HOWE CORRESPONDENCE OF UNIPOTENT CHARACTERS FOR A FINITE SYMPLECTIC/EVEN-ORTHOGONAL DUAL PAIR

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ABSTRACT. In this paper we give a complete and explicit description of the Howe correspondence of unipotent characters for a finite reductive dual pair of a symplectic group and an even orthogonal group in terms of the Lusztig parametrization. That is, the conjecture by Aubert-Michel-Rouquier is confirmed.

CONTENTS

1. INTRODUCTION

1.1. Let $\omega^{\psi}_{\mathrm{Sp}_{2N}}$ denote the character of the Weil representation (*cf.* [Gér77]) of a finite symplectic group $\mathrm{Sp}_{2N}(q)$ with respect to a nontrivial additive character ψ of a finite field $\mathbf{F}_q$ of characteristic $p \neq 2$. Let $(\mathbf{G},\mathbf{G}')$ be one of the following three basic types of reductive dual pairs in Sp_{2N}:

(1) two general linear groups $(\mathrm{GL}_n,\mathrm{GL}_{n'})$;
(2) two unitary groups $(\mathrm{U}_n,\mathrm{U}_{n'})$;
(3) one symplectic group and one orthogonal group $(\mathrm{Sp}_{2n},\mathrm{O}^{\epsilon}_{n'})$

where $\epsilon = +$ or $-$. Now $\omega^{\psi}_{\mathrm{Sp}_{2N}}$ is regarded as a character of $G\times G'$ and denoted by $\omega^{\psi}_{\mathbf{G},\mathbf{G}'}$ via the homomorphisms $G\times G' \to G\cdot G' \hookrightarrow \mathrm{Sp}_{2N}(q)$ where G, G' denote the finite groups of rational points of $\mathbf{G},\mathbf{G}'$ respectively. Then $\omega^{\psi}_{\mathbf{G},\mathbf{G}'}$ is decomposed as a sum of irreducible characters

$$\omega^{\psi}_{\mathbf{G},\mathbf{G}'} = \sum_{\rho\in\mathscr{E}(G),\ \rho'\in\mathscr{E}(G')} m_{\rho,\rho'}\rho\otimes\rho'$$

where each $m_{\rho,\rho'}$ is a non-negative integer, and $\mathscr{E}(G)$ denotes the set of irreducible characters of G (i.e., the set of the characters of irreducible representations of G). Then it

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establishes a relation

$$\Theta_{\mathbf{G},\mathbf{G}'} = \{ (\rho,\rho') \in \mathcal{E}(G) \times \mathcal{E}(G') \mid m_{\rho,\rho'} \neq 0 \}$$

between $\mathcal{E}(G)$ and $\mathcal{E}(G')$ which is called the *Howe correspondence* (or Θ*-correspondence*) for the dual pair $(\mathbf{G},\mathbf{G}')$. The main task is to describe the correspondence explicitly.

1.2. It is known that $\mathcal{E}(G)$ is partitioned as a disjoint union

$$\mathcal{E}(G) = \bigcup_{(s)\subset (G^*)^0} \mathcal{E}(G)_s$$

of *Lusztig series* $\mathcal{E}(G)_s$ indexed by the conjugacy classes (s) of semisimple elements in the connected component $(G^*)^0$ of the dual group G^* of G. Elements in $\mathcal{E}(G)_1$ are called *unipotent characters*. Lusztig shows that there exists a bijection

$$\mathfrak{L}_s \colon \mathcal{E}(G)_s \longrightarrow \mathcal{E}(C_{G^*}(s))_1$$

where $C_{G^*}(s)$ is the centralizer in G^* of s (*cf.* [Lus77]). For a semisimple element s we can define three groups $G^{(0)}, G^{(1)}, G^{(2)}$ so that there is a natural bijection

$$\mathcal{E}(C_{G^*}(s))_1 \simeq \mathcal{E}(G^{(0)} \times G^{(1)} \times G^{(2)})_1$$

(*cf.* [Pan19a] subsection 6.2). Then we have a (modified) Lusztig correspondence

$$\begin{aligned} \Xi_s \colon \mathcal{E}(G)_s &\to \mathcal{E}(G^{(0)} \times G^{(1)} \times G^{(2)})_1, \\ \rho &\mapsto \rho^{(0)} \otimes \rho^{(1)} \otimes \rho^{(2)} \end{aligned}$$

where $\rho^{(j)} \in \mathcal{E}(G^{(j)})_1$ for $j = 0,1,2$. Moreover, we have the corresponding decomposition $s = s^{(0)} \times s^{(1)} \times s^{(2)}$.

Recall that a class function on G is called *uniform* if it is a linear combination of the Deligne-Lusztig virtual characters $R_{T,\theta}$. For a class function f on G, let $f^\sharp$ denote its projection on the subspace of uniform class functions.

Now let $(\mathbf{G},\mathbf{G}')$ be a dual pair and suppose that $\rho \in \mathcal{E}(G)_s$, $\rho' \in \mathcal{E}(G')_{s'}$ for some s, s'. For simplicity in this subsection we assume that the orthogonal group is even for a symplectic/orthogonal dual pair. Then one can show that

- both $\mathbf{G}^{(0)}, \mathbf{G}'^{(0)}$ are products of general linear groups or unitary groups;
- both $\mathbf{G}^{(1)}, \mathbf{G}'^{(1)}$ are classical groups of the same type;
- $(\mathbf{G}^{(2)}, \mathbf{G}'^{(2)})$ forms a reductive dual pair of either two general linear groups, two unitary groups, or one symplectic group and one even orthogonal group.

It is known that unipotent characters are preserved in the Howe correspondence for the dual pair $(\mathbf{G}^{(2)}, \mathbf{G}'^{(2)})$ (*cf.* [AM93]). Then one can show that $\rho\otimes\rho'$ occurs in $\omega^\psi_{\mathbf{G},\mathbf{G}'}$ (i.e., $m_{\rho,\rho'} \neq 0$) if and only if the following conditions are satisfied:

(1) $s^{(0)} = s'^{(0)}$, $\mathbf{G}^{(0)} \simeq \mathbf{G}'^{(0)}$ and $\rho^{(0)} = \rho'^{(0)}$;
(2) $\mathbf{G}^{(1)} \simeq \mathbf{G}'^{(1)}$ and $\rho^{(1)} = \rho'^{(1)}$;
(3) $\rho^{(2)} \otimes \rho'^{(2)}$ occurs in $\omega_{\mathbf{G}^{(2)},\mathbf{G}'^{(2)},1}$

where $\omega_{\mathbf{G}^{(2)},\mathbf{G}'^{(2)},1}$ denotes the unipotent part of $\omega^\psi_{\mathbf{G}^{(2)},\mathbf{G}'^{(2)}}$, i.e, the following diagram

$$\begin{array}{ccc} \rho & \xrightarrow{\Theta_{\mathbf{G},\mathbf{G}'}} & \rho' \\ \Xi_s \downarrow & & \downarrow \Xi_{s'} \\ \rho^{(0)} \otimes \rho^{(1)} \otimes \rho^{(2)} & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Theta_{\mathbf{G}^{(2)},\mathbf{G}'^{(2)}}} & \rho'^{(0)} \otimes \rho'^{(1)} \otimes \rho'^{(2)} \end{array} \tag{1.1}$$

commutes. Therefore we can reduce the Howe correspondence $\Theta_{\mathbf{G},\mathbf{G}'}$ of general irreducible characters to the correspondence $\Theta_{\mathbf{G}^{(2)},\mathbf{G}'^{(2)}}$ of irreducible unipotent characters.

Remark 1.2. (1) If the pair $(\mathbf{G},\mathbf{G}')$ consists of two general linear groups or two unitary groups, then all the irreducible characters of G and G' are uniform and so the above commutative diagram can be read off from the result in [AMR96] théorème 2.6 (*cf.* [Pan19b] theorem 3.10).

(2) If $(\mathbf{G},\mathbf{G}')$ consists of a symplectic group and an orthogonal group, using the decomposition of $\omega^\sharp_{\mathbf{G},\mathbf{G}'}$ in [Sri79] and [Pan21], the commutativity of the diagram (under proper choices of Ξ_s and $\Xi_{s'}$) is proved in [Pan19a]. Unlike the cases of general linear groups or unitary groups, most of the irreducible characters of symplectic groups or orthogonal groups are not uniform. This is the main difference and difficulty for studying the correspondence for symplectic/orthogonal dual pairs.

1.3. So now we focus on the correspondences of irreducible unipotent characters for symplectic/even-orthogonal dual pairs. First we review some results on the classification of the irreducible unipotent characters by Lusztig in [Lus77], [Lus81] and [Lus82]. Let

$$\Lambda=\binom{A}{B}=\binom{a_1,a_2,\ldots,a_{m_1}}{b_1,b_2,\ldots,b_{m_2}}$$

denote a *reduced symbol*, i.e., an ordered pair of two finite subsets A,B of non-negative integers such that $0\notin A\cap B$. Note that we always assume that $a_1>a_2>\cdots>a_{m_1}$ and $b_1>b_2>\cdots>b_{m_2}$. The rank and the defect of a symbol Λ (denoted by $\mathrm{rk}(\Lambda)$ and $\mathrm{def}(\Lambda)$ respectively) are defined in (2.1). Let $\mathcal{S}$ denote the set of reduced symbols, and let $\mathcal{S}_{n,d}$ denote the set of reduced symbols of rank n and defect d. Then we define the following sets of symbols associated to $\mathbf{G}$:

$$\begin{aligned}\mathcal{S}_{\mathrm{Sp}_{2n}}&=\{\Lambda\in\mathcal{S}\mid\mathrm{rk}(\Lambda)=n,\ \mathrm{def}(\Lambda)\equiv1\pmod 4\};\\ \mathcal{S}_{\mathrm{O}^+_{2n}}&=\{\Lambda\in\mathcal{S}\mid\mathrm{rk}(\Lambda)=n,\ \mathrm{def}(\Lambda)\equiv0\pmod 4\};\\ \mathcal{S}_{\mathrm{O}^-_{2n}}&=\{\Lambda\in\mathcal{S}\mid\mathrm{rk}(\Lambda)=n,\ \mathrm{def}(\Lambda)\equiv2\pmod 4\}.\end{aligned}\tag{1.3}$$

Then Lusztig gives a parametrization of the set of irreducible unipotent characters $\mathcal{E}(G)_1$ by the set of symbols $\mathcal{S}_{\mathbf{G}}$. The irreducible character parametrized by a symbol Λ will be denoted by ρ_Λ.

For a symbol $\Lambda=\binom{a_1,a_2,\ldots,a_{m_1}}{b_1,b_2,\ldots,b_{m_2}}$, we associate it a *bi-partition*

$$\Upsilon(\Lambda)=\begin{bmatrix}a_1-(m_1-1),a_2-(m_1-2),\ldots,a_{m_1-1}-1,a_{m_1}\\ b_1-(m_2-1),b_2-(m_2-2),\ldots,b_{m_2-1}-1,b_{m_2}\end{bmatrix}.\tag{1.4}$$

Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ where $\epsilon=+$ or $-$. For $\Lambda\in\mathcal{S}_{\mathbf{G}}$, $\Lambda'\in\mathcal{S}_{\mathbf{G}'}$, we write $\Upsilon(\Lambda)=\left[\begin{smallmatrix}\lambda\\ \mu\end{smallmatrix}\right]$ and $\Upsilon(\Lambda')=\left[\begin{smallmatrix}\lambda'\\ \mu'\end{smallmatrix}\right]$. Then we define a relation $\mathcal{B}_{\mathbf{G},\mathbf{G}'}$ on $\mathcal{S}_{\mathbf{G}}\times\mathcal{S}_{\mathbf{G}'}$ by

$$\begin{aligned}\mathcal{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}&=\{(\Lambda,\Lambda')\in\mathcal{S}_{\mathrm{Sp}_{2n}}\times\mathcal{S}_{\mathrm{O}^+_{2n'}}\mid\mu\preccurlyeq\lambda',\ \mu'\preccurlyeq\lambda,\ \mathrm{def}(\Lambda')=-\mathrm{def}(\Lambda)+1\};\\ \mathcal{B}_{\mathrm{Sp}_{2n},\mathrm{O}^-_{2n'}}&=\{(\Lambda,\Lambda')\in\mathcal{S}_{\mathrm{Sp}_{2n}}\times\mathcal{S}_{\mathrm{O}^-_{2n'}}\mid\lambda'\preccurlyeq\mu,\ \lambda\preccurlyeq\mu',\ \mathrm{def}(\Lambda')=-\mathrm{def}(\Lambda)-1\}\end{aligned}\tag{1.5}$$

where the relation $\lambda\preccurlyeq\mu$ on partitions is given in (2.10). Moreover, we define

$$\mathcal{D}_{\mathrm{Sp}_{2n},\mathrm{O}^-_{2n'}}=\mathcal{D}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}=\mathcal{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}\cap(\mathcal{S}_{n,1}\times\mathcal{S}_{n',0}).\tag{1.6}$$

Then it is proved in [Pan21] that

$$\omega^\sharp_{\mathbf{G},\mathbf{G}',1}=\frac{1}{2}\sum_{(\Sigma,\Sigma')\in\mathcal{D}_{\mathbf{G},\mathbf{G}'}} R^{\mathbf{G}}_{\Sigma}\otimes R^{\mathbf{G}'}_{\Sigma'}=\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{\mathbf{G},\mathbf{G}'}}\rho^\sharp_\Lambda\otimes\rho^\sharp_{\Lambda'} \tag{1.7}$$

where $R^{\mathbf{G}}_{\Sigma}$ and $R^{\mathbf{G}'}_{\Sigma'}$ are the almost characters given in Subsection 3.3 and Subsection 3.2.

In this article, we can go a step further to remove the uniform projection and obtain an explicit description in terms of Lusztig's symbols of the Howe correspondence of unipotent characters for a symplectic/even-orthogonal dual pair:

Theorem 1.8. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ *where* $\epsilon=+$ *or* $-$. *Then*

$$\omega_{\mathbf{G},\mathbf{G}',1}=\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{\mathbf{G},\mathbf{G}'}}\rho_\Lambda\otimes\rho_{\Lambda'},$$

i.e., $(\rho_\Lambda,\rho_{\Lambda'})$ *occurs in* $\Theta_{\mathbf{G},\mathbf{G}'}$ *if and only if* $(\Lambda,\Lambda')\in\mathcal{B}_{\mathbf{G},\mathbf{G}'}$.

Remark 1.9. In [AMR96] théorème 5.5, théorème 3.10 and conjecture 3.11, Aubert, Michel and Rouquier give an explicit description (in terms of partitions or bi-partitions) of the correspondence of unipotent characters for a dual pair of either two general linear groups or two unitary groups, and they have a conjecture on the description of the correspondence for a symplctic/even-orthogonal dual pair. A comparison between the theorem above and their conjecture is in Subsection 3.5.

Combining the theorem and the commutativity between Howe correspondence and Lusztig correspondence in (1.1), we obtain a complete description of the whole Howe correspondence of irreducible characters for any finite reductive dual pair. Some applications of the description can be found in [Pan19a] and [Pan20].

1.4. The contents of the paper are organized as follows. In Section 2, we recall the definition and basic properties of symbols introduced by Lusztig. Then we discuss the relations $\mathcal{D}_{Z,Z'}$ and $\mathcal{B}_{Z,Z'}$ which play the important roles in our main results. In Section 3, we recall the Lusztig's parametrization of irreducible unipotent characters of a symplectic group or an even orthogonal group. Then we state our main theorems in Subsection 3.4. In Section 4, we provide several properties of cells of a symplectic group or an even orthogonal group. These properties will be used in the proof of our main result: Theorem 1.8 in last two sections.

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2. SYMBOLS AND BI-PARTITIONS

In the first part of this section we recall the notion of "symbols" and "bi-partitions" from [Lus77] §3.

2.1. Symbols. A *symbol* is an ordered pair

$$\Lambda=\binom{A}{B}=\binom{a_1,a_2,\ldots,a_{m_1}}{b_1,b_2,\ldots,b_{m_2}}$$

of two finite subsets A,B (possibly empty) of non-negative integers. We always assume that elements in A,B are written respectively in strictly decreasing order, i.e., $a_1>a_2>\cdots>a_{m_1}$ and $b_1>b_2>\cdots>b_{m_2}$. A symbol is called *degenerate* if $A=B$, and it is called *non-degenerate* otherwise. The *cardinality*, *size*, *rank* and *defect* of a symbol $\Lambda=\binom{A}{B}$ are defined by

$$
\begin{aligned}
|\Lambda| &= |A|+|B|,\\
\operatorname{size}(\Lambda) &= (|A|,|B|),\\
\operatorname{rank}(\Lambda) &= \sum_{i=1}^{m_1} a_i+\sum_{i=1}^{m_2} b_i-\left\lfloor\left(\frac{|A|+|B|-1}{2}\right)^2\right\rfloor,\\
\operatorname{def}(\Lambda) &= |A|-|B|
\end{aligned}
\tag{2.1}
$$

where $|X|$ denotes the cardinality of a finite set X. For a symbol Λ, let Λ^* (resp. Λ_*) denote the first row (resp. second row) of Λ, i.e., $\Lambda=\binom{\Lambda^*}{\Lambda_*}$. For a symbol $\Lambda=\binom{A}{B}$, we define its *transpose* $\Lambda^{\rm t}=\binom{B}{A}$. A symbol $\binom{A}{B}$ is called *reduced* if $0\notin A\cap B$. If both Λ^*,Λ_* are the empty set, then Λ is denoted by $\binom{-}{-}$ or just $\emptyset$.

We define an equivalence relation on symbols generated by

$$
\begin{pmatrix} a_1,a_2,\ldots,a_{m_1}\\ b_1,b_2,\ldots,b_{m_2}\end{pmatrix}\sim\begin{pmatrix} a_1+1,a_2+1,\ldots,a_{m_1}+1,0\\ b_1+1,b_2+1,\ldots,b_{m_2}+1,0\end{pmatrix}.
$$

It is not difficult to see that ranks and defects are invariant on an equivalence class of symbols. Moreover, each equivalence class contains a unique reduced symbol. In the remaining part of this article, a symbol is always assumed to be reduced unless specified otherwise.

A symbol Λ_1 is called a *subsymbol* of another symbol Λ_2, denoted by $\Lambda_1\subset\Lambda_2$, if $\Lambda_1^*\subset\Lambda_2^*$ and $(\Lambda_1)_*\subset(\Lambda_2)_*$. If $\Lambda_1\subset\Lambda_2$, we define the *symbol substraction* by

$$
\Lambda_2\smallsetminus\Lambda_1=\begin{pmatrix}\Lambda_2^*\smallsetminus\Lambda_1^*\\ (\Lambda_2)_*\smallsetminus(\Lambda_1)_*\end{pmatrix}.
$$

For two symbols Λ_1,Λ_2, we define their *union* and *intersection* by

$$
\Lambda_1\cup\Lambda_2=\begin{pmatrix}\Lambda_1^*\cup\Lambda_2^*\\ (\Lambda_1)_*\cup(\Lambda_2)_*\end{pmatrix},\qquad \Lambda_1\cap\Lambda_2=\begin{pmatrix}\Lambda_1^*\cap\Lambda_2^*\\ (\Lambda_1)_*\cap(\Lambda_2)_*\end{pmatrix}.
$$

2.2. Special symbols. A symbol

$$
Z=\begin{pmatrix} a_1,a_2,\ldots,a_{m+1}\\ b_1,b_2,\ldots,b_m\end{pmatrix}
\tag{2.2}
$$

of defect 1 is called *special* if $a_1\geq b_1\geq a_2\geq b_2\geq\cdots\geq a_m\geq b_m\geq a_{m+1}$; similarly a symbol

$$
Z=\begin{pmatrix} a_1,a_2,\ldots,a_m\\ b_1,b_2,\ldots,b_m\end{pmatrix}
\tag{2.3}
$$

of defect 0 is called *special* if $a_1\geq b_1\geq a_2\geq b_2\geq\cdots\geq b_{m-1}\geq a_m\geq b_m$. For a special symbol Z, we define its subsymbol of "*singles*" $Z_{\rm I}=Z\smallsetminus\binom{Z^*\cap Z_*}{Z^*\cap Z_*}$. The *degree* of a special symbol Z is defined to be

$$
\deg(Z)=\begin{cases}\frac{|Z_{\rm I}|-1}{2}, & \text{if } Z \text{ has defect 1;}\\ \frac{|Z_{\rm I}|}{2}, & \text{if } Z \text{ has defect 0.}\end{cases}
$$

For a subsymbol $M\subset Z_{\rm I}$, we denote

$$
\Lambda_M=(Z\smallsetminus M)\cup M^{\rm t},
\tag{2.4}
$$

i.e., Λ_M is the symbol obtained from Z by switching the row position of entries in M and keeping other entries unchanged. Note that $\Lambda_\emptyset = Z$ and $\Lambda_{Z_{\mathrm{I}}} = Z^{\mathrm{t}}$.

Example 2.5. The symbol $Z = \binom{4,3}{3,2}$ is a special symbol of rank 8 and defect 0. Now $Z_{\mathrm{I}} = \binom{4}{2}$ and so $\deg(Z) = 1$. Then we have

M	$\binom{-}{-}$	$\binom{4}{-}$	$\binom{-}{2}$	$\binom{4}{2}$
Λ_M	$\binom{4,3}{3,2}$	$\binom{3}{4,3,2}$	$\binom{4,3,2}{3}$	$\binom{3,2}{4,3}$

If Z is a special symbol of rank n and defect 1, we define

$$\begin{aligned} \mathscr{S}_Z^{\mathrm{Sp}_{2n}} &= \{\Lambda_M \mid M \subset Z_{\mathrm{I}},\ |M| \text{ even}\} \subset \mathscr{S}_{\mathrm{Sp}_{2n}}, \\ \mathscr{S}_{Z,1} &= \mathscr{S}_Z^{\mathrm{Sp}_{2n}} \cap \mathscr{S}_{n,1}; \end{aligned} \tag{2.6}$$

if Z is a special symbol of rank n and defect 0, we define

$$\begin{aligned} \mathscr{S}_Z^{\mathrm{O}^+_{2n}} &= \{\Lambda_M \mid M \subset Z_{\mathrm{I}},\ |M| \text{ even}\} \subset \mathscr{S}_{\mathrm{O}^+_{2n}}, \\ \mathscr{S}_Z^{\mathrm{O}^-_{2n}} &= \{\Lambda_M \mid M \subset Z_{\mathrm{I}},\ |M| \text{ odd}\} \subset \mathscr{S}_{\mathrm{O}^-_{2n}}, \\ \mathscr{S}_{Z,0} &= \mathscr{S}_Z^{\mathrm{O}^+_{2n}} \cap \mathscr{S}_{n,0}. \end{aligned} \tag{2.7}$$

It is not difficult to see that

$$|\mathscr{S}_Z^{\mathbf{G}}| = \begin{cases} 2^{2\deg(Z)}, & \text{if } \mathbf{G} = \mathrm{Sp}_{2n}; \\ 2^{2\deg(Z)-1}, & \text{if } \mathbf{G} = \mathrm{O}^\epsilon_{2n} \text{ and } \deg(Z) > 0; \\ 1, & \text{if } \mathbf{G} = \mathrm{O}^+_{2n} \text{ and } \deg(Z) = 0; \\ 0, & \text{if } \mathbf{G} = \mathrm{O}^-_{2n} \text{ and } \deg(Z) = 0. \end{cases}$$

Moreover, we have

$$\begin{aligned} \mathscr{S}_{\mathrm{Sp}_{2n}} &= \bigcup_{Z \text{ special, } \mathrm{rk}(Z)=n,\ \mathrm{def}(Z)=1} \mathscr{S}_Z^{\mathrm{Sp}_{2n}}, \\ \mathscr{S}_{\mathrm{O}^\epsilon_{2n}} &= \bigcup_{Z \text{ special, } \mathrm{rk}(Z)=n,\ \mathrm{def}(Z)=0} \mathscr{S}_Z^{\mathrm{O}^\epsilon_{2n}}. \end{aligned}$$

If the context is clear, $\mathscr{S}_Z^{\mathbf{G}}$ will be just denoted by $\mathscr{S}_Z$.

For $\Lambda_{M_1}, \Lambda_{M_2} \in \mathscr{S}_Z$, we define an addition

$$\Lambda_{M_1} + \Lambda_{M_2} = \Lambda_N \quad \text{where } N = (M_1 \cup M_2) \smallsetminus (M_1 \cap M_2). \tag{2.8}$$

Note that $\Lambda + Z = \Lambda$ and $\Lambda + \Lambda = Z$ for any $\Lambda \in \mathscr{S}_Z$. Both $\mathscr{S}_Z^{\mathrm{Sp}_{2n}}$ and $\mathscr{S}_Z^{\mathrm{O}^+_{2n}}$ are closed under the addition with identity element $\Lambda_\emptyset = Z$. This gives $\mathscr{S}_Z^{\mathrm{Sp}_{2n}}, \mathscr{S}_Z^{\mathrm{O}^+_{2n}}$ a vector space structure over the field $\mathbf{F}_2$ with two elements. On the other hand, if $\Lambda_1 \in \mathscr{S}_Z^{\mathrm{O}^+_{2n}}$ and $\Lambda_2 \in \mathscr{S}_Z^{\mathrm{O}^-_{2n}}$, it is easy to check that $\Lambda_1 + \Lambda_2 \in \mathscr{S}_Z^{\mathrm{O}^-_{2n}}$; moreover, if $\Lambda_1, \Lambda_2 \in \mathscr{S}_Z^{\mathrm{O}^-_{2n}}$, then $\Lambda_1 + \Lambda_2 \in \mathscr{S}_Z^{\mathrm{O}^+_{2n}}$.

Example 2.9. (1) The symbol $Z = \binom{2,0}{1}$ is a special symbol of rank 2, defect 1 and degree 1. Now $Z_{\mathrm{I}} = Z$ and there are 4 subsymbols M of Z_{I} of an even number of entries, namely $\binom{-}{-}, \binom{2}{1}, \binom{0}{1}, \binom{2,0}{-}$. The corresponding Λ_M are $\binom{2,0}{1}, \binom{1,0}{2}, \binom{2,1}{0}, \binom{-}{2,1,0}$. Therefore,

$$\mathscr{S}_Z = \mathscr{S}_Z^{\mathrm{Sp}_4} = \left\{\binom{2,0}{1}, \binom{1,0}{2}, \binom{2,1}{0}, \binom{-}{2,1,0}\right\}.$$

The addition table of $\mathscr{S}_Z$ is

$+$	$\binom{2,0}{1}$	$\binom{1,0}{2}$	$\binom{2,1}{0}$	$\binom{-}{2,1,0}$
$\binom{2,0}{1}$	$\binom{2,0}{1}$	$\binom{1,0}{2}$	$\binom{2,1}{0}$	$\binom{-}{2,1,0}$
$\binom{1,0}{2}$	$\binom{1,0}{2}$	$\binom{2,0}{1}$	$\binom{-}{2,1,0}$	$\binom{2,1}{0}$
$\binom{2,1}{0}$	$\binom{2,1}{0}$	$\binom{-}{2,1,0}$	$\binom{2,0}{1}$	$\binom{1,0}{2}$
$\binom{-}{2,1,0}$	$\binom{-}{2,1,0}$	$\binom{2,1}{-}$	$\binom{1,0}{2}$	$\binom{2,0}{1}$

(2) The symbol $Z=\binom{3,1}{2,0}$ is a special symbol of rank 4, defect 0 and degree 2. Now $Z_I=Z$ and there are 16 subsymbols M of Z_I. Half of them have an even number of entries, and the other half have an odd number of entries. Then we see that

$$\mathscr{S}_Z^{O_8^+}=\{\tbinom{3,1}{2,0},\tbinom{2,0}{3,1},\tbinom{3,0}{2,1},\tbinom{2,1}{3,0},\tbinom{1,0}{3,2},\tbinom{3,2}{1,0},\tbinom{3,2,1,0}{-},\tbinom{-}{3,2,1,0}\},$$
$$\mathscr{S}_Z^{O_8^-}=\{\tbinom{3,2,1}{0},\tbinom{0}{3,2,1},\tbinom{3,1,0}{2},\tbinom{2}{3,1,0},\tbinom{3,2,0}{1},\tbinom{1}{3,2,0},\tbinom{2,1,0}{3},\tbinom{3}{2,1,0}\}.$$

2.3. Bi-partitions. For a partition $\lambda=[\lambda_1,\lambda_2,\ldots,\lambda_k]$ with $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k\geq 0$, define $|\lambda|=\lambda_1+\lambda_2+\cdots+\lambda_k$. For two partitions $\lambda=[\lambda_1,\ldots,\lambda_k]$, $\mu=[\mu_1,\ldots,\mu_l]$, we may assume that $k=l$ by adding several 0's if necessary, then we denote

$$\lambda\preccurlyeq\mu\quad\text{if } \mu_1\geq\lambda_1\geq\mu_2\geq\lambda_2\geq\cdots\geq\mu_k\geq\lambda_k. \tag{2.10}$$

Let $\mathscr{P}_2(n)$ denote the set of *bi-partitions* $\left[\begin{smallmatrix}\lambda\\ \mu\end{smallmatrix}\right]$ of n, i.e., the set of ordered pair of two partitions λ,μ such that $|\lambda|+|\mu|=n$. It is easy to check that the mapping Υ in (1.4) induces a bijection

$$\Upsilon\colon\mathscr{S}_{n,d}\longrightarrow\begin{cases}\mathscr{P}_2(n-(\frac{d-1}{2})(\frac{d+1}{2})), & \text{if } d \text{ is odd;}\\ \mathscr{P}_2(n-(\frac{d}{2})^2), & \text{if } d \text{ is even.}\end{cases}$$

2.4. The relations $\mathscr{B}_{Z,Z'}$ and $\mathscr{D}_{Z,Z'}$. Let $(G,G')=(\mathrm{Sp}_{2n},O^\epsilon_{2n'})$ where $\epsilon=+$ or $-$. Recall that a relation $\mathscr{B}_{G,G'}$ between $\mathscr{S}_G$ and $\mathscr{S}_{G'}$, and a relation $\mathscr{D}_{\mathrm{Sp}_{2n},O^+_{2n'}}$ between $\mathscr{S}_{n,1}$ and $\mathscr{S}_{n',0}$ are defined in (1.5) and (1.6). Let Z,Z' be special symbols of ranks n,n' and defects 1, 0 respectively. Define a relation $\mathscr{B}_{Z,Z'}$ between $\mathscr{S}_Z^G$ and $\mathscr{S}_{Z'}^{G'}$, and a relation $\mathscr{D}_{Z,Z'}$ between $\mathscr{S}_{Z,1}$ and $\mathscr{S}_{Z',0}$ by

$$\begin{aligned}\mathscr{B}_{Z,Z'}&=\mathscr{B}_{G,G'}\cap(\mathscr{S}_Z^G\times\mathscr{S}_{Z'}^{G'}),\\ \mathscr{D}_{Z,Z'}&=\mathscr{D}_{\mathrm{Sp}_{2n},O^+_{2n'}}\cap(\mathscr{S}_{Z,1}\times\mathscr{S}_{Z',0}).\end{aligned} \tag{2.11}$$

It is not difficult to see that

$$\mathscr{B}_{G,G'}=\bigcup_{Z,Z'}\mathscr{B}_{Z,Z'}\qquad\text{and}\qquad\mathscr{D}_{G,G'}=\bigcup_{Z,Z'}\mathscr{D}_{Z,Z'} \tag{2.12}$$

where the disjoint union $\bigcup_{Z,Z'}$ is taken over all special symbols Z,Z' of ranks n,n' and defects 1, 0 respectively.

