supercuspidal representations are regular when p is not too small. In this paper, we define and construct *regular supercuspidal types*, which are constructed by the same data (S, θ) as Kaletha's construction of regular supercuspidal representations. The regular supercuspidal representation constructed by (S, θ) contains

the regular supercuspidal type constructed by the same (S, θ) . We prove that the Hecke algebra associated with the type constructed by (S, θ) is isomorphic to the group algebra of the quotient of $S(F)$ by the unique maximal compact subgroup. This result is proved independent of [Yu01, Conjecture 0.2] and the work of [Mor93].

We sketch the outline of this paper. In Section 3, we review Yu's construction of supercuspidal representations briefly. In Section 4, we prove [Yu01, Conjecture 0.2]. In Section 5, we review Kaletha's construction of regular supercuspidal representations, define regular supercuspidal types, and determine the structure of Hecke algebras associated with regular supercuspidal types.

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2. NOTATION AND ASSUMPTIONS

Let F be a non-archimedean local field of residue characteristic p , k_F be its residue field, and G be a connected reductive group over F that splits over a tamely ramified extension of F . We denote by $Z(G)$ the center of G and by G_{der} the derived subgroup of G . We assume that p is an odd prime.

We denote by $\mathcal{B}(G, F)$ the enlarged Bruhat–Tits building of G over F . If T is a maximal, maximally split torus of $G_E := G \times_F E$ for a field extension E over F , then $\mathcal{A}(T, E)$ denotes the apartment of T inside the Bruhat–Tits building $\mathcal{B}(G_E, E)$ of G_E over E . For any $y \in \mathcal{B}(G, F)$, we denote by $[y]$ the projection of y on the reduced building and by $G(F)_y$ (resp. $G(F)_{[y]}$) the subgroup of $G(F)$ fixing y (resp. $[y]$). For $y \in \mathcal{B}(G, F)$ and $r \in \widetilde{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{r+ \mid r \in \mathbb{R}_{\geq 0}\}$, we write $G(F)_{y,r}$ for the Moy–Prasad filtration subgroup of $G(F)$ of depth r [MP94, MP96].

Suppose that K is a subgroup of $G(F)$ and $g \in G(F)$. We denote gKg^{-1} by gK . If ρ is a smooth representation of K , ${}^g\rho$ denotes the representation $x \mapsto \rho(g^{-1}xg)$ of gK . If $\text{Hom}_{K \cap {}^gK}({}^g\rho, \rho)$ is non-zero, we say g intertwines ρ .

3. REVIEW OF YU'S CONSTRUCTION

In this section, we recall Yu's construction of supercuspidal representations and supercuspidal types of $G(F)$ [Yu01].

An input for Yu's construction of supercuspidal representations of $G(F)$ is a tuple $(\vec{G}, y, \vec{r}, \rho_{-1}, \vec{\phi})$ where

- D1:** $\vec{G} = (G^0 \subsetneq G^1 \subsetneq \dots \subsetneq G^d = G)$ is a sequence of twisted Levi subgroups of G that split over a tamely ramified extension of F , i.e., there exists a tamely ramified extension E of F such that G_E^i is split for $0 \leq i \leq d$, and $(G_E^0 \subsetneq G_E^1 \subsetneq \dots \subsetneq G_E^d = G_E)$ is a split Levi sequence in G_E in the sense of [Yu01, Section 1]; we assume that $Z(G^0)/Z(G)$ is anisotropic;
- D2:** y is a point in $\mathcal{B}(G^0, F) \cap \mathcal{A}(T, E)$ whose projection on the reduced building of $G^0(F)$ is a vertex, where T is a maximal torus of G^0 (hence of G^i) whose splitting field E is a tamely ramified extension of F ; we denote by $\Phi(G^i, T, E)$ the corresponding root system of G^i for $0 \leq i \leq d$;
- D3:** $\vec{r} = (r_0, \dots, r_d)$ is a sequence of real numbers satisfying

$$\begin{cases} 0 < r_0 < r_1 < \dots < r_{d-1} \leq r_d & (d > 0), \\ 0 \leq r_0 & (d = 0); \end{cases}$$

- D4:** ρ_{-1} is an irreducible representation of $G^0(F)_{[y]}$ such that $\rho_{-1} \upharpoonright_{G^0(F)_{y,0}}$ is the inflation of a cuspidal representation of $G^0(F)_{y,0}/G^0(F)_{y,0+}$;

- D5:** $\vec{\phi} = (\phi_0, \dots, \phi_d)$ is a sequence of characters, where ϕ_i is a character of $G^i(F)$; we assume that ϕ_i is trivial on $G^i(F)_{y,r_i+}$ but non-trivial on $G^i(F)_{y,r_i}$ for $0 \leq i \leq d-1$. If $r_{d-1} < r_d$, we assume that ϕ_d is trivial on $G^d(F)_{y,r_d+}$ but non-trivial on $G^d(F)_{y,r_d}$, otherwise we assume that $\phi_d = 1$. Moreover, we assume that ϕ_i is G^{i+1} -generic of depth r_i relative to y in the sense of [Yu01, Section. 9] for $0 \leq i \leq d-1$.

Using the datum, we define

$$\begin{cases} K^i = G^0(F)_{[y]} G^1(F)_{y,r_0/2} \cdots G^i(F)_{y,r_{i-1}/2}, \\ K_+^i = G^0(F)_{y,0+} G^1(F)_{y,(r_0/2)+} \cdots G^i(F)_{y,(r_{i-1}/2)+} \end{cases}$$

for $0 \leq i \leq d$. We also define subgroups J^i, J_+^i of G for $1 \leq i \leq d$ as follows. For $\alpha \in \Phi(G, T, E)$, let $U_\alpha = U_{T,\alpha}$ denote the root subgroup of G corresponding to α . We set $U_0 = T$. For $x \in \mathcal{B}(G, F)$, $\alpha \in \Phi(G, T, E) \cup \{0\}$, and $r \in \mathbb{R}_{\geq 0}$, let $U_\alpha(E)_{x,r}$ denote the Moy–Prasad filtration subgroup of $U_\alpha(E)$ of depth r [MP94, MP96]. We define

$$\begin{cases} J^i = G(F) \cap \langle U_\alpha(E)_{y,r_{i-1}}, U_\beta(E)_{y,r_{i-1}/2} \mid \alpha \in \Phi(G^{i-1}, T, E) \cup \{0\}, \beta \in \Phi(G^i, T, E) \setminus \Phi(G^{i-1}, T, E) \rangle, \\ J_+^i = G(F) \cap \langle U_\alpha(E)_{y,r_{i-1}}, U_\beta(E)_{y,(r_{i-1}/2)+} \mid \alpha \in \Phi(G^{i-1}, T, E) \cup \{0\}, \beta \in \Phi(G^i, T, E) \setminus \Phi(G^{i-1}, T, E) \rangle \end{cases}$$

for $1 \leq i \leq d$. As explained in [Yu01, Section 1], J^i and J_+^i are independent of the choice of a maximal torus T of G^0 so that T splits over a tamely ramified extension E of F and $y \in \mathcal{A}(T, E)$.

Yu constructed irreducible representations ρ_i and ρ'_i of K^i for $0 \leq i \leq d$ inductively. First, we put $\rho'_0 = \rho_{-1}, \rho_0 = \rho'_0 \otimes \phi_0$.