The following three lemmas are from [Pan21] corollary 5.1, lemma 2.5, lemma 2.6:

Lemma 2.13. *Let Z,Z' be special symbols of ranks n,n' and defects* 1, 0 *respectively. Then $\mathscr{B}_{Z,Z'}\neq\emptyset$ if and only if $\mathscr{D}_{Z,Z'}\neq\emptyset$.*

Lemma 2.14. *Let Z,Z' be special symbols of size $(m+1,m),(m',m')$ respectively for some non-negative integers m,m'. If $\mathscr{D}_{Z,Z'}\neq\emptyset$, then either $m'=m$ or $m'=m+1$.*

Lemma 2.15. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ *where* $\epsilon=+$ *or* $-$. *Let* Z,Z' *be two special symbols of ranks* n,n' *and sizes* $(m+1,m),(m',m')$ *respectively where* $m'=m,m+1$. *Let*

$$\Lambda=\binom{a_1,a_2,\ldots,a_{m_1}}{b_1,b_2,\ldots,b_{m_2}}\in\mathscr{S}_Z^{\mathbf{G}},\qquad \Lambda'=\binom{c_1,c_2,\ldots,c_{m'_1}}{d_1,d_2,\ldots,d_{m'_2}}\in\mathscr{S}_{Z'}^{\mathbf{G}'}.$$

Then $(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}$ *if and only if one of the following conditions is satisfied:*

$$\begin{cases} m'_1=m_2,\ a_i>d_i,\ d_i\geq a_{i+1},\ c_i\geq b_i,\ b_i>c_{i+1} \text{ for each } i, & \text{if } \epsilon=+,\ m'=m;\\ m'_1=m_2+1,\ a_i\geq d_i,\ d_i>a_{i+1},\ c_i>b_i,\ b_i\geq c_{i+1} \text{ for each } i, & \text{if } \epsilon=+,\ m'=m+1;\\ m'_1=m_2-1,\ d_i\geq a_i,\ a_i>d_{i+1},\ b_i>c_i,\ c_i\geq b_{i+1} \text{ for each } i, & \text{if } \epsilon=-,\ m'=m;\\ m'_1=m_2,\ d_i>a_i,\ a_i\geq d_{i+1},\ b_i\geq c_i,\ c_i>b_{i+1} \text{ for each } i, & \text{if } \epsilon=-,\ m'=m+1.\end{cases}$$

Example 2.16. Consider the dual pair $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_4,\mathrm{O}_8^+)$, and $Z=\binom{2,0}{1}$, $Z'=\binom{3,1}{2,0}$. Now $\mathscr{S}_Z^{\mathrm{Sp}_4}$ and $\mathscr{S}_{Z'}^{\mathrm{O}_8^+}$ are given in Example 2.9. Then by Lemma 2.15, it is not difficult to see that $\mathscr{B}_{Z,Z'}$ is given by

$\mathscr{B}_{Z,Z'}$	$\binom{3,1}{2,0}$	$\binom{2,0}{3,1}$	$\binom{3,0}{2,1}$	$\binom{2,1}{3,0}$	$\binom{3,2}{1,0}$	$\binom{1,0}{3,2}$	$\binom{3,2,1,0}{-}$	$\binom{-}{3,2,1,0}$
$\binom{2,0}{1}$	✓							
$\binom{1,0}{2}$					✓			
$\binom{2,1}{0}$			✓					
$\binom{-}{2,1,0}$							✓	

Here a check mark "✓" in row $\Lambda\in\mathscr{S}_Z^{\mathrm{Sp}_4}$ and column $\Lambda'\in\mathscr{S}_{Z'}^{\mathrm{O}_8^+}$ means that $(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}$. We also see that $\mathscr{D}_{Z,Z'}=\mathscr{B}_{Z,Z'}\smallsetminus\{(\binom{-}{2,1,0},\binom{3,2,1,0}{-})\}$. Note that $(Z,Z')\in\mathscr{D}_{Z,Z'}$.

3. Finite Howe Correspondence of Unipotent Characters

In the first part of this section we review the parametrization of (irreducible) unipotent characters of a symplectic group or an even orthogonal group by Lusztig in [Lus81] and [Lus82]. A comparison of our main result and the conjecture in [AMR96] is in the final subsection.

3.1. Deligne-Lusztig virtual characters. If $\mathbf{G}$ is connected, let $R_{\mathbf{T},\theta}=R^{\mathbf{G}}_{\mathbf{T},\theta}$ denote the *Deligne-Lusztig virtual character* of G with respect to a rational maximal torus $\mathbf{T}$ and an irreducible character $\theta\in\mathscr{E}(T)$ where $T=\mathbf{T}^F$. If $\mathbf{G}=\mathrm{O}^\epsilon_n$, we define

$$R^{\mathrm{O}^\epsilon_n}_{\mathbf{T},\theta}=\mathrm{Ind}^{\mathrm{O}^\epsilon_n(q)}_{\mathrm{SO}^\epsilon_n(q)}R^{\mathrm{SO}^\epsilon_n}_{\mathbf{T},\theta}.$$

Let $\mathscr{V}(G)$ denote the space of class functions on G which is an inner product space with an orthonormal basis $\mathscr{E}(G)$. Let $\mathscr{V}(G)^\sharp$ denote the subspace of $\mathscr{V}(G)$ spanned by all Deligne-Lusztig virtual characters of G. For $f\in\mathscr{V}(G)$, the orthogonal projection $f^\sharp$ of f over $\mathscr{V}(G)^\sharp$ is called the *uniform projection* of f, and f is called *uniform* if $f^\sharp=f$.

If $\mathbf{G}$ is connected, it is well-known that the regular character $\mathrm{Reg}_{\mathbf{G}}$ of G is uniform (*cf.* [Car85] corollary 7.5.6). Because $\mathrm{Reg}_{\mathrm{O}^\epsilon}=\mathrm{Ind}^{\mathrm{O}^\epsilon}_{\mathrm{SO}^\epsilon}(\mathrm{Reg}_{\mathrm{SO}^\epsilon})$, we see that $\mathrm{Reg}_{\mathrm{O}^\epsilon}$ is also uniform. Therefore, we have

$$\rho(1)=\langle\rho,\mathrm{Reg}_{\mathbf{G}}\rangle_{\mathbf{G}}=\langle\rho^\sharp,\mathrm{Reg}_{\mathbf{G}}\rangle_{\mathbf{G}}=\rho^\sharp(1). \tag{3.1}$$

In particular, $\rho^\sharp\neq 0$ for any $\rho\in\mathscr{E}(G)$.

3.2. Unipotent characters of $\mathrm{Sp}_{2n}(q)$. From [Lus77] theorem 8.2, there exists a bijective parametrization $\mathscr{S}_{\mathrm{Sp}_{2n}} \to \mathscr{E}(\mathrm{Sp}_{2n})_1$ denoted by $\Lambda \mapsto \rho_\Lambda$. It is know that there is a one-to-one correspondence between the set $\mathscr{P}_2(n)$ and the set $\mathscr{E}(W_n)$ for the Weyl group W_n of Sp_{2n} (*cf.* [GP00] theorem 5.5.6). Then for a symbol $\Sigma \in \mathscr{S}_{n,1}$, we can associate a uniform function R_Σ on $\mathrm{Sp}_{2n}(q)$ given by $R_\Sigma = R_\chi$ where $R_\chi = R^{\mathbf{G}}_\chi$ is defined in [Pan21] subsection 3.2, and $\chi \in \mathscr{E}(W_n)$ associated to $\Upsilon(\Sigma)$ where Υ is the bijection $\mathscr{S}_{n,1} \to \mathscr{P}_2(n)$ given in (1.4).

For a special symbol Z of rank n and defect 1, let $\mathscr{V}_Z = \mathscr{V}(\mathbf{G})_Z$ denote the subspace spanned by $\{\rho_\Lambda \mid \Lambda \in \mathscr{S}_Z\}$. It is known that $\{R_\Sigma \mid \Sigma \in \mathscr{S}_{Z,1}\}$ forms an orthonormal basis for the uniform projection $\mathscr{V}_Z^\sharp$ of the space $\mathscr{V}_Z$. The following proposition is modified from [Lus81] theorem 5.8:

Proposition 3.2 (Lusztig). *Let $\mathbf{G} = \mathrm{Sp}_{2n}$, Z a special symbol of rank n and defect 1. For $\Sigma \in \mathscr{S}_{Z,1}$, we have*

$$\langle R_\Sigma, \rho_\Lambda\rangle_{\mathbf{G}} = \begin{cases} (-1)^{\langle \Sigma,\Lambda\rangle} 2^{-\deg(Z)}, & \text{if } \Lambda \in \mathscr{S}_Z; \\ 0, & \text{otherwise} \end{cases}$$

where $\langle,\rangle\colon \mathscr{S}_{Z,1} \times \mathscr{S}_Z \to \mathbf{F}_2$ is given by $\langle \Lambda_N, \Lambda_M\rangle = |N \cap M|$ (mod 2).

From the proposition we see that if $\rho_\Lambda \in \mathscr{S}_Z$, then

$$\rho_\Lambda^\sharp = \frac{1}{2^{\deg(Z)}} \sum_{\Sigma \in \mathscr{S}_{Z,1}} (-1)^{\langle \Sigma,\Lambda\rangle} R_\Sigma;$$

and if $\Sigma \in \mathscr{S}_{Z,1}$, then

$$R_\Sigma = \frac{1}{2^{\deg(Z)}} \sum_{\Lambda \in \mathscr{S}_Z} (-1)^{\langle \Sigma,\Lambda\rangle} \rho_\Lambda.$$

Example 3.3. Let $Z = \binom{2,0}{1}$, a special symbol of rank 2, degree 1 and defect 1. Now the table of $(-1)^{\langle \Sigma,\Lambda\rangle}$ for $\Sigma \in \mathscr{S}_{Z,1}$ and $\Lambda \in \mathscr{S}_Z$ is

	$\binom{2,0}{1}$	$\binom{2,1}{0}$	$\binom{1,0}{2}$	$\binom{-}{2,1,0}$
$\binom{2,0}{1}$	1	1	1	1
$\binom{2,1}{0}$	1	1	−1	−1
$\binom{1,0}{2}$	1	−1	1	−1

In the leftmost column are all $\Sigma \in \mathscr{S}_{Z,1}$ and in the topmost row are all $\Lambda \in \mathscr{S}_Z$. Therefore, we have

$$\begin{aligned} R_{\binom{2,0}{1}} &= \frac{1}{2}\big[\rho_{\binom{2,0}{1}} + \rho_{\binom{2,1}{0}} + \rho_{\binom{1,0}{2}} + \rho_{\binom{-}{2,1,0}}\big], \\ R_{\binom{2,1}{0}} &= \frac{1}{2}\big[\rho_{\binom{2,0}{1}} + \rho_{\binom{2,1}{0}} - \rho_{\binom{1,0}{2}} - \rho_{\binom{-}{2,1,0}}\big], \\ R_{\binom{1,0}{2}} &= \frac{1}{2}\big[\rho_{\binom{2,0}{1}} - \rho_{\binom{2,1}{0}} + \rho_{\binom{1,0}{2}} - \rho_{\binom{-}{2,1,0}}\big]. \end{aligned}$$

3.3. Unipotent characters of $\mathrm{O}^\epsilon_{2n}(q)$. From [Lus77] theorem 8.2, we know that there exists a bijective parametrization $\mathscr{S}_{\mathrm{O}^\epsilon_{2n}} \to \mathscr{E}(\mathrm{O}^\epsilon_{2n}(q))_1$ by $\Lambda \mapsto \rho_\Lambda$. It is also known that $\rho_{\Lambda^t} = \rho_\Lambda \cdot \mathrm{sgn}$.

For a special symbol Z of rank n and defect 0, as in the symplectic case, let $\mathscr{V}_Z$ denote the subspace spanned by $\{\rho_\Lambda \mid \Lambda \in \mathscr{S}_Z\}$.

- If Z is degenerate, i.e., $\deg(Z)=0$, then $\mathscr{S}_Z^{O^-_{2n}}=\emptyset$, $\mathscr{S}_Z^{O^+_{2n}}=\{Z\}$ and $\rho_Z=R_Z^{O^+_{2n}}$, i.e., $\mathscr{V}(O^+_{2n})_Z=\mathscr{V}(O^+_{2n})^\sharp_Z$ is one-dimensional.
- If Z is non-degenerate, i.e., $\deg(Z)\geq 1$, then $\Sigma\in\mathscr{S}_{Z,0}$ if and only if $\Sigma^t\in\mathscr{S}_{Z,0}$. It is known that $R_{\Sigma^t}^{O^\epsilon_{2n}}=\epsilon R_\Sigma^{O^\epsilon_{2n}}$ (*cf.* [Pan21] subsection 3.4). Let $\bar{\mathscr{S}}_{Z,0}$ denote a complete set of representatives of cosets $\{\Sigma,\Sigma^t\}$ in $\mathscr{S}_{Z,0}$, then $\{\frac{1}{\sqrt{2}}R_\Sigma^{O^\epsilon_{2n}}\mid\Sigma\in\bar{\mathscr{S}}_{Z,0}\}$ forms an orthonormal basis for $\mathscr{V}_Z^\sharp$.

The following proposition is a modification for O^ϵ_{2n} from [Lus82] theorem 3.15:

Proposition 3.4 (Lusztig). *Let* $\mathbf{G}=O^\epsilon_{2n}$ *where* $\epsilon=+$ *or* $-$, *Z a non-degenerate special symbol of rank n and defect* 0. *For any* $\Sigma\in\mathscr{S}_{Z,0}$, *we have*

$$\langle R_\Sigma^{\mathbf{G}},\rho_\Lambda\rangle_{\mathbf{G}}=\begin{cases}(-1)^{\langle\Sigma,\Lambda\rangle}2^{-(\deg(Z)-1)}, & \text{if } \Lambda\in\mathscr{S}_Z;\\ 0, & \text{otherwise}\end{cases}$$

where $\langle,\rangle\colon\mathscr{S}_{Z,0}\times\mathscr{S}_Z\to\mathbf{F}_2$ *by* $\langle\Lambda_M,\Lambda_N\rangle=|M\cap N|\pmod 2$.

From the proposition we see that if $\rho_\Lambda\in\mathscr{S}_Z$ (with Z non-degenerate), then

$$\rho_\Lambda^\sharp=\frac{1}{2^{\deg(Z)}}\sum_{\Sigma\in\bar{\mathscr{S}}_{Z,0}}(-1)^{\langle\Sigma,\Lambda\rangle}R_\Sigma^{\mathbf{G}};$$

and if $\Sigma\in\mathscr{S}_{Z,0}$, then

$$R_\Sigma^{\mathbf{G}}=\frac{1}{2^{\deg(Z)-1}}\sum_{\Lambda\in\mathscr{S}_Z}(-1)^{\langle\Sigma,\Lambda\rangle}\rho_\Lambda.$$

3.4. Strategy of the proof of the main result. Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},O^\epsilon_{2n'})$ where $\epsilon=+$ or $-$. All the efforts in this article are to remove the uniform projection of both sides of identity (1.7). The proof will be divided into two stages (Section 5 and Section 6):

- To recover the relation between $\omega_{\mathbf{G},\mathbf{G}',1}$ and $\sum_{(\Lambda,\Lambda')\in\mathscr{B}_{\mathbf{G},\mathbf{G}'}}\rho_\Lambda\otimes\rho_{\Lambda'}$ from the uniform projection, we will use the technique learned from [KS05], pp.436–438. That is, we reduce the problem into a system of linear equations. To write down these equations, we need the theory of "*cells*" by Lusztig from [Lus81] theorem 5.6 and [Lus82] proposition 3.13. The variables of the linear system are the multiplicities of those $\rho_\Lambda\otimes\rho_{\Lambda'}$ occurring in $\omega_{\mathbf{G},\mathbf{G}',1}$. The solutions must be non-negative integers, that is the reason why we are almost able to solve the equations. This means that little information is lost after taking the uniform projection. Due to the disconnectedness of O^ϵ_{2n}, irreducible characters $\rho_{\Lambda'},\rho_{\Lambda'^t}$ are not distinguishable by Deligne-Lusztig virtual characters. So in the first stage we can only conclude that $\rho_\Lambda\otimes\rho_{\Lambda'}$ or $\rho_\Lambda\otimes\rho_{\Lambda'^t}$ occur in $\omega_{\mathbf{G},\mathbf{G}',1}$ if and only if (Λ,Λ') or (Λ,Λ'^t) occur in $\mathscr{B}_{\mathbf{G},\mathbf{G}'}$.
- Because the Howe correspondence and the parametrization $\Lambda\mapsto\rho_\Lambda$ are both compatible with parabolic induction, the ambiguity in the first stage can be removed once the correspondence of unipotent cuspidal characters is fixed. The proof of the theorem is in Subsection 6.1 for $\mathrm{def}(\Lambda')>0$, and in Subsection 6.3 for $\mathrm{def}(\Lambda')=0$.

3.5. The conjecture by Aubert-Michel-Rouquier. In Theorem 1.8, we describe the Howe correspondence of unipotent characters in terms of Lusztig's "symbols"; the conjecture in [AMR96] p.383 describes the correspondence in terms of "bi-partitions". The main difference between these two descriptions is that a bi-partition does not contain the information of the "defect" of a symbol which is controlled by the unipotent cuspidal characters. Therefore, the description in [AMR96] p.383 needs to specify the correspondence of unipotent cuspidal characters first. Now we want to make the comparison more explicit.

In our convention, we always assume that the defect of a symbol for a symplectic group (resp. split even orthogonal group, non-split even orthogonal group) is 1 (mod 4) (resp. 0 (mod 4), 2 (mod 4)). Our convention is different from the original one in [Lus77] p.134 where the defect of a symbol is always assumed to be non-negative. In particular, the unique unipotent cuspidal character ζ_k of $\mathrm{Sp}_{2k(k+1)}(q)$ by our convention is parametrized by $\zeta_k = \rho_{\Lambda_k}$ where

$$\Lambda_k = \begin{cases} \binom{2k,2k-1,\ldots,1,0}{-}, & \text{if } k \text{ is even;} \\ \binom{-}{2k,2k-1,\ldots,1,0}, & \text{if } k \text{ is odd.} \end{cases} \tag{3.5}$$

Note that $\mathrm{def}(\Lambda_k) = (-1)^k(2k+1)$.

Example 3.6. Suppose that $\mathbf{G} = \mathrm{Sp}_{2k(k+1)+2t}$, $\mathbf{L} = \mathrm{Sp}_{2k(k+1)} \times \mathbf{T}_t$ where $\mathbf{T}_t$ is t-copies of GL_1. The irreducible constituents ρ_Λ in $R^{\mathbf{G}}_{\mathbf{L}}(\zeta_k)$ are parametrized by

$$\mathscr{S}_{k(k+1)+t,(-1)^k(2k+1)} = \{\, \Lambda \mid \mathrm{def}(\Lambda) = (-1)^k(2k+1),\ \Upsilon(\Lambda) \in \mathscr{P}_2(t)\,\}.$$

For the cases that $k = 0, 1, 2$ and $t = 0, 1, 2$ those Λ are given by the table:

k \ t		0	1		2				
k	$\Upsilon(\Lambda)$	$\left[{- \atop -}\right]$	$\left[{1 \atop -}\right]$	$\left[{- \atop 1}\right]$	$\left[{2 \atop -}\right]$	$\left[{1,1 \atop -}\right]$	$\left[{1 \atop 1}\right]$	$\left[{- \atop 2}\right]$	$\left[{- \atop 1,1}\right]$
0	Λ	$\binom{0}{-}$	$\binom{1}{-}$	$\binom{1,0}{1}$	$\binom{2}{-}$	$\binom{2,1}{0}$	$\binom{2,0}{1}$	$\binom{1,0}{2}$	$\binom{2,1,0}{2,1}$
1	Λ	$\binom{-}{2,1,0}$	$\binom{1}{3,2,1,0}$	$\binom{-}{3,1,0}$	$\binom{2}{3,2,1,0}$	$\binom{2,1}{4,3,2,1,0}$	$\binom{1}{4,2,1,0}$	$\binom{-}{4,1,0}$	$\binom{-}{3,2,0}$
2	Λ	$\binom{4,3,2,1,0}{-}$	$\binom{5,3,2,1,0}{-}$	$\binom{5,4,3,2,1,0}{1}$	$\binom{6,3,2,1,0}{-}$	$\binom{5,4,2,1,0}{-}$	$\binom{6,4,3,2,1,0}{1}$	$\binom{5,4,3,2,1,0}{2}$	$\binom{6,5,4,3,2,1,0}{2,1}$

For $k \geq 1$, let $\zeta^{\mathrm{I}}_k, \zeta^{\mathrm{II}}_k$ be the unipotent cuspidal characters of $\mathrm{O}^{\epsilon_k}_{2k^2}(q)$ where $\epsilon_k = (-1)^k$ such that $(\zeta_k, \zeta^{\mathrm{II}}_k)$ and $(\zeta_k, \zeta^{\mathrm{I}}_{k+1})$ occur in the Howe correspondence (*cf.* [AM93]). Then we have $\zeta^{\mathrm{I}}_k = \rho_{\Lambda'_k}$ and $\zeta^{\mathrm{II}}_k = \rho_{\Lambda'^{\mathrm{t}}_k}$ where

$$\Lambda'_k = \begin{cases} \binom{2k-1,2k-2,\ldots,1,0}{-}, & \text{if } k \text{ is even;} \\ \binom{-}{2k-1,2k-2,\ldots,1,0}, & \text{if } k \text{ is odd.} \end{cases} \tag{3.7}$$

Note that $\mathrm{def}(\Lambda'_k) = (-1)^k 2k$.

Example 3.8. Suppose that $\mathbf{G} = \mathrm{O}^{\epsilon_k}_{2k^2+2t}$ where $\epsilon_k = (-1)^k$, $\mathbf{L} = \mathrm{O}^{\epsilon_k}_{2k^2} \times \mathbf{T}_t$ where $\mathbf{T}_t$ is t-copies of GL_1. For $k \geq 1$, the irreducible constituents ρ_Λ in each $R^{\mathbf{G}}_{\mathbf{L}}(\zeta^{\mathrm{I}}_k), R^{\mathbf{G}}_{\mathbf{L}}(\zeta^{\mathrm{II}}_k)$ are parametrized respectively by:

$$\mathscr{S}_{k^2+t,(-1)^k 2k} = \{\, \Lambda \mid \mathrm{def}(\Lambda) = (-1)^k 2k,\ \Upsilon(\Lambda) \in \mathscr{P}_2(t)\,\},$$
$$\mathscr{S}_{k^2+t,(-1)^{k+1} 2k} = \{\, \Lambda \mid \mathrm{def}(\Lambda) = (-1)^{k+1} 2k,\ \Upsilon(\Lambda) \in \mathscr{P}_2(t)\,\}.$$

For the cases that $k=0,1,2$ and $t=0,1,2$ those Λ are given by the table:

		t	0	1		2				
k		$\Upsilon(\Lambda)$	$\begin{bmatrix}-\\-\end{bmatrix}$	$\begin{bmatrix}1\\-\end{bmatrix}$	$\begin{bmatrix}-\\1\end{bmatrix}$	$\begin{bmatrix}2\\-\end{bmatrix}$	$\begin{bmatrix}1,1\\-\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}-\\2\end{bmatrix}$	$\begin{bmatrix}-\\1,1\end{bmatrix}$
0		Λ	$\binom{-}{-}$	$\binom{1}{0}$	$\binom{0}{1}$	$\binom{2}{0}$	$\binom{2,1}{1,0}$	$\binom{1}{1}$	$\binom{0}{2}$	$\binom{1,0}{2,1}$
1	I	Λ	$\binom{-}{1,0}$	$\binom{1}{2,1,0}$	$\binom{-}{2,0}$	$\binom{2}{2,1,0}$	$\binom{2,1}{3,2,1,0}$	$\binom{1}{3,1,0}$	$\binom{-}{3,0}$	$\binom{-}{2,1}$
	II	Λ	$\binom{1,0}{-}$	$\binom{2,0}{-}$	$\binom{2,1,0}{1}$	$\binom{3,0}{-}$	$\binom{2,1}{-}$	$\binom{3,1,0}{1}$	$\binom{2,1,0}{2}$	$\binom{3,2,1,0}{2,1}$
2	I	Λ	$\binom{3,2,1,0}{-}$	$\binom{4,2,1,0}{-}$	$\binom{4,3,2,1,0}{1}$	$\binom{5,2,1,0}{-}$	$\binom{4,3,1,0}{-}$	$\binom{5,3,2,1,0}{1}$	$\binom{4,3,2,1,0}{2}$	$\binom{5,4,3,2,1,0}{2,1}$
	II	Λ	$\binom{-}{3,2,1,0}$	$\binom{1}{4,3,2,1,0}$	$\binom{-}{4,2,1,0}$	$\binom{2}{4,3,2,1,0}$	$\binom{2,1}{5,4,3,2,1,0}$	$\binom{1}{5,3,2,1,0}$	$\binom{-}{5,2,1,0}$	$\binom{-}{4,3,1,0}$

Following the notation in [AMR96], let $\Theta_{\zeta_k,\zeta_k^{\mathrm{II}}}$ and $\Theta_{\zeta_k,\zeta_{k+1}^{\mathrm{I}}}$ be the mappings between bi-partitions defined in [AMR96] p.383. (Note that ζ_k is denoted by λ_k in [AMR96], etc.) Moreover, let x_i, x_i^* and X, X^* be the notations used in [AMR96] p.383. Then [AMR96] conjecture 3.11 describes the correspondence in terms of bi-partitions "$\phi\boxtimes\psi$" by

- the relation $\Theta_{\zeta_k,\zeta_k^{\mathrm{II}}}$ is given by $\phi\boxtimes\psi\mapsto X^*(\phi)\boxtimes X\psi$; and
- the relation $\Theta_{\zeta_k,\zeta_{k+1}^{\mathrm{I}}}$ is given by $\phi\boxtimes\psi\mapsto X\phi\boxtimes X^*(\psi)$.

Proposition 3.9. *Keep the notations as above. If we apply the identification*

$$\begin{bmatrix}\phi\\\psi\end{bmatrix}\mapsto\begin{cases}\phi\boxtimes\psi, & \text{if } \mathrm{def}(\Lambda)>0;\\ \psi\boxtimes\phi, & \text{if } \mathrm{def}(\Lambda)\leq 0\end{cases}\tag{3.10}$$

where $\left[\begin{smallmatrix}\phi\\\psi\end{smallmatrix}\right]=\Upsilon(\Lambda)$, *then the statement in Theorem 1.8 is equivalent to the statement in* [AMR96] *conjecture 3.11.*

Proof. For $k\geq 0$, we define

$$\mathscr{B}_{\zeta_k,\zeta_k^{\mathrm{II}}}=\left\{(\Lambda,\Lambda')\in\bigcup_{n,n'\geq 0}\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}_{2n'}^{\epsilon_k}}\mid \mathrm{def}(\Lambda)=(-1)^k(2k+1)\right\};$$

$$\mathscr{B}_{\zeta_k,\zeta_{k+1}^{\mathrm{I}}}=\left\{(\Lambda,\Lambda')\in\bigcup_{n,n'\geq 0}\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}_{2n'}^{\epsilon_{k+1}}}\mid \mathrm{def}(\Lambda)=(-1)^k(2k+1)\right\}.$$

Now we want to show that the description of Howe correspondence in terms of $\mathscr{B}_{\zeta_k,\zeta_k^{\mathrm{II}}}$ and $\mathscr{B}_{\zeta_k,\zeta_{k+1}^{\mathrm{I}}}$ is equivalent to the description in terms of $\Theta_{\zeta_k,\zeta_k^{\mathrm{II}}}$ and $\Theta_{\zeta_k,\zeta_{k+1}^{\mathrm{I}}}$ respectively under the identification in (3.10).

For two symbols Λ,Λ' we write $\left[\begin{smallmatrix}\phi\\\psi\end{smallmatrix}\right]=\Upsilon(\Lambda)$ and $\left[\begin{smallmatrix}\phi'\\\psi'\end{smallmatrix}\right]=\Upsilon(\Lambda')$. First we consider the correspondences $\mathscr{B}_{\zeta_k,\zeta_k^{\mathrm{II}}}$ and $\Theta_{\zeta_k,\zeta_k^{\mathrm{II}}}$.

(1) Suppose that k is even. Then $\epsilon_k=+$. Now $(\Lambda,\Lambda')\in\mathscr{B}_{\zeta_k,\zeta_k^{\mathrm{II}}}$ if and only if
  - $\mathrm{def}(\Lambda)=2k+1$ and $\mathrm{def}(\Lambda')=-2k$
  - $\psi'\preccurlyeq\phi$ and $\psi\preccurlyeq\phi'$

  Now $\mathrm{def}(\Lambda)>0$ and $\mathrm{def}(\Lambda')\leq 0$, so by (3.10) we have the identifications $\left[\begin{smallmatrix}\phi\\\psi\end{smallmatrix}\right]\mapsto\phi\boxtimes\psi$ and $\left[\begin{smallmatrix}\phi'\\\psi'\end{smallmatrix}\right]\mapsto\psi'\boxtimes\phi'$. Then it is not difficult to see that the condition $\psi'\preccurlyeq\phi$ and $\psi\preccurlyeq\phi'$ is equivalent to the condition $\psi'\boxtimes\phi'=x_i^*(\phi)\boxtimes x_j(\psi)$ for some $i,j\geq 0$.

(2) Suppose that k is odd. Then $\epsilon_k=-$. Now $(\Lambda,\Lambda')\in\mathscr{B}_{\zeta_k,\zeta_k^{\mathrm{II}}}$ if and only if
  - $\mathrm{def}(\Lambda)=-2k-1$ and $\mathrm{def}(\Lambda')=2k$

- $\phi' \preccurlyeq \psi$ and $\phi \preccurlyeq \psi'$

Now $\operatorname{def}(\Lambda) < 0$ and $\operatorname{def}(\Lambda') > 0$, so we have $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] \mapsto \psi \boxtimes \phi$ and $\left[\begin{smallmatrix}\phi'\\ \psi'\end{smallmatrix}\right] \mapsto \phi' \boxtimes \psi'$. Then the condition $\phi' \preccurlyeq \psi$ and $\phi \preccurlyeq \psi'$ is equivalent to the condition $\phi' \boxtimes \psi' = x_i^*(\psi) \boxtimes x_j(\phi)$ for some $i, j \geq 0$.

Next we consider the correspondences $\mathscr{B}_{\zeta_k,\zeta^{\mathrm{I}}_{k+1}}$ and $\Theta_{\zeta_k,\zeta^{\mathrm{I}}_{k+1}}$.

(3) Suppose that k is even. Then $\epsilon_{k+1} = -$. Now $(\Lambda, \Lambda') \in \mathscr{B}_{\zeta_k,\zeta^{\mathrm{I}}_{k+1}}$ if and only if
- $\operatorname{def}(\Lambda) = 2k+1$ and $\operatorname{def}(\Lambda') = -2(k+1)$
- $\phi' \preccurlyeq \psi$ and $\phi \preccurlyeq \psi'$

Now $\operatorname{def}(\Lambda) > 0$ and $\operatorname{def}(\Lambda') \leq 0$, so we have $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] \mapsto \phi \boxtimes \psi$ and $\left[\begin{smallmatrix}\phi'\\ \psi'\end{smallmatrix}\right] \mapsto \psi' \boxtimes \phi'$. Then the condition $\phi' \preccurlyeq \psi$ and $\phi \preccurlyeq \psi'$ is equivalent to the condition $\psi' \boxtimes \phi' = x_i(\phi) \boxtimes x_j^*(\psi)$ for some $i, j \geq 0$.

(4) Suppose that k is odd. Then $\epsilon_{k+1} = +$. Now $(\Lambda, \Lambda') \in \mathscr{B}_{\zeta_k,\zeta^{\mathrm{I}}_{k+1}}$ if and only if
- $\operatorname{def}(\Lambda) = -2k-1$ and $\operatorname{def}(\Lambda') = 2(k+1)$
- $\psi' \preccurlyeq \phi$ and $\psi \preccurlyeq \phi'$

Now $\operatorname{def}(\Lambda) < 0$ and $\operatorname{def}(\Lambda') > 0$, so we have $\left[\begin{smallmatrix}\phi\\ \psi\end{smallmatrix}\right] \mapsto \psi \boxtimes \phi$ and $\left[\begin{smallmatrix}\phi'\\ \psi'\end{smallmatrix}\right] \mapsto \phi' \boxtimes \psi'$. Then the condition $\psi' \preccurlyeq \phi$ and $\psi \preccurlyeq \phi'$ is equivalent to the condition $\phi' \boxtimes \psi' = x_i(\psi) \boxtimes x_j^*(\phi)$ for some $i, j \geq 0$.

Hence the proposition is proved. □

4. Cells for a Symplectic Group or an Even Orthogonal Group

In this section, we provide several technical lemmas which are needed in the next two sections.

4.1. Consecutive pairs. Let $\mathbf{G} = \mathrm{Sp}_{2n}$ or $\mathrm{O}^{\epsilon}_{2n}$ where $\epsilon = +$ or $-$, and let Z be a special symbol of rank n, Z_{I} the subsymbol of singles of Z.