Suppose that ρ_{i-1} and ρ'_{i-1} are already constructed, and $\rho'_{i-1} \upharpoonright_{G^{i-1}(F)_{y,r_{i-1}}}$ is 1-isotypic. In [Yu01, Section 11], Yu defined a representation ϕ'_{i-1} of K^i using the theory of Weil representation. This representation only depends on ϕ_{i-1} . If $r_{i-1} < r_i$, $\phi'_{i-1} \upharpoonright_{G^i(F)_{y,r_i}}$ is 1-isotypic. Let $\inf(\rho'_{i-1})$ be the inflation of ρ'_{i-1} via the map $K^i = K^{i-1} J^i \rightarrow K^{i-1} J^i / J^i \simeq K^{i-1} / G^{i-1}(F)_{y,r_{i-1}}$. Now we define $\rho'_i = \inf(\rho'_{i-1}) \otimes \phi'_{i-1}$, which is trivial on $G^i(F)_{y,r_i}$ if $r_{i-1} < r_i$. Finally, we define $\rho_i = \rho'_i \otimes \phi_i$.

We explain the construction of ϕ'_{i-1} . Let $\langle \cdot, \cdot \rangle_i$ be a pairing on J^i / J_+^i defined by $\langle a, b \rangle_i = \hat{\phi}_{i-1}(aba^{-1}b^{-1})$. Here, $\hat{\phi}_{i-1}$ denotes an extension of $\phi_{i-1} \upharpoonright_{K^0 G^{i-1}(F)_{y,0}}$ to $K^0 G^{i-1}(F)_{y,0} G(F)_{y,(r_{i-1}/2)+}$ defined in [Yu01, Section 4]. The pairing is well-defined because by [BT72, Proposition 6.4.44], $[J^i, J^i]$ is contained in J_+^i . Note that since the order of every element in J^i / J_+^i divides p , we can regard J^i / J_+^i as an \mathbb{F}_p -vector space. By [Yu01, Lemma 11.1], this pairing is non-degenerate on J^i / J_+^i . In addition, by the construction of $\hat{\phi}_{i-1}$, for $j \in J_+^i$, j^p is contained in $\text{Ker}(\hat{\phi}_{i-1})$. Therefore, the order of every element in $\hat{\phi}_{i-1}(J_+^i)$ divides p , and since $\hat{\phi}_{i-1}(J_+^i)$ is a non-trivial subgroup of \mathbb{C}^\times , this implies that $\hat{\phi}_{i-1}(J_+^i)$ is isomorphic to \mathbb{F}_p . Hence we can regard $\langle \cdot, \cdot \rangle_i$ as a non-degenerate \mathbb{F}_p -valued pairing on J^i / J_+^i and J^i / J_+^i as a symplectic space over \mathbb{F}_p . For a symplectic space $(V, \langle \cdot, \cdot \rangle)$ over \mathbb{F}_p , we define the Heisenberg group $V^\#$ of V to be the set $V \times \mathbb{F}_p$ with the group law $(v, a)(w, b) = (v + w, a + b + \frac{1}{2} \langle v, w \rangle)$. Yu constructed a canonical isomorphism $j: J^i / (J_+^i \cap \text{Ker}(\hat{\phi}_{i-1})) \rightarrow (J^i / J_+^i) \times J_+^i / (J_+^i \cap \text{Ker}(\hat{\phi}_{i-1})) \simeq (J^i / J_+^i)^\#$ in [Yu01, Proposition 11.4]. Combining this isomorphism and the map $K^{i-1} \rightarrow \text{Sp}(J^i / J_+^i)$ induced by the conjugation, we define $K^{i-1} \ltimes J^i \rightarrow K^{i-1} \ltimes (J^i / (J_+^i \cap \text{Ker}(\hat{\phi}_{i-1}))) \rightarrow \text{Sp}(J^i / J_+^i) \ltimes (J^i / J_+^i)^\#$. Combining this map and the Weil representation of $\text{Sp}(J^i / J_+^i) \ltimes (J^i / J_+^i)^\#$ associated with the central character $\hat{\phi}_{i-1}$, we construct a representation $\tilde{\phi}_{i-1}$ of $K^{i-1} \ltimes J^i$. Let $\inf(\phi_{i-1})$ be the inflation of ϕ_{i-1} via the map $K^{i-1} \ltimes J^i \rightarrow K^{i-1}$, then $\inf(\phi_{i-1}) \otimes \tilde{\phi}_{i-1}$ factors through the map $K^{i-1} \ltimes J^i \rightarrow K^{i-1} J^i = K^i$. We define ϕ'_{i-1} be the representation of K^i whose inflation to $K^{i-1} \ltimes J^i$ is $\inf(\phi_{i-1}) \otimes \tilde{\phi}_{i-1}$.

For open subgroups K_1, K_2 of $G^i(F)$ and a representation ρ of K_1 , let $\text{ind}_{K_1}^{K_2} \rho$ denote the compactly induced representation of K_2 .

Theorem 3.1 ([Yu01, Theorem 15.1]). *The representation $\pi_i = \text{ind}_{K^i}^{G^i(F)} \rho_i$ of $G^i(F)$ is an irreducible supercuspidal representation of depth r_i for $0 \leq i \leq d$.*

For the proof of this theorem, Yu uses [Yu01, Proposition 14.1] and [Yu01, Theorem 14.2], which are pointed out in [Fin19] to be false. However, Fintzen uses an alternative approach in [Fin19] and proves [Yu01, Theorem 15.1] without using [Yu01, Proposition 14.1] or [Yu01, Theorem 14.2].

Next, we review Yu's construction of supercuspidal types. We start with a datum $(\vec{G}, y, \vec{r}, \circ\rho_{-1}, \vec{\phi})$ satisfying **D1**, **D2**, **D3**, **D5** and $\circ\mathbf{D4}$:

$\circ\mathbf{D4}$: $\circ\rho_{-1}$ is an irreducible representation of $G^0(F)_y$ such that $\circ\rho_{-1} \upharpoonright_{G^0(F)_{y,0}}$ is the inflation of a cuspidal representation of $G^0(F)_{y,0}/G^0(F)_{y,0+}$,

instead of **D4**. We then follow the above construction replacing K^i with

$${}^\circ K^i = G^0(F)_y G^1(F)_{y, r_0/2} \cdots G^i(F)_{y, r_{i-1}/2},$$

and construct an irreducible representation ${}^\circ \rho_i$ of ${}^\circ K^i$.

Proposition 3.2. *Let $(\vec{G}, y, \vec{r}, \rho_{-1}, \vec{\phi})$ be a datum satisfying **D1**, **D2**, **D3**, **D4**, **D5** and ${}^\circ \rho_{-1}$ be an irreducible representation of $G^0(F)_y$ which is contained in $\rho_{-1} \upharpoonright_{G^0(F)_y}$. Then $(\vec{G}, y, \vec{r}, {}^\circ \rho_{-1}, \vec{\phi})$ satisfies **D1**, **D2**, **D3**, **{}^\circ D4**, **D5** and the representation ${}^\circ \rho_i$ constructed above is an \mathfrak{s}_i -type in the sense of [BK98], where \mathfrak{s}_i is the inertial equivalence class of $[G^i, \pi_i]_{G^i}$.*

Proof. Note that ${}^\circ K^i$ is the unique maximal compact subgroup of K^i , and ${}^\circ \rho_i$ is contained in $\rho_i \upharpoonright_{{}^\circ K^i}$. Then this proposition follows from [BK98, Proposition 5.4]. \square

Types obtained in this way are called *tame supercuspidal types*. In the following, we write $K^d, K_+^d, {}^\circ K^d, \rho_d, {}^\circ \rho_d, \pi_d$ simply by $K, K_+, {}^\circ K, \rho, {}^\circ \rho, \pi$, respectively.