A pair $\binom{s}{t} \subset Z_{\mathrm{I}}$ is called *consecutive* if there is no other entries in Z lying between s and t i.e., there is no entry x in Z such that $s < x < t$ or $t < x < s$. For a set of (disjoint) consecutive pairs Ψ_0 in Z_{I}, we define:

$$
\begin{aligned}
\mathscr{S}_{Z,\Psi_0} &= \{\, \Lambda_M \mid M \leq \Psi_0 \,\}, \\
\mathscr{S}_Z^{\Psi_0} = \mathscr{S}_Z^{\mathbf{G},\Psi_0} &= \{\, \Lambda_M \in \mathscr{S}_Z^{\mathbf{G}} \mid M \subset Z_{\mathrm{I}} \smallsetminus \Psi_0 \,\}
\end{aligned}
\tag{4.1}
$$

where $M \leq \Psi_0$ means that M is a subset of pairs in Ψ_0 and is regarded as a subsymbol of Z_{I}. If $\Psi_0 = \emptyset$, it is clear that $\mathscr{S}_{Z,\Psi_0} = \{Z\}$ and $\mathscr{S}_Z^{\Psi_0} = \mathscr{S}_Z^{\mathbf{G}}$. If $M \leq \Psi_0$, then $|M^*| = |M_*|$ and hence $\operatorname{def}(\Lambda_M) = \operatorname{def}(Z)$. Therefore $\mathscr{S}_{Z,\Psi_0} \subset \mathscr{S}_{Z,1}$ if $\operatorname{def}(Z) = 1$; and $\mathscr{S}_{Z,\Psi_0} \subset \mathscr{S}_{Z,0}$ if $\operatorname{def}(Z) = 0$. Suppose that $\delta = \deg(Z)$ and δ_0 is the number of pairs in Ψ_0. Then it is not difficult to see that

$$
\begin{aligned}
|\mathscr{S}_{Z,\Psi_0}| &= 2^{\delta_0}, \\
|\mathscr{S}_Z^{\Psi_0}| &= \begin{cases} 2^{2(\delta-\delta_0)}, & \text{if } \operatorname{def}(Z) = 1; \\ 2^{2(\delta-\delta_0)-1}, & \text{if } \operatorname{def}(Z) = 0, \text{ and } \delta > \delta_0; \\ 1, & \text{if } \operatorname{def}(Z) = 0,\ \delta = \delta_0, \text{ and } \epsilon = +; \\ 0, & \text{if } \operatorname{def}(Z) = 0,\ \delta = \delta_0, \text{ and } \epsilon = -. \end{cases}
\end{aligned}
\tag{4.2}
$$

Note that if $\Lambda_1 \in \mathscr{S}_Z^{\Psi_0}$ and $\Lambda_2 \in \mathscr{S}_{Z,\Psi_0}$, then $\Lambda_1 + \Lambda_2$ is in $\mathscr{S}_Z^{\mathbf{G}}$.

Remark 4.3. Note that $\mathscr{S}_Z^{\mathbf{G},\Psi_0}$ is always a subset of $\mathscr{S}_Z^{\mathbf{G}}$; and $\mathscr{S}_{Z,\Psi_0}$ is a subset of $\mathscr{S}_Z^{\mathbf{G}}$ if $\mathbf{G} = \mathrm{Sp}_{2n}$ or O_{2n}^+. However, $\mathscr{S}_{Z,\Psi_0}$ is a subset of $\mathscr{S}_Z^{\mathrm{O}_{2n}^+}$ even if $\mathbf{G} = \mathrm{O}_{2n}^-$.

Lemma 4.4. *Let $\mathbf{G} = \mathrm{Sp}_{2n}$ or O_{2n}^+, Z a special symbol of rank n, Ψ_0 a set of consecutive pairs in Z_{I}. Then $\mathscr{S}_{Z,\Psi_0} \cap \mathscr{S}_Z^{\mathbf{G},\Psi_0} = \{Z\}$.*

Proof. Because now we assume that $\mathbf{G} = \mathrm{Sp}_{2n}$ or O_{2n}^+, both $\mathscr{S}_{Z,\Psi_0}$ and $\mathscr{S}_Z^{\mathbf{G},\Psi_0}$ are subsets of $\mathscr{S}_Z^{\mathbf{G}}$. Suppose that $\Lambda_M \in \mathscr{S}_{Z,\Psi_0} \cap \mathscr{S}_Z^{\mathbf{G},\Psi_0}$ for some $M \subset Z_{\mathrm{I}}$. From (4.1), we see that the only possible M is the empty set, and so $\Lambda_M = Z$. □

Example 4.5. Let $\mathbf{G} = \mathrm{O}_8^\epsilon$, and let $Z = \binom{3,1}{2,0}$. Now $Z_{\mathrm{I}} = Z$, and $\left\{\binom{1}{0}\right\}$, $\left\{\binom{1}{2}\right\}$, $\left\{\binom{3}{2}\right\}$, and $\left\{\binom{3}{2},\binom{1}{0}\right\}$ are the possible nonempty set of consecutive pairs Ψ_0 in Z_{I}. Then we have

ϵ	Ψ_0	$\left\{\binom{1}{0}\right\}$	$\left\{\binom{1}{2}\right\}$	$\left\{\binom{3}{2}\right\}$	$\left\{\binom{3}{2},\binom{1}{0}\right\}$
	$\mathscr{S}_{Z,\Psi_0}$	$\left\{\binom{3,1}{2,0},\binom{3,0}{2,1}\right\}$	$\left\{\binom{3,1}{2,0},\binom{3,2}{1,0}\right\}$	$\left\{\binom{3,1}{2,0},\binom{2,1}{3,0}\right\}$	$\left\{\binom{3,1}{2,0},\binom{3,2}{1,0},\binom{3,0}{2,1},\binom{2,0}{3,1}\right\}$
$+$	$\mathscr{S}_Z^{\mathbf{G},\Psi_0}$	$\left\{\binom{3,1}{2,0},\binom{2,1}{3,0}\right\}$	$\left\{\binom{3,1}{2,0},\binom{1,0}{3,2}\right\}$	$\left\{\binom{3,1}{2,0},\binom{3,0}{2,1}\right\}$	$\left\{\binom{3,1}{2,0}\right\}$
$-$	$\mathscr{S}_Z^{\mathbf{G},\Psi_0}$	$\left\{\binom{3,2,1}{0},\binom{1}{3,2,0}\right\}$	$\left\{\binom{3,1,0}{2},\binom{1}{3,2,0}\right\}$	$\left\{\binom{3,1,0}{2},\binom{3}{2,1,0}\right\}$	$\emptyset$

Let us give an example to see how to compute this table. If $\Psi_0 = \left\{\binom{1}{0}\right\}$ and $\epsilon = -$, then the possible subsymbols M of $Z_{\mathrm{I}} \smallsetminus \Psi_0 = \binom{3}{2}$ with odd number of entries are $\binom{-}{2}$ and $\binom{3}{-}$, and so the possible Λ_M are $\binom{3,2,1}{0}$ and $\binom{1}{3,2,0}$. Hence $\mathscr{S}_Z^{\mathbf{G},\Psi_0} = \left\{\binom{3,2,1}{0},\binom{1}{3,2,0}\right\}$. Note that $\mathscr{S}_{Z,\Psi_0}$ does not depend on ϵ.

Lemma 4.6. *Let Ψ_0 be a set of consecutive pairs in Z_{I}. Suppose that $\Lambda_1, \Lambda_1' \in \mathscr{S}_Z^{\mathbf{G},\Psi_0}$ and $\Lambda_2, \Lambda_2' \in \mathscr{S}_{Z,\Psi_0}$. If $\rho_{\Lambda_1+\Lambda_2} = \rho_{\Lambda_1'+\Lambda_2'}$, then $\Lambda_1 = \Lambda_1'$ and $\Lambda_2 = \Lambda_2'$.*

Proof. If $\rho_{\Lambda_1+\Lambda_2} = \rho_{\Lambda_1'+\Lambda_2'}$, then $\Lambda_1 + \Lambda_2 = \Lambda_1' + \Lambda_2'$. Note that $\Lambda + Z = \Lambda$ and $\Lambda + \Lambda = Z$ for any $\Lambda \in \mathscr{S}_Z^{\mathbf{G}}$. Therefore we have

$$\Lambda_1 + \Lambda_1' = \Lambda_1 + \Lambda_2 + \Lambda_1' + \Lambda_2 = \Lambda_1' + \Lambda_2' + \Lambda_1' + \Lambda_2 = \Lambda_2 + \Lambda_2'.$$

(1) Suppose that $\mathbf{G} = \mathrm{Sp}_{2n}$ or O_{2n}^+. Note that both $\mathscr{S}_{Z,\Psi_0}$ and $\mathscr{S}_Z^{\mathbf{G},\Psi_0}$ are closed under addition and $\mathscr{S}_{Z,\Psi_0} \cap \mathscr{S}_Z^{\mathbf{G},\Psi_0} = \{Z\}$ by Lemma 4.4.
(2) Suppose that $\mathbf{G} = \mathrm{O}_{2n}^-$. Now $\mathscr{S}_{Z,\Psi_0} \subset \mathscr{S}_{\mathrm{O}_{2n}^+}$ and is still closed under addition. Moreover, $\Lambda_1 + \Lambda_1' \in \mathscr{S}_Z^{\mathrm{O}_{2n}^+,\Psi_0}$, and $\mathscr{S}_{Z,\Psi_0} \cap \mathscr{S}_Z^{\mathrm{O}_{2n}^+,\Psi_0} = \{Z\}$ by Lemma 4.4, again.

Therefore, for both (1) and (2), we conclude that $\Lambda_1 + \Lambda_1' = Z$, i.e., $\Lambda_1 = \Lambda_1' + Z = \Lambda_1'$. Similarly, we have $\Lambda_2 = \Lambda_2'$. □

4.2. Cells. We first recall the notion of "cells" by Lusztig from [Lus81] and [Lus82]. Let Z be a special symbol with symbol of singles Z_{I}, and let $\delta = \deg(Z)$. Then we have $|Z_{\mathrm{I}}| = 2\delta + \mathrm{def}(Z)$ from Subsection 2.2.

(1) If Z is of defect 1, then an *arrangement* of Z_{I} is defined to be a partition Φ of the $2\delta + 1$ singles in Z_{I} into δ (disjoint) pairs and one isolated element such that each pair contains one entry in the first row and one entry in the second row of Z_{I}.
(2) If Z is of defect 0, then an *arrangement* of Z_{I} is defined to be a partition Φ of the 2δ singles in Z_{I} into δ pairs such that each pair contains one entry in the first row and one entry in the second row of Z_{I}.

A set Ψ of some pairs (possibly empty) in Φ is called a *subset of pairs* of Φ and is denoted by $\Psi\leq\Phi$. Note that if $\Psi\leq\Phi$, then Ψ does not contain the isolated element in the arrangement Φ. A subset of pairs Ψ of an arrangement Φ of $Z_{\rm I}$ can be regarded a subsymbol of $Z_{\rm I}$, and as usual let Ψ^* (resp. Ψ_*) denote the set of entries in the first (resp. second) row in Ψ.

Example 4.7. The symbol $Z=\binom{4,2,0}{3,1}$ is a special symbol of rank 6 and defect 1, and $Z_{\rm I}=Z$. The following are all possible arrangements of $Z_{\rm I}$:

$$\Phi_1=\{\tbinom{4}{3},\tbinom{2}{1},\tbinom{0}{-}\},\qquad \Phi_2=\{\tbinom{4}{1},\tbinom{2}{3},\tbinom{0}{-}\},\qquad \Phi_3=\{\tbinom{4}{-},\tbinom{2}{3},\tbinom{0}{1}\},$$
$$\Phi_4=\{\tbinom{4}{-},\tbinom{2}{1},\tbinom{0}{3}\},\qquad \Phi_5=\{\tbinom{4}{3},\tbinom{2}{-},\tbinom{0}{1}\},\qquad \Phi_6=\{\tbinom{4}{1},\tbinom{2}{-},\tbinom{0}{3}\}.$$

Each Φ_i has 4 subsets of pairs, for example,

$$\{\Psi\mid\Psi\leq\Phi_1\}=\{\emptyset,\{\tbinom{4}{3}\},\{\tbinom{2}{1}\},\{\tbinom{4}{3},\tbinom{2}{1}\}\}.$$

Each Ψ is regarded as a subsymbol of $Z_{\rm I}$, and so we have

$$\{\Psi\mid\Psi\leq\Phi_1\}=\{\emptyset,\tbinom{4}{3},\tbinom{2}{1},\tbinom{4,2}{3,1}\}.$$

For a subset of pairs Ψ of an arrangement Φ of $Z_{\rm I}$, recall that the following uniform class function on G is defined in [Lus81]:

$$R_{\underline{c}}=R_{\underline{c}(Z,\Phi,\Psi)}=\sum_{\Psi'\leq\Phi}(-1)^{|(\Phi\smallsetminus\Psi)\cap\Psi'^*|}R_{\Lambda_{\Psi'}}\tag{4.8}$$

where $\Lambda_{\Psi'}=(Z\smallsetminus\Psi')\cup\Psi'^{\rm t}$ is defined as in (2.4), and $(\Phi\smallsetminus\Psi)\cap\Psi'^*$ is understood to be the set of entries $((\Phi\smallsetminus\Psi)^*\cup(\Phi\smallsetminus\Psi)_*)\cap\Psi'^*$. Note that $\operatorname{def}(\Lambda_{\Psi'})=\operatorname{def}(Z)$, and $R_{\Lambda_{\Psi'}}=R^{\rm G}_{\Lambda_{\Psi'}}$ is given in Subsection 3.2 and Subsection 3.3.

Remark 4.9. Our notation is slightly different from that in [Lus81] and [Lus82]. More precisely, the uniform class function $R_{\underline{c}(Z,\Phi,\Psi)}$ in (4.8) is denoted by $R(\underline{c}(Z,\Phi,\Phi\smallsetminus\Psi))$ in [Lus81] and [Lus82].

For a subset of pairs Ψ of an arrangement Φ of $Z_{\rm I}$,

- if $\operatorname{def}(Z)=1$, we define

$$C_{\Phi,\Psi}=\{\Lambda_M\in\mathscr{S}_Z\mid |M\cap\Psi'|\equiv|(\Phi\smallsetminus\Psi)\cap\Psi'^*|\pmod 2\text{ for all }\Psi'\leq\Phi\};\tag{4.10}$$

- if $\operatorname{def}(Z)=0$, we define

$$C_{\Phi,\Psi}=\{\Lambda_M\mid M\subset Z_{\rm I},\ |M\cap\Psi'|\equiv|(\Phi\smallsetminus\Psi)\cap\Psi'^*|\pmod 2\text{ for all }\Psi'\leq\Phi\}.\tag{4.11}$$

Such a set $C_{\Phi,\Psi}$ is called a *cell*. From the definition it is not difficult to see that a symbol Λ_M is in $C_{\Phi,\Psi}$ if and only if the subsymbol M of $Z_{\rm I}$ satisfies the following two conditions:

- M contains either none or two entries of each pair in Ψ; and
- M contains exactly one entry of each pair in $\Phi\smallsetminus\Psi$.

In particular, it is clear from the definition that if Ψ consists of all pairs in Φ, then we have

$$C_{\Phi,\Psi}=\{\Lambda_M\mid M\leq\Phi\}.\tag{4.12}$$

Remark 4.13. (1) Suppose that Z is of rank n and defect 1 and $\Lambda_M\in C_{\Phi,\Psi}\subset\mathscr{S}_Z^{{\rm Sp}_{2n}}$ for some Φ,Ψ. The requirement that $|M|$ is even (*cf.* (2.6)) implies that M must contain the isolated element in the arrangement Φ if $\Phi\smallsetminus\Psi$ consists of an odd number of pairs; and M does not contain the isolated element if $\Phi\smallsetminus\Psi$ consists of an even number of pairs.

(2) Suppose that Z is of rank n and defect 0. We shall see in Lemma 4.32 that $C_{\Phi,\Psi}\subset \mathscr{S}_Z^{\mathrm{O}^+_{2n}}$ if $\Phi\smallsetminus\Psi$ consists of an even number of pairs; and $C_{\Phi,\Psi}\subset\mathscr{S}_Z^{\mathrm{O}^-_{2n}}$ if $\Phi\smallsetminus\Psi$ consists of an odd number of pairs.

Example 4.14. Suppose that $Z=\binom{4,2,0}{3,1}$, $\Phi=\{\binom{4}{-},\binom{2}{3},\binom{0}{1}\}$ and $\Psi=\{\binom{0}{1}\}$. There are four possible $M\subset Z_{\mathrm{I}}$ that satisfies the condition in (4.10), namely, $\binom{4,2,0}{1}$, $\binom{4,2}{-}$, $\binom{4,0}{3,1}$ and $\binom{4}{3}$, and resulting Λ_M are $\binom{1}{4,3,2,0}$, $\binom{0}{4,3,2,1}$, $\binom{3,2,1}{4,0}$ and $\binom{3,2,0}{4,1}$ respectively, i.e.,

$$C_{\Phi,\Psi}=\left\{\binom{1}{4,3,2,0},\binom{0}{4,3,2,1},\binom{3,2,1}{4,0},\binom{3,2,0}{4,1}\right\}.$$

Lemma 4.15. *Suppose that both $\Lambda_{M_1},\Lambda_{M_2}$ are in $C_{\Phi,\Psi}$ for some arrangement Φ of Z_{I} and some $\Psi\leq\Phi$. Then*

$$|M_1\cap\Psi'|\equiv|M_2\cap\Psi'|\pmod 2$$

for any $\Psi'\leq\Phi$.

Proof. Suppose that $\Lambda_{M_1},\Lambda_{M_2}\in C_{\Phi,\Psi}$. Then by (4.10) or (4.11) we have

$$|M_1\cap\Psi'|\equiv|(\Phi\smallsetminus\Psi)\cap\Psi'^*|\equiv|M_2\cap\Psi'|\pmod 2$$

for all subsets Ψ' of pairs of Φ. □

Lemma 4.16. *Let Z be a special symbol, Φ an arrangement of Z_{I}, Ψ_0 a subset of consecutive pairs, and $\Lambda\in\mathscr{S}_Z^{\Psi_0}$. If $\Lambda\in C_{\Phi,\Psi}$ for some $\Psi\leq\Phi$, then $\Psi_0\leq\Psi$.*

Proof. Suppose that $\Lambda=\Lambda_M$ for some $M\subset Z_{\mathrm{I}}\smallsetminus\Psi_0$, i.e., $M\cap\Psi_0=\emptyset$. From the rule before Remark 4.13, the assumption $\Lambda\in C_{\Phi,\Psi}$ implies that M contains exactly one entry from each pair in $\Phi\smallsetminus\Psi$. Therefore we must have $\Psi_0\leq\Psi$. □

4.3. Cells for a symplectic group. In this subsection, let $\mathbf{G}=\mathrm{Sp}_{2n}$, and let Z be a special symbol of rank n and defect 1.

Lemma 4.17. *Let Z be a special symbol of defect 1, Φ a fixed arrangement of Z_{I}, Ψ,Ψ' subsets of pairs of Φ. Then*

(i) $|C_{\Phi,\Psi}|=2^{\deg(Z)}$;
(ii) *if $\Psi\neq\Psi'$, then $C_{\Phi,\Psi}\cap C_{\Phi,\Psi'}=\emptyset$;*
(iii) $\mathscr{S}_Z=\bigcup_{\Psi\leq\Phi}C_{\Phi,\Psi}$.

Proof. Let z_0 denote the isolated element in Φ. Suppose that Λ_M is an element of $C_{\Phi,\Psi}$. From the conditions before Remark 4.13, we can write $M=M_1\cup M_2$ where M_1 consists of exactly one element from each pair of $\Phi\smallsetminus\Psi$ and possibly z_0 so that $|M_1|$ is even, and M_2 consists of some pairs from Ψ.

Let $\delta=\deg(Z)$. Suppose that Ψ consists of δ' pairs for some $\delta'\leq\delta$. So we have $2^{\delta'}$ possible choices for M_2. We have $2^{\delta-\delta'}$ choices when we chose one element from each pair in $\Phi\smallsetminus\Psi$ and we have two choices to choose z_0 or not. However, the requirement that $|M_1|$ is even implies that the possible choices of M_1 is exactly $2^{\delta-\delta'}$. Thus the total choices for M is $2^{\delta'}\cdot 2^{\delta-\delta'}=2^\delta$ and hence (i) is proved.

Suppose $\Psi\neq\Psi'$ and $\Lambda_M\in C_{\Phi,\Psi}\cap C_{\Phi,\Psi'}$ for some $M\subset Z_{\mathrm{I}}$. Without loss of generality, we may assume that $\Psi\not\subseteq\Psi'$, so there is a pair $\binom{s}{t}\in\Phi$ such that $\binom{s}{t}\in\Psi$ and $\binom{s}{t}\notin\Psi'$. By the two conditions before Remark 4.13, $\Lambda_M\in C_{\Phi,\Psi}$ implies that $|M\cap\binom{s}{t}|=0$ or 2, and $\Lambda_M\in C_{\Phi,\Psi'}$ implies that $|M\cap\binom{s}{t}|=1$. We get a contradiction and hence (ii) is proved.

We know that $|\mathscr{S}_Z|=2^{2\delta}$ from Subsection 2.2, and we have 2^δ choices of Ψ for a fixed arrangement Φ. Therefore (iii) follows from (i) and (ii) directly. □

Proposition 4.18. *Let* $\mathbf{G}=\mathrm{Sp}_{2n}$, Z *a special symbol of rank* n *and defect* 1, Φ *an arrangement of* $Z_{\rm I}$ *and* $\Psi\leq\Phi$. *Then*

$$R_{\underline{c}(Z,\Phi,\Psi)}=\sum_{\Lambda\in C_{\Phi,\Psi}}\rho_\Lambda.$$

In particular, the class function $\sum_{\Lambda\in C_{\Phi,\Psi}}\rho_\Lambda$ *is uniform.*

Proof. Let Λ_M be a symbol in $C_{\Phi,\Psi}$. From Proposition 3.2, we have

$$\langle\rho_{\Lambda_M},R_{\Lambda_{\Psi'}}\rangle=\frac{1}{2^\delta}(-1)^{|M\cap\Psi'|}$$

where $\delta=\deg(Z)$ and $\Psi'\leq\Phi$. Then by (4.8) and (4.10), we have

$$\langle\rho_{\Lambda_M},R_{\underline{c}(Z,\Phi,\Psi)}\rangle=\frac{1}{2^\delta}\sum_{\Psi'\leq\Phi}(-1)^{|(\Phi\smallsetminus\Psi)\cap\Psi'^*|}(-1)^{|M\cap\Psi'|}=\frac{1}{2^\delta}\sum_{\Psi'\leq\Phi}1=1.$$

This means ρ_Λ occurs with multiplicity one in $R_{\underline{c}(Z,\Phi,\Psi)}$ for each $\Lambda\in C_{\Phi,\Psi}$. From [Lus81] theorem 5.6 we know that $R_{\underline{c}(Z,\Phi,\Psi)}$ is a sum of 2^δ distinct irreducible characters of G and $C_{\Phi,\Psi}$ has also 2^δ elements. As all the ρ_Λ are non-isomorphic, the result follows. □

Example 4.19. Let $Z=\binom{2,0}{1}$, a special symbol of rank 2 and defect 1. Now $Z_{\rm I}=Z$ has two possible arrangements Φ, namely $\{\binom{2}{1},\binom{0}{-}\}$ and $\{\binom{0}{1},\binom{2}{-}\}$; and each arrangement Φ has two subsets of pairs Ψ, namely the only pair in Φ and the empty symbol $\binom{-}{-}$. So we have the following table:

Φ	Ψ	$R_{\underline{c}(Z,\Phi,\Psi)}$	$\sum_{\Lambda\in C_{\Phi,\Psi}}\rho_\Lambda$
$\{\binom{2}{1},\binom{0}{-}\}$	$\binom{2}{1}$	$R_{\binom{2,0}{1}}+R_{\binom{1,0}{2}}$	$\rho_{\binom{2,0}{1}}+\rho_{\binom{1,0}{2}}$
	$\binom{-}{-}$	$R_{\binom{2,0}{1}}-R_{\binom{1,0}{2}}$	$\rho_{\binom{2,1}{0}}+\rho_{\binom{-}{2,1,0}}$
$\{\binom{0}{1},\binom{2}{-}\}$	$\binom{0}{1}$	$R_{\binom{2,0}{1}}+R_{\binom{2,1}{0}}$	$\rho_{\binom{2,0}{1}}+\rho_{\binom{2,1}{0}}$
	$\binom{-}{-}$	$R_{\binom{2,0}{1}}-R_{\binom{2,1}{0}}$	$\rho_{\binom{1,0}{2}}+\rho_{\binom{-}{2,1,0}}$

The equality between $R_{\underline{c}(Z,\Phi,\Psi)}$ and $\sum_{\Lambda\in C_{\Phi,\Psi}}\rho_\Lambda$ can easily be seen from the identities in Example 3.3.

Remark 4.20. Note that in [Lus81] theorem 5.6, the cardinality q of the base field is assumed to be large, however, according the comment by the end of [Lus82] the restriction is removed by a result of Asai.

Remark 4.21. If Ψ consists of all pairs in Φ, then $(\Phi\smallsetminus\Psi)\cap\Psi'^*=\emptyset$ and $(-1)^{|(\Phi\smallsetminus\Psi)\cap\Psi'^*|}=1$ for any $\Psi'\leq\Phi$, and by (4.12) the identity in Proposition 4.18 becomes

$$\sum_{\Psi'\leq\Phi}R_{\Lambda_{\Psi'}}=\sum_{\Psi'\leq\Phi}\rho_{\Lambda_{\Psi'}}.$$

Lemma 4.22. *Suppose that* Z *is a special symbol of defect* 1 *with* $Z_{\rm I}=\binom{s_1,s_2,\ldots,s_{\delta+1}}{t_1,t_2,\ldots,t_\delta}$ *where* $\delta=\deg(Z)$. *Let* Φ_1,Φ_2 *be two arrangements of* $Z_{\rm I}$ *given by*

$$\Phi_1=\left\{\binom{s_1}{t_1},\binom{s_2}{t_2},\cdots,\binom{s_\delta}{t_\delta},\binom{s_{\delta+1}}{-}\right\},\qquad \Phi_2=\left\{\binom{s_1}{-},\binom{s_2}{t_1},\binom{s_3}{t_2},\cdots,\binom{s_{\delta+1}}{t_\delta}\right\}.$$

Then for any $\Psi_1\leq\Phi_1$ *and any* $\Psi_2\leq\Phi_2$, *we have* $|C_{\Phi_1,\Psi_1}\cap C_{\Phi_2,\Psi_2}|=1$.

Proof. Let $\Psi_1\leq\Phi_1$ and $\Psi_2\leq\Phi_2$. Suppose that Λ_M is in the intersection $C_{\Phi_1,\Psi_1}\cap C_{\Phi_2,\Psi_2}$ for some $M\subset Z_{\rm I}$ with $|M|$ even. From the two conditions before Remark 4.13, we have the following inferences:

(1) if $s_i \in M$ and $\binom{s_i}{t_i} \leq \Psi_1$, then $t_i \in M$;
(2) if $s_i \notin M$ and $\binom{s_i}{t_i} \leq \Psi_1$, then $t_i \notin M$;
(3) if $s_i \in M$ and $\binom{s_i}{t_i} \not\leq \Psi_1$, then $t_i \notin M$;
(4) if $s_i \notin M$ and $\binom{s_i}{t_i} \not\leq \Psi_1$, then $t_i \in M$;
(5) if $t_i \in M$ and $\binom{s_{i+1}}{t_i} \leq \Psi_2$, then $s_{i+1} \in M$;
(6) if $t_i \notin M$ and $\binom{s_{i+1}}{t_i} \leq \Psi_2$, then $s_{i+1} \notin M$;
(7) if $t_i \in M$ and $\binom{s_{i+1}}{t_i} \not\leq \Psi_2$, then $s_{i+1} \notin M$;
(8) if $t_i \notin M$ and $\binom{s_{i+1}}{t_i} \not\leq \Psi_2$, then $s_{i+1} \in M$

for $i = 1, \ldots, \delta$. This means that for any fixed Ψ_1, Ψ_2, the set M is uniquely determined by the "initial condition" whether s_1 belongs to M or not. So now there are two possible choices of M one of which contains s_1 and the other does not. Moreover, from (1)–(8) above, it is easy to see that both possible choices of M are complement subsets to each other in $Z_{\rm I}$, i.e., the two possible choices of M form a partition of $Z_{\rm I}$. Moreover, among the two possible choices of M, there is only one whose cardinality is even, and hence the lemma is proved. □

Lemma 4.23. *Let* Z *be a special symbol of defect* 1*, and let* Φ_1, Φ_2 *be the two arrangements of* $Z_{\rm I}$ *given in Lemma 4.22. For any given* $\Lambda \in \mathcal{S}_Z$*, there exist* $\Psi_1 \leq \Phi_1$ *and* $\Psi_2 \leq \Phi_2$ *such that* $C_{\Phi_1,\Psi_1} \cap C_{\Phi_2,\Psi_2} = \{\Lambda\}$.

Proof. Let $\Lambda \in \mathcal{S}_Z$, and let Φ_1, Φ_2 be the two arrangements given in Lemma 4.22. By (iii) of Lemma 4.17, there is a subset of pairs Ψ_i of Φ_i such that $\Lambda \in C_{\Phi_i,\Psi_i}$ for $i = 1, 2$, i.e., $\Lambda \in C_{\Phi_1,\Psi_1} \cap C_{\Phi_2,\Psi_2}$. Then the lemma follows from Lemma 4.22 immediately. □

Example 4.24. Let $Z = \binom{6,4,2,0}{5,3,1}$. Then Z is a special symbol of defect 1 and degree 3, and $Z_{\rm I} = Z$. Now $\Phi_1 = \{\binom{6}{5}, \binom{4}{3}, \binom{2}{1}, \binom{0}{-}\}$ and $\Phi_2 = \{\binom{4}{5}, \binom{2}{3}, \binom{0}{1}, \binom{6}{-}\}$ are two arrangements in Lemma 4.22. We have the following table:

	$\binom{-}{-}$	$\binom{4}{5}$	$\binom{2}{3}$	$\binom{0}{1}$	$\binom{4,2}{5,3}$	$\binom{2,0}{3,1}$	$\binom{4,0}{5,1}$	$\binom{4,2,0}{5,3,1}$
$\binom{-}{-}$	$\binom{-}{6,5,4,3,2,1,0}$	$\binom{6,5}{4,3,2,1,0}$	$\binom{6,5,4,3}{2,1,0}$	$\binom{6,5,4,3,2,1}{0}$	$\binom{4,3}{6,5,2,1,0}$	$\binom{2,1}{6,5,4,3,0}$	$\binom{4,3,2,1}{6,5,0}$	$\binom{6,5,2,1}{4,3,0}$
$\binom{6}{5}$	$\binom{5,4,3,2,1,0}{6}$	$\binom{6,4,3,2,1,0}{5}$	$\binom{6,2,1,0}{5,4,3}$	$\binom{6,0}{5,4,3,2,1}$	$\binom{5,2,1,0}{6,4,3}$	$\binom{5,4,3,0}{6,2,1}$	$\binom{5,0}{6,4,3,2,1}$	$\binom{6,4,3,0}{5,2,1}$
$\binom{4}{3}$	$\binom{3,2,1,0}{6,5,4}$	$\binom{6,5,3,2,1,0}{4}$	$\binom{6,5,4,2,1,0}{3}$	$\binom{6,5,4,0}{3,2,1}$	$\binom{4,2,1,0}{6,5,3}$	$\binom{3,0}{6,5,4,2,1}$	$\binom{4,0}{6,5,3,2,1}$	$\binom{6,5,3,0}{4,2,1}$
$\binom{2}{1}$	$\binom{1,0}{6,5,4,3,2}$	$\binom{6,5,1,0}{4,3,2}$	$\binom{6,5,4,3,1,0}{2}$	$\binom{6,5,4,3,2,0}{1}$	$\binom{4,3,1,0}{6,5,2}$	$\binom{2,0}{6,5,4,3,1}$	$\binom{4,3,2,0}{6,5,1}$	$\binom{6,5,2,0}{4,3,1}$
$\binom{6,4}{5,3}$	$\binom{5,4}{6,3,2,1,0}$	$\binom{6,4}{5,3,2,1,0}$	$\binom{6,3}{5,4,2,1,0}$	$\binom{6,3,2,1}{5,4,0}$	$\binom{5,3}{6,4,2,1,0}$	$\binom{5,4,2,1}{6,3,0}$	$\binom{5,3,2,1}{6,4,0}$	$\binom{6,4,2,1}{5,3,0}$
$\binom{4,2}{3,1}$	$\binom{3,2}{6,5,4,1,0}$	$\binom{6,5,3,2}{4,1,0}$	$\binom{6,5,4,2}{3,1,0}$	$\binom{6,5,4,1}{3,2,0}$	$\binom{4,2}{6,5,3,1,0}$	$\binom{3,1}{6,5,4,2,0}$	$\binom{4,1}{6,5,3,2,0}$	$\binom{6,5,3,1}{4,2,0}$
$\binom{6,2}{5,1}$	$\binom{5,4,3,2}{6,1,0}$	$\binom{6,4,3,2}{5,1,0}$	$\binom{6,2}{5,4,3,1,0}$	$\binom{6,1}{5,4,3,2,0}$	$\binom{5,2}{6,4,3,1,0}$	$\binom{5,4,3,1}{6,2,0}$	$\binom{5,1}{6,4,3,2,0}$	$\binom{6,4,3,1}{5,2,0}$
$\binom{6,4,2}{5,3,1}$	$\binom{5,4,1,0}{6,3,2}$	$\binom{6,4,1,0}{5,3,2}$	$\binom{6,3,1,0}{5,4,2}$	$\binom{6,3,2,0}{5,4,1}$	$\binom{5,3,1,0}{6,4,2}$	$\binom{5,4,2,0}{6,3,1}$	$\binom{5,3,2,0}{6,4,1}$	$\binom{6,4,2,0}{5,3,1}$

In the leftmost column are all 8 possible subset of pairs $\Psi_1 \leq \Phi_1$, and in the topmost row are all 8 possible $\Psi_2 \leq \Phi_2$. The 8 symbols in the row indexed by Ψ_1 are the elements in C_{Φ_1,Ψ_1}, and the 8 symbols in the column indexed by Ψ_2 are the elements in C_{Φ_2,Ψ_2}. For example if $\Psi_1 = \binom{6}{5}$, then

$$C_{\Phi_1,\Psi_1} = \left\{\binom{5,4,3,2,1,0}{6}, \binom{6,4,3,2,1,0}{5}, \binom{6,2,1,0}{5,4,3}, \binom{6,0}{5,4,3,2,1}, \binom{5,2,1,0}{6,4,3}, \binom{5,4,3,0}{6,2,1}, \binom{5,0}{6,4,3,2,1}, \binom{6,4,3,0}{5,2,1}\right\}.$$

From the table we can conclude that $|C_{\Phi_1,\Psi_1} \cap C_{\Phi_2,\Psi_2}| = 1$ for any $\Psi_1 \leq \Phi_1$ and any $\Psi_2 \leq \Phi_2$. Note that the 64 symbols in the above table are all the symbols in this $\mathcal{S}_Z$. Here we give an example to show how to compute this table following the rule in the proof of Lemma 4.22. Suppose that $\Psi_1 = \{\binom{2}{1}\}$ and $\Psi_2 = \{\binom{4}{5}, \binom{2}{3}\}$, and suppose that $\Lambda_M \in C_{\Phi_1,\Psi_1} \cap C_{\Phi_2,\Psi_2}$ for some $M \subset Z_{\rm I}$ with $|M|$ even.

- If $6\in M$, now $\binom{6}{5}\not\leq\Psi_1$, so $5\notin M$ by (3); now $\binom{4}{5}\leq\Psi_2$, so $4\notin M$ by (6); now $\binom{4}{3}\not\leq\Psi_1$, so $3\in M$ by (4); now $\binom{2}{3}\leq\Psi_2$, so $2\in M$ by (5); now $\binom{2}{1}\leq\Psi_1$, so $1\in M$ by (1); now $\binom{1}{0}\not\leq\Psi_2$, so $0\notin M$ by (7), then we obtain $M=\binom{6,2}{3,1}$.
- If $6\notin M$, now $\binom{6}{5}\not\leq\Psi_1$, so $5\in M$ by (4); now $\binom{4}{5}\leq\Psi_2$, so $4\in M$ by (5); now $\binom{4}{3}\not\leq\Psi_1$, so $3\notin M$ by (3); now $\binom{2}{3}\leq\Psi_2$, so $2\notin M$ by (6); now $\binom{2}{1}\leq\Psi_1$, so $1\notin M$ by (2); now $\binom{1}{0}\not\leq\Psi_2$, so $0\in M$ by (8), then we obtain $M=\binom{4,0}{5}$.

Now $\binom{6,2}{3,1},\binom{4,0}{5}$ are the only two subsymbols M of $Z_{\rm I}$ satisfying the two conditions before (4.12) for both $\Psi_1=\{\binom{2}{1}\}\leq\Phi_1=\{\binom{6}{5},\binom{4}{3},\binom{2}{1},\binom{0}{-}\}$ and $\Psi_2=\{\binom{4}{5},\binom{2}{3}\}\leq\Phi_2=\{\binom{6}{-},\binom{4}{5},\binom{2}{3},\binom{0}{1}\}$. However, we need $|M|$ to be even to make $\Lambda_M\in\mathscr{S}_Z$. So we conclude that $\binom{4,3,1,0}{6,5,2}=\Lambda_{\binom{6,2}{3,1}}$ is the only symbol in this intersection $C_{\Phi_1,\Psi_1}\cap C_{\Phi_2,\Psi_2}$.

Lemma 4.25. *Let Z be a special symbol of defect 1, and let Λ_1,Λ_2 be two distinct symbols in $\mathscr{S}_Z$. There exists an arrangement Φ of $Z_{\rm I}$ with two subsets of pairs Ψ_1,Ψ_2 such that $\Lambda_i\in C_{\Phi,\Psi_i}$ for $i=1,2$ and $C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_2}=\emptyset$.*

Proof. Suppose that $\Lambda_1=\Lambda_{M_1}$ and $\Lambda_2=\Lambda_{M_2}$ for $M_1,M_2\subset Z_{\rm I}$. Because $M_1\neq M_2$ and both $|M_1|$ and $|M_2|$ are even, it is clear that we can find a pair $\Psi=\binom{s}{t}$ such that one of M_1,M_2 contains exactly one of the two elements s,t and the other set contains either both s,t or none, i.e.,

$$|M_1\cap\Psi|\not\equiv|M_2\cap\Psi|\pmod 2). \tag{4.26}$$

Let Φ be any arrangement of $Z_{\rm I}$ that contains Ψ as a subset of pairs. By (iii) of Lemma 4.17, we know that $\Lambda_{M_1}\in C_{\Phi,\Psi_1}$ and $\Lambda_{M_2}\in C_{\Phi,\Psi_2}$ for some subsets of pairs Ψ_1,Ψ_2 of Φ. Then by Lemma 4.15 and (4.26) we see that $\Psi_1\neq\Psi_2$. Finally, by (ii) of Lemma 4.17, we know that $C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_2}=\emptyset$. □

Example 4.27. Let $Z=\binom{6,4,2,0}{5,3,1}$, and keep the notation in Example 4.24. Let Λ,Λ' be distinct symbols in $\mathscr{S}_Z$. Then Λ,Λ' must be in different rows or different columns in the table in Example 4.24. If Λ,Λ' are in different rows, then we let $\Phi=\Phi_1$ and we see that there are two different subsets of pairs $\Psi_1,\Psi_1'\leq\Phi$ such that $\Lambda\in C_{\Phi,\Psi_1}$, $\Lambda'\in C_{\Phi,\Psi_1'}$ and of course $C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_1'}=\emptyset$. If Λ,Λ' are in different columns, then we let $\Phi=\Phi_2$ and we have two different columns $C_{\Phi,\Psi_2},C_{\Phi,\Psi_2'}$ containing Λ,Λ' respectively, and with empty intersection.

We need stronger versions of Lemma 4.23 and Lemma 4.25.

Lemma 4.28. *Let Z be a special symbol of defect 1, Φ an arrangement of $Z_{\rm I}$, Ψ a subset of pairs in Φ, Ψ_0 a set of consecutive pairs in $Z_{\rm I}$ such that $\Psi_0\leq\Psi$. Then*

$$C_{\Phi,\Psi}=(C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0})+\mathscr{S}_{Z,\Psi_0}:=\{\Lambda_1+\Lambda_2\mid\Lambda_1\in(C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0}),\ \Lambda_2\in\mathscr{S}_{Z,\Psi_0}\}.$$

Proof. Let Λ_M be an element in $C_{\Phi,\Psi}$ for some $M\subset Z_{\rm I}$. Then $M=M_1\cup M_2$ where $M_1=M\cap(Z_{\rm I}\smallsetminus\Psi_0)$ and $M_2=M\cap\Psi_0$. And we have $\Lambda_M=\Lambda_{M_1}+\Lambda_{M_2}$ since $M_1\cap M_2=\emptyset$ (*cf.* (2.8)). From the requirement of M before Remark 4.13, M needs to contain either none or two entries from each pair in Ψ_0, we see that M_2 is a subset of pairs in Ψ_0, i.e., $M_2\leq\Psi_0$ and hence $\Lambda_{M_2}\in\mathscr{S}_{Z,\Psi_0}$. Now M_1 also satisfies the condition in (4.10), and so $\Lambda_{M_1}\in C_{\Phi,\Psi}$. Moreover, $M_1\subset Z_{\rm I}\smallsetminus\Psi_0$ and $|M_1|$ is even, so we have $\Lambda_{M_1}\in\mathscr{S}_Z^{\Psi_0}$. Then $\Lambda_{M_1}\in C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0}$. On the other hand, if $\Lambda_{M_3}\in C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0}$ and $\Lambda_{M_4}\in\mathscr{S}_{Z,\Psi_0}$ for some M_3,M_4, then it is obvious that $\Lambda_{M_3}+\Lambda_{M_4}=\Lambda_{M_3\cup M_4}\in C_{\Phi,\Psi}$. □

Example 4.29. Suppose that $Z=\binom{4,2,0}{3,1}$, $\Phi=\{\binom{4}{-},\binom{2}{3},\binom{0}{1}\}$ and $\Psi_0=\Psi=\{\binom{0}{1}\}$. Then

$$\mathscr{S}_{Z,\Psi_0}=\left\{\binom{4,2,0}{3,1},\binom{4,2,1}{3,0}\right\},\qquad \mathscr{S}_Z^{\Psi_0}=\left\{\binom{4,2,0}{3,1},\binom{4,3,0}{2,1},\binom{3,2,0}{4,1},\binom{0}{4,3,2,1}\right\}$$

Now by Example 4.14, we see that

$$C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0}=\left\{\binom{3,2,0}{4,1},\binom{0}{4,3,2,1}\right\}.$$

Now

$$\binom{3,2,0}{4,1}+\binom{4,2,0}{3,1}=\binom{3,2,0}{4,1},\qquad \binom{0}{4,3,2,1}+\binom{4,2,0}{3,1}=\binom{0}{4,3,2,1},$$
$$\binom{3,2,0}{4,1}+\binom{4,2,1}{3,0}=\binom{3,2,1}{4,0},\qquad \binom{0}{4,3,2,1}+\binom{4,2,1}{3,0}=\binom{1}{4,3,2,0},$$

i.e., we do have $C_{\Phi,\Psi}=(C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0})+\mathscr{S}_{Z,\Psi_0}$.

Lemma 4.30. *Let Z be a special symbol of defect 1, and let Ψ_0 be a set of consecutive pairs in $Z_{\rm I}$. For any given $\Lambda\in\mathscr{S}_Z^{\Psi_0}$, there exist two arrangements Φ_1,Φ_2 of $Z_{\rm I}$ with subsets of pairs Ψ_1,Ψ_2 respectively such that $\Psi_0\leq\Psi_i$ for $i=1,2$, and*

$$C^\natural_{\Phi_1,\Psi_1}\cap C^\natural_{\Phi_2,\Psi_2}=\{\Lambda\}$$

where $C^\natural_{\Phi_i,\Psi_i}:=C_{\Phi_i,\Psi_i}\cap\mathscr{S}_Z^{\Psi_0}$.

Proof. Because Ψ_0 is a set of consecutive pairs in $Z_{\rm I}$, the symbol Z' given by $Z'=Z\smallsetminus\Psi_0$ is still a special symbol of the same defect and $Z'_{\rm I}=Z_{\rm I}\smallsetminus\Psi_0$. Because $\Lambda\in\mathscr{S}_Z^{\Psi_0}$, we can write $\Lambda=\Lambda'\cup\Psi_0$ (*cf.* Subsection 2.1) for a unique $\Lambda'\in\mathscr{S}_{Z'}$. Write $Z'_{\rm I}=\binom{s'_1,s'_2,\ldots,s'_{\delta_1+1}}{t'_1,t'_2,\ldots,t'_{\delta_1}}$ and define

$$\Phi'_1=\left\{\binom{s'_1}{t'_1},\binom{s'_2}{t'_2},\cdots,\binom{s'_{\delta_1}}{t'_{\delta_1}},\binom{s'_{\delta_1+1}}{-}\right\}\quad\text{and}\quad\Phi'_2=\left\{\binom{s'_1}{-},\binom{s'_2}{t'_1},\binom{s'_3}{t'_2},\cdots,\binom{s'_{\delta_1+1}}{t'_{\delta_1}}\right\}.$$

By Lemma 4.23, we know that there exist sets of pairs Ψ'_1,Ψ'_2 of Φ'_1,Φ'_2 respectively such that

$$C_{\Phi'_1,\Psi'_1}\cap C_{\Phi'_2,\Psi'_2}=\{\Lambda'\}.$$

Now Ψ_0 itself can be regarded as an arrangement of itself, so from (4.12) we have

$$C_{\Psi_0,\Psi_0}=\{\,\Lambda_N\in\mathscr{S}_{\Psi_0}\mid N\leq\Psi_0\,\}.$$

Now let $\Phi_i=\Phi'_i\cup\Psi_0$, $\Psi_i=\Psi'_i\cup\Psi_0$ for $i=1,2$, so we have $\Psi_0\leq\Psi_i\leq\Phi_i$ for $i=1,2$. From Lemma 4.28, we can see that

$$C_{\Phi_i,\Psi_i}=\{\,\Lambda_1\cup\Lambda_2\mid\Lambda_1\in C_{\Phi'_i,\Psi'_i},\ \Lambda_2\in C_{\Psi_0,\Psi_0}\,\}.$$

Therefore

$$C_{\Phi_1,\Psi_1}\cap C_{\Phi_2,\Psi_2}=\{\,\Lambda'\cup\Lambda_2\mid\Lambda_2\in C_{\Psi_0,\Psi_0}\,\},$$

and hence

$$C^\natural_{\Phi_1,\Psi_1}\cap C^\natural_{\Phi_2,\Psi_2}=C_{\Phi_1,\Psi_1}\cap C_{\Phi_2,\Psi_2}\cap\mathscr{S}_Z^{\Psi_0}=\{\Lambda'\cup\Psi_0\}=\{\Lambda\}.$$

□

Lemma 4.31. *Let Z be a special symbol of defect 1, and let Ψ_0 be a set of consecutive pairs in $Z_{\rm I}$. Let Λ_1,Λ_2 be two distinct symbols in $\mathscr{S}_Z^{\Psi_0}$. There exists an arrangement Φ of $Z_{\rm I}$ with two subsets of pairs Ψ_1,Ψ_2 such that $\Psi_0\leq\Psi_i$ and $\Lambda_i\in C_{\Phi,\Psi_i}$ for $i=1,2$, and $C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_2}=\emptyset$.*

Proof. Let Z' be defined as in the proof of the previous lemma, i.e., $Z'=Z\smallsetminus\Psi_0$. Then we known that $\Lambda_i=\Lambda'_i\cup\Psi_0$ for $\Lambda'_i\in\mathcal{S}_{Z'}$. Clearly, Λ'_1,Λ'_2 are distinct. Then by Lemma 4.25, we know that there is an arrangement Φ' of Z' with subsets of pairs Ψ'_1,Ψ'_2 such that $\Lambda'_i\in C_{\Phi',\Psi'_i}$ for $i=1,2$ and $C_{\Phi',\Psi'_1}\cap C_{\Phi',\Psi'_2}=\emptyset$. Let $\Phi_i=\Phi'_i\cup\Psi_0$, $\Psi_i=\Psi'_i\cup\Psi_0$ for $i=1,2$. Then as in the proof of the previous lemma, we can see that

$$C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_2}=\{\,\Lambda_1\cup\Lambda_2\mid\Lambda_1\in C_{\Phi'_1,\Psi'_1}\cap C_{\Phi'_2,\Psi'_2},\ \Lambda_2\in C_{\Psi_0,\Psi_0}\,\}=\emptyset.$$

□

It is clear that if $\Psi_0=\emptyset$, then Lemma 4.30 and Lemma 4.31 are reduced to Lemma 4.23 and Lemma 4.25 respectively.

4.4. Cells for an even orthogonal group. In this subsection, let $\mathbf{G}=\mathrm{O}^\epsilon_{2n}$ for $\epsilon=+$ or $-$, Z a special symbol of rank n and defect 0, Φ an arrangement of Z_{I}, and $\Psi\leq\Phi$.

Lemma 4.32. *If $\Phi\smallsetminus\Psi$ consists of an even number of pairs, then $C_{\Phi,\Psi}\subset\mathcal{S}_Z^{\mathrm{O}^+_{2n}}$; on the other hand, if $\Phi\smallsetminus\Psi$ consists of an odd number of pairs, then $C_{\Phi,\Psi}\subset\mathcal{S}_Z^{\mathrm{O}^-_{2n}}$.*

Proof. Suppose that $\Lambda_M\in C_{\Phi,\Psi}$ for some $M\subset Z_{\mathrm{I}}$. Then from the condition before Remark 4.13, we know that M contains exactly one element from each pair in $\Phi\smallsetminus\Psi$ and contains either none or two elements in each pair in Ψ. This implies that $|M|$ is odd if $\Phi\smallsetminus\Psi$ consists of odd number of pairs; and $|M|$ is even if $\Phi\smallsetminus\Psi$ consists of even number of pairs. Hence the lemma follows from the definition in (2.7). □

Example 4.33. Suppose that $Z=\binom{5,3,1}{4,2,0}$, $\Phi=\{\binom{5}{4},\binom{3}{2},\binom{1}{0}\}$, and $\Psi=\{\binom{5}{4},\binom{1}{0}\}$. Now Z is a special symbol of rank 9 and defect 0, and $Z_{\mathrm{I}}=Z$. To construct a subsymbol M of Z_{I} such that $\Lambda_M\in C_{\Phi,\Psi}$, we need to choose one element from each pair in $\Phi\smallsetminus\Psi=\{\binom{3}{2}\}$ and choose a subset of pairs of Ψ. Hence we have 8 possible subsets M, namely, $\binom{3}{-}$, $\binom{3,1}{0}$, $\binom{5,3}{4}$, $\binom{5,3,1}{4,0}$, $\binom{-}{2}$, $\binom{1}{2,0}$, $\binom{5}{4,2}$, $\binom{5,1}{4,2,0}$. Hence

$$C_{\Phi,\Psi}=\left\{\binom{5,1}{4,3,2,0},\binom{5,0}{4,3,2,1},\binom{4,1}{5,3,2,0},\binom{4,0}{5,3,2,1},\binom{5,3,2,1}{4,0},\binom{5,3,2,0}{4,1},\binom{4,3,2,1}{5,0},\binom{4,3,2,0}{5,1}\right\}.$$

Note that $\Phi\smallsetminus\Psi$ consists of one pair, so $C_{\Phi,\Psi}\subset\mathcal{S}_Z^{\mathrm{O}^-_{18}}$.

Lemma 4.34. *Let Z be a special symbol of rank n and defect 0, Φ a fixed arrangement of Z_{I}, and Ψ,Ψ' subsets of pairs of Φ. Suppose that* $\deg(Z)\geq 1$. *Then*

(i) *$\Lambda\in C_{\Phi,\Psi}$ if and only if $\Lambda^{\mathrm{t}}\in C_{\Phi,\Psi}$;*
(ii) *$|C_{\Phi,\Psi}|=2^{\deg(Z)}$;*
(iii) *if $\Psi\neq\Psi'$, then $C_{\Phi,\Psi}\cap C_{\Phi,\Psi'}=\emptyset$;*
(iv) *we have*

$$\mathcal{S}_Z^{\mathrm{O}^+_{2n}}=\bigcup_{\Psi\leq\Phi,\ \#(\Phi\smallsetminus\Psi)\text{ even}}C_{\Phi,\Psi}\quad\text{and}\quad\mathcal{S}_Z^{\mathrm{O}^-_{2n}}=\bigcup_{\Psi\leq\Phi,\ \#(\Phi\smallsetminus\Psi)\text{ odd}}C_{\Phi,\Psi}$$

where $\#(\Phi\smallsetminus\Psi)$ means the number of pairs in $\Phi\smallsetminus\Psi$.

Proof. Suppose that $\Lambda_M\in C_{\Phi,\Psi}$ for some $M\subset Z_{\mathrm{I}}$. Then it is easy to check that $(\Lambda_M)^{\mathrm{t}}=\Lambda_{Z_{\mathrm{I}}\smallsetminus M}$. It is clear that M satisfies the condition that it consists of exactly one element from each pair in $\Phi\smallsetminus\Psi$ and a subset of pairs of Ψ if and only if $Z_{\mathrm{I}}\smallsetminus M$ satisfies the same condition. Hence (i) is proved.

Let $\delta=\deg(Z)$. From the conditions before Remark 4.13, we can write $M=M_1\cup M_2$ where M_1 consists of exactly one element from each pair of $\Phi\smallsetminus\Psi$, and M_2 consists of some pairs from Ψ. Suppose that Ψ contains δ' pairs for some $\delta'\leq\delta$. So we have $2^{\delta'}$ possible choices for M_2 and $2^{\delta-\delta'}$ choices for M_1. Thus the total choices for M is $2^{\delta'}\cdot 2^{\delta-\delta'}=2^{\delta}$ and hence (ii) is proved.

The proof of (iii) is similar to that of Lemma 4.17.

For any fixed arrangement Φ of $Z_{\rm I}$, we have

$$\mathscr{S}_Z^{{\rm O}^+_{2n}}\cup\mathscr{S}_Z^{{\rm O}^-_{2n}}=\bigcup_{\Psi\leq\Phi}C_{\Phi,\Psi}$$

by the same argument of the proof of Lemma 4.17. Then (iv) follows from Lemma 4.32 immediately. □

Let $\mathbf{G}={\rm O}^\epsilon_{2n}$ where $\epsilon=+$ or $-$, and let Z be a special symbol of rank n and defect 0. A subset of pairs Ψ of an arrangement Φ of $Z_{\rm I}$ is called *admissible* for Φ if $\#(\Phi\smallsetminus\Psi)$ is even when $\epsilon=+$; and $\#(\Phi\smallsetminus\Psi)$ is odd when $\epsilon=-$.

Proposition 4.35. *Let $\mathbf{G}={\rm O}^\epsilon_{2n}$, Z a special symbol of rank n and defect 0, Φ an arrangement of $Z_{\rm I}$ with an admissible subset of pairs Ψ. Then*

$$R^{{\rm O}^\epsilon}_{\underline{c}(Z,\Phi,\Psi)}=\sum_{\Lambda\in C_{\Phi,\Psi}}\rho_\Lambda.$$

Proof. If $\epsilon=+$ and Z is of degree 0, i.e., Z is degenerate, then it is clear that $\Phi=\Psi=\emptyset$, $\underline{c}(Z,\Phi,\Psi)=C_{\Phi,\Psi}=\{Z\}$ and $R^{{\rm O}^+}_Z=\rho_Z$. If $\epsilon=-$ and Z degenerate, then $\underline{c}(Z,\Phi,\Psi)=C_{\Phi,\Psi}=\emptyset$. So the proposition holds if Z is degenerate.

Now suppose that $\delta=\deg(Z)\geq 1$. Let $C^{{\rm SO}^\epsilon}_{\Phi,\Psi}$ be a subset of $C_{\Phi,\Psi}$ such that $C^{{\rm SO}^\epsilon}_{\Phi,\Psi}$ contains exactly one element from each pair $\{\Lambda,\Lambda^{\rm t}\}\subset C_{\Phi,\Psi}$. Therefore $|C^{{\rm SO}^\epsilon}_{\Phi,\Psi}|=2^{\delta-1}$ by (ii) of Lemma 4.34. By the argument in the proof of Proposition 4.18, we can show that $\langle\rho^{{\rm SO}^\epsilon}_\Lambda,R^{{\rm SO}^\epsilon}_{\underline{c}(Z,\Phi,\Psi)}\rangle_{{\rm SO}^\epsilon}=1$ for every $\Lambda\in C_{\Phi,\Psi}$. Moreover, we know that $R^{{\rm SO}^\epsilon}_{\underline{c}(Z,\Phi,\Psi)}$ is a sum of $2^{\delta-1}$ distinct irreducible characters of ${\rm SO}^\epsilon(q)$ by [Lus82] proposition 3.13. Thus we have

$$R^{{\rm SO}^\epsilon}_{\underline{c}(Z,\Phi,\Psi)}=\sum_{\Lambda\in C^{{\rm SO}^\epsilon}_{\Phi,\Psi}}\rho^{{\rm SO}^\epsilon}_\Lambda,$$

and then

$$R^{{\rm O}^\epsilon}_{\underline{c}(Z,\Phi,\Psi)}={\rm Ind}^{{\rm O}^\epsilon}_{{\rm SO}^\epsilon}R^{{\rm SO}^\epsilon}_{\underline{c}(Z,\Phi,\Psi)}=\sum_{\Lambda\in C^{{\rm SO}^\epsilon}_{\Phi,\Psi}}{\rm Ind}^{{\rm O}^\epsilon}_{{\rm SO}^\epsilon}\rho^{{\rm SO}^\epsilon}_\Lambda=\sum_{\Lambda\in C^{{\rm SO}^\epsilon}_{\Phi,\Psi}}(\rho_\Lambda+\rho_{\Lambda^{\rm t}})=\sum_{\Lambda\in C_{\Phi,\Psi}}\rho_\Lambda.$$

□

Lemma 4.36. *Let Λ_1,Λ_2 be two symbols in $\mathscr{S}_Z^{\mathbf{G}}$ such that $\Lambda_1\neq\Lambda_2,\Lambda_2^{\rm t}$. There exists an arrangement Φ of $Z_{\rm I}$ with admissible subsets of pairs Ψ_1,Ψ_2 such that $\Lambda_i,\Lambda_i^{\rm t}\in C_{\Phi,\Psi_i}$ for $i=1,2$ and $C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_2}=\emptyset$.*

Proof. Suppose that $\Lambda_1=\Lambda_{M_1}$ and $\Lambda_2=\Lambda_{M_2}$ for $M_1,M_2\subset Z_{\rm I}$. The assumption that $\Lambda_1\neq\Lambda_2,\Lambda_2^{\rm t}$ means that $M_1\neq M_2$ and $M_1\neq(Z_{\rm I}\smallsetminus M_2)$. Then it is clear that we can find a pair $\Psi=\binom{s}{t}$ in $Z_{\rm I}$ such that one of M_1,M_2 contains exactly one of the two elements s,t and the other set contains either both s,t or none, i.e.,

$$|M_1\cap\Psi|\not\equiv|M_2\cap\Psi|\pmod 2). \tag{4.37}$$

Let Φ be any arrangement of $Z_{\rm I}$ that contains Ψ as a subset of pairs. By (iv) of Lemma 4.34, we know that $\Lambda_{M_1}\in C_{\Phi,\Psi_1}$ and $\Lambda_{M_2}\in C_{\Phi,\Psi_2}$ for some subsets of pairs Ψ_1,Ψ_2 of Φ. Then by Lemma 4.3 and (4.37) we see that $\Psi_1\neq\Psi_2$. Finally, by (iii) of Lemma 4.34, we know that $C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_2}=\emptyset$. □

Example 4.38. Let $Z = \binom{5,3,1}{4,2,0}$. Then Z is a special symbol of rank 9 and defect 0, and $Z_{\mathrm{I}} = Z$. Let $\Phi_1 = \{\binom{5}{4}, \binom{3}{2}, \binom{1}{0}\}$, $\Phi_2 = \{\binom{5}{0}, \binom{3}{4}, \binom{1}{2}\}$ be two arrangements of Z_{I}.

(1) Suppose that $\epsilon = +$. Then we have the following table:

	$\binom{5}{0}$	$\binom{3}{4}$	$\binom{1}{2}$	$\binom{5,3,1}{4,2,0}$
$\binom{5}{4}$	$\binom{4,3,2,1,0}{5}, \binom{5}{4,3,2,1,0}$	$\binom{5,3,2,1,0}{4}, \binom{4}{5,3,2,1,0}$	$\binom{5,1,0}{4,3,2}, \binom{4,3,2}{5,1,0}$	$\binom{5,3,2}{4,1,0}, \binom{4,1,0}{5,3,2}$
$\binom{3}{2}$	$\binom{5,4,3}{2,1,0}, \binom{2,1,0}{5,4,3}$	$\binom{5,4,2,1,0}{3}, \binom{3}{5,4,2,1,0}$	$\binom{5,4,3,1,0}{2}, \binom{2}{5,4,3,1,0}$	$\binom{5,4,2}{3,1,0}, \binom{3,1,0}{5,4,2}$
$\binom{1}{0}$	$\binom{5,4,3,2,1}{0}, \binom{0}{5,4,3,2,1}$	$\binom{5,4,0}{3,2,1}, \binom{3,2,1}{5,4,0}$	$\binom{5,4,3,2,0}{1}, \binom{1}{5,4,3,2,0}$	$\binom{5,4,1}{3,2,0}, \binom{3,2,0}{5,4,1}$
$\binom{5,3,1}{4,2,0}$	$\binom{5,2,1}{4,3,0}, \binom{4,3,0}{5,2,1}$	$\binom{5,3,0}{4,2,1}, \binom{4,2,1}{5,3,0}$	$\binom{4,3,1}{5,2,0}, \binom{5,2,0}{4,3,1}$	$\binom{5,3,1}{4,2,0}, \binom{5,3,1}{4,2,0}$

In the leftmost column are the four subsets of odd number of pairs Ψ_1 of Φ_1 (so that $\#(\Phi_1 \smallsetminus \Psi_1)$ is even), and in the topmost row are the four subsets of odd number of pairs Ψ_2 of Φ_2. The row indexed by Ψ_1 is the cell C_{Φ_1,Ψ_1} and the column indexed by Ψ_2 is the cell C_{Φ_2,Ψ_2}, and we see that $C_{\Phi_1,\Psi_1} \cap C_{\Phi_2,\Psi} = \{\Lambda, \Lambda^{\mathrm{t}}\}$ for some $\Lambda \in \mathscr{S}_Z^{\mathrm{O}_{18}^+}$. Note that the 32 symbols in the table are all elements in $\mathscr{S}_Z^{\mathrm{O}_{18}^+}$.

(2) Suppose that $\epsilon = -$. Then we have the following table:

	$\binom{-}{-}$	$\binom{5,3}{4,0}$	$\binom{3,1}{4,2}$	$\binom{5,1}{2,0}$
$\binom{-}{-}$	$\binom{5,4,3,2,1,0}{-}, \binom{-}{5,4,3,2,1,0}$	$\binom{3,2,1,0}{5,4}, \binom{5,4}{3,2,1,0}$	$\binom{5,4,1,0}{3,2}, \binom{3,2}{5,4,1,0}$	$\binom{5,4,3,2}{1,0}, \binom{1,0}{5,4,3,2}$
$\binom{5,3}{4,2}$	$\binom{5,2,1,0}{4,3}, \binom{4,3}{5,2,1,0}$	$\binom{4,2,1,0}{5,3}, \binom{5,3}{4,2,1,0}$	$\binom{5,3,1,0}{4,2}, \binom{4,2}{5,3,1,0}$	$\binom{4,3,1,0}{5,2}, \binom{5,2}{4,3,1,0}$
$\binom{3,1}{2,0}$	$\binom{5,4,3,0}{2,1}, \binom{2,1}{5,4,3,0}$	$\binom{3,0}{5,4,2,1}, \binom{5,4,2,1}{3,0}$	$\binom{5,4,2,0}{3,1}, \binom{3,1}{5,4,2,0}$	$\binom{5,4,3,1}{2,0}, \binom{2,0}{5,4,3,1}$
$\binom{5,1}{4,0}$	$\binom{4,3,2,1}{5,0}, \binom{5,0}{4,3,2,1}$	$\binom{5,3,2,1}{4,0}, \binom{4,0}{5,3,2,1}$	$\binom{5,3,2,0}{4,1}, \binom{4,1}{5,3,2,0}$	$\binom{4,3,2,0}{5,1}, \binom{5,1}{4,3,2,0}$

Now the leftmost column are the four subsets of even number of pairs Ψ_1 of Φ_1, and the topmost row are the four subsets of even number of pairs Ψ_2 of Φ_2. The 32 symbols in the table are all elements in $\mathscr{S}_Z^{\mathrm{O}_{18}^-}$.