4. HECKE ALGEBRA ISOMORPHISM

Let K be a compact open subgroup of $G(F)$ and (ρ, W) be an irreducible representation of K . We define a Hecke algebra $\mathcal{H}(G(F), \rho)$ associated with (K, ρ) as in [BK98, Section 2] and write $\check{\mathcal{H}}(G(F), \rho)$ for $\mathcal{H}(G(F), \check{\rho})$, where $\check{\rho}$ is the contragredient of ρ . So, $\check{\mathcal{H}}(G(F), \rho)$ is the \mathbb{C} -vector space of compactly supported functions $\Phi : G(F) \rightarrow \text{End}_{\mathbb{C}}(W)$ satisfying

$$\Phi(k_1 g k_2) = \rho(k_1) \circ \Phi(g) \circ \rho(k_2), \quad k_i \in K, g \in G(F),$$

and for $\Phi_1, \Phi_2 \in \check{\mathcal{H}}(G(F), \rho)$, the product $\Phi_1 * \Phi_2$ is defined by

$$(\Phi_1 * \Phi_2)(x) = \int_{G(F)} \Phi_1(y) \circ \Phi_2(y^{-1}x) dy.$$

Here, we normalize the Haar measure of $G(F)$ so that $\text{vol}(K) = 1$. If \mathfrak{s} is an inertial equivalence class of a cuspidal pair and (K, ρ) is an \mathfrak{s} -type, the Bernstein block associated with \mathfrak{s} is equivalent to the category of $\mathcal{H}(G(F), \rho)$ -modules [BK98, Theorem 4.3]. We restrict our attention to Hecke algebras associated with tame supercuspidal types, defined in Section 3.

Let $(\vec{G}, y, \vec{r}, \rho_{-1}, \vec{\phi})$ be a datum satisfying **D1**, **D2**, **D3**, **D4**, **D5** and ${}^\circ \rho_{-1}$ be an irreducible representation of $G^0(F)_y$ which is contained in $\rho_{-1} \upharpoonright_{G^0(F)_y}$. We construct a $[G, \pi]_G$ -type $({}^\circ K, {}^\circ \rho)$ as in Section 3.

We define

$$\text{supp}(\check{\mathcal{H}}(G(F), {}^\circ \rho)) = \bigcup_{f \in \check{\mathcal{H}}(G(F), {}^\circ \rho)} \text{supp}(f),$$

where $\text{supp}(f)$ denotes the support of f . We call it *the support of $\check{\mathcal{H}}(G(F), {}^\circ \rho)$* . Note that

$$\text{supp}(\check{\mathcal{H}}(G(F), {}^\circ \rho)) = \{g \in G(F) \mid g \text{ intertwines } {}^\circ \rho\}.$$

Proposition 4.1. *The support of $\check{\mathcal{H}}(G(F), {}^\circ \rho)$ is contained in ${}^\circ K G^0(F)_{[y]} {}^\circ K$. Moreover, an element $g \in G^0(F)_{[y]}$ intertwines ${}^\circ \rho$ if and only if g intertwines ${}^\circ \rho_{-1}$.*

Remark 4.2. By [Yu01, Remark 3.5], $G^0(F)_{[y]}$ normalizes ${}^\circ K$, so ${}^\circ K G^0(F)_{[y]} {}^\circ K = G^0(F)_{[y]} {}^\circ K$.

Proposition 4.1 follows easily from [Yu01, Corollary 15.5]. Indeed, according to [Yu01, Corollary 15.5], if $g \in G(F)$ intertwines ${}^\circ \rho$, then $g \in {}^\circ K G^0(F) {}^\circ K$, and an element $g \in G^0(F)$ intertwines ${}^\circ \rho$ if and only if g intertwines ${}^\circ \rho_{-1}$. Moreover, by using the argument in the proof of [MP96, Proposition 6.6], we can prove that if $g \in G^0(F)$ intertwines ${}^\circ \rho_{-1}$, then $g \in G^0(F)_{[y]}$. However [Yu01, Corollary 15.5] relies on [Yu01, Proposition 14.1] and [Yu01, Theorem 14.2], which are pointed out in [Fin19] to be false. In the following, we prove Proposition 4.1 using an argument by Fintzen in [Fin19, Theorem 3.1], which does not rely on [Yu01, Proposition 14.1] or [Yu01, Theorem 14.2].

For the first claim, it is enough to show that if $g \in G(F)$ intertwines ${}^\circ \rho$, then $g \in {}^\circ K G^0(F)_{[y]} {}^\circ K$. The first step is the following Lemma.

Lemma 4.3. *If $g \in G(F)$ intertwines ${}^\circ \rho$, then $g \in {}^\circ K G^0(F) {}^\circ K$.*

Proof. This follows from [Yu01, Proposition 4.1] and [Yu01, Proposition 4.4]. Note that [Yu01, Proposition 4.4] is also true if we replace ρ_i with ${}^\circ \rho$. \square

Next, we prove the following Lemma.

Lemma 4.4. *If $g \in G^0(F)$ intertwines ${}^\circ\rho$, then $g \in G^0(F)_{[y]}$.*

Proof. Let f be a nonzero element of $\text{Hom}_{{}^\circ K \cap {}^g({}^\circ K)}({}^g({}^\circ\rho), {}^\circ\rho)$. We write V_f for the image of f . By [Yu01, Proposition 4.4], ${}^\circ\rho \upharpoonright_{K_+}$ is θ_d -isotypic, where

$$\theta_d = \prod_{j=0}^d \hat{\phi}_j \upharpoonright_{K_+}.$$

This implies that $G^0(F)_{y,0+} (\subset K_+)$ acts on ${}^\circ\rho$ by θ_d , and ${}^g G^0(F)_{y,0+}$ acts on ${}^g({}^\circ\rho)$ by ${}^g\theta_d$. For $h \in G^0(F)_{y,0} \cap {}^g G^0(F)_{y,0+}$ and $0 \leq j \leq d$,

$${}^g\hat{\phi}_j(h) = \hat{\phi}_j(g^{-1}hg) = \phi_j(g^{-1}hg) = \phi_j(g)^{-1}\phi_j(h)\phi_j(g) = \phi_j(h) = \hat{\phi}_j(h).$$

Hence, $G^0(F)_{y,0} \cap {}^g G^0(F)_{y,0+}$ acts on ${}^g({}^\circ\rho)$ by θ_d . Therefore, if we let

$$U'_{-1} = (G^0(F)_{y,0} \cap {}^g G^0(F)_{y,0+}) G^0(F)_{y,0+},$$

U'_{-1} acts on V_f by θ_d .

From the construction, ${}^\circ\rho$ is decomposed as

$${}^\circ\rho = \bigotimes_{i=-1}^d V_i,$$

where V_{-1} is the inflation of ${}^\circ\rho_{-1}$ via the map

$${}^\circ K = G^0(F)_y J^1 \cdots J^d \rightarrow G^0(F)_y J^1 \cdots J^d / J^1 \cdots J^d \simeq G^0(F)_y / G^0(F)_{y,r_0},$$

V_i is the inflation of ϕ'_i via the map

$${}^\circ K = {}^\circ K^{i+1} J^{i+2} \cdots J^d \rightarrow {}^\circ K^{i+1} J^{i+2} \cdots J^d / J^{i+2} \cdots J^d = {}^\circ K^{i+1} / G^{i+1}(F)_{y,r_{i+1}},$$

for $0 \leq i \leq d-2$, $V_{d-1} = \phi'_{d-1}$, and $V_d = \phi_d$.