Lemma 4.39. *Let Z be a special symbol of defect 0, Φ an arrangement of Z_{I}, Ψ an admissible subset of pairs, Ψ_0 an set of consecutive pairs in Z_{I} such that $\Psi_0 \leq \Psi$. Then*

$$C_{\Phi,\Psi} = (C_{\Phi,\Psi} \cap \mathscr{S}_Z^{\Psi_0}) + \mathscr{S}_{Z,\Psi_0} := \{\Lambda_1 + \Lambda_2 \mid \Lambda_1 \in (C_{\Phi,\Psi} \cap \mathscr{S}_Z^{\Psi_0}),\ \Lambda_2 \in \mathscr{S}_{Z,\Psi_0}\}.$$

Proof. The proof is similar to that of Lemma 4.28. □

Lemma 4.40. *Let Z be a special symbol of defect 0, and let Ψ_0 be a set of consecutive pairs in Z_{I}. Let Λ_1, Λ_2 be two symbols in $\mathscr{S}_Z^{\Psi_0}$ such that $\Lambda_1 \neq \Lambda_2, \Lambda_2^{\mathrm{t}}$. There exists an arrangement Φ of Z_{I} with subsets Ψ_1, Ψ_2 such that $\Psi_0 \leq \Psi_i$ and $\Lambda_i, \Lambda_i^{\mathrm{t}} \in C_{\Phi,\Psi_i}$ for $i = 1, 2$, and $C_{\Phi,\Psi_1} \cap C_{\Phi,\Psi_2} = \emptyset$.*

Proof. The proof of the lemma is similar to that of Lemma 4.31 except that we need to apply Lemma 4.36 instead of Lemma 4.25. □

If $\Psi_0 = \emptyset$, then $\mathscr{S}_Z^{\mathbf{G},\Psi_0} = \mathscr{S}_Z^{\mathbf{G}}$ and Lemma 4.40 is reduced to Lemma 4.36.

5. A System of Linear Equations

The purpose of this section is to prove Theorem 5.3. Two special cases are verified in Subsection 5.2 and Subsection 5.3. The general case is proved in Subsection 5.4. In this section, let $(\mathbf{G}, \mathbf{G}') = (\mathrm{Sp}_{2n}, \mathrm{O}_{2n'}^{\epsilon})$ where $\epsilon = +$ or $-$, and let Z, Z' be special symbols of ranks n, n' and defects 1, 0 respectively. Let $\delta = \deg(Z)$ and $\delta' = \deg(Z')$.

5.1. Decomposition with respect to special symbols. Recall that $\mathcal{V}_Z$, $\mathcal{V}_{Z'}$ are subspaces spanned by $\{\rho_\Lambda \mid \Lambda\in\mathcal{S}_Z\}$, $\{\rho_{\Lambda'}\mid\Lambda'\in\mathcal{S}_{Z'}\}$ respectively. Let $\omega_{Z,Z'}$ denote the orthogonal projection of $\omega_{\mathbf{G},\mathbf{G}',1}$ over $\mathcal{V}_Z\otimes\mathcal{V}_{Z'}$. Then by Proposition 3.2 and Proposition 3.4 we have

$$\omega_{\mathbf{G},\mathbf{G}',1}=\sum_{Z,Z'}\omega_{Z,Z'}\quad\text{and}\quad\omega^\sharp_{\mathbf{G},\mathbf{G}',1}=\sum_{Z,Z'}\omega^\sharp_{Z,Z'}$$

where Z,Z' run over all special symbols of rank n,n' and defect $1,0$ respectively. Moreover, because $\mathcal{B}_{\mathbf{G},\mathbf{G}'}=\bigcup_{Z,Z'}\mathcal{B}_{Z,Z'}$, we have

$$\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{\mathbf{G},\mathbf{G}'}}\rho_\Lambda\otimes\rho_{\Lambda'}=\sum_{Z,Z'}\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{Z,Z'}}\rho_\Lambda\otimes\rho_{\Lambda'}.$$

Now (1.7) implies that

$$\omega^\sharp_{Z,Z'}=\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{Z,Z'}}\rho^\sharp_\Lambda\otimes\rho^\sharp_{\Lambda'}. \tag{5.1}$$

Then, for any uniform class function $f\in\mathcal{V}^\sharp_Z\otimes\mathcal{V}^\sharp_{Z'}$, we have

$$\langle f,\omega_{Z,Z'}\rangle=\langle f,\omega^\sharp_{Z,Z'}\rangle=\Big\langle f,\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{Z,Z'}}\rho^\sharp_\Lambda\otimes\rho^\sharp_{\Lambda'}\Big\rangle=\Big\langle f,\sum_{(\Lambda,\Lambda')\in\mathcal{B}_{Z,Z'}}\rho_\Lambda\otimes\rho_{\Lambda'}\Big\rangle. \tag{5.2}$$

Now the candidates of the uniform class functions are those construct from the cells described in Section 4.2.

Theorem 5.3. *Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ where $\epsilon=+$ or -1, and let Z,Z' be special symbols of rank n,n' and defect $1,0$ respectively. Then $\rho_\Lambda\otimes\rho_{\Lambda'}$ or $\rho_\Lambda\otimes\rho_{\Lambda'^{\mathrm{t}}}$ occurs in $\omega_{Z,Z'}$ if and only if (Λ,Λ') or $(\Lambda,\Lambda'^{\mathrm{t}})$ occurs in $\mathcal{B}_{Z,Z'}$.*

For Theorem 5.3 there is nothing to prove if $\mathcal{B}_{Z,Z'}=\emptyset$, so we assume that $\mathcal{B}_{Z,Z'}\neq\emptyset$. Then we have $\mathcal{D}_{Z,Z'}\neq\emptyset$ by Lemma 2.13. Now we define

$$\begin{aligned}D_{Z'}&:=\{\Lambda\in\mathcal{S}_{Z,1}\mid(\Lambda,Z')\in\mathcal{D}_{Z,Z'}\};\\D_Z&:=\{\Lambda'\in\mathcal{S}_{Z',0}\mid(Z,\Lambda')\in\mathcal{D}_{Z,Z'}\}.\end{aligned}$$

It is proved in [Pan21] proposition 6.4 that there are subsets of consecutive pairs Ψ_0,Ψ'_0 in $Z_{\mathrm{I}},Z'_{\mathrm{I}}$ respectively such that $D_{Z'}=\mathcal{S}_{Z,\Psi_0}$ and $D_Z=\mathcal{S}_{Z',\Psi'_0}$ (*cf.* (4.1)). Then Ψ_0,Ψ'_0 are called the *core* of $\mathcal{D}_{Z,Z'}$ in $Z_{\mathrm{I}},Z'_{\mathrm{I}}$ respectively.

Suppose that $\mathcal{D}_{Z,Z'}\neq\emptyset$, and let Ψ_0,Ψ'_0 be the cores of $\mathcal{D}_{Z,Z'}$ in $Z_{\mathrm{I}},Z'_{\mathrm{I}}$ respectively. Define

$$\mathcal{B}^\natural_{Z,Z'}=\mathcal{B}_{Z,Z'}\cap(\mathcal{S}_Z^{\Psi_0}\times\mathcal{S}_{Z'}^{\Psi'_0}).$$

Then it is not difficult to check that

$$\mathcal{B}_{Z,Z'}=\{(\Lambda_1+\Lambda_2,\Lambda'_1+\Lambda'_2)\mid(\Lambda_1,\Lambda_2)\in\mathcal{B}^\natural_{Z,Z'},\ \Lambda_2\in\mathcal{S}_{Z,\Psi_0},\ \Lambda'_2\in\mathcal{S}_{Z',\Psi'_0}\} \tag{5.4}$$

(*cf.* [Pan21] (6.4) and (8.1)) and $\mathcal{B}^\natural_{Z,Z'}$ is an one-to-one subrelation of $\mathcal{B}_{Z,Z'}$. Then from the proofs of [Pan21] proposition 7.17, we know that either $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)+1$ or $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)$.

Example 5.5. Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{18},\mathrm{O}^+_{18})$, $Z=\binom{5,3,1}{3,1}\in\mathcal{S}_{\mathrm{Sp}_{18}}$, and $Z'=\binom{5,3,1}{4,2,0}\in\mathcal{S}_{\mathrm{O}^+_{18}}$. Then $Z_{\mathrm{I}}=\binom{5}{-}$, $\deg(Z)=0$, $|\mathcal{S}_Z|=1$; $Z'_{\mathrm{I}}=Z'$, $\deg(Z')=3$, $|\mathcal{S}_{Z'}^{\mathrm{O}^+_{18}}|=2^5$. All the symbols in $\mathcal{S}_{Z'}^{\mathrm{O}^+_{18}}$ are listed in Example 4.38. It is not difficult to check that

$$\mathcal{B}_{Z,Z'}=\mathcal{D}_{Z,Z'}=\Big\{(Z,\Lambda')\mid\Lambda'=\tbinom{5,3,1}{4,2,0},\tbinom{5,3,0}{4,2,1},\tbinom{4,3,1}{5,2,0},\tbinom{4,3,0}{5,2,1},\tbinom{5,2,1}{4,3,0},\tbinom{5,2,0}{4,3,1},\tbinom{4,2,1}{5,3,0},\tbinom{4,2,0}{5,3,1}\Big\}.$$

Therefore,

$$D_{Z'}=\{Z\}=\mathscr{S}_{Z,\Psi_0},$$
$$D_Z=\left\{\binom{5,3,1}{4,2,0},\binom{5,3,0}{4,2,1},\binom{4,3,1}{5,2,0},\binom{4,3,0}{5,2,1},\binom{5,2,1}{4,3,0},\binom{5,2,0}{4,3,1},\binom{4,2,1}{5,3,0},\binom{4,2,0}{5,3,1}\right\}=\mathscr{S}_{Z',\Psi'_0},$$

i.e., $\Psi_0=\emptyset$ and $\Psi'_0=\left\{\binom{5}{4},\binom{3}{2},\binom{1}{0}\right\}$. Now $\mathscr{S}_Z^{\Psi_0}=\{Z\}$, $\mathscr{S}_{Z'}^{\Psi'_0}=\{Z'\}$, and $\mathscr{B}^\natural_{Z,Z'}=\{(Z,Z')\}$.

A non-empty relation $\mathscr{B}_{Z,Z'}$ (or $\mathscr{D}_{Z,Z'}$) is called *one-to-one* if $\Psi_0=\Psi'_0=\emptyset$, which means that $D_{Z'}=\{Z\}$ and $D_Z=\{Z'\}$. If $\mathscr{B}_{Z,Z'}$ is one-to-one, then from above we know that either $\deg(Z')=\deg(Z)+1$ or $\deg(Z')=\deg(Z)$. Then Theorem 5.3 will be proved in Subsection 5.2 for the case that $\mathscr{B}_{Z,Z'}$ is one-to-one and $\deg(Z')=\deg(Z)+1$; and it will be proved in Subsection 5.3 for the case that $\mathscr{B}_{Z,Z'}$ is one-to-one and $\deg(Z')=\deg(Z)$.

5.2. Special case I. In this subsection, let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ where $\epsilon=+$ or $-$, Z,Z' special symbols of ranks n,n' and defects $1,0$ respectively, and we assume that $\mathscr{B}_{Z,Z'}$ is one-to-one and $\deg(Z')=\deg(Z)+1$, i.e., $\delta'=\delta+1$. We write

$$Z_{\mathrm{I}}=\begin{pmatrix}a_1,a_2,\dots,a_{\delta+1}\\ b_1,b_2,\dots,b_\delta\end{pmatrix},\qquad Z'_{\mathrm{I}}=\begin{pmatrix}c_1,c_2,\dots,c_{\delta'}\\ d_1,d_2,\dots,d_{\delta'}\end{pmatrix},\tag{5.6}$$

and define

$$\begin{aligned}\theta\colon\{a_1,\dots,a_{\delta+1}\}\cup\{b_1,\dots,b_\delta\}&\rightarrow\{c_1,\dots,c_{\delta+1}\}\cup\{d_1,\dots,d_{\delta+1}\}\\ a_i&\mapsto d_i\\ b_i&\mapsto c_{i+1}\end{aligned}\tag{5.7}$$

for each i. Note that c_1 is not in the image of θ. Then θ induces an injective map (still denoted by θ)

$$\begin{aligned}\theta\colon\mathscr{S}_Z&\rightarrow\mathscr{S}_{Z'}\\ \Lambda_M&\mapsto\begin{cases}\Lambda_{\theta(M)}, & \text{if }\epsilon=+;\\ \Lambda_{\binom{c_1}{-}\cup\theta(M)}, & \text{if }\epsilon=-\end{cases}\end{aligned}\tag{5.8}$$

where $M\subset Z_{\mathrm{I}}$ with $|M|$ even. Note that now $|\mathscr{S}_Z|=2^{2\delta}$, $|\mathscr{S}_{Z'}|=2^{2\delta+1}$ and

$$\mathscr{S}_{Z'}=\{\,\theta(\Lambda),\theta(\Lambda)^{\mathrm{t}}\mid\Lambda\in\mathscr{S}_Z\,\}.$$

Lemma 5.9. *Suppose that $\mathscr{B}_{Z,Z'}$ is one-to-one and* $\deg(Z')=\deg(Z)+1$. *Then*

$$\mathscr{B}_{Z,Z'}=\{\,(\Lambda,\theta(\Lambda))\mid\Lambda\in\mathscr{S}_Z\,\}.$$

Proof. This lemma is essentially [Pan21] lemma 6.15. Note that in [Pan21] lemma 6.15, we assumed that both Z,Z' are regular, i.e., we assumed that $Z=Z_{\mathrm{I}}$ and $Z'=Z'_{\mathrm{I}}$. However, this assumption is not necessary for the lemma. □

If $\Phi=\left\{\binom{s_1}{t_1},\dots,\binom{s_\delta}{t_\delta},\binom{s_{\delta+1}}{-}\right\}$ is an arrangement of Z_{I} (i.e., $\{s_1,\dots,s_{\delta+1}\}$ (resp. $\{t_1,\dots,t_\delta\}$) is a permutation of $\{a_1,\dots,a_{\delta+1}\}$ (resp. $\{b_1,\dots,b_\delta\}$)), then

$$\theta(\Phi):=\left\{\binom{\theta(t_1)}{\theta(s_1)},\dots,\binom{\theta(t_\delta)}{\theta(s_\delta)},\binom{c_1}{\theta(s_{\delta+1})}\right\}$$

is an arrangement of Z'_{I}. If $\Psi=\left\{\binom{s_{i_1}}{t_{i_1}},\dots,\binom{s_{i_k}}{t_{i_k}}\right\}$ is a subset of pairs of Φ, we define $\theta(\Psi)$ as follows:

(1) if either $\epsilon=-$ and $|\Phi\smallsetminus\Psi|$ is odd, or $\epsilon=+$ and $|\Phi\smallsetminus\Psi|$ is even, let

$$\theta(\Psi)=\left\{\binom{\theta(t_{i_1})}{\theta(s_{i_1})},\dots,\binom{\theta(t_{i_k})}{\theta(s_{i_k})}\right\};$$

(2) if either $\epsilon=-$ and $|\Phi\smallsetminus\Psi|$ is even, or $\epsilon=+$ and $|\Phi\smallsetminus\Psi|$ is odd, let

$$\theta(\Psi)=\left\{\binom{\theta(t_{i_1})}{\theta(s_{i_1})},\ldots,\binom{\theta(t_{i_k})}{\theta(s_{i_k})},\binom{c_1}{\theta(s_{\delta+1})}\right\}.$$

Then $\theta(\Psi)$ is an admissible subset of pairs of $\theta(\Phi)$, i.e., $\#(\theta(\Phi)\smallsetminus\theta(\Psi))$ is even if $\epsilon=+$; $\#(\theta(\Phi)\smallsetminus\theta(\Psi))$ is odd if $\epsilon=-$.

Lemma 5.10. *Suppose that $\mathscr{B}_{Z,Z'}$ is one-to-one and* $\deg(Z')=\deg(Z)+1$*. Let Φ be an arrangement of $Z_{\rm I}$, and Ψ be a subset of pairs of Φ. Then*

$$C_{\theta(\Phi),\theta(\Psi)}=\{\,\theta(\Lambda),\theta(\Lambda)^{\rm t}\mid\Lambda\in C_{\Phi,\Psi}\,\}$$

where $C_{\Phi,\Psi}$ is defined in (4.10).

Proof. As above, let $\delta=\deg(Z)$, and let $s_{\delta+1}$ be the isolated element in Φ. Suppose that $\Lambda_M\in C_{\Phi,\Psi}$ for some $M\subset Z_{\rm I}$ and $\theta(\Lambda_M)=\Lambda_{M'}$ for some $M'\subset Z'_{\rm I}$. From the rules before Remark 4.13, we know that M contains exactly one element from each pair of $\Phi\smallsetminus\Psi$ and contains some subset of pairs in Ψ. Moreover, M contains the isolated element $s_{\delta+1}$ if and only if $|\Phi\smallsetminus\Psi|$ is odd. Then

(1) if $\epsilon=+$ and $|\Phi\smallsetminus\Psi|$ is even, then $s_{\delta+1}\notin M$ and $M'=\theta(M)$;
(2) if $\epsilon=+$ and $|\Phi\smallsetminus\Psi|$ is odd, then $s_{\delta+1}\in M$ and $M'=\theta(M)$;
(3) if $\epsilon=-$ and $|\Phi\smallsetminus\Psi|$ is even, then $s_{\delta+1}\notin M$ and $M'=\binom{c_1}{-}\cup\theta(M)$;
(4) if $\epsilon=-$ and $|\Phi\smallsetminus\Psi|$ is odd, then $s_{\delta+1}\in M$ and $M'=\binom{c_1}{-}\cup\theta(M)$.

It is easy to see from the definition above that for each case above M' consists of exactly one element from each pair in $\theta(\Phi)\smallsetminus\theta(\Psi)$ and a subset of pairs in $\theta(\Psi)$, i.e., $\Lambda_{M'}\in C_{\theta(\Phi),\theta(\Psi)}$.

From (i) of Lemma 4.34, we know that

$$\theta(\Lambda)\in C_{\theta(\Phi),\theta(\Psi)}\text{ if and only if }\theta(\Lambda)^{\rm t}\in C_{\theta(\Phi),\theta(\Psi)}.$$

So we have

$$\{\,\theta(\Lambda),\theta(\Lambda)^{\rm t}\mid\Lambda\in C_{\Phi,\Psi}\,\}\subseteq C_{\theta(\Phi),\theta(\Psi)}.\tag{5.11}$$

Now $|C_{\Phi,\Psi}|=2^{\delta}$ by Lemma 4.17. Because Z' is of degree $\delta+1$, $|C_{\theta(\Phi),\theta(\Psi)}|=2^{\delta+1}$ by Lemma 4.34. Hence both sets in (5.11) have the same cardinality $2^{\delta+1}$, they must be the same. □

Example 5.12. As in Example 2.16, let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_4,\mathrm{O}_8^+)$, and $Z=\binom{2,0}{1}$, $Z'=\binom{3,1}{2,0}$. Now $Z_{\rm I}=Z$, $Z'_{\rm I}=Z'$, $\mathscr{B}_{Z,Z'}$ is one-to-one, $\deg(Z')=2=\deg(Z)+1$, $|\mathscr{S}_Z|=4$, and $|\mathscr{S}_{Z'}|=8$. Note that $Z'=Z^{\rm t}\cup\binom{3}{-}$, and in fact if $(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}$, then $\Lambda'=\theta(\Lambda)=\Lambda^{\rm t}\cup\binom{3}{-}$. Now $\Phi=\left\{\binom{2}{1},\binom{0}{-}\right\}$ is an arrangement of $Z_{\rm I}$, and $\theta(\Phi)=\left\{\binom{1}{2},\binom{3}{0}\right\}$ is an arrangement of $Z'_{\rm I}$. Let $\Psi=\left\{\binom{2}{1}\right\}$. Then by definition $\theta(\Psi)=\theta(\Phi)$, and it is easy to verify that

$$\begin{aligned}C_{\Phi,\Psi}&=\left\{\binom{2,0}{1},\binom{1,0}{2}\right\}\\ C_{\theta(\Phi),\theta(\Psi)}&=\left\{\binom{3,1}{2,0},\binom{3,2}{1,0},\binom{1,0}{3,2},\binom{2,0}{3,1}\right\}=\{\,\theta(\Lambda),\theta(\Lambda)^{\rm t}\mid\Lambda\in C_{\Phi,\Psi}\,\}\end{aligned}$$

Proposition 5.13. *Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^{\epsilon}_{2n'})$ where $\epsilon=+$ or $-$, and let Z,Z' be special symbols of ranks n,n' and defects $1,0$ respectively. Suppose that $\mathscr{B}_{Z,Z'}$ is one-to-one and* $\deg(Z')=\deg(Z)+1$*. Then*

$$\omega_{Z,Z'}=\sum_{\Lambda\in\mathscr{S}_Z}\rho_\Lambda\otimes\rho_{f(\Lambda)}$$

where $f(\Lambda)$ is either equal to $\theta(\Lambda)$ or $\theta(\Lambda)^{\rm t}$ (but not both).

Proof. Because we assume that $\mathscr{B}_{Z,Z'}$ is one-to-one and $\deg(Z')=\deg(Z)+1$, we know that

$$\mathscr{B}_{Z,Z'}=\{\,(\Lambda,\theta(\Lambda))\mid\Lambda\in\mathscr{S}_Z\,\}$$

by Lemma 5.9, and (5.1) becomes

$$\omega^\sharp_{Z,Z'}=\sum_{\Lambda\in\mathscr{S}_Z}\rho^\sharp_\Lambda\otimes\rho^\sharp_{\theta(\Lambda)}. \tag{5.14}$$

For $\Lambda,\Lambda'\in\mathscr{S}_Z$, define

$$x_{\Lambda,\Lambda'}=\langle\rho_\Lambda\otimes\rho_{\theta(\Lambda')}+\rho_\Lambda\otimes\rho_{\theta(\Lambda')^{\rm t}},\omega_{Z,Z'}\rangle,$$

the sum of multiplicities of $\rho_\Lambda\otimes\rho_{\theta(\Lambda')}$ and $\rho_\Lambda\otimes\rho_{\theta(\Lambda')^{\rm t}}$ in $\omega_{Z,Z'}$. So we need to show that $x_{\Lambda,\Lambda'}=1$ if $\Lambda=\Lambda'$ and $x_{\Lambda,\Lambda'}=0$ otherwise.

Now suppose that Φ,Φ' are any two arrangements of $Z_{\rm I}$, and $\Psi\leq\Phi,\Psi'\leq\Phi'$. Then by Proposition 4.18 and Proposition 4.35, the class function

$$\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda_1\in C_{\theta(\Phi'),\theta(\Psi')}}\rho_\Lambda\otimes\rho_{\Lambda_1}$$

on $G\times G'$ is unform. Then by Lemma 5.10, we have

$$\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda_1\in C_{\theta(\Phi'),\theta(\Psi')}}\rho_\Lambda\otimes\rho_{\Lambda_1}=\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}(\rho_\Lambda\otimes\rho_{\theta(\Lambda')}+\rho_\Lambda\otimes\rho_{\theta(\Lambda')^{\rm t}}).$$

Then by (5.14), equation (5.2) becomes

$$\begin{aligned}\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}x_{\Lambda,\Lambda'}&=\Big\langle\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}(\rho_\Lambda\otimes\rho_{\theta(\Lambda')}+\rho_\Lambda\otimes\rho_{\theta(\Lambda')^{\rm t}}),\omega_{Z,Z'}\Big\rangle\\&=\Big\langle\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}(\rho_\Lambda\otimes\rho_{\theta(\Lambda')}+\rho_\Lambda\otimes\rho_{\theta(\Lambda')^{\rm t}}),\omega^\sharp_{Z,Z'}\Big\rangle\\&=\Big\langle\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}(\rho_\Lambda\otimes\rho_{\theta(\Lambda')}+\rho_\Lambda\otimes\rho_{\theta(\Lambda')^{\rm t}}),\sum_{\Lambda''\in\mathscr{S}_Z}\rho^\sharp_{\Lambda''}\otimes\rho^\sharp_{\theta(\Lambda'')}\Big\rangle\\&=\Big\langle\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}(\rho_\Lambda\otimes\rho_{\theta(\Lambda')}+\rho_\Lambda\otimes\rho_{\theta(\Lambda')^{\rm t}}),\sum_{\Lambda''\in\mathscr{S}_Z}\rho_{\Lambda''}\otimes\rho_{\theta(\Lambda'')}\Big\rangle.\end{aligned}$$

From the definition of θ we know that $\theta(\Lambda'')\neq\theta(\Lambda')^{\rm t}$ for any $\Lambda'',\Lambda'\in\mathscr{S}_Z$. For a symbol $\Lambda''\in\mathscr{S}_Z$ to contribute a multiplicity in the above identity, we need $\Lambda''=\Lambda$ and $\Lambda''=\Lambda'$ for some $\Lambda\in C_{\Phi,\Psi}$ and some $\Lambda'\in C_{\Phi',\Psi'}$, i.e., Λ'' must be in the intersection $C_{\Phi,\Psi}\cap C_{\Phi',\Psi'}$. Therefore, the above equation becomes

$$\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}x_{\Lambda,\Lambda'}=|C_{\Phi,\Psi}\cap C_{\Phi',\Psi'}| \tag{5.15}$$

for any two arrangements Φ,Φ' of $Z_{\rm I}$ with any $\Psi\leq\Phi$ and any $\Psi'\leq\Phi'$.

Suppose that Λ_1,Λ_2 are distinct symbols in $\mathscr{S}_Z$. Then by Lemma 4.25, there exists an arrangement Φ with two subsets of pairs Ψ_1,Ψ_2 such that $\Lambda_i\in C_{\Phi,\Psi_i}$ for $i=1,2$ and $C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_2}=\emptyset$. Because each $x_{\Lambda,\Lambda'}$ is a non-negative integer, from equation (5.15) we conclude that $x_{\Lambda_1,\Lambda_2}=0$ for any distinct $\Lambda_1,\Lambda_2\in\mathscr{S}_Z$.

For any $\Lambda\in\mathscr{S}_Z$, by Lemma 4.23, there exist two arrangements Φ_1,Φ_2 of $Z_{\rm I}$ with subsets of pairs Ψ_1,Ψ_2 respectively such that $C_{\Phi_1,\Psi_1}\cap C_{\Phi_2,\Psi_2}=\{\Lambda\}$. Because we know $x_{\Lambda,\Lambda'}=0$ if $\Lambda\neq\Lambda'$, equation (5.15) is reduced to $x_{\Lambda,\Lambda}=1$. Therefore, exactly one of $\rho_\Lambda\otimes\rho_{\theta(\Lambda)},\rho_\Lambda\otimes\rho_{\theta(\Lambda)^{\rm t}}$ occurs in $\omega_{Z,Z'}$.

For $\Lambda\in\mathscr{S}_Z$, let $f(\Lambda)$ be either $\theta(\Lambda)$ or $\theta(\Lambda)^t$ such that $\rho_\Lambda\otimes\rho_{f(\Lambda)}$ occurs in $\omega_{Z,Z'}$. Then we just show the character $\overline{\omega}_{Z,Z'}$ defined by

$$\overline{\omega}_{Z,Z'}=\sum_{\Lambda\in\mathscr{S}_Z}\rho_\Lambda\otimes\rho_{f(\Lambda)}.$$

is a sub-character of $\omega_{Z,Z'}$, i.e., $\omega_{Z,Z'}-\overline{\omega}_{Z,Z'}$ is a non-negative integral combination of irreducible characters of $G\times G'$. Note that $\rho_{\theta(\Lambda)}$ and $\rho_{\theta(\Lambda)^t}$ are different by a sign character of $\mathrm{O}^\epsilon_{2n'}(q)$, so they have the same degree. Therefore $\overline{\omega}_{Z,Z}$ and $\sum_{\Lambda\in\mathscr{S}_Z}\rho_\Lambda\otimes\rho_{\theta(\Lambda)}$ have the same degree. By (5.14) and (3.1), we see that $\omega_{Z,Z'}$ and $\overline{\omega}_{Z,Z'}$ have the same degree. Therefore $\overline{\omega}_{Z,Z'}=\omega_{Z,Z'}$ and the proposition is proved. □

5.3. Special case II. In this subsection, let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ where $\epsilon=+$ or $-$, Z,Z' special symbols of ranks n,n' and defects $1,0$ respectively, and we assume that $\mathscr{B}_{Z,Z'}$ is one-to-one and $\deg(Z')=\deg(Z)$, i.e., $\delta'=\delta$. Write $Z_{\mathrm{I}},Z'_{\mathrm{I}}$ as in (5.6), and define

$$\begin{aligned}\theta\colon\{c_1,\ldots,c_\delta\}\cup\{d_1,\ldots,d_\delta\}&\rightarrow\{a_1,\ldots,a_{\delta+1}\}\cup\{b_1,\ldots,b_\delta\}\\ c_i&\mapsto b_i\\ d_i&\mapsto a_{i+1}\end{aligned}\tag{5.16}$$

for each i. Note that a_1 is not in the image of θ. Then θ induces an injective map

$$\begin{aligned}\theta\colon\mathscr{S}_{Z'}&\rightarrow\mathscr{S}_Z\\ \Lambda_{M'}&\mapsto\begin{cases}\Lambda_{\theta(M')}, & \text{if } \epsilon=+;\\ \Lambda_{\binom{a_1}{-}\cup\theta(M')}, & \text{if } \epsilon=-\end{cases}\end{aligned}\tag{5.17}$$

where $M'\subset Z'_{\mathrm{I}}$ with $|M'|$ even if $\epsilon=+$; and $|M'|$ odd if $\epsilon=-$. Note that now $|\mathscr{S}_Z|=2^{2\delta}$, $|\mathscr{S}_{Z'}|=2^{2\delta-1}$, and

$$\mathscr{S}_Z=\{\,\theta(\Lambda')\mid\Lambda'\in\mathscr{S}_{Z'}^{\mathrm{O}^+_{2n'}}\cup\mathscr{S}_{Z'}^{\mathrm{O}^-_{2n'}}\,\}.$$

Lemma 5.18. *Suppose that $\mathscr{B}_{Z,Z'}$ is one-to-one and* $\deg(Z')=\deg(Z)$. *Then*

$$\mathscr{B}_{Z,Z'}=\{\,(\theta(\Lambda'),\Lambda')\mid\Lambda'\in\mathscr{S}_{Z'}\,\}.$$

Proof. The proof is similar to that of Lemma 5.9. □

If $\Phi'=\left\{\binom{s'_1}{t'_1},\ldots,\binom{s'_{\delta'}}{t'_{\delta'}}\right\}$ is an arrangement of Z'_{I}, then

$$\theta(\Phi')=\left\{\binom{a_1}{-},\binom{\theta(t'_1)}{\theta(s'_1)},\ldots,\binom{\theta(t'_{\delta'})}{\theta(s'_{\delta'})}\right\}\tag{5.19}$$

is an arrangement of Z_{I}. If $\Psi'=\left\{\binom{s'_{i_1}}{t'_{i_1}},\ldots,\binom{s'_{i_k}}{t'_{i_k}}\right\}$ is an admissible subset of pairs of Φ', then

$$\theta(\Psi')=\left\{\binom{\theta(t'_{i_1})}{\theta(s'_{i_1})},\ldots,\binom{\theta(t'_{i_k})}{\theta(s'_{i_k})}\right\}\tag{5.20}$$

is a subset of pairs in $\theta(\Phi')$.

Lemma 5.21. *Suppose that $\mathscr{B}_{Z,Z'}$ is one-to-one and* $\deg(Z')=\deg(Z)$. *Let Φ' be an arrangement of Z'_{I}, and let Ψ' be an admissible subset of pairs in Φ'. Then*

$$C_{\theta(\Phi'),\theta(\Psi')}=\{\,\theta(\Lambda)\mid\Lambda\in C_{\Phi',\Psi'}\,\}.$$

Proof. Suppose that $\Lambda_{M'}\in C_{\Phi',\Psi'}$ for some $M'\subset Z'_{\mathrm{I}}$, and $\theta(\Lambda_{M'})=\Lambda_M$ for some M. We know that $M'=M'_1\cup M'_2$ where M'_1 consists of exactly one element from each pair in $\Phi'\smallsetminus\Psi'$ and $M'_2\leq\Psi'$.

(1) First suppose that $\epsilon=+$. Then $|M'|$ is even. From the above definition, we see that $M=\theta(M')$. Hence M contains one element from each pair in $\theta(\Phi')\smallsetminus\theta(\Psi')$ and contains a subset of pairs in $\theta(\Psi')$, and $|M|$ is even. Therefore $\theta(\Lambda_{M'})$ is in $C_{\theta(\Phi'),\theta(\Psi')}$.
(2) Next suppose that $\epsilon=-$. Now $|M'|$ is odd, and we see that $M=\binom{a_1}{-}\cup\theta(M')$ from (5.17). Now again M consists one element from each pair in $\theta(\Phi')\smallsetminus\theta(\Psi')$ and contains a subset of pairs in $\theta(\Psi')$, and $|M|$ is even. Therefore $\theta(\Lambda_{M'})$ is in $C_{\theta(\Phi'),\theta(\Psi')}$.