Let T be a maximal torus of G^0 so that T splits over a tamely ramified extension E of F and $y, g \cdot y \in \mathcal{A}(T, E)$. Such a torus exists by the fact that any two points of $\mathcal{B}(G^0, F)$ is contained in an apartment of $\mathcal{B}(G^0, F)$ and by the discussion in the beginning of [Yu01, Section 2]. We define

$$U_i = G(F) \cap \langle U_\alpha(E)_{y,r_i/2} \mid \alpha \in \Phi(G^{i+1}, T, E) \setminus \Phi(G^i, T, E), \alpha(y - g \cdot y) < 0 \rangle,$$

for $0 \leq i \leq d-1$. Since U_i is contained in J^{i+1} , the action of U_i on V_j is trivial for $-1 \leq j \leq i-1$. For, $i+1 \leq j$, U_i is contained in ${}^\circ K^j$, which acts on V_j by $\phi'_j \upharpoonright_{{}^\circ K^j} = \phi_j \otimes \tilde{\phi}_j$. Here, the action of ${}^\circ K^j$ by $\tilde{\phi}_j$ factors through the map ${}^\circ K^j \rightarrow \text{Sp}(J^{j+1}/J_+^{j+1})$ induced by the conjugation. By [BT72, Proposition 6.4.44],

$$[J^{j+1}, U_i] \subset [J^{j+1}, G^j(F)_{y,0+}] \subset J_+^{j+1}.$$

Therefore, the action of U_i by $\tilde{\phi}_j$ is trivial, hence U_i acts on V_j by ϕ_j . For $j = i$, U_i acts on V_i by the Heisenberg representation $\tilde{\phi}_i$ of J^{i+1}/J_+^{i+1} . Putting these arguments together, we see that the action of U_i by ${}^\circ\rho$ is

$$(\otimes_{j=-1}^{i-1} \text{Id}_{V_j}) \otimes \tilde{\phi}_i \otimes (\otimes_{j=i+1}^d \phi_j).$$

On the other hand, for $\alpha \in \Phi(G^{i+1}, T, E) \setminus \Phi(G^i, T, E)$ which satisfies $\alpha(y - g \cdot y) < 0$, we have

$$\begin{aligned} g^{-1} U_\alpha(E)_{y,r_i/2} &= g^{-1} U_\alpha(E)_{g \cdot y, (r_i/2) - \alpha(y - g \cdot y)} \\ &= U_{g^{-1}\alpha}(E)_{y, (r_i/2) - \alpha(y - g \cdot y)} \\ &\subset U_{g^{-1}\alpha}(E)_{y, (r_i/2) +}. \end{aligned}$$

Here, $g^{-1}\alpha$ denotes the character $t \mapsto \alpha(gt g^{-1})$ of $g^{-1}T$, and $U_{g^{-1}\alpha} = U_{g^{-1}T, g^{-1}\alpha}$ denotes the corresponding root subgroup. Since J_+^{i+1} is independent of the choice of a maximal torus, and $g^{-1}T$ is a maximal torus of G^0 so that $g^{-1}T$ splits over E and $y \in \mathcal{A}(g^{-1}T, E)$, $g^{-1}U_i \subset J_+^{i+1} \subset K_+^{i+1} \subset K_+^d$. As ${}^\circ\rho \upharpoonright_{K_+^d}$ is θ_d -isotypic, U_i acts on V_f by ${}^g\theta_d \upharpoonright_{U_i}$.

By the construction of $\hat{\phi}_j$ in [Yu01, Section 4], $\hat{\phi}_j$ is trivial on

$$G(F) \cap \langle U_{g^{-1}\alpha}(E)_{y, (r_j/2) +} \mid \alpha \in \Phi(G, T, E) \setminus \Phi(G^j, T, E) \rangle$$

for $0 \leq j \leq d-1$. Therefore, for $j \leq i$, $g^{-1}U_i$ is contained in $\text{Ker}(\hat{\phi}_j)$. This implies that

$${}^g\theta_d \upharpoonright_{U_i} = \prod_{j=i+1}^d {}^g\hat{\phi}_j \upharpoonright_{U_i} = \prod_{j=i+1}^d {}^g\phi_j \upharpoonright_{U_i} = \prod_{j=i+1}^d \phi_j \upharpoonright_{U_i}.$$

Hence, U_i acts on V_f by

$$(\otimes_{j=-1}^i \text{Id}_{V_j}) \otimes (\otimes_{j=i+1}^d \phi_j).$$

Comparing the action of U_i by ${}^\circ\rho$ and the action of U_i on V_f , we conclude that V_f is contained in $V_{-1} \otimes (\otimes_{i=0}^{d-1} V_i^{U_i}) \otimes V_d$.

We study the subspace $V_i^{U_i}$ for $0 \leq i \leq d-1$. Recall that V_i is the space of the Weil representation of $\text{Sp}(J^{i+1}/J_+^{i+1}) \ltimes (J^{i+1}/J_+^{i+1})^\#$. We write $W^{i+1} = J^{i+1}/J_+^{i+1}$. We define the subspace $(W^{i+1})_1$ to be the image of U_i in W^{i+1} , $(W^{i+1})_2$ to be the image of

$$G(F) \cap \langle U_\alpha(E)_{y,r_i/2} \mid \alpha \in \Phi(G^{i+1}, T, E) \setminus \Phi(G^i, T, E), \alpha(y - g \cdot y) = 0 \rangle$$

in W^{i+1} , and $(W^{i+1})_3$ to be the image of

$$G(F) \cap \langle U_\alpha(E)_{y,r_i/2} \mid \alpha \in \Phi(G^{i+1}, T, E) \setminus \Phi(G^i, T, E), \alpha(y - g \cdot y) > 0 \rangle$$

in W^{i+1} . Note that $(W^{i+1})_k$ is written by V_k in [Yu01, Section 13] for $1 \leq k \leq 3$. By [Yu01, Lemma 13.6], $(W^{i+1})_1, (W^{i+1})_3$ are totally isotropic subspaces of the symplectic space $(W^{i+1}, \langle \cdot, \cdot \rangle_{i+1})$ and

$$(W^{i+1})_1^\perp = (W^{i+1})_1 \oplus (W^{i+1})_2, (W^{i+1})_3^\perp = (W^{i+1})_2 \oplus (W^{i+1})_3.$$

Let P_{i+1} be the maximal parabolic subgroup of $\text{Sp}(W^{i+1})$ that preserves $(W^{i+1})_1$. Then we obtain the natural map

$$\iota: P_{i+1} \rightarrow \text{Sp}\left((W^{i+1})_1^\perp / (W^{i+1})_1\right) \simeq \text{Sp}\left((W^{i+1})_2\right).$$

We write $(\tilde{\phi}_i)_2$ for the Weil representation of $\text{Sp}\left((W^{i+1})_2\right) \ltimes ((W^{i+1})_2)^\#$ associated with the central character $\hat{\phi}_i$. By [Gér77, Theorem 2.4 (b)], the restriction of $\tilde{\phi}_i$ from $\text{Sp}\left(W^{i+1}\right) \ltimes (W^{i+1})^\#$ to $P_{i+1} \ltimes (W^{i+1})^\#$ is given by

$$\text{ind}_{P_{i+1} \ltimes ((W^{i+1})_1 \oplus ((W^{i+1})_2)^\#)}^{P_{i+1} \ltimes (W^{i+1})^\#} (\tilde{\phi}_i)_2 \otimes (\chi_{i+1} \ltimes 1).$$