Now we conclude that $\{\,\theta(\Lambda)\mid\Lambda\in C_{\Phi',\Psi'}\,\}\subseteq C_{\theta(\Phi'),\theta(\Psi')}$. Since both sets have the same cardinality 2^δ, they must be equal. □

Corollary 5.22. *Suppose that* $\mathscr{B}_{Z,Z'}$ *is one-to-one and* $\deg(Z')=\deg(Z)$. *Let* Φ' *be an arrangement of* $Z'_{\rm I}$ *and* θ *given in (5.17). Then*

$$\theta(\mathscr{S}_{Z'})=\bigcup_{\textit{admissible }\Psi'\leq\Phi'}C_{\theta(\Phi'),\theta(\Psi')}$$

Proof. From Lemma 4.34, we know that

$$\mathscr{S}_{Z'}=\bigcup_{\text{admissible }\Psi'\leq\Phi'}C_{\Phi',\Psi'}.$$

Then the corollary follows from Lemma 5.21 immediately. □

Proposition 5.23. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ *where* $\epsilon=+$ *or* $-$, *and let* Z,Z' *be special symbols of ranks* n,n' *and defects* $1,0$ *respectively. Suppose that* $\mathscr{B}_{Z,Z'}$ *is one-to-one and* $\deg(Z')=\deg(Z)$. *Then*

$$\omega_{Z,Z'}=\sum_{\Lambda'\in\mathscr{S}_{Z'}}\rho_{\theta(\Lambda')}\otimes\rho_{f'(\Lambda')}$$

where $f'(\Lambda')$ *is equal to either* Λ' *or* $\Lambda'^{\rm t}$ *(but not both).*

Proof. The proof is similar to that of Proposition 5.13. Because we assume that $\mathscr{B}_{Z,Z'}$ is one-to-one and $\deg(Z')=\deg(Z)$, we know that

$$\mathscr{B}_{Z,Z'}=\{\,(\theta(\Lambda'),\Lambda')\mid\Lambda'\in\mathscr{S}_{Z'}\,\}$$

by Lemma 5.18, and (5.1) becomes

$$\omega^\sharp_{Z,Z'}=\sum_{\Lambda'\in\mathscr{S}_{Z'}}\rho^\sharp_{\theta(\Lambda')}\otimes\rho^\sharp_{\Lambda'}. \tag{5.24}$$

For $\Lambda\in\mathscr{S}_Z$ and $\Lambda'\in\mathscr{S}_{Z'}$, define

$$x_{\Lambda,\Lambda'}=\langle\rho_\Lambda\otimes\rho_{\Lambda'},\omega_{Z,Z'}\rangle.$$

Then each $x_{\Lambda,\Lambda'}$ is a non-negative integer. For an arrangement Φ of $Z_{\rm I}$ with a subset of pairs Ψ, and an arrangement Φ' of $Z'_{\rm I}$ with an admissible subset of pairs Ψ', the class function

$$\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}\rho_\Lambda\otimes\rho_{\Lambda'}$$

on $G\times G'$ is uniform by Proposition 4.18 and Proposition 4.35. Then, by (5.24), we have

$$\begin{aligned}\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}x_{\Lambda,\Lambda'}&=\Big\langle\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}\rho_\Lambda\otimes\rho_{\Lambda'},\omega_{Z,Z'}\Big\rangle\\&=\Big\langle\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}\rho_\Lambda\otimes\rho_{\Lambda'},\sum_{\Lambda''\in\mathscr{S}_{Z'}}\rho_{\theta(\Lambda'')}\otimes\rho_{\Lambda''}\Big\rangle.\end{aligned}$$

For a symbol $\Lambda''\in\mathscr{S}_{Z'}$ to contribute a multiplicity, we need both

- $\Lambda''=\Lambda'$ for some $\Lambda'\in C_{\Phi',\Psi'}$, i.e., $\theta(\Lambda'')=\theta(\Lambda')$ for some $\theta(\Lambda')\in C_{\theta(\Phi'),\theta(\Psi')}$, and
- $\theta(\Lambda'')=\Lambda$ for some $\Lambda\in C_{\Phi,\Psi}$,

i.e., we need $\theta(\Lambda'')$ to be in the intersection $C_{\Phi,\Psi}\cap C_{\theta(\Phi'),\theta(\Psi')}$ by Lemma 5.21. Therefore,

$$\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}} x_{\Lambda,\Lambda'}=|C_{\Phi,\Psi}\cap C_{\theta(\Phi'),\theta(\Psi')}| \tag{5.25}$$

for any arrangements Φ of $Z_{\rm I}$ with any $\Psi\leq\Phi$, and any arrangement Φ' of $Z'_{\rm I}$ with any admissible $\Psi'\leq\Phi'$.

Now let $\Phi=\theta(\Phi'')$ and $\Psi=\theta(\Psi'')$ for some arrangement Φ'' of $Z'_{\rm I}$ with some admissible $\Psi''\leq\Phi''$. Then by Lemma 5.21, (5.25) becomes

$$\sum_{\Lambda''\in C_{\Phi'',\Psi''}}\sum_{\Lambda'\in C_{\Phi',\Psi'}} x_{\theta(\Lambda''),\Lambda'}=|C_{\theta(\Phi''),\theta(\Psi'')}\cap C_{\theta(\Phi'),\theta(\Psi')}|=|C_{\Phi'',\Psi''}\cap C_{\Phi',\Psi'}| \tag{5.26}$$

for any two arrangements Φ',Φ'' of $Z'_{\rm I}$ and any admissible $\Psi'\leq\Phi'$, $\Psi''\leq\Phi''$. Suppose that Λ'_1,Λ'_2 are symbols in $\mathscr{S}_{Z'}$ such that $\Lambda'_1\neq\Lambda'_2,(\Lambda'_2)^{\rm t}$. Then by Lemma 4.36, there exists an arrangement Φ' of $Z'_{\rm I}$ with two subsets of pairs Ψ'_1,Ψ'_2 such that $\Lambda'_i,(\Lambda'_i)^{\rm t}\in C_{\Phi',\Psi'_i}$ for $i=1,2$ and $C_{\Phi',\Psi'_1}\cap C_{\Phi',\Psi'_2}=\emptyset$. Because each $x_{\theta(\Lambda''),\Lambda'}$ is a non-negative integer, from equation (5.26) we conclude that $x_{\theta(\Lambda'_1),\Lambda'_2}=x_{\theta(\Lambda'_1),(\Lambda'_2)^{\rm t}}=0$.

Clearly there is an arrangement Φ' of $Z'_{\rm I}$ such that $\theta(\Phi')$ in (5.19) is the arrangement Φ_2 of $Z_{\rm I}$ in Lemma 4.22. For any given $\Lambda'\in\mathscr{S}_{Z'}$, by Lemma 4.23 there exist an arrangements Φ_1 of $Z_{\rm I}$ with a subset of pairs Ψ_1, and a subset of pairs Ψ_2 of $\theta(\Phi')$ such that $C_{\Phi_1,\Psi_1}\cap C_{\theta(\Phi'),\Psi_2}=\{\theta(\Lambda')\}$. Moreover, by Corollary 5.22 we know that $\Psi_2=\theta(\Psi')$ for some admissible $\Psi'\leq\Phi'$, i.e.,

$$\{\theta(\Lambda')\}=C_{\Phi_1,\Psi_1}\cap C_{\theta(\Phi'),\theta(\Psi')}. \tag{5.27}$$

Because we know $x_{\theta(\Lambda'_1),\Lambda'_2}=0$ for any $\Lambda'_1\neq\Lambda'_2,(\Lambda'_2)^{\rm t}$, by (5.27), equation (5.25) is reduced to

$$x_{\theta(\Lambda'),\Lambda'}+x_{\theta(\Lambda'),\Lambda'^{\rm t}}=1.$$

For $\Lambda'\in\mathscr{S}_{Z'}$, let $f'(\Lambda')$ be either Λ' or $\Lambda'^{\rm t}$ such that $\rho_{\theta(\Lambda')}\otimes\rho_{f'(\Lambda')}$ occurs in $\omega_{Z,Z'}$. We just show that the character $\sum_{\Lambda'\in\mathscr{S}_{Z'}}\rho_{\theta(\Lambda')}\otimes\rho_{f'(\Lambda')}$ is a sub-character of $\omega_{Z,Z'}$. By the same argument in the last paragraph of proof of Proposition 5.13, we conclude that

$$\omega_{Z,Z'}=\sum_{\Lambda'\in\mathscr{S}_{Z'}}\rho_{\theta(\Lambda')}\otimes\rho_{f'(\Lambda')}.$$

□

5.4. The general case. Now let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^{\epsilon}_{2n'})$ where $\epsilon=+$ or $-$, and let Z,Z' be special symbols of ranks n,n' and defects $1,0$ respectively. Suppose that $\mathscr{D}_{Z,Z'}\neq\emptyset$, and let Ψ_0,Ψ'_0 denote the cores of $\mathscr{D}_{Z,Z'}$ in $Z_{\rm I},Z'_{\rm I}$ respectively. Let Φ,Φ' be arrangements of $Z_{\rm I},Z'_{\rm I}$ with subsets of pairs Ψ,Ψ' respectively such that $\Psi_0\leq\Psi\leq\Phi$ and $\Psi'_0\leq\Psi'\leq\Phi'$. It is known that either $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)+1$ or $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)$.

(1) Suppose that $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)+1$. Let θ be given as in (5.7) (with $Z_{\rm I}$ replaced by $Z_{\rm I}\smallsetminus\Psi_0$ and $Z'_{\rm I}$ replaced by $Z'_{\rm I}\smallsetminus\Psi'_0$), and so we have a mapping $\theta\colon\mathscr{S}_Z^{\Psi_0}\rightarrow\mathscr{S}_{Z'}^{\Psi'_0}$ given as in (5.8). Now $\Phi\smallsetminus\Psi_0$ is an arrangement of $Z_{\rm I}\smallsetminus\Psi_0$, and $\theta(\Phi\smallsetminus\Psi_0)$ is an arrangement of $Z'_{\rm I}\smallsetminus\Psi'_0$. Now $\Psi\smallsetminus\Psi_0$ is a subset of pairs of $\Phi\smallsetminus\Psi_0$, and we define

$$\bar\theta(\Phi)=\theta(\Phi\smallsetminus\Psi_0)\cup\Psi'_0,\qquad \bar\theta(\Psi)=\theta(\Psi\smallsetminus\Psi_0)\cup\Psi'_0. \tag{5.28}$$

It is easy to see that $\bar\theta(\Phi)$ is an arrangement of $Z'_{\rm I}$, $\bar\theta(\Psi)$ is a subset of pairs of $\bar\theta(\Phi)$, and $\Psi'_0\leq\bar\theta(\Psi)$.

(2) Suppose that $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)$. Let θ is given as in (5.16), and so we have a mapping $\theta\colon\mathscr{S}_{Z'}^{\Psi'_0}\to\mathscr{S}_Z^{\Psi_0}$. Now $\Phi'\smallsetminus\Psi'_0$ is an arrangement of $Z'_{\rm I}\smallsetminus\Psi'_0$, and $\theta(\Phi'\smallsetminus\Psi'_0)$ is an arrangement of $Z_{\rm I}\smallsetminus\Psi_0$ Then we define

$$\bar\theta(\Phi')=\theta(\Phi'\smallsetminus\Psi'_0)\cup\Psi_0,\qquad \bar\theta(\Psi')=\theta(\Psi'\smallsetminus\Psi'_0)\cup\Psi_0. \tag{5.29}$$

Similarly, $\bar\theta(\Phi')$ is an arrangement of $Z_{\rm I}$, $\bar\theta(\Psi')$ is a subset of pairs of $\bar\theta(\Phi')$, and $\Psi_0\leq\bar\theta(\Psi')$.

Lemma 5.30. *Keep the above setting.*

(i) *Suppose that* $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)+1$. *Let* Φ *be an arrangement of* $Z_{\rm I}$, *and let* Ψ *be a subset of pairs in* Φ. *Then*

$$C_{\bar\theta(\Phi),\bar\theta(\Psi)}=\{\,\theta(\Lambda_1)+\Lambda_2,\theta(\Lambda_1)^{\rm t}+\Lambda_2\mid\Lambda_1\in C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0},\ \Lambda_2\in\mathscr{S}_{Z',\Psi'_0}\,\}.$$

(ii) *Suppose that* $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)$. *Let* Φ' *be an arrangement of* $Z'_{\rm I}$, *and let* Ψ' *be an admissible subset of pairs in* Φ'. *Then*

$$C_{\bar\theta(\Phi'),\bar\theta(\Psi')}=\{\,\theta(\Lambda_1)+\Lambda_2\mid\Lambda_1\in C_{\Phi',\Psi'}\cap\mathscr{S}_{Z'}^{\Psi'_0},\ \Lambda_2\in\mathscr{S}_{Z,\Psi_0}\,\}.$$

Proof. First suppose that $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)+1$. Now $\Psi'_0\leq\bar\theta(\Psi)\leq\bar\theta(\Phi)$, so by Lemma 4.28, we have

$$C_{\bar\theta(\Phi),\bar\theta(\Psi)}=\{\,\Lambda'_1+\Lambda_2\mid\Lambda'_1\in C_{\theta(\Phi),\theta(\Psi)}\cap\mathscr{S}_{Z'}^{\Psi'_0},\ \Lambda_2\in\mathscr{S}_{Z',\Psi'_0}\,\}.$$

We know that the relation $\mathscr{D}_{Z,Z'}\cap(\mathscr{S}_Z^{\Psi_0}\times\mathscr{S}_{Z'}^{\Psi'_0})$ is one-to-one, so by Lemma 5.10, we see that

$$C_{\bar\theta(\Phi),\bar\theta(\Psi)}\cap\mathscr{S}_{Z'}^{\Psi'_0}=\{\,\theta(\Lambda_1),\theta(\Lambda_1)^{\rm t}\mid\Lambda_1\in C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0}\,\}$$

and hence the lemma is proved.

The proof of (ii) is similar. □

Example 5.31. (1) Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{12},\mathrm{O}^+_{14})$, $Z=\binom{4,2,0}{3,1}$, $Z'=\binom{5,2,0}{4,2,0}$. Then $Z_{\rm I}=Z$, $Z'_{\rm I}=\binom{5}{4}$, $|\mathscr{S}_Z|=16$, and $|\mathscr{S}_{Z'}|=2$. It is not difficult to check that

$$\begin{aligned}\mathscr{B}_{Z,Z'}&=\left\{(\Lambda,Z')\mid\Lambda=\tbinom{4,2,0}{3,1},\tbinom{4,3,0}{2,1},\tbinom{4,2,1}{3,0},\tbinom{4,3,1}{2,0}\right\};\\ D_Z&=\{Z'\}=\mathscr{S}_{Z',\Psi'_0};\\ D_{Z'}&=\left\{\tbinom{4,2,0}{3,1},\tbinom{4,3,0}{2,1},\tbinom{4,2,1}{3,0},\tbinom{4,3,1}{2,0}\right\}=\mathscr{S}_{Z,\Psi_0},\end{aligned}$$

i.e., $\Psi_0=\left\{\binom{2}{3},\binom{0}{1}\right\}$, $\Psi'_0=\emptyset$. Now Ψ_0 is regarded as the subsymbol $\binom{2,0}{3,1}$ of $Z_{\rm I}$, so $Z_{\rm I}\smallsetminus\Psi_0=\binom{4}{-}$, $Z'_{\rm I}\smallsetminus\Psi'_0=\binom{5}{4}$, and hence $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)+1=1$. Then $\mathscr{S}_Z^{\Psi_0}=\{Z\}$ and $\mathscr{S}_{Z'}^{\Psi'_0}=\{Z',Z'^{\rm t}\}$, the mapping $\theta\colon\mathscr{S}_Z^{\Psi_0}\to\mathscr{S}_{Z'}^{\Psi'_0}$ is just $Z\mapsto Z'$, and so $\mathscr{B}^\natural_{Z,Z'}=\{(Z,Z')\}$. The only arrangement Φ of $Z_{\rm I}$ containing Ψ_0 is Ψ_0 itself, and so now $\Psi_0=\Psi=\Phi=\left\{\binom{2}{3},\binom{0}{1}\right\}$. Now $\bar\theta(\Phi)=\left\{\binom{5}{4}\right\}$ and $\bar\theta(\Psi)=\left\{\binom{5}{4}\right\}$ by (5.28). We have

$$C_{\Phi,\Psi}=\left\{\tbinom{4,2,0}{3,1},\tbinom{4,3,0}{2,1},\tbinom{4,2,1}{3,0},\tbinom{4,3,1}{2,0}\right\},\qquad C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0}=\{Z\}$$

$$C_{\bar\theta(\Phi),\bar\theta(\Psi)}=\{Z',Z'^{\rm t}\},$$

and the conclusion in (i) of Lemma 5.30 is clearly verified.

(2) Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{14},\mathrm{O}^+_{14})$, $Z=\binom{5,2,0}{3,1}$, $Z'=\binom{5,2,0}{4,2,0}$. Then $Z_{\mathrm{I}}=Z$, $Z'_{\mathrm{I}}=\binom{5}{4}$, $|\mathscr{S}_Z|=16$, and $|\mathscr{S}_{Z'}|=2$. It is not difficult to check that

$$\begin{aligned}
\mathscr{B}_{Z,Z'}&=\left\{(\Lambda,Z'),(\Lambda,Z'^{\mathrm{t}})\mid\Lambda=\tbinom{4,2,0}{3,1},\tbinom{4,3,0}{2,1},\tbinom{4,2,1}{3,0},\tbinom{4,3,1}{2,0}\right\};\\
D_Z&=\{Z',Z'^{\mathrm{t}}\}=\mathscr{S}_{Z',\Psi'_0};\\
D_{Z'}&=\left\{\tbinom{4,2,0}{3,1},\tbinom{4,3,0}{2,1},\tbinom{4,2,1}{3,0},\tbinom{4,3,1}{2,0}\right\}=\mathscr{S}_{Z,\Psi_0},
\end{aligned}$$

i.e., $\Psi_0=\left\{\binom{2}{3},\binom{0}{1}\right\}$, $\Psi'_0=\left\{\binom{5}{4}\right\}$. Now $Z_{\mathrm{I}}\smallsetminus\Psi_0=\binom{4}{-}$, $Z'_{\mathrm{I}}\smallsetminus\Psi'_0=\emptyset$, and hence $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)=0$. Then $\mathscr{S}_Z^{\Psi_0}=\{Z\}$ and $\mathscr{S}_{Z'}^{\Psi'_0}=\{Z'\}$, the mapping $\theta\colon\mathscr{S}_{Z'}^{\Psi'_0}\to\mathscr{S}_Z^{\Psi_0}$ is just $Z'\mapsto Z$, and so $\mathscr{B}^\natural_{Z,Z'}=\{(Z,Z')\}$. The only arrangement Φ' of Z'_{I} containing Ψ'_0 is Ψ'_0 itself, and so now $\Psi'_0=\Psi'=\Phi'=\left\{\binom{5}{4}\right\}$. Now $\bar\theta(\Phi')=\bar\theta(\Psi')=\left\{\binom{2}{3},\binom{0}{1}\right\}$ by (5.29). We have

$$C_{\Phi',\Psi'}=\{Z',Z'^{\mathrm{t}}\},\qquad C_{\Phi',\Psi'}\cap\mathscr{S}_{Z'}^{\Psi'_0}=\{Z'\}$$
$$C_{\bar\theta(\Phi'),\bar\theta(\Psi')}=\left\{\tbinom{4,2,0}{3,1},\tbinom{4,3,0}{2,1},\tbinom{4,2,1}{3,0},\tbinom{4,3,1}{2,0}\right\},$$

and the conclusion in (ii) of Lemma 5.30 is verified.

Proof of Theorem 5.3. Let $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$, and let Z,Z' be special symbols of rank n,n', defects $1,0$ respectively. Let Ψ_0,Ψ'_0 be the cores in $Z_{\mathrm{I}},Z'_{\mathrm{I}}$ of $\mathscr{D}_{Z,Z'}$, and δ_0,δ'_0 the numbers of pairs in Ψ_0,Ψ'_0 respectively. From (5.4), we have

$$\sum_{(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}}\rho_\Lambda\otimes\rho_{\Lambda'}=\sum_{(\Lambda_1,\Lambda'_1)\in\mathscr{B}^\natural_{Z,Z'}}\sum_{\Lambda_2\in D_{Z'}}\sum_{\Lambda'_2\in D_Z}\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\Lambda'_1+\Lambda'_2}.$$

Note that $D_{Z'}=\{\Lambda_M\mid M\leq\Psi_0\}$ and $D_Z=\{\Lambda_N\mid N\leq\Psi'_0\}$ by proposition 6.4 in [Pan21]. From the proofs of [Pan21] proposition 7.17, we know that either $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)+1$ or $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)$.

(1) Suppose that $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)+1$. Then from the discussion before Lemma 5.30 and Lemma 5.9, we have a injective map $\theta\colon\mathscr{S}_Z^{\Psi_0}\to\mathscr{S}_{Z'}^{\Psi'_0}$, and

$$\mathscr{B}^\natural_{Z,Z'}=\{(\Lambda,\theta(\Lambda))\mid\Lambda\in\mathscr{S}_Z^{\Psi_0}\}.\tag{5.32}$$

For two arrangements Φ,Φ' of Z_{I} with a subset of pairs Ψ,Ψ' respectively such that $\Psi_0\leq\Psi$ and $\Psi_0\leq\Psi'$, by Lemma 4.28 and Proposition 4.18 the class function

$$\sum_{\Lambda\in C_{\Phi,\Psi}}\rho_\Lambda=\sum_{\Lambda_1\in C^\natural_{\Phi,\Psi}}\sum_{\Lambda_2\in\mathscr{S}_{Z,\Psi_0}}\rho_{\Lambda_1+\Lambda_2}$$

on G is uniform where $C^\natural_{\Phi,\Psi}=C_{\Phi,\Psi}\cap\mathscr{S}_Z^{\Psi_0}$, similarly by Lemma 5.30, the class function

$$\sum_{\Lambda'\in C_{\bar\theta(\Phi'),\bar\theta(\Psi')}}\rho_{\Lambda'}=\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}}\sum_{\Lambda'_2\in\mathscr{S}_{Z',\Psi'_0}}\rho_{\theta(\Lambda'_1)+\Lambda'_2}+\rho_{\theta(\Lambda'_1)^{\mathrm{t}}+\Lambda'_2}$$

on G' is uniform where $C^\natural_{\Phi',\Psi'}=C_{\Phi',\Psi'}\cap\mathscr{S}_Z^{\Psi_0}$.

For $\Lambda_1,\Lambda'_1\in\mathscr{S}_Z^{\Psi_0}$, we define

$$x_{\Lambda_1,\Lambda'_1}=\frac{1}{2^{\delta_0+\delta'_0}}\sum_{\Lambda_2\in\mathscr{S}_{Z,\Psi_0}}\sum_{\Lambda'_2\in\mathscr{S}_{Z',\Psi'_0}}\langle\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\theta(\Lambda'_1)+\Lambda'_2}+\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\theta(\Lambda'_1)^{\mathrm{t}}+\Lambda'_2},\omega_{Z,Z'}\rangle.$$

Note that $|\mathscr{S}_{Z,\Psi_0}|=2^{\delta_0}$ and $|\mathscr{S}_{Z',\Psi'_0}|=2^{\delta'_0}$ by (4.2). Now by (5.2), (5.4) and (5.32), we have

$$\begin{aligned}&\sum_{\Lambda_1\in C^\natural_{\Phi,\Psi}}\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}} x_{\Lambda_1,\Lambda'_1}\\ &=\frac{1}{2^{\delta_0+\delta'_0}}\Bigg\langle\sum_{\Lambda_1\in C^\natural_{\Phi,\Psi}}\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}}\sum_{\Lambda_2\in\mathscr{S}_{Z,\Psi_0}}\sum_{\Lambda'_2\in\mathscr{S}_{Z',\Psi'_0}}(\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\theta(\Lambda'_1)+\Lambda'_2}+\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\theta(\Lambda'_1)^\mathrm{t}+\Lambda'_2}),\\ &\qquad\sum_{\Lambda''_1\in\mathscr{S}_Z^{\Psi_0}}\sum_{\Lambda''_2\in\mathscr{S}_{Z,\Psi_0}}\sum_{\Lambda'''_2\in\mathscr{S}_{Z',\Psi'_0}}\rho_{\Lambda''_1+\Lambda''_2}\otimes\rho_{\theta(\Lambda''_1)+\Lambda'''_2}\Bigg\rangle.\end{aligned}$$

For a symbol $\Lambda''_1\in\mathscr{S}_Z^{\Psi_0}$ to contribute a multiplicity in the above identity, by Lemma 4.6, we need $\Lambda''_1=\Lambda_1$ and $\Lambda''_1=\Lambda'_1$ for some $\Lambda_1\in C^\natural_{\Phi,\Psi}$ and some $\Lambda'_1\in C^\natural_{\Phi',\Psi'}$, i.e., Λ''_1 must be in the intersection $C^\natural_{\Phi,\Psi}\cap C^\natural_{\Phi',\Psi'}$. Then

$$\sum_{\Lambda_1\in C^\natural_{\Phi,\Psi}}\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}} x_{\Lambda_1,\Lambda'_1}=\frac{|C^\natural_{\Phi,\Psi}\cap C^\natural_{\Phi',\Psi'}|}{2^{\delta_0+\delta'_0}}\sum_{\Lambda_2\in\mathscr{S}_{Z,\Psi_0}}\sum_{\Lambda'_2\in\mathscr{S}_{Z',\Psi'_0}}1=|C^\natural_{\Phi,\Psi}\cap C^\natural_{\Phi',\Psi'}| \tag{5.33}$$

for any arrangements Φ,Φ' of Z_I with subsets of pairs Ψ,Ψ' respectively such that $\Psi_0\leq\Psi$ and $\Psi_0\leq\Psi'$. Note that as in the proof of Proposition 5.13, $\theta(\Lambda''_1)\neq\theta(\Lambda'_1)^\mathrm{t}$ for any $\Lambda'_1,\Lambda''_1\in\mathscr{S}_Z^{\Psi_0}$.

Suppose that Λ_1,Λ_2 are distinct symbols in $\mathscr{S}_Z^{\Psi_0}$. By Lemma 4.31, there exists an arrangement Φ of Z_I with two subsets of pairs Ψ_1,Ψ_2 such that $\Psi_0\leq\Psi_i$, $\Lambda_i\in C_{\Phi,\Psi_i}$ for $i=1,2$ and $C_{\Phi,\Psi_1}\cap C_{\Phi,\Psi_2}=\emptyset$. Then $C^\natural_{\Phi,\Psi_1}\cap C^\natural_{\Phi,\Psi_2}=\emptyset$. Because each x_{Λ_1,Λ'_1} is a non-negative integer, from (5.33) we conclude that $x_{\Lambda_1,\Lambda_2}=0$ for any distinct $\Lambda_1,\Lambda_2\in\mathscr{S}_Z^{\Psi_0}$.

Finally, for any $\Lambda\in\mathscr{S}_Z^{\Psi_0}$, by Lemma 4.30, there exist two arrangements Φ_1,Φ_2 of Z_I with subsets of pairs Ψ_1,Ψ_2 respectively such that $\Psi_0\leq\Psi_i$ for $i=1,2$ and

$$C^\natural_{\Phi_1,\Psi_1}\cap C^\natural_{\Phi_2,\Psi_2}=C_{\Phi_1,\Psi_1}\cap C_{\Phi_2,\Psi_2}\cap\mathscr{S}_Z^{\Psi_0}=\{\Lambda\}.$$

Because we know $x_{\Lambda_1,\Lambda_2}=0$ if $\Lambda_1\neq\Lambda_2$, equation (5.33) is now reduced to $x_{\Lambda,\Lambda}=1$.

Suppose that $\Lambda\in\mathscr{S}_Z$ and $\Lambda'\in\mathscr{S}_{Z'}$ such that $(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}$. We can write $\Lambda=\Lambda_1+\Lambda_2$ for $\Lambda_1\in\mathscr{S}_Z^{\Psi_0}$ and $\Lambda_2\in\mathscr{S}_{Z,\Psi_0}$ and similarly write $\Lambda'=\Lambda'_1+\Lambda'_2$ for $\Lambda'_1\in\mathscr{S}_{Z'}^{\Psi_0}$ and $\Lambda'_2\in\mathscr{S}_{Z',\Psi'_0}$ such that $(\Lambda_1,\Lambda'_1)\in\mathscr{B}^\natural_{Z,Z'}$. Then we have shown that either $\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\Lambda'_1+\Lambda'_2}$ or $\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\Lambda'^\mathrm{t}_1+\Lambda'_2}$ occurs in $\omega_{Z,Z'}$. Note that $\Lambda'^\mathrm{t}=(\Lambda'_1+\Lambda'_2)^\mathrm{t}=\Lambda'^\mathrm{t}_1+\Lambda'_2$ by [Pan21] lemma 2.1. Hence we conclude that either $\rho_\Lambda\otimes\rho_{\Lambda'}$ or $\rho_\Lambda\otimes\rho_{\Lambda'^\mathrm{t}}$ occurs in $\omega_{Z,Z'}$.

(2) Suppose that $\deg(Z'\smallsetminus\Psi'_0)=\deg(Z\smallsetminus\Psi_0)$. Then from the discussion before Lemma 5.30 and Lemma 5.18, we have a injective map $\theta\colon\mathscr{S}_{Z'}^{\Psi'_0}\to\mathscr{S}_Z^{\Psi_0}$ and

$$\mathscr{B}^\natural_{Z,Z'}=\{(\theta(\Lambda'),\Lambda')\mid\Lambda'\in\mathscr{S}_{Z'}^{\Psi'_0}\}.$$

For an arrangement Φ of Z_I with a subset of pairs Ψ such that $\Psi_0\leq\Psi$, and an arrangement Φ' of Z'_I with an admissible subset of pairs Ψ' such that $\Psi'_0\leq\Psi'$, the class function

$$\sum_{\Lambda\in C_{\Phi,\Psi}}\sum_{\Lambda'\in C_{\Phi',\Psi'}}\rho_\Lambda\otimes\rho_{\Lambda'}=\sum_{\Lambda_1\in C^\natural_{\Phi,\Psi}}\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}}\sum_{\Lambda_2\in\mathscr{S}_{Z,\Psi_0}}\sum_{\Lambda'_2\in\mathscr{S}_{Z',\Psi'_0}}\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\Lambda'_1+\Lambda'_2}$$

on $G\times G'$ is uniform by Proposition 4.18 and Proposition 4.35.

For $\Lambda_1\in\mathscr{S}_Z^{\Psi_0}$ and $\Lambda'_1\in\mathscr{S}_{Z'}^{\Psi'_0}$, define

$$x_{\Lambda_1,\Lambda'_1}=\frac{1}{2^{\delta_0+\delta'_0}}\sum_{\Lambda_2\in\mathscr{S}_{Z,\Psi_0}}\sum_{\Lambda'_2\in\mathscr{S}_{Z',\Psi'_0}}\langle\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\Lambda'_1+\Lambda'_2},\omega_{Z,Z'}\rangle.$$

Then

$$\sum_{\Lambda_1\in C^\natural_{\Phi,\Psi}}\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}}x_{\Lambda_1,\Lambda'_1}=\frac{1}{2^{\delta_0+\delta'_0}}\Bigg\langle\sum_{\Lambda_1\in C^\natural_{\Phi,\Psi}}\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}}\sum_{\Lambda_2\in\mathscr{S}_{Z,\Psi_0}}\sum_{\Lambda'_2\in\mathscr{S}_{Z',\Psi'_0}}\rho_{\Lambda_1+\Lambda_2}\otimes\rho_{\Lambda'_1+\Lambda'_2},$$
$$\sum_{\Lambda''_1\in\mathscr{S}_{Z'}^{\Psi'_0}}\sum_{\Lambda''_2\in\mathscr{S}_{Z,\Psi_0}}\sum_{\Lambda'''_2\in\mathscr{S}_{Z',\Psi'_0}}\rho_{\theta(\Lambda''_1)+\Lambda''_2}\otimes\rho_{\Lambda''_1+\Lambda'''_2}\Bigg\rangle.$$

By the same argument in (1) and in the proof of Proposition 5.23 (in particular, (5.25)) we conclude that

$$\sum_{\Lambda_1\in C^\natural_{\Phi,\Psi}}\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}}x_{\Lambda_1,\Lambda'_1}=|C^\natural_{\Phi,\Psi}\cap C^\natural_{\bar\theta(\Phi'),\bar\theta(\Psi')}| \tag{5.34}$$

for any arrangement Φ of $Z_{\rm I}$ with subset of pairs Ψ such that $\Psi_0\leq\Psi$, and any arrangement Φ' of $Z'_{\rm I}$ with admissible subset of pairs Ψ' such that $\Psi'_0\leq\Psi'$.