Here, we regard $(\tilde{\phi}_i)_2$ be a representation of $P_{i+1} \ltimes ((W^{i+1})_1 \oplus ((W^{i+1})_2)^\#)$ by defining the action of $(W^{i+1})_1$ to be trivial and defining the action of P_{i+1} to be the composition of ι and $(\tilde{\phi}_i)_2$. The character χ_{i+1} of P_{i+1} is χ^{E+} of [Gér77, Lemma 2.3 (d)], which factors through the natural map $P_{i+1} \rightarrow \text{GL}((W^{i+1})_1)$. Since $(W^{i+1})_3$ is a complete system of representatives for

$$(P_{i+1} \ltimes (W^{i+1})^\#) / (P_{i+1} \ltimes ((W^{i+1})_1 \oplus ((W^{i+1})_2)^\#)),$$

as a representation of $P_{i+1} \ltimes ((W^{i+1})_1 \oplus ((W^{i+1})_2)^\#)$,

$$\text{ind}_{P_{i+1} \ltimes ((W^{i+1})_1 \oplus ((W^{i+1})_2)^\#)}^{P_{i+1} \ltimes (W^{i+1})^\#} (\tilde{\phi}_i)_2 \otimes (\chi_{i+1} \ltimes 1) \simeq \bigoplus_{v_3 \in (W^{i+1})_3} v_3(\tilde{\phi}_i)_2 \otimes (\chi_{i+1} \ltimes 1).$$

Since $(W^{i+1})_1$ acts on $(\tilde{\phi}_i)_2$ trivially, $(W^{i+1})_1$ acts $v_3(\tilde{\phi}_i)_2$ by

$$v_1 \mapsto \hat{\phi}_i(v_3^{-1}v_1v_3v_1^{-1}) = \langle v_3^{-1}, v_1 \rangle_{i+1}.$$

We note that $(W^{i+1})_3^\perp = (W^{i+1})_2 \oplus (W^{i+1})_3$. Hence, for every element $v_3 \in (W^{i+1})_3$ there exists $v_1 \in (W^{i+1})_1$ such that $\langle v_3^{-1}, v_1 \rangle_{i+1} \neq 0$. Therefore,

$$\left(\text{ind}_{P_{i+1} \ltimes ((W^{i+1})_1 \oplus ((W^{i+1})_2)^\#)}^{P_{i+1} \ltimes (W^{i+1})^\#} (\tilde{\phi}_i)_2 \otimes (\chi_{i+1} \ltimes 1) \right)^{\{1\} \ltimes ((W^{i+1})_1 \times \{0\})} \simeq (\tilde{\phi}_i)_2 \otimes \chi_{i+1}$$

as a representation of P_{i+1} .

The image of U_i via the special isomorphism constructed in [Yu01, Proposition 11.4] is $(W^{i+1})_1 \times \{0\} \subset (W^{i+1})^\#$. Therefore P_{i+1} acts on $V_i^{U_i}$ by $(\tilde{\phi}_i)_2 \otimes \chi_{i+1}$.

We define

$$U_{-1} = G(F) \cap \langle U_\alpha(E)_{y,0} \mid \alpha \in \Phi(G^0, T, E), \alpha(y - g \cdot y) < 0 \rangle.$$

Then, U'_{-1} is contained in $U_{-1}G^0(F)_{y,0+}$. By [BT72, Proposition 6.4.44],

$$[J^{i+1}, G(F)_{y,0+}] \subset J_+^{i+1},$$

so the image of $G^0(F)_{y,0+}$ in $\mathrm{Sp}(W^{i+1})$ is trivial. Also, by [BT72, Proposition 6.4.44],

$$[G^{i+1}(F)_{y,f(1,2)}, U_{-1}] \subset G^{i+1}(F)_{y,f_1},$$

where $f(1,2)$ and f_1 are functions on $\Phi(G^{i+1}, T, E) \cup \{0\}$ defined by

$$f_{(1,2)}(\alpha) = \begin{cases} r_i & (\alpha \in \Phi(G^i, T, E) \cup \{0\}) \\ \frac{r_i}{2} & (\alpha \in \Phi(G^{i+1}, T, E) \setminus \Phi(G^i, T, E), \alpha(y - g \cdot y) \leq 0) \\ (\frac{r_i}{2})_+ & (\alpha \in \Phi(G^{i+1}, T, E) \setminus \Phi(G^i, T, E), \alpha(y - g \cdot y) > 0) \end{cases},$$

$$f_1(\alpha) = \begin{cases} r_i & (\alpha \in \Phi(G^i, T, E) \cup \{0\}) \\ \frac{r_i}{2} & (\alpha \in \Phi(G^{i+1}, T, E) \setminus \Phi(G^i, T, E), \alpha(y - g \cdot y) < 0) \\ (\frac{r_i}{2})_+ & (\alpha \in \Phi(G^{i+1}, T, E) \setminus \Phi(G^i, T, E), \alpha(y - g \cdot y) \geq 0) \end{cases}$$

and for $h = f_{(1,2)}, f_1$,

$$G^{i+1}(F)_{y,h} = G(F) \cap \langle U_\alpha(E)_{y,h(\alpha)} \mid \alpha \in \Phi(G^{i+1}, T, E) \cup \{0\} \rangle.$$

Note that the image of $G^{i+1}(F)_{y,f(1,2)}$ (resp. $G^{i+1}(F)_{y,f_1}$) in W^{i+1} is $(W^{i+1})_1 \oplus (W^{i+1})_2$ (resp. $(W^{i+1})_1$). Therefore, the image of U_{-1} in $\mathrm{Sp}(W^{i+1})$ is contained in P_{i+1} , and the image of U_{-1} via the map

$$\iota: P_{i+1} \rightarrow \mathrm{Sp}\left((W^{i+1})_1^\perp / (W^{i+1})_1\right) \simeq \mathrm{Sp}\left((W^{i+1})_2\right)$$

is trivial. Moreover, since U_{-1} is a pro- p subgroup of $G^0(F)$, the image in $\mathrm{GL}((W^{i+1})_1)$ under the natural map $P_{i+1} \rightarrow \mathrm{GL}((W^{i+1})_1)$ is a p -group, hence contained in the commutator subgroup of $\mathrm{GL}((W^{i+1})_1)$. Therefore, χ_{i+1} is trivial on the image of U_{-1} . These arguments imply that $\tilde{\phi}_i(U'_{-1})$ is trivial for $0 \leq i \leq d-1$, so the action of U'_{-1} on $(\otimes_{i=0}^{d-1} V_i^{U_i}) \otimes V_d$ is θ_d -isotypic. Since U'_{-1} acts on V_f by θ_d and V_f is contained in $V_{-1} \otimes (\otimes_{i=0}^{d-1} V_i^{U_i}) \otimes V_d$, this implies that $V_{-1}^{U'_{-1}}$ is nonzero. As $[y]$ is a vertex, if $g \notin G^0(F)_{[y]}$, the image of U'_{-1} in $G^0(F)_{y,0}/G^0(F)_{y,0+}$ is a unipotent radical of a proper parabolic subgroup of $G^0(F)_{y,0}/G^0(F)_{y,0+}$. This contradicts that ${}^\circ\rho_{-1} \upharpoonright_{G^0(F)_{y,0}}$ is the inflation of a cuspidal representation of $G^0(F)_{y,0}/G^0(F)_{y,0+}$. Therefore we obtain that $g \in G^0(F)_{[y]}$. \square

We prove the second claim of Proposition 4.1 using an argument in [Yu01, Proposition 4.6]. Since $G^0(F)_{[y]}$ is contained in K and V_i is a restriction of a representation of K to ${}^\circ K$ for $0 \leq i \leq d$, if $g \in G^0(F)_{[y]}$ intertwines ${}^\circ\rho_{-1}$, then g intertwines ${}^\circ\rho$. We prove that the converse is true. Assume that $g \in G^0(F)_{[y]}$ intertwines ${}^\circ\rho$. Since V_d is a restriction of a character of $G(F)$, g also intertwines $\otimes_{j=-1}^{d-1} V_j$. If $d = 0$ it implies that g intertwines ${}^\circ\rho_{-1}$. Suppose $d \geq 1$. We prove that g intertwines $\otimes_{j=-1}^{d-2} V_j$. Let f be a nonzero element of $\mathrm{Hom}_{{}^\circ K}({}^g({}^\circ\rho), {}^\circ\rho)$. We write $f = \sum_j f'_j \otimes f''_j$, where $f'_j \in \mathrm{End}_\mathbb{C}(\otimes_{i=-1}^{d-2} V_i)$ and $f''_j \in \mathrm{End}_\mathbb{C}(V_{d-1})$. We may assume that $\{f'_j\}$ is a linearly independent set.

Since the action of J^d on $\otimes_{i=-1}^{d-2} V_i$ is trivial, for $x \in J^d$ we obtain

$$\sum_j f'_j \otimes (f''_j \circ ({}^g\phi'_{d-1})(x)) = \sum_j f'_j \otimes (\phi'_{d-1}(x) \circ f''_j).$$

The linearly independence of $\{f'_j\}$ implies that $f''_j \in \mathrm{Hom}_{J^d}({}^g\phi'_{d-1}, \phi'_{d-1})$. By [Yu01, Proposition 12.3], $\mathrm{Hom}_{J^d}({}^g\phi'_{d-1}, \phi'_{d-1})$ is 1-dimensional, so we may assume that there is only one j . We write $f = f' \otimes f''$, where $f' \in \mathrm{End}_\mathbb{C}(\otimes_{i=-1}^{d-2} V_i)$ and $f'' \in \mathrm{Hom}_{J^d}({}^g\phi'_{d-1}, \phi'_{d-1})$. Since $g \in G^0(F)_{[y]} \subset K$, $\mathrm{Hom}_K({}^g\phi'_{d-1}, \phi'_{d-1}) = \mathrm{End}_K(\phi'_{d-1})$ is 1-dimensional, and it is a subspace of $\mathrm{Hom}_{J^d}({}^g\phi'_{d-1}, \phi'_{d-1})$, which is also 1-dimensional. Therefore $\mathrm{Hom}_{J^d}({}^g\phi'_{d-1}, \phi'_{d-1}) = \mathrm{Hom}_K({}^g\phi'_{d-1}, \phi'_{d-1})$, and f'' is K -equivariant. This implies that f' is a nonzero element in $\mathrm{Hom}_{{}^\circ K}({}^g(\otimes_{j=-1}^{d-2} V_j), \otimes_{j=-1}^{d-2} V_j)$, and g intertwines $\otimes_{j=-1}^{d-2} V_j$. Then, an inducting argument implies that g intertwines ${}^\circ\rho_{-1}$.

Remark 4.5. In the recent work [FKS21], a modification of Yu's construction called the twisted Yu construction was given by Fintzen, Kaletha, and Spice. The modification is obtained by twisting the data in the original construction by a sign character ϵ defined in [FKS21, 4.1]. They showed that the validity of [Yu01, Proposition 14.1] and [Yu01, Theorem 14.2] can be restored if we use the twisted construction instead of the original construction (see [FKS21, Corollary 4.1.11] and

[FKS21, Corollary 4.1.12]). Hence, the validity of [Yu01, Corollary 15.5] can also be restored, and we can prove Proposition 4.1 directly in this case.

We now prove [Yu01, Conjecture 0.2].

Theorem 4.6. *There is a support-preserving algebra isomorphism*

$$\check{\mathcal{H}}(G(F), {}^\circ\rho) \simeq \check{\mathcal{H}}(G^0(F), {}^\circ\rho_{-1}).$$

Here, we say that an isomorphism $\eta : \check{\mathcal{H}}(G(F), {}^\circ\rho) \rightarrow \check{\mathcal{H}}(G^0(F), {}^\circ\rho_{-1})$ is support-preserving if for every $f \in \check{\mathcal{H}}(G(F), {}^\circ\rho)$, $\text{supp}(f) = {}^\circ K \text{supp}(\eta(f)) {}^\circ K$.

Remark 4.7. According to [Mis19, Corollary 6.3], the center of $\check{\mathcal{H}}(G(F), {}^\circ\rho)$ is isomorphic to the center of $\check{\mathcal{H}}(G^0(F), {}^\circ\rho_{-1})$. In particular, [Yu01, Conjecture 0.2] holds if these Hecke algebras are commutative. However, these Hecke algebras are not commutative in general (see [HV15, 4.4]). Hence, our result is more general than [Mis19, Corollary 6.3].

Proof of Theorem 4.6. We set

$$G_{\circ\rho}^0 = \{g \in G^0(F)_{[y]} \mid g \text{ intertwines } {}^\circ\rho\} = \{g \in G^0(F)_{[y]} \mid g \text{ intertwines } {}^\circ\rho_{-1}\}.$$

The second equation follows from Proposition 4.1. By Proposition 4.1 and Remark 4.2, the support of $\check{\mathcal{H}}(G(F), {}^\circ\rho)$ is $G_{\circ\rho}^0 {}^\circ K$, and the support of $\check{\mathcal{H}}(G^0(F), {}^\circ\rho_{-1})$ is $G_{\circ\rho}^0 {}^\circ K^0$. Fix a complete system of representatives $\{g_i\}_i \subset G_{\circ\rho}^0$ for

$$G_{\circ\rho}^0 {}^\circ K / {}^\circ K = G_{\circ\rho}^0 {}^\circ K^0 / {}^\circ K^0 = G_{\circ\rho}^0 / (G_{\circ\rho}^0 \cap {}^\circ K^0).$$

For each g_i , $\text{Hom}_{{}^\circ K^0}({}^{g_i}({}^\circ\rho_{-1}), {}^\circ\rho_{-1})$ is 1-dimensional. We fix a basis $(T_{g_i})_{-1}$ of this space. We define

$$T_{g_i} = (T_{g_i})_{-1} \otimes \phi'_0(g_i) \otimes \dots \otimes \phi'_d(g_i).$$

The element T_{g_i} is a basis of the 1-dimensional vector space $\text{Hom}_{{}^\circ K}({}^{g_i}({}^\circ\rho), {}^\circ\rho)$. We define $f_{g_i} \in \check{\mathcal{H}}(G(F), {}^\circ\rho)$ and $(f_{g_i})_{-1} \in \check{\mathcal{H}}(G^0(F), {}^\circ\rho_{-1})$ by

$$\begin{aligned} f_{g_i}(x) &= \begin{cases} T_{g_i} \circ {}^\circ\rho(k) & (x = g_i k, k \in {}^\circ K) \\ 0 & (\text{otherwise}), \end{cases} \\ (f_{g_i})_{-1}(x) &= \begin{cases} (T_{g_i})_{-1} \circ {}^\circ\rho_{-1}(k) & (x = g_i k, k \in {}^\circ K^0) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