Now let $\Phi=\bar\theta(\Phi'')$ and $\Psi=\bar\theta(\Psi'')$ for some arrangement Φ'' of $Z'_{\rm I}$ with some admissible subset of pairs Ψ''. Then by Lemma 5.30, (5.34) becomes

$$\sum_{\Lambda''_1\in C^\natural_{\Phi'',\Psi''}}\sum_{\Lambda'_1\in C^\natural_{\Phi',\Psi'}}x_{\theta(\Lambda''_1),\Lambda'_1}=|C^\natural_{\bar\theta(\Phi''),\bar\theta(\Psi'')}\cap C^\natural_{\bar\theta(\Phi'),\bar\theta(\Psi')}|=|C^\natural_{\Phi'',\Psi''}\cap C^\natural_{\Phi',\Psi'}| \tag{5.35}$$

for any two arrangements Φ',Φ'' of $Z'_{\rm I}$ and any admissible subsets of pairs Ψ',Ψ'' containing Ψ'_0 respectively. Suppose Λ'_1,Λ'_2 are symbols in $\mathscr{S}_{Z'}^{\Psi'_0}$ such that $\Lambda'_1\neq\Lambda'_2,\Lambda'^{\rm t}_2$. Then by Lemma 4.40, there exists an arrangement Φ' of $Z'_{\rm I}$ with two subsets of pairs Ψ'_1,Ψ'_2 containing Ψ'_0 such that $\Lambda'_i,\Lambda'^{\rm t}_i\in C_{\Phi',\Psi'_i}$ for $i=1,2$ and $C_{\Phi',\Psi'_1}\cap C_{\Phi',\Psi'_2}=\emptyset$. Because each $x_{\theta(\Lambda''),\Lambda'}$ is a non-negative integer, from equation (5.35) we conclude that

$$x_{\theta(\Lambda'_1),\Lambda'_2}=x_{\theta(\Lambda'_1),\Lambda'^{\rm t}_2}=0.$$

Clearly there is an arrangement Φ' of $Z'_{\rm I}$ such that $\bar\theta(\Phi')$ in (5.29) is the arrangement Φ_2 in Lemma 4.30. For any given $\Lambda'_1\in\mathscr{S}_{Z'}^{\Psi'_0}$, then $\theta(\Lambda'_1)\in\mathscr{S}_Z^{\Psi_0}$, and by Lemma 4.30, there exist an arrangement Φ_1 of $Z_{\rm I}$ with a subset of pairs Ψ_1, and a subset of pairs Ψ_2 of $\bar\theta(\Phi')$ given above such that $\Psi_0\leq\Psi_1$, $\Psi_0\leq\Psi_2$, and $C^\natural_{\Phi_1,\Psi_1}\cap C^\natural_{\bar\theta(\Phi'),\Psi_2}=\{\theta(\Lambda'_1)\}$. Similar to the argument in the proof of Proposition 5.23, we see that $\Psi_2=\bar\theta(\Psi')$ for some admissible Ψ' such that $\Psi'_0\leq\Psi'\leq\Phi'$. Therefore we have

$$C^\natural_{\Phi_1,\Psi_1}\cap C^\natural_{\bar\theta(\Phi'),\bar\theta(\Psi')}=\{\theta(\Lambda'_1)\}.$$

Because we know $x_{\theta(\Lambda_1),\Lambda_2}=0$ for any $\Lambda_1\neq\Lambda_2,\Lambda_2^{\rm t}$, (5.34) is now reduced to

$$x_{\theta(\Lambda'_1),\Lambda'_1}+x_{\theta(\Lambda'_1),\Lambda'^{\rm t}_1}=1.$$

By the same argument in the last paragraph of (1), we conclude that if $(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}$, then either $\rho_\Lambda\otimes\rho_{\Lambda'}$ or $\rho_\Lambda\otimes\rho_{\Lambda'^{\rm t}}$ occurs in $\omega_{Z,Z'}$.

For two cases (1) and (2), and for each $(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}$, define $\widetilde{\Lambda}'$ to be either Λ' or $\Lambda'^{\rm t}$ such that $\rho_\Lambda\otimes\rho_{\widetilde{\Lambda}'}$ occurs in $\omega_{Z,Z'}$. Therefore, $\sum_{(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}}\rho_\Lambda\otimes\rho_{\widetilde{\Lambda}'}$ is a sub-character of $\omega_{Z,Z'}$. Then by the same argument in the last paragraph of the proof of Proposition 5.13, we conclude that

$$\omega_{Z,Z'}=\sum_{(\Lambda,\Lambda')\in\mathscr{B}_{Z,Z'}}\rho_\Lambda\otimes\rho_{\widetilde{\Lambda}'}.$$

□

From the above proof, we conclude the following corollary:

Corollary 5.36. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ *where* $\epsilon=+$ *or* $-$, $\rho_\Lambda\in\mathscr{E}(G)_1$, $\rho_{\Lambda'}\in\mathscr{E}(G')_1$ *such that* $\Lambda'\neq\Lambda'^{\rm t}$. *Then exactly one of* $(\Lambda,\Lambda'),(\Lambda,\Lambda'^{\rm t})$ *occurs in* $\mathscr{B}_{\mathbf{G},\mathbf{G}'}$ *if and only if exactly one of* $\rho_\Lambda\otimes\rho_{\Lambda'},\rho_\Lambda\otimes\rho_{\Lambda'^{\rm t}}$ *occurs in the correspondence.*

6. Symbol Correspondence and Parabolic Induction

The purpose of this section is to remove the ambiguity of Theorem 5.3, i.e., to provide a proof of Theorem 1.8. The proof for the case $\mathrm{def}(\Lambda')\neq 0$ is in Subsection 6.1 (Proposition 6.4); and the proof for the case $\mathrm{def}(\Lambda')=0$ is in Subsection 6.3 (Proposition 6.20).

6.1. Properties of the parametrization. For a symbol

$$\Lambda=\binom{a_1,a_2,\ldots,a_{m_1}}{b_1,b_2,\ldots,b_{m_2}}\in\mathscr{S}, \tag{6.1}$$

we define Ω^+_Λ to be the set consisting of the following types of symbols:

(I) $\binom{a_1,\ldots,a_{i-1},a_i+1,a_{i+1},\ldots,a_{m_1}}{b_1,b_2,\ldots,b_{m_2}}$ for $i=1,\ldots,m_1$ if $a_{i-1}>a_i+1$ (the condition is empty if $i=1$);

(II) $\binom{a_1,a_2,\ldots,a_{m_1}}{b_1,\ldots,b_{j-1},b_j+1,b_{j+1},\ldots,b_{m_2}}$ for $j=1,\ldots,m_2$ if $b_{j-1}>b_j+1$ (the condition is empty if $j=1$);

(III) $\binom{a_1+1,a_2+1,\ldots,a_{m_1}+1,1}{b_1+1,b_2+1,\ldots,b_{m_2}+1,0}$ if $a_{m_1}\neq 0$;

(IV) $\binom{a_1+1,a_2+1,\ldots,a_{m_1}+1,0}{b_1+1,b_2+1,\ldots,b_{m_2}+1,1}$ if $b_{m_2}\neq 0$.

Clearly, if $\Lambda'\in\Omega^+_\Lambda$, then $\mathrm{rank}(\Lambda')=\mathrm{rank}(\Lambda)+1$ and $\mathrm{def}(\Lambda')=\mathrm{def}(\Lambda)$.

We also define

$$\Omega^-_\Lambda=\{\,\Lambda'\in\mathscr{S}\mid\Lambda\in\Omega^+_{\Lambda'}\,\}.$$

Therefore, if $\Lambda'\in\Omega^-_\Lambda$, then $\mathrm{rank}(\Lambda')=\mathrm{rank}(\Lambda)-1$ and $\mathrm{def}(\Lambda')=\mathrm{def}(\Lambda)$. It is not difficult to see that Ω^-_Λ consists of symbols of the following types:

(I') $\binom{a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_{m_1}}{b_1,b_2,\ldots,b_m}$ for $i=1,\ldots,m_1-1$ if $a_i>a_{i+1}+1$;
$\binom{a_1,\ldots,a_{m_1-1},a_{m_1}-1}{b_1,b_2,\ldots,b_m}$ if $a_{m_1}\geq 1$ and $(a_{m_1},b_{m_2})\neq(1,0)$;

(II') $\binom{a_1,a_2,\ldots,a_{m_1}}{b_1,\ldots,b_{j-1},b_j-1,b_{j+1},\ldots,b_{m_2}}$ for $j=1,\ldots,m_2-1$ if $b_j>b_{j+1}+1$;
$\binom{a_1,a_2,\ldots,a_{m_1}}{b_1,\ldots,b_{m_2-1},b_{m_2}-1}$ if $b_{m_2}\geq 1$ and $(a_{m_1},b_{m_2})\neq(0,1)$;

(III') $\binom{a_1-1,a_2-1,\ldots,a_{m_1-1}-1}{b_1-1,b_2-1,\ldots,b_{m_2-1}-1}$ if $(a_{m_1},b_{m_2})=(1,0)$ or $(0,1)$.

Example 6.2. Suppose that $\Lambda=\binom{4,2,1}{3,0}$. Then we have

$$\Omega^+_\Lambda=\left\{\binom{5,2,1}{3,0},\binom{4,3,1}{3,0},\binom{4,2,1}{4,0},\binom{4,2,1}{3,1},\binom{5,3,2,1}{4,1,0}\right\},$$
$$\Omega^-_\Lambda=\left\{\binom{3,2,1}{3,0},\binom{4,2,1}{2,0},\binom{3,1}{2}\right\}.$$

Note that $\binom{3,1}{2}\sim\binom{4,2,0}{3,0}$ (*cf.* Subsection 2.1).

Recall that in Subsection 3.2 and Subsection 3.3, we have a parametrization $\mathscr{S}_{\mathbf{G}}\to\mathscr{E}(G)_1$ by $\Lambda\mapsto\rho_\Lambda$ for $\mathbf{G}=\mathrm{Sp}_{2n}$ or $\mathrm{O}^\epsilon_{2n'}$. The parametrization also satisfies the following conditions:

- The unique unipotent cuspidal character ζ_k of $\mathrm{Sp}_{2k(k+1)}(q)$ is parametrized by $\zeta_k=\rho_{\Lambda_k}$ where Λ_k is given in (3.5).
- For $k\geq 1$, the two unipotent cuspidal characters $\zeta^{\mathrm{I}}_k,\zeta^{\mathrm{II}}_k$ of $\mathrm{O}^{\epsilon_k}_{2k^2}(q)$ where $\epsilon_k=(-1)^k$ are parametrized by $\zeta^{\mathrm{I}}_k=\rho_{\Lambda'_k}$ and $\zeta^{\mathrm{II}}_k=\rho_{\Lambda'^{\mathrm{t}}_k}$ where Λ'_k is given in (3.7). For $k=0$, the trivial character $\mathbf{1}_{\mathrm{O}^+_0}$ of $\mathrm{O}^+_0(q)$ is also a unipotent cuspidal character and is parametrized by the empty symbol $\binom{-}{-}$.
- $\mathbf{1}_{\mathrm{O}^+_2}=\rho_{\binom{1}{0}}$ and $\mathrm{sgn}_{\mathrm{O}^+_2}=\rho_{\binom{0}{1}}$.
- The following branching rule holds:

$$\mathrm{Ind}^{\mathrm{Sp}_{2(n+1)}(q)}_{P_n}(\rho_\Lambda\otimes\mathbf{1})=\sum_{\Lambda''\in\Omega^+_\Lambda}\rho_{\Lambda''};$$
$$\mathrm{Ind}^{\mathrm{O}^+_{2(n'+1)}(q)}_{P'_{n'}}(\rho_{\Lambda'}\otimes\mathbf{1})=\sum_{\Lambda'''\in\Omega^+_{\Lambda'}}\rho_{\Lambda'''}\tag{6.3}$$

where P_n (resp. $P'_{n'}$) is a parabolic subgroup of $\mathrm{Sp}_{2(n+1)}(q)$ (resp. $\mathrm{O}^+_{2(n'+1)}(q)$) with Levi factor $\mathrm{Sp}_{2n}(q)\times\mathrm{GL}_1(q)$ (resp. $\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)$). In particular, the defects are preserved by parabolic induction.

Proposition 6.4. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^\epsilon_{2n'})$ *where* $\epsilon=+$ *or* $-$, $\rho_\Lambda\in\mathscr{E}(G)_1$, $\rho_{\Lambda'}\in\mathscr{E}(G')_1$. *Suppose that* $\mathrm{def}(\Lambda')\neq 0$. *Then* $\rho_\Lambda\otimes\rho_{\Lambda'}$ *occurs in* $\omega_{\mathbf{G},\mathbf{G}',1}$ *if and only if* $(\Lambda,\Lambda')\in\mathscr{B}_{\mathbf{G},\mathbf{G}'}$.

Proof. If $(\Lambda,\Lambda')\in\mathscr{B}_{\mathbf{G},\mathbf{G}'}$, then from the definition in (1.5) we know that

$$\mathrm{def}(\Lambda')=\begin{cases}-\mathrm{def}(\Lambda)+1, & \text{if } \epsilon=+;\\ -\mathrm{def}(\Lambda)-1, & \text{if } \epsilon=-.\end{cases}\tag{6.5}$$

We also know that $\mathrm{def}(\Lambda'^{\mathrm{t}})=-\mathrm{def}(\Lambda')\neq\mathrm{def}(\Lambda')$ since we assume that $\mathrm{def}(\Lambda')\neq 0$. Now suppose that $\rho_\Lambda\otimes\rho_{\Lambda'}$ occurs in $\omega_{\mathbf{G},\mathbf{G}',1}$.

(1) First suppose that both $\rho_\Lambda,\rho_{\Lambda'}$ are cuspidal. Then we know that $n=k(k+1)$, and $n'=k^2$ or $(k+1)^2$ for some non-negative integer k.

  (a) Suppose that $n=k(k+1)$ and $n'=(k+1)^2$ (and $\epsilon=(-1)^{k+1}$). Then we know that $\zeta_k\otimes\zeta^{\mathrm{I}}_{k+1}$ occurs in $\omega_{\mathbf{G},\mathbf{G}',1}$ from [AM93] theorem 5.2. Now $\zeta_k=\rho_{\Lambda_k}$, $\zeta^{\mathrm{I}}_{k+1}=\rho_{\Lambda'_{k+1}}$ where Λ_k,Λ'_{k+1} are given in (3.5) and (3.7), and we have

$$\Lambda'_{k+1}=\begin{cases}\binom{-}{2k+1}\cup\Lambda^{\mathrm{t}}_k, & \text{if } k \text{ is even};\\ \binom{2k+1}{-}\cup\Lambda^{\mathrm{t}}_k, & \text{if } k \text{ is odd}.\end{cases}$$

  Hence $\mathrm{def}(\Lambda'_{k+1})=-\mathrm{def}(\Lambda_k)+1$ if $\epsilon=+$; and $\mathrm{def}(\Lambda'_{k+1})=-\mathrm{def}(\Lambda_k)-1$ if $\epsilon=-$.

(b) Suppose that $n = k(k+1)$ and $n' = k^2$ (and $\epsilon = (-1)^k$). Then we know that $\zeta_k \otimes \zeta_k^{\rm II}$ occurs in $\omega_{{\bf G},{\bf G}',1}$ from [AM93] theorem 5.2. Now $\zeta_k = \rho_{\Lambda_k}$, $\zeta_k^{\rm II} = \rho_{\Lambda_k'^{\rm t}}$, and

$$\Lambda_k'^{\rm t} = \begin{cases} \Lambda_k^{\rm t} \smallsetminus \binom{-}{2k}, & \text{if } k \text{ is even (hence } \epsilon = +); \\ \Lambda_k^{\rm t} \smallsetminus \binom{2k}{-}, & \text{if } k \text{ is odd (hence } \epsilon = -). \end{cases}$$

Hence $\mathrm{def}(\Lambda_k'^{\rm t}) = -\mathrm{def}(\Lambda_k)+1$ if $\epsilon = +$; and $\mathrm{def}(\Lambda_k'^{\rm t}) = -\mathrm{def}(\Lambda_k)-1$ if $\epsilon = -$.

(2) Next, suppose that Λ is not cuspidal and $\mathrm{def}(\Lambda) = 4d+1$ for some integer d. If $\epsilon = +$, we also assume that $d \neq 0$. This assumption implies that $\mathrm{def}(\Lambda') \neq 0$. Then ρ_Λ lies in the parabolic induced character $\mathrm{Ind}_{P_{k(k+1)}}^{\mathrm{Sp}_{2n}(q)}(\rho_{\Lambda_k} \otimes {\bf 1})$ where $P_{k(k+1)}$ is the parabolic subgroup whose Levi factor is $\mathrm{Sp}_{2k(k+1)}(q) \times T$ and T is a split torus of rank $n - k(k+1)$ and Λ_k is given in (3.5) for some non-negative integer k such that $k(k+1) < n$.

It is well-known that the theta correspondence is compatible with the parabolic induction (*cf.* [AMR96] théorème 3.7), so now $\rho_{\Lambda'}$ lies in the parabolic induced character $\mathrm{Ind}_{P'_{k'^2}}^{\mathrm{O}^\epsilon_{2n'}(q)}(\rho_{\bar\Lambda'_{k'}} \otimes {\bf 1})$ where $P'_{k'^2}$ is the parabolic subgroup whose Levi factor is $\mathrm{O}^\epsilon_{2k'^2}(q) \times T'$, k' is equal to k or $k+1$ depending on ϵ, T' is a split torus of rank $n' - k'^2$, and $\bar\Lambda'_{k'}$ is $\Lambda'_{k'}$ or $\Lambda'^{\rm t}_{k'}$ given in (3.7). Now the defects of $\Lambda_k, \bar\Lambda'_{k'}$ satisfy (6.5) by (1). Moreover, we know that $\mathrm{def}(\Lambda) = \mathrm{def}(\Lambda_k)$ and $\mathrm{def}(\Lambda') = \mathrm{def}(\bar\Lambda'_{k'})$ from the remark before the proposition.

Now we have shown that if $\rho_\Lambda \otimes \rho_{\Lambda'}$ occurs in $\omega_{{\bf G},{\bf G}',1}$ and $\mathrm{def}(\Lambda') \neq 0$, then $\mathrm{def}(\Lambda'^{\rm t}) \neq \mathrm{def}(\Lambda') = -\mathrm{def}(\Lambda) + \epsilon$, and hence $(\Lambda, \Lambda'^{\rm t}) \notin \mathscr{B}_{{\bf G},{\bf G}'}$. Then by Theorem 5.3, we must have $(\Lambda, \Lambda') \in \mathscr{B}_{{\bf G},{\bf G}'}$.

On the other hand, suppose that $(\Lambda, \Lambda') \in \mathscr{B}_{{\bf G},{\bf G}'}$ and $\mathrm{def}(\Lambda') \neq 0$. Then by definition of $\mathscr{B}_{{\bf G},{\bf G}'}$ we have $(\Lambda, \Lambda'^{\rm t}) \notin \mathscr{B}_{{\bf G},{\bf G}'}$. Then by the above argument, we see that $\rho_\Lambda \otimes \rho_{\Lambda'^{\rm t}}$ does not occurs in $\omega_{{\bf G},{\bf G}',1}$. Then by Theorem 5.3, we must have $\rho_\Lambda \otimes \rho_{\Lambda'}$ occurs in $\omega_{{\bf G},{\bf G}',1}$. □

Corollary 6.6. *Let* $({\bf G},{\bf G}') = (\mathrm{Sp}_{2n}, \mathrm{O}^-_{2n'})$, $\rho_\Lambda \in \mathscr{E}(G)_1$, $\rho_{\Lambda'} \in \mathscr{E}(G')_1$. *Then* $\rho_\Lambda \otimes \rho_{\Lambda'}$ *occurs in* $\omega_{{\bf G},{\bf G}',1}$ *if and only if* $(\Lambda, \Lambda') \in \mathscr{B}_{{\bf G},{\bf G}'}$.

Proof. Let $\Lambda' \in \mathscr{S}_{\mathrm{O}^-_{2n'}}$. Then $\mathrm{def}(\Lambda') \equiv 2 \pmod 4$ from (1.3), so $\mathrm{def}(\Lambda') \neq 0$. Hence the corollary follows the previous proposition immediately. □

6.2. Branching rule and symbol correspondence. From now on we consider the case which is not settled by Proposition 6.4, i.e., $\epsilon = +$, $\mathrm{def}(\Lambda) = 1$ and $\mathrm{def}(\Lambda') = 0$. Define

$$\mathscr{B}^+ = \bigcup_{n,n' \geq 0} \mathscr{B}_{\mathrm{Sp}_{2n}, \mathrm{O}^+_{2n'}}.$$

For a symbol Λ of defect 1 and a set Ω' of symbols of defect 0, we define two subsets of Ω' by

$$\Theta_\Lambda(\Omega') = \{\, \Lambda' \in \Omega' \mid (\Lambda, \Lambda') \in \mathscr{B}^+ \,\},$$
$$\Theta^*_\Lambda(\Omega') = \{\, \Lambda' \in \Omega' \mid (\Lambda, \Lambda'^{\rm t}) \in \mathscr{B}^+ \text{ and } (\Lambda, \Lambda') \notin \mathscr{B}^+ \,\}.$$

Similarly, for a symbol Λ' of defect 0 and a set Ω of symbols of defect 1, we define

$$\Theta_{\Lambda'}(\Omega) = \{\, \Lambda \in \Omega \mid (\Lambda, \Lambda') \in \mathscr{B}^+ \,\},$$
$$\Theta^*_{\Lambda'}(\Omega) = \{\, \Lambda \in \Omega \mid (\Lambda, \Lambda'^{\rm t}) \in \mathscr{B}^+ \text{ and } (\Lambda, \Lambda') \notin \mathscr{B}^+ \,\}.$$

Example 6.7. Let $\Lambda=\binom{8,5,1}{6,2}$ and $\Lambda'=\binom{7,4,1}{8,5,1}$. By Lemma 2.15, it is easy to check that $(\Lambda,\Lambda')\in\mathscr{B}^+$ and

$$\begin{aligned}\Omega^+_{\Lambda'}&=\left\{\tbinom{8,4,1}{8,5,1},\tbinom{7,5,1}{8,5,1},\tbinom{7,4,2}{8,5,1},\tbinom{8,5,2,1}{9,6,2,0},\tbinom{7,4,1}{9,5,1},\tbinom{7,4,1}{8,6,1},\tbinom{7,4,1}{8,5,2},\tbinom{8,5,2,0}{9,6,2,1}\right\},\\ \Theta_\Lambda(\Omega^+_{\Lambda'})&=\left\{\tbinom{8,4,1}{8,5,1},\tbinom{7,5,1}{8,5,1},\tbinom{7,4,2}{8,5,1}\right\},\\ \Theta^*_\Lambda(\Omega^+_{\Lambda'})&=\left\{\tbinom{7,4,1}{9,5,1},\tbinom{7,4,1}{8,6,1},\tbinom{7,4,1}{8,5,2}\right\}.\end{aligned}$$

Lemma 6.8. *Suppose that $(\rho_\Lambda,\rho_{\Lambda'_1})$ occurs in the Howe correspondence, $(\Lambda,\Lambda'_1)\in\mathscr{B}^+$, and $\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})=\emptyset$. For any $\Lambda'\in\Omega^+_{\Lambda'_1}$, if $(\rho_\Lambda,\rho_{\Lambda'})$ occurs in the Howe correspondence, then $(\Lambda,\Lambda')\in\mathscr{B}^+$.*

Proof. Suppose that $\Lambda'\in\Omega^+_{\Lambda'_1}$ and $(\rho_\Lambda,\rho_{\Lambda'})$ occurs in the Howe correspondence. Then we know by Theorem 5.3 that (Λ,Λ') or $(\Lambda,\Lambda'^{\rm t})$ is in $\mathscr{B}^+$. But $\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})=\emptyset$ means that there is no $\Lambda''\in\Omega^+_{\Lambda'_1}$ such that $(\Lambda,\Lambda''^{\rm t})\in\mathscr{B}^+$ and $(\Lambda,\Lambda'')\notin\mathscr{B}^+$. Hence we have $(\Lambda,\Lambda')\in\mathscr{B}^+$. □

Lemma 6.9. *Suppose that $(\rho_{\Lambda_1},\rho_{\Lambda'})$ occurs in the Howe correspondence, $(\Lambda_1,\Lambda')\in\mathscr{B}^+$, and $\Theta^*_{\Lambda'}(\Omega^+_{\Lambda_1})=\emptyset$. For any $\Lambda\in\Omega^+_{\Lambda_1}$, if $(\rho_\Lambda,\rho_{\Lambda'})$ occurs in the Howe correspondence, then $(\Lambda,\Lambda')\in\mathscr{B}^+$.*

Proof. The proof is similar to that of Lemma 6.8. □

Lemma 6.10. *Let Λ,Λ' be symbols of defects $1,0$ respectively such that $(\Lambda,\Lambda')\in\mathscr{B}^+$. Then*

$$|\Theta_\Lambda(\Omega^+_{\Lambda'})|=1+|\Theta_{\Lambda'}(\Omega^-_\Lambda)|\quad\textit{and}\quad|\Theta_{\Lambda'}(\Omega^+_\Lambda)|=1+|\Theta_\Lambda(\Omega^-_{\Lambda'})|.$$

Proof. We will only prove the first equality since the proof for the second one is similar. Write $\Lambda=\binom{a_1,a_2,\dots,a_{m+1}}{b_1,b_2,\dots,b_m}$, $\Lambda'=\binom{c_1,c_2,\dots,c_{m'}}{d_1,d_2,\dots,d_{m'}}$ for some m,m', and suppose that $(\Lambda,\Lambda')\in\mathscr{B}^+$. We know that $m'=m,m+1$ by Lemma 2.14. It is clear that the symbol

$$\Lambda''_0:=\begin{pmatrix}c_1+1,c_2,\dots,c_{m'}\\ d_1,d_2,\dots,d_{m'}\end{pmatrix}$$

is in $\Omega^+_{\Lambda'}$ and $(\Lambda,\Lambda''_0)\in\mathscr{B}^+$ by Lemma 2.15, i.e., $\Lambda''_0\in\Theta_\Lambda(\Omega^+_{\Lambda'})$. Now we want to prove the lemma by constructing an injective map $\Theta_{\Lambda'}(\Omega^-_\Lambda)\to\Theta_\Lambda(\Omega^+_{\Lambda'})$ given by $\Lambda_1\mapsto\Lambda'_1$ such that the only element in $\Theta_\Lambda(\Omega^+_{\Lambda'})$ not in the image of the map is Λ''_0.

First suppose that $m'=m+1$. Now we consider the following possible types of elements in Ω^-_Λ and $\Omega^+_{\Lambda'}$:

(1) Consider the following two symbols

$$\Lambda_1=\begin{pmatrix}a_1,\dots,a_{i-1},a_i-1,a_{i+1},\dots,a_{m+1}\\ b_1,b_2,\dots,b_m\end{pmatrix},\quad \Lambda'_1=\begin{pmatrix}c_1,c_2,\dots,c_{m+1}\\ d_1,\dots,d_{i-1},d_i+1,d_{i+1},\dots,d_{m+1}\end{pmatrix}$$

for some $i=1,\dots,m$. If $\Lambda_1\in\Theta_{\Lambda'}(\Omega^-_\Lambda)$, i.e., $\Lambda_1\in\Omega^-_\Lambda$ and $(\Lambda_1,\Lambda')\in\mathscr{B}^+$, then we have

- $a_i-1>a_{i+1}$ (because Λ_1 is a symbol);
- $d_{i-1}>a_i$ (because $(\Lambda,\Lambda')\in\mathscr{B}^+$ by Lemma 2.15);
- $a_i-1\geq d_i$ (because $(\Lambda_1,\Lambda')\in\mathscr{B}^+$ by Lemma 2.15),

which imply

- $d_{i-1}>d_i+1$;
- $a_i\geq d_i+1$,

i.e., Λ'_1 is a symbol and $\Lambda'_1\in\Theta_\Lambda(\Omega^+_{\Lambda'})$. On the other hand, it is also not difficult to see that $\Lambda'_1\in\Theta_\Lambda(\Omega^+_{\Lambda'})$ implies $\Lambda_1\in\Theta_{\Lambda'}(\Omega^-_\Lambda)$.

(2) Consider

$$\Lambda_1=\binom{a_1,\ldots,a_m,a_{m+1}-1}{b_1,b_2,\ldots,b_m},\quad \Lambda_1'=\binom{c_1,c_2,\ldots,c_{m+1}}{d_1,\ldots,d_m,d_{m+1}+1}.$$

If $\Lambda_1\in\Theta_{\Lambda'}(\Omega_\Lambda^-)$, then we have $d_m>a_{m+1}$ and $a_{m+1}-1\geq d_{m+1}$, which implies that $d_m>d_{m+1}+1$ and $a_{m+1}\geq d_{m+1}+1$, i.e., $\Lambda_1'\in\Theta_\Lambda(\Omega_{\Lambda'}^+)$. One the other hand, $\Lambda_1'\in\Theta_\Lambda(\Omega_{\Lambda'}^+)$ also implies $\Lambda_1\in\Theta_{\Lambda'}(\Omega_\Lambda^-)$.

(3) Consider

$$\Lambda_1=\binom{a_1,\ldots,a_{m+1}}{b_1,\ldots,b_{i-1},b_i-1,b_{i+1},\ldots,b_m},\quad \Lambda_1'=\binom{c_1,\ldots,c_i,c_{i+1}+1,c_{i+2},\ldots,c_{m+1}}{d_1,d_2,\ldots,d_{m+1}}$$

for some $i=1,\ldots,m$. By the similar argument in (1) or (2), we see that $\Lambda_1\in\Theta_{\Lambda'}(\Omega_\Lambda^-)$ if and only if $\Lambda_1'\in\Theta_\Lambda(\Omega_{\Lambda'}^+)$.

(4) Because now $m'=m+1$, any $\Lambda_1\in\Omega_\Lambda^-$ of type (III') in Subsection 6.1 is not in $\Theta_{\Lambda'}(\Omega_\Lambda^-)$ by Lemma 2.14, moreover, any $\Lambda_1'\in\Omega_{\Lambda'}^+$ of type (III) or (IV) in Subsection 6.1 is not in $\Theta_\Lambda(\Omega_{\Lambda'}^+)$.

Hence the lemma is proved for the case that $m'=m+1$.

Next suppose that $m'=m$. Except for the situations similar to those considered above, there is another possibility, i.e., the case that $(a_{m+1},b_m)=(1,0)$ or $(0,1)$. Let

$$\Lambda_1=\binom{a_1-1,\ldots,a_m-1}{b_1-1,\ldots,b_{m-1}-1}.$$

Then we have $\Lambda_1\in\Omega_\Lambda^-$ and $(\Lambda_1,\Lambda')\in\mathscr{B}^+$. Note that now $d_m\geq a_{m+1}$ and $c_m\geq b_m$ since $(\Lambda,\Lambda')\in\mathscr{B}^+$. So $d_m\geq 1$ if $(a_{m+1},b_m)=(1,0)$; $c_m\geq 1$ if $(a_{m+1},b_m)=(0,1)$. Let

$$\Lambda_1'=\begin{cases}\binom{c_1+1,\ldots,c_m+1,0}{d_1+1,\ldots,d_m+1,1}, & \text{if } (a_{m+1},b_m)=(1,0);\\ \binom{c_1+1,\ldots,c_m+1,1}{d_1+1,\ldots,d_m+1,0}, & \text{if } (a_{m+1},b_m)=(0,1).\end{cases}$$

It is easy to check that $\Lambda_1'\in\Omega_{\Lambda'}^+$ and $(\Lambda,\Lambda_1')\in\mathscr{B}^+$. On the other hand, $\Lambda_1'\in\Theta_\Lambda(\Omega_{\Lambda'}^+)$ also implies that $\Lambda_1\in\Theta_{\Lambda'}(\Omega_\Lambda^-)$. Again, for $m'=m$, we still have an injective mapping from $\Theta_{\Lambda'}(\Omega_\Lambda^-)$ to $\Theta_\Lambda(\Omega_{\Lambda'}^+)$ given by $\Lambda_1\mapsto\Lambda_1'$ with only one extra element Λ_0'' not in the image. Hence the lemma is proved. □

Lemma 6.11. *Let Λ, Λ' be symbols of sizes $(m+1,m),(m',m')$ respectively, and suppose that $(\Lambda,\Lambda')\in\mathscr{B}^+$.*

(i) *If $m'=m+1$, then $\Theta_\Lambda(\Omega_{\Lambda'}^-)\neq\emptyset$.*

(ii) *If $m'=m$ and $\Theta_{\Lambda'}(\Omega_\Lambda^-)=\emptyset$, then $m=0$ and $\Lambda=\binom{0}{-}$.*

Proof. Write $\Lambda=\binom{a_1,\ldots,a_{m+1}}{b_1,\ldots,b_m}$, $\Lambda'=\binom{c_1,\ldots,c_{m'}}{d_1,\ldots,d_{m'}}$. First suppose that $m'=m+1$, and we define

$$\Lambda_1'=\begin{cases}\binom{c_1-1,\ldots,c_m-1}{d_1-1,\ldots,d_m-1}, & \text{if } (c_{m+1},d_{m+1})=(1,0) \text{ or } (0,1);\\ \binom{c_1,\ldots,c_m,c_{m+1}-1}{d_1,\ldots,d_{m+1}}, & \text{if } c_{m+1}\geq 1 \text{ and } (c_{m+1},d_{m+1})\neq(1,0);\\ \binom{c_1,\ldots,c_{m+1}}{d_1,\ldots,d_m,d_{m+1}-1}, & \text{if } d_{m+1}\geq 1 \text{ and } (c_{m+1},d_{m+1})\neq(0,1).\end{cases}$$

It is easy to see that $\Lambda_1'\in\Omega_{\Lambda'}^-$. Moreover, the assumption that $(\Lambda,\Lambda')\in\mathscr{B}^+$ implies that $(\Lambda,\Lambda_1')\in\mathscr{B}^+$ by Lemma 2.15. Thus (i) is proved.