Then as vector spaces,

$$\check{\mathcal{H}}(G(F), {}^\circ\rho) = \bigoplus_i \mathbb{C} f_{g_i}, \quad \check{\mathcal{H}}(G^0(F), {}^\circ\rho_{-1}) = \bigoplus_i \mathbb{C} (f_{g_i})_{-1}$$

and $f_{g_i} \mapsto (f_{g_i})_{-1}$ gives a support preserving vector space isomorphism $\check{\mathcal{H}}(G(F), {}^\circ\rho) \rightarrow \check{\mathcal{H}}(G^0(F), {}^\circ\rho_{-1})$. We will show that it is an algebra isomorphism. Let g_{i_1}, g_{i_2} be representatives, and take g_{i_3} so that $g_{i_1} g_{i_2} \in g_{i_3} {}^\circ K^0$. We simply write g_1, g_2, g_3 for $g_{i_1}, g_{i_2}, g_{i_3}$, respectively. For $x \in G$, we have

$$\begin{aligned} (f_{g_1} * f_{g_2})(x) &= \int_G f_{g_1}(y) \circ f_{g_2}(y^{-1}x) dy \\ &= \int_{{}^\circ K} T_{g_1} \circ {}^\circ\rho(k) \circ f_{g_2}(k^{-1}g_1^{-1}x) dk \\ &= \begin{cases} T_{g_1} \circ T_{g_2} \circ {}^\circ\rho((g_1 g_2)^{-1} g_3 k') & (x = g_3 k', k' \in {}^\circ K) \\ 0 & (\text{otherwise}) \end{cases} \\ &= c \cdot f_{g_3}(x), \end{aligned}$$

where c satisfies

$$c \cdot T_{g_3} = T_{g_1} \circ T_{g_2} \circ {}^\circ\rho((g_1 g_2)^{-1} g_3).$$

By a similar calculation, we obtain

$$(f_{g_1})_{-1} * (f_{g_2})_{-1} = c_{-1} \cdot (f_{g_3})_{-1},$$

where c_{-1} satisfies

$$c_{-1} \cdot (T_{g_3})_{-1} = (T_{g_1})_{-1} \circ (T_{g_2})_{-1} \circ {}^\circ\rho_{-1}((g_1 g_2)^{-1} g_3).$$

By the definition of T_{g_i} ,

$$\begin{aligned}
& T_{g_1} \circ T_{g_2} \circ {}^\circ \rho((g_1 g_2)^{-1} g_3) \\
&= ((T_{g_1})_{-1} \circ (T_{g_2})_{-1} \circ {}^\circ \rho_{-1}((g_1 g_2)^{-1} g_3)) \otimes \left(\bigotimes_{j=0}^d \phi'_j(g_1) \circ \phi'_j(g_2) \circ \phi'_j((g_1 g_2)^{-1} g_3) \right) \\
&= ((T_{g_1})_{-1} \circ (T_{g_2})_{-1} \circ {}^\circ \rho_{-1}((g_1 g_2)^{-1} g_3)) \otimes \left(\bigotimes_{j=0}^d \phi'_j(g_3) \right) \\
&= c_{-1} \cdot (T_{g_3})_{-1} \otimes \left(\bigotimes_{j=0}^d \phi'_j(g_3) \right) \\
&= c_{-1} \cdot T_{g_3}.
\end{aligned}$$

This implies that $c = c_{-1}$ and the isomorphism constructed above is an algebra isomorphism. \square

5. HECKE ALGEBRAS FOR REGULAR SUPERCUSPIDAL TYPES

Firstly, we review the definition and the construction of regular supercuspidal representations. In this section, we assume that p is odd, is not a bad prime for G , and does not divide the order of the fundamental group of G_{der} . These assumptions are needed for the existence of a Howe factorization.

Let (S, θ) be a regular tame elliptic pair, i.e., S is a tame elliptic maximal torus of G , and θ is a character of $S(F)$ which satisfy the conditions in [Kal19, Definition 3.7.5]. Let $[y]$ be the point of reduced building of G over F which is associated to S in the sense of the paragraph above [Kal19, Lemma 3.4.3] and chose $y \in \mathcal{B}(G, F)$ such that the projection of y on the reduced building is $[y]$.

From (S, θ) , we define a sequence of twisted Levi subgroups

$$\vec{G} = (S = G^{-1} \subset G^0 \subsetneq \dots \subsetneq G^d = G)$$

in G and a sequence of real numbers $\vec{r} = (0 = r_{-1}, r_0, \dots, r_d)$ as in [Kal19, 3.6].

A Howe factorization of (S, θ) is a sequences of characters $\vec{\phi} = (\phi_{-1}, \dots, \phi_d)$, where ϕ_i is a character of $G^i(F)$ satisfying

$$\theta = \prod_{i=-1}^d \phi_i \upharpoonright_{S(F)}$$

and some additional technical conditions (see [Kal19, Definition 3.6.2]). By [Kal19, Proposition 3.6.7], (S, θ) has a Howe factorization. We take a Howe factorization $\vec{\phi}$. Using the pair (S, ϕ_{-1}) , we define an irreducible representation ρ_{-1} of $G^0(F)_{[y]}$ as follows.

Let G_y° be the reductive quotient of the special fiber of the connected parahoric group scheme of G^0 associated to y and S° be the reductive quotient of the special fiber of the connected Néron model of S . Then $S^\circ \subset G_y^\circ$ is an elliptic maximal torus. The restriction of ϕ_{-1} to $S(F)_0$ factors through a character $\bar{\phi}_{-1} : S^\circ(k_F) \rightarrow \mathbb{C}^\times$.

Let $\kappa_{(S, \bar{\phi}_{-1})} = \pm R_{S^\circ, \bar{\phi}_{-1}}$ be the irreducible cuspidal representation of $G_y^\circ(k_F)$ arising from the Deligne–Lusztig construction applied to S° and $\bar{\phi}_{-1}$ [DL76, Section 1]. We identify it with its inflation to $G^0(F)_{y,0}$. We can extend $\kappa_{(S, \bar{\phi}_{-1})}$ to a representation $\kappa_{(S, \phi_{-1})}$ of $S(F)G^0(F)_{y,0}$ [Kal19, 3.4.4]. Now we define $\rho_{-1} = \text{ind}_{S(F)G^0(F)_{y,0}}^{G^0(F)_{[y]}} \kappa_{(S, \phi_{-1})}$.

Proposition 5.1. *Let (S, θ) be a regular tame elliptic pair and $G^i, r_i, \phi_i, \rho_{-1}$ be as above. Then, $((G^i)_{i=0}^d, y, (r_i)_{i=0}^d, \rho_{-1}, (\phi_i)_{i=0}^d)$ satisfies **D1**, **D2**, **D3**, **D4**, **D5**.*

Proof. This is a part of [Kal19, Proposition 3.7.8]. \square

Using this datum, we construct an irreducible supercuspidal representation $\pi_{(S, \theta)}$ of $G(F)$, which only depends on (S, θ) . An irreducible supercuspidal representation of $G(F)$ obtained in this way is called *regular*.

Next, we define *regular supercuspidal types*. Let (S, θ) be a regular tame elliptic pair and define $G^i, r_i, \phi_i, \kappa_{(S, \phi_{-1})}$ as above. We define

$${}^\circ\rho_{-1} = \text{ind}_{G^0(F)_y \cap S(F)G^0(F)_{y,0}}^{G^0(F)_y} {}^\circ\kappa_{(S, \phi_{-1})},$$

where ${}^\circ\kappa_{(S, \phi_{-1})}$ denotes the restriction of $\kappa_{(S, \phi_{-1})}$ to $G^0(F)_y \cap S(F)G^0(F)_{y,0}$.

Proposition 5.2. *The representation ${}^\circ\rho_{-1}$ is an irreducible representation of $G^0(F)_y$.*

Proof. This is essentially the same as [Kal19, Proposition 3.4.20], which is the same as the one for [DR09, Lemma 4.5.1].

Since $G^0(F)_y \cap S(F)G^0(F)_{y,0}$ contains $G^0(F)_{y,0}$ and $\kappa_{(S, \phi_{-1})} \upharpoonright_{G^0(F)_{y,0}} = \kappa_{(S, \bar{\phi}_{-1})}$ is irreducible, ${}^\circ\kappa_{(S, \phi_{-1})}$ is also irreducible. Therefore it is enough to show that if $g \in G^0(F)_y$ intertwines ${}^\circ\kappa_{(S, \phi_{-1})}$, then $g \in G^0(F)_y \cap S(F)G^0(F)_{y,0}$. Suppose $g \in G^0(F)_y$ intertwines ${}^\circ\kappa_{(S, \phi_{-1})}$, then g also intertwines $\kappa_{(S, \bar{\phi}_{-1})}$. Hence, by [DL76, Theorem 6.8], there is $h \in G^0(F)_{y,0}$ so that $\text{Ad}(hg)(S^\circ, \bar{\phi}_{-1}) = (S^\circ, \bar{\phi}_{-1})$. By [Kal19, Lemma 3.4.5], there is $l \in G^0(F)_{y,0+}$ so that $\text{Ad}(lhg)(S, \phi_{-1} \upharpoonright_{S(F)_0}) = (S, \phi_{-1} \upharpoonright_{S(F)_0})$. Thus $lhg \in S(F)$ by the regularity of θ , and $g \in G^0(F)_y \cap S(F)G^0(F)_{y,0}$. \square

This proposition implies that $((G^i)_{i=0}^d, y, (r_i)_{i=0}^d, {}^\circ\rho_{-1}, (\phi_i)_{i=0}^d)$ satisfies **D1**, **D2**, **D3**, **${}^\circ$ D4**, **D5**. We construct a $[G, \pi_{(S, \theta)}]_G$ -type $({}^\circ K, {}^\circ\rho)$ from this datum. We call types obtained in this way *regular supercuspidal types*.

Remark 5.3. Let (S, θ) be a regular tame elliptic pair. Let ρ_{-1} be the representation of $G^0(F)_{[y]}$ and ${}^\circ\rho_{-1}$ be the representation of $G^0(F)_y$ defined as above. If ρ' is an irreducible representation of $G^0(F)_y$ which is contained in $\rho_{-1} \upharpoonright_{G^0(F)_y}$ then there is $g \in G^0(F)_{[y]}$ so that ρ' is equivalent to $g({}^\circ\rho_{-1})$. Now,

$$\rho' \simeq g({}^\circ\rho_{-1}) \simeq \text{ind}_{G^0(F)_y \cap gS(F)g^{-1}G^0(F)_{y,0}}^{G^0(F)_y} {}^\circ\kappa_{(gSg^{-1}, g\phi)}$$

and $(gSg^{-1}, g\phi)$ is a regular tame elliptic pair. Therefore, the $[G, \pi_{(S, \theta)}]_G$ -type constructed by the datum $((G^i)_{i=0}^d, y, (r_i)_{i=0}^d, \rho', (\phi_i)_{i=0}^d)$ is regular supercuspidal.

We determine the structure of Hecke algebras associated with regular supercuspidal types. So, let (S, θ) be a regular tame elliptic pair and $({}^\circ K, {}^\circ\rho)$ be as above. We consider the Hecke algebra $\mathcal{H}(G, {}^\circ\rho)$ associated with the type $({}^\circ K, {}^\circ\rho)$.

Proposition 5.4. *The support of $\check{\mathcal{H}}(G(F), {}^\circ\rho)$ is $S(F){}^\circ K$.*

Proof. By Proposition 4.1, it is enough to show that $g \in G^0(F)_{[y]}$ intertwines ${}^\circ\rho_{-1}$ if and only if $g \in S(F)G^0(F)_y$. If $g \in S(F)G^0(F)_y$, it is obvious that g intertwines ${}^\circ\rho_{-1}$. Conversely, suppose that $g \in G^0(F)_{[y]}$ intertwines ${}^\circ\rho_{-1}$. By the construction of ${}^\circ\rho_{-1}$, there exists $g_0 \in G^0(F)_y$ so that gg_0 intertwines ${}^\circ\kappa_{(S, \phi_{-1})}$. Then, as in the proof of Proposition 5.2, we conclude that $gg_0 \in S(F)G^0(F)_{y,0}$, and so $g \in S(F)G^0(F)_y$. \square

We define $S(F)_b = {}^\circ K \cap S(F) = G^0(F)_y \cap S(F)$, which is the unique maximal compact subgroup of $S(F)$.

Corollary 5.5. *The algebra $\check{\mathcal{H}}(G(F), {}^\circ\rho)$ is isomorphic to the group algebra $\mathbb{C}[S(F)/S(F)_b]$ of $S(F)/S(F)_b$.*

Proof. Since ${}^\circ\rho_{-1}$ is the restriction of $\text{ind}_{S(F)G^0(F)_{y,0}}^{S(F)G^0(F)_y} \kappa_{(S, \phi_{-1})}$ to $G^0(F)_y$, we can extend ${}^\circ\rho_{-1}$ to $S(F)G^0(F)_y$. Therefore, we can extend ${}^\circ\rho$ to $S(F){}^\circ K$. Then, the claim follows from a general proposition below. \square

Proposition 5.6. *Let K be a compact open subgroup of $G(F)$ and (ρ, W) be an irreducible representation of K . Assume that the intertwiner*

$$K' = \{g \in G(F) \mid g \text{ intertwines } \rho\}$$

is a group which normalizes K , and ρ extends to a representation ρ' of K' . Then, the Hecke algebra $\mathcal{H}(G(F), \rho)$ associated with (K, ρ) is isomorphic to the group algebra $\mathbb{C}[K'/K]$ of K'/K .

Proof. We can prove the claim by using the same argument in the proof of [BK98, Proposition 5.6]. For $[k'] = k'K \in K'/K$, we define $f_{[k']} \in \check{\mathcal{H}}(G(F), \rho)$ by

$$f_{[k']}(x) = \begin{cases} \rho'(x) & (x \in [k']) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, as a vector space

$$\check{\mathcal{H}}(G(F), \rho) = \bigoplus_{[k'] \in K'/K} \mathbb{C}f_{[k']},$$

and for $[k'_1], [k'_2] \in K'/K$, $f_{[k'_1]} * f_{[k'_2]} = f_{[k'_1 k'_2]}$. Hence, $\check{\mathcal{H}}(G(F), \rho)$ is isomorphic to the group algebra of K'/K . \square

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