Next suppose that $m' = m$ and $m \geq 1$, and we define

$$\Lambda_1 = \begin{cases} \begin{pmatrix} a_1 - 1, \dots, a_m - 1 \\ b_1 - 1, \dots, b_{m-1} - 1 \end{pmatrix}, & \text{if } (a_{m+1}, b_m) = (1,0) \text{ or } (0,1); \\ \begin{pmatrix} a_1, \dots, a_m, a_{m+1} - 1 \\ b_1, \dots, b_m \end{pmatrix}, & \text{if } a_{m+1} \geq 1 \text{ and } (a_{m+1}, b_m) \neq (1,0); \\ \begin{pmatrix} a_1, \dots, a_{m+1} \\ b_1, \dots, b_{m-1}, b_m - 1 \end{pmatrix}, & \text{if } b_m \geq 1 \text{ and } (a_{m+1}, b_m) \neq (0,1). \end{cases}$$

It is easy to see that $\Lambda_1 \in \Omega^-_\Lambda$. Moreover, the assumption that $(\Lambda, \Lambda') \in \mathscr{B}^+$ implies that $(\Lambda_1, \Lambda') \in \mathscr{B}^+$ by Lemma 2.15. Therefore we conclude that if $m \geq 1$, then $\Theta_{\Lambda'}(\Omega^-_\Lambda) \neq \emptyset$. Next suppose that $m' = m$ and $m = 0$, i.e., $\Lambda = \binom{a_1}{-}$ and $\Lambda' = \binom{-}{-}$ for some $a_1 \geq 0$. If $a_1 \geq 1$, then $\Lambda_1 = \binom{a_1 - 1}{-} \in \Theta_{\Lambda'}(\Omega^-_\Lambda)$ and hence $\Theta_{\Lambda'}(\Omega^-_\Lambda) \neq \emptyset$. ☐

Example 6.12. Let $\Lambda = \binom{a}{-}$ and $\Lambda'_1 = \binom{c}{d}$ be symbols of sizes $(1,0)$ and $(1,1)$ respectively such that $(\Lambda, \Lambda'_1) \in \mathscr{B}^+$. So now we have $a \geq d$ by Lemma 2.15. Now by the definition we have $\binom{c+1}{d}, \binom{c}{d+1} \in \Omega^+_{\Lambda'_1}$; moreover, we also have $\binom{c+1,1}{d+1,0} \in \Omega^+_{\Lambda'_1}$ if $c \neq 0$; and $\binom{c+1,0}{d+1,1} \in \Omega^+_{\Lambda'_1}$ if $d \neq 0$. Clearly, we have $\binom{c+1}{d} \in \Theta_\Lambda(\Omega^+_{\Lambda'_1})$. If $\Lambda' = \binom{c+1,1}{d+1,0}$ or $\binom{c+1,0}{d+1,1}$, then Λ' is of size $(2,2)$ and hence $(\Lambda, \Lambda'), (\Lambda, \Lambda'^{\mathrm{t}}) \notin \mathscr{B}^+$ by Lemma 2.14.

Suppose that $\Theta^*_\Lambda(\Omega^+_{\Lambda'_1}) \neq \emptyset$. Then we must have $\Lambda'' := \binom{c}{d+1} \in \Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$, i.e., $(\Lambda, \Lambda''^{\mathrm{t}}) \in \mathscr{B}^+$ and $(\Lambda, \Lambda'') \notin \mathscr{B}^+$, which imply $a \geq c$ and $a = d$. Let

$$\Lambda'_2 = \begin{cases} \binom{d+1}{c-1}, & \text{if } c \geq 1; \\ \binom{d}{c}, & \text{if } c = 0. \end{cases}$$

Clearly, $\Lambda''^{\mathrm{t}} \in \Omega^+_{\Lambda'_2}$ and $(\Lambda, \Lambda'_2) \in \mathscr{B}^+$.

(1) If $c \geq 1$, then $\Theta_\Lambda(\Omega^+_{\Lambda'_2}) = \left\{\binom{d+2}{c-1}, \binom{d+1}{c}\right\}$.
(2) If $c = 0$, then $d \geq 1$ (since we always consider reduced symbols) and hence $a = d \geq c+1$. Therefore $\Theta_\Lambda(\Omega^+_{\Lambda'_2}) = \left\{\binom{d+1}{c}, \binom{d}{c+1}\right\}$.

For both cases (1) and (2), we conclude that $\Theta^*_\Lambda(\Omega^+_{\Lambda'_2}) = \emptyset$.

Now we want to show that the above example is in fact a general phenomena:

Lemma 6.13. *Let Λ, Λ'_1 be symbols of sizes $(m+1, m), (m+1, m+1)$ respectively such that $(\Lambda, \Lambda'_1) \in \mathscr{B}^+$. Suppose that $\Lambda'' \in \Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$. Then there exists a symbol Λ'_2 such that $\Lambda''^{\mathrm{t}} \in \Omega^+_{\Lambda'_2}$, $(\Lambda, \Lambda'_2) \in \mathscr{B}^+$, and $\Theta^*_\Lambda(\Omega^+_{\Lambda'_2}) = \emptyset$.*

Proof. Write $\Lambda = \binom{a_1, \dots, a_{m+1}}{b_1, \dots, b_m}$, $\Lambda'_1 = \binom{c_1, \dots, c_{m+1}}{d_1, \dots, d_{m+1}}$. Because now $(\Lambda, \Lambda'_1) \in \mathscr{B}^+$, by Lemma 2.15 we have

$$\begin{aligned} a_1 \geq d_1 > a_2 \geq d_2 > \cdots > a_{m+1} \geq d_{m+1}, \\ c_1 > b_1 \geq c_2 > b_2 \geq \cdots \geq c_m > b_m \geq c_{m+1}. \end{aligned} \tag{6.14}$$

Let $\Lambda'' \in \Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$, i.e, $\Lambda'' \in \Omega^+_{\Lambda'_1}$, $(\Lambda, \Lambda''^{\mathrm{t}}) \in \mathscr{B}^+$, and $(\Lambda, \Lambda'') \notin \mathscr{B}^+$. If Λ'', as an element of $\Omega^+_{\Lambda'_1}$, is of type (III) or (IV) in Subsection 6.2, then $\Lambda'', \Lambda''^{\mathrm{t}}$ are of size $(m+2, m+2)$ and hence $(\Lambda, \Lambda''^{\mathrm{t}}) \notin \mathscr{B}^+$ by Lemma 2.14. Therefore, Λ'' must be of type (I) or (II):

(1) Suppose that

$$\Lambda'' = \begin{pmatrix} c_1, \dots, c_{k-1}, c_k + 1, c_{k+1}, \dots, c_{m+1} \\ d_1, \dots, d_{m+1} \end{pmatrix}$$

for some k such that $c_{k-1} > c_k + 1$. If $k = 1$, then $(\Lambda, \Lambda'_1) \in \mathscr{B}^+$ implies $(\Lambda, \Lambda'') \in \mathscr{B}^+$ and we get a contradiction. So we must have $k \geq 2$. Because now $(\Lambda, \Lambda'') \notin \mathscr{B}^+$ and $(\Lambda, \Lambda''^{\mathrm{t}}) \in \mathscr{B}^+$, we have

- $b_{k-1} = c_k$
- $a_i \geq c_i > a_{i+1}$ for $i \neq k$ and $a_k \geq c_k + 1 > a_{k+1}$; $d_j > b_j \geq d_{j+1}$ for each j.

Then $d_{k-1} > a_k \geq c_k + 1 = b_{k-1} + 1 \geq d_k + 1$ and hence

$$\Lambda'_2 := \binom{d_1, \ldots, d_{k-2}, d_{k-1} - 1, d_k, \ldots, d_{m+1}}{c_1, \ldots, c_{k-1}, c_k + 1, c_{k+1}, \ldots, c_{m+1}}$$

is a symbol. It is easy to see that $\Lambda''^{\mathrm{t}} \in \Omega^+_{\Lambda'_2}$ and $(\Lambda, \Lambda'_2) \in \mathscr{B}^+$. Moreover, for any

$$\Lambda' = \binom{c'_1, \ldots, c'_{m+1}}{d'_1, \ldots, d'_{m+1}} \in \Omega^+_{\Lambda'_2},$$

we have $d'_k \geq c_k + 1$. Because now $b_{k-1} = c_k < d'_k$, we have $(\Lambda, \Lambda'^{\mathrm{t}}) \notin \mathscr{B}^+$. Therefore $\Theta^*_\Lambda(\Omega^+_{\Lambda'_2}) = \emptyset$.

(2) Suppose that

$$\Lambda'' = \binom{c_1, \ldots, c_{m+1}}{d_1, \ldots, d_{l-1}, d_l + 1, d_{l+1}, \ldots, d_{m+1}}$$

for some $l \geq 2$ such that $d_{l-1} > d_l + 1$. Because now $(\Lambda, \Lambda'') \notin \mathscr{B}^+$ and $(\Lambda, \Lambda''^{\mathrm{t}}) \in \mathscr{B}^+$, we have

- $a_l = d_l$;
- $a_i \geq c_i > a_{i+1}$ for each i; $b_j \geq d_{j+1} > b_{j+1}$ for $j \neq l-1$ and $b_{l-1} \geq d_l + 1 > b_l$.

Then $c_{l-1} > b_{l-1} \geq d_l + 1 = a_l + 1 \geq c_l + 1$ and hence

$$\Lambda'_2 := \binom{d_1, \ldots, d_{l-1}, d_l + 1, d_{l+1}, \ldots, d_{m+1}}{c_1, \ldots, c_{l-2}, c_{l-1} - 1, c_l, \ldots, c_{m+1}}$$

is a symbol. It is easy to see that $\Lambda''^{\mathrm{t}} \in \Omega^+_{\Lambda'_2}$ and $(\Lambda, \Lambda'_2) \in \mathscr{B}^+$. Moreover, by the same argument in (1), we can see that $\Theta^*_\Lambda(\Omega^+_{\Lambda'_2}) = \emptyset$.

(3) Suppose that

$$\Lambda'' = \binom{c_1, \ldots, c_{m+1}}{d_1 + 1, d_2, \ldots, d_{m+1}}.$$

Then we have

- $a_1 = d_1$
- $a_i \geq c_i > a_{i+1}$ and $b_i \geq d_{i+1} > b_{i+1}$ for each i, and $d_1 + 1 > b_1$.

If $m = 0$, this is just Example 6.12. So now we assume that $m \geq 1$. Note that c_{m+1}, d_{m+1} can not both be 0. We can define

$$\Lambda'_2 = \begin{cases} \binom{d_1+1, d_2, \ldots, d_m, d_{m+1}-1}{c_1, \ldots, c_{m+1}}, & \text{if } d_{m+1} \geq 1 \text{ and } (c_{m+1}, d_{m+1}) \neq (0,1); \\ \binom{d_1+1, d_2, \ldots, d_{m+1}}{c_1, \ldots, c_m, c_{m+1}-1}, & \text{if } c_{m+1} \geq 1 \text{ and } (c_{m+1}, d_{m+1}) \neq (1,0); \\ \binom{d_1, d_2-1 \ldots, d_m-1}{c_1-1, \ldots, c_m-1}, & \text{if } (c_{m+1}, d_{m+1}) = (1,0) \text{ or } (0,1). \end{cases}$$

For above three possible situations, it is easy to see that $\Lambda''^{\mathrm{t}} \in \Omega^+_{\Lambda'_2}$ and $(\Lambda, \Lambda'_2) \in \mathscr{B}^+$. By the similar argument in (1) we can see that $\Theta^*_\Lambda(\Omega^+_{\Lambda'_2}) = \emptyset$.

□

Example 6.15. Let $\Lambda=\binom{8,5,1}{6,2}$ and $\Lambda'_1=\binom{7,4,1}{8,3,0}$. Then $(\Lambda,\Lambda'_1)\in\mathscr{B}^+$,

$$\Theta_\Lambda(\Omega^+_{\Lambda'_1})=\left\{\binom{8,4,1}{8,3,0},\binom{7,5,1}{8,3,0},\binom{7,4,2}{8,3,0},\binom{7,4,1}{8,4,0},\binom{7,4,1}{8,3,1}\right\}\quad\text{and}\quad\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})=\left\{\binom{7,4,1}{9,3,0}\right\}.$$

Now $\Lambda''=\binom{7,4,1}{9,3,0}\in\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$. So let $\Lambda'_2=\binom{8,2}{6,3}$ as given in (3) of the proof of the previous lemma, and we have $\Lambda''^{\mathrm{t}}\in\Omega^+_{\Lambda'_2}$, $(\Lambda,\Lambda'_2)\in\mathscr{B}^+$,

$$\Theta_\Lambda(\Omega^+_{\Lambda'_2})=\Omega^+_{\Lambda'_2}=\left\{\binom{9,2}{6,3},\binom{8,3}{6,3},\binom{8,2}{7,3},\binom{8,2}{6,4},\binom{9.3,1}{7,4,0},\binom{9,3,0}{7,4,1}\right\}\quad\text{and}\quad\Theta^*_\Lambda(\Omega^+_{\Lambda'_2})=\emptyset.$$

Lemma 6.16. *Let Λ,Λ'_1 be symbols of sizes $(m+1,m),(m,m)$ respectively such that $m\geq 1$ and $(\Lambda,\Lambda'_1)\in\mathscr{B}^+$. Suppose that $\Lambda''\in\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$. Then there exists a symbol Λ'_2 such that $\Lambda''^{\mathrm{t}}\in\Omega^+_{\Lambda'_2}$, $(\Lambda,\Lambda'_2)\in\mathscr{B}^+$ and $\Theta^*_\Lambda(\Omega^+_{\Lambda'_2})=\emptyset$.*

Proof. Write $\Lambda=\binom{a_1,\dots,a_{m+1}}{b_1,\dots,b_m}$, $\Lambda'_1=\binom{c_1,\dots,c_m}{d_1,\dots,d_m}$. Because now $(\Lambda,\Lambda'_1)\in\mathscr{B}^+$, by Lemma 2.15 we have

$$\begin{aligned}&a_1>d_1\geq a_2>d_2\geq\cdots\geq a_m>d_m\geq a_{m+1},\\&c_1\geq b_1>c_2\geq b_2>\cdots>c_m\geq b_m.\end{aligned}\tag{6.17}$$

Let $\Lambda''\in\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$.

(1) Suppose that

$$\Lambda''=\begin{pmatrix}c_1,\dots,c_{k-1},c_k+1,c_{k+1},\dots,c_m\\d_1,\dots,d_m\end{pmatrix}$$

for some k such that $c_{k-1}>c_k+1$. The proof for this case is similar to (1) in the proof of Lemma 6.13.

(2) Suppose that

$$\Lambda''=\begin{pmatrix}c_1,\dots,c_m\\d_1,\dots,d_{l-1},d_l+1,d_{l+1},\dots,d_m\end{pmatrix}$$

for some l such that $d_{l-1}>d_l+1$. The proof for this case is similar to (2) in the proof of Lemma 6.13.

(3) Suppose that

$$\Lambda''=\begin{pmatrix}c_1+1,\dots,c_m+1,1\\d_1+1,\dots,d_m+1,0\end{pmatrix}.$$

So we have $c_m\geq 1$. The assumptions $(\Lambda,\Lambda'')\notin\mathscr{B}^+$ and $(\Lambda,\Lambda''^{\mathrm{t}})\in\mathscr{B}^+$ imply that

- $b_m=0$;
- $a_i>c_i\geq a_{i+1}$ for each i; and $d_j\geq b_j>d_{j+1}$ for $j=1,\dots,m-1$.

Because Λ is reduced and now $b_m=0$, we must have $a_{m+1}\neq 0$. Hence $d_m\geq a_{m+1}\geq 1$. Let

$$\Lambda'_2=\begin{pmatrix}d_1+1,\dots,d_{m-1}+1,d_m,0\\c_1+1,\dots,c_m+1,1\end{pmatrix}.$$

It is easy to see that $\Lambda''^{\mathrm{t}}\in\Omega^+_{\Lambda'_2}$ and $(\Lambda,\Lambda'_2)\in\mathscr{B}^+$. Moreover, for any

$$\Lambda'=\begin{pmatrix}c'_1,\dots,c'_{m+1}\\d'_1,\dots,d'_{m+1}\end{pmatrix}\in\Omega^+_{\Lambda'_2},$$

we have $d'_{m+1}\geq 1$. Because now $b_m=0<d'_{m+1}$, we have $(\Lambda,\Lambda'^{\mathrm{t}})\notin\mathscr{B}^+$. Therefore $\Theta^*_\Lambda(\Omega^+_{\Lambda'_2})=\emptyset$.

(4) Suppose that

$$\Lambda''=\binom{c_1+1,\dots,c_m+1,0}{d_1+1,\dots,d_m+1,1}.$$

The proof is similar to (3) above.

□

Lemma 6.18. *Let Λ_1,Λ' be symbols of sizes $(m+1,m),(m',m')$ respectively, and suppose that $(\Lambda_1,\Lambda')\in\mathscr{B}^+$. Suppose that $\Lambda''\in\Theta^*_{\Lambda'}(\Omega^+_{\Lambda_1})$. Then there exists a symbol Λ_2 such that $\Lambda''\in\Omega^+_{\Lambda_2}$, $(\Lambda_2,\Lambda')\in\mathscr{B}^+$, and $\Theta^*_{\Lambda'^{\mathrm{t}}}(\Omega^+_{\Lambda_2})=\emptyset$.*

Proof. We know that $m'=m,m+1$. Then the proof is similar to those of Lemma 6.13 and Lemma 6.16. □

6.3. Branching rule and Howe correspondence. For $\rho\in\mathscr{E}(\mathrm{Sp}_{2n}(q))$, let Ω^+_ρ denote the set of irreducible constituents of the parabolic induced character $\mathrm{Ind}_{P_n}^{\mathrm{Sp}_{2(n+1)}(q)}(\rho\otimes\mathbf{1})$, then, if $n\geq 1$, we also define

$$\Omega^-_\rho=\{\,\rho_1\in\mathscr{E}(\mathrm{Sp}_{2(n-1)}(q))\mid\rho\in\Omega^+_{\rho_1}\,\}$$

where P_n is a parabolic subgroup of $\mathrm{Sp}_{2(n+1)}(q)$ whose Levi factor is $\mathrm{Sp}_{2n}(q)\times\mathrm{GL}_1(q)$. The analogous definition also applies to $\rho'\in\mathscr{E}(\mathrm{O}^+_{2n'}(q))$.

For $\rho\in\mathscr{E}(\mathrm{Sp}_{2n}(q))$ and a set of irreducible characters $\Omega'\subset\mathscr{E}(\mathrm{O}^+_{2n'}(q))$, we define

$$\Theta_\rho(\Omega')=\{\,\rho'\in\Omega'\mid(\rho,\rho')\text{ occurs in the Howe correspondence}\,\}.$$

Similarly, for $\rho'\in\mathscr{E}(\mathrm{O}^+_{2n'}(q))$ and a set $\Omega\subset\mathscr{E}(\mathrm{Sp}_{2n}(q))$, we define

$$\Theta_{\rho'}(\Omega)=\{\,\rho\in\Omega\mid(\rho,\rho')\text{ occurs in the Howe correspondence}\,\}.$$

The following lemma is extracted from the proof of [AMR96] théorème 3.7:

Lemma 6.19. *Let $\rho\in\mathscr{E}(\mathrm{Sp}_{2n}(q))$ and $\rho'\in\mathscr{E}(\mathrm{O}^+_{2n'}(q))$ such that (ρ,ρ') occurs in the Howe correspondence. Then*

$$|\Theta_\rho(\Omega^+_{\rho'})|=1+|\Theta_{\rho'}(\Omega^-_\rho)|\quad\text{and}\quad|\Theta_{\rho'}(\Omega^+_\rho)|=1+|\Theta_\rho(\Omega^-_{\rho'})|.$$

Proof. Suppose that $\rho\in\mathscr{E}(\mathrm{Sp}_{2n}(q))$, $\rho'\in\mathscr{E}(\mathrm{O}^+_{2n'}(q))$ and (ρ,ρ') occurs in the Howe correspondence. Note that the Howe correspondence for symplctic/orthogonal dual pair is of multiplicity one (*cf.* [MVW87] p.97). By Frobenius reciprocity, we have

$$\begin{aligned}|\Theta_\rho(\Omega^+_{\rho'})|&=\left\langle\omega_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2(n'+1)}},\rho\otimes R^{\mathrm{O}^+_{2(n'+1)}(q)}_{\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)}(\rho'\otimes 1)\right\rangle\\&=\left\langle(1\otimes{}^*R^{\mathrm{O}^+_{2(n'+1)}(q)}_{\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)})(\omega_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2(n'+1)}}),\rho\otimes\rho'\otimes 1\right\rangle.\end{aligned}$$

Here $R^{\mathrm{O}^+_{2(n'+1)}(q)}_{\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)}$ denotes the Lusztig induction, and now it is just the usual parabolic induction. From [AMR96] p.382, we know that

$$\begin{aligned}&\left(1\otimes{}^*R^{\mathrm{O}^+_{2(n'+1)}(q)}_{\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)}\right)(\omega_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2(n'+1)}})\\&\qquad=\omega_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}\otimes\mathbf{1}+R^{\mathrm{Sp}_{2n}(q)\times\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)}_{\mathrm{Sp}_{2(n-1)}(q)\times\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)\times\mathrm{GL}_1(q)}(\omega_{\mathrm{Sp}_{2(n-1)},\mathrm{O}^+_{2n'}}\otimes R_{\mathrm{GL}_1})\end{aligned}$$

where R_{GL_1} denotes the character of the representation of $\mathrm{GL}_1(q)\times\mathrm{GL}_1(q)$ on $\mathbb{C}(\mathrm{GL}_1(q))$. By our assumption, we have

$$\langle\omega_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}\otimes\mathbf{1},\rho\otimes\rho'\otimes\mathbf{1}\rangle=1.$$

Moreover, by Frobenius reciprocity again, we have

$$\begin{aligned}\Big\langle R^{\mathrm{Sp}_{2n}(q)\times\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)}_{\mathrm{Sp}_{2(n-1)}(q)\times\mathrm{O}^+_{2n'}(q)\times\mathrm{GL}_1(q)\times\mathrm{GL}_1(q)}(\omega_{\mathrm{Sp}_{2(n-1)},\mathrm{O}^+_{2n'}}\otimes R_{\mathrm{GL}_1}),\rho\otimes\rho'\otimes\mathbf{1}\Big\rangle \\ =\Big\langle \omega_{\mathrm{Sp}_{2(n-1)},\mathrm{O}^+_{2n'}},{}^*R^{\mathrm{Sp}_{2n}(q)}_{\mathrm{Sp}_{2(n-1)}(q)}(\rho)\otimes\rho'\Big\rangle=|\Theta_{\rho'}(\Omega^-_{\rho})|.\end{aligned}$$

Hence the first equality is proved. The proof of the second equality is similar. □

Proposition 6.20. *Let* $(\mathbf{G},\mathbf{G}')=(\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'})$, $\rho_\Lambda\in\mathscr{E}(G)_1$, $\rho_{\Lambda'}\in\mathscr{E}(G')_1$ *for some symbols* Λ,Λ' *of defects* $1,0$ *respectively. Then* $\rho_\Lambda\otimes\rho_{\Lambda'}$ *occurs in* $\omega_{\mathbf{G},\mathbf{G}',1}$ *if and only if* $(\Lambda,\Lambda')\in\mathscr{B}_{\mathbf{G},\mathbf{G}'}$.

Proof. We are going to prove the proposition by induction on $n+n'$. First we consider the cases that $n'=0$. The Howe correspondence for the dual pair $(\mathrm{Sp}_{2n}(q),\mathrm{O}^+_0(q))$ is given by $\mathbf{1}_{\mathrm{Sp}_{2n}}\otimes\mathbf{1}_{\mathrm{O}^+_0}$ and we know that $\mathbf{1}_{\mathrm{Sp}_{2n}}=\rho_{\binom{n}{-}}$, $\mathbf{1}_{\mathrm{O}^+_0}=\rho_{\binom{-}{-}}$. It is clear that

$$\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_0}=\left\{\left(\tbinom{n}{-},\tbinom{-}{-}\right)\right\}.$$

Hence the proposition holds for $n'=0$ (and any non-negative integer n). Next we consider the cases that $n=0$. The Howe correspondence for the dual pair $(\mathrm{Sp}_0(q),\mathrm{O}^+_{2n'}(q))$ is given by $\mathbf{1}_{\mathrm{Sp}_0}\otimes\mathbf{1}_{\mathrm{O}^+_{2n'}}$, and we know that $\mathbf{1}_{\mathrm{Sp}_0}=\rho_{\binom{0}{-}}$, $\mathbf{1}_{\mathrm{O}^+_{2n'}}=\rho_{\binom{n'}{0}}$. It is clear that

$$\mathscr{B}_{\mathrm{Sp}_0,\mathrm{O}^+_{2n'}}=\left\{\left(\tbinom{0}{-},\tbinom{n'}{0}\right)\right\}.$$

Hence the proposition holds for $n=0$ (and any non-negative integer n').

Now suppose that $(\Lambda,\Lambda')\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$ and $\mathrm{def}(\Lambda')=0$ for some positive n,n', and write $\Lambda=\binom{a_1,\ldots,a_{m+1}}{b_1,\ldots,b_m}$, $\Lambda'=\binom{c_1,\ldots,c_{m'}}{d_1,\ldots,d_{m'}}$ for some nonnegative integers m,m'. It is known that $m'=m,m+1$ by Lemma 2.14.

(1) Suppose that $m'=m+1$. By Lemma 6.11, we know that $\Theta_\Lambda(\Omega^-_{\Lambda'})\neq\emptyset$. Let $\Lambda'_1\in\Theta_\Lambda(\Omega^-_{\Lambda'})$, i.e., $\Lambda'\in\Omega^+_{\Lambda'_1}$ and $(\Lambda,\Lambda'_1)\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2(n'-1)}}$. Then, by induction hypothesis, $(\rho_\Lambda,\rho_{\Lambda'_1})$ occurs in the Howe correspondence and we have

$$|\Theta_{\rho_{\Lambda'_1}}(\Omega^-_{\rho_\Lambda})|=|\Theta_{\Lambda'_1}(\Omega^-_\Lambda)|. \tag{6.21}$$

Now by Lemma 6.19 and Lemma 6.10, we have the equality

$$|\Theta_{\rho_\Lambda}(\Omega^+_{\rho_{\Lambda'_1}})|=|\Theta_\Lambda(\Omega^+_{\Lambda'_1})|. \tag{6.22}$$

(a) Suppose that Λ'_1 is of type (I') or (II') in Subsection 6.1. Then Λ'_1 is of size $(m+1,m+1)$.

(i) First suppose that $\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})=\emptyset$. For any $\Lambda''\in\Omega^+_{\Lambda'_1}$, by Lemma 6.8, if $(\rho_\Lambda,\rho_{\Lambda''})$ occurs in the Howe correspondence, then we have $(\Lambda,\Lambda'')\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$.

(ii) Next suppose that $\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})\neq\emptyset$. For any $\Lambda'''\in\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$, we know that $(\Lambda,\Lambda'''^{\mathrm{t}})\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$ and $(\Lambda,\Lambda''')\notin\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$ by the definition. By Lemma 6.13, we can find $\Lambda'_2\in\mathscr{S}_{\mathrm{O}^+_{2(n'-1)}}$ such that $\Lambda'''^{\mathrm{t}}\in\Omega^+_{\Lambda'_2}$, $(\Lambda,\Lambda'_2)\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2(n'-1)}}$ and $\Theta^*_\Lambda(\Omega^+_{\Lambda'_2})=\emptyset$. By (i), with Λ'_1 replaced by Λ'_2 and the fact $(\Lambda,\Lambda''')\notin\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$, we see that $(\rho_\Lambda,\rho_{\Lambda'''})$ does not occur in the Howe correspondence. In short, we have shown that for any $\Lambda'''\in\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$, $(\rho_\Lambda,\rho_{\Lambda'''})$ does not occur in the Howe correspondence. So now for any $\Lambda''\in\Omega^+_{\Lambda'_1}$, if $(\rho_\Lambda,\rho_{\Lambda''})$ occurs in the Howe correspondence and $(\Lambda,\Lambda'')\notin\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$, we must have $(\Lambda,\Lambda''^{\mathrm{t}})\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$, i.e., $\Lambda''\in\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})$ and we get a contradiction. Therefore, for any $\Lambda''\in\Omega^+_{\Lambda'_1}$, the occurrence of $(\rho_\Lambda,\rho_{\Lambda''})$ in the Howe correspondence implies that $(\Lambda,\Lambda'')\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$.

Hence for both (i) and (ii), by (6.22), the condition $\Lambda''\in\Theta_\Lambda(\Omega^+_{\Lambda'_1})$ also implies that $(\rho_\Lambda,\rho_{\Lambda''})$ occurs in the Howe correspondence. In particular, since $\Lambda'\in\Omega^+_{\Lambda'_1}$ and $(\Lambda,\Lambda')\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_{2n'}}$, we have that $(\rho_\Lambda,\rho_{\Lambda'})$ occurs in the Howe correspondence.

(b) Suppose that Λ'_1 is of type (III') in Subsection 6.1. Then Λ'_1 is of size (m,m).
   (i) First suppose that $\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})=\emptyset$. The proof is exactly the same as in (a.i).
   (ii) Suppose that $\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})\neq\emptyset$. First suppose that $m=0$. This means that $\Lambda=\binom{n}{-}$ for some $n\geq 0$, $\Lambda'_1=\binom{-}{-}$, $\Lambda'=\binom{1}{0}$ and $\Theta^*_\Lambda(\Omega^+_{\Lambda'_1})=\{\binom{0}{1}\}$. It is well known that $\rho_{\binom{n}{-}}=\mathbf{1}_{\mathrm{Sp}_{2n}}$, $\rho_{\binom{1}{0}}=\mathbf{1}_{\mathrm{O}^+_2}$, $(\mathbf{1}_{\mathrm{Sp}_{2n}},\mathbf{1}_{\mathrm{O}^+_2})$ occurs in the Howe correspondence, and $(\binom{n}{-},\binom{1}{0})\in\mathscr{B}_{\mathrm{Sp}_{2n},\mathrm{O}^+_2}$, i.e., the proposition is true for this case. Next suppose that $m\geq 1$. Then the proof is similar to that of (a.ii). The only difference is that we need to apply Lemma 6.16 instead of Lemma 6.13.

(2) Suppose that $m'=m$. Since the case that $m=0$ is just the case that $n'=0$, we assume that $m\geq 1$. By Lemma 6.11, we know that $\Theta_{\Lambda'}(\Omega^-_\Lambda)\neq\emptyset$. Let $\Lambda_1\in\Theta_{\Lambda'}(\Omega^-_\Lambda)$, i.e., $\Lambda\in\Omega^+_{\Lambda_1}$ and $(\Lambda_1,\Lambda')\in\mathscr{B}_{\mathrm{Sp}_{2(n-1)},\mathrm{O}^+_{2n}}$. Then $(\rho_{\Lambda_1},\rho_{\Lambda'})$ occurs in the Howe correspondence by induction hypothesis. The remaining proof is similar to that of (1). Note that we need to apply Lemma 6.18 instead of Lemma 6.13 and Lemma 6.16.

□

Proof of Theorem 1.8. The theorem is just the combination of Proposition 6.4 and Proposition 6.20. □

